第六次作业

1、证明：令 $f_1(x) = a_0 + a_1x + a_2x^2 + \dots + a_n x^{n-1}$

$f_2(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}$

假设 $f_1(x), f_2(x) \in U$. 则 $a_0+a_1+\dots+a_{n-1}=0$. $b_0+b_1+\dots+b_{n-1}=0$.

对于任意实数 α 和 β. 则有

$$\alpha f_1(x)+\beta f_2(x) = (\alpha a_0+\beta b_0)+(\alpha a_1+\beta b_1)x+\dots+(\alpha a_{n-1}+\beta b_{n-1})x^{n-1}$$

又 $$\alpha f_1(1)+\beta f_2(1) = \alpha a_0+\beta b_0+\alpha a_1+\beta b_1+\dots+\alpha a_{n-1}+\beta b_{n-1}$$
$$= \alpha(a_0+a_1+\dots+a_{n-1})+\beta(b_0+b_1+\dots+b_{n-1})$$
$$= \alpha\cdot 0+\beta\cdot 0=0.$$

故 $\alpha f_1(x)+\beta f_2(x)\in U$

而 U 是一个子空间. 且有 $\dim(U)=n-1$.

由于 $\dim(U)=n-1$. 所以 U 的补空间 U' 的为维为1.

U 的补空间 $U'=\{a_0 \mid a_0\in R\}$ （所有常数项构成的子空间）.

2. 解：由题 $U=\begin{bmatrix}1&4&5\\2&-1&1\\3&3&6\\6&6&12\end{bmatrix}$, $W=\begin{bmatrix}1&2\\-1&-1\\1&4\\1&5\end{bmatrix}$.

令 $\alpha_1=[1,2,3,6]^T$, $\alpha_2=[4,-1,3,6]^T$, $\alpha_3=[5,1,6,12]^T$.

$\beta_1=[1,-1,1,1]^T$, $\beta_2=[2,-1,4,5]^T$.

$\because$ β_1 与 β_2 线性无关.

$\therefore \dim(W)=2$

又 $\because \alpha_3=\alpha_1+\alpha_2$

$\alpha_1\ \alpha_2\ \alpha_3$ 线性相关, α_1, α_2 线性无关

$\therefore \dim(U)=2$

令 $$A=[\alpha_1,\alpha_2,\beta_1,\beta_2]=\left[\begin{array}{cc:cc}1&4&-1&-2\\2&-1&1&1\\3&3&-1&-4\\6&6&-1&-5\end{array}\right]\xrightarrow{\text{初等变换}}\begin{bmatrix}1&4&-1&-2\\0&9&-3&5\\0&0&1&3\\0&0&0&0\end{bmatrix}$$

$r(A)=3$

则 $\dim(U+W)=3$，且 α_1,β_1,β_2 为 $U+W$ 的一组基.

$\dim(U\cap W)=\dim U+\dim W-\dim(U+W)=2+2-3=1$.

令 $\alpha\in U\cap W$. 则 $\alpha=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=\cancel{k_4\beta_4}\,k_4\beta_1+k_5\beta_2$

解得 $-3x_4[1\ -1\ 1\ 1]^T+x_4(2\ -1\ 4\ 5)^T=x_4[-1\ 2\ 1\ 2]^T$

3. 解：由题 $x+y+z+w=0$，则有 $\begin{cases} \alpha_1 = (-1,1,0,0)^T \\ \alpha_2 = (-1,0,1,0)^T \\ \alpha_3 = (-1,0,0,1)^T \end{cases}$ 为 U 的一组基.

$x-y+z-w=0$. $\begin{cases} \beta_1 = (1,1,0,0)^T \\ \beta_2 = (-1,0,1,0)^T \\ \beta_3 = (1,0,0,1)^T \end{cases}$ 为 W 的一组基.

故 $\dim(U)=3$，$\dim(W)=3$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$，则 $\begin{matrix} e_1 = (-1,0,1,0)^T \\ e_2 = (0,-1,0,+1)^T \end{matrix}$ 是 $U\cap W$ 的一组基

故 $\dim(U\cap W)=2$

则 $\dim(U+W) = \dim(U)+\dim(W) - \dim(U\cap W) = 3+3-2=4$.

$\because U+W=\mathbb{R}^4$

$\therefore U+W$ 的基可取为 $\mathbb{R}^4$ 的标准基. 即

$(1,0,0,0)$，$(0,1,0,0)$. $(0,0,0,1)$ $(0,0,0,1)$.

