CS492: Probabilistic Programming

Denotational Semantics of Probabilistic Programs

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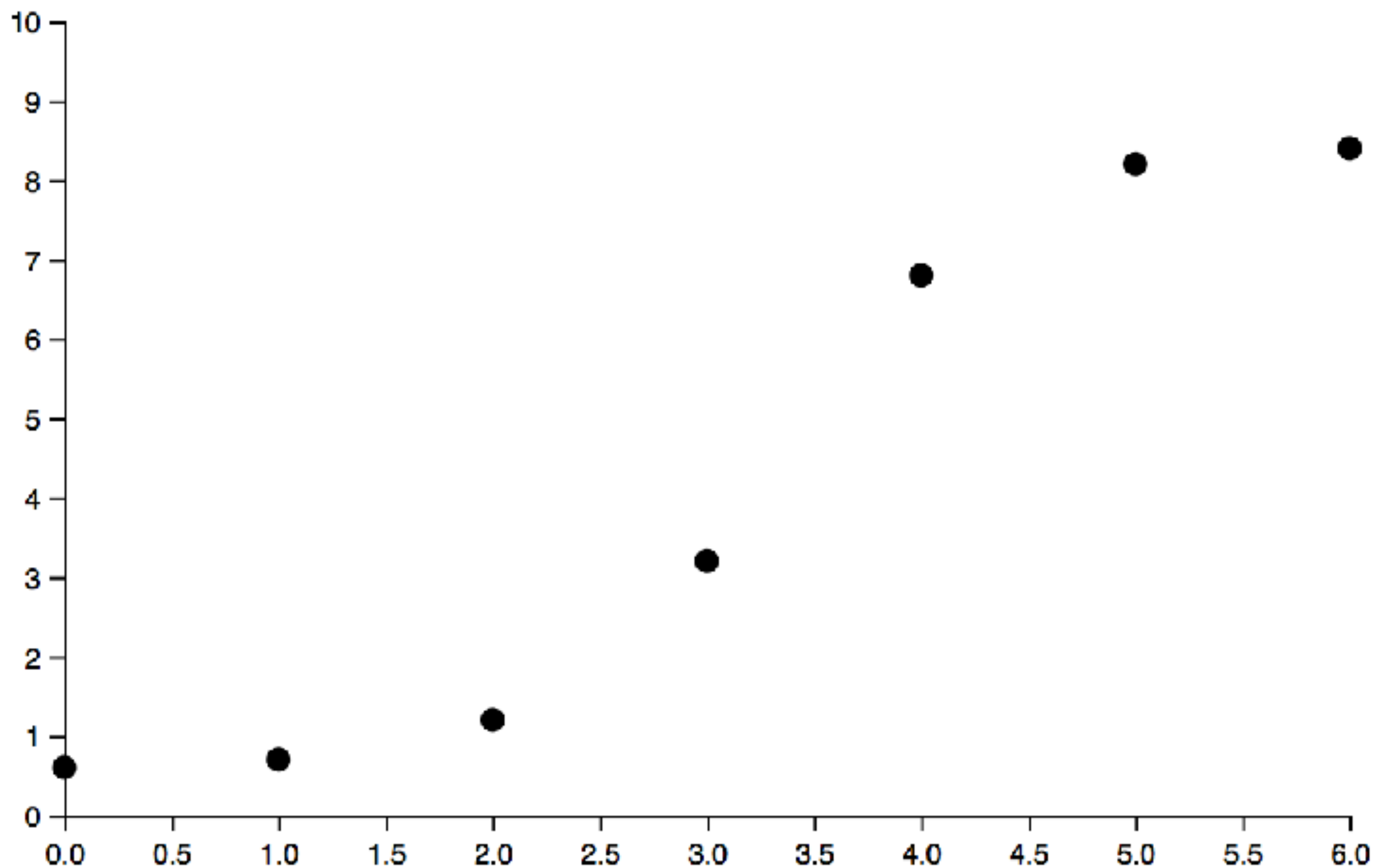
KAIST

