CS492: Probabilistic Programming

Markov Chain Monte Carlo

Hongseok Yang
KAIST

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Really about: Metropolis-Hastings algorithm

```
(doquery :lmh induce-fn [ints2 outs2])
```

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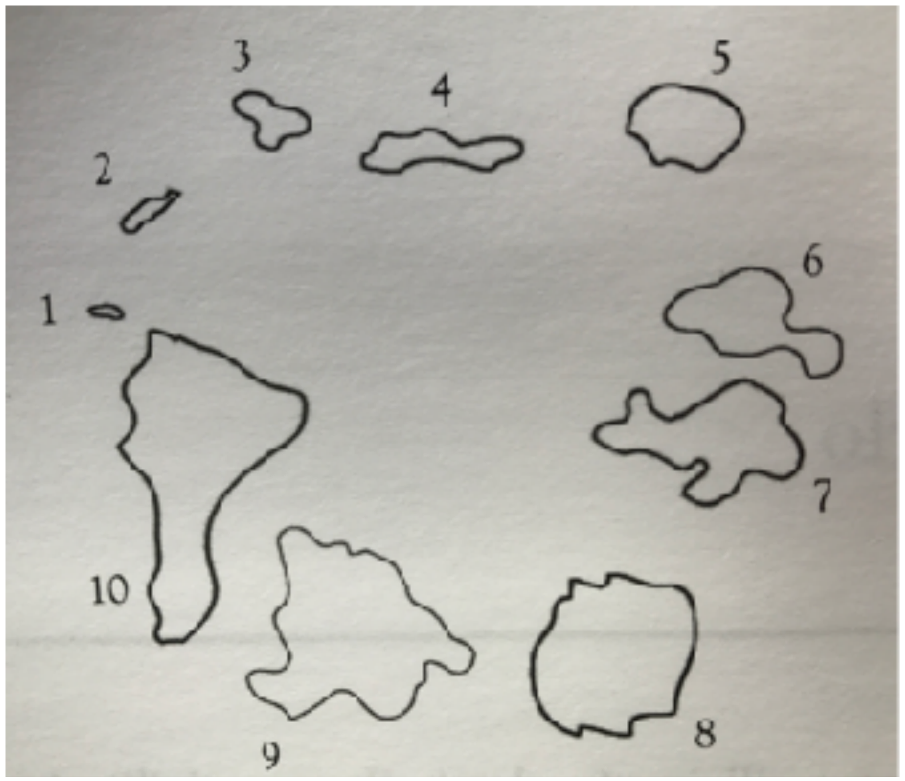
Lightweight Metropolis Hastings algorithm* (LMH).

* Wingate, Stuhlmuller and Goodman's paper at AISTATS 2011

Learning outcome

- Can explain Metropolis-Hastings algorithm.
- Can say when this algo. is correct.
- Can develop an instance of the algorithm.

