Stellar Evolution – Hints to exercises – Chapter 3

3.1 **Conceptual questions**

(a) Read parahtaph 3.1 of the lecture notes.

(b) Degenareracy is very important in neutron stars, white dwarfs and in the cores of red giants. It also plays a role in low-mass main sequence stars.

(c) See the left panel of Figure 3.2 of the lecture notes. When the density increases the electrons are 'pushed' to higher p-states, because of their higher momenta they exert more pressure.

(d) p_{F} increases when the density increases. As $\upsilon_{\max} = p_{\mathrm{F}}/m_{\mathrm{e}}$ approaches c, the particles become relativistic. Also read paragraph 3.3.5 of the lecture notes.

(e) Read the final section of paragraph 3.3.5: importance of electron degeneracy in stars.

3.2 **Mean molecular weight**

See Section 3.3.3. The mean molecular weight μ is the average mass (in m_{u}) per particle. Assume $m_e \approx 0$, and assume full ionization. The total number of particles contributed by an atom of species i is $N_i = Z_i + 1$. Let n_i be the number/cm^3 of nuclei of species i. Then

$$\mu = \frac{\text{total mass}/m_{\mathrm{u}}}{\text{number of particles}} = \frac{\sum_i A_i n_i}{\sum N_i n_i} = \frac{\sum_i X_i}{\sum (1+Z_i) X_i / A_i}$$

which gives eq. (3.23).

3.3 **The $\rho - T$ plane**

Consider the equations of state for each of these regions: ideal gas P_{i} eq. (3.21), radiation P_{rad} eq. (3.44), NR degeneracy P_{NR} eq. (3.35) and ER degeneracy P_{ER} eq. (3.37).

For the transitions between the regions solve $P_{\mathrm{i}} = P_{\mathrm{rad}}$, $P_{\mathrm{i}} = P_{\mathrm{NR}}$, $P_{\mathrm{i}} = P_{\mathrm{ER}}$ and $P_{\mathrm{NR}} = P_{\mathrm{ER}}$. The solutions are given in Section 3.3.7.

3.4 **The pressure of a gas of free particles**

(a) See the start of Section 3.3.

(b) Follow the hints given in Section 3.3.2.

(c) See Section 3.3.1.

(d) See Section 3.3.4.

(e) See Section 3.3.5.

(f) See Section 3.3.5.

(g) See Section 3.3.5.

(h) Hint: use $\epsilon = pc$, $U = \int_0^\infty pc\, n(p) dp$ and $\displaystyle\int_0^\infty \frac{x^3}{\exp(x) - 1} dp = \frac{\pi^4}{15}$.

3.5 **Adiabatic derivatives**

(a) For an ideal gas $U = \frac{3}{2}NkT$ and $P = NkT/V$. Insert these in the first law of thermodynamics:

$$dU = -PdV \quad \Rightarrow \quad d(\frac{3}{2}NkT) = -\frac{NkT}{V}dV \quad \Rightarrow \quad \frac{3}{2}\frac{dT}{T} = -\frac{dV}{V} \quad \Rightarrow$$
$$d\ln T = -\frac{2}{3}d\ln V \quad \Rightarrow \quad T \propto V^{-\frac{2}{3}} \quad \Rightarrow \quad PV \propto V^{-\frac{2}{3}} \quad \Rightarrow \quad P \propto V^{-\frac{5}{3}} \quad \Rightarrow$$
$$P \propto \rho^{\frac{5}{3}} \tag{1}$$

(b)

$$P \propto \rho^{\frac{5}{3}} \Rightarrow P \propto \frac{P^{\frac{5}{3}}}{T} \Rightarrow T \propto P^{\frac{2}{5}} \Rightarrow d\ln T = \frac{2}{5}d\ln P \Rightarrow \frac{d\ln T}{d\ln P} = \frac{2}{5} \tag{2}$$

(c) $\nabla_{\mathrm{ad}} = \left(\frac{d\log T}{d\log P}\right)_{\mathrm{ad}}$ can be interpreted as a gradient because $\log P$ increases monotonically inside a star in HE, so P can be considered as a coordinate, like r or m.

(d) Hints:
- Write down the equation of state for the mixture in differential form, eq. (3.48), and express χ_T and χ_ρ in terms of β. This should give: $d\log P = \beta\, d\log\rho + (4 - 3\beta)\, d\log T$.
- Write down the internal energy in differential form, and again express this in terms of β. This should give: $(\rho/P)\, du = (12 - \frac{21}{2}\beta)\, d\log T - 3(1-\beta)\, d\log\rho$.
- Apply the first law of thermodynamics to derive the relation between $d\log T$ and $d\log\rho$ for an adiabatic process, again expressed in β.
- Finally eliminate $d\log T$ from the two relations.

(e) $\beta = 0 \Rightarrow \nabla_{\mathrm{ad}} = \frac{4}{3}$ (radiation dominates)
$\beta = 1 \Rightarrow \nabla_{\mathrm{ad}} = \frac{5}{3}$ (ideal gas pressure dominates)

3.6 **Ionization effects**

(a) The fraction E_C/kT, eq. (3.70) is $\ll 1$ for normal stars. In planets it is substantial and can even dominate (solidifaction). For a degenerate gas

$$\frac{E_c}{p_F^2/2m} \propto n^{-1/3},$$

so the effect becomes less important for higher densities. Electrons in stars behave like an ideal gas, but the ions can form a crystallization in sufficiently cool WD.

(b) The Saha equation (which gives that the ions start to recombine at very high densities) is not valid for high densities. For very high densities the 'potential wells' of the ions start to overlap, which leads to fewer bound states. Therefore the energy needed to kick an electron from a bound to a free state becomes lower and eventually zero.

(c) See Section 3.5.1. By compressing an ideal gas adiabatically, the work done is all converted into heat. If the gas is partially ionized, a certain amount of the work done is used to ionize the gas further and only a part is left for heating the gas. In other words, a certain increase of pressure will lead to a larger temperature increase for an ideal gas than for an partially ionized gas.