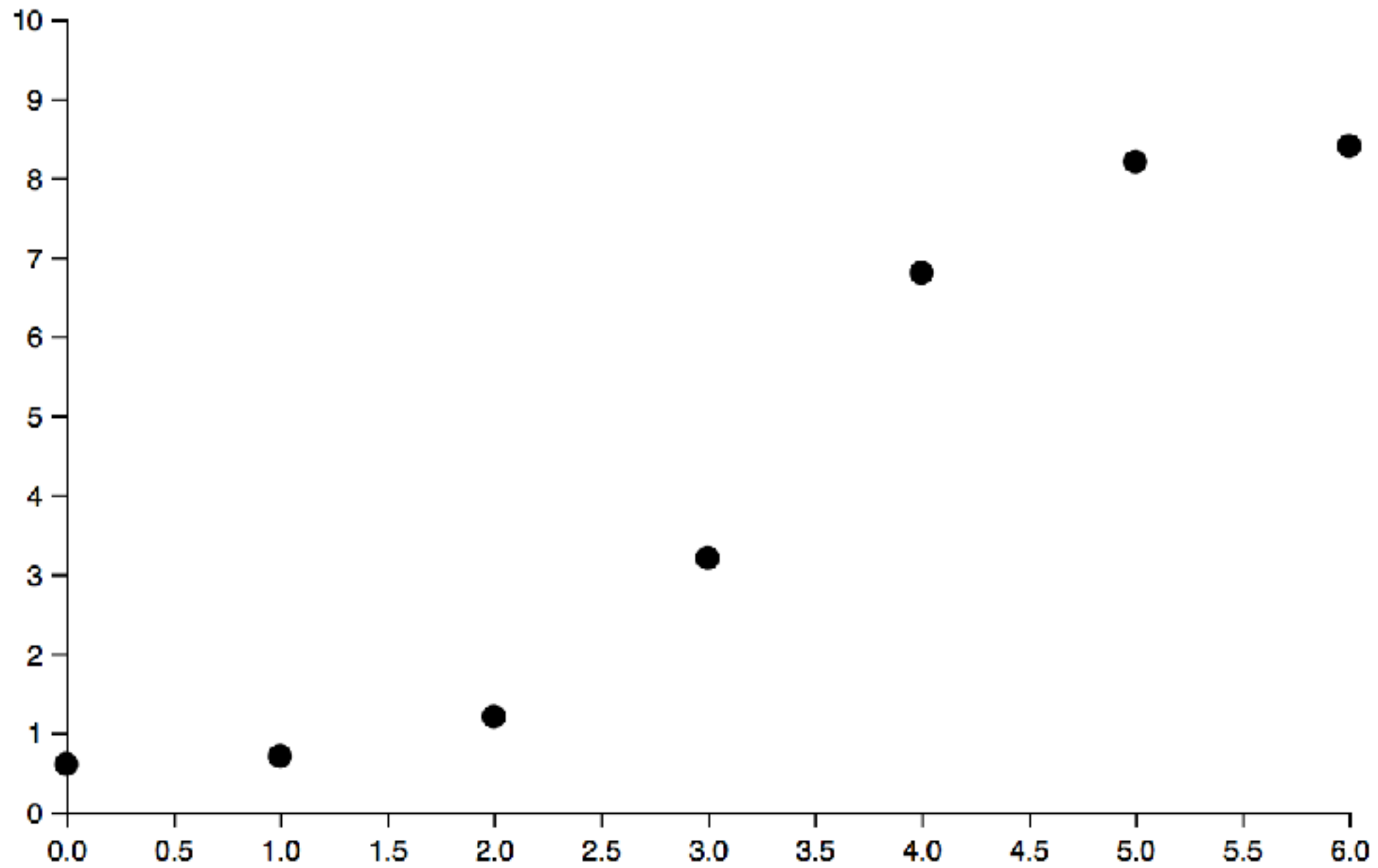


CS492: Probabilistic Programming

