I should like to express my gratitude to Dr H. London for drawing my attention, several years ago, to this problem, to Professor R. Peierls and Dr D. Shoenberg for their critical interest, and to Mr R. G. Chambers for help with the experiments.

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Size effect variation of the electrical conductivity of metals

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This paper records experiments and theoretical work concerned with the variation of conductivity with size in metals.

Experimental results for the conductivity in thin wires of pure sodium of varying diameter in the absence of a magnetic field and also in the presence of longitudinal and transverse magnetic fields are given.

Using the general statistical theory of metals the variation of resistance with size in the case of conductivity wires of square cross-section is calculated for comparison with the first set of experiments. A theoretical investigation follows of the alteration in conductivity produced in metallic films by the application of transverse magnetic fields, and this is compared with the corresponding experimental results obtained on the sodium cylinders.

1. Introduction

Towards the end of the last century, it was known that very thin films of metal exhibited a higher specific electrical resistance than the same metal in bulk. In particular, Miss I. Stone (1898) carried out a detailed experimental study of films of silver deposited by the 'Rochelle salt process'. J. J. Thomson (1901) was the first to suggest that the source of the phenomenon lay in the limitation of the mean free

path of the conduction electrons by the geometry of the specimen when its size became sufficiently small to be of the order of the mean free path due to electron-lattice collisions. However, when we remark that the mean free path of electrons in, say, bulk silver is $\sim 10^{-5}$ cm. at 0° C, it becomes obvious that very thin films of metal will be required for experiments at ambient temperatures. This limitation in turn calls forth a number of subsidiary resistive effects, often time dependent, which arise from the mode of preparation of such thin films. These effects, although themselves of interest (see Andrew 1949 for references), naturally tend to obscure the pure geometrical limitation of free path which it is desired to examine. On the other hand, the advent of low temperatures has enabled fresh work to be undertaken on relatively large specimens, since the conductivity of pure metals increases greatly at low temperatures. Lovell (1936) examined films of rubidium at temperatures around 70° K, attained by pumping liquid oxygen and nitrogen. These films were still rather thin (~ 40 Å), since the increase of mean free path in bulk at such a temperature is only about four times the room-temperature values. However, Andrew (1949) has recently extended the field to the liquid-hydrogen and liquid-helium region (20° K and below). Pure tin, as used by Andrew, may drop in electrical resistance at temperatures $\sim 2°$ K to lower than 10^{-4} of its room temperature value, indicating a bulk electronic mean free path $\sim 10^{-2}$ cm. Consequently, appreciable size-effect variation may now be expected to occur in specimens as thick as $100\,\mu$ (10^{-2} cm.). Andrew examined pure mercury in capillaries down to $\sim 6\,\mu$ diameter and tin in foils as thin as $\sim 3\,\mu$ and found generally satisfactory agreement with theory. Part of the latter he developed himself to give an approximate solution for the size-effect in cylinders, previous theoretical work (Thomson 1901; Lovell 1936; Fuchs 1938) having been applied to planar models ('films').

All the theoretical investigations have assumed the simple isotropic free-electron model of a metal. Since neither tin nor mercury specifically are simple metals, it was felt worth while to undertake work on sodium which as an alkali metal might be expected *a priori* to approximate to the ideal metal and has also been found (MacDonald & Mendelssohn 1948, 1950; MacDonald 1950) through detailed low-temperature work to obey almost perfectly the free-electron model. Size-effect variation has been observed at temperatures between 20 and 4·2° K on very pure sodium in capillaries between 15 and $66\,\mu$ diameter, and theory has been developed, extending Fuchs's analysis, for comparison with these experiments. A new phenomenon has also been observed; the application of transverse and longitudinal magnetic fields produces a marked variation of electrical resistance under these conditions. In particular, a longitudinal magnetic field produces a monotonic decrease of the resistance which tends towards the bulk value for large magnetic fields. These effects would be expected to arise if one considers the alteration produced in the shape of the electron orbits in relation to the size of the specimen.

The discovery of the effect has occasioned theoretical investigation by Sondheimer (1949) and Chambers (1950).* We ourselves have considered the case of a transverse magnetic field applied in the plane of a film, since this situation appeared to us closest to the experimental conditions involved. A preliminary note appeared

* We are grateful to Mr Chambers for letting us see his manuscript before publication.

in *Nature* (Sarginson & MacDonald 1949). Good qualitative agreement is found with the experiments, but it appears still desirable to obtain experimental data in the future on sodium *films* in magnetic fields in order to test the theory more closely.

The investigation also provides some information on the variation of the Hall field across the film.

2. Experiments. By D. K. C. MacDonald

2·1. *Preparation of specimens*

Very pure sodium, obtained through the courtesy of Messrs British Thomson-Houston Co. Ltd., was used throughout. Experiments on 'large' specimens (MacDonald & Mendelssohn 1948, 1950) showed this to have a residual resistance

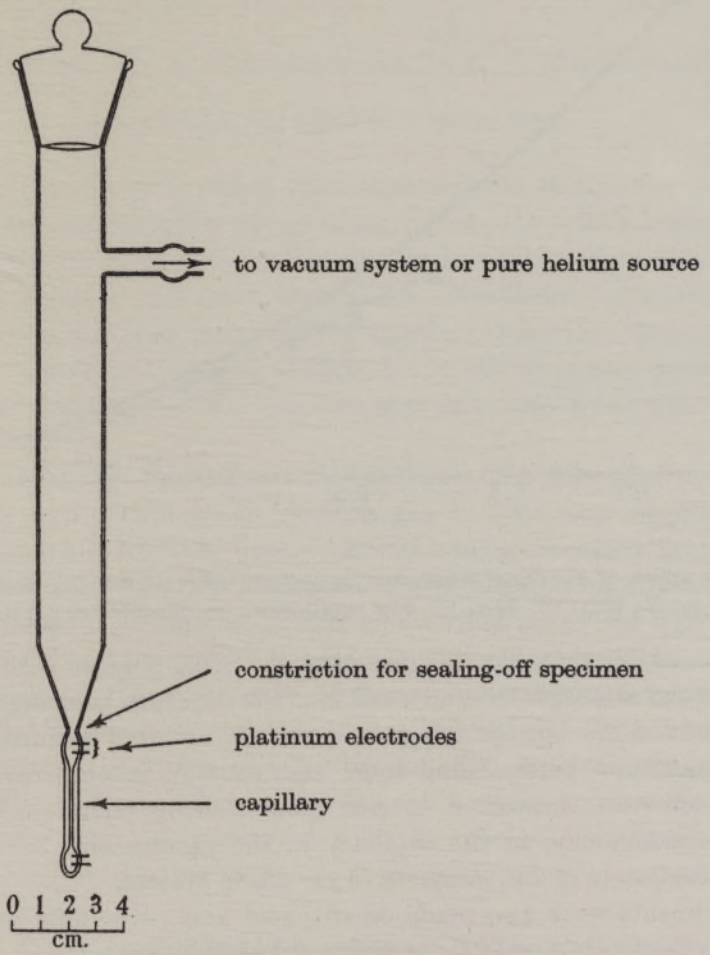

Figure 1. Typical capillary mould.

$\sim 7{\cdot}2 \times 10^{-4}$ of its room-temperature value when measured in liquid helium (4·2° K). The specimens were prepared by melting sodium *in vacuo* in a soft-glass system (see figure 1) and then pressing them into soft-glass capillary moulds by pure helium gas at 1 atm. pressure. The capillary specimens were provided with platinum potential and current electrodes at either end. After solidification the specimens were cracked off at the constriction and then immediately sealed with a suitable compound.

