CSC418: Assignment #1

Due on Tuesday, February 9, 2016

Liam Jackson

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Problem 1

a) Position Function: $x(t) = at \qquad y(t) = -\frac{1}{2}gt^2 + bt + h$
$r(t) =< x(t), y(t) >$
$r(t) =< at, -\frac{1}{2}gt^2 + bt + h >$
$T(t) = r'(t) =< a, -gt + b >$
The normal vector will be orthogonal to the tangent vector. Given any vector (a,b), the orthogonal vector will be (-b, a). So the normal vector will be $N(t) =< gt - b, a >$
We can prove this by taking the dot product of T(t) and N(t). The dot product of two orthogonal vectors will be 0. $T(t) \cdot N(t) =< a, -gt + b > \cdot < gt - b, a >= agt - ab - agt + ab = 0$
b) Impact is when y(t) = 0 for t > 0
$0 = -\frac{1}{2}gt^2 + bt + h$
Using quadratic formula with $a = -\frac{1}{2}g, b = b, c = h$
$t_i = \dfrac{-b \pm \sqrt{b^2 + 2gh}}{-g}$
$x(t_i) = a * t_i = a * \dfrac{-b \pm \sqrt{b^2 + 2gh}}{-g}$
$y(t_i) = 0$
$x'(t_i) = a$
$y'(t_i) = -g * t_i + b = -g * \dfrac{-b \pm \sqrt{b^2 + 2gh}}{-g} + b$

Problem 2

a) A translation can be generalized by a matrix multiplication of the form $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ uniform scaling can be generalize by a matrix multiplation of the form $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 & s * t_x \\ 0 & s & s * t_y \\ 0 & 0 & 1 \end{bmatrix}$
So no, they are not commutative.
b) A rotation matrix can be written as $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ If we use θ to represent one angle and ϕ to represent the other, the multiplication can be written as $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix} = \begin{bmatrix} cos\theta cos\phi - sin\theta sin\phi & -sin\phi cos\theta - cos\phi sin\theta \\ cos\phi sin\theta + sin\phi cos\theta & -sin\phi sin\theta + cos\phi cos\theta \end{bmatrix}$
We can note that swapping θ and ϕ do not change the result at all, so we know that they are commutative.
c) A shear with respect to the x axis can be written as $\begin{bmatrix} 1 & h \\ 1 & 1 \end{bmatrix}$ And we've established what uniform scaling looks like. So then $\begin{bmatrix} 1 & h \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s & sh \\ s & s \end{bmatrix}$ and $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & h \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s & sh \\ s & s \end{bmatrix}$ So they are commutative.

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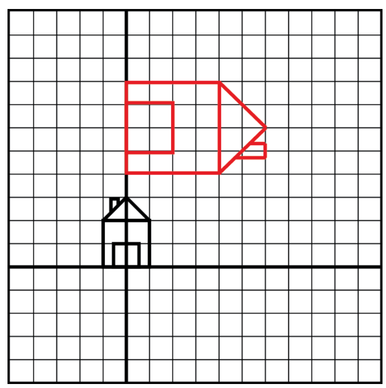
d) We've established what shear with respect to the x axis is and non uniform scalings looks like $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$ So then $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 1 & h \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s_x & s_x h \\ s_y & s_y \end{bmatrix}$ and $\begin{bmatrix} 1 & h \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} = \begin{bmatrix} s_x & s_y h \\ s_x & s_y \end{bmatrix}$ which results in two different matrices. So as long as you scale with two different factors, a shear with respect to the x axis and non-uniform scaling is not commutative.

Problem 3

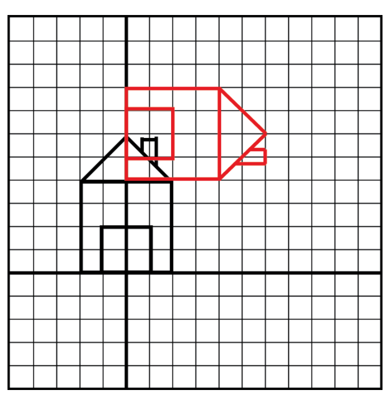
Simply by looking at the 2 different houses, we can notice they differ by 4 factors. We notice the red house is 2 times bigger than the black house. We notice it is rotated 270 degrees clockwise. We notice it is flipped along the y-axis. And we notice it has been translated upwards and to the right. For the sake of explanation, I've recreated the figure in my own imaging software, as shown below. First things first, we want to flip

Figure 1: The starting house



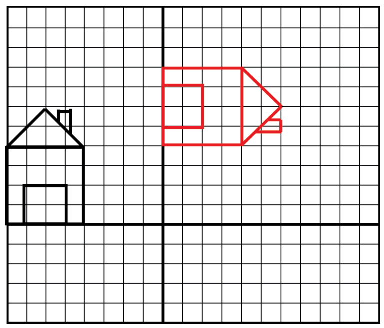
it along the y-axis. We can do this by scaling the house with respect to x by -1. We also want to scale it uniformly by 2. This results in the matrix as such $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Applying this matrix to the black house will result in the following output.