Line fitting



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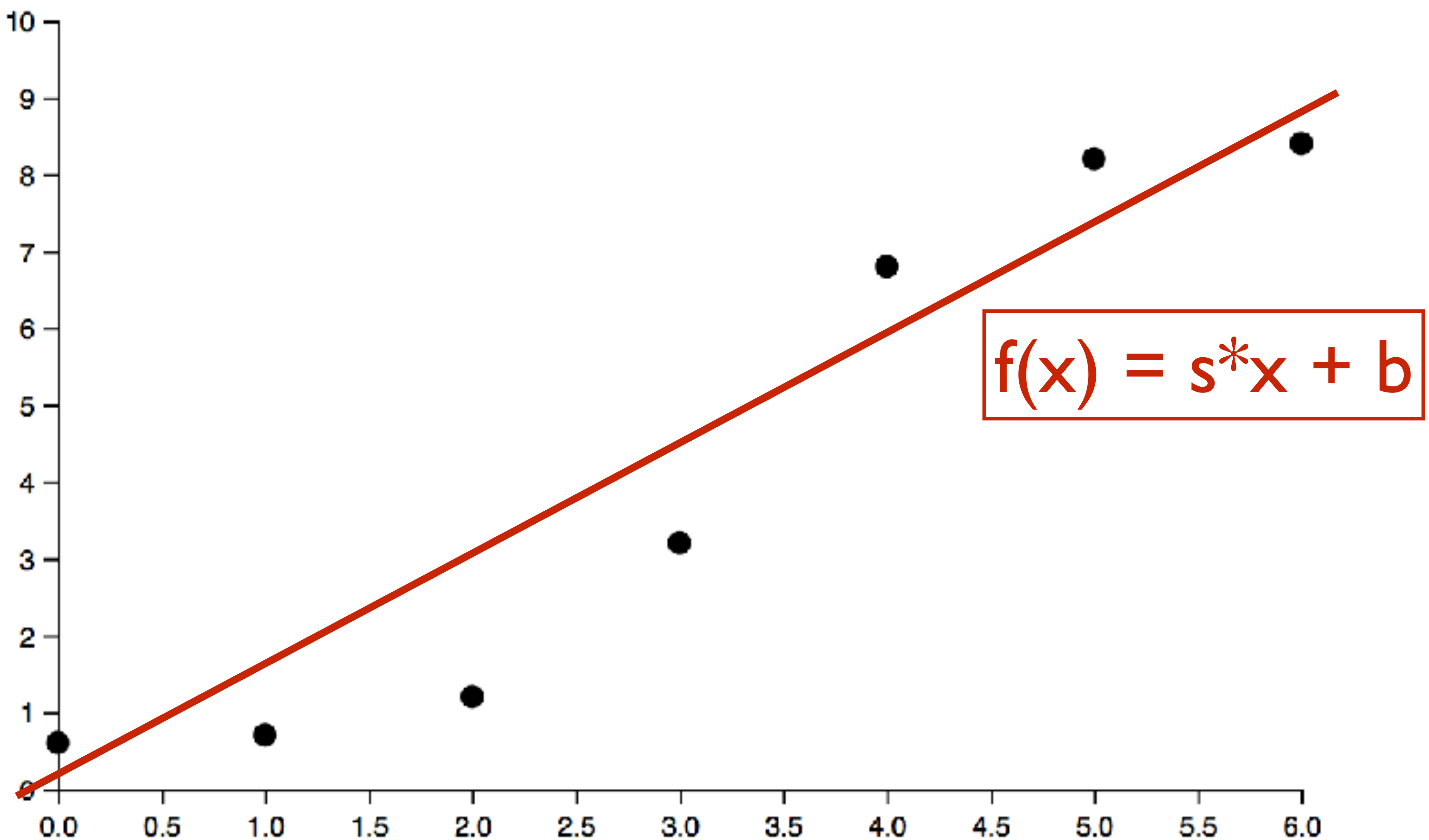

