8.3

14

> If there is just one team, obviously it does not need to compete to, hence $R(1) = 0$;
> Since each time we complete a round, the number of teams will be cut into half, we have
> $R(n) = R(\frac{n}{2}) + 1$

28

a)

> For each query, if the two answers are same, then the first person does not lie, meaning we can process recursively on the half set that we know number x is in. If the two answers are different, then the fist person must lie since he can only lie once. In this situation, we know he can not lie anymore, so we query third time to get the right answer. Since he lies only once, we do not need to ask twice after he lies. If he lies in the last query, we need to use a total of $2logn + 1$ queries to find x.

b)

> Assume the 4 generated sets are A, B, C, D, then we can ask if $x \in A \cup B$ and if $x \in A \cup C$ to eliminate $n/4$ elements. We can prove it by list all the situations.
>
> 1. If the answers to both queries are *yes*, then we have $x \in A \cup B$ and $x \in A \cup C$, hence $x \in A \cup B \cup C$ since at least one answer is true. So we can eliminate set D.
> 2. If the answer to first query is *yes* and second is *no*, then we have $x \in A \cup B$ and $x \notin A \cup C$. If both are true, then $x \in A \cup B - A \cup C$, $x \in B$. If first is true and second is false, then $x \in A \cup B$ and $x \in A \cup C$, $x \in (A \cup B) \cap (A \cup C)$, $x \in A$. If first is false and second is true, then $x \notin A \cup B$ and $x \notin A \cup C$, $x \in D$. Hence we have $x \in A \cup B \cup D$, eliminating C.
> 3. If the answer to first query is *no* and second is *yes*, we can follow the process of 2 to eliminate B.
> 4. If the answers to both queries are *no*, then we have $x \notin A \cup B$ and $x \notin A \cup C$. If both are true, then $x \in D$. If first is true (false) and second is false(true), then $x \in B \cup C$. Hence we have $x \in B \cup C \cup D$, eliminating set A.

c)

> Since each time we eliminate $n/4$ elements in the set, we get a problem to find x based on a set with $3n/4$ elements by querying twice. Hence $f(n) = f(\frac{3}{4}n) + 2$

d)

> $f(n) = f(\frac{3}{4}n) + 2 = f((\frac{3}{4})^2 n) + 4 = f((\frac{3}{4})^3 n) + 6 = \cdots = 2log_{\frac{3}{4}} n \approx 4.8logn$

e)

> The naive way is more efficient. Since using two quries, the naive way eliminates half of elements, while way (b) only eliminates $\frac{1}{4}$ elements.

8.4

16

> For egg and plain, the factors are both $x^2 + x^3 + \dots$ For salty, the factor is $x^2 + x^3$. Hence the generating function is $(x^2 + x^3 + \dots)^2 (x^2 + x^3) = x^6 (1 + x + x^2 + \dots)^2 (1 + x) = \frac{x^6 (1+x)}{(1-x)^2}$. Hence we need to find the coefficient of x^6. The coefficient of x^6 in $\frac{1}{(1-x)^2}$ is 7, the coefficient of x^5 in $\frac{1}{(1-x)^2}$ is 6. So the answer is 13.

24

a)

> The factor of x_1 is $x^3 + x^4 + \dots$ the factor of x_2 is $x + x^2 + x^3 + x4 + x^5$, the factor of x_3 is $1 + x + x^2 + x^3 + x^4$, the factor of x_4 is $x + x^2 + x^3 + \dots$
> So the generating function is $(x^3 + x^4 + \dots)(x + x^2 + x^3 + x4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \dots) = \frac{x^5 (1+x+x^2+x^3+x^4)^2}{(1-x)^2}$

b)

> From the generating function, we need to find the coefficient of x^2 in $\frac{(1+x+x^2+x^3+x^4)^2}{(1-x)^2}$, which is same as in $\frac{1+2x+3x^2}{(1-x)^2}$ since the higher are invalid. Hence $1 \times 3 + 2 \times 2 + 3 \times 1 = 10$

36

> Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$ be the generating function for the sequence $\{a_k\}$. Then we have $a_k x^k = 3a_{k-1} x^k + 4^{k-1} x^k$ by multiplying both sides by x^k. Then we have
>
> $$\begin{aligned} G(x) - 1 = \sum_{k=1}^{\infty} a_k x^k &= 3 \sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=1}^{\infty} 4^{k-1} x^k \\ &= 3x \sum_{k=0}^{\infty} a_k x^k + x \sum_{k=0}^{\infty} 4^k x^k \\ &= 3xG(x) + \frac{x}{1-4x} \end{aligned}$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1}{1-4x} = \sum_{k=0}^{\infty} 4^k a^k$$

Hence we have

$$a_k = 4^k$$