CS492: Probabilistic Programming

Denotational Semantics of Probabilistic Programs

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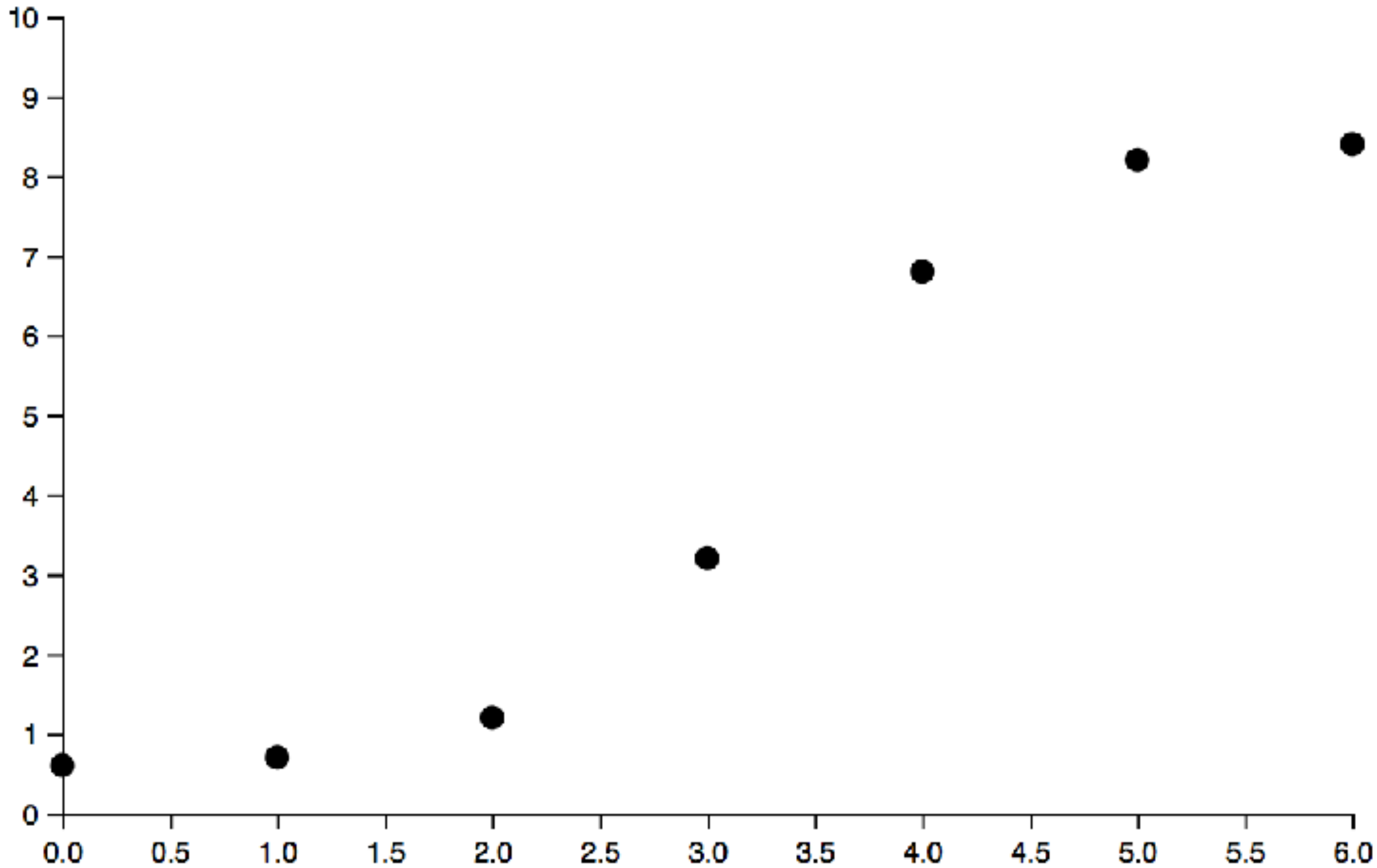
KAIST