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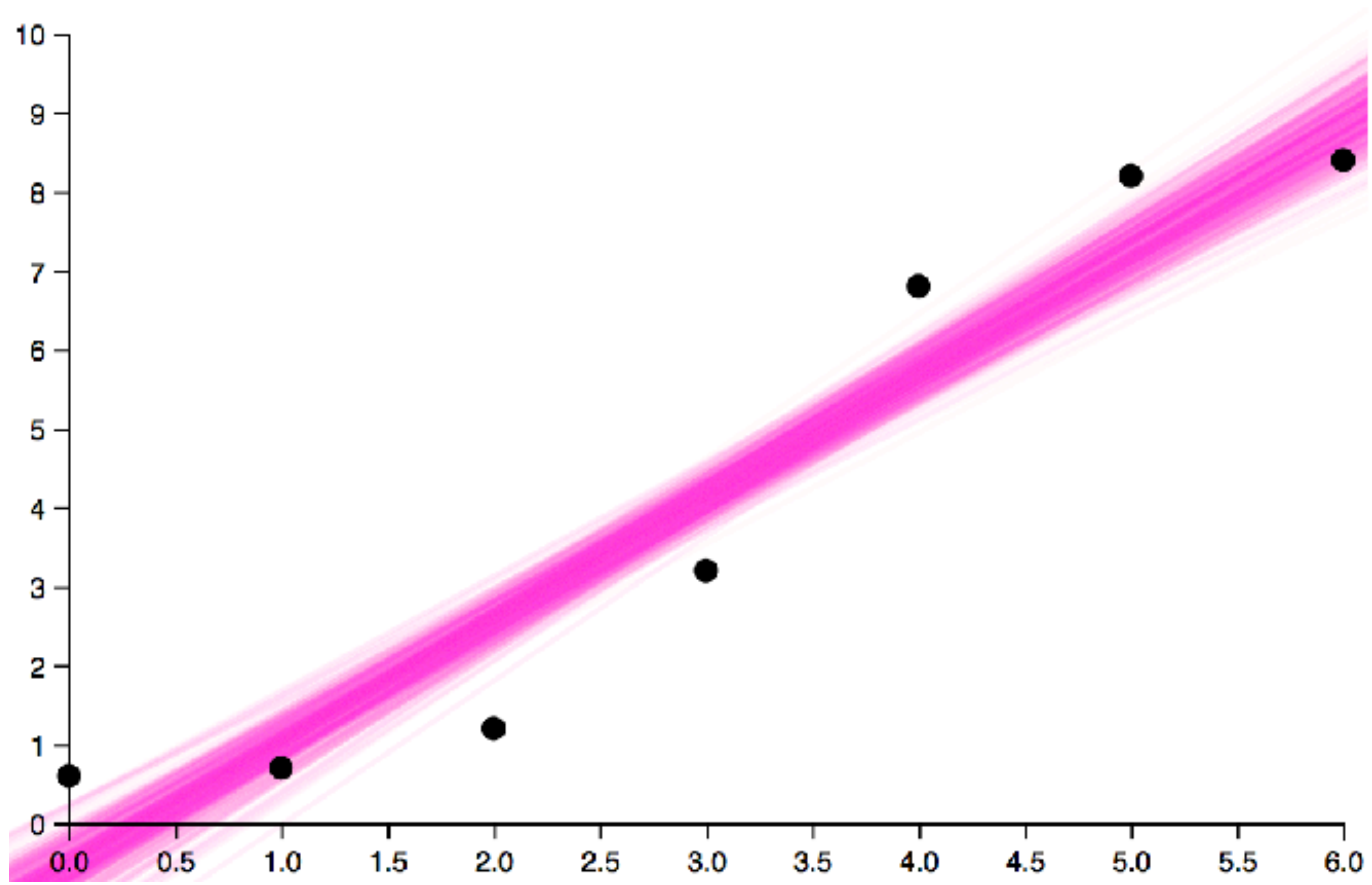
Samples from posterior



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Should define distr. on functions. Not easy.

