Generalizing Shallow Water Simulations with Dispersive Surface Waves

ABSTRACT

This paper introduces a novel method for simulating large bodies of water as a height field. At the start of each time step, we partition the waves into a bulk flow (which approximately satisfies the assumptions of the shallow water equations) and surface waves (which approximately satisfy the assumptions of Airy wave theory). We then solve the two wave regimes separately using appropriate state-of-the-art techniques, and re-combine the resulting wave velocities at the end of each step. This strategy leads to the first heightfield wave model capable of simulating complex interactions between both deep and shallow water effects, like the waves from a boat wake sloshing up onto a beach, or a dam break producing wave interference patterns and eddies. We also analyze the numerical dispersion created by our method and derive an exact correction factor for waves at a constant water depth, giving us a numerically perfect re-creation of theoretical water wave dispersion patterns.

1 INTRODUCTION

The motion of water is well-described by the incompressible Euler equations

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0$$

where u is the water velocity, $D\mathbf{u}/Dt$ is the material derivative, ρ is water density, p is pressure, and g is acceleration due to gravity. Numerically approximating this equation is prohibitively expensive for the animation of large bodies of water, so researchers reduce the complexity by assuming the water surface takes the form of a height field, where the water height and velocity are both just functions of 2D spatial coordinates, instead of 3D. The most common reductions of these equations used in computer graphics are Airy wave theory and shallow water approximations.

Airy wave theory [Airy 1841] assumes that the motion is a po- tential flow (velocity is the gradient of some potential, $\mathbf{u} = \nabla\phi$) and that the wave amplitude a is small relative to the wavelength λ ($a \ll \lambda$, or equivalently $ka \ll 1$ for wavenumber $k = 2\pi/\lambda$). These assumptions produce linearized water wave equations which are used extensively throughout the computer graphics literature [Can- abal et al. 2016; Tessendorf 2004a]. This linear wave theory produces

waves with an angular frequency ω depending on wavenumber k and water depth h

$$\omega(k, h) = \sqrt{gk \tanh(kh)}. \quad (2)$$

This particular dependence of ω on k is called the dispersion re- lationship, and it tells us how the frequency varies with different wavelengths. It also effectively prescribes the surface wavespeed, which is equal to ω / k . Accurately reproducing this dispersion rela- tion is essential for many water-related phenomena like raindrop ripples and particular interference patterns in the wake behind boats, as illustrated in Figure 1. One drawback to this approach, however, is that its $ka \ll 1$ assumption prevents the wave heights from having a major effect on the fluid domain boundaries. For example, it cannot model water waves sloshing up a sloped beach, or spilling over a dam and filling a basin, because these scenarios require the water domain to change shape depending on the motion of the surface waves.

Another popular technique for simplifying water dynamics is to assume that the depth of the water h is much smaller than the wavelength ($h \ll \lambda$, or equivalently $kh \ll 1$). This assumption gives rise to the shallow water equations (SWE), which do allow waves to slosh around and spill over terrain. However, this long wavelength / shallow depth assumption is only appropriate in exceptionally limited scenarios, because SWE produces a dispersion relationship

$$\omega(k, h) = k\sqrt{gh}, \quad (3)$$

which is drastically different from Equation 2 when $h \boxed{?} \lambda$. Notably, SWE is unable to produce the signature water wave interference patterns described above, and instead produces ripples similar to acoustic shock waves.

Although both of these methods for animating water are the- oretically sound and offer robust and efficient implementations, they both have restrictive assumptions that lead to visually obvious breakdowns in common scenarios: Airy wave simulators have to avoid flooding scenarios and gently sloped solid obstacles; shallow water solvers either add procedural textures to approximate a more appropriate dispersion relation for deeper water [Chentanez and Müller 2010], or they fail to convincingly animate short wavelengths altogether.

Interestingly, both of these simulation techniques excel where the other fails. The purpose of this paper is to introduce a principled unification of the two fluid regimes together into a single model that lacks all of the failure modes described above. We do this by decomposing the water into two regimes of water motion: a bulk flow that is best described by the shallow water assumption $h \ll \lambda$, and the remaining surface waves that fail the shallow water test but still obey Airy wave theory. The bulk flow naturally includes most of the fluid's mass and momentum (it includes the biggest waves with the longest wavelengths), and so we simulate it using a shallow water solver capable of simulating flooding and convective eddies. Conversely, the surface waves contain all of the surface details and high frequencies that are essential for producing convincing water wave animations, so we simulate them with a highly detailed Airy wave solver. Re-computing the decomposition at each timestep allows water motion to continuously shift between shallow and deep regimes and furthermore it permits the fluid domain to significantly change over time.

In summary this paper offers the following contributions:

- The first height field method capable of simulating both Airy wave interference patterns and large bulk motions like flood- ing and 2D convective eddies in the same simulation;

- A practical algorithm for producing a spatially-varying de- composition of shallow and surface flows;

- A derivation of surface wave numerical dispersion errors, and a novel scheme for canceling them exactly in a constant depth;

- Exact volume conservation and real-time performance appro- priate for games and other interactive scenarios.

  The remainder of this paper is organized as follows: Section 2 dis- cusses related work, and Section 3 provides additional background from fluid dynamics and introduces the theoretical framework be- hind our method. Section 4 describes our algorithm in detail, in- cluding the decomposition, the shallow water solver, the modified Airy solver, and implementation details. Section 5 evaluates our method and discusses results and future work, Section 6 concludes the paper.