Stellar Evolution – Hints to exercises – Chapter 2

2.1 **Density profile**

(a) $m(r) = 4\pi\rho_c r^3[\frac{1}{3} - \frac{1}{5}(r/R)^2]$

(b) $M = \frac{8}{15}\pi\rho_c R^3$

(c) Hint: density = mass/volume

2.2 **Hydrostatic equilibrium**

(a) Gravity, pressure gradient.

(b) See Section 2.2.

(c) Hint: take limit for $dr \to 0$.

(d) Hint: use $P(R) = 0$ and the fact that $r \leq R$.

(e) Hint: calculate $m(r)$, substitute it directly in the equation for hydrostatic equilibrium and integrate from the surface to the center to obtain P_c. Answer: $P_c = \frac{15}{16\pi} GM^2/R^4$.

2.3 **The virial theorem**

(a) Use eq. (2.20) and the appropriate expression for $m(r)$. You should get:

for constant density: $E_{\rm gr} = -\frac{3}{5}\frac{GM^2}{R}$

for $\rho(r)$ as in Ex. 2.1: $E_{\rm gr} = -\frac{5}{7}\frac{GM^2}{R}$

(b) Hint: The internal kinetic energy per particle for an ideal gas is $\frac{3}{2}kT$ and the number of particles per unit volume can be written as $\rho/(\mu m_u)$, where μ is the mean molecular weight expressed in atomic units and m_u the mass of one atomic unit.

Now express the kinetic internal energy per volume. Integrate this over the full star.

(c) $E_{{\rm int},\odot} = \frac{3}{2}(k/m_u\mu)M\langle T\rangle \approx 2.1 \times 10^{48}$ erg, assuming $\mu = 0.6$.

$E_{{\rm gr},\odot} \approx -2.3 \times 10^{48}$ erg.

$E_{{\rm tot},\odot} \approx -2 \times 10^{47}$ erg, and the Sun would be (barely) bound. (Note that these estimates are very rough and can only be trusted to an order of magnitude!)

(d) -

(e) For the left-hand side of equation (2.40) substitute P using the ideal gas law: $P = \rho kT/(\mu m_u)$, now compare with equation (2.39).

(f) Use equation (2.40).

(g) Use (c) to estimate $\Delta E_{\rm int}$ and the virial theorem to estimate $\Delta E_{\rm tot}$.

2.4 **Conceptual questions**

(a) *Use the virial theorem to explain why stars are hot, i.e. have a high internal temperature and therefore radiate energy.*

For the gravitational energy $E_{\rm gr}$ of a star we know that

$$E_{\rm gr} = -\alpha\frac{GM^2}{R} < 0, \qquad (1)$$

were $\alpha > 0$ depends on the density distribution $\rho(r)$. Using the virial theorem $E_{\mathrm{int}} = -0.5E_{\mathrm{gr}} > 0$, therefore $T > 0$. Stellar gas is hot, therefore it must radiate, i.e. lose energy form its surface.

(b) *What are the consequences of energy loss for the star, especially for its temperature?*

Energy loss: $\frac{dE_{\mathrm{tot}}}{dt} = -L < 0$. Using the virial theorem:

$$\begin{aligned} \frac{dE_{\mathrm{gr}}}{dt} &= -2L < 0 \quad \text{star contracts} \\ \frac{dE_{\mathrm{int}}}{dt} &= L > 0 \quad \text{star heats up} \end{aligned} \tag{2}$$

The energy liberated from contraction is used for heating the star, the other half is radiated away.

(c) *Most stars are in thermal equilibrium. What is compensating for the energy loss?*

Energy liberated in nuclear reactions, $L = \frac{dE_{\mathrm{nuc}}}{dt}$.

(d) *What happens to a star in thermal equilibrium (and in hydrostatic equilibrium) if the energy production by nuclear reactions in a star drops (slowly enough to maintain hydrostatic equilibrium)?*

If $|dE_{\mathrm{nuc}}/dt| < L$ then $E_{\mathrm{tot}} \downarrow$. Using the virial theorem (which we can use because the star is still in hydrostatic equilibrium), also $|E_{\mathrm{gr}}| \uparrow$ (the star contracts) and $E_{\mathrm{int}} \uparrow$ (the star heats up).

(e) *Why does this have a stabilizing effect? On what time scale does the change take place.*

The temperature increase will accelerate the nuclear reactions until $L = |dE_{\mathrm{nuc}}/dt|$ again. The changes take place on the Kelvin Helmholtz timescale.

(f) *What happens if hydrostatic equilibrium is violated, e.g. by a sudden increase of the pressure.*

The virial theorem does not hold any longer and d^2r/dt^2 in the equation of motion is not zero anymore. An increase in pressure will lift the gas elements outwards, i.e. the star expands. In a normal star the expansion will lead to a shallower pressure gradient. At the same time the gravitional acceleration is reduced. If hydrostatical equilibrium can be restored or not depends on the equation of state (see coming lectures). In a normal main sequence star equilibrium is restored on the hydrodynamical timescale.

In extreme cases the star will explode. (For example when nuclear burning starts in a degenerate star, leading to a thermal nuclear runaway which suddenly lifts the degeneracy, such that the pressure becomes temperature dependent, which leads to an enormous increase of pressure; see coming lectures).