一：27、28、29、30、31、33、34、36、38、41.

二：23、24、27、30、

作业四

27、由过渡矩阵定义，设 α, β 为两组基，有 $\alpha = \beta A$，$\beta = \alpha B$.

则 $\alpha = \alpha(BA)$. 所以 $BA = I$，即 A, B 可逆且互为逆矩阵.

28、令 $t = x-1$，则 $x = t+1$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$f(x) = a_0 + a_1(t+1) + a_2(t+1)^2 + \cdots a_n(t+1)^n$$

$$= (a_0 + a_1 + \cdots a_n) + (a_1 + 2a_2 + 3a_3 + \cdots na_n)t + (a_2 + 3a_3 + 6a_4 + C_5^3 a_5 + \cdots)t^2$$

$$+ \cdots a_n t^n$$

$\therefore 1, x-1, (x-1)^2 \cdots (x-1)^n$ 是 $R[x]_{n+1}$ 的一组基，且坐标为 $[\sum_1^n a_i, \sum_2^n C_i^1 a_i, \sum_3^n C_i^2 a_i \cdots, a_n]$

29、正定性：$(\alpha, \alpha) = (\alpha, \alpha)_1 + (\alpha, \alpha)_2$. 由于分别都是内积，即 $(\alpha, \alpha)_1 \geq 0$ $(\alpha, \alpha)_2 \geq 0$

$\forall \alpha, \beta, \gamma \in V$

$\forall k, l \in R$

$\therefore (\alpha, \alpha) \geq 0$

线性：$(k\alpha + l\beta, \gamma) = (k\alpha + l\beta, \gamma)_1 + (k\alpha + l\beta, \gamma)_2$

$$= k(\alpha, \gamma)_1 + l(\beta, \gamma)_1 + k(\alpha, \gamma)_2 + l(\beta, \gamma)_2$$

$$= k((\alpha, \gamma)_1 + (\alpha, \gamma)_2) + l((\beta, \gamma)_1 + (\beta, \gamma)_2)$$

$$= k(\alpha, \gamma) + l(\beta, \gamma)$$

对称：$(\alpha, \beta) = (\alpha, \beta)_1 + (\alpha, \beta)_2 = (\beta, \alpha)_1 + (\beta, \alpha)_2 = (\beta, \alpha)$

$(\alpha, \beta) = (\beta, \alpha)_1 + (\beta, \alpha)_2$ 也是 V 的一种内积.

30、正定性. $(x, x) = ax_1x_1 + bx_1x_2 + bx_2x_1 + cx_2x_2$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

$$= x_2^2 \left(a\left(\frac{x_1}{x_2}\right)^2 + 2b\left(\frac{x_1}{x_2}\right) + c \right) \geq 0$$ 恒成立.

有 $x_2^2 \geq 0$，则 $a > 0$ 且 $\Delta = (2b)^2 - 4ac \leq 0 \Rightarrow b^2 \leq ac$.

又正定只有 $x = 0$ 时才可为零，$\therefore b^2 < ac$。线性、对称性显然.

综上，有 $a > 0$ 且 $b^2 < ac$.

31、 正定．$(f,f) = f(0)f(0) + f(\frac{\pi}{2})f(\frac{\pi}{2})$.

$f, g, h \in V$

$k, l \in R$.

$= a^2 + b^2 \geq 0$. 且仅当 $a=b=0$ 时等号成立，也即 f 是 V 中零点时.

线性．$(kf+lg, h) = (kf(0)+lg(0))h(0) + (kf(\frac{\pi}{2})+lg(\frac{\pi}{2}))h(\frac{\pi}{2})$.

$= k[f(0)h(0) + f(\frac{\pi}{2})h(\frac{\pi}{2})] + l[g(0)h(0) + g(\frac{\pi}{2})h(\frac{\pi}{2})]$.

$= k(f,h) + l(g,h)$.

对称．$(f,g) = f(0)g(0) + f(\frac{\pi}{2})g(\frac{\pi}{2}) = (g,f)$. $\therefore$ 是内积.

