PHY651: PARTICLE INTERACTIONS

3. **Particle Interactions:** **[7 hours]**

   3.1 Introduction to Feynman diagrams, Feynman rules

   3.2 Particle exchange: forces and potentials, calculation of amplitudes

   3.3 Range of forces,

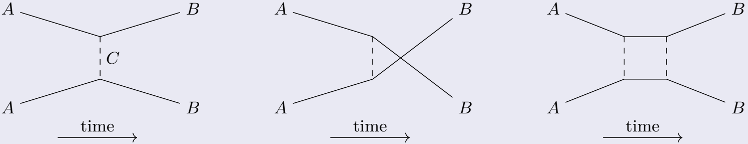
   3.4 Yukawa potential,

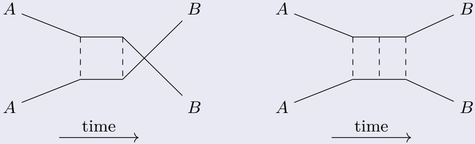
   3.5 Observable quantities: cross sections and decay rates: amplitudes, cross-sections and unstable states.

PARTICLE INTERACTIONS

LOOP DIAGRAMS

- As I explained already, there can be infinite numbers of Feynman diagrams that contribute to a process/interaction.
- For most of the purposes, considering only the low order diagrams suffice because generally the contibution from the higher order diagrams are much smaller.
- Here are some examples for the same simple interaction $A + A \to B + B$ for a scalar field we considered in the last lecture:

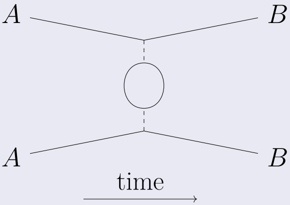




And so on ...

Loop Diagrams

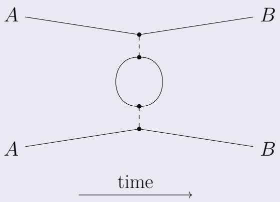
- There is another class of valid Feynman diagrams that involves loops, which can be formed, for example, by a virtual particle “decaying” into virtual particles that again annihilate to create another virtual particle. We will next deal with such example for $A + A \to B + B$.



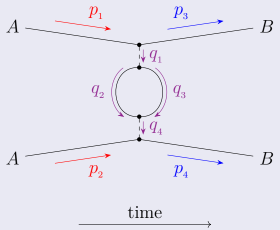
- We next try to evaluate $\mathcal{M}$ for the above diagram.

FEYNMAN RULES EXAMPLE: LOOP DIAGRAM

0. **Draw the Feynman Diagram:**

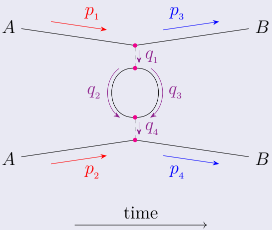


1. **Notation/Labeling:**



FEYNMAN RULES EXAMPLE: LOOP DIAGRAM

2. **Write Down the Coupling Constant for each vertex:**



$$(-ig) \times (-ig) \times (-ig) \times (-ig) = (-ig)^4 = g^4$$

3. **Write down the propagator for each internal line:**

   We have four internal lines and the corresponding propagators are:

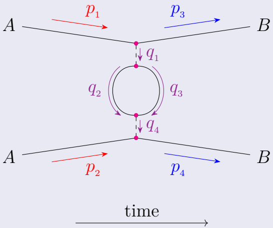
$$\frac{i}{q_1^2 - m_C^2 c^2}\,,\ \frac{i}{q_4^2 - m_C^2 c^2}\,,\ \frac{i}{q_2^2 - m_A^2 c^2} \quad \text{and} \quad \frac{i}{q_3^2 - m_B^2 c^2}$$

   Thus the integrand so far is:

$$g^4 \frac{i}{q_1^2 - m_C^2 c^2} \frac{i}{q_2^2 - m_A^2 c^2} \frac{i}{q_3^2 - m_B^2 c^2} \frac{i}{q_4^2 - m_C^2 c^2}\,,$$

Feynman Rules Example: Loop diagram

4. **Momentum conservation: Introduce a $\delta-$function for momentum conservation at each vertex:**



We have four vertices and so we need to write four $\delta-$functions

$$(2\pi)^4\delta^{(4)}(p_1 - q_1 - p_3)$$
$$(2\pi)^4\delta^{(4)}(q_1 - q_2 - q_3)$$
$$(2\pi)^4\delta^{(4)}(q_2 + q_3 - q_4)$$
$$(2\pi)^4\delta^{(4)}(p_2 + q_4 - p_4)$$

Thus the integrand becomes:

$$(2\pi)^{16} g^4 \frac{\delta^{(4)}(p_1 - q_1 - p_3)\delta^{(4)}(q_1 - q_2 - q_3)\delta^{(4)}(q_2 + q_3 - q_4)\delta^{(4)}(p_2 + q_4 - p_4)}{(q_1^2 - m_C^2 c^2)(q_2^2 - m_A^2 c^2)(q_3^2 - m_B^2 c^2)(q_4^2 - m_C^2 c^2)}$$

Particle Interactions

Feynman Rules Example: Loop diagram

5. **Integrate over all internal momenta q_i:**

$$\int \cancel{(2\pi)^{16}} g^4 \frac{\delta^{(4)}(p_1 - q_1 - p_3)\delta^{(4)}(q_1 - q_2 - q_3)\delta^{(4)}(q_2 + q_3 - q_4)\delta^{(4)}(p_2 + q_4 - p_4)}{(q_1^2 - m_C^2 c^2)(q_2^2 - m_A^2 c^2)(q_3^2 - m_B^2 c^2)(q_4^2 - m_C^2 c^2)} \frac{d^4 q_1}{\cancel{(2\pi)^4}} \frac{d^4 q_2}{\cancel{(2\pi)^4}} \frac{d^4 q_3}{\cancel{(2\pi)^4}} \frac{d^4 q_4}{\cancel{(2\pi)^4}}$$

From the first $\delta-$ everything except $q_1 = p_1 - p_3$ are left out and so for everything except $q_4 = p_4 - p_2$ from the last one. So we are left with:

$$\int g^4 \frac{\delta^{(4)}(p_1 - p_3 - q_2 - q_3)\delta^{(4)}(q_2 + q_3 - (p_4 - p_2))}{[(p_1 - p_3)^2 - m_C^2 c^2](q_2^2 - m_A^2 c^2)(q_3^2 - m_B^2 c^2)[(p_4 - p_2)^2 - m_C^2 c^2]} d^4 q_2 d^4 q_3$$

Further integrating w.r.t. q_2 keeping in mind that due to $\delta^{(4)}(q_2 + q_3 - (p_4 - p_2))$, everything except $q_2 = p_4 - p_2 - q_3$ is left out. So we now have

$$g^4 \int \frac{\delta^{(4)}(p_1 - p_3 - p_4 + p_2 + q_3 - q_3) d^4 q_3}{[(p_1 - p_3)^2 - m_C^2 c^2][(p_4 - p_2 - q_3)^2 - m_A^2 c^2](q_3^2 - m_B^2 c^2)[(p_4 - p_2)^2 - m_C^2 c^2]}$$

FEYNMAN RULES EXAMPLE: LOOP DIAGRAM

Thus we obtain

$$g^4 \int \frac{\delta^{(4)}(p_1 + p_2 - p_3 - p_4) d^4 q_3}{[(p_1 - p_3)^2 - m_C^2 c^2][(p_4 - p_2 - q_3)^2 - m_A^2 c^2](q_3^2 - m_B^2 c^2)[(p_4 - p_2)^2 - m_C^2 c^2]}$$

4. **Cancel the $\delta-$function term $(2\pi)^4 \delta^{(4)}(P_f - P_i)$ to get $i\mathcal{M}$:**

Remember: $S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathcal{M}$

$$i\mathcal{M} =$$

$$\left(\frac{g}{2\pi}\right)^4 \int \frac{d^4 q_3}{[(p_1 - p_3)^2 - m_C^2 c^2][(p_1 - p_3 - q_3)^2 - m_A^2 c^2](q_3^2 - m_B^2 c^2)[(p_4 - p_2)^2 - m_C^2 c^2]}$$

What is the nature of $\mathcal{M}$ at large q_3?
Think of $d^4 q_3$ as the 4D volume element: $q^3 dq d\Omega$ analogus to $r^2 dr \sin\theta d\theta d\phi$ in 3D.
At large q_3, $\mathcal{M} \propto \int d^4 q_3 / q_4^3$. Thus

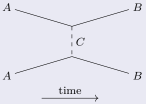
$$\mathcal{M} \propto \int_0^\infty \frac{q_3^3}{q_3^4} dq_3 = \int_0^\infty \frac{dq_3}{q_3} dq_3 = \infty$$

RENORMALIZATION

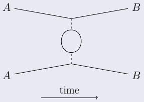
1. We see an example of how contribution for some Feynman diagrams diverge. The loop diagram example we considered diverges as $q \to \infty$, and such a divergence is called ultra-violet divergence. Divergences can sometimes occur as $q \to 0$, which fall under the category of infra-red divergence.
2. In a Feynman diagram, one can access / know about how "bad" the divergence is, by having a look at the quantity

$$D = 4L - 2I \tag{1}$$

where L is the number of loops and I is the number of propagators (internal lines). If $D < 0$, the diagram converges and if $D \geq$ it does not. As such, for



$D = 4 \times 0 - 2 \times 1 = -2$ and thus the integral is finite, and for



$D = 4 \times 1 - 2 \times 1 = 2$ and the diagram diverges.

Renormalization

3. Now, let's try to write D in another form. Let's consider an $n-$ order Feynman diagram (i.e., having n vertices), with E external lines, I internal lines and L loops. Out of I internal momenta, there are n delta functions for momentum conservation at n vertices and one more momentum conservation relation for total momentum conservation to be satisfied. Thus, the number of independent integration momenta (and hence the number of loops) in such a diagram is:

$$L = I - n + 1 \tag{2}$$

4. If we consider ϕ^r-theory, there are r lines at each vertex which may be internal lines or external lines. So we will have $r \times n$ lines. But 2 vertices share one internal line. Thus we will have,

$$r\,n = E + 2I \tag{3}$$

5. From equations 1, 2 and 3,

$$D = 4 - E + n(r-4) \tag{4}$$

RENORMALIZATION

6. We next look at a few scenarios:
   - For $r = 2$, the field is free field that does not have a vertex or interaction.
   - For $r = 3$, $D = 4 - E - n$. We will have only a small finite number of diagrams with $E + n < 4$ i.e., $D \geq 0$. Such fields are said to be super-normalizable.
   - For $r = 4$, $D = 4 - E$. We will have $D \geq 0$ for diagrams with $E \geq 4$ and thus have a finite number of diagrams which diverge.
   - For $r > 4$, $D = 4 - E + n \times$ *positive integer*. So $D \geq 0$ for infinitely large number of cases, as for a fixed set of external lines (E) increasing n increases D. Such a theory / field is nonrenormalizable.
7. The unwanted / nonphysical singularities are taken care of by the process of regularization and renormalization.
   - **Regularization:** You introduce some parameter(s) to make the divergent quantities in the integral finite. You will obtain the final answer for the “bare quantities” of the integral as a function of these parameters.
   - **Renormalization:** You apply certain conditions so that the physical quantities you want to obtain depend only on other physical quantities (not “bare quantities”).

We will discuss about these methods in later lectures

Particle Interactions

Feynman Rules for QED

For Quantum Electrodynamics, we are essentially dealing with interactions between charged particles which are mediated by photons. You can follow more or less similar recipe for QED, but there are a few differences.

0. **Draw valid Feynman diagram:** For a particular process of interest, draw a Feynman diagram with the minimum number of vertices (lowest order diagram which is the leading order too).
1. **Notation/Labeling:** Label the particles, four-momenta for all lines, both external (p_i) and internal (q_i), as we did for scalar field theory. Also, include their corresponding spins (s_i).
2. **Coupling Constant:** At each vertex write a factor of $ig_e\gamma^\mu$. The symbol g_e represents the dimensionless coupling constant and is related to the fine-structure constant by

$$\alpha = \frac{g_e^2}{4\pi}$$

and γ^μ is the Dirac gamma matrix.

FEYNMAN RULES FOR QED

3. **Propagators:** Propagators associated with the internal lines can be of two types.
    - Photon propagator connects two vertices $ig_e\gamma^\mu$ and $ig_e\gamma^\nu$. Thus it is in a form such that it contracts the indices μ and ν:

      $$\text{Photon Propagator:} \quad -\frac{ig_{\mu\nu}}{q^2}$$

    - Fermion Propagator has the form:

      $$\text{Fermion Propagator:} \quad \frac{i(\not{q} + m)}{q^2 - m^2c^2}$$

    where $\not{q} \equiv \gamma^\mu q_\mu$.
4. **Momentum Conservation:** For each vertex, write down the $\delta-$function of the form $(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3)$ which ensures the four-momentum conservation at that vertex.

Feynman Rules for QED

5. **Integration over q:** For each internal line write a factor

$$\frac{d^4q}{(2\pi)^4}$$

and integrate over all internal momenta.

6. **Cancel the $\delta-$function:** The result will have a $\delta-$function,

$$(2\pi)^4\delta^{(4)}(p_1 + ...p_n)$$

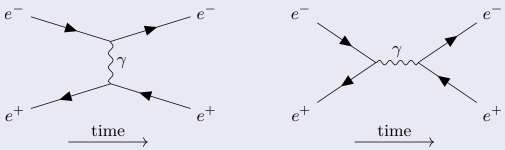
which enforces the energy-momentum conservation. Drop it out, and what remains is $i\mathcal{M}$ for the considered process.

FEYNMAN RULES FOR QED: BHABHA SCATTERING EXAMPLE

We consider one example of QED interaction- Bhabha Scattering

$$e^- + e^+ \to e^- + e^+$$

which has two lowest order Feynman diagrams:

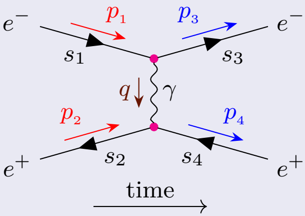


The first diagram is $t-$channel and the second one $s-$channel. We will use Feynman rules to obtain $\mathcal{M}$ for these diagrams, starting with the first diagram.

Particle Interactions

Feynman Rules for QED: Bhabha Scattering Example

1. **Label the Feynman diagram:** Particles, internal and external momenta, and spins.



2. **Coupling constants:** At each vertex write a factor $ig_e\gamma^\mu$. The consant $g_e = \sqrt{4\pi\ \alpha}$.
   For the evalualtion of the matrix element for the coupling, we follow the fermion lines backward to have factors:

$$\overline{u}_3\ ig_e\ \gamma^\mu u_1 \text{ for the fermion line at the top and}$$
$$\overline{\nu}_2\ ig_e\ \gamma^\nu \nu_4 \text{ for the fermion line at the bottom and}$$

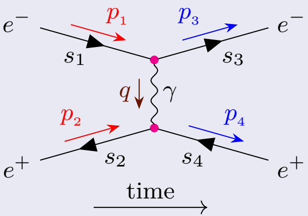
Symbols u_i are the spinors $u^{(s_i)}$ for spins s_i and satisfy Dirac equations.

Particle Interactions

Feynman Rules for QED: Bhabha Scattering Example

Keep in mind, we will work on every fermion line **backwards**, using fermion in $\to u$, fermion out $\to \overline{u}$, anti-fermion in $\to \overline{\nu}$ and anti-fermion out ν.

3. **Propagators:** We have only one internal line (a photon line). Write the propagator corresponding to it.



$$-\frac{ig_{\mu\nu}}{q^2}$$

4. **Momentum Conservation:** We have two vertices and hence two delta functions for momentum conservation:

$$(2\pi)^4\delta^{(4)}(p_1 - p_3 - q) \text{ for the upper vertex and}$$

$$(2\pi)^4\delta^{(4)}(p_2 + q - p_4) \text{for the lower vertex.}$$

FEYNMAN RULES FOR QED: BHABHA SCATTERING EXAMPLE

5. Integration over q: We now add a factor

$$\frac{d^4q}{(2\pi)^4}$$

and integrate over q to obtain:

$$\int [\overline{u}_3\ ig_e\ \gamma^\mu u_1] \left(-\frac{ig_{\mu\nu}}{q^2}\right) [\overline{v}_2\ ig_e\ \gamma^\nu v_4]$$
$$\times\ \delta^{(4)}(p_1 - p_3 - q)(2\pi)^4\delta^{(4)}(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

Because of the first $\delta-$ function, everything except $q = p_1 - p_3$ are left out and we will have:

$$[\overline{u}_3\ ig_e\ \gamma^\mu u_1]\,\frac{-ig_{\mu\nu}}{(p_1 - p_3)^2}\,[\overline{v}_2\ ig_e\ \gamma^\nu v_4]\,(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

FEYNMAN RULES FOR QED: BHABHA SCATTERING EXAMPLE

6. **Cancel the delta function:** Cancel the delta function $(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ to obtain $i\mathcal{M}_t$

$$i\mathcal{M}_t = [\overline{u}_3\ ig_e\ \gamma^{\mu}u_1]\,\frac{-ig_{\mu\nu}}{(p_1 - p_3)^2}\,[\overline{v}_2\ ig_e\ \gamma^{\nu}v_4]$$
$$\therefore\, \mathcal{M}_t = -\,[\overline{u}_3\ ig_e\ \gamma^{\mu}u_1]\,\frac{g_{\mu\nu}}{\underbrace{(p_1 - p_3)^2}_{t}}\,[\overline{v}_2\ ig_e\ \gamma^{\nu}v_4]$$

Proceed similarly to obtain $\mathcal{M}_s$, that is the second diagram. The total amplitude is then

$$\mathcal{M} = \mathcal{M}_t - \mathcal{M}_s$$

Note the $-$ve sign! It is for anti-symmetrization, which is required if two of the contributing Feynman diagrams differ only on the exchange of two incoming/outgoing particles.