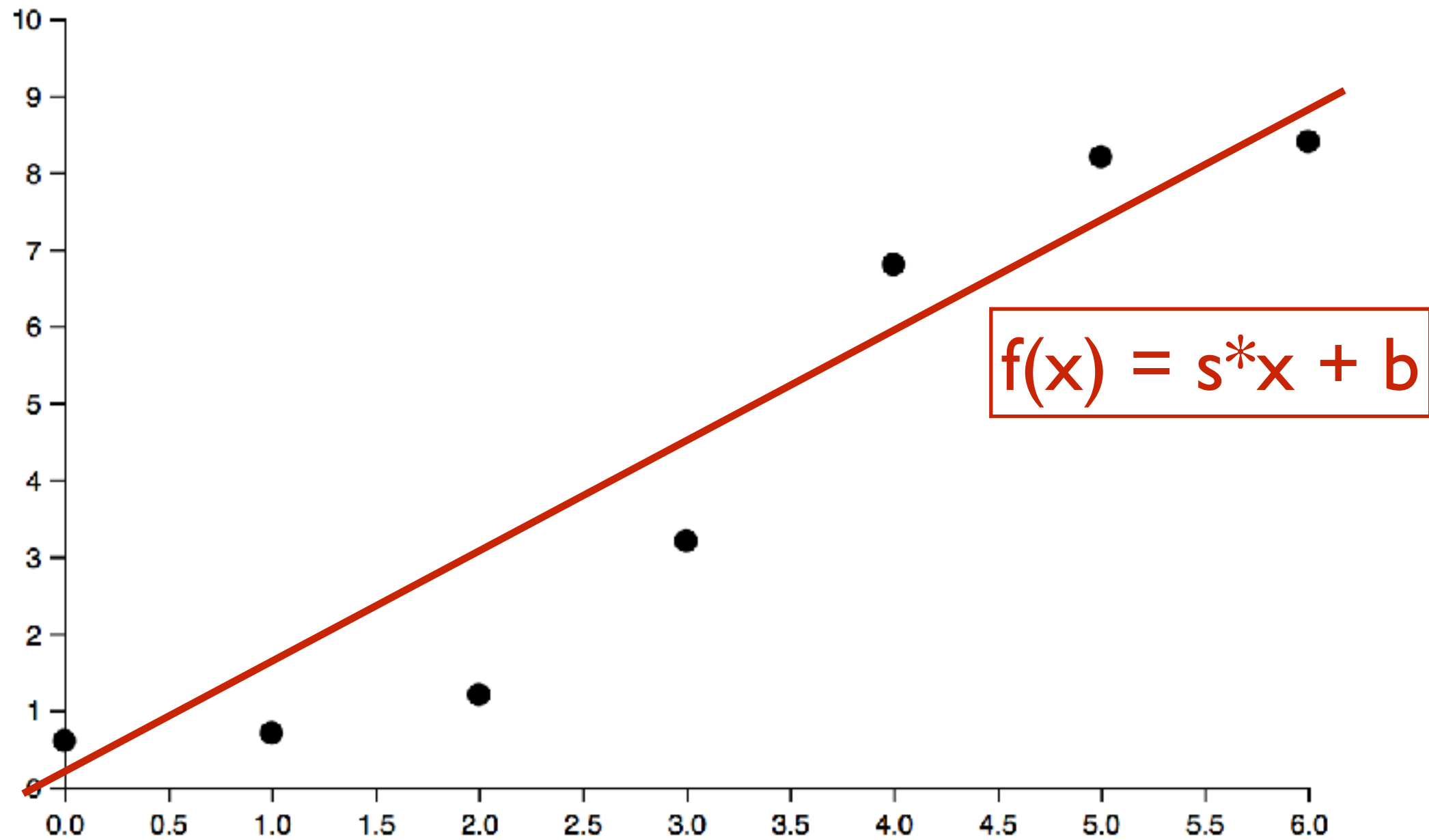
Denotational Semantics of Probabilistic Programs