It has been found that capillaries down to $\sim 30\mu$ diameter can be filled quite readily with sodium, but very considerable difficulty was experienced with the narrower capillaries. Since sodium 'wets' clean glass this difficulty is not to be ascribed to surface-tension effects as was the case in Andrew's experiments on mercury. This was in fact verified, since pressurizing with helium at ~ 4 atm. did not aid in trying to fill 20μ capillaries. It was ultimately found that, owing to the high chemical activity of the sodium, the utmost cleanliness and freedom from any form of contamination were essential to permit filling. Thus, when a specimen was allowed

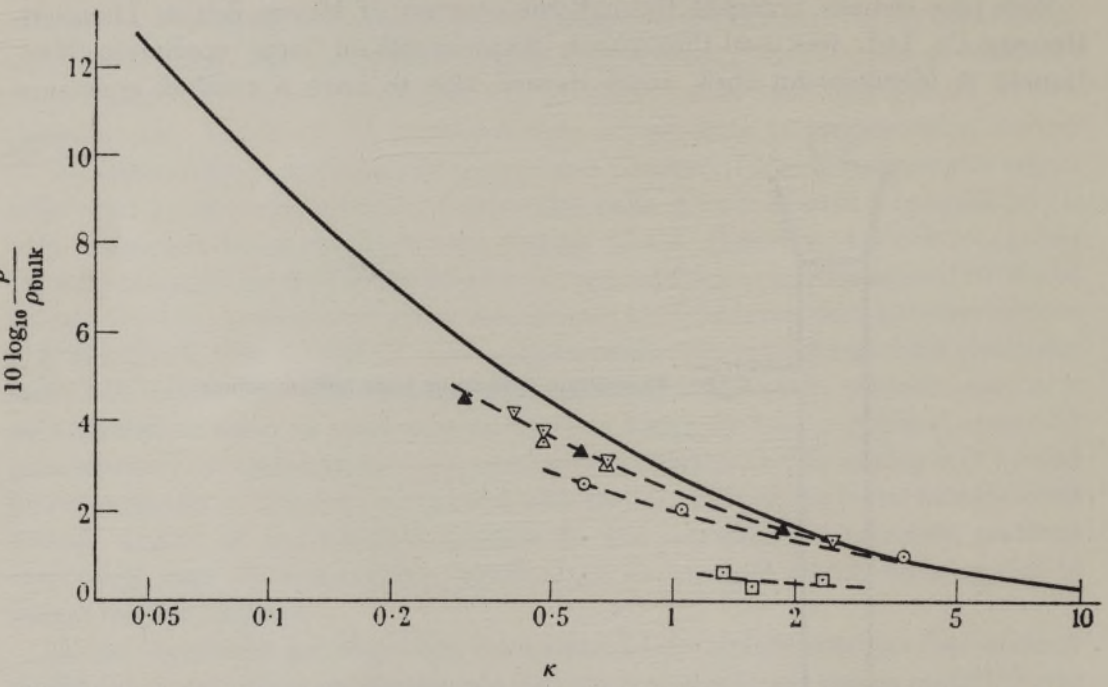

FIGURE 2. Size effect of electrical resistance in sodium wires of varying diameter: ▲, 15μ; △, ▽, 20μ; ⊙, 30μ; ⊡, 66μ capillaries. —, theoretical curve.

to lie for an hour after glass-blowing (the open end being plugged with cotton-wool), moisture from the atmosphere condensed into the capillary creating a microscopic deposit sufficient to prevent the subsequent passage of liquid sodium, although the mould was 'baked out' before filling under high vacuum at temperatures $\sim 150°$ C. Success was achieved through a specific glass-blowing technique* designed to prevent any condensation in the capillary in the process and by attaching the apparatus immediately it was complete to the pump system.

Some experiments were also made on tin, gold and silver foils and these were rolled down to thicknesses $\sim 6\mu$ from spectrographically pure rod or wire obtained from Messrs Johnson, Matthey and Co. Ltd. The rollers were carefully cleaned before use to avoid any surface contamination.

2·2. *Experimental methods*

Measurements were carried out in both a Simon expansion helium liquefier (Simon 1936) and also a simple hydrogen cryostat for some measurements between ~ 11 and

* I am indebted to Mr D. G. Saxton for his patience and trouble in this aspect of the work.

20° K. Temperatures in both cases were measured with a helium gas thermometer. The electrical resistance was measured in some cases with a Tinsley potentiometer and in the wider specimens with a high-impedance galvanometer amplifier (MacDonald, 1947) using negative feed-back.

2·3. *Experimental results*

The experimental observations on the simple size effect are shown in figure 2. The results on the various specimens are plotted as the broken lines with the experimental points; the theoretical curve (to be discussed below) is shown as the full line.* Figures 5 and 6 show the results obtained for the change of conductivity under transverse and longitudinal magnetic fields which are to be compared with the theory.

3. Theory. By K. Sarginson and D. K. C. MacDonald

3·1. *Simple size-effect in a square wire*

3·11. *General*

Thomson (1901) and Lovell (1936) both analyzed the increase of resistance in a thin film using the concept of limitation of mean free path by the dimensions of the conductor; the method is fundamentally similar to that used in the elementary kinetic theory analysis of transport phenomena. Nordheim (1931, 1934), in considering the case of a thin wire, essentially adapted an expression appropriate to the flow of gas down a tube of diameter comparable to the molecular mean free path. Andrew (1949) treated the case of a thin wire approximately by an extension of the Thomson-Lovell method.

Fuchs (1938), however, approached the problem of a thin film using the full statistical theory appropriate to an electron gas in dynamical equilibrium. This method dates principally from the time of Lorentz and has become a familiar feature of the modern theory of metals. The derivation of a (probability) distribution function of velocity and position yielding statistical equilibrium under the influence of the applied (electric 'E' and magnetic 'H') fields and the electron-lattice collisions is basic to this treatment. The expression of this requirement leads in general to an integro-differential equation for the distribution function, N. This is known as the Boltzmann equation. Given the distribution function as a solution, the electric current (and hence the electrical conductivity) and the heat current (if any) may be derived, essentially as velocity moments. In the case of limitation by size, the scattering at the physical edges enters as appropriate boundary conditions to be imposed on the solution.

More recently, Dingle (1950)† has been dealing in this way with a *cylindrical* wire, while Chambers (1950) has used generalized kinetic theory methods to treat the same problem. We, however, have extended Fuchs's treatment to the analysis of a *square* wire of side a. In all these treatments the electron-lattice collisions are assumed to be

* The abscissa is the ratio of the specimen diameter as determined by the bore of the capillary to the electron mean free path in bulk metal. The latter is known from detailed low temperature experiments on a 'large' specimen of the same material.

† We have not yet had the opportunity of reading Dr Dingle's paper.

sufficiently represented by a simple isotropic relaxation time, τ, throughout. This corresponds precisely to the assumption mentioned earlier of the idealized quasi-free electron model of a metal. The introduction of this relaxation time, related to the bulk conductivity, σ by the equation $\sigma = \frac{\nu e^2 \tau}{m}$ (cf. Wilson 1936) (where ν is the number of free electrons per unit volume), reduces the Boltzmann equation to a partial differential equation. It is also generally assumed, following Lorentz, that the distribution function, N, under the applied fields differs only slightly from that, N_0, existing under true equilibrium conditions ($E = H \equiv 0$). This corresponds to the fact that, in metals, the average energy gained by an electron from the fields between collisions with the lattice is very small compared with the intrinsic Fermi energy. Finally, it is assumed that no restriction need be made on the magnitude of H in the solution of the Boltzmann equation. This corresponds to neglect of possible quantization of the electron orbits in the magnetic field (cf. Țițeica 1935; Wilson 1936), and is therefore in harmony with a purely classical treatment of the metal. Since sodium in bulk exhibits a very small magneto-resistive effect it would appear that the neglect of quantization is then justifiable, but caution must be exercised when considering more complex metals.

3·12. *Detailed analysis*

Let the sides of the square cross-section of the wire be a. Choose axes Ox, Oy, Oz with Ox, Oy along sides of the square and Oz along one edge of the wire.

The appropriate Boltzmann equation is

$$v_x \frac{\partial n}{\partial x} + v_y \frac{\partial n}{\partial y} + \frac{n}{\tau} = \frac{eE}{m} \frac{v_z}{v} \frac{\partial N_0}{\partial v}, \tag{1}$$

where $n = N - N_0$,

$N(x, y, z, v_x, v_y, v_z)$ is the actual distribution function,

$N_0(v_x, v_y, v_z)$ being that with $E = 0$,

and v is the velocity of the electrons on the Fermi surface.

The general solution of (1) is

$$n = \frac{eE}{m} \frac{\tau v_z}{v} \frac{\partial N_0}{\partial v} \left(1 - e^{-x/\tau v_x} f\left(\frac{y}{v_y} - \frac{x}{v_x}\right)\right), \tag{2a}$$

or equally well

$$n = \frac{eE}{m} \frac{\tau v_z}{v} \frac{\partial N_0}{\partial v} \left(1 - e^{-y/\tau v_y} \phi\left(\frac{x}{v_x} - \frac{y}{v_y}\right)\right), \tag{2b}$$

where f and ϕ are arbitrary functions to be determined from the boundary conditions. If we assume inelastic electron scattering at the boundaries, then we require that n be independent of the ratio $v_x : v_y : v_z$ at $x = 0, a$; $y = 0, a$ for those electrons *leaving* the boundaries. This can only be realized if $n = 0$ for these conditions. The complete distribution function can then be most simply represented by the scheme of table 1 in velocity space.

TABLE 1

	v_y
$n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{-y/\tau v_y})$, $y<-\frac{v_y}{v_x}(a-x)$ $n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{(a-x)/\tau v_x})$, $y>-\frac{v_y}{v_x}(a-x)$	$n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{-y/\tau v_y})$, $y<\frac{v_y}{v_x}x$. $n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{-x/\tau v_x})$, $y>\frac{v_y}{v_x}x$
$n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{(a-x)/\tau v_x})$, $(a-x)<\frac{v_x}{v_y}(a-y)$ $n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{(a-y)/\tau v_y})$, $(a-x)>\frac{v_x}{v_y}(a-y)$	$n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{-x/\tau v_x})$, $x<-\frac{v_x}{v_y}(a-y)$ $n=\frac{eE\tau}{m}\frac{v_z}{v}\frac{\partial N_0}{\partial v}(1-e^{(a-y)/\tau v_y})$, $x>-\frac{v_x}{v_y}(a-y)$

v_x

It is obvious that each quadrant in velocity space will contribute equally to the total current J_z; thus

$$J_z=\frac{4}{a^2}\iiint_{\substack{v_y>0\\ v_x/v_y>1}}\left(\frac{e^2E\tau}{mv}\frac{\partial N_0}{\partial v}v_z^2\int_{x=0}^{a}\left\{\int_0^{(v_y/v_x)x}(1-e^{-y/\tau v_y})\,dy+\int_{(v_y/v_x)x}^{a}(1-e^{-x/\tau v_x})\,dy\right\}dx\right)dv_x dv_y dv_z$$

$$+\frac{4}{a^2}\iiint_{\substack{v_x>0\\ v_y/v_x>1}}\left(\frac{e^2E\tau}{mv}\frac{\partial N_0}{\partial v}v_z^2\int_{y=0}^{a}\left\{\int_0^{(v_x/v_y)y}(1-e^{-x/\tau v_x})\,dx+\int_{(v_x/v_y)y}^{a}(1-e^{-y/\tau v_y})\,dx\right\}dy\right)dv_x dv_y dv_z \quad (3)$$

$$=\frac{4}{a^2}\left(a^2\iiint_{\substack{v_x>0\\ v_y>0}}\frac{e^2E\tau}{mv}\frac{\partial N_0}{\partial v}v_z^2\,dv_x dv_y dv_z-2\iiint_{0<v_x<v_y}\frac{e^2E\tau}{mv}\frac{\partial N_0}{\partial v}v_z^2\left\{a\tau v_y(1-e^{-a/\tau v_y})+a\tau v_x(1+e^{-a/\tau v_y})-2\tau^2v_xv_y(1-e^{-a/\tau v_y})\right\}dv_x dv_y dv_z\right). \quad (4)$$

Thus

$$\frac{\sigma}{\sigma_{\text{bulk}}}\equiv\frac{\rho_{\text{bulk}}}{\rho}$$

$$=1-\frac{2\iiint_{0<v_x<v_y}v_z^2\left(\frac{\tau v_y}{a}\{1-e^{-a/\tau v_y}\}+\frac{\tau v_x}{a}\{1+e^{-a/\tau v_y}\}-\frac{2\tau^2v_xv_y}{a^2}\{1-e^{-a/\tau v_y}\}\right)dv_x dv_y dv_z}{\iiint_{\substack{v_x>0\\ v_y>0}}v_z^2\,dv_x dv_y dv_z}. \quad (5)$$

The method to be adopted in evaluating (5) depends on the value of the parameter $\kappa \equiv a/l$, where l, the mean free path, $= v\tau$, by definition.

For κ large (i.e. incipient size effect), we may express (5) as

$$\frac{\sigma}{\sigma_{\text{bulk}}} = 1 - \frac{6}{\pi}\left(\frac{\pi}{8\kappa} - \frac{2}{15\kappa^2}\right) + \frac{12}{\pi\kappa}\int_{\theta=0}^{\frac{1}{2}\pi}\int_{\phi=0}^{\frac{1}{2}\pi} e^{-\kappa/(\sin\theta\cos\phi)} \cos\phi \sin^2\theta \cos^2\theta \, d\theta \, d\phi$$

$$+ \frac{12}{\pi\kappa^2}\int_0^{\frac{1}{2}\pi} (\tfrac{1}{2} e^{-\sqrt{2}\kappa/(\sin\theta)} - e^{-\kappa/(\sin\theta)}) \sin^3\theta \cos^2\theta \, d\theta. \quad (6)$$

By then considering upper bounds for the latter integral terms we have for $\kappa \geqslant 4$, with an upper limit of error $\sim 2/1000$,

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx 1 - 0{\cdot}75/\kappa + 0{\cdot}254/\kappa^2. \quad (7)$$

It is interesting to compare (7) with the expression obtained by Fuchs for a *film* of thickness a under the same condition, namely,

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx 1 - 0{\cdot}375/\kappa + \ldots. \quad (7a)$$

Comparison of the terms in $1/\kappa$ in (7) and (7a) shows the independently additive limitation imposed by the two boundaries in the square wire in the limit of incipient size effect.

On the other hand, for small κ (intense size effect) proceeding from (5) and (6) we may obtain an expansion in powers of κ, which finally yields the approximate formula for $\kappa \ll 1$:

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx \tfrac{3}{4}\kappa \log_e\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \tfrac{1}{2}\kappa(\sqrt{2}-1)$$

$$= 1{\cdot}115\kappa. \quad (8)$$

It is again interesting to compare Fuchs's corresponding expression for a thin *film*:

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx \frac{3\kappa}{4}\log_e\left(\frac{1}{\kappa}\right). \quad (8a)$$

The much more rapid decrease of conductivity in the case of the 'wire' is immediately evident.