Foundational question

Measure theory provides a foundation for modern probability theory.

But it doesn't support higher-order fns well.

$$\mathrm{ev} : (\mathbb{R} \to_m \mathbb{R}) \times \mathbb{R} \to \mathbb{R}, \qquad \mathrm{ev}(f,x) = f(x).$$

[Aumann 61] ev is not measurable no matter which σ-algebra is used for $\mathbb{R} \to_m \mathbb{R}$.

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Denotational semantics: Compositional method. Answers a deep Q.

Learning outcome

- Can define a denotational semantics for a simple programming language.
- Can use measure theory to interpret prob. prog. with continuous distributions.
- Can use quasi-Borel space to interpret higher-order prob. prog. with conditioning.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.
2. Baby measure theory.
   PL with cont. distribution.
3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.
4. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

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Denotational semantics

- Defines formal meanings of programs.
- Interprets type and expr. mathematically.
  - Type as space (e.g. set, measurable space).
  - Expr. as good function between spaces.
- Used for justifying compiler optimisation, inference algorithms and language constructs.

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First-order PL with discrete random choices

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t  ::=  bool | rational | dist[bool] | dist[rational]

e  ::=  c | x | (p e ... e) | (let [x e] e) | (if e e e)

c  ::=  true | false | 0 | 1 | ...

p  ::=  sample | flip | poisson | and | + | ...
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Only primitive functions can be applied.

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Interpret type as set and expr. as function.

Types mean sets

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x:bool, y:bool $\vdash$ (if x y y): bool

x:bool, y:rational $\vdash$ (if x y y) : rational

x:rational $\vdash$ (sample (flip x)) : bool

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[Q] Define the interpretation recursively.

Compiler optimisation

Show the following equations:

$[\![\Gamma \vdash (\texttt{if true } e_1\ e_2) : t]\!] = [\![\Gamma \vdash e_1 : t]\!]$

$[\![\Gamma \vdash (\texttt{sample (flip (+ 0.1 0.2))}) : \texttt{bool}]\!]$
$= [\![\Gamma \vdash (\texttt{sample (flip 0.3)}) : \texttt{bool}]\!]$

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First-order PL with discrete random choices

and continuous

t ::= bool | ~~rational~~ real | dist[bool] | dist[~~rational~~ real]

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| normal | uniform-continuous | …

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[Try] Interpret $[\![\mathtt{t}]\!]$ as a set.

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[Sol] Use measure theory. $[\![\mathtt{t}]\!]$ as a measurable space, and $[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ as a measurable function.

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$X = \{0, 1, 2\}$.

Define $p : X \to [0,1]$. E.g., $p = [0.4, 0.4, 0.2]$.

Lifted $p : 2^X \to [0,1]$ by $p(A) = \sum_{x \in A} p(x)$.

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Uncountable sum.
Typically ∞.

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Pick a good collection $\Sigma \subseteq 2^X$.

Define p : Σ→[0,1] with some care.

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σ-algebra

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probability measure

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Σ is a <u>σ-algebra</u> if it contains X, and is closed under countable union and set subtraction.

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$p : \Sigma \to [0,1]$ is a <u>probability measure</u> if $p(X)=1$ and $p(\biguplus_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} p(A_n)$ for all disjoint A_n's.

(X,Σ,p) is a <u>probability space</u> if …

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\times\mathbb{B}, \{ A \times B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R}-A) \text{ finite or countable} \})$.
4. $(\mathbb{R}, \{ (r,s] \mid r<s \})$.

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\text{x}\mathbb{B}, \{ A\text{x}B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
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[Q] Convert them to measurable spaces.

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[Q] Convert them to measurable spaces.

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B}\mathrm{x}\mathbb{B}, \sigma\{ A\mathrm{x}B \mid A\in 2^{\mathbb{B}} \text{ and } B\in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A\subseteq\mathbb{R} \mid A \text{ or } (\mathbb{R}\text{-}A) \text{ finite or countable } \})$.
4. $(\mathbb{R}, \sigma\{ (r,s] \mid r<s \})$.

Closure exists.
$\sigma(\Pi)$ smallest σ-algebra containing Π.

(X, Σ), (Y, Θ) - mBle spaces.

Product σ-algebra: $\Sigma \otimes \Theta = \sigma\{A x B \mid A \in \Sigma, B \in \Theta\}$.

Product space: $(X,\Sigma) \mathrm{x}_m (Y, \Theta) = (X x Y, \Sigma \otimes \Theta)$.

Borel σ-algebra on $\mathbb{R}$: $\mathfrak{B} = \sigma\{(r,s] \mid r<s\}$.

Borel space: $(\mathbb{R}, \mathfrak{B})$.

(X, Σ), (Y, Θ) - mBle spaces.

$Pr(\Sigma) = \ . \ . \ .$

Probability space: $Pr(X,\Sigma) = (Pr(X), Pr(\Sigma))$

[Q] What is $Pr(\Sigma)$?

(X, Σ), (Y, Θ) - mBle spaces.

$Pr(\Sigma) = \sigma\{ \{p \mid p(A)<r\} \mid A\in\Sigma, r\in\mathbb{R} \}$.

Probability space: $Pr(X,\Sigma) = (Pr(X), Pr(\Sigma))$

[Q] What is $Pr(\Sigma)$?

Types mean mBle spaces

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$$[\![\texttt{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\texttt{dist[bool]}]\!] = \mathrm{Pr}([\![\texttt{bool}]\!])$$

Types mean mBle spaces

$$[\![\text{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\texttt{real}]\!] = \ \ldots$$

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[Q] Fill in …

Types mean mBle spaces

$$[\![\mathtt{bool}]\!] = (\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\mathtt{real}]\!] = (\mathbb{R}, \mathfrak{B})$$

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$$[\![\mathtt{dist}[\mathtt{real}]]\!] = \mathrm{Pr}([\![\mathtt{real}]\!])$$

[Q] Fill in …

Types mean mBle spaces

$$[\![\mathtt{x}_1{:}\mathtt{t}_1, \ldots, \mathtt{x}_n{:}\mathtt{t}_n]\!] = (X, \Sigma)$$

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$$[\![x_1{:}t_1, \ldots, x_n{:}t_n]\!] = (X, \Sigma)$$

where

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$X = \ldots$

$\Sigma = \ldots$

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$X = \{\text{map } \eta \text{ from } \{x_1,\ldots,x_n\} \mid \eta(x_i) \in X_i \text{ for all } i\}$

$\Sigma = \sigma\{ \{\eta \mid \eta(x_i) \in A_i \text{ for all } i\} \mid A_i \in \Sigma_i \text{ for all } i\}$

[Q] Fill in …

(X, Σ), (Y, Θ) - mBle spaces.

$f{:}X \to Y$ is measurable (denoted $f{:}X \to_m Y$) if $f^{-1}(A) \in \Sigma$ for all $A \in \Theta$.

Exprs mean mBle fns

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a mBle fn from $[\![\Gamma]\!]$ to $\Pr[\![\mathtt{t}]\!]$.

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$[\![\texttt{y:real} \vdash \texttt{(sample (norm y 1))} : \texttt{real}]\!]\eta(A)$

$= \int_A \text{density-norm}(s \mid \eta(y), 1)\ ds.$

Defined recursively. Complex but doable.

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.
2. Baby measure theory.
   PL with cont. distribution.
3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.
4. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

Prob. PL with HO fns and continuous random choices

t ::= bool | real | dist[bool] | dist[real] | $(t_1,\ldots,t_n)\to t$

e ::= c | x | (fn [x ... x] e) | (e e ... e) | (if e e e)

c ::= true | false | 0 | 1 | 2 | and | + | ...
| sample | flip | normal | ...

