Causal Reconstruction Kernels for Consistent Signal Recovery

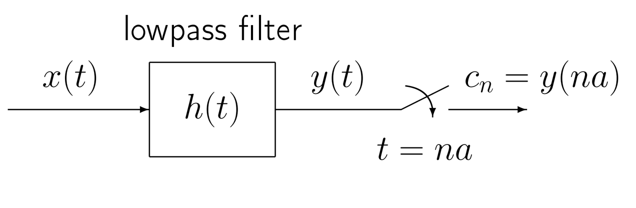
Volker Pohl, Fanny Yang and Holger Boche
Lehrstuhl für Theoretische Informationstechnik, Technische Universität München

Motivation – Causal Reconstruction

Causal signal reconstruction is crucial
- in on-line applications
- feedback loops in control systems
- to reduce border effects in image processing, etc.

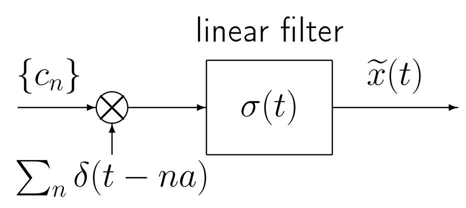
Framework – Shift Invariant Sampling

Non-ideal Acquisition



$$c_n = y(na) = \int_{-\infty}^{\infty} h(\tau)\, x(na-\tau)\, \mathrm{d}\tau$$

Shift-Invariant Reconstruction



$$\widetilde{x}(t) = \textstyle\sum_{n\in\mathbb{Z}} c_n\, \sigma(t-na)$$

- Subordinate signal space: arbitrary Hilbert space $\mathcal{H}$
- The sampling process can be described as the evaluation inner products

$$c_n = y(na) = \langle x, s_n\rangle\ , \qquad n \in \mathbb{Z}$$

- Sampling functions $s_n \in \mathcal{H}$ have the form $s_n = (\mathrm{T}_a^n s)(t) = s(t-na)$

or in general $\quad s_n = \mathrm{U}^n s \quad$ with $\begin{cases} \text{generator } s \in \mathcal{H} \\ \mathrm{U} \text{ unitary operator on } \mathcal{H} \end{cases}$

$\Rightarrow$ Sequence $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ of sampling functions forms a stationary sequence in $\mathcal{H}$
$\Rightarrow$ Sequence $\boldsymbol{s}$ is characterized by its corresponding spectral density $\Phi_{\boldsymbol{s}} \in L^1(\mathbb{T})$

$$\langle s_n, s_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n-m)\theta}\, \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta$$

- Sampling space: $\mathcal{S} := \overline{\mathrm{span}}\{s_n : n \in \mathbb{Z}\}$
- $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for the sampling space $\mathcal{S}$
 $\Leftrightarrow \quad 0 < A \leq \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) \leq B < \infty$ for a.e. $\theta \in [-\pi, \pi)$

M. Unser and A. Aldroubi, "A General Sampling Theory for Nonideal Acquisition Devices," *IEEE Trans. Signal Process.*, vol. 42, no. 11, pp. 2915–2925, Nov. 1994.
T. Michaeli, V. Pohl, and Y. C. Eldar, "U-Invariant Sampling: Extrapolation and Causal Interpolation from Generalized Samples," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2085–2100, May 2011.

Reconstruction – Ideal World

Assumptions
- Let $x \in \mathcal{S}$ be an arbitrary signal
- All past and future signal samples $c_n = \langle x, s_n\rangle$, $n \in \mathbb{Z}$ are known

Goal
Signal reconstruction of the form

$$\widetilde{x}(t) = \textstyle\sum_{n\in\mathbb{Z}} \langle x, s_n\rangle\, \sigma_n(t)$$

such that
- $\widetilde{x}(t) = x(t)$ for all $x \in \mathcal{S}$ (Perfect reconstruction)
- $\langle \widetilde{x}, s_n\rangle = \langle x, s_n\rangle$ for all $n \in \mathbb{Z}$ (Consistency)

Solution
A well known result from frame theory states that the problem is solved by the dual Riesz basis $\{\sigma_n\}_{n\in\mathbb{Z}}$ of $\{s_n\}_{n\in\mathbb{Z}}$, given by

$$\sigma_n(t) = (\mathrm{U}^n\sigma)(t) \tag{1}$$

$$\text{with}\quad \sigma(t) = \sum_{k\in\mathbb{Z}} \alpha_k\, s_{-k}(t) \quad\text{and}\quad \alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{-\mathrm{i}k\theta}}{\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{d}\theta.$$

Often, in reality only the past signal samples are known!