Hongseok Yang
KAIST

Line fitting



Line fitting



Anglican program

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(defquery lin-regression [])
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(defquery lin-regression []  
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        b (sample (normal 0 6))  
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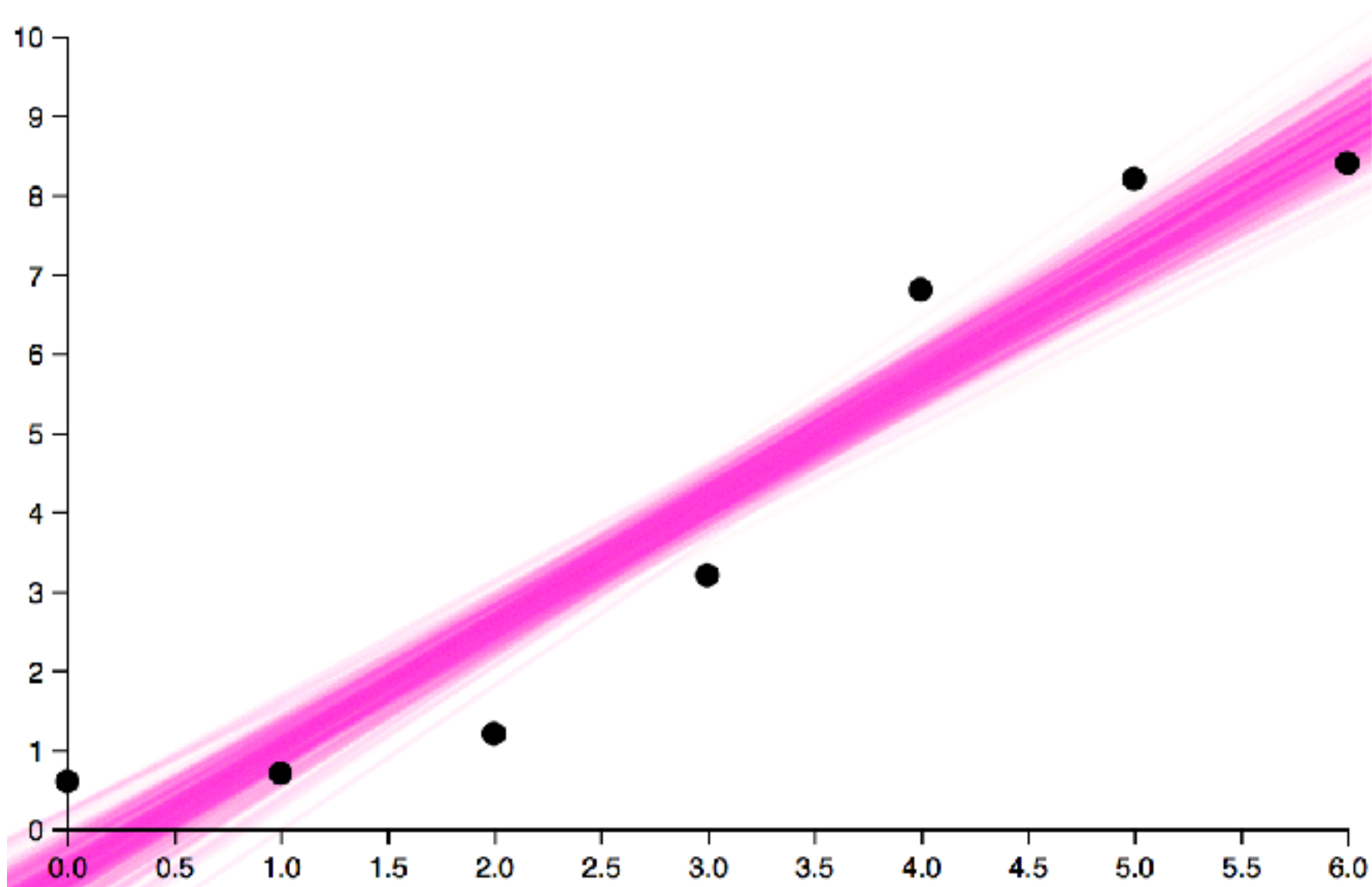
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Samples from posterior




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[Q] Which posterior?