$h(t) = 3(\cos t\cos 7 - \sin t\sin 7) + 4(\sin t\cos 9 + \cos t\sin 9)$.

$= (3\cos 7 + 4\sin 9)\cos t + (4\cos 9 - 3\sin 7)\sin t$.

则长度为 $\sqrt{a^2+b^2} = \sqrt{(3\cos 7+4\sin 9)^2 + (4\cos 9 - 3\sin 7)^2} = \sqrt{3^2(\cos^2 7+\sin^2 7) + 4(\sin^2 9+\cos^2 9)}$

$= 5$.

33、 该不等式成立，证明，对任意 $x=(x_1 \cdots x_n)^T$，$y=(y_1 \cdots y_n)^T \in R^n$，显然有 $(R^n, R, +, \cdot)$

定义 $(x,y) = \sum a_i x_i y_i$ 其中 $a_i > 0$，$a_i \in R$

由于、 1° $(x,x) = \sum a_i x_i^2 \geq 0$，且等号成立当且仅当 $x=0$

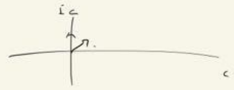
2° $(kx+ly, z) = \sum a_i (kx_i + ly_i) z_i = k\sum a_i x_i z_i + l\sum a_i y_i z_i$.

$= k(x,z) + l(y,z)$.

3° $(x,y) = \sum a_i x_i y_i = (y,x)$. $\therefore$ 是一个内积

$\therefore$ 有 $(x,y)^2 \leq (x,x)(y,y)$.

$\Rightarrow (\sum a_i x_i y_i)^2 \leq (\sum a_i x_i^2)(\sum a_i y_i^2)$.



由于，2维线性空间，即 $(a+bi \in V, R+\cdot)$.

34, (1) 要求 $(i, 1+i)=0$ 且 $(i,i)=1$, $(1+i,1+i)=1$.

则内积：$(\alpha_1+\beta_1 i, \alpha_2+\beta_2 i) = (\alpha_1+\alpha_1 i+(\beta_1-\alpha_1)i, \alpha_2+\alpha_2 i+(\beta_2-\alpha_2)i)$

$= \alpha_1\alpha_2(1+i,1+i)+(\beta_1-\alpha_1)\alpha_2(i, 1+i)+\alpha_1(\beta_2-\alpha_2)(1+i,i)+(\beta_1-\alpha_1)(\beta_2-\alpha_2)(i,i)$

$= \alpha_1\alpha_2+(\beta_1-\alpha_1)(\beta_2-\alpha_2) = 2\alpha_1\alpha_2+\beta_1\beta_2-\alpha_1\beta_2-\alpha_2\beta_1$

则存在。

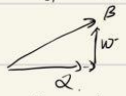
(2). 有 $(e_1,e_1)=1$. $(e_1, e_1+e_2)=(e_1,e_1)+(e_1,e_2)=0$. $\therefore (e_1,e_2)=-1$.

$(e_1,e_1+e_2+e_3)=0=(e_1,e_1)+(e_1,e_2)+(e_1,e_3) \Rightarrow (e_1,e_3)=0$. $(e_1+e_2, e_1+e_2)=1=(e_1,e_1)+2(e_1,e_2)+(e_2,e_2)$

$\therefore (e_2,e_2)=2$. $(e_1+e_2, e_1+e_2+e_3)=1+(e_1,e_3)+(e_2,e_3)$ $\therefore (e_2,e_3)=-1$.

$(e_1+e_2+e_3, e_1+e_2+e_3)-1=(e_3,e_1+e_2+e_3)+0=(e_1,e_3)+(e_2,e_3)+(e_3,e_3)=0-1+(e_3,e_3)$ $\therefore (e_3,e_3)=2$.

$\therefore ((x_1,x_2,x_3),(y_1,y_2,y_3)) = x_1y_1+2x_2y_2+2x_3y_3-x_1y_2-x_2y_1-x_2y_3-x_3y_2$.

