Advanced counting

Applications of Recurrence Relations

It depends.

Solving Linear Recurrence Relations

Theorem 1

> Let c_1 and c_2 be real numbers. Suppose that $r_2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $an = \alpha_1 r_1^n + \alpha_2 r_2^n (\forall n = 0, 1, 2, \dots)$, where $\alpha 1$ and $\alpha 2$ are constants.
>
> $$a_0 = C_0 = \alpha_1 + \alpha_2$$
> $$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$$
>
> $$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2 = \alpha_1 (r_1 - r_2) + C_0 r_2$$
>
> $$\begin{cases} \alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2} \\ \alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2} \end{cases}$$

Problem. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$

Solution.

> The characteristic equation of the recurrence relation is $r^2 - r - 2$. Its roots are $r = 2$ and $r = -1$. Hence, the solution to this recurrence relation is
>
> $$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$
>
> for some constants α_1 and $alpha_2$. Using the initial conditions, it follows that
>
> $$a_0 = 2 = \alpha_1 + \alpha_2$$
> $$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$$
>
> Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with
>
> $$a_n = 3 \cdots 2^n - (-1)^n$$

Theorem 2

> Let c_1 and c_2 be real numbers with $\underline{c_2 \neq 0}$.Suppose that $r_2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $an = \alpha_1 r_0^n + \alpha_2 n r_0^n, \forall n = 0, 1, 2, \ldots$, where α1 and α2 are constants.

Problem. What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$.

Solution.

> The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence the solution (to this recurrence relation) is
>
> $$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$
>
> for some constant α_1 and α_2. Using the initial conditions, it follows that
>
> $$1 = a_0 = \alpha_1$$
> $$6 = a_1 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$$
>
> Solving these two equations shows that $\alpha_1 = \alpha_2 = 1$. Hence, the solution to the recurrence relation and initial conditions is
>
> $$a_n = 3^n + n3^n$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5

> If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$
>
> then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n(h)\}$ is a solution of the associated homogeneous recurrence relation
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Theorem 6

> Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$
>
> , where $c_1, c_2, \ldots, c_k$ are real numbers, and

> $$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$
>
> , where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s $\underline{is not a root of the characteristic equation of the associated linear homogeneous recurrence relation}$, there is a particular solution of the form
>
> $$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$
>
> . When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form
>
> $$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

Divide-and-Conquer Algorithms and Recurrence Relations

Theorem 1

> $f(n) = af(n/b) + c$ whenever n is divisible by b, b is an interger greater than 1, and c is a positive real number. Then
>
> $$f(n) is \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$
>
> $$f(n) = C_1 n^{\log_b a} + C_2 \quad \text{if } n = b^k \text{ and } a \neq 1$$

Generating Functions

Theorem 1

> Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
>
> $$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x) + g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Definition 2

> Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by
>
> $$\binom{u}{k} = \begin{cases} u(u-1)\ldots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

Theorem x

> $$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Theorem 2

> Let x be a real number with $|x| < 1$ and let u be a real number. Then
>
> $$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Using Generating Functions to Solve Recurrence Relations

Solve the recurrence relation $a_k = 3a_{k-1} \forall k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

> Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. First note that
>
> $$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$$
>
> Using the recurrence relation, we see that
>
> $$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2 \end{aligned}$$
>
> Because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,
>
> $$\begin{aligned} G(x) - 3xG(x) = (1-3x)G(x) = 2 \\ G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k \end{aligned}$$
>
> Consequently, $a_k = 2 \cdot 3^k$

Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1} \ \forall k = 1, 2, 3, \ldots$ and initial condition $a_1 = 9$.

> Assume $a_0 = 1$, then $a_1 = 8a_0 + 10^0 = 9$, which is consistent with our original initial condition
> We multiply both sides of the recurrence relation by x^n to obtain
>
> $$a_n x^n = 8a_{n-1} x^n + 10^{n-1} x^n$$
>
> Let $G(x) = \sum_{k=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$
\begin{aligned}
G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8G(x) + \frac{x}{1-10x}
\end{aligned}
$$

Solving for G(x) shows that

$$
\begin{aligned}
G(x) &= \frac{1-9x}{(1-8x)(1-10x)} \\
&= \frac{1}{2}(\frac{1}{1-8x} + \frac{1}{1-10x}) \\
&= \frac{1}{2}(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n) \\
&= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n
\end{aligned}
$$

Consequently, we have shown that

$$
a_n = \frac{1}{2}(8^n + 10^n)
$$

Graphs and Graph Models

when the context is clear, we will use the term graph to refer only to undirected graphs.

Simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.

Multigraph: Graph that may have multiple edges connecting the same vertices.

Directed multigraphs: Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex.

Multiplicity: When there are m directed edges, each associated to an ordered pair of vertices (u, v), we say that (u, v) is an edge of multiplicity m.

TABLE 1 Graph Terminology.			
Type	***Edges***	***Multiple Edges Allowed?***	***Loops Allowed?***
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

K_n: Complete Graphs, a simple graph that contains exactly one edge between each pair of distinct vertices.

C_n: Cycle, as it says.

W_n: Wheel, add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C.

Q_n: n-Cube, a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

Bipartite Graph: A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

$K_{m,n}$: Complete Bipartite Graphs, a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

M: Matching, a subset of the set E of edges of the graph such that no two edges are incident with the same vertex.

Maximum Matching: a matching with the largest number of edges.

Complete matching from V_1 to V_2: a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) with $|M| = |V_1|$

Definition 1

> A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Definition 2

A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Graph Terminology and Special Types of Graphs

Definition 1 Adjacent

> Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Definition 2 N(v)

> The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = U_{v \in A} N(v)$.

Theorem 1

> Let G = (V, E) be an undirected graph with m edges. Then $2m = \sum_{v \in V} deg(v)$

Theorem 2

> An undirected graph has an even number of vertices of odd degree.

Definition 3 Adjacent in UG

> When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.

Theorem 3

> Let G = (V, E) be a graph with directed edges. Then
>
> $$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

Theorem 4

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Theorem 5

The bipartite graph $G = (V, E)$ with bipartition $(V1, V2)$ has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Representing Graphs and Graph Isomorphism

Connectivity

Definition 1

> An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

Theorem 1

> There is a simple path between every pair of distinct vertices of a connected undirected graph.

Definition 2

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

Definition 3

> cut vertices (or articulation points), the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components.

> cut edge or bridge, an edge whose removal produces a graph with more connected components than in the original graph

Definition 4

> A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

> A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Euler and Hamilton Paths

Theorem 1

> A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem 2

> A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Theorem 3

> If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Theorem 4

> If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

A tree with n vertices has $n - 1$ edges.

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

A full m-ary tree with
(i) n vertices has $i = (n - 1)/m$ internal vertices and $l = [(m - 1)n + 1]/m$ leaves, (ii)
iinternalverticeshasn=mi+1verticesandl=(m−1)i+1leaves,
(iii) l leaves has $n = (ml - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.

There are at most mh leaves in an m-ary tree of height h.

If an m-ary tree of height h has l leaves, then $h \geq \lceil \log m\ l \rceil$. If the m-ary tree is full and bal- anced, then $h = \lceil \log m\ l \rceil$. (We are using the ceiling function here. Recall that $\lceil x \rceil$ is the small- est integer greater than or equal to x.)