Good king Markov puzzle*

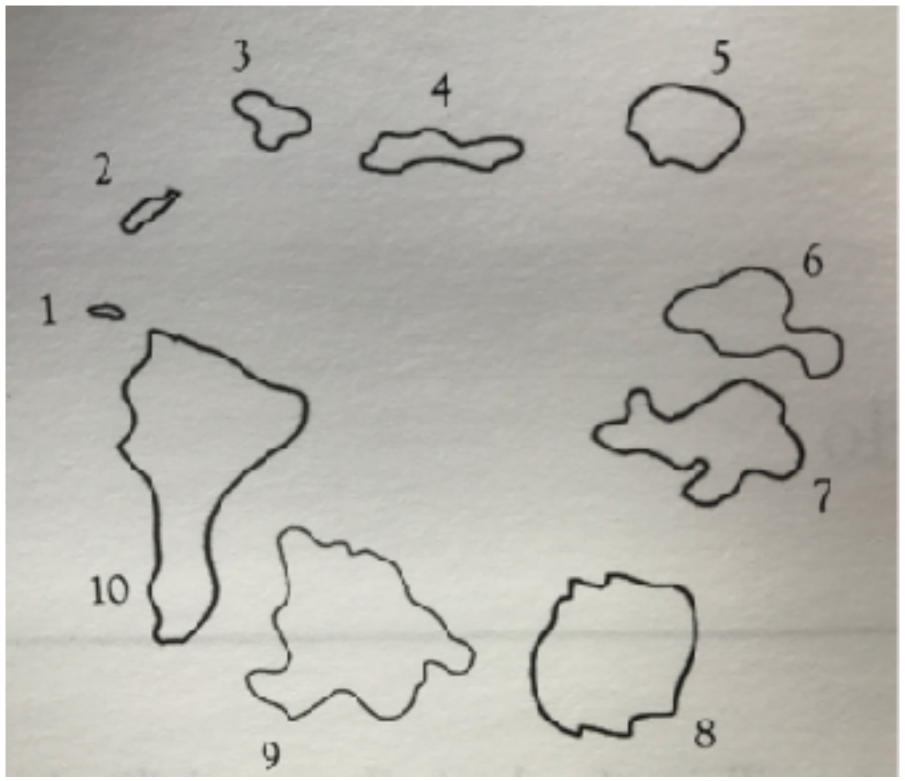


Markov rules 10 islands.

100i people live in island i.

* Borrowed from McElreath's book "Statistical Rethinking"

Good king Markov puzzle*



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100i people live in island i.

King loves his people and wants to visit each island in proportion to its population size.

[Q] Find an algorithm. No scheduling nor bookkeeping. Can move adjacent islands only.

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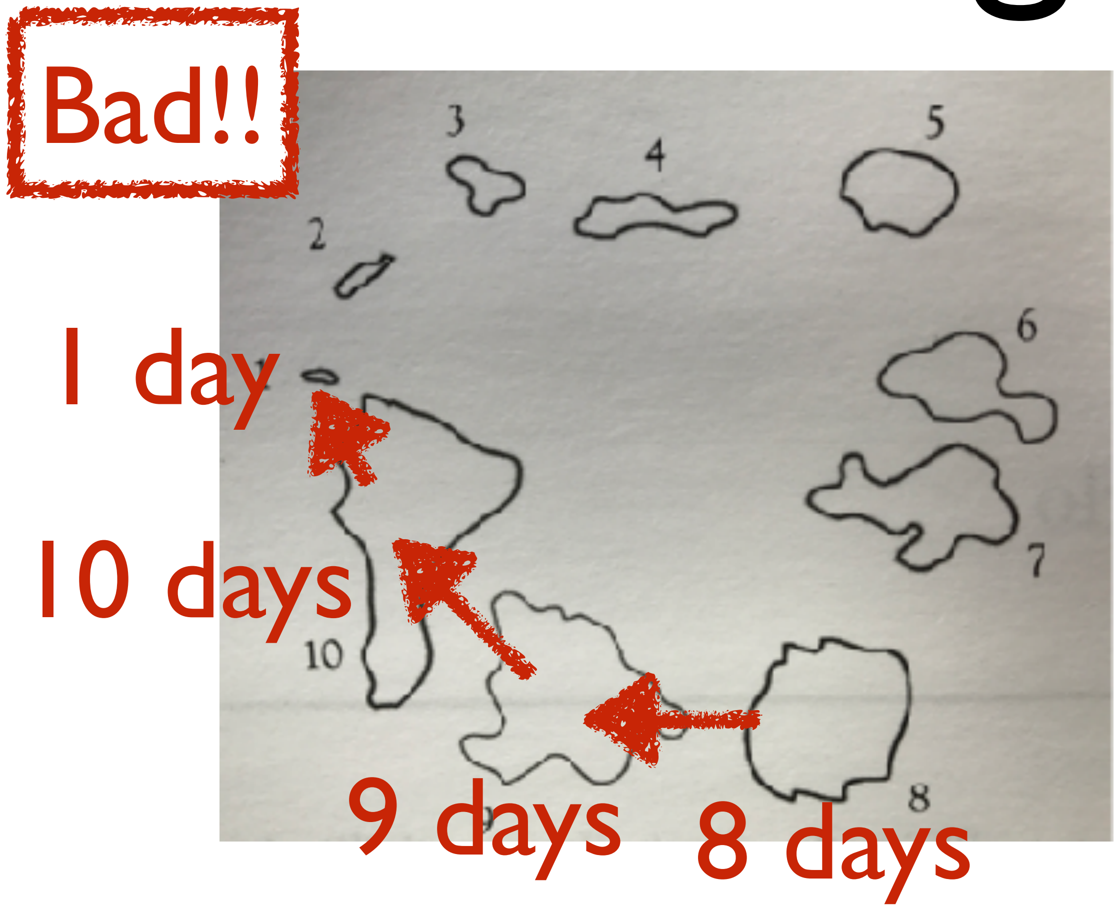