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Inference algo. gives
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Should define distr. on
functions. Not easy.

Foundational question

Measure theory provides a foundation for modern probability theory.

But it doesn't support higher-order fns well.

$$\text{ev} : (\mathbb{R} \rightarrow_m \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}, \quad \text{ev}(f, x) = f(x).$$

[Aumann 61] ev is not measurable no matter which σ -algebra is used for $\mathbb{R} \rightarrow_m \mathbb{R}$.

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Denotational semantics:
Compositional method.
Answers a deep Q.

Learning outcome

- Can define a denotational semantics for a simple programming language.
- Can use measure theory to interpret prob. prog. with continuous distributions.
- Can use quasi-Borel space to interpret higher-order prob. prog. with conditioning.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".

Plan for the rest

1. Denotational semantics.
PL with discrete random choices.
2. Baby measure theory.
PL with cont. distribution.
3. Quasi-Borel space (QBS).
PL with cont. distr. & higher-order (HO) fns.
4. SFinKer monad on QBS.
PL with cont. distr., HO fns & conditioning.

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Denotational semantics

- Defines formal meanings of programs.
- Interprets type and expr. mathematically.
 - Type as space (e.g. set, measurable space).
 - Expr. as good function between spaces.
- Used for justifying compiler optimisation, inference algorithms and language constructs.

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First-order PL with discrete random choices

$t ::= \text{bool} \mid \text{rational} \mid \text{dist}[\text{bool}] \mid \text{dist}[\text{rational}]$

$e ::= c \mid x \mid (p \ e \ \dots \ e) \mid (\text{let } [x \ e] \ e) \mid (\text{if } e \ e \ e)$

$c ::= \text{true} \mid \text{false} \mid 0 \mid 1 \mid \dots$

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Only primitive functions can be applied.

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[Q] Denotational semantics of this PL?

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[Q] Denotational semantics of this PL?

Interpret type as set and expr. as function.

Types mean sets

$\llbracket t \rrbracket$ is the meaning of t .

$\llbracket \text{bool} \rrbracket = \dots$

$\llbracket \text{rational} \rrbracket = \dots$

$\llbracket \text{dist}[\text{bool}] \rrbracket = \dots$

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[Q] Fill in \dots

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$$\llbracket \text{dist}[\text{bool}] \rrbracket = \text{DiscProb}(\llbracket \text{bool} \rrbracket)$$

$$\llbracket \text{dist}[\text{rational}] \rrbracket = \text{DiscProb}(\llbracket \text{rational} \rrbracket)$$

[Q] Fill in . . .

Typed expressions

$$x_1:t_1, x_2:t_2, \dots, x_n:t_n \vdash e : t$$

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typing context Γ


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- Denotes a t -typed expr. e under Γ .

$x:\text{bool}, y:\text{bool} \vdash (\text{if } x \ y \ y) : \text{bool}$

$x:\text{bool}, y:\text{rational} \vdash (\text{if } x \ y \ y) : \text{rational}$

$x:\text{rational} \vdash (\text{sample } (\text{flip } x)) : \text{bool}$

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$$\begin{aligned} \text{I. } & \llbracket x_1:t_1, \dots, x_n:t_n \rrbracket = \\ & \{\text{map } \eta \text{ from } \{x_1, \dots, x_n\} \mid \eta(x_i) \in \llbracket t_i \rrbracket \text{ for all } i\} \end{aligned}$$

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[Q] Define the interpretation recursively.