36, 对于 β, α, 如图,  设 β 到 α 投影为 $a\alpha$.

则 $(a\alpha, \beta-a\alpha)=0=(a\alpha, \beta)+(a\alpha, -a\alpha) = a(\alpha\beta)+a(\overline{-a})(\alpha\alpha)$

$\Rightarrow a(\alpha\beta)=-a(\overline{-a})(\alpha\alpha)$

$\Rightarrow \bar{a}=\frac{(\alpha\beta)}{(\alpha\alpha)} \Rightarrow a=\overline{\left(\frac{(\alpha\beta)}{(\alpha\alpha)}\right)}=\frac{\overline{(\alpha\beta)}}{(\alpha\alpha)}=\frac{(\beta\alpha)}{(\alpha\alpha)}$.

$\beta=\frac{(\beta\alpha)}{(\alpha\alpha)}\alpha+\left(\beta-\frac{(\beta\alpha)}{(\alpha\alpha)}\alpha\right)$.

$\Rightarrow \|\beta\|^2=\left\|\frac{(\beta\alpha)}{(\alpha\alpha)}\alpha\right\|^2+\|w\|^2 \geq \left\|\frac{(\beta\alpha)}{(\alpha\alpha)}\alpha\right\|^2=\left|\frac{(\beta\alpha)}{(\alpha\alpha)}\right|^2(\alpha\alpha)=\frac{|(\beta\alpha)|^2}{|(\alpha\alpha)|^2}(\alpha\alpha)$

$\Rightarrow |(\beta\alpha)|^2 \leq (\alpha\alpha)(\beta\beta) \Rightarrow |(\beta\alpha)| \leq \|\alpha\|\|\beta\|$. 等号成立当仅当 $\alpha\beta$ 垂直或线性相关。

则三角不等式. $(\alpha+\beta)^2=\alpha^2+\beta^2+2(\alpha,\beta) \leq \alpha^2+\beta^2+2\|\alpha\|\|\beta\|$.

38, (d1) $d(\alpha,\beta) = \|\alpha-\beta\| = \|\beta-\alpha\| = d(\beta,\alpha)$

(d2) $d(\alpha,\beta) = \|\alpha-\beta\| \geq 0$. 且 若 $\alpha=\beta$ 时 $d(\alpha,\beta)=0$. 若 $\alpha\neq\beta$, 则 $\|\alpha-\beta\|\neq 0$ ∴ 当仅当 $\alpha=\beta$ 时 等号成立.

(d3) $d(\alpha,\beta)+d(\beta,\gamma) = \|\alpha-\beta\|+\|\beta-\gamma\|$

$= \sqrt{(\alpha-\beta,\alpha-\beta)}+\sqrt{(\beta-\gamma,\beta-\gamma)}$

$= \sqrt{\quad}$

41, (1) 正定 $(A,A) = tr(AA^*) = \sum_{i,j=1}^{n} a_{ij}^2$

线性 $(kA+lB,C) = tr((kA+lB)C^*) = k\,tr(AC^*)+l\,tr(BC^*) = k(A,C)+l(B,C)$

对称 $(A,B) = tr(AB^*) = \sum_{i,j=1}^{n} a_{ij}b_{ij} = \sum_{i,j=1}^{n} b_{ij}a_{ij} = tr(BA^*) = (B,A)$.

(2) 长度为 $\sqrt{(A,A)} = \sqrt{tr(AA^*)} = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}$ 所有元素平方和为1的向量都是单位向量.

(3) $(E_{ij},E_{kl}) = tr(E_{ij}E_{kl}^*)$ 其值为1 当仅当 $i=k$, $j=l$. 所以是标准正交基

(4) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$.

第二章.

23, 任意取 $\alpha_1,\alpha_2\cdots\alpha_n$ 是V的一组基 对应于 $\beta_1\cdots\beta_m$ 是V的一组基 设此映射 σ.

则, $\forall\alpha\in V = [\alpha_1\cdots\alpha_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ 对应于 $[\beta_1\cdots\beta_m]\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$.