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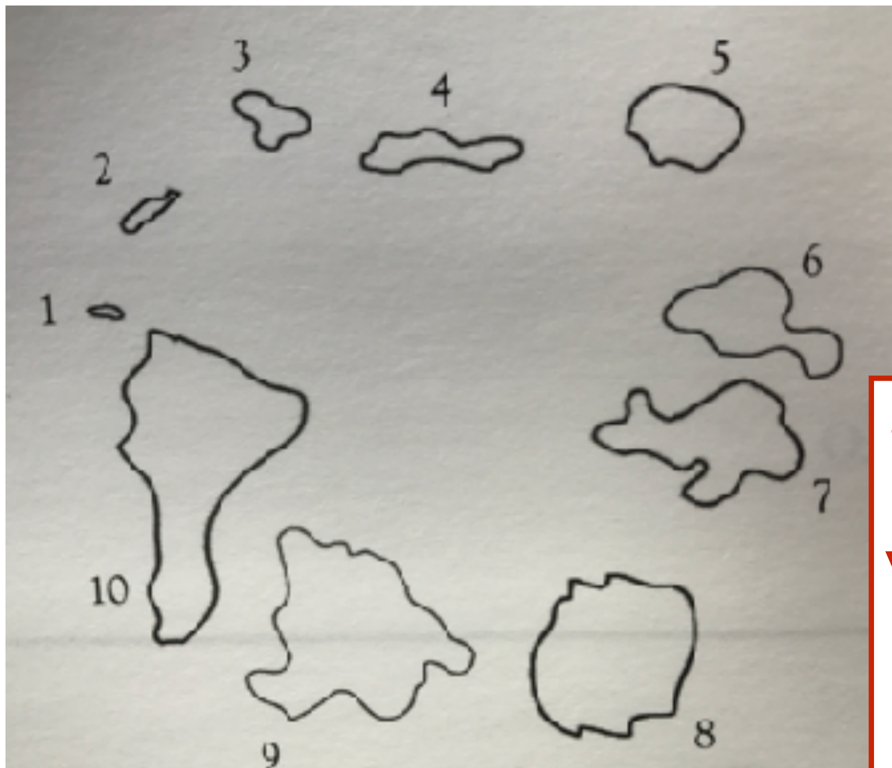

Markov rules 10 islands.

100i people live in island i.

i ~ discrete(1,2,…,10).
Visit i.
Repeat.

Bad!!

King loves his people and wants to visit each island in proportion to its population size.