Line fitting



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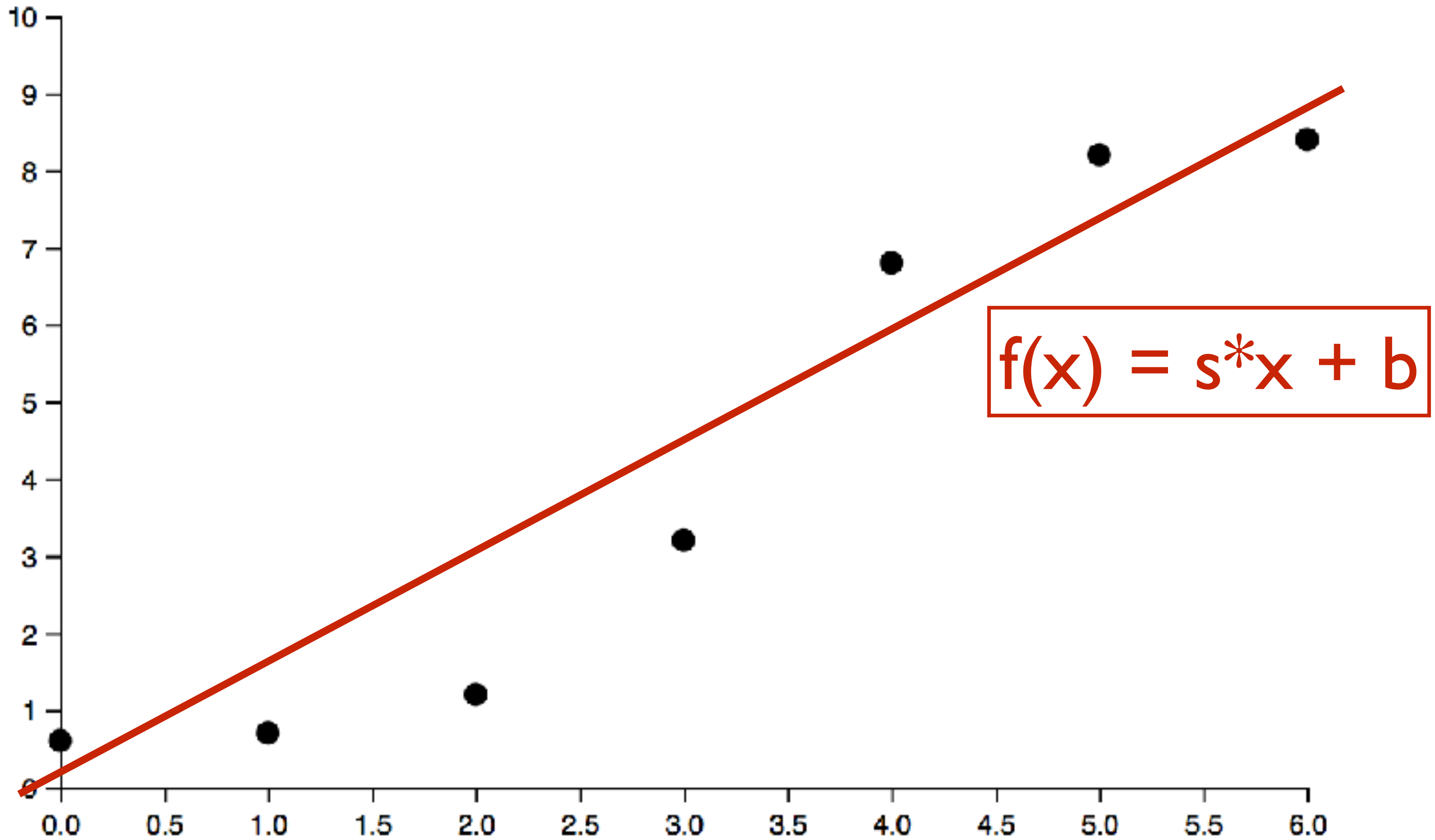