We further note that Andrew's (1949) formula for a cylindrical wire of diameter b under the condition $\kappa' \equiv b/l, \ll 1$, is simply

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx \kappa'. \quad (8b)$$

If now we set $b = \dfrac{2}{\sqrt{\pi}}a$ so that the cross-sectional areas in Andrew's and our cases are equal we have from (8b)

$$\frac{\sigma}{\sigma_{\text{bulk}}} \approx 1{\cdot}128\kappa, \quad (8c)$$

in very close agreement with (8).

Finally, for values of $\kappa \sim 1$, we may continue from (5) to obtain single integrals tolerably convenient for direct numerical integration. We find

$$\frac{\sigma}{\sigma_{\text{bulk}}} = 1 - \frac{3}{\pi}\left(\frac{\pi}{4\kappa} - \frac{4}{15\kappa^2}\right) + \frac{12}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{\pi}{4\kappa}\cos\theta - \frac{1}{3\kappa}\sin\theta - \frac{1}{3\kappa^2}\sin 2\theta\right) e^{-\kappa/(\cos\theta)} \sin^3\theta\, d\theta$$

$$+\frac{12}{\pi}\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left\{\frac{\cos\theta}{\kappa}\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\cos^{-1}(\cot\theta)\right) + \frac{\cot^2\theta}{2\kappa}\sqrt{(-\cos 2\theta)} - \frac{1}{3\kappa}\left(\sin\theta - \frac{(-\cos 2\theta)^{\frac{3}{2}}}{\sin^2\theta}\right)\right.$$

$$\left. - \frac{2}{3\kappa^2}\left(\sin\theta\cos\theta - \frac{\cos\theta}{\sin^2\theta}(-\cos 2\theta)^{\frac{3}{2}}\right)\right\} e^{-\kappa/(\cos\theta)} \sin^3\theta\, d\theta. \quad (9)$$

For $\kappa = 1$ we find $\sigma/\sigma_{\text{bulk}} = 0{\cdot}517$, and for $\kappa = 0{\cdot}33$ we find $\sigma/\sigma_{\text{bulk}} = 0{\cdot}277$.

3·13. *Comparison with experiment*

In figure 2, using (7), (8) and (9), we have plotted $10\log_{10}\rho/\rho_{\text{bulk}}$ against κ. The experimental data obtained at various temperatures between 4·2 and 20·4° K on cylindrical specimens of diameter 15, 20 (two specimens), 30 and 66 μ are also plotted thereon; the diameters, b, have been converted to an effective a using the relation $a = \frac{\sqrt{\pi}}{2}b$ as above.

We notice immediately that all the experimental points lie below, or on, the theoretical curve, suggesting strongly that no impurity additional to that in a bulk specimen is present which would yield an artificially high ('residual') resistance at low temperatures. The general feature that all the results tend to lie below the theoretical curve suggests that the hypothesis of entirely diffuse scattering at the boundaries is not generally fulfilled. Furthermore, the uniform trend towards closer agreement with theory with reduction of diameter suggests strongly that the degree of elastic scattering is a function of the specimen size.

Two alternatives present themselves. On the one hand, it is possible that this is a property inherent in the size of the specimen itself, such that the uniformity of the crystalline structure of the surface depends in some way significantly on the volume of the sample. In order to examine such a hypothesis it would first be necessary to determine the surface conditions required to provide effective elastic scattering in a particular case. In the first place, if the surface were presumed irregular within a range less than the 'extent' of the electron then it seems evident that wholly diffuse scattering must ensue. If we turn to the uncertainty relation $\Delta x \Delta p_x \sim h$ to determine Δx, we might set $\Delta p \sim p$, where p is the magnitude of the momentum on the Fermi surface. In this case $\Delta x \sim \lambda$, the de Broglie wave-length, and $\lambda \approx 7 \times 10^{-8}$ cm. for sodium, assuming one free electron per atom. Alternatively, one might set $\Delta p \sim \sqrt{(2mkT)}$; for $T \sim 10°$ K this yields $\Delta x \approx 4 \times 10^{-6}$ cm. Both of these values are, however, very small compared with the size of our specimens, and therefore a *correlation* in this sense seems very improbable.

On the other hand, since the specimens were formed in drawn glass capillaries it appears possible that the intrinsic surface irregularities of the glass might be correlated with the diameter to which the capillary is drawn and hence impressed on the metal surface.

3·2. *Influence of a transverse magnetic field on the electrical conductivity of a film*

3·21. *The distribution function*

In this section we investigate the change in conductivity produced in a thin film under the action of a transverse magnetic field in the plane of the film. Let a right-handed set of axes be taken so that the x-axis is in the direction of the applied electric field E, the y-axis is in the direction of the magnetic field H, and the z-axis is perpendicular to the plane of the film. The Hall field will be in the direction of the z-axis. It will not be constant as in cases hitherto considered (e.g. Sondheimer 1949; Chambers 1950), but will vary in magnitude across the film. Let, then, the Hall field be $F(z)+\frac{El}{r}$, where $\frac{El}{r}$ is the Hall field in bulk metal, r being the radius of the classical orbit of a free electron in the magnetic field. Further, in velocity space, we introduce polar angles χ, ψ such that $v_x = v\sin\chi\sin\psi$, $v_y = v\cos\chi$, $v_z = v\sin\chi\cos\psi$. The Boltzmann equation now reads

$$v\sin\chi\cos\psi\frac{\partial N}{\partial z}+\frac{eH}{mc}\frac{\partial N}{\partial\psi}-\frac{1}{\tau}\left(-N+\frac{1}{4\pi}\iint N(v,z)\,d\omega\right)$$
$$=\frac{e}{m}\frac{\partial N}{\partial v}\left(E\sin\chi\sin\psi+\left\{F(z)+\frac{El}{r}\right\}\sin\chi\cos\psi\right), \quad (10)$$

where the integral involved is to be taken over the Fermi surface, $d\omega$ being an element of solid angle.

The term $\frac{1}{\tau}\left(-N+\frac{1}{4\pi}\iint N(v,z)\,d\omega\right)$ gives the net change in N per unit time due to collisions. τ is the probability per unit time of an electron being scattered, and we assume that this has the same value as in bulk metal. In the case of bulk metal the integral term vanishes on the grounds of symmetry and τ may then be identified with the relaxation time (cf. Wilson 1936, p. 159).