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Function type.

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Function type.
General fn decl. and app.

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Function type.
General fn decl. and app.
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Function type.
General fn decl. and app.
General constants.

Measure theory insufficient due to HO fns.

Use a new foundation of probability theory based on quasi-Borel spaces.

Interpret $[\![\mathtt{t}]\!]$ as a quasi-Borel space (QBS), and $[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ as a QBS morphism.

High-level idea:
Random variable first.

Random variable α in X

Random variable α in X

$$\alpha : \Omega \to X$$

- X - set of values.
- Ω - set of random seeds.
- Random seed generator.

Random variable α in X in measure theory

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1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$

Random variable α in X in measure theory

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1. $\Sigma \subseteq 2^{\Omega}$, $\Theta \subseteq 2^{X}$
2. $\mu : \Sigma \to [0,1]$

Random variable α in X in measure theory

$\alpha : \Omega \to X$ is a random element if $\alpha^{-1}(A) \in \Sigma$ for all $A \in \Theta$

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Random variable α in X in quasi-Borel spaces

$$\alpha : \Omega \to X$$

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Random variable α in X in quasi-Borel spaces

$$\alpha : \mathbb{R} \to X$$

- X - set of values.
- $\mathbb{R}$ - set of random seeds.
- Random seed generator.

1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$

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in quasi-Borel spaces

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Random variable α in X
in quasi-Borel spaces

$\alpha : \mathbb{R} \to X$ is a random variable
if $\alpha \in M$

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1. $\mathbb{R}$ as random source
2. Borel subsets $\mathfrak{B} \subseteq 2^{\mathbb{R}}$
3. $M \subseteq [\mathbb{R} \to X]$

- Measure theory:
  - Measurable space $(X, \Theta \subseteq 2^X)$.
  - Random variable is an induced concept.
- QBS:
  - Quasi-Borel space $(X, M \subseteq [\mathbb{R} \to X])$.
  - M is the set of random variables.

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Quasi-Borel space - set with random variables.

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such that M has enough random variables.

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1. M contains all constant functions.
2. $(\alpha \circ \beta) \in M$ for all $\alpha \in M$ and mBle $\beta : \mathbb{R} \to \mathbb{R}$.

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$$(X, M \subseteq [\mathbb{R} \to X])$$

such that M has enough random variables.

1. M contains all constant functions.
2. $(\alpha \circ \beta) \in M$ for all $\alpha \in M$ and mBle $\beta : \mathbb{R} \to \mathbb{R}$.
3. If $\mathbb{R} = \biguplus_{i \in \mathbb{N}} R_i$ with $R_i \in \mathfrak{B}$ and $\alpha_1, \alpha_2, \ldots \in M$, then $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}} \in M$.
   Here $(\alpha_i \text{ when } R_i)_{i \in \mathbb{N}}(r) = \alpha_i(r)$ for all $r \in R_i$.

[Q] Pick a non-QBS.

1. $(\mathbb{R},\ \{\alpha:\mathbb{R}\to\mathbb{R} \mid \alpha$ is a constant function$\})$.
2. $(\mathbb{R},\ [\mathbb{R}\to\mathbb{R}])$.
3. $(\mathbb{R},\ \{\alpha:\mathbb{R}\to\mathbb{R} \mid \alpha$ measurable wrt. $\mathfrak{B}\})$.

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[Q] Turn it into a QBS.

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Standard way of converting a set to a QBS.

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3. $\boxed{(\mathbb{R}, \{\alpha:\mathbb{R}\to\mathbb{R} \mid \alpha \text{ measurable wrt. } \mathfrak{B}\}).}$

Standard way of converting a mBle space to a QBS.

(QBS) morphism

(X,M), (Y,N) - QBSes.