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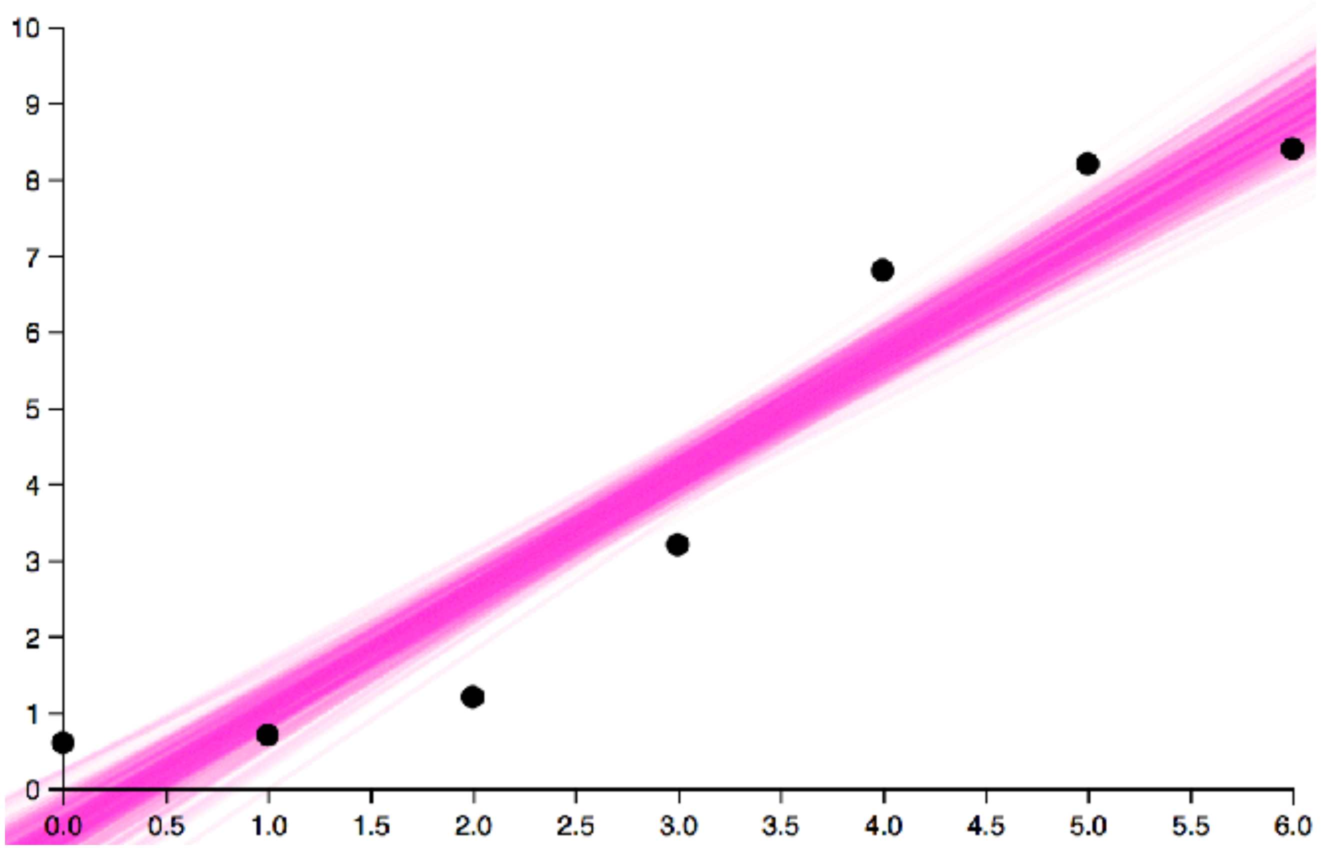
Samples from posterior



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Should define distr. on functions. Not easy.

Foundational question

Measure theory provides a foundation for modern probability theory.

But it doesn't support higher-order fns well.

$$\mathrm{ev} : (\mathbb{R} \to_m \mathbb{R}) \times \mathbb{R} \to \mathbb{R}, \qquad \mathrm{ev}(f,x) = f(x).$$

[Aumann 61] ev is not measurable no matter which σ-algebra is used for $\mathbb{R} \to_m \mathbb{R}$.

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Denotational semantics: Compositional method. Answers a deep Q.

Learning outcome

- Can define a denotational semantics for a simple programming language.
- Can use measure theory to interpret prob. prog. with continuous distributions.
- Can use quasi-Borel space to interpret higher-order prob. prog. with conditioning.

References

1. A convenient category for higher-order probability theory. Heunen et al. LICS'17.
2. Commutative semantics for probabilistic programs. Staton. ESOP'17.
3. Reynolds's "Theories of Programming Languages".
4. Billingsley's "Probability and Measure".

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.
2. Baby measure theory.
   PL with cont. distribution.
3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.
4. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.

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Denotational semantics

- Defines formal meanings of programs.
- Interprets type and expr. mathematically.
  - Type as space (e.g. set, measurable space).
  - Expr. as good function between spaces.
- Used for justifying compiler optimisation, inference algorithms and language constructs.

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First-order PL with discrete random choices

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t ::= bool | rational | dist[bool] | dist[rational]

e ::= c | x | (p e ... e) | (let [x e] e) | (if e e e)

c ::= true | false | 0 | 1 | ...

p ::= sample | flip | poisson | and | + | ...
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Only primitive functions can be applied.

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Interpret type as set and expr. as function.

Types mean sets

$[\![\mathtt{t}]\!]$ is the meaning of $\mathtt{t}$.

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$[\![\mathtt{dist[bool]}]\!] = \mathrm{DiscProb}([\![\mathtt{bool}]\!])$

$[\![\mathtt{dist[rational]}]\!] = \mathrm{DiscProb}([\![\mathtt{rational}]\!])$

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- Γ is a finite map from variables to types.

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x:bool, y:bool $\vdash$ (if x y y): bool

x:bool, y:rational $\vdash$ (if x y y) : rational

x:rational $\vdash$ (sample (flip x)) : bool

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[Q] Define the interpretation recursively.