Compiler optimisation

Show the following equations:

$$\llbracket \Gamma \vdash (\text{if true } e_1 \ e_2) : t \rrbracket = \llbracket \Gamma \vdash e_1 : t \rrbracket$$

$$\begin{aligned} &\llbracket \Gamma \vdash (\text{sample } (\text{flip } (+ \ 0.1 \ 0.2)) : \text{bool}) \rrbracket \\ &= \llbracket \Gamma \vdash (\text{sample } (\text{flip } 0.3)) : \text{bool} \rrbracket \end{aligned}$$

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$e ::= c \mid x \mid (p \ e \ \dots \ e) \mid (\text{let } [x \ e] \ e) \mid (\text{if } e \ e \ e)$

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First-order PL with discrete random choices and continuous

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 $\quad \mid \text{normal} \mid \text{uniform-continuous} \mid \dots$

How to define denotational semantics?

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$$t ::= \text{bool} \mid \text{real} \mid \text{dist}[\text{bool}] \mid \text{dist}[\text{real}]$$

[Try] Interpret $\llbracket t \rrbracket$ as a set.

How to define denotational semantics?

Types as spaces and expressions as functions.

$$t ::= \text{bool} \mid \text{real} \mid \text{dist}[\text{bool}] \mid \text{dist}[\text{real}]$$

[Try] Interpret $\llbracket t \rrbracket$ as a set. Then, we get stuck
since $\llbracket \text{dist}[\text{real}] \rrbracket$ can't be $\text{DiscProb}(\llbracket \text{real} \rrbracket)$.

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[Try] Interpret $\llbracket t \rrbracket$ as a set. Then, we get stuck since $\llbracket \text{dist}[\text{real}] \rrbracket$ can't be $\text{DiscProb}(\llbracket \text{real} \rrbracket)$.

[Sol] Use measure theory. $\llbracket t \rrbracket$ as a measurable space, and $\llbracket \Gamma \vdash e : t \rrbracket$ as a measurable function.

How to specify prob. p ?

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$X = \{0, 1, 2\}$.

Define $p : X \rightarrow [0, 1]$. E.g., $p = [0.4, 0.4, 0.2]$.

Lifted $p : 2^X \rightarrow [0, 1]$ by $p(A) = \sum_{x \in A} p(x)$.

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Uncountable sum.
Typically ∞ .

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Pick a good collection $\Sigma \subseteq 2^X$.

Define $p : \Sigma \rightarrow [0, 1]$ with some care.

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σ -algebra

Pick a **good** collection $\Sigma \subseteq 2^X$.

Define **p** : $\Sigma \rightarrow [0, 1]$ with some **care**.
probability measure

Let $\Sigma \subseteq 2^X$.

Σ is a σ -algebra if it contains X , and is closed under countable union and set subtraction.

(X, Σ) is a measurable space if Σ is a σ -algebra.

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Σ is a σ -algebra if it contains X , and is closed under countable union and set subtraction.

(X, Σ) is a measurable space if Σ is a σ -algebra.

$p : \Sigma \rightarrow [0, 1]$ is a probability measure if $p(X) = 1$ and $p(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} p(A_n)$ for all disjoint A_n 's.

(X, Σ, p) is a probability space if ...

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B} \times \mathbb{B}, \{ A \times B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R} - A) \text{ finite or countable} \})$.
4. $(\mathbb{R}, \{ (r, s] \mid r < s \})$.

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Closure exists.

$\sigma(\Pi)$ smallest σ -algebra containing Π .

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

Product σ -algebra: $\Sigma \otimes \Theta = \sigma\{A \times B \mid A \in \Sigma, B \in \Theta\}$.

Product space: $(X, \Sigma) \times_m (Y, \Theta) = (X \times Y, \Sigma \otimes \Theta)$.

Borel σ -algebra on \mathbb{R} : $\mathfrak{B} = \sigma\{(r, s] \mid r < s\}$.

Borel space: $(\mathbb{R}, \mathfrak{B})$.

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

$\text{Pr}(\Sigma) = \dots$

Probability space: $\text{Pr}(X, \Sigma) = (\text{Pr}(X), \text{Pr}(\Sigma))$

[Q] What is $\text{Pr}(\Sigma)$?