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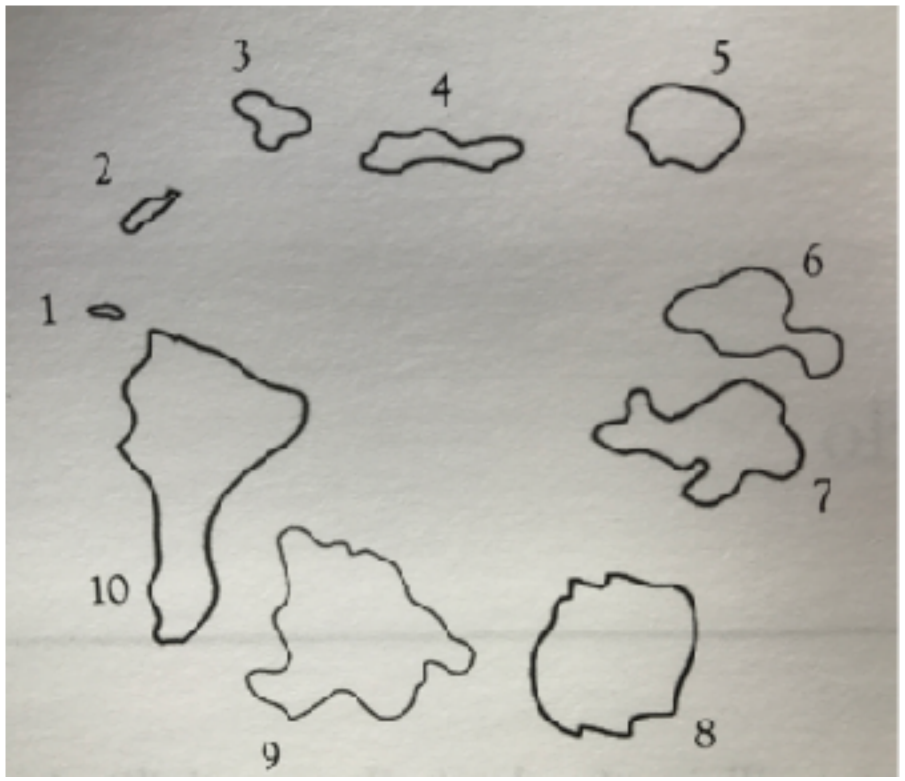

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Solution

k_n — island that the king visits at step n.

Repeat the following steps:

1. Flip a coin with prob. 0.5. If head, pick next k' clockwise. If tail, use k' counterclockwise.
2. $\alpha := \min(1, k'/k_n)$.
3. Flip a coin with prob. α. If head, $k_{n+1} := k'$. Otherwise, $k_{n+1} := k_n$.

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3. Flip a coin with prob. α. If head, $k_{n+1} := k'$. Otherwise, $k_{n+1} := k_n$.

[Q] Why correct? What does correctness even mean?

Sequence by the algo.: $k_1, \ k_2, \ \ldots, \ k_n, \ \ldots$

Sequence by the algo.: k_1, k_2, …, k_n, …

[Strong convergence] For any $f : \{1,..,10\} \rightarrow \mathbb{R}$, $(\sum_{j \leq n} f(k_j))/n \longrightarrow \mathbb{E}_{p(i)}[f(i)]$ as $n \longrightarrow \infty$ with prob. 1, where $p(i)=i/55$, target prob. for visiting island i.

Intuitively, frequency represents probability.

Sequence by the algo.: k_1, k_2, …, k_n, …

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Holds because 1) the random move of the algo. has p as invariant; 2) the algo. can move between any two islands in finitely many steps.

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[Q] Prove 1) and 2).

Metropolis algorithm

Goal: Generate samples from target $r(x)/Z$, where $Z=\int r(x)dx$, the normalising constant.

Parameter: Conditional distribution $q(x'|x)$.

- Should be symmetric $q(x'|x) = q(x|x')$.
- Represents a random move.
- Called proposal kernel.

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[Q] What are r(-) and q(-|-) in King Markov?

Metropolis algorithm

Target $r(x)/Z$. Symmetric proposal $q(x'|x)$.

1. initialise x_1 randomly; $n:=1$
2. repeat:
   a) $x' \sim q(x'|x_n)$; $\alpha := \min(1, r(x')/r(x_n))$
   b) $u \sim \text{uniform}(0,1)$
   c) $x_{n+1} :=$ (if $(u \leq \alpha)$ then x' else x_n); $n:=n+1$

Metropolis

Noisy greedy exploration.