We assume, as usual, that N can be expressed as $N = N_0 + n$, where n is small. Then

$$v\sin\chi\cos\psi\frac{\partial n}{\partial z}+\frac{eH}{mc}\frac{\partial n}{\partial\psi}+\frac{n}{\tau}$$
$$=\frac{e}{m}\frac{\partial N_0}{\partial v}\left(E\sin\chi\sin\psi+F(z)\sin\chi\cos\psi+\frac{El}{r}\sin\chi\cos\psi\right)+\frac{1}{4\pi}\iint n(v,z)\,d\omega.$$

Let $n=\frac{1}{v}\frac{e}{m}\frac{\partial N_0}{\partial v}\phi(v_x,v_y,v_z,z)$, and $\frac{mcv}{eH}=r$, the orbit radius, giving

$$\sin\chi\cos\psi\frac{\partial\phi}{\partial z}+\frac{1}{r}\frac{\partial\phi}{\partial\psi}+\frac{\phi}{l}$$
$$=E\sin\chi\sin\psi+F(z)\sin\chi\cos\psi+\frac{El}{r}\sin\chi\cos\psi+\frac{1}{4\pi l}\iint\phi(v,z)\,d\omega. \quad (11)$$

Let the excess electron charge density at the point z be $\rho(z)$, then

$$\rho(z)=-e\iiint n\,dv_x dv_y dv_z$$
$$=\frac{2e^2m^2v}{h^3}\iint\phi(v,z)\,d\omega, \quad (12a)$$

and Poisson's equation requires

$$\frac{\partial F(z)}{\partial z} = 4\pi\rho = \frac{8\pi e^2 m^2 v}{h^3}\iint \phi(v,z)\, d\omega. \tag{12b}$$

Thus ϕ must satisfy

$$\sin\chi\cos\psi\frac{\partial\phi}{\partial z}+\frac{1}{r}\frac{\partial\phi}{\partial\psi}+\frac{\phi}{l} = E\sin\chi\sin\psi+\frac{El}{r}\sin\chi\cos\psi+F(z)\sin\chi\cos\psi+\gamma l\frac{\partial F}{\partial z},$$

where $\gamma = \dfrac{h^3}{32\pi^2 e^2 l^2 m^2 v}$, $\sim 10^{-11}$, when $l \approx 45\times10^{-4}$ cm., $v \approx 10^8$ cm./sec. (the appropriate values for our sodium metal at $\sim 4°$ K). Letting $z/r = \sin\chi\sin\eta - \xi$ and $\psi = \eta$, we have

$$\frac{1}{r}\frac{\partial\phi}{\partial\eta}+\frac{\phi}{l} = E\sin\chi\sin\eta + \frac{El}{r}\sin\chi\cos\eta + F(r\sin\chi\sin\eta - r\xi)\sin\chi\cos\eta + \gamma l F'(r\sin\chi\sin\eta - r\xi),$$

which has as a general solution

$$\phi = El\sin\chi\sin\eta + r\,e^{-(r/l)\eta}\int^{\eta}\{F(r\sin\chi\sin t - r\xi)\sin\chi\cos t + \gamma l F'(r\sin\chi\sin t - r\xi)\}\,e^{(r/l)t}\,dt, \tag{13}$$

where the lower limit of integration is arbitrary.

The solutions which satisfy the assumed boundary conditions, namely diffuse scattering at the edges, are set out in table 2 according to their regions of validity in co-ordinate and velocity space. The value of the inverse sine in the expressions is to be taken in the range $-\pi/2$ to $+\pi/2$.

In the solutions, A is a constant to be determined by the condition that $F(z)$ must vanish at $z = 0, d$. From table 2, when $z = 0$, $\phi = A$ for $v_z > 0$. From (12a), the excess density of electrons leaving $z = 0$ is therefore $\dfrac{2e^2m^2v}{h^3}2\pi A$, and the total excess density of electrons at $z = 0$ is twice this. Thus

$$\left(\frac{\partial F(z)}{\partial z}\right)_{z=0} = \frac{32\pi^2e^2m^2v}{h^3}A = \frac{A}{\gamma l^2};$$

we note that A is an odd function of H.

The function Φ_2 gives the distribution at points in the film for values of χ and ψ such that electrons scattered from either edge could not directly reach the point with such values of χ and ψ. The range concerned is determined by the geometry of the film coupled with the appropriate classical orbits. The arbitrary limit of integration in this function is determined by the condition that Φ_2 must assume the same value at $\psi = -\frac{1}{2}\pi$ and $\psi = \frac{3}{2}\pi$ for a given χ and z.

3·22. *The Hall field*

In order to obtain a complete solution for the conductivity it is necessary to determine the Hall field as a function of z and H universally. In principle this

TABLE 2. SOLUTION OF THE TRANSPORT EQUATION

I: $z<r\sin\chi(1+\sin\psi)$, $d-z>r\sin\chi(1-\sin\psi)$

$$\underline{\phi = El\sin\chi\sin\psi+\Phi_1},$$

where

$$\Phi_1 = r\,e^{-(r/l)\psi}\int_{\sin^{-1}\left(\sin\psi-\frac{z}{r\sin\chi}\right)}^{\psi}[F(z-r\sin\chi(\sin\psi-\sin t))\sin\chi\cos t + l\gamma F'(z-r\sin\chi(\sin\psi-\sin t))]\,e^{(r/l)t}\,dt - \left[El\left(\sin\chi\sin\psi-\frac{z}{r}\right)-A\right]\exp\left[-\frac{r}{l}\left(\psi-\sin^{-1}\left(\sin\psi-\frac{z}{r\sin\chi}\right)\right)\right],$$

when $-\frac{1}{2}\pi<\psi<\frac{3}{2}\pi$

II: $z>r\sin\chi(1+\sin\psi)$, $d-z>r\sin\chi(1-\sin\psi)$

$$\underline{\phi = El\sin\chi\sin\psi+\Phi_2},$$

where

$$\Phi_2 = r\,e^{-(r/l)\psi}\int_{-\frac{1}{2}\pi}^{\psi}[F(z-r\sin\chi(\sin\psi-\sin t))\sin\chi\cos t+l\gamma F'(z-r\sin\chi(\sin\psi-\sin t))]\,e^{(r/l)t}\,dt + \frac{re^{-(r/l)\psi}}{e^{(r/l)2\pi}-1}\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi}[F(z-r\sin\chi(\sin\psi-\sin t))\sin\chi\cos t + l\gamma F'(z-r\sin\chi(\sin\psi-\sin t))]\,e^{(r/l)t}\,dt,$$

when $-\frac{1}{2}\pi<\psi<\frac{3}{2}\pi$

III: $z<r\sin\chi(1+\sin\psi)$, $d-z<r\sin\chi(1-\sin\psi)$ (this domain exists only when $d<2r$)

$$\underline{\phi = El\sin\chi\sin\psi+\Phi_3},$$

where $\Phi_3=\Phi_1$, when

$$\sin^{-1}\left(\frac{z}{r\sin\chi}-1\right)<\psi<\pi+\sin^{-1}\left(\frac{d-z}{r\sin\chi}-1\right),$$

and $\Phi_3=\Phi_4$, when

$$\pi+\sin^{-1}\left(\frac{d-z}{r\sin\chi}-1\right)<\psi<\frac{3\pi}{2} \quad\text{and}\quad -\frac{\pi}{2}<\psi<\sin^{-1}\left(\frac{z}{r\sin\chi}-1\right).$$

IV: $z>r\sin\chi(1+\sin\psi)$, $d-z<r\sin\chi(1-\sin\psi)$

$$\underline{\phi = El\sin\chi\sin\psi+\Phi_4},$$

where

$$\Phi_4 = re^{-(r/l)(\psi+2\pi)}\int_{\pi-\sin^{-1}\left(\sin\psi+\frac{d-z}{r\sin\chi}\right)}^{\psi+2\pi}[F(z-r\sin\chi(\sin\psi-\sin t))\sin\chi\cos t + l\gamma F'(z-r\sin\chi(\sin\psi-\sin t))]\,e^{(r/l)t}\,dt - \left[A+El\left(\sin\chi\sin\psi+\frac{d-z}{r}\right)\right]\exp\left[-\frac{r}{l}\left(\psi+\pi+\sin^{-1}\left(\sin\psi+\frac{d-z}{r\sin\chi}\right)\right)\right]$$

when $-\frac{1}{2}\pi<\psi<\frac{1}{2}\pi$, and

$$\Phi_4 = r\,e^{-(r/l)\psi}\int_{\pi-\sin^{-1}\left(\sin\psi+\frac{d-z}{r\sin\chi}\right)}^{\psi}[F(z-r\sin\chi(\sin\psi-\sin t))\sin\chi\cos t + l\gamma F'(z-r\sin\chi(\sin\psi-\sin t))]\,e^{(r/l)t}\,dt - \left[A+El\left(\sin\chi\sin\psi+\frac{d-z}{r}\right)\right]\exp\left[-\frac{r}{l}\left(\psi-\pi+\sin^{-1}\left(\sin\psi+\frac{d-z}{r\sin\chi}\right)\right)\right],$$

when $\frac{1}{2}\pi<\psi<\frac{3}{2}\pi$.

function would be obtained from the analytic expression of the vanishing everywhere of the component of the current density in the direction Oz.