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

$$\Pr(\Sigma) = \sigma\{ \{p \mid p(A) < r\} \mid A \in \Sigma, r \in \mathbb{R} \}.$$

Probability space: $\Pr(X, \Sigma) = (\Pr(X), \Pr(\Sigma))$

[Q] What is $\Pr(\Sigma)$?

Types mean mBle spaces

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$$\llbracket \text{bool} \rrbracket = (\mathbb{B}, 2^{\mathbb{B}})$$

$$\llbracket \text{dist}[\text{bool}] \rrbracket = \text{Pr}(\llbracket \text{bool} \rrbracket)$$

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[Q] Fill in ...

Types mean mBle spaces

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$$\llbracket \text{real} \rrbracket = (\mathbb{R}, \mathfrak{B})$$

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$$[[x_1:t_1, \dots, x_n:t_n]] = (X, \Sigma)$$

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where

$$(X_i, \Sigma_i) = [[t_i]]$$

$$X = \dots$$

$$\Sigma = \dots$$

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$$\llbracket x_1:t_1, \dots, x_n:t_n \rrbracket = (X, \Sigma)$$

where

$$(X_i, \Sigma_i) = \llbracket t_i \rrbracket$$

$$X = \{\text{map } \eta \text{ from } \{x_1, \dots, x_n\} \mid \eta(x_i) \in X_i \text{ for all } i\}$$

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$$\Sigma = \sigma\{ \{\eta \mid \eta(x_i) \in A_i \text{ for all } i\} \mid A_i \in \Sigma_i \text{ for all } i\}$$

[Q] Fill in ...

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

$f: X \rightarrow Y$ is measurable (denoted $f: X \rightarrow_m Y$) if $f^{-1}(A) \in \Sigma$ for all $A \in \Theta$.

Exprs mean mBle fns

$\llbracket \Gamma \vdash e : t \rrbracket$ is a **mBle** fn from $\llbracket \Gamma \rrbracket$ to **Pr** $\llbracket t \rrbracket$.

Exprs mean mBle fns

$\llbracket \Gamma \vdash e : t \rrbracket$ is a mBle fn from $\llbracket \Gamma \rrbracket$ to $\text{Pr}[\llbracket t \rrbracket]$.

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$= \int_A \text{density-norm}(s \mid \eta(y), 1) \, ds.$

Defined recursively. Complex but doable.

Plan for the rest

1. Denotational semantics.
PL with discrete random choices.
2. Baby measure theory.
PL with cont. distribution.
3. Quasi-Borel space (QBS).
PL with cont. distr. & higher-order (HO) fns.
4. SFinKer monad on QBS.
PL with cont. distr., HO fns & conditioning.

Prob. PL with HO fns and continuous random choices

$t ::= \text{bool} \mid \text{real} \mid \text{dist}[\text{bool}] \mid \text{dist}[\text{real}] \mid (t_1, \dots, t_n) \rightarrow t$

$e ::= c \mid x \mid (\text{fn } [x \dots x] e) \mid (e \ e \dots e) \mid (\text{if } e \ e \ e)$

$c ::= \text{true} \mid \text{false} \mid 0 \mid 1 \mid 2 \mid \text{and} \mid + \mid \dots$
 $\mid \text{sample} \mid \text{flip} \mid \text{normal} \mid \dots$

Prob. PL with **H_O** fns and continuous random choices

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Function type.

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Function type.

General fn decl. and app.

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Function type.
General fn decl. and app.
General constants.

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Function type.
General fn decl. and app.
General constants.

Measure theory insufficient due to HO fns.

Use a new foundation of probability theory based on quasi-Borel spaces.

Interpret $\llbracket t \rrbracket$ as a quasi-Borel space (QBS), and $\llbracket \Gamma \vdash e : t \rrbracket$ as a QBS morphism.

High-level idea:
Random variable first.

Random variable α in X

Random variable α in X

$$\alpha : \Omega \rightarrow X$$

- X - set of values.
- Ω - set of random seeds.
- Random seed generator.

Random variable α in X in measure theory

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$$I. \Sigma \subseteq 2^\Omega, \Theta \subseteq 2^X$$

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$$1. \Sigma \subseteq 2^\Omega, \Theta \subseteq 2^X$$

$$2. \mu : \Sigma \rightarrow [0, 1]$$

Random variable α in X in measure theory

$\alpha : \Omega \rightarrow X$ is a random element
if $\alpha^{-1}(A) \in \Sigma$ for all $A \in \Theta$

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$\begin{aligned} 1. \Sigma &\subseteq 2^\Omega, \Theta \subseteq 2^X \\ 2. \mu &: \Sigma \rightarrow [0, 1] \end{aligned}$
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Random variable α in X in quasi-Borel spaces

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$$\alpha : \mathbb{R} \rightarrow X$$

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1. \mathbb{R} as random source
2. Borel subsets $\mathcal{B} \subseteq 2^{\mathbb{R}}$

Random variable α in X in quasi-Borel spaces

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Random variable α in X in quasi-Borel spaces

$\alpha : \mathbb{R} \rightarrow X$ is a random variable
if $\alpha \in M$

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- Measure theory:
 - Measurable space $(X, \Theta \subseteq 2^X)$.
 - Random variable is an induced concept.
- QBS:
 - Quasi-Borel space $(X, M \subseteq [\mathbb{R} \rightarrow X])$.
 - M is the set of random variables.

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such that M has enough random variables.

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I. **M contains all constant functions.**

Quasi-Borel space - set with random variables.

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such that M has **enough** random variables.

1. M contains all constant functions.
2. $(\alpha \circ \beta) \in M$ for all $\alpha \in M$ and measurable $\beta: \mathbb{R} \rightarrow \mathbb{R}$.

Quasi-Borel space - set with random variables.

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such that M has **enough** random variables.

1. M contains all constant functions.
2. $(\alpha \circ \beta) \in M$ for all $\alpha \in M$ and measurable $\beta: \mathbb{R} \rightarrow \mathbb{R}$.
3. If $\mathbb{R} = \biguplus_{i \in \mathbb{N}} R_i$ with $R_i \in \mathfrak{B}$ and $\alpha_1, \alpha_2, \dots \in M$,
then $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}} \in M$.

Here $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}}(r) = \alpha_i(r)$ for all $r \in R_i$.

[Q] Pick a non-QBS.

1. $(\mathbb{R}, \{\alpha: \mathbb{R} \rightarrow \mathbb{R} \mid \alpha \text{ is a constant function}\})$.
2. $(\mathbb{R}, [\mathbb{R} \rightarrow \mathbb{R}])$.
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$\{(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}} \mid \alpha_i \text{ constant fn and } R_i \in \mathfrak{B}\}$

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Standard way of converting a set to a QBS.

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Standard way of converting a set to a QBS.

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Standard way of converting a mBle space to a QBS.

(QBS) morphism

$(X, M), (Y, N)$ - QBSes.

$f : X \rightarrow Y$ is a morphism if $(f \circ \alpha) \in N$ for all $\alpha \in M$.

Maps random elements to random elements.

We will write $f : X \rightarrow_q Y$.

[Th] QBSes support higher-order functions well.
(The category of QBSes is cartesian closed.)

[Q] What are the sets of random variables?

1. Product: $(X, \mathcal{M}) \times_q (Y, \mathcal{N}) = (Z, \mathcal{O})$.

- $Z = X \times Y$, $\pi_1(x, y) = x$, $\pi_2(x, y) = y$.

- $\mathcal{O} = ???$

2. Fn space: $[(X, \mathcal{M}) \rightarrow_q (Y, \mathcal{N})] = (Z, \mathcal{O})$

- $Z = \{ f \mid f : X \rightarrow_q Y \}$, $\text{ev}(f, x) = f(x)$.

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[Q] What are the sets of random elements?

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- $Z = X \times Y$, $\pi_1(x, y) = x$, $\pi_2(x, y) = y$.
- $O = \{ r \mapsto (\alpha(r), \beta(r)) \mid \alpha \in M \text{ and } \beta \in N \}$.