∴ $\sigma(\alpha) = [\alpha_1\cdots\alpha_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}A = [\beta_1\cdots\beta_m]\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

有 $\beta_1 = [\alpha_1\cdots\alpha_n]\begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1n} \end{bmatrix}\cdots\beta_m = [\alpha_1\cdots\alpha_n]\begin{bmatrix} A_{m1} \\ \vdots \\ A_{mn} \end{bmatrix}$. 是一一对应的.

显然 即单又满.

又, $a,b\in V$
$\begin{cases} \sigma(a+b) = \sigma(a)+\sigma(b) \\ \sigma(ka) = k\sigma(a). \end{cases}$ ∴是线性的. ∴ $Hom(V,V) \cong F^{m\times n}$.

24. proof. $\phi: V \to V^{**}$ $\phi: \alpha = \sum_{1}^{n} x_i \alpha_i \to \sigma(\alpha) \in V^{**}$.

$(\sigma(\alpha))(f) \triangleq f(\alpha)$ $f \in V^*$.

check ϕ is linear. check ϕ 单射.

check. $\begin{cases} \phi(\alpha+\beta) = \sigma(\alpha) + \sigma(\beta) \\ \phi(k\alpha) = k\sigma(\alpha) \end{cases}$ $\phi: V \to V^{**}$.

$(\phi(\alpha+\beta))(f) \triangleq f(\alpha+\beta) = f(\alpha) + f(\beta)$ $\forall f \in V^*$

$= (\phi(\alpha))(f) + (\phi(\beta))(f)$

$= (\phi(\alpha) + \sigma(\beta))(f)$.

$\Rightarrow \phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta)$

$(\phi(k\alpha))(f) \xlongequal{\forall f \in V^*} f(k\alpha) = kf(\alpha) = k(\sigma(\alpha))(f)$.

$= (k\phi(\alpha))(f) \Rightarrow \phi(k\alpha) = k(\sigma\alpha)$

check ϕ 单射. 即 $(\phi(\alpha))(f) = f(\alpha)$ $\forall f \in V^*$

if $\phi(\alpha) = 0$. $\Rightarrow \forall f \in V^*$ $\underline{(\phi(\alpha))(f)} = 0(f) = 0$.

$\Rightarrow$ $f(\alpha) = 0$ $\forall f \in V^*$

$\Rightarrow \alpha = 0$. $\Rightarrow \phi$ 单.

综上, ϕ 线性且单射, 是同构变换,

取 V 的一组基 $\alpha_1 \cdots \alpha_n$. 有对偶基 $\alpha_1^* \cdots \alpha_n^* \in V^*$, $\alpha_1^{**} \cdots \alpha_n^{**} \in V^{**}$ 则 $\phi(\alpha_i) = \alpha_i^{**}$.

$\therefore$ 逆射 ϕ^{-1}: $\alpha_i^{**} \to \alpha_i$. $i \in (1, n)$

27. (1). 显然，每 σ 一次就多一个0，所以幂零指数为 n.

矩阵为 $\begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix}$

(2). 不一定，如. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ 不是同构变换.

(3). 不一定，如. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = D$. $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

则 $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ 不可逆.

30. $(\alpha_1+\alpha_2, \beta) = (\alpha_1, \beta) + (\alpha_2, \beta) = 0$. 其中 $\alpha_1, \alpha_2, \alpha \in W$, $k \in F$.

$(k\alpha, \beta) = k(\alpha, \beta) = 0$, $\therefore$ W 是 V 的子空间.

又. $\dim W = n - \dim U$,

check $U \cap W = 0$. 令. $\alpha \in W$ 且 $\alpha \in U$.

则. $(\alpha, \alpha) = 0$. 又. $(\alpha, \alpha) \geq 0$ 且 只有 $\alpha = 0$ 时 才可等于零 $\therefore$ 不成立.

$\therefore U \cap W = 0$, 综上, $V = U \oplus W$.