作业十. 5 做过.

1, pf. 对 n 作归纳法. $n=1$ 成立. 设对所有小于 n 的都成立.

check 对 n 成立. 取 λ_1 为 A 的一个特征值. 相应特征向量 α_1. 单位化 e_1.

则 $Ae_1=\lambda_1 e_1$. 将 e_1 扩充为 C^n 一个标准正交基 $e_1 \cdots e_n$ let $U=(e_1 \cdots e_n)$ 是酉阵

$Ae_2 = ()e_1+()e_2 \cdots ()e_n$.

$\vdots$

$Ae_n = ()e_1 \cdots +()e_n$.

$$A(e_1 \cdots e_n) = (Ae_1 \cdots Ae_n) = (e_1 \cdots e_n)\begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

对右下角 $n-1$ 阶矩阵 提出 C_1. $AU_1 = U_1 \begin{bmatrix} \lambda_1 & * \; * & * \cdots \\ 0 & & \\ 0 & \boxed{C_1} & \\ \vdots & & \end{bmatrix}$

对 C_1 存在 $n-1$ 阶酉阵 U_2 s.t. $U_2^* C_1 U = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$.

set $U = U_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}$. 显然是酉阵.

$$U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}^* U_1^* A U_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}^* \begin{bmatrix} \lambda_1 & * \\ 0 & \\ \vdots & C_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & * \\ 0 & U_2^* C U_2 \end{bmatrix}.$$

2, (1) 由于 $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \\ -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} = LR$

$\therefore |\lambda I - A| = \lambda^2 |\lambda I - RL|$

$RL = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

$\therefore |\lambda I - A| = \lambda^2(\lambda+1)(\lambda+2)$.

$\therefore A^2(A+I)(A+2I) = 0$.

$A^2 + 3A + 2I = 0$. $A^2 = -3A - 2I$.

$\therefore A^6 = (-3A-2I)^3 = (-3A-2I)(9A^2+12A+4I)$.

$= (-3A-2I)(-27A+12A-18I+4I)$.

$= (-3A-2I)(-15A-14I)$.

$= 45A^2+30A+52A+28I = 82A+28I-135A-90I$

$= -53A-62I$.

$$= \begin{bmatrix} 96 & -95 & 95 & -96 \\ 64 & 63 & -63 & -64 \\ 64 & -64 & 64 & -64 \\ -32 & -32 & 32 & -32 \end{bmatrix}.$$

(2). 由上一问.特征方程.得. $J = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & -2 \end{bmatrix}.$

3, (1). $A = \begin{bmatrix} a_1b_1+x & a_1b_2 & a_1b_3 \cdots & a_1b_n \\ a_2b_1 & +x & & \\ a_3b_1 & & \ddots & \\ \vdots & & & \\ a_nb_1 & & & \end{bmatrix} = \begin{bmatrix} a_1b_1+x & & \cdots & \\ -\frac{a_2}{a_1}x. & x & 0 & \cdots\ 0 \\ -\frac{a_3}{a_1}x. & 0 & x & 0 \cdots \\ & & & \ddots \\ & & & x \end{bmatrix}$

$= (a_1b_1+x)\cdot x^{n-1} + (-1)\cdot a_1b_2\cdot\left(-\frac{a_2}{a_1}x\cdot x^{n-2}\right) + a_1b_3\cdot\left[x\cdot(-1)\left(-\frac{a_3}{a_1}x\cdot x^{n-3}\right)\right]\cdots$

$= \left(x+\sum_1^n a_ib_i\right)x^{n-1}.$

(2). $|A| = |xI_n+\alpha\beta^T| = x^{n-1}|x+\beta^T\alpha| = (x+\beta^T\alpha)x^{n-1}.$

(3). 显然 $\alpha\beta^T$ 的秩 ≤ 1. ∴特征值有 $n-1$ 个为 0. 1个为 $\sum_1^n a_ib_i$.

则 $xI+\alpha\beta^T$ 特征值有 $n-1$ 个为 x, 1个为 $x+\sum_1^n a_ib_i$.

∴ $|A| = (x+\beta^T\alpha)x^{n-1}.$

4, 如图是上三角阵.且前两个相等.后两个相等.

则 其可相似于. $\begin{bmatrix} \alpha & u & & \\ & \alpha & & \\ & & \beta & v \\ & & & \beta \end{bmatrix}.$

若 $\alpha=0$. 则 $u=0$. 否则 $u=1$.

若 $\beta=0$ 则 $v=0$ 否则 $v=1$.

10, $\because \begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\-1\end{bmatrix}=0$. $\therefore$ 是正交向量.

$\therefore$ A是对称矩阵 有 $A=\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}0 & \\ & 1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}^{-1}$.

$=\frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$.