2. Fn space: $[(X, M) \rightarrow_q (Y, N)] = (Z, O)$

- $Z = \{ f \mid f : X \rightarrow_q Y \}$, $\text{ev}(f, x) = f(x)$
- $O = \{ g : \mathbb{R} \rightarrow Z \mid r \mapsto g(\gamma(r))(\alpha(r)) \in N \text{ for all } \gamma : \mathbb{R} \rightarrow_m \mathbb{R} \text{ and } \alpha \in M \}$.

Why works?

$$[\text{NO}] \text{ ev} : (\mathbb{R} \rightarrow_m \mathbb{R}) \times_m \mathbb{R} \rightarrow_m \mathbb{R}$$

vs

$$[\text{YES}] \text{ ev} : (\mathbb{R} \rightarrow_q \mathbb{R}) \times_q \mathbb{R} \rightarrow_q \mathbb{R}$$

Why works?

$$[\text{NO}] \text{ ev} : (\mathbb{R} \rightarrow_{\mathbf{m}} \mathbb{R}) \mathbf{x}_{\mathbf{m}} \mathbb{R} \rightarrow_{\mathbf{m}} \mathbb{R}$$

vs

$$[\text{YES}] \text{ ev} : (\mathbb{R} \rightarrow_{\mathbf{q}} \mathbb{R}) \mathbf{x}_{\mathbf{q}} \mathbb{R} \rightarrow_{\mathbf{q}} \mathbb{R}$$

Because the QBS product is more permissive.

Types mean mBle spaces

$$\llbracket \text{bool} \rrbracket = \text{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

$$\llbracket \text{dist}[\text{bool}] \rrbracket = \text{Pr}_q(\llbracket \text{bool} \rrbracket)$$

Types mean mBle spaces



Conversion of
mBle space to QBS

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QBS prob. space

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$$\llbracket \text{real} \rrbracket = \dots$$

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QBS prob. space


$$\llbracket \text{dist}[\text{real}] \rrbracket = \dots$$

$$\llbracket (t_1, t_2) \rightarrow t \rrbracket = \dots$$

[Q] Fill in ...

Types mean mBle spaces

Conversion of
mBle space to QBS



$$[[\text{bool}]] = \text{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

$$[[\text{real}]] = \text{MStoQBS}(\mathbb{R}, \mathfrak{B})$$

$$[[\text{dist}[\text{bool}]]] = \text{Pr}_q([[\text{bool}]])$$

$$[[\text{dist}[\text{real}]]] = \text{Pr}_q([[\text{real}]])$$

QBS prob. space



$$[(t_1, t_2) \rightarrow t] = \dots$$

[Q] Fill in ...

Types mean mBle spaces

Conversion of
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QBS prob. space



$$[(t_1, t_2) \rightarrow t] = [[t_1]] \times_q [[t_2]] \rightarrow_q \text{Pr}_q([t])$$

[Q] Fill in ...

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$$\llbracket x_1:t_1, \dots, x_n:t_n \rrbracket = (X, M)$$

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Types mean QBSes

$$\llbracket \mathbf{x}_1:t_1, \dots, \mathbf{x}_n:t_n \rrbracket = (X, M)$$

where

$$(X_i, M_i) = \llbracket t_i \rrbracket$$

$$X = \{\eta \mid \eta(\mathbf{x}_i) \in X_i \text{ for all } i\}$$

$$M = \{r \mapsto (\mathbf{x}_i \mapsto \alpha_i(r)) \mid \alpha_i \in M_i \text{ for all } i\}$$

[Q] Fill in ...

Exprs mean QBS morphisms

$\llbracket \Gamma \vdash e : t \rrbracket$ is a QBS **morphism** from $\llbracket \Gamma \rrbracket$ to $\text{Pr}_q\llbracket t \rrbracket$.

We couldn't cover:

1. QBS probability space.
2. SFinKer Monad on QBSes and semantics of conditioning.

If you want to know about them, look at:

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".