4. 证明

$$\begin{tikzcd} X_1 \arrow[r, "\sigma_1"] \arrow[rr, bend right, "\sigma"'] & XA \arrow[r] & XAB \end{tikzcd}$$

$W = \ker(\sigma) \subseteq \mathbb{F}^n$

$U = \mathrm{Im}(\sigma_1|_W) \subseteq \mathbb{F}^m$

由维数定理

$\dim(W) = \dim(\ker(\sigma)) = \dim(\mathbb{F}^n) - \dim(\mathrm{Im}(\sigma)) = n - r(AB)$

$\dim(U) = \dim(W) - \dim(\ker(\sigma_1|_W))$.

设 $v\in\ker(\sigma_1|_W)$，$v\in W$，$\sigma_1(v)=0$ 则 $\ker(\sigma_1|_W)\subseteq\ker(\sigma_1)$

$v\in\ker(\sigma_1)$，$\sigma_1(v)=0 \Rightarrow \sigma(v)=0$，$v\in W$，$v\in\ker(\sigma_1|_W)$ 则 $\ker(\sigma_1)\subseteq\ker(\sigma_1|_W)$

$\therefore \ker(\sigma_1|_W) = \ker(\sigma_1)$

$\dim(U) = \dim(W) - \dim(\ker(\sigma_1))$

$\quad = n - r(AB) - (n - r(A))$

$\quad = r(A) - r(AB)$

5. 解: $\partial(1, x, x^2, \dots, x^{n-1}) = (0, 1, 2x, \dots, (n-1)x^{n-2})$

$$= (1, x, x^2, \dots, x^{n-1}) \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

故 ∂ 在基 $\{1, x, x^2, \dots, x^{n-1}\}$ 下的矩阵

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

即 $\det(A) = 0$
$\operatorname{tr}(A) = 0$

$\partial(1, (x-a), (x-a)^2, \dots, (x-a)^{n-1}) = (0, 1, 2(x-a), \dots, (n-1)(x-a)^{n-2})$

$= (1, (x-a), \dots, (x-a)^{n-1}) \cdot B$

$B = A$.

即 $\det(B) = \det(A) = 0$
$\operatorname{tr}(B) = \operatorname{tr}(A) = 0$.

6. 证明: (1) 设 $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$

$\because f(0) = 0$

$\therefore f(x) = a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$.

取 $\forall f_1(x), f_2(x) \in U$, $\alpha, \beta \in \mathbb{R}$. 则有

$\alpha f_1(x) + \beta f_2(x) \in U$; $\alpha f_1(0) + \beta f_2(0) = (\alpha f_1 + \beta f_2)(0) = 0$

从而 U 是 V 的子空间, $\{x, x^2, \dots, x^{n-1}\}$ 是 U 的一组基

(2) 当 $n=3$ 时

$\dim U^\perp = n - \dim U = 3 - 2 = 1$

设 $U^\perp$ 中的一组基为 $ax^2 + bx + c$

$\langle x, ax^2+bx+c \rangle = \int_0^1 x(ax^2+bx+c)dx = \frac{1}{4}a + \frac{1}{3}b + \frac{c}{2} = 0$

$\langle x^2, ax^2+bx+c \rangle = \int_0^1 x^2(ax^2+bx+c)dx = \frac{1}{5}a + \frac{1}{4}b + \frac{c}{3} = 0$

解得 $\begin{cases} 3a = 10c \\ b = -4c \cdots \end{cases}$ 取 $a = 10$ 则 $U^T = [10x^2 - 12x + 3]$

7. 证明：(1) $\forall k_1, k_2 \in \mathbb{R}$, $\alpha_1, \alpha_2 \in V$

$$\sigma(k_1\alpha_1 + k_2\alpha_2) = k_1\alpha_1 + k_2\alpha_2 - 2\langle k_1\alpha_1 + k_2\alpha_2, \alpha_0\rangle \alpha_0$$
$$= k_1\alpha_1 + k_2\alpha_2 - 2k_1\langle \alpha_1, \alpha_0\rangle\alpha_0 - 2k_2\langle \alpha_2, \alpha_0\rangle \alpha_0$$
$$= k_1(\alpha_1 - 2\langle \alpha_1, \alpha_0\rangle \alpha_0) + k_2(\alpha_2 - 2\langle \alpha_2, \alpha_0\rangle \alpha_0)$$
$$= k_1\sigma(\alpha_1) + k_2\sigma(\alpha_2)$$

∴ σ 是线性变换.

