Stellar Evolution – Hints to exercises – Chapter 7

7.1 **General understanding of the stellar evolution equations**

Most answers can be found in the lecture notes, Section 7.1 and 7.3.

7.2 **Dynamical Stability**

(a) See Section 7.4.

(b) See Section 7.5.1.

(c) Stars dominated by radiation pressure (high T / low ρ) or stars where the electrons are extremely relativistic and degenerate. Small disturbances can lead to either collapse or explosion of the star.

(d) Partial ionization can lead to $\gamma_{\rm ad} < 4/3$.

(e) Fe photo-disintegration can occur in all massive stars that form an Fe core after going through successive nuclear burning cycles, and initiates the collapse of the core and a supernova explosion. Pair creation can occur in extremely massive stars.

7.3 **Mass radius relation for degenerate stars**

(a) $P_c = b\,\dfrac{GM^2}{R^4}$ and also $P_c = K_{\rm NR}\left(\dfrac{\rho_c}{\mu_e}\right)^{5/3} = K_{\rm NR}\left(\dfrac{a\,\bar{\rho}}{\mu_e}\right)^{5/3} = K_{\rm NR}\left(\dfrac{a}{\mu_e}\,\dfrac{3M}{4\pi R^3}\right)^{5/3}$.

Equating the two expressions gives $R = \dfrac{K_{\rm NR}}{bG}\left(\dfrac{3a}{4\pi\mu_e}\right)^{5/3} M^{-1/3}$.

Therefore $R \propto M^{-1/3}$: the more massive the white dwarf, the smaller its radius. (Note that a and b in this case correspond to the values for a $n = 1.5$ polytrope, and can be found in Table 4.1.)

(b) A silimar derivation as above shows that the radius drops out, and $M = \left(\dfrac{K_{\rm NR}}{bG}\right)^{3/2}\left(\dfrac{3a}{4\pi\mu_e}\right)^2$ is constant. Taking a and b for an $N = 3$ polytrope (Table 4.1) gives the Cahndrasekhar mass, $M_{\rm Ch} = 1.46\,(2/\mu_e)^2\,M_\odot$.

(c) SNe Ia are thought to occur in binaries where a white dwarf with a mass lower than $M_{\rm Ch}$ accretes from a companion star. As its mass increases and it radius decreases, the white dwarf becomes more dense, until all the electrons become extremely relativistic degenarate. The electron pressure cannot support more weight than $M_{\rm Ch}$ and the star collapses, ...

7.4 **Main-sequence homology relations**

(b) Low-mass stars with $M \lesssim 0.25\,M_\odot$. Stars with $M \gtrsim 1\,M_\odot$ also have a roughly similar density dirtribution.

(c) An ideal gas is assumed, which breaks down for the highest-mass stars where radiation pressure is important. Radiative energy transport is assumed, which is not valid for low-mass stars which have large convective envelopes. κ is assumed to be constant which is in reality a function of density and temperature.

(d) Use $a = 1$ and $b = -3.5$ in equation (6.30). Use equation (6.30) and (6.31) to calculate the relation.

(e) $R \propto M^{0.7} \Rightarrow L \propto \mu^{7.5} M^{5.15}$. Opacity that follows Kramers' law is mostly due to free-free absorption. This is the main source of opacity inside low-mass stars. In more massive stars electron scattering is the main opacity source.

7.5 Central behaviour of the stellar structure equations

(a) For a variable A write $\mathrm{d}A/\mathrm{d}r = \mathrm{d}A/\mathrm{d}m \cdot \mathrm{d}m/\mathrm{d}r$ with $\mathrm{d}m/\mathrm{d}r = 4\pi r^2 \rho$, etc. Apply this to eqs. (7.12)–(7.15).

(b) Write each quantity A as a Taylor expansion around the centre

$$A(r) = A(0) + \left.\frac{\mathrm{d}A}{\mathrm{d}r}\right|_{r=0} r + \frac{1}{2}\left.\frac{\mathrm{d}^2A}{\mathrm{d}r^2}\right|_{r=0} r^2 + \frac{1}{6}\left.\frac{\mathrm{d}^3A}{\mathrm{d}r^3}\right|_{r=0} r^3 + \ldots$$

and keep only the lowest-order term with r^n.

