# Anfängerpraktikum Part II: OSZ

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(Dated: August 23, 2022)

Filters and an oscillating circuit are described by an oscilloscope by fitting measured data to theory functions. These fit parameters are then compared to their theoretical values and to make a conclusion about the accuracy of the theory. Additionally, certain edge cases of the high- and low-pass filter are explained in detail.

**Contents**

# 1. THEORY

## 1.1. Transfer function and phase shift of a high-pass filter

If a capacitor, $C$, and resistor, $R$, are connected in series—a so called RC-element—, a frequency-dependent voltage divider can be obtained. For output voltages, $V_{\mathrm{out}}$, tapped at resistor a high-pass filter can be obtained, see figure 1.
Here, the capacitor describes a frequency-dependant reactance, $Z_{\mathrm{C}}$, of magnitude $1/iwC$ for sinusoidal AC voltages. With the voltage divider formula the transfer function—a dimensionless, frequency-dependant quantity—,

$$g(f) = \left|\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}\right| = \frac{f/f_{\mathrm{G}}}{\sqrt{1+(f/f_{\mathrm{G}})^2}} \quad , \tag{1}$$

with the input voltage, $V_{\mathrm{in}}$, and cutt-off frequency,

$$f_{\mathbf{G}} = \frac{\omega_{\mathrm{G}}}{2\pi} = \frac{1}{2\pi RC} \quad , \tag{2}$$

can be derived.
For the phase-shift, $\varphi$, between $V_{\mathrm{in}}$ and $V_{\mathrm{out}}$ the following applies:

$$\varphi = \arctan\left(\frac{f_{\mathrm{G}}}{f}\right) \quad , \tag{3}$$

where $\varphi = +\pi/4$ at the cutt-off frequency.

## 1.2. Oscillating circuit

An oscillating circuit contains a capacitor, a resistor and an inductor. An oscillation occurs when the electric energy of the capacitor is periodically converted into magnetic energy of the inductor, and vice versa.
This circuit can be described by the following differential equation:

$$V_{\mathrm{in}} = L\ddot{Q} + R\dot{Q} + \frac{Q}{C} \quad . \tag{4}$$

In the homogeneous case , $V_{\mathrm{in}} = 0$, we get,

$$Q(t) = Q_0 \cos\left(\sqrt{(2\pi f_0)^2 - \delta^2}\cdot t\right) e^{-\delta t} \quad , \tag{5}$$

with the eigenfrequency,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \quad , \tag{6}$$

and the damping constant,

$$\delta = \frac{R}{2L} \quad . \tag{7}$$

Equation (5) has to its oscillating part an additional damping part, which comes from the ohmic resistor converting electric energy into heat, thereby damping the oscillation.

### 1.2.1. Series connection

If we choose an AC voltage of the form $V_{\mathrm{in}}(t) = V_{0,\mathrm{in}} sin(\omega t)$ to be the input voltage of the circuit and differentiate equation (4), we get the differential equation

$$\frac{dV_{\mathrm{in}}}{dt} = L\ddot{I} + R\dot{I} + \frac{I}{C} \tag{8}$$

that can be solved by $I(t) = I_0 \cos(\omega t - \Phi)$, with

$$I_0 = \frac{V_{0,\mathrm{in}}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \text{and} \tag{9}$$

$$\Phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{10}$$

If the output voltage is meassured at the meassuring resistor, $V_{\mathrm{out}} = R_{\mathrm{m}} I(t)$, the transfer function and phase-shift of the series connection are given by

$$\begin{aligned} g_{\mathrm{s}}(f) = \left|\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}\right| &= \frac{R_{\mathrm{m}}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \\ &= \frac{R_{\mathrm{m}}}{L} \cdot \frac{\omega}{\sqrt{4\delta^2\omega^2 + (\omega^2 - \omega_0^2)^2}} \quad \text{and} \end{aligned} \tag{11}$$

$$\begin{aligned} \varphi_{\mathrm{s}} &= \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \\ &= \arctan\left(\frac{\omega^2 - \omega_0^2}{2\delta\omega}\right) \quad , \end{aligned} \tag{12}$$

so the resonance frequency is

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad . \tag{13}$$

### 1.2.2. Parallel connection

In a parallel oscillating circuit the resonance frequency, $f_{\mathrm{res}}$, can be approximated for small damping constants to

$$f_{\mathrm{res}} \approx f_0 \approx \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \quad . \tag{14}$$

### 1.2.3. Bandwidth and quality of a resonance curve

The bandwidth,

$$B_\omega = 2\delta = \frac{R}{L} \quad , \tag{15}$$

of a series connection is defined as the difference between to frequencies at $1/\sqrt{2}$ times the height of the maximum

value of a resonance curve.
The quality,

$$Q = \frac{\omega_0}{B_\omega} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad , \tag{16}$$

of a series connection is defined as the ratio between the resonance frequency and the bandwidth.

## 2. METHODS

### 2.1. High-Pass Filter

On a set-up basic high-pass filter circuit (1), an oscilloscope is used to measure the output voltage and the phase-shift. While the input voltage is set to a constant peak value, the input frequency $f$ is chosen to be ranging from 0,1 kHz to 100 kHz.
In order to achieve the anticipated cut-off frequency between 1kHz and 10kHz, components with the following values where chosen:

- Capacitor with magnitude 5,6 nF ± 10 %,
- Ohmic resistor with magnitude 4,7 kΩ ± 1 %.

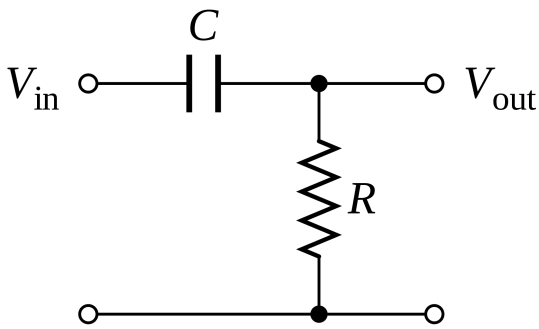


FIG. 1: Basic circuit of a high-pass filