What is Linear Viscoelasticity?

Viscoelastic materials are those for which the relationship between stress and strain depends on time. Linear viscoelastic materials are those for which there is a linear relationship between stress and strain (at any given time). Linear viscoelasticity is a theory describing the behaviour of such ideal materials.

Strain-time curves for various constant stresses are shown in the following figure for a linear viscoelastic material. At any given time, say t_1, the strain is proportional to stress, so that the strain due to $3\sigma_o$ at t_1 is three times the strain due to σ_o at the same time.
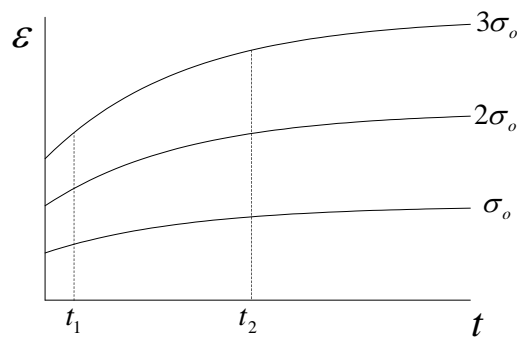

Linear viscoelasticity is a reasonable approximation to the time-dependent behaviour of polymers, and metals and ceramics at relatively low temperatures and under relatively low stress.

## Some observed phenomena in real materials

In the figure below is shown the typical response of a viscoelastic material to a creep-recovery test, that is, to a constant load and to the subsequent removal of that load. It first suffers an instantaneous strain upon loading. This may include elastic and permanent plastic strain. The strain then increases over time. This time-dependent response is known as *creep*. The strain usually increases with an ever decreasing strain rate. A steady state, that is one of constant strain rate, is usually reached (if the load is applied for long enough) in metals at high temperatures, but many materials whose response is linear often do not (e.g. many plastics, metals at low stress levels, etc.). Some of the strain that accumulates during creep will be recoverable upon

unloading and some will not. We cannot detect the ratio of recoverable to permanent creep strain until the load is removed[1].
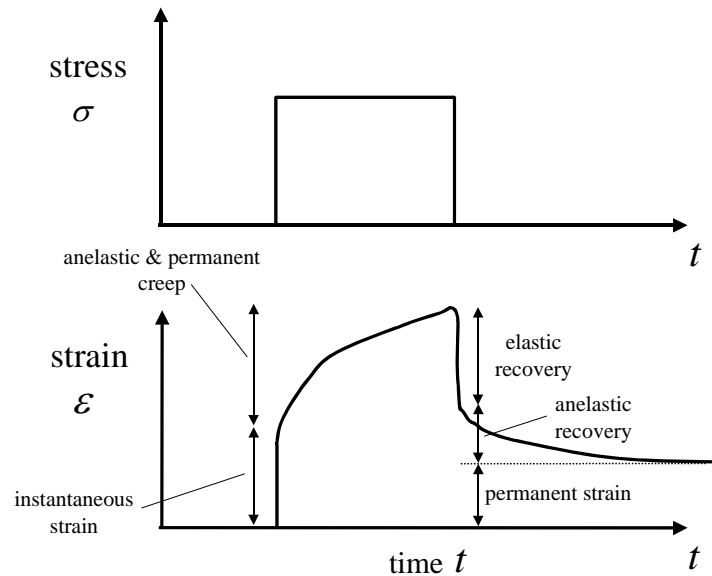
When unloaded, the elastic (instantaneous) strain is recovered immediately. More strain is then recovered over time - this is known as anelastic recovery (or "delayed elastic recovery", for obvious reasons). This anelastic strain is usually very small for metals, but may be significant in polymeric materials. A permanent strain may then be left in the material - this will include the instantaneous plastic strain and the permanent creep strain.

In the next figure is shown the typical response of a viscoelastic material to a relaxation test, that is, to a constant strain.

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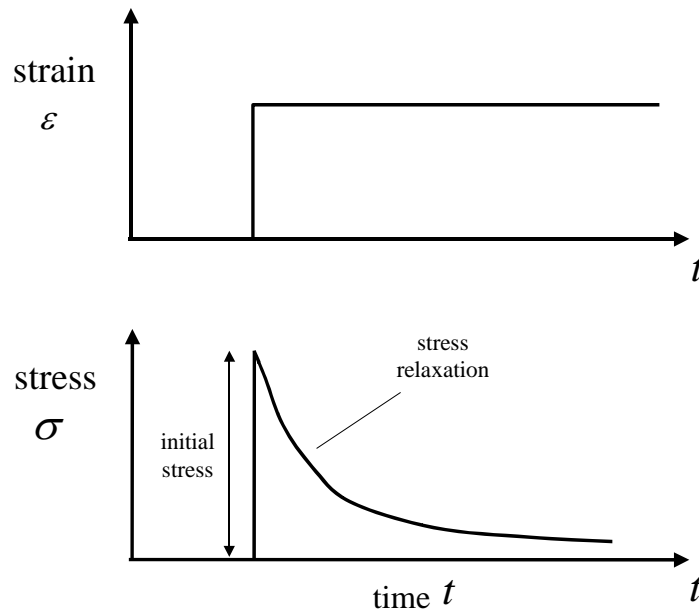
[1] There is some evidence that the initial, transient, creep strain is in the main recoverable, and that the later steady-state creep is permanent, though this is not true in general.

Mechanical (rheological) models

The word viscoelastic is derived from the words "viscous" + "elastic", that is, a viscoelastic material exhibits both elastic and viscous behaviour. We can build up a theory of linear viscoelasticity by considering simple linear elements such as the (elastic) linear spring and the (viscous) linear dash pot[2]. We will start with simple models and increase the complexity until we have an infinite number of elements. In general, the more elements we have, the more accurate will our model be in describing the real response of real materials. That said, the more complex the model, the more material parameters there are which need to be evaluated by experiment - the determination of a large number of material parameters might be a difficult, if not an impossible, task.

We will first examine these models with an emphasis on the physical responses of the spring and dash pot elements. Later on we will re-examine them from a more mathematical perspective, using the Laplace transform. We will only look at 1-dimensional models. We will examine ways of extending our analysis to 3-dimensions later on.

We can gauge the viability of our models by seeing whether they predict the observed responses of real models to some simple loads/strains. In particular, do they predict the creep, recovery and relaxation stress/strain curves discussed above?

### The Linear Elastic Solid (the "linear spring")

The simplest way to create a model of a material is to suppose that it consists of nothing but a linear spring of stiffness E. The constitutive equation for the linear elastic solid is then simply

$$\varepsilon = \frac{1}{E}\sigma$$

The creep and recovery behaviour of the material to a suddenly applied load σ_o is as shown, $\varepsilon = \sigma_o / E$. An important point to make, even though it is obvious and self-evident, is that the spring reacts instantly to the load, and when the load is removed it again reacts instantly. The response may also be written in the form

$$\varepsilon = \sigma_o J$$

and J is known as the *compliance* (the inverse of the stiffness).

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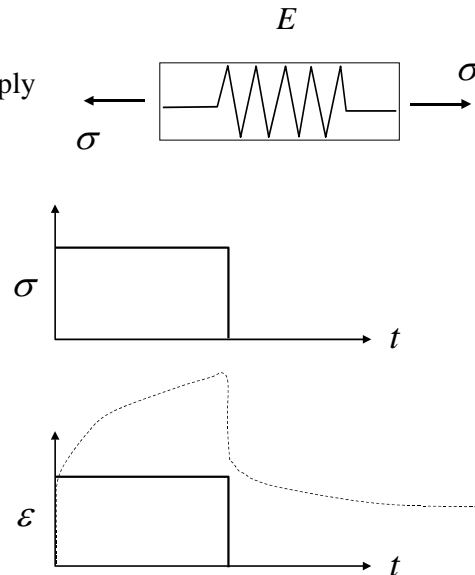
[2] A non-linear theory can be developed by including non-linear dash pots (strain rate not linearly proportional to stress).

The response of our ideal material is obviously not very representative of the response to a real material (which is indicated by the dotted line). There is no creep stage, anelastic recovery or permanent creep strain.

The Linear Viscous Fluid (the "linear dash pot")

First, consider an ideal incompressible viscous fluid (an incompressible "Newtonian" fluid) bounded by a movable upper plate and a fixed lower plate, as shown. Flow of the fluid may be induced by the application of a shear stress τ to the upper plate. The fluid in contact with the upper plate will "stick" to it and move quickly, but the fluid at the lower fixed plate will not move. A velocity gradient is thus established and it can be shown (and verified experimentally) that the velocity gradient is related to the applied shear by a constant η, the viscosity of the fluid. Thus

$$\frac{dv}{dy} = \frac{1}{\eta}\tau$$

Now $v = d(u_x)/dt$, and shear strain is related to the displacements through $\gamma = d(u_x)/dy$. Thus

$$\frac{dv}{dy} = \frac{d}{dy}\left(\frac{du_x}{dt}\right) = \frac{d}{dt}\left(\frac{du_x}{dy}\right) = \frac{d\gamma}{dt}$$

and it follows that the shear strain rate is proportional to the shear stress,

$$\dot{\gamma} = \frac{1}{\eta}\tau$$

This idea of viscous fluid flow is used in the analysis of viscoelastic materials. We can imagine a bar of material acting like a fluid when under tension. In that case we have normal stress and strain rates, $\dot{\varepsilon} = \sigma/\eta$.

Consider now a dash pot (a piston moving in a viscous fluid of viscosity η). For this dash pot we assume the above ideal fluid behaviour

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma$$

That is, the larger the stress, the faster the material strains. The strain due to a suddenly applied load σ_o may be obtained by integrating this expression; we have (assuming zero initial strain)

$$\varepsilon = \frac{\sigma_o}{\eta}t$$

The strain is seen to increase linearly and without bound so long as the stress is applied. Note that there is *no movement* of the dash pot at the onset of load. It takes time for the strain to build up. When the load is removed, there is

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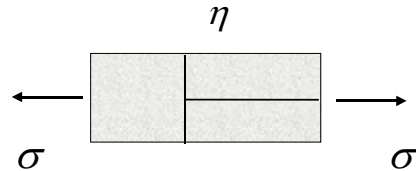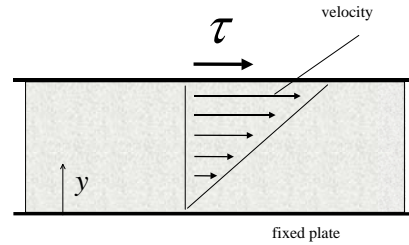
no stress to move the piston back through the fluid, so that any strain built up is permanent.

Note that the strain is proportional to stress, as it must be for a linear material. Again, this linear relationship between stress and strain may be emphasised by rewriting the response in the form

$$\varepsilon(t) = \sigma_o J(t), \quad J(t) = t/\eta$$

The compliance J is now a function of time, and is called the *creep compliance function*[3].

The creep and recovery response of the material is as shown. The slope of the creep-line is σ_o/η. There is no instantaneous strain, creep of ever-decreasing strain rate, elastic recovery or anelastic recovery, but there is a permanent strain (all the creep strain is permanent).
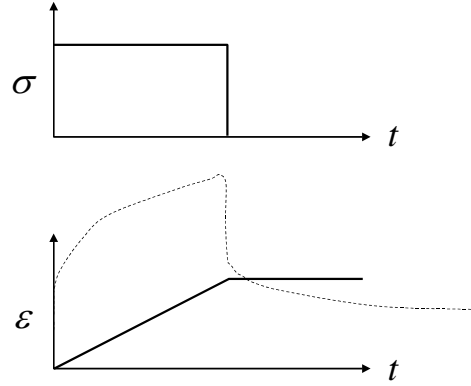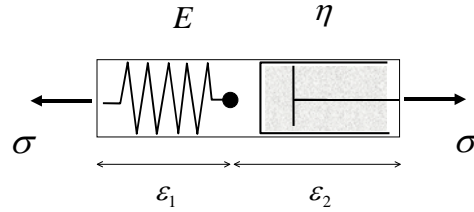
The Maxwell Model

The Maxwell model consists of a spring and a dash pot in series. We can divide the total strain into two separate strains, one for the spring (ε_1) and one for the dash pot (ε_2). The stress will be the same in both elements. We thus have the three equations in four unknowns ($\varepsilon, \varepsilon_1, \varepsilon_2, \sigma$)

$$\varepsilon_1 = \frac{1}{E}\sigma$$

$$\dot{\varepsilon}_2 = \frac{1}{\eta}\sigma$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

We can in principal eliminate ε_1 and ε_2, and be left with one constitutive equation relating ε to σ. We will use Laplace transforms to accomplish this for more complicated models, but here we can solve the equations quite easily. Differentiating the first and third equations, and putting the first and second into the third, gives the constitutive relation for the Maxwell model:

$$\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \frac{1}{\eta}\sigma$$

We put this in the standard form (stress on left, strain on right, increasing order of derivatives from left to right, coefficient of σ is 1)

$$\boxed{\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}}$$

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[3] The creep compliance function $J(t)$ is often known simply as the "creep function".

What is the response of this model to a sudden load σ_o? Physically, we know that the spring will stretch immediately and that the dash pot will take time to react. Thus we will have an initial strain $\varepsilon(0) = \sigma_o / E$. We then expect the dash pot to take up the stress, and so for the strain to increase linearly with slope σ_o / η.

The stress-rate is zero[4] so our constitutive law becomes

$$\dot{\varepsilon} = \frac{\sigma_o}{\eta} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{\eta} t + C$$

$$\rightarrow \quad \varepsilon(t) = \sigma_o \left(\frac{1}{\eta} t + \frac{1}{E} \right)$$

We again emphasise the linearity of the response by writing the above in terms of a creep compliance function:

$$\varepsilon(t) = \sigma_o J(t) \quad \text{where} \quad J(t) = t/\eta + 1/E$$

Now when we remove the load, the spring will again react immediately, but the dash pot will have no tendency to recover. Thus there is an immediate elastic recovery σ_o / E, with the "creep" strain due to the dash pot remaining[5].
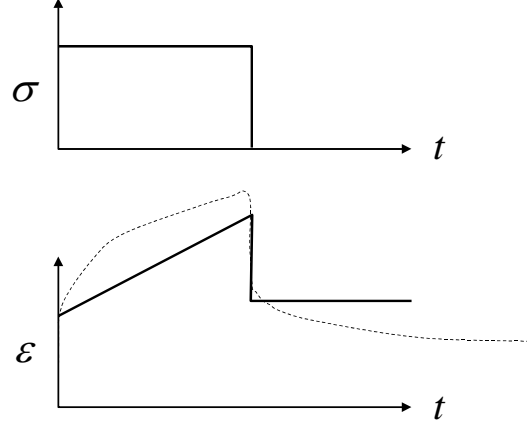
The full creep and recovery response is as shown above. There is creep, but not of the ever-decreasing strain-rate type, there is no delayed elasticity, but there is the elastic response, the permanent strain, and we seem to be getting nearer the real thing.

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[4] There is a jump in stress from zero to σ_o when the load is applied, implying an infinite stress-rate $\dot{\sigma}$. We're not really interested in this jump because we can predict the corresponding jump in strain from the physical response of a linear spring. We are more interested in what happens just "after" the load is applied. In that sense, when we speak of initial strains and stress-rates, we mean their values at 0^+, just to the right of $t = 0$; the stress-rate is zero from 0^+ on.

We can though, if we need to, deal with the sudden jump in stress by integrating the constitutive equation across the point $t = 0$ as follows:

$$(E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \int_{-\Delta\tau}^{+\Delta\tau}\dot{\sigma}(t)dt = E\int_{-\Delta\tau}^{+\Delta\tau}\dot{\varepsilon}(t)dt$$

$$\rightarrow (E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \left[\sigma(+\Delta\tau) - \sigma(-\Delta\tau)\right] = E\left[\varepsilon(+\Delta\tau) - \varepsilon(-\Delta\tau)\right]$$

In the limit as $\Delta\tau \rightarrow 0$, the integral tends to zero (σ is finite), the values of stress and strain at 0^-, i.e. in the limit as $\Delta\tau \rightarrow 0$ from the left, are zero. All that remains are the values to the right, at 0^+. Thus we have $\sigma(0^+) = E\varepsilon(0^+)$, as expected. We can deal with this sudden loading and unloading much easier by using the Laplace transform (see later), in which case the stress is defined for all time, including $t < 0$, by using the unit (Heaviside) step function, i.e. the stress can be written $\sigma(t) = \sigma_o H(t)$.

[5] As with the sudden loading, the sudden unloading can be dealt with in a more mathematical fashion by integrating the constitutive equation across the point of unloading.

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The Kelvin Model

We now move onto another two-element model, the Kelvin model, which consists of a spring and dash pot in parallel. It is assumed there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash pot. However, here there is no reason why the stresses in both portions should be the same. We thus have the following three equations:

$$\varepsilon = \frac{1}{E}\sigma_1$$

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma_2$$

$$\sigma = \sigma_1 + \sigma_2$$

Eliminating σ_1, σ_2 then leaves us with the constitutive law

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$

which is in standard form.
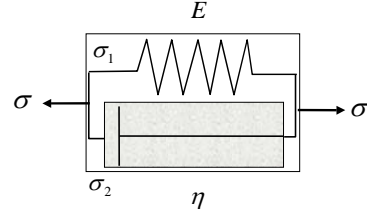
If we apply a sudden load σ_0 to the Kelvin model, the spring will want to stretch immediately, but is held back by the dash pot, which cannot react immediately. All the stress is thus initially taken up by the dash pot (there is no stress in the spring because if there was there would have to be at least some strain). Our creep curve thus starts with an initial slope σ_o / η.

Some strain then starts to take place, and so the stress starts to decrease in the dash pot and increase in the spring - the stress is being transferred from the dash pot to the spring. The slope of the creep curve is now σ_2 / η, where σ_2 is the stress in the dash pot - thus the slope of the creep curve is ever decreasing as σ_2 decreases. In the limit when $\sigma_2 = 0$ (which will happen after an infinite amount of time!), the spring will carry all the stress and thus the maximum strain is σ_o / E.
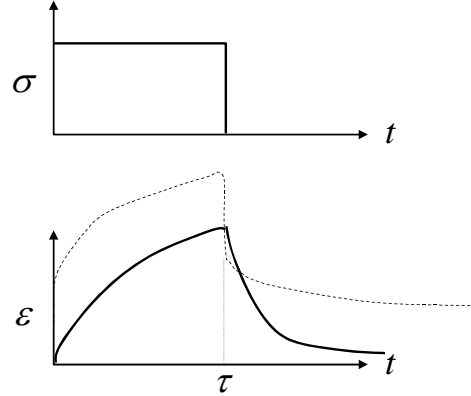
We have a first order non-homogeneous ODE with the initial condition $\varepsilon(0) = 0$, which we can solve to get the response

$$\varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right)$$

which agrees with the above physical reasoning.

Now when we unload, the spring will want to contract but again the dash pot will hold it back. But the spring will eventually pull the dash pot back to its original zero position given time (we expect full recovery). Thus suppose we unload at time $t = \tau$.

The strain at this time is $\varepsilon(\tau) = (\sigma_o / E)\left(1 - e^{-(E/\eta)\tau}\right)$. Our constitutive law, with zero stress, reduces to $0 = E\varepsilon + \eta\dot{\varepsilon}$. If we solve this equation we get

$$\varepsilon(t) = Ce^{-(E/\eta)t}$$

but this t is measured from the point where "zero load" begins. If we want to measure time from the onset of load, we must replace this t with $t - \tau$. Using the aforementioned "initial" condition then leads to

$$\varepsilon(t) = \frac{\sigma_o}{E} e^{-(E/\eta)t}\left(e^{(E/\eta)\tau} - 1\right), \quad t > \tau$$

The creep and recovery response is as shown above. We have a transient-type creep and anelastic recovery, but no instantaneous or permanent strain.

Mechanical models revisited (the Laplace Transform)

We will soon look at a more realistic model, a three-element model known as the *standard linear model*. We will need to use the Laplace transform method to obtain the constitutive law for this more complex model. As an introduction to this, let's first derive again the constitutive laws for the simpler Maxwell and Kelvin models using the Laplace transform.

The Laplace Transform

We will use extensively the formula for the Laplace transform of the derivative of a function[6]:

$$L(\dot{f}) = s\bar{f} - f(0)$$
$$L(\ddot{f}) = s^2\bar{f} - sf(0) - \dot{f}(0), \qquad \text{etc.}$$

where s is the transform variable, the overbar denotes the Laplace transform of the function, and $f(0)$ is the value of the function at time $t = 0$. To be more mathematically precise, the Laplace transform is defined in such a way that $f(0)$ refers to $t = 0^-$, that is, just "to the left" of time zero. Some other important Laplace transforms are summarised in the following table (α is a constant):

$f(t)$	$\bar{f}(s)$
α	α / s
$H(t)$	$1/s$
$\delta(t - \tau)$	$e^{-\tau s}$
$\dot{\delta}(t)$	s
$e^{-\alpha t}$	$1/(\alpha + s)$
$\left(1 - e^{-\alpha t}\right)/\alpha$	$1/s(\alpha + s)$
$t/\alpha - \left(1 - e^{-\alpha t}\right)/\alpha^2$	$1/s^2(\alpha + s)$
t^n	$n!/s^{1+n}, \quad n = 0, 1, \cdots$

TABLE: Laplace Transforms

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[6] This rule actually only works if the function and its derivatives are continuous, although the derivative of the function being transformed may be piecewise continuous. Discontinuities in the function or its derivatives introduce additional terms.

A useful formula is the time-shifting formula, which is (this is used and made clearer below)

$$L\left[f(t-\tau)H(t-\tau)\right]=e^{-\tau s}\bar{f}(s)$$

The Maxwell Model

The three equations that need to be solved in the Maxwell model are

$$\varepsilon_1 = \frac{1}{E}\sigma, \quad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \quad \varepsilon = \varepsilon_1 + \varepsilon_2$$

Taking Laplace transforms gives

$$\bar{\varepsilon}_1 = \frac{1}{E}\bar{\sigma}, \quad s\bar{\varepsilon}_2 = \frac{1}{\eta}\bar{\sigma}, \quad \bar{\varepsilon} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2$$

and we have assumed that the strain ε_2 is zero at $t = 0^-$. We have reduced our three differential equations to three algebraic equations, which may now be solved to get

$$\bar{\sigma} + \frac{\eta}{E}s\bar{\sigma} = \eta s\bar{\varepsilon}$$

Transforming back then gives the same result as before:

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}$$

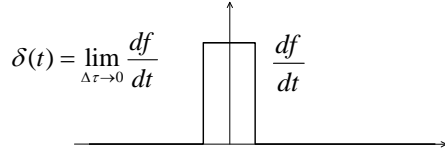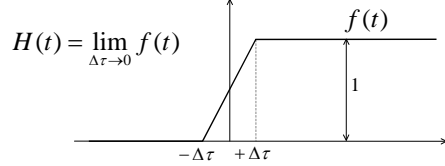Now we examine the response to a sudden load. When using the Laplace transform, we write the load as $\sigma(t) = \sigma_o H(t)$, where $H(t)$ is the unit (Heaviside) step function ($=1$ for $t > 0$, $=0$ for $t < 0$) Also, $dH(t)/dt = \delta(t)$, the impulse function (this can be seen from the accompanying figure). Thus our constitutive law is

$$\sigma_o H(t) + \frac{\eta}{E}\sigma_o \delta(t) = \eta\dot{\varepsilon}$$

Using the Laplace transform gives

$$\sigma_o / s + \frac{\eta}{E}\sigma_o = \eta s\bar{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\sigma_o}{E}\frac{1}{s} + \frac{\sigma_o}{\eta}\frac{1}{s^2} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{E}H(t) + \frac{\sigma_o}{\eta}t,$$

which is the same result as before (we have used $\varepsilon(0) = \varepsilon(0^-) = 0$). Subsequent unloading, at time $t = \tau$ say, can be dealt with most conveniently by superimposing another load $\sigma(t) = -\sigma_o H(t-\tau)$ onto the first. Putting this into the constitutive equation and using the Laplace transform gives

$$\bar{\varepsilon} = -\frac{\sigma_o}{\eta}\frac{1}{s^2}e^{-\tau s} - \frac{\sigma_o}{E}\frac{1}{s}e^{-\tau s}$$

Transforming back, again using the time-shifting rule, gives

$$\varepsilon(t) = -\frac{\sigma_o}{\eta}(t-\tau)H(t-\tau) - \frac{\sigma_o}{E}H(t-\tau)$$

Adding this to the strain due to the first load then gives the expected result

$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_o}{E} + \dfrac{\sigma_o}{\eta}t, & 0 < t < \tau \\[2ex] \dfrac{\sigma_o}{\eta}\tau, & \tau < t \end{cases}$$

The Kelvin Model

Taking Laplace transforms of the three equations for the Kelvin model gives $\bar{\sigma} = E\bar{\varepsilon} + \eta s \bar{\varepsilon}$, which yields $\sigma = E\varepsilon + \eta\dot{\varepsilon}$. The response to a load $\sigma(t) = \sigma_o H(t)$ is then

$$\sigma_o H(t) = E\varepsilon + \eta\dot{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\sigma_o}{\eta}\frac{1}{s(E/\eta + s)} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right)$$

The response to another load of magnitude $\sigma(t) = -\sigma_o H(t - \tau)$ is now

$$-\sigma_o H(t-\tau) = E\varepsilon + \eta\dot{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = -\frac{\sigma_o}{\eta}\frac{e^{-\tau s}}{s(E/\eta + s)} \quad \rightarrow \quad \varepsilon(t) = -\frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)(t-\tau)}\right)H(t-\tau)$$

The response to both loads now gives the complete creep and recovery response:

$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right), & 0 < t < \tau \\[2ex] \dfrac{\sigma_o}{E}e^{-(E/\eta)t}\left(e^{(E/\eta)\tau} - 1\right), & t > \tau \end{cases}$$

The Standard Linear Model

We are now in a position to look at a more complicated (and realistic) model, the standard linear model. This model consists of a spring in series with a Kelvin unit. Upon loading we expect the left-hand spring to stretch immediately. The dash pot then takes up the stress, transferring the load to the second spring as it slowly opens over time. Upon unloading we expect the left-hand spring to contract immediately and for the right-hand spring to slowly contract, being held back by the dash pot.
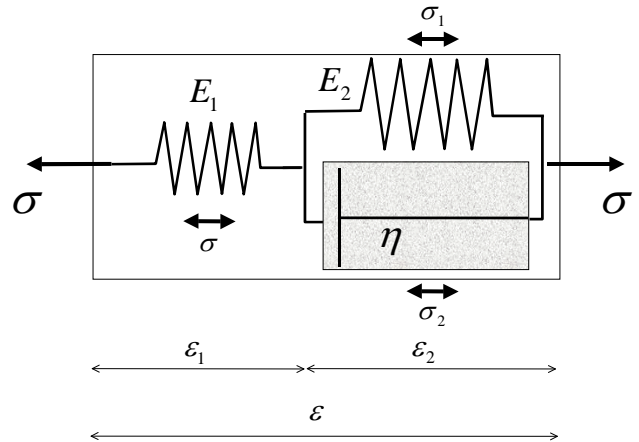
The equations for this model are, from the figure,

$$\sigma = \sigma_1 + \sigma_2$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

$$\sigma = E_1 \varepsilon_1$$

$$\sigma_1 = E_2 \varepsilon_2$$

$$\sigma_2 = \eta \dot{\varepsilon}_2$$

We can eliminate the four unknowns from these five equations using the Laplace transform, and we have

$$(E_1 + E_2)\bar{\sigma} + \eta s \bar{\sigma} = E_1 E_2 \bar{\varepsilon} + E_1 \eta s \bar{\varepsilon}$$

which transforms back to (in standard form)

$$\boxed{\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1 \eta}{E_1 + E_2}\dot{\varepsilon}}$$

The response to a load $\sigma(t) = \sigma_o H(t)$ is then

$$(E_1 + E_2)\sigma_o H(t) + \eta \sigma_o \delta(t) = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon}$$

$$\rightarrow \quad \bar{\varepsilon} = \frac{\sigma_o}{E_1}\frac{1}{((E_2/\eta) + s)} + \frac{\sigma_o(E_1 + E_2)}{E_1 \eta}\frac{1}{s((E_2/\eta) + s)}$$

$$\rightarrow \quad \varepsilon(t) = \sigma_o J(t)$$

and the creep compliance is

$$J(t) = \frac{1}{E_1}e^{-(E_2/\eta)t} + \frac{E_1 + E_2}{E_1 E_2}\left(1 - e^{-(E_2/\eta)t}\right)$$

Note that $\varepsilon(0) = \sigma_o / E_1$ as expected.

For recovery we have to superimpose an opposite load onto the first, at time τ say. We have

$$-(E_1 + E_2)\sigma_o H(t - \tau) - \eta \sigma_o \delta(t - \tau) = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon}$$

$$\rightarrow \quad \bar{\varepsilon} = -\frac{\sigma_o}{E_1}\frac{1}{(E_2/\eta + s)}e^{-\tau s} - \sigma_o\left(\frac{E_1 + E_2}{E_1 \eta}\right)\frac{1}{s(E_2/\eta + s)}e^{-\tau s}$$
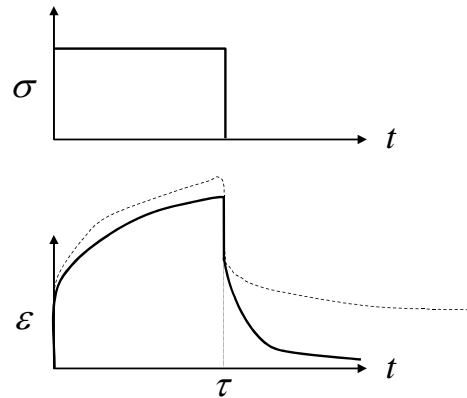
$$\rightarrow \quad \varepsilon(t) = -\sigma_o H(t - \tau)\left\{\frac{1}{E_1}e^{-(E_2/\eta)(t-\tau)} + \left(\frac{E_1 + E_2}{E_1 E_2}\right)\left(1 - e^{-(E_2/\eta)(t-\tau)}\right)\right\}$$

The response after time τ is then

$$\varepsilon(t) = \frac{\sigma_o}{E_2}e^{-(E_2/\eta)t}\left(e^{(E_2/\eta)\tau} - 1\right)$$

This is, as expected, simply the recovery response of the Kelvin unit to the right of the model.

The full response is a shown. This seems to be fairly close now to the response of a real material, although there is no permanent strain left.

The Creep Compliance and the Relaxation Modulus

We have already seen the creep compliance function $J(t)$. It is the strain due to unit stress:

$$\varepsilon(t) = \sigma_o J(t), \qquad \varepsilon(t) = J(t) \text{ when } \sigma_o = 1$$

Similarly, we can define the *relaxation modulus*, $E(t)$. This is the stress due to unit strain, i.e.

$$\sigma(t) = \varepsilon_o E(t), \qquad \sigma(t) = E(t) \text{ when } \varepsilon_o = 1$$

The relaxation modulus describes the response of a material in a stress-relaxation (constant strain) test. It can be evaluated using the same method we used to evaluate the creep compliance. In many simple mechanical models, the relaxation modulus will contain an exponential function of the form (as shown)

$$e^{-t/t_R}$$

in one of its terms. The value of t_R is called the *relaxation time*, and is a measure of how quickly the material relaxes.
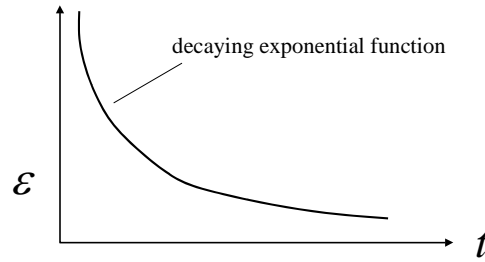
One of the problems with simple models such as the Maxwell model is that they only exhibit a single relaxation time. However, real materials are better modelled by many relaxation times, since their relaxation behaviour over the short term usually differs from their relaxation behaviour in the long term. For example the relaxation modulus of a material might look like

$$E(t) = a_1 e^{-t/t_{R1}} + a_2 e^{-t/t_{R2}} + a_3 e^{-t/t_{R3}} + \cdots$$

Similarly, the creep compliance often contains exponential functions. For example, the creep compliance for the Kelvin model is

$$J(t) = (1/E)\left(1 - e^{-(E/\eta)t}\right)$$

The time quantity $t_R = \eta/E$ is another characteristic property of the model, and it is known as the *retardation time*. The larger the retardation time, the less strain that occurs (which can be seen by plotting $J(t)$ for various t_R's.

Non-constant Loading

We can evaluate the response of our models to non-constant loading (or straining).

Example
As an example, consider the two loads shown. The total load is the same in both, but load (1) is applied more gradually. Let us consider the Maxwell model, for which the constitutive
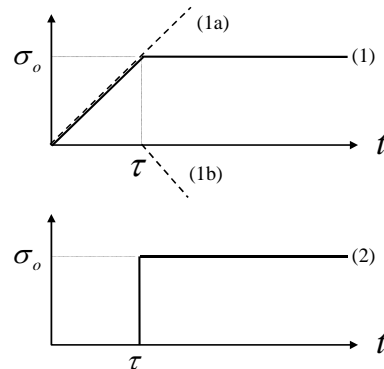
relation is

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}$$

Load (1) can be thought of as consisting of the two loads (1a) $\sigma = (\sigma_o/\tau)t$ and (1b) $\sigma = -(\sigma_o/\tau)(t-\tau)H(t-\tau)$ applied at time $t = \tau$. Load (2) consists of a constant load applied at time $t = \tau$.

For load (1a) we have

$$\frac{\sigma_o}{\tau}t + \frac{\eta}{E}\frac{\sigma_o}{\tau} = \eta\dot{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\sigma_o}{\eta\tau}\frac{1}{s^3} + \frac{1}{E}\frac{\sigma_o}{\tau}\frac{1}{s^2} \quad \rightarrow \quad \frac{\varepsilon(t)}{\sigma_o} = \frac{1}{E\tau}t + \frac{1}{2\eta\tau}t^2$$

which gives the response for $t < \tau$.

For load (1b) we have [note: $L\big((t-\tau)\delta(t-\tau)\big) = 0$]

$$\eta\dot{\varepsilon} = -\frac{\sigma_o}{\tau}(t-\tau)H(t-\tau) - \frac{\eta}{E}\frac{\sigma_o}{\tau}\big[(t-\tau)\delta(t-\tau) + H(t-\tau)\big]$$

$$\rightarrow \quad s\bar{\varepsilon} = -\frac{\sigma_o}{\eta\tau}e^{-\tau s}\frac{1}{s^2} - \frac{1}{E}\frac{\sigma_o}{\tau}e^{-\tau s}\frac{1}{s}$$

$$\rightarrow \quad \bar{\varepsilon} = -\frac{\sigma_o}{\eta\tau}e^{-\tau s}\frac{1}{s^3} - \frac{1}{E}\frac{\sigma_o}{\tau}e^{-\tau s}\frac{1}{s^2}$$

$$\rightarrow \quad \frac{\varepsilon(t)}{\sigma_o} = H(t-\tau)\left\{-\frac{1}{2\eta\tau}(t-\tau)^2 - \frac{1}{E\tau}(t-\tau)\right\}$$

The response after time τ is then given by adding the two results, and we get

$$\frac{\varepsilon(t)}{\sigma_o} = \frac{1}{E} + \frac{1}{\eta}(t - \frac{\tau}{2})$$

For load (2) we have (from our earlier work on the Maxwell model)

$$\frac{\varepsilon^{(2)}(t)}{\sigma_o} = \frac{1}{E} + \frac{1}{\eta}(t-\tau), \quad t > \tau$$

which is less than the strain due to load (1).

This example illustrates two points. First, it shows that material has a "memory". It remembers the previous loading history, responding differently to different loading histories. Second, it shows that the rate of loading is important in viscoelastic materials. Our result agrees with an observed phenomenon: the strain in viscoelastic materials is larger for stresses which grow gradually to their final value, rather than when applied more quickly[7].

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[7] Actually, for the Maxwell model, if we applied the second load at time $t = \tau/2$, so that the total stress applied in (1) and (2) was the same, we would have obtained the same response after time τ. However, this isn't the case in general.

Generalized Models

More complicated models can be constructed by considering more and more elements. These are usually either of the form of the generalized Maxwell model or the generalized Kelvin chain, shown below. Such models will contain many parameters and will exhibit a whole array of relaxation and retardation times.

The generalized Maxwell model consists of N different Maxwell units in parallel, each unit with different parameter values. The absence of the isolated spring would ensure fluid-type behaviour, whereas the absence of the isolated dash pot would ensure an impact (instantaneous) response.

The generalised Kelvin chain consists of a chain of Kelvin units and again the isolated spring may be omitted if fluid-type response is required.

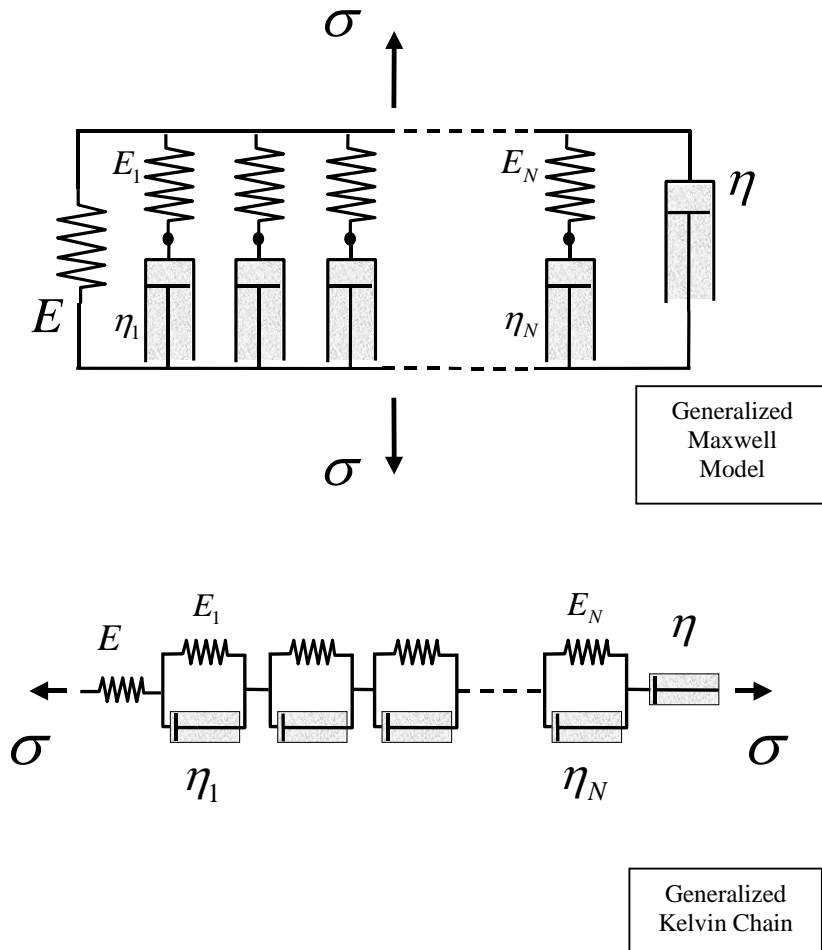It is evident that we will, in general, have a constitutive equation of the form
$$p_o\sigma + p_1\dot\sigma + p_2\ddot\sigma + p_3\dddot\sigma + p_4\sigma^{(IV)} + \cdots = q_o\varepsilon + q_1\dot\varepsilon + q_2\ddot\varepsilon + q_3\dddot\varepsilon + q_4\varepsilon^{(IV)} + \cdots$$
which can be written in the short hand form
$$\mathbf{P}\sigma = \mathbf{Q}\varepsilon$$
and $\mathbf{P}$ and $\mathbf{Q}$ are the linear differential operators

$$\mathbf{P} = \sum_{i=0}^{n} p_i \frac{\partial^i}{\partial t^i}, \qquad \mathbf{Q} = \sum_{i=0}^{n} q_i \frac{\partial^i}{\partial t^i}$$

Generalized
Maxwell
Model

Generalized
Kelvin Chain

Relationship between Creep and Relaxation

We can take the Laplace transform of the general constitutive equation $\mathbf{P}\sigma = \mathbf{Q}\varepsilon$
to get
$$\left(p_o + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \cdots\right)\bar{\sigma} = \left(q_o + q_1 s + q_2 s^2 + q_3 s^3 + q_4 s^4 + \cdots\right)\bar{\varepsilon}$$
which can also be written in the contracted form
$$P(s)\bar{\sigma} = Q(s)\bar{\varepsilon}$$
where P and Q are the polynomials
$$P(s) = \sum_{i=0}^{n} p_i s^i, \qquad Q(s) = \sum_{i=0}^{n} q_i s^i$$

Now the Laplace transforms of the creep compliance $\left(J(t) \to \bar{J}(s)\right)$ and relaxation modulus $\left(E(t) \to \bar{E}(s)\right)$ can be written in terms of these polynomials as follows. First, the strain due to a unit load $\sigma = H(t)$ is $J(t)$. Since $\bar{\sigma} = 1/s$, substitution into the above equation gives
$$\bar{J}(s) = \frac{P(s)}{sQ(s)}$$
Similarly, the stress due to a unit strain $\varepsilon = H(t)$ is $E(t)$. Thus
$$\bar{E}(s) = \frac{Q(s)}{sP(s)}$$

It follows that

$$\bar{J}(s)\bar{E}(s) = \frac{1}{s^2}$$

Thus, for a linear viscoelastic material, there is a unique and simple relationship between the creep and relaxation behaviour.

Hereditary Integrals

If we know the creep compliance function (or the relaxation modulus) we can completely characterise the response of a linear viscoelastic material. Of course the creep compliance function doesn't have to be derived from particular spring or dash pot models. We could just as well derive a creep function from experimental data.

Now the strain due to a step load is given by $\varepsilon(t) = \sigma_o J(t)$. Also, the strain due to a second load, $\Delta\sigma$ say, applied at some later time τ, is $\varepsilon(t) = \Delta\sigma J(t - \tau)$. The total strain is evidently
$$\varepsilon(t) = \sigma_o J(t) + \Delta\sigma J(t - \tau)$$
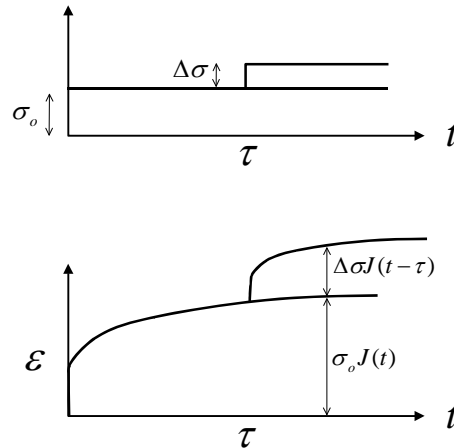
Note that we can only add responses like this because of the linearity of the material. Because the material is linear we have that the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause". This is known as the Boltzmann Superposition Principle.

We can generalise the addition of two loads to the case of an infinite number of applied loads of infinitesimal magnitude, $d\sigma_i$, in which case

$$\varepsilon(t) = \sigma_o J(t) + \sum_{i=1}^{\infty} d\sigma_i J(t - \tau_i)$$

$$= \sigma_o J(t) + \int_0^t J(t - \tau)\left(\frac{d\sigma}{dt}\right)_{t=\tau} d\tau$$

This integral is known as a hereditary integral. It is another form of the constitutive equation for a linear viscoelastic material. We can use it to evaluate the response of a material due to any arbitrary load. For example, considering again the gradually applied load studied above,

$$\sigma(t) = \frac{\sigma_o}{\tau'}t, \quad t < \tau'$$

$$\sigma(t) = \sigma_o, \quad t > \tau'$$

(we introduce the dash on the τ to avoid confusion with the variable of integration). We can now write (for the response after time τ'), substituting in the creep compliance function for the Maxwell model,

$$\varepsilon(t) = \sigma_o J(t) + \int_0^{\tau'} J(t - \tau)\left(\frac{d\sigma}{dt}\right)_{t=\tau} d\tau + \int_{\tau'}^t J(t - \tau)\left(\frac{d\sigma}{dt}\right)_{t=\tau} d\tau$$

$$= \frac{\sigma_o}{\tau'}\int_0^{\tau'}\left[\frac{t-\tau}{\eta} + \frac{1}{E}\right]d\tau$$

$$= \sigma_o\left[\frac{t - \tau'/2}{\eta} + \frac{1}{E}\right]$$

which is the same as we obtained before, but it was much easier using the hereditary integral than using the Laplace transform.