Reconstruction – Real World

New Assumption
Let $\boldsymbol{c}_0 = \{c_0, c_{-1}, \dots\}$ be the past signal samples known at $t = t_0$

Goal

Naive Solution

$$\widetilde{x}_-(t) = \sum_{n=0}^{\infty} c_{-n}\sigma_{-n}(t)\ , \quad t \leq t_0$$

based on the non-causal dual frame (1).

However this reconstruction is not perfect, i.e. $\widetilde{x}_-(t) \neq x_-(t)$, because we need the dual Riesz basis $\{\zeta_{-n}\}_{n=0}^{\infty}$ of $\{s_{-n}\}_{n=0}^{\infty}$!

Main Result – Causal Dual Riesz Basis

Theorem Let $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ be a stationary sequence in a Hilbert space $\mathcal{H}$ which is a Riesz basis for $\mathcal{S} = \overline{\mathrm{span}}\{s_n : n \in \mathbb{Z}\}$ and let $\Phi_{\boldsymbol{s}}$ be the spectral density of $\boldsymbol{s}$. Then $\boldsymbol{s}_0 = \{s_{-n}\}_{n=0}^{\infty}$ is a Riesz basis for $\mathcal{S}_0 = \overline{\mathrm{span}}\{s_{-n} : n = 0, 1, 2, \dots\}$ and the corresponding dual Riesz basis $\{\zeta_{-n}\}_{n=0}^{\infty}$ is given by

$$\zeta_{-n} = \sum_{k=0}^{\infty} \widehat{\psi}_n(k)\, s_{-k}\ , \quad n = 0, 1, 2, \dots$$

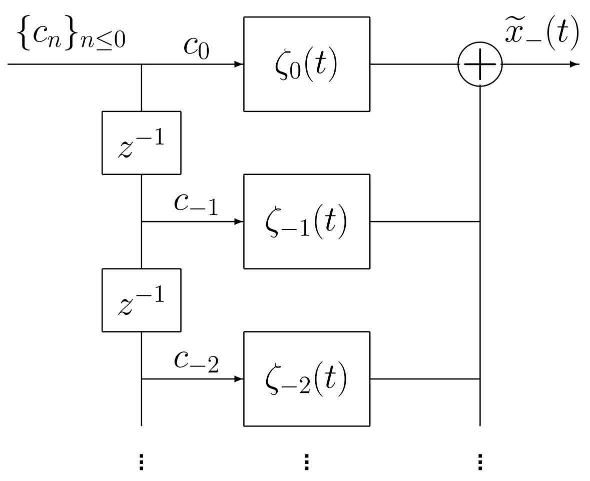
$$\text{with}\quad \widehat{\psi}_n(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_n(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{-\mathrm{i}k\theta}\, \mathrm{d}\theta\ , \quad n = 0, 1, 2, \dots$$

$\psi_n \in H^2$ are defined as

$$\psi_n(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\Phi_{\boldsymbol{s}}^+(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{P}_+\left[\frac{\mathrm{e}^{\mathrm{i}n\theta}}{\Phi_{\boldsymbol{s}}^-(\mathrm{e}^{\mathrm{i}\theta})}\right]\ , \quad n = 0, 1, 2, \dots$$

and wherein $\Phi_{\boldsymbol{s}}^+$ and $\Phi_{\boldsymbol{s}}^-$ are the spectral factors of $\Phi_{\boldsymbol{s}}$.