(2) $\forall \alpha, \beta \in V$

$$(\sigma(\alpha), \sigma(\beta)) = (\alpha - 2(\alpha, \alpha_0)\alpha_0, \beta - 2(\beta, \alpha_0)\alpha_0)$$
$$= (\alpha, \beta) - 2(\alpha, \alpha_0)(\alpha_0, \beta) - 2(\beta, \alpha_0)(\alpha, \alpha_0)$$
$$+ 4(\alpha, \alpha_0)(\beta, \alpha_0)(\alpha_0, \alpha_0)$$
$$= (\alpha, \beta)$$

故 σ 为正交变换.

证明:

8. $\sigma((x,y)^T) = A(x,y)^T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix}$

$= (ax-by) + (bx+ay)i$

$= ax + ayi + bxi - by$

$= (a+bi)(x+yi)$

$\sigma((x,y)^T) = A(x,y)^T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} ax+by \\ -bx+ay \end{bmatrix}$

$= (ax+by) + ayi - bxi$

$= (a-bi)(x+yi)$

9. 解: X是二阶实矩阵且 $\operatorname{tr}X=0$. 则 X 可表示为 $X=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

则 $\dim(V)=3$.

$\sigma(X) = B^TX - X^TB,\ X\in V$

$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \begin{bmatrix} a & c \\ b & -a \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{pmatrix} 0 & b-a-c \\ a+c-b & 0 \end{pmatrix} \in V$

取 $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ 为 V的基

$\sigma(A_1) = \begin{pmatrix} 0 & 0-1-0 \\ 1+0-0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0\cdot A_1 - A_2 + A_3$

$\sigma(A_2) = \begin{pmatrix} 0 & 1-0-0 \\ 0+0+1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0\cdot A_1 + A_2 - A_3$

$\sigma(A_3) = \begin{pmatrix} 0 & 0-0-1 \\ 0+1-0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0A_1 - A_2 + A_3$

$(\sigma(A_1), \sigma(A_2), \sigma(A_3)) = (A_1\ A_2\ A_3)\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

$\therefore C = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

第7题，

$(x,y)=\overline{(y,x)}$

$(x,ay)=\overline{(ay,x)}=\overline{a(y,x)}=\bar{a}(x,y)$

(2) 证明. 对欧式空间中任意元素 α,β.

$$
\begin{aligned}
\text{有}\langle\sigma(\alpha),\sigma(\beta)\rangle &= (\alpha-2(\alpha,\alpha_0)\alpha_0,\ \beta-2(\beta,\alpha_0)\alpha_0) \\
&=(\alpha,\beta)-(\alpha,2(\beta,\alpha_0)\alpha_0)-(2(\alpha,\alpha_0)\alpha_0,\beta) \\
&\quad +4((\alpha,\alpha_0)\alpha_0,(\beta,\alpha_0)\alpha_0) \\
&=(\alpha,\beta)\underbrace{-2\overline{(\alpha_0,\beta)}(\alpha,\alpha_0)-2\overline{(\alpha_0,\alpha)}(\beta,\alpha_0)+4\overline{(\alpha_0,\alpha)}\,\overline{(\alpha_0,\beta)}(\alpha_0,\alpha_0)}_{\shortparallel} \\
&=(\alpha,\beta)
\end{aligned}
$$

$2(\beta,\alpha_0)(\alpha,\alpha_0)-2(\alpha,\alpha_0)(\beta,\alpha_0)+4(\alpha,\alpha_0)(\beta,\alpha_0)=0$

第9题 补充

$$C=\begin{bmatrix}0&0&0\\-1&1&-1\\1&-1&1\end{bmatrix}$$，其特征值 $\lambda_1=2$, $\lambda_{2,3}=0$

当 $\lambda_1=2$ 时, 对应特征向量 $u_1=(0,\frac{1}{\sqrt2},-\frac{1}{\sqrt2})^T$

当 $\lambda_{2,3}=0$ 时, 对应特征向量 $u_2=(0,\frac{1}{\sqrt2},\frac{1}{\sqrt2})^T$, $u_3=(\frac{2}{\sqrt6},-\frac{1}{\sqrt6},\frac{1}{\sqrt6})^T$

$$
\begin{aligned}
\therefore\text{有}\ (\sigma(A_1)\ \sigma(A_2)\ \sigma(A_3)) &= (A_1\ A_2\ A_3)(u_1\ u_2\ u_3)\begin{bmatrix}2&&\\&0&\\&&0\end{bmatrix} \\
&=(A_1\ A_2\ A_3)\begin{bmatrix}0&0&\frac{2}{\sqrt6}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\end{bmatrix}\begin{bmatrix}2&&\\&0&\\&&0\end{bmatrix} \\
&=\left[\frac{1}{\sqrt2}A_2-\frac{1}{\sqrt2}A_3,\ \frac{1}{\sqrt2}A_2+\frac{1}{\sqrt2}A_3,\ \frac{2}{\sqrt6}A_1-\frac{1}{\sqrt6}A_2+\frac{1}{\sqrt6}A_3\right]\begin{bmatrix}2&&\\&0&\\&&0\end{bmatrix}
\end{aligned}
$$

$$E_1=\frac{1}{\sqrt2}A_2-\frac{1}{\sqrt2}A_3=\begin{bmatrix}0&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&0\end{bmatrix}$$

$$E_2=\frac{1}{\sqrt2}A_2+\frac{1}{\sqrt2}A_3=\begin{bmatrix}0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&0\end{bmatrix}$$

$$E_3=\frac{2}{\sqrt6}A_1-\frac{1}{\sqrt6}A_2+\frac{1}{\sqrt6}A_3=\begin{bmatrix}\frac{2}{\sqrt6}&-\frac{1}{\sqrt6}\\\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}\end{bmatrix}$$

10. 解：(1) A 为正规矩阵，则 $AA^H=A^HA$ 且 A 可酉对角化

即 ∃ 酉阵 U，使 $U^HAU=D$，D 为对角矩阵。

∴ $U^H(A-xI)U=D-xI$，

其中 $(D-xI)$ 为对角矩阵。

∴ $A-xI$ 可酉对角化，$A-xI$ 为正规矩阵

(2) $(Ax)^H(Ax)=x^HA^HAx=x^HA\cdot A^Hx=(A^Hx)^H(A^Hx)$

∴ $(Ax)^H(Ax)=(A^Hx)^H(A^Hx)$

∴ Ax 与 A^Hx 长度相同

(3) ∃ 酉阵 U 使得 $U^HAU=D$，D 为对角阵。

U 中的每一列都为 A 的特征向量

∵ A 为正规矩阵，即 $A^HA=AA^H$

∴ A^H 也为正规矩阵

∴ $(U^HAU)^H=D^H=U^HA^HU$

∴ U 中的每列都为 A^H 的特征向量

∴ A 中的任一特征向量都是 A^H 的特征向量。

(4). 设 $Ax_i=\lambda_ix_i$

$Ax_j=\lambda_jx_j$，$\lambda_i\neq\lambda_j$

$$\begin{aligned}\lambda_i(x_i,x_j)&=\langle\lambda_ix_i,x_j\rangle\\&=\langle Ax_i,x_j\rangle\\&=\langle x_i,A^Hx_j\rangle\\&=\langle x_i,\lambda_j^Hx_j\rangle\\&=\lambda_j\langle x_i,x_j\rangle\end{aligned}$$

∵ $\lambda_i\neq\lambda_j$，∴ $\langle x_j,x_j\rangle=0$

∴ A 属于不同特征值的特征向量正交。

11. 证明:(1) A为正规矩阵，则A可酉对角化

即 $U^H A U = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$(U^H A U)^H = U^H A^H U = D^H = \begin{bmatrix} \lambda_1^H & & & \\ & \lambda_2^H & & \\ & & \ddots & \\ & & & \lambda_n^H \end{bmatrix}$

$\because \lambda_1, \lambda_2, \cdots, \lambda_n$ 全为实数.

$\therefore D^H = D \iff A^H = A$. 即A为Hermite矩阵

(2) A为酉阵 $\iff A^H A = I$

"$\Rightarrow$" $Ax = \lambda x$. $(Ax)^H (Ax) = x^H A^H A x = x^H x$

$(Ax)^H (Ax) = (\lambda x)^H (\lambda x) = \lambda^H \lambda \cdot x^H \cdot x$

$\therefore \lambda^H \lambda = 1$ 即 $\|\lambda\| = 1$.

"$\Leftarrow$" 由定义可知.A为酉阵.

(3) A是幂等阵 $\iff A^2 = A$

"$\Rightarrow$" $Ax = \lambda x$,

$A \cdot Ax = \lambda Ax \Rightarrow A^2 x = \lambda^2 x$

$\Rightarrow Ax = \lambda^2 x = \lambda x$

$\therefore \lambda^2 = \lambda$. 则 $\lambda = 0$ 或 $\lambda = 1$.