We can also derive easily a corresponding hereditary integral in terms of the relaxation modulus:

$$\sigma(t) = \varepsilon_o E(t) + \int_0^t E(t - \tau)\left(\frac{d\varepsilon}{dt}\right)_{t=\tau} d\tau$$
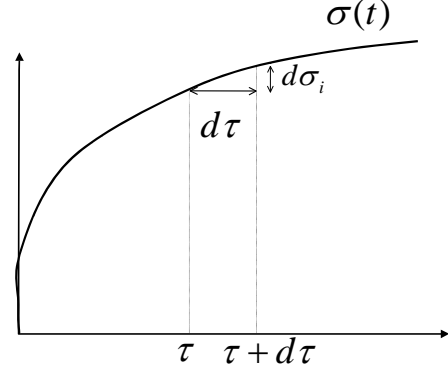
This module covers the 1-dimensional modelling of linear viscoelastic materials, in particular the derivation of 1-dimensional constitutive equations describing linear viscoelastic materials. Linear viscoelastic models can be constructed from mechanical elements, linear springs and viscous dashpots. It should be emphasised here that linear viscoelastic models can be constructed without recourse to such elements. Indeed, sophisticated three dimensional modelling of viscoelastic materials is best done using hereditary integrals – the creep compliance and relaxation modulus functions in these integrals can be obtained from experiment and no knowledge of mechanical elements is necessary. That said, the study of mechanical models forms a good introduction and foundation to viscoelastic modelling.

The most important slides in this module are 3, 4, 34, 50 and 51

(1) Slides 1-8: These introductory slides discuss the different experimental observations (creep, recovery, etc. & the creep-recovery diagrams on slide 4) which must be predicted by any model. *Linear* viscoelasticity is also defined here. [**Lecture Notes pages 21-22**].

(2) Slides 9-19: The basic Maxwell and Kelvin models are discussed here. The main things here are the derivation of the constitutive laws and the response of the material/model to a standard creep-recovery test. The response is derived in these slides by solving 1st order differential equations, by separating variables. The response of the models is also explained *physically* (by examining the mechanical elements which constitute the material), not just derived using mathematics. [**Lecture Notes pages 23-28**]

(3) Slides 20-30: Here, the two basic models are revisited, and a more complicated model is examined. The constitutive equations and the model-responses are now derived using the Laplace Transform, which is a much more efficient method than trying to solve the differential equations directly. [**Lecture Notes pages 28-31**]

(4) Slides 31-38: Two important functions – the Creep Compliance and the Relaxation Modulus – associated with linear viscoelastic materials are defined here. These are crucial functions which completely characterise the behaviour of the material. Each model has its own set of functions, which can be derived by considering the response to creep-recovery and relaxation tests. Associated with these functions are the *relaxation* and *retardation* times, but you will not be assessed on these. [**Lecture Notes page 32**]

(5) Slides 39-40: These slides show how to analyse cases where the loading is not constant. You will not be expected to analyse non-constant problems using the Laplace transform, as here – it is much better to use the hereditary integral (see later) [**Lecture Notes page 32-33**]

(6) Slides 41-46: These slides show how more complicated models an be constructed, by considering more and more mechanical elements. One should be able to infer the general physical response of the generalised models by simply examining them – without doing any calculations (see the exam questions below). The Creep Compliance and Relaxation Modulus functions for these types of model will be many-termed, as on slides 42-43, but the details are unimportant. The important thing to note is that the constitutive equation of these models can always be represented in the differential form given on slide 44. By taking the

Laplace transform of this equation, a relationship between the (transformed) Creep Compliance and Relaxation Modulus functions can be established. [**Lecture Notes page 34-35**]

(7) Slides 47-53: These slides discuss the crucial concept of the hereditary integral. [**Lecture Notes page 35-36**]

Problems

1.
The constitutive equation for a "Maxwell" material is

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}$$

Such basic viscoelastic laws are used in a wide range of applications, for example to model the behaviour of the tectonic plates (the plates which float on and travel independently over the mantle of the earth, and which are responsible for earthquakes, volcanoes, etc.)

(a) Derive this constitutive law by assuming the response of each point in the material is similar to that of a dashpot and spring in series.

(b) Derive an expression for the creep curve of this material, that is, the strain as a function of time due to some constant load σ_o - say clearly, by using an argument based on the physical response of the spring/dash-pot, how you determine any constant of integration.

(c) What then is the Creep Compliance of a Maxwell material? [yes, that simple]

(d) Derive the Relaxation Modulus for this material

(e) What general characteristics of a viscoelastic material (we're talking about response due to loading and unloading here) does the Maxwell model *not* account for.

[note: do not use the Laplace transform in this problem]

2.
[In the following, show the details]

(a) Using the relevant constitutive law, calculate the relaxation modulus $E(t)$ for the Kelvin model (use any method you like)

(b) From your knowledge of the response of springs and dash pots, is your result in (a) what you would expect ? Explain.

(c) check your result for (a) by using the formula $\bar{J}(s)\bar{E}(s) = 1/s^2$ (get $\bar{J}(s)$ from the notes).

3.
[In the following, show the details]

(a) Derive the constitutive relation (in standard form) for the following three-element model using the Laplace transform.

(b) Derive the creep compliance $J(t)$ by considering a suddenly applied load.
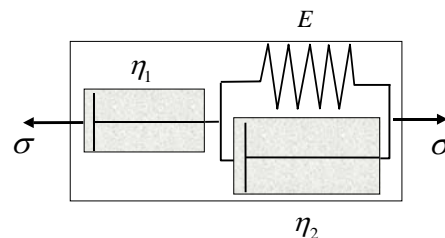
4.

Use the hereditary integral form of the constitutive equation for a linear viscoelastic material, i.e.

$$\varepsilon(t) = \sigma_o J(t) + \int_0^t J(t-\tau)\left(\frac{d\sigma}{dt}\right)_{t=\tau} d\tau,$$

to evaluate the response of a material with creep compliance function

$$J(t) = \ln(t+1)$$

to a load $\sigma(t) = \sigma_o(1+Bt)$. Sketch $J(t)$, which of course gives the strain response due to a unit load $\sigma(t) = 1$. Sketch also the load $\sigma(t)$ and the calculated strain $\varepsilon(t)$.

[note: $\int_0^t \ln[(b-x)+1]dx = -(b-t+1)\ln(b-t+1) + (b+1)\ln(b+1) - t$]

5. What do the terms *Creep Compliance* and *Relaxation Modulus* mean?

6. Derive the formula $\bar{J}(s)\bar{E}(s) = 1/s^2$ by using the differential form of the constitutive equation for a linear elastic material, where the overbar indicates the Laplace transform of a function, and s is the transform variable.

7. Derive the hereditary integral

$$\varepsilon(t) = \sigma_o J(t) + \int_0^t J(t-\tau)\left(\frac{d\sigma}{dt}\right)_{t=\tau} d\tau$$