Overall Causal Reconstruction Scheme



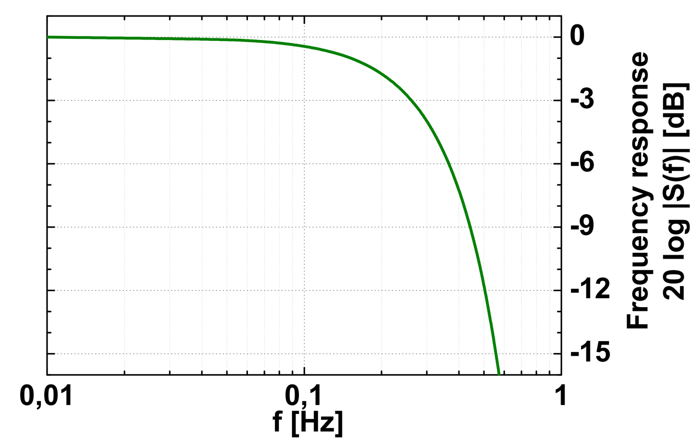
- Reconstructed past signal

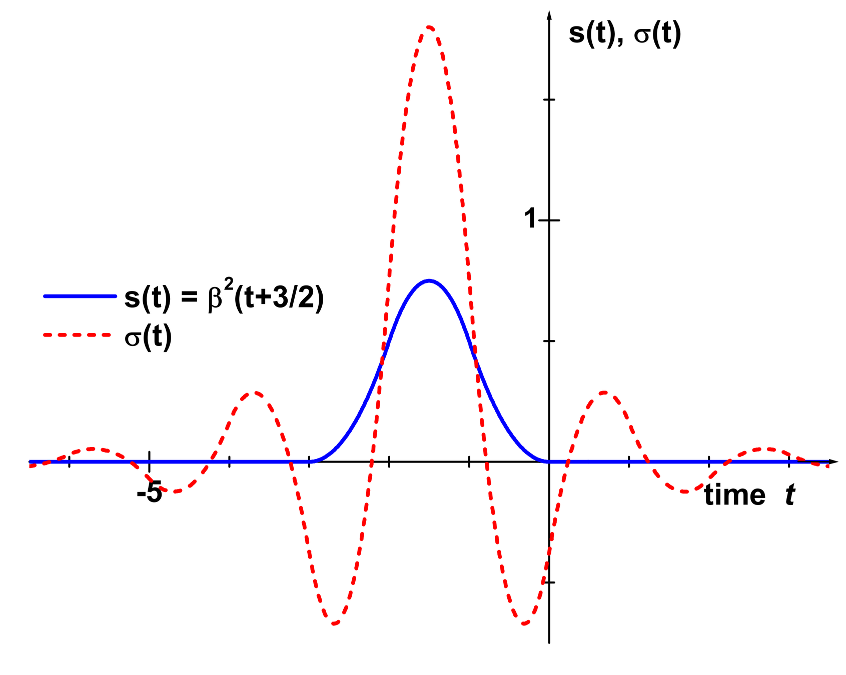
$$\widetilde{x}_-(t) = \textstyle\sum_{n\leq 0} c_{-n}\, \zeta_{-n}(t)\ , \quad t \leq t_0$$

- Each reconstruction kernel $\zeta_{-n}(t)$ has a different shape
- Each sampling value corresponds to the weighting of the respective kernel

Example – Causal Spline Reconstruction

Practical Assumptions
- Shift-invariant sampling in $L^2(\mathbb{R})$ with period $a = 1$: $(\mathrm{U}^n s)(t) = s(t-n)$
- Impulse response $s(t)$ is a B-spline of 2nd degree as a model of a non-ideal lowpass





$s(t)$: generator of the sampling functions
$\sigma(t)$: generator of the non-causal dual basis

Causal versus Non-causal Reconstruction Kernels

- Dashed lines: truncated non-causal reconstruction kernels

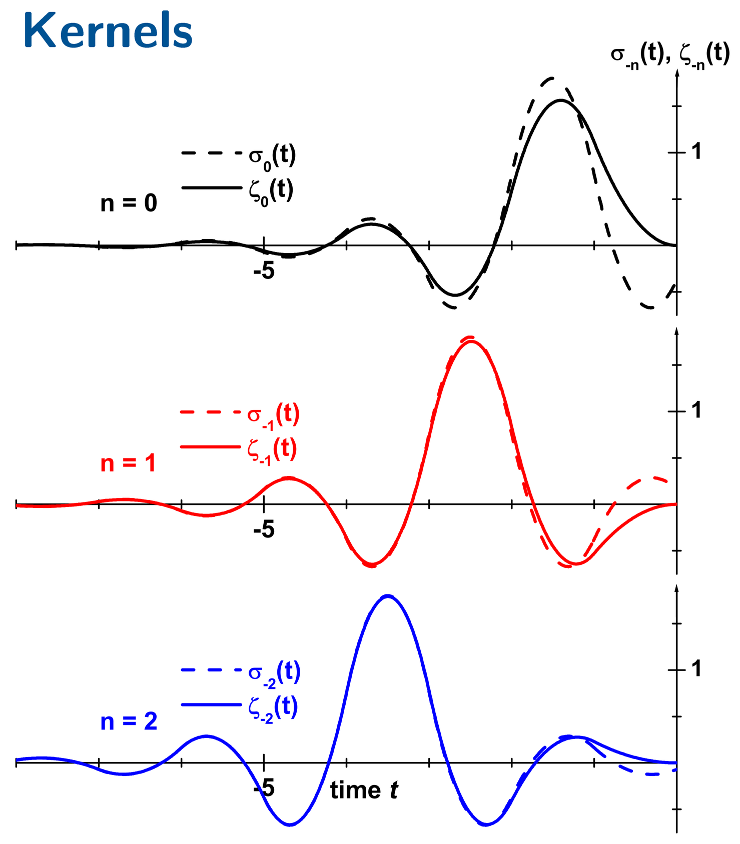
$$\sigma_{-n}(t) = \sum_{k\in\mathbb{Z}} \alpha_k\, s_{-k}(t+n)$$