$f : X \to Y$ is a morphism if $(f \circ \alpha) \in N$ for all $\alpha \in M$.

Maps random elements to random elements.

We will write $f : X \to_q Y$.

[Th] QBSes support higher-order functions well.
(The category of QBSes is cartesian closed.)

[Q] What are the sets of random variables?

1. Product: $(X,M) \times_q (Y,N) = (Z,O)$.
   - $Z = X \times Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.
   - $O = ???$
2. Fn space: $[(X,M) \to_q (Y,N)] = (Z,O)$
   - $Z = \{ f \mid f : X \to_q Y \}$, $ev(f,x) = f(x)$.
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[Q] What are the sets of random elements?

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   - $O = \{ r \mapsto (\alpha(r),\beta(r)) \mid \alpha \in M \text{ and } \beta \in N \}$.
2. Fn space: $[(X,M) \rightarrow_q (Y,N)] = (Z,O)$
   - $Z = \{ f \mid f : X \rightarrow_q Y \}$, $ev(f,x) = f(x)$
   - $O = \{ g : \mathbb{R} \rightarrow Z \mid r \mapsto g(\gamma(r))(\alpha(r)) \in N$ for all $\gamma : \mathbb{R} \rightarrow_m \mathbb{R}$ and $\alpha \in M \}$.

Why works?

[NO] ev : $(\mathbb{R} \to_m \mathbb{R}) \times_m \mathbb{R} \to_m \mathbb{R}$

vs

[YES] ev : $(\mathbb{R} \to_q \mathbb{R}) \times_q \mathbb{R} \to_q \mathbb{R}$

Why works?

[NO] ev : $(\mathbb{R} \to_m \mathbb{R}) \times_m \mathbb{R} \to_m \mathbb{R}$

vs

[YES] ev : $(\mathbb{R} \to_q \mathbb{R}) \times_q \mathbb{R} \to_q \mathbb{R}$

Because the QBS product is more permissive.

Types mean QBSes

$$[\![\mathtt{bool}]\!] = \mathrm{MStoQBS}(\mathbb{B}, 2^{\mathbb{B}})$$

$$[\![\mathtt{dist[bool]}]\!] = \mathrm{Pr_q}([\![\mathtt{bool}]\!])$$

Types mean QBSes

Conversion of
mBle space to QBS

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QBS prob. space

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QBS prob. space

$$[\![\texttt{dist[real]}]\!] = \ldots$$

$$[\![\texttt{(t}_1\texttt{,t}_2\texttt{)}\!\rightarrow\!\texttt{t}]\!] = \ldots$$

[Q] Fill in …

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Conversion of mBle space to QBS

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$[\![\text{real}]\!] = \mathrm{MStoQBS}(\mathbb{R}, \mathfrak{B})$

$[\![\text{dist[bool]}]\!] = \mathrm{Pr}_q([\![\text{bool}]\!])$

QBS prob. space

$[\![\text{dist[real]}]\!] = \mathrm{Pr}_q([\![\text{real}]\!])$

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$$[\![\text{dist[real]}]\!] = \text{Pr}_q([\![\text{real}]\!])$$

$$[\![(t_1,t_2)\longrightarrow t]\!] = [\![t_1]\!] \times_q [\![t_2]\!] \rightarrow_q \text{Pr}_q([\![t]\!])$$

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where

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$X = \{\eta \mid \eta(x_i) \in X_i \text{ for all } i\}$

$M = \{r \longmapsto (x_i \longmapsto \alpha_i(r)) \mid \alpha_i \in M_i \text{ for all } i\}$

[Q] Fill in …

Exprs mean QBS morphisms

$[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ is a QBS morphism from $[\![\Gamma]\!]$ to $\mathrm{Pr}_q[\![\mathtt{t}]\!]$.

We couldn't cover:

1. QBS probability space.
2. SFinKer Monad on QBSes and semantics of conditioning.

If you want to know about them, look at:

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".