"$\Leftarrow$" 由(1) $\exists\ U^H A U = D$.

又$\because$A的特征值只能为0或1

$\therefore D^2 = D$

$\therefore A = UDU^H = UD^2U^H = A^2$

$\therefore$ A为幂等阵.

(4) $U^H A A^H U = (U^H A U)(U^H A^H U)$

$= D \cdot D^H$

$= \begin{bmatrix} |\lambda_1|^2 & & & \\ & |\lambda_2|^2 & & \\ & & \ddots & \\ & & & |\lambda_n|^2 \end{bmatrix}$

对非正规矩阵不成立.

例：令 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$AA^H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ 特征值为 1, 3

$A^H A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ 特征值 $\frac{3 \pm \sqrt{5}}{2}$.

12. 证明：(1) 由(11)题第3问可知 若A为幂等阵，则特征值只能为0或1.

再由(11)题第1问可知若A的特征值均为实数 则A为 Hermite矩阵

(2) 若 $A^3=A^2$

$\exists$ 酉阵U 使得 $U^HAU=D$

$U^HA^3U=D^3 \Rightarrow D^3=D^2$

$U^HA^2U=D^2$

$\therefore$ D中对角元素均为1或0

$\therefore D^2=D$. $A^2=A$.

(3) A是Hermite阵. 且 $A^k=I$.

则 $A^H=A$

$Ax=\lambda x$

$A^{k-1}Ax=\lambda A^{k-1}x$

$A^kx=\lambda\cdot\lambda^{k-1}x$

$A^k=I$, $x=\lambda^kx$

$\therefore (\lambda^k-1)x=0$

$\because x\neq 0$

$\therefore \lambda^k=1$

$\therefore \lambda=\pm1$

$\therefore D^2=I$

$\therefore A^2=UD^2U^H=I$.

17. 取$\forall \alpha_1,\alpha_2 \in \mathbb{R}^n$，$\|w\|=1$

解〈1〉 $\langle \sigma(\alpha_1),\sigma(\alpha_2)\rangle = \langle \alpha_1 - a(\alpha_1,w)w,\ \alpha_2 - a(\alpha_2,w)w\rangle$

$= \langle \alpha_1 - a(\alpha_1,w)w,\ \alpha_2\rangle - (\alpha_1 - a(\alpha_1,w)w,\ a(\alpha_2,w)w\rangle$

$= \langle \alpha_1,\alpha_2\rangle - a\langle \alpha_1,w\rangle\langle w,\alpha_2\rangle - a^*\overline{\langle \alpha_2,w\rangle}\langle \alpha_1,w\rangle$

$+ a\cdot a^*\langle \alpha_1,w\rangle\overline{\langle \alpha_2,w\rangle}\langle w,w\rangle$

$= \langle \alpha_1,\alpha_2\rangle - a(\alpha_1,w)(w,\alpha_2) - a\langle w,\alpha_2\rangle\langle \alpha_1,w\rangle$

$+|a|^2\langle \alpha_1,w\rangle\langle w,\alpha_2\rangle$

$= \langle \alpha_1,\alpha_2\rangle - (|a|^2 - a^* - a)\langle \alpha_1,w)(w,\alpha_2)$

令$\langle \alpha_1,\alpha_2\rangle - (|a|^2 - a^* - a)\langle \alpha_1,w)\langle w,\alpha_2\rangle = \langle \alpha_1,\alpha_2\rangle$

即 $a^2-2a=0 \Rightarrow a=0$ 或 $a=2$ 时为正交变换.

若 w为任意向量.

则有 $\langle \sigma(\alpha_1),\sigma(\alpha_2)\rangle = \langle \alpha_1\ \alpha_2\rangle - (|a|^2\|w\|^2 - a^* - a)\langle \alpha_1,w\rangle(w,\alpha_2)$

则 $|a|^2\|w\|^2 - a^* - a = 0$

① 当$\|w\|^2=0$，即 $w=0$时. $\langle \alpha_1,w\rangle = \langle w,\alpha_2\rangle = 0$. a可取任意值

② 当w不为0时. $a^2w^2 - 2a = 0 \Rightarrow a = \frac{2}{w^2}$ 或 $a=0$.

14. 解: $B=\begin{pmatrix}A\\A\end{pmatrix}=\begin{pmatrix}U\\U\end{pmatrix}DV^H$

$$\begin{pmatrix}U\\U\end{pmatrix}^H\begin{pmatrix}U\\U\end{pmatrix}=(U^H\ \ U^H)\begin{pmatrix}U\\U\end{pmatrix}=U^HU+U^HU=2I$$

$\therefore\ B=\frac{1}{\sqrt{2}}\begin{pmatrix}U\\U\end{pmatrix}\cdot\sqrt{2}D\cdot V^H$

解:

15. $A\in\mathbb{C}^{n\times n}$ 为可逆矩阵.