- Solid lines: causal reconstruction kernels

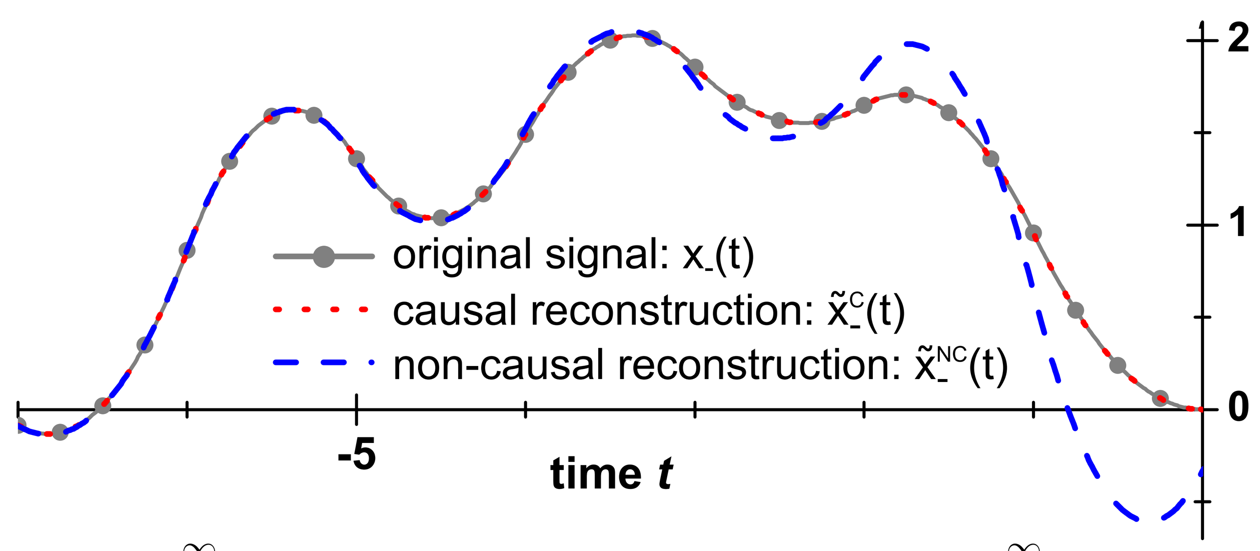
$$\zeta_{-n}(t) = \sum_{k=0}^{\infty} \widehat{\psi}_n(k)\, s_{-k}(t)$$

- For large n, the causal kernel $\zeta_{-n}(t)$ converges to non-causal kernel $\sigma_{-n}(t)$

$$\zeta_{-n} \to \sigma_{-n} \quad \text{as} \quad n \to \infty$$



Signal Reconstruction – Causal versus Non-causal



$$\widetilde{x}_-^{\mathrm{NC}}(t) = \sum_{n=0}^{\infty} \langle x, s_{-n}\rangle\, \sigma_{-n}(t) \qquad\text{and}\qquad \widetilde{x}_-^{\mathrm{C}}(t) = \sum_{n=0}^{\infty} \langle x, s_{-n}\rangle\, \zeta_{-n}(t)$$

- Significant differences close to the border
- Causal and non-causal reconstruction coincide at the distant past $t \to -\infty$

{volker.pohl, fanny.yang, boche}@tum.de