- For the mass $m(r)$, use $\mathrm{d}m/\mathrm{d}r = 4\pi r^2\rho$. Verify that $\mathrm{d}m/\mathrm{d}r$ and $\mathrm{d}^2m/\mathrm{d}r^2$ both vanish at $r = 0$, and that $\mathrm{d}^3m/\mathrm{d}r^3 = 8\pi\rho_c$. Since $m(0) = 0$ you get

  $$m(r) = \frac{4}{3}\pi\rho_c\, r^3 + \ldots$$

  which is simply the mass of a small sphere with constant density ρ_c.
- For the luminosity $l(r)$, it is easier to use $\mathrm{d}l/\mathrm{d}m = \epsilon \equiv \epsilon_{\mathrm{nuc}} - \epsilon_\nu + \epsilon_{\mathrm{gr}}$, and use a Taylor expansion in m rather than r. This gives $l(r) = \epsilon_c\, m(r) + \ldots$ because $l(0) = 0$. Then combine with the expansion for $m(r)$:

  $$l(r) = \frac{4}{3}\pi\rho_c\, \epsilon_c\, r^3 + \ldots$$

- For the pressure $P(r)$, use $\mathrm{d}P/\mathrm{d}r = Gm\rho/r^2$. Verify that $\mathrm{d}P/\mathrm{d}r = 0$ and $\mathrm{d}^2P/\mathrm{d}r^2 = -\frac{4}{3}\pi G\rho_c^2$ at $r = 0$. At the centre $P(0) = P_c$ so that

  $$P(r) = P_c - \frac{2}{3}\pi G\rho_c^2\, r^2 + \ldots$$

- For the temperature $T(r)$ we have to distinguish between radiative and convective transport. In case of *radiative transport*, take eq. (5.16) for $\mathrm{d}T/\mathrm{d}r$ which you can write as

  $$\frac{\mathrm{d}(T^4)}{\mathrm{d}r} = -\frac{3}{4\pi ac}\,\frac{\kappa\rho l}{r^2}$$

  which vanishes in the centre, since $l \propto r^3$ and both κ_c and ρ_c are finite. The second derivative of T^4 in the centre is

  $$\frac{\mathrm{d}^2(T^4)}{\mathrm{d}r^2} = -\frac{3\kappa_c\rho_c}{4\pi ac}\left(\frac{1}{r^2}\frac{\mathrm{d}l}{\mathrm{d}r} - \frac{2}{r^3}\,l\right)_{r=0} = -\frac{\kappa_c\,\rho_c^2\,\epsilon_c}{ac}$$

  so that

  $$T^4(r) = T_c^4 - \frac{\kappa_c\,\rho_c^2\,\epsilon_c}{2ac}\, r^2 + \ldots$$

- In case of *convective energy transport* write $\mathrm{d}T/\mathrm{d}r = -(T/P)\,\nabla_{\mathrm{ad}}\,\mathrm{d}P/\mathrm{d}r$ which can be written as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}r} = \frac{\nabla_{\mathrm{ad}}}{P}\,\frac{\mathrm{d}P}{\mathrm{d}r}$$

  which vanishes in the centre because $\mathrm{d}P/\mathrm{d}r = 0$. The second derivative of $\ln T$ in the centre equals $\mathrm{d}^2\ln T/\mathrm{d}r^2 = (\nabla_{\mathrm{ad}}/P)_c\,\mathrm{d}^2P/\mathrm{d}r^2 = (\nabla_{\mathrm{ad}}/P)_c\,\frac{4}{3}\pi G\rho_c^2$, so that

$$\ln T(r) = \ln T_c - \frac{2\pi G\,\rho_c^2\,\nabla_{\mathrm{ad,c}}}{3P_c}\,r^2 + \ldots$$