A的奇异值分解 $A=UDV^H$，等式两边同时取逆 $A^{-1}=(UDV^H)^{-1}$

即 $A^{-1}=VD^{-1}U^H$

16. 证明：由 $\lambda_i = \sigma_i^2$，则

$$A^T A u_i = \sigma_i^2 u_i$$

等式左乘 u_i^T

$$u_i^T A^T A u_i = u_i^T \sigma_i^2 u_i$$

$$\Rightarrow (Au_i)^T (Au_i) = \sigma_i^2 u_i^T u_i$$

若特征向量 u_i 的为单位长度，即得 $\sigma_i = \|Au_i\|$

17.

17题:

证明: $A = UDV^H$，其中 U是酉阵，$D = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$

$\langle Au_i, Au_j \rangle = (Au_j)^H Au_i = \lambda_i \lambda_j u_j^H u_i$

$\because u_1, u_2, \ldots, u_n$ 是特征向量，$u_j^H u_i = 0$，$j \neq i$

$\therefore \langle Au_i, Au_j \rangle = 0$

即 $\{Au_1, Au_2, \ldots, Au_r\}$ 相互正交，

又 $\because Au_i \in col(A)$

$\therefore dim(col(A)) \geq r$

又 $\because A = UDV^H = \sqrt{\lambda_1} u_1 v_1^H + \sqrt{\lambda_2} u_2 v_2^H + \ldots + \sqrt{\lambda_r} u_r v_r^H$

$\therefore dim(col(A)) \leq r$

综上 $dim(col(A)) = r$.

$\{Au_1, Au_2, \ldots, Au_r\}$ 是 col(A) 的一组正交基.

A 有 r 个非零奇异值. $rank(A) = r$.

18.题.

解: $A = UDV^H$，其中 $D = diag(d_1, d_2, \ldots, d_n, 0, \ldots 0)$.

令 $\begin{bmatrix} 0 & A^H \\ A & 0 \end{bmatrix}$ 的特征向量为 w，即 $\begin{bmatrix} 0 & A^H \\ A & 0 \end{bmatrix} w = \lambda w$，令 $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

则有 $\begin{bmatrix} 0 & A^H \\ A & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \begin{bmatrix} A^H w_2 \\ A w_1 \end{bmatrix} = \begin{bmatrix} \lambda w_1 \\ \lambda w_2 \end{bmatrix}$

$\therefore \begin{matrix} A^H w_2 = \lambda w_1 \\ A w_1 = \lambda w_2 \end{matrix} \Rightarrow \begin{matrix} AA^H w_2 = \lambda^2 w_2 \\ A^H A w_1 = \lambda^2 w_1 \end{matrix}$

$\because A = UDV^H \Rightarrow A^H A = VD^H DV^H \Rightarrow A^H A V = V \begin{bmatrix} |d_1|^2 & & & \\ & |d_2|^2 & & \\ & & \ddots & \\ & & & |d_n|^2 \\ & & & & 0 \end{bmatrix}$

$\therefore A^H A v_i = |d_i|^2 v_i \quad (i = 1, 2, \ldots n)$

$\because AA^H = UDD^H U^H \Rightarrow AA^H U = U \begin{bmatrix} |d_1|^2 & & & \\ & |d_2|^2 & & \\ & & \ddots & \\ & & & |d_n|^2 \\ & & & & 0 \end{bmatrix}$

$\therefore A^H A u_i = |d_i|^2 u_i \quad (i = 1, 2, \ldots n)$

综上 $\begin{cases} w_2 = u_i \\ w_1 = \cancel{\ } v_i \end{cases} \quad \therefore w = \begin{bmatrix} v_i \\ u_i \end{bmatrix}$

$A = UDV^H \Rightarrow AV = UD$

$A^H = VDU^H \Rightarrow A^H U = VD$

$D = diag(\sigma_1, \sigma_2, \ldots, \sigma_n, 0, \ldots 0)$

$\begin{bmatrix} 0 & A^H \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \sigma_i \begin{bmatrix} v_i \\ u_i \end{bmatrix} (1 \leq i \leq n)$

其中 v_i 是V的第i列

u_i 是U的第i列

σ_i 是第i个奇异值.