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   a) $x' \sim q(x'|x_n)$; $\alpha := \min(1, r(x')/r(x_n))$

      ≥ 1 for better x'

      < 1 for worse x'

   b) $u \sim \text{uniform}(0,1)$
   c) $x_{n+1} := (\text{if } (u \leq \alpha) \text{ then } x' \text{ else } x_n)$; $n{:=}n{+}1$

Metropolis algorithm

Noisy greedy exploration.
No need to know Z.

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[Q1] How is it related to our sol. for King Markov?

Metropolis Algorithm

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[Q2] Does each step preserve $r(x)/Z$ as invariant?

Metropolis-Hastings

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No need to know Z.

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[Q3] Use it & do posterior inference. Assume $x \in \mathbb{R}^m$.

$r(x) = p(y|x)p(x)$
$q(x'|x) = \text{normal}(x, \varepsilon \times ID)$

Noisy greedy exploration.
No need to know Z.

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[Q4] Instantiate this algo. for Anglican programs.

```
(let [x1  (sample (normal 0 1))
      x2  (if (> x1 0)
              1
              (sample (normal (* x1 x1) 4)))]

  (observe (normal x1 1) 2)
  (observe (normal x2 1) 3)
  (list x1 x2))
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Option 1: Use density formula for all random vars.

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Option 1: Use density formula for all random vars.

Can't handle loops.

```
(let [p (sample (uniform-continuous 0.1 0.9))
      n (loop [i 0]
          (if (sample (flip p)) i (recur (+ i 1))))]
 (observe (normal n 3) 5)
 p)
```

```
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Option 2: Use distribution on execution traces.

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Option 2: Use distribution on execution traces.

But difficult to find symmetric q.

Metropolis Hastings algorithm

Target $r(x)/Z$. ~~Symmetric~~ proposal $q(x'|x)$.

1. initialise x_1 randomly; $n{:=}1$

$$\frac{r(x')\times q(x_n|x')}{r(x_n)\times q(x'|x_n)}$$

2. repeat:
   a) $x' \sim q(x'|x_n)$; $\alpha := \min(1, \cancel{r(x')/r(x_n)})$
   b) $u \sim \text{uniform}(0,1)$
   c) $x_{n+1} := (\text{if } (u \leq \alpha) \text{ then } x' \text{ else } x_n)$; $n{:=}n+1$

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   b) $u \sim \text{uniform}(0,1)$

   c) $x_{n+1} :=$ (if $(u \leq \alpha)$ then x' else x_n); $n{:=}n+1$

[Q1] Develop q for Anglican programs.

Metropolis Hastings algorithm

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[Q2] Noisy greedy exploration. Find a (relative) obj.

Hastings
Metropolis algorithm

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[Q3] Does each step preserve $r(x)/Z$ as invariant?

Recap of the MH algo.

- Generate samples from unnormalised $r(x)$. No need to know $Z=\int r(x)dx$.
- Noisy greedy exploration using $q(x'|x)$.

Guarantees informally

[Thm1] Each step of MH has r/Z as inv. dist.

Guarantees informally

MH samples: $x_1, \; x_2, \; x_3, \; \ldots, \; x_n, \; \ldots$

[Thm2] For all $f{:}X \to \mathbb{R}$ with $\mathbb{E}_{r(x)/Z}[f(x)]$ defined,

$\sum_{i \leq n} f(x_i)/n \longrightarrow \mathbb{E}_{r(x)/Z}[f(x)]$ as $n \longrightarrow \infty$ with prob. 1,

if the MH with q is r/Z-irreducible.

Guarantees informally

The estimate converges to the right value.

MH samples: x_1, x_2, x_3, …, x_n, …

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MH moves well. For any r/Z-possible x, x', the MH can go from x to x' with non-zero prob.

Guarantees informally

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Consequence of a general result in ergodic theory.
Thm1 plays a crucial role in the proof.

Reference

I looked at Chapters 5 and 6 of Robert & Casella's "Monte Carlo Statistical Methods".

Not recommended for general reading.

But details and pointers can be found there.