Compiler optimisation

Show the following equations:

$[\![\Gamma \vdash (\texttt{if true } e_1\ e_2) : t]\!] = [\![\Gamma \vdash e_1 : t]\!]$

$[\![\Gamma \vdash (\texttt{sample (flip (+ 0.1 0.2))}) : \texttt{bool}]\!]$
$= [\![\Gamma \vdash (\texttt{sample (flip 0.3)}) : \texttt{bool}]\!]$

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First-order PL with discrete and continuous random choices

t ::= bool | ~~rational~~ real | dist[bool] | dist[~~rational~~ real]

e ::= c | x | (p e … e) | (let [x e] e) | (if e e e)

c ::= true | false | 0 | 1 | …

p ::= sample | flip | poisson | and | + | …
| normal | uniform-continuous | …

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$$\mathtt{t} \;::=\; \mathrm{bool} \mid \mathrm{real} \mid \mathrm{dist[bool]} \mid \mathrm{dist[real]}$$

[Try] Interpret $[\![\mathtt{t}]\!]$ as a set.

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[Sol] Use measure theory. $[\![\mathtt{t}]\!]$ as a measurable space, and $[\![\Gamma \vdash \mathtt{e} : \mathtt{t}]\!]$ as a measurable function.

How to specify prob. p?

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$X = \{0, 1, 2\}$.

Define $p : X \to [0,1]$. E.g., $p = [0.4, 0.4, 0.2]$.

Lifted $p : 2^X \to [0,1]$ by $p(A) = \sum_{x \in A} p(x)$.

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Uncountable sum.
Typically ∞.

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~~Define~~

Pick a good collection $\Sigma \subseteq 2^X$.

Define $p : \Sigma \to [0,1]$ with some care.

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σ-algebra

Pick a good collection $\Sigma \subseteq 2^X$.

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probability measure

Let $\Sigma \subseteq 2^X$.

Σ is a <u>σ-algebra</u> if it contains X, and is closed under countable union and set subtraction.

(X, Σ) is a <u>measurable space</u> if Σ is a σ-algebra.

$p : \Sigma \to [0,1]$ is a <u>probability measure</u> if $p(X)=1$ and $p(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} p(A_n)$ for all disjoint A_n's.

(X,Σ,p) is a <u>probability space</u> if …

[Q] What are not measurable spaces?

1. $(\mathbb{B}, 2^{\mathbb{B}})$.
2. $(\mathbb{B} \mathrm{x} \mathbb{B}, \{ A\mathrm{x}B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R}\text{-}A) \text{ countable} \})$.
4. $(\mathbb{R}, \{ (r,s] \mid r<s \})$.

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2. $(\mathbb{B} \times \mathbb{B}, \{ A \times B \mid A \in 2^{\mathbb{B}} \text{ and } B \in 2^{\mathbb{B}} \})$.
3. $(\mathbb{R}, \{ A \subseteq \mathbb{R} \mid A \text{ or } (\mathbb{R} - A) \text{ countable} \})$.
4. $(\mathbb{R}, \{ (r,s] \mid r < s \})$.

[Q] Convert them to measurable spaces.

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Closure exists.
$\sigma(\Pi)$ smallest σ-algebra containing Π.

(X, Σ), (Y, Θ) - mBle spaces.

Product σ-algebra: $\Sigma \otimes \Theta = \sigma\{A x B \mid A \in \Sigma, B \in \Theta\}$.

Product space: $(X,\Sigma) x_m (Y, \Theta) = (X x Y, \Sigma \otimes \Theta)$.

Borel σ-algebra on $\mathbb{R}$: $\mathfrak{B} = \sigma\{(r,s] \mid r<s\}$.

Borel space: $(\mathbb{R}, \mathfrak{B})$.

$(X, \Sigma), (Y, \Theta)$ - mBle spaces.

$Pr(\Sigma) = \ldots$

Probability space: $Pr(X,\Sigma) = (Pr(X), Pr(\Sigma))$

[Q] What is $Pr(\Sigma)$?

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$Pr(\Sigma) = \sigma\{ \{p \mid p(A)<r\} \mid A\in\Sigma, r\in\mathbb{R} \}$.

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$f:X \to Y$ is measurable (denoted $f:X \to_m Y$) if $f^{-1}(A) \in \Sigma$ for all $A \in \Theta$.

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$= \int_A$ density-norm(s | η(y), 1) ds.

Defined recursively. Complex but doable.

Plan for the rest

1. Denotational semantics.
   PL with discrete random choices.

2. Baby measure theory.
   PL with cont. distribution.

3. Quasi-Borel space (QBS).
   PL with cont. distr. & higher-order (HO) fns.

4. SFinKer monad on QBS.
   PL with cont. distr., HO fns & conditioning.