Figure 2: House after scaling and reflection



We can now translate the house in such a method that rotating it will result in perfect alignment of the house and its destination. The red house bottom center is 6 units above the x-axis, attached to the y-axis. So we want the black house to be 6 units to the left of the y-axis and attached to the x-axis. So we apply a translation with respect to x by -6 units. This can be done by the following matrix. $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Applying this to the house will result in the following output. Finally, from here, all we have to do is rotate the

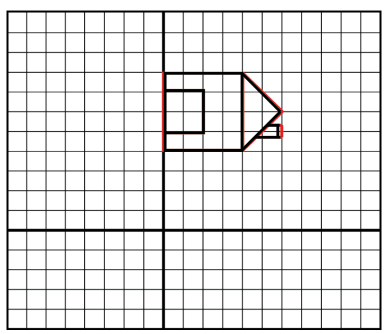
Figure 3: House after translation



house by 270 degrees. This can be done with the following matrix. $\begin{bmatrix} cos270 & -sin270 & 0 \\ sin270 & cos270 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying this matrix to the house will then result in the following output.

Figure 4: House after rotation



Putting these 4 matrices together in reverse order will be $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 4

a) A method I learned previously to solve for affine transformations can be used here. If the matrix doesn't work for the points given, I then know it's a projective homography and can use a different method. The method to solve affine homographies is as so. I can write a matrix of points as

```
x = [0 1 0 1];
y = [0 0 1 1];
xp = [-4 -3 1 0];
yp = [2 0 -7 -5];

P = [x(1) y(1) 0 0 1 0;
    0 0 x(1) y(1) 0 1;
    x(2) y(2) 0 0 1 0;
    0 0 x(2) y(2) 0 1;
    x(3) y(3) 0 0 1 0;
    0 0 x(3) y(3) 0 1];
pPrime = [xp(1) yp(1) xp(2) yp(2) xp(3) yp(3)]';
T = inv(P)*pPrime;
T = [T(1) T(2) T(5); T(3) T(4) T(6); 0 0 1];
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I can test these out on the points and determine that it works and is in fact an affine transformation.
b) the point (1, 2) gets mapped to (7,-18)
c) As we determined in part a, this is in fact an affine matrix.

Problem 5

Given the matrix $\begin{bmatrix} 0 & 2 & 6 \\ -3 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}$ We need to decompose it into a translation, then rotation, then scaling.
Because we know transformations and their order, we know it will be of the form:
$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x cos(\theta) & -s_x sin(\theta) & 0 \\ s_y sin(\theta) & s_y cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} =$
$\begin{bmatrix} s_x cos(\theta) & -s_x sin(\theta) & t_x s_x cos(\theta) - t_y s_x sin(\theta) \\ s_y sin(\theta) & s_y cos(\theta) & t_x s_y sin(\theta) + t_y s_y cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$ From here we can solve it as a system of linear equations.
$s_x * cos(\theta) = 0$
$-s_x * sin(\theta) = 2$
$s_y * sin(\theta) = -3$
$s_y * cos(\theta) = 0$
$t_x * s_x * cos(\theta) - t_y * s_x sin(\theta) = 6$
$t_x * s_y * sin(\theta) + t_y * s_y cos(\theta) = -6$
Since we know s_x is non-zero, because $-s_x * sin(\theta)$ is non-zero, then we know $cos(\theta)$ is zero, so $\theta = 90, 270; sin(\theta) = \pm 1$. This means $s_x = 2, s_y = 3$ or $s_x = -2, s_y = -3$ Assuming the scaling is positive, then $sin(\theta) = -1$, then $t_x * 2 * 0 - t_y * 2 * -1 = 6 \longrightarrow 2t_y = 6 \longrightarrow t_y = 3$
$t_x * 3 * -1 + 3 * 3 * 0 = -6 \longrightarrow -3t_x = -6 \longrightarrow t_x = 2$ Now that we know all the variables, we can put them

back in the matrix to confirm our answer. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ Which does in fact equal $\begin{bmatrix} 0 & 2 & 6 \\ -3 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}$