This, however, leads to an integral equation involving multiple integrals which examination shows to be intractable without very prolonged numerical work. In the later work we have therefore been forced to approximate by the use of the bulk Hall field throughout which leads to an over-estimation of the conductivity in general.

However, it has been possible to evaluate approximately the Hall field and its variation across the film in the case of small magnetic fields as follows.

We expand the Hall field and the distribution function in powers of $1/r$, and determine the coefficient of $1/r$ in the Hall field. We denote the complete Hall field by

$$\frac{1}{r}F_1(z)+\frac{1}{r^3}F_3(z)+\ldots$$

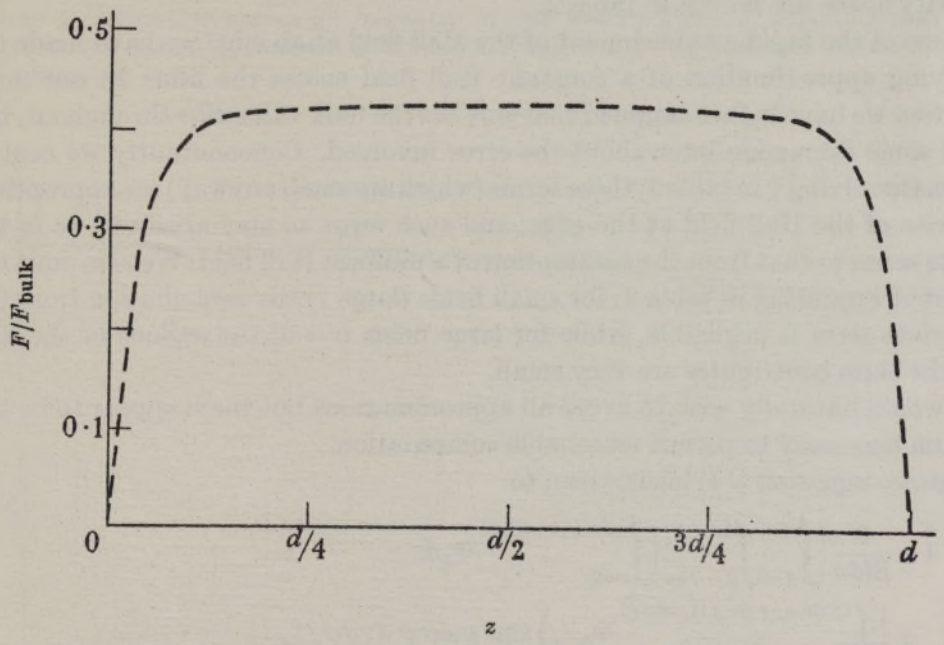

FIGURE 3. Variation of Hall field, F, across a thin film of thickness $d = 0{\cdot}64 \times$ mean free path, assuming diffuse scattering at the boundaries.

(remembering that it is an odd function of H), and arrive at an implicit equation for F_1 neglecting the curvature of the orbits. In dimensionless form this reads

$$\int_0^{\xi} G(\eta)\,\alpha(\xi-\eta)\,d\eta+\int_{\xi}^{d_1} G(\eta)\,\alpha(\eta-\xi)\,d\eta+\int_0^{\xi}\gamma G'(\eta)\,\beta(\xi-\eta)\,d\eta-\int_{\xi}^{d_1}\gamma G'(\eta)\,\beta(\eta-\xi)\,d\eta$$
$$+(\lambda+\tfrac{1}{6}\xi)\,\alpha(\xi)+(\lambda+\tfrac{1}{6}(d_1-\xi))\,\alpha(d_1-\xi)-\tfrac{1}{3}(2-e^{-\xi}-e^{-(d-\xi)})=0,$$

where $l\xi = z$, $ElG(\eta) = F_1(l\eta)$, $ld_1 = d$, $\dfrac{\lambda El^2}{r} = A$, $\alpha(x) = \displaystyle\int_1^{\infty} e^{-xt}\frac{dt}{t^3}$, $\beta(x) = \displaystyle\int_1^{\infty} e^{-xt}\frac{dt}{t^2}$.

The equation was solved by numerical methods. Since the Hall field assumes, at most, its bulk value El/r, $\gamma G'(\eta)$ will be negligible except possibly for *very* small values of η. We have therefore neglected the third and fourth integrals in the calculation which we have carried out for $d_1 = 0{\cdot}64$ (corresponding to $l \approx 45\mu$, $d \approx 29\mu$). The calculated value for λ is 0·1, and the variation of the Hall field is shown in figure 3.

An estimate made for the case also of $d_1 = 2(d = 90\mu)$ showed that the Hall field then attains a value $\sim 0{\cdot}7$ of the bulk value.

The rapid rise of the Hall field as one leaves an edge appears somewhat remarkable to us, and at present we cannot suggest any obvious physical reason for this behaviour, which one might expect could be expressed in terms of some 'characteristic depth'.

3·23. *Determination of the electrical conductivity*

In order to evaluate the electric current, and hence the conductivity, we require

$$J_x = \frac{e}{d}\int_0^d dz \iiint N v_x dv_x dv_y dv_z, \tag{14}$$

where the expressions for the distribution function appropriate to the various regions of velocity space are set out in table 2.

In view of the rapid establishment of the Hall field at an edge we have made the simplifying approximation of a constant Hall field across the film. In our main calculation we have in fact assumed that this has the bulk value El/r throughout, but we add some discussion later about the error involved. Concomitantly we neglect the terms involving γ in table 2; these terms (which are small anyway) are appropriate to the rise of the Hall field at the edge, and such error as may arise will be in the opposite sense to that from the assumption of a uniform Hall field. We also omit the constant A appearing in table 2; for small fields (large r) the contribution from the appropriate term is negligible, while for large fields ($r \to 0$) the regions of the film where the term contributes are very small.

One would naturally wish to avoid all approximations but these appear to be the minimum necessary to permit reasonable computation.

In extenso equation (14) leads, then, to

$$\begin{aligned}\frac{J_x}{J_{\text{bulk}}} = 1 + \frac{3}{Eld\pi}\Bigg\{ & \int_{\chi=0}^{\alpha}\int_{\psi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\int_{z=0}^{r\sin\chi(1+\sin\psi)}\Phi_1 dz\right. \\ & \left.+\frac{1}{2}\int_{z=r\sin\chi(1+\sin\psi)}^{2r\sin\alpha - r\sin\chi(1-\sin\psi)}\Phi_2 dz\right)\sin^2\chi\sin\psi\, d\chi d\psi \\ & + \int_{\chi=\alpha}^{\frac{1}{2}\pi}\left(\int_{\psi=-\frac{1}{2}\pi}^{\sin^{-1}\left(\frac{2\sin\alpha}{\sin\chi}-1\right)}\int_{z=0}^{r\sin\chi(1+\sin\psi)}\Phi_1 dz\sin\psi\, d\psi\right. \\ & \left.+\int_{\psi=\sin^{-1}\left(\frac{2\sin\alpha}{\sin\chi}-1\right)}^{\frac{1}{2}\pi}\int_{z=0}^{2r\sin\alpha}\Phi_1 dz\sin\psi\, d\psi\right) \\ & + \int_{\psi=\frac{1}{2}\pi}^{\pi+\sin^{-1}\left(\frac{2\sin\alpha}{\sin\chi}-1\right)}\int_{z=0}^{2r\sin\alpha - r\sin\chi(1-\sin\psi)}\Phi_1 dz\sin\psi\, d\psi\Bigg)\sin^2\chi d\chi\Bigg\}\end{aligned} \tag{15}$$

for the case of a film such that $d < 2r$, and we write $d = 2r\sin\alpha$, for convenience.

Also

$$\begin{aligned}\frac{J_x}{J_{\text{bulk}}} = 1 + \frac{3}{Eld\pi}\Bigg\{ & \int_{\chi=0}^{\frac{1}{2}\pi}\int_{\psi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{z=0}^{r\sin\chi(1+\sin\psi)}\Phi_1 dz\sin\psi\sin^2\chi\, d\psi\, d\chi \\ & + \frac{1}{2}\int_{\chi=0}^{\frac{1}{2}\pi}\int_{\psi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{z=r\sin\chi(1+\sin\psi)}^{d-r\sin\chi(1-\sin\psi)}\Phi_2 dz\sin\psi\sin^2\chi\, d\psi d\chi\Bigg\}\end{aligned} \tag{16}$$

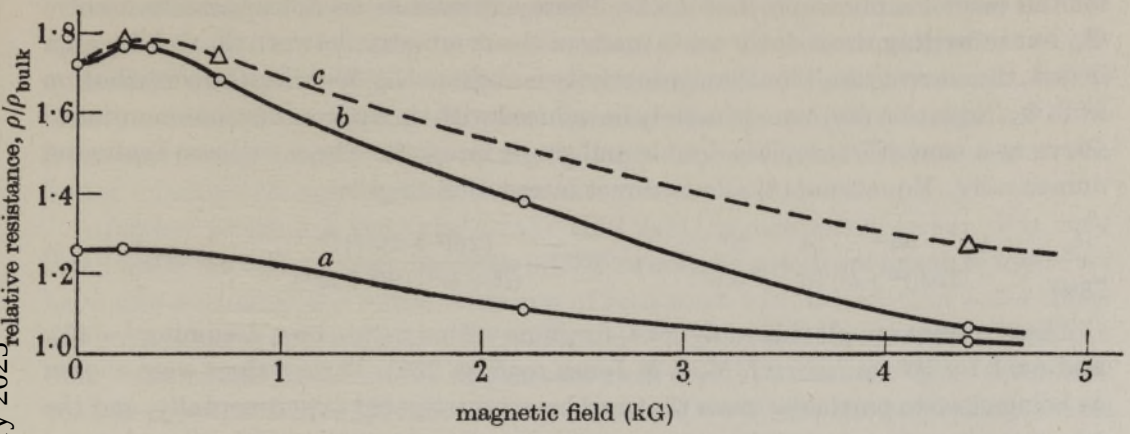

FIGURE 4. Theoretical curves for resistance of thin sodium films. *a*, 58μ thickness; *b*, 29μ thickness (Hall field assumed bulk value throughout); *c*, 29μ thickness (estimated correct curve—see text).

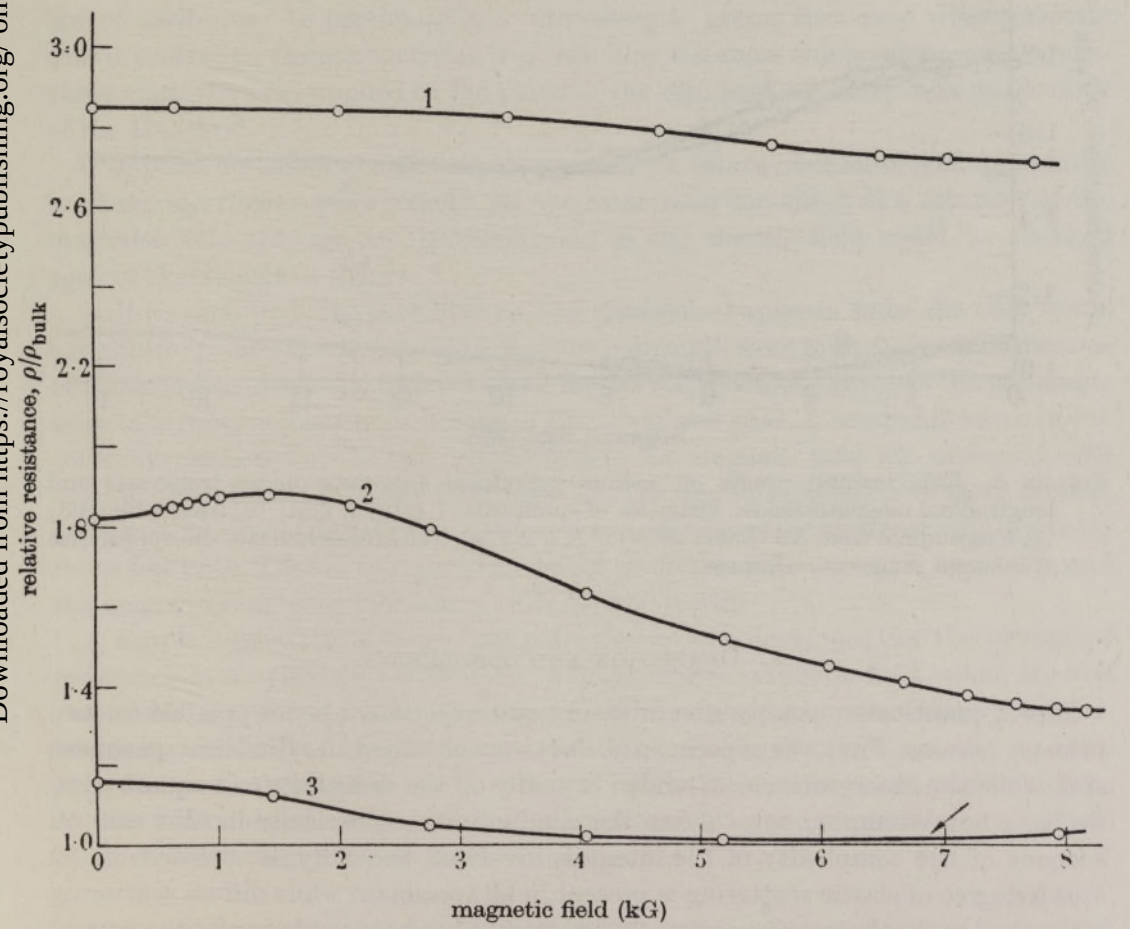

FIGURE 5. Experimental results on sodium cylindrical specimens under a transverse magnetic field. Diameters: 1, 20μ; 2, 30μ; 3, 66μ. The arrow indicates the appearance of inherent magneto-resistance. All results at $T = 4{\cdot}2°$ K ($l \simeq 44\,\mu$).

for the case of a film such that $d > 2r$. These expressions do not appear to involve Φ_4, but in writing them down use is made of the symmetry between Φ_1 and Φ_4, and, in fact, the correct result for the conductivity is obtained by doubling the contribution from Φ_1. Equation (15) can ultimately be reduced with the approximations mentioned above to a sum of irreducible double and single integrals. These we have evaluated numerically. Equation (16) allows direct integration to give

$$\frac{J_x}{J_{\text{bulk}}} = 1 - \frac{9lr^2}{32d(r^2+l^2)}\left\{\frac{4}{3} + \frac{2l^2}{l^2+r^2}(1-e^{-(2r/l)\pi}) + \frac{(26l^2+24r^2)\,l^2}{(l^2+4r^2)\,(9l^2+4r^2)}(1+e^{-(2r/l)\pi})\right\}. \quad (17)$$

These results are plotted in figure 4, for films with $d \approx 29\mu$, 58μ, assuming $l = 45\mu$ and $v \approx 1{\cdot}1 \times 10^8$ cm./sec. (cf. Mott & Jones 1936, p. 268). These values were chosen as being close to particular cases that had been investigated experimentally, and the corresponding experimental curves appear on figure 5, together with other data on figure 6.

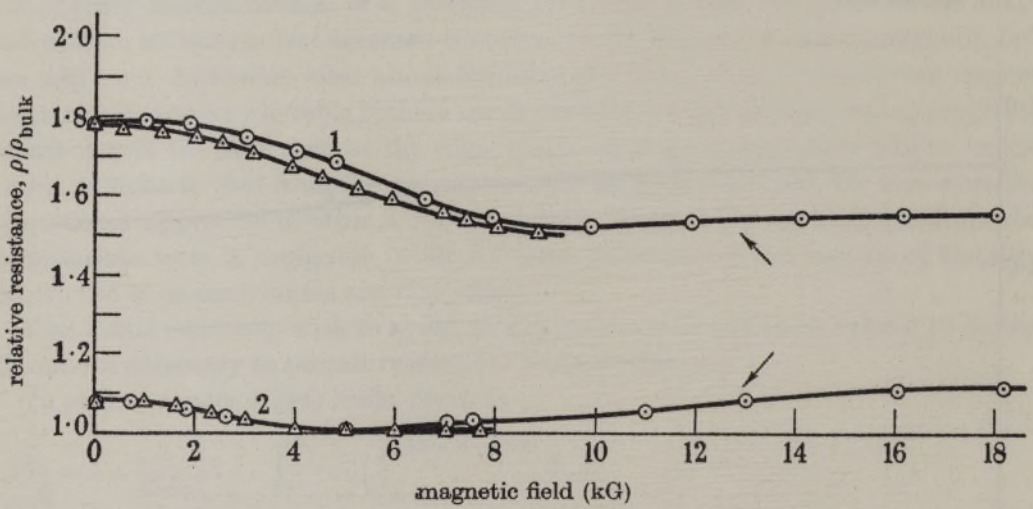

FIGURE 6. Experimental results on sodium cylindrical specimens under transverse and longitudinal magnetic fields. Diameter of specimens: 1, 30μ; 2, 66μ. ⊙, transverse field; △, longitudinal field. All results at $\sim 11°$ K ($l \simeq 37\mu$). The arrows indicate the appearance of inherent magneto-resistance.

4. Discussion and conclusion

Direct quantitative comparison of theory and experiment is not possible for two primary reasons. First, the experimental data were obtained on cylindrical specimens and, while the theory may be extended formally on the same lines to a square wire, we have not attempted to evaluate the conductivity numerically in that case on account of the complexity of the integrals involved. Secondly, it is clear from §2 that a degree of elastic scattering is present in all specimens while diffuse scattering is assumed in the theory. Consequently no attempt has been made to plot theory and experiment on the same graph. It is clear that the qualitative features of the experiments are well predicted by the theory, in particular, the rapid reduction of the initial 'hump' as the specimen size is increased. The very much quicker descent

of the resistance with magnetic field given by the theory in the case of the thinner specimen is due considerably to the geometry, since the constricting effect of the cylindrical structure will obviously become stronger as the size is reduced; this is clearly evidenced by the experimental data on the 20μ cylinder in figure 5. The fractional rate of decrease in the relatively large 66μ specimen shows in fact much better quantitative agreement between theory and experiment.

A further problem is the value of the Hall field, as mentioned earlier. For very small fields the value chosen has little effect, as may be seen from figure 5, where we have also evaluated the initial variation of resistance with a Hall field = 0·4 times the bulk value as determined in §3·22. As the field increases the deviation from the bulk value causes the resistance to decrease more slowly; we have roughly estimated that at 4000 G the Hall field should have attained $\sim 0{\cdot}8$ time the bulk value and on this basis computed a value for the resistance as shown. This value, coupled with the 'corrected' part of the curve for small fields, gives one a fair idea of the overall correct curve for a film specimen.

It should be observed that the Hall field and its variation will in fact alter with the law of scattering. In particular it is interesting to notice that even with perfectly elastic scattering the conductivity in a thin film will show a dependence on a transverse magnetic field applied in the plane of the film because of the non-uniformity of the Hall field.

It appears desirable to make experiments in the future on films of sodium in order to check the theory more closely. At the same time the effect of a rotation of the magnetic field through 90° perpendicular to the electric field could be checked against Sondheimer's theory.

Sodium (and possibly pure lithium and potassium) appears to be the only metal suitable for these experiments in view of the higher inherent bulk magneto-resistance of other metals, even the heavier alkali metals (cf. MacDonald 1950). Experiments were in fact carried out on thin foils of tin, silver and gold. A small difference in the correct sense between the two orientations of the magnetic field was observed with tin, but the inherent magneto-resistance was the dominating factor in these metals, and also with silver and gold the residual resistance was rather high resulting in a short mean free path. Thus in the case of silver the mean free path at $\sim 4°$ K was $\sim 7\mu$, and the magneto-resistance increase $\sim 13\,\%$ for about 8 kG.

A simple approximate mean free-path theory was developed for the change of resistance in a cylindrical wire under a longitudinal magnetic field which showed reasonable agreement with experiments. While this paper was being prepared, however, we heard of Mr Chambers's detailed work on this problem, and we have therefore not considered it worth while to include the analysis here, particularly as Mr Chambers has also carried out a collateral set of experiments in a longitudinal magnetic field on our 30μ sodium specimen.

We should like to thank Professor M. H. L. Pryce and Mr J. L. Olsen for helpful discussions in the early stages of the investigation. This work was carried out during the tenure of an I.C.I. Research Fellowship by one of us (D.K.C.MacD.).

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Liquid helium films
II. The flow of the film

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The rate of flow of the helium II film has been found to vary slightly with the height of the film and the pressure head. Most of the variation can be ascribed to changes in the thickness of the film, but a small residual variation at small pressure heads suggests that at least part of the flow is subject to frictional retardation. Measurements of the damping of the film oscillations enable a small upper limit to be put on the frictional retardation of the remaining part, but a similar method applied to flow through wide capillaries indicates that in this case most of the flow is opposed by frictional forces. The results are discussed in conjunction with the measurements of Allen & Misener on flow through capillaries and an attempt made to formulate some of the general features of the flow of the superfluid component.

1. Introduction

A very satisfactory qualitative explanation of the anomalous transport properties of liquid helium II is provided by the two-fluid theories of Tisza (1938, 1947) and Landau (1941), who assume that the liquid is a mixture of a normal component and a superfluid component. The normal component possesses a normal viscosity and carries most or all of the heat content of the liquid, whereas the superfluid component flows with appreciable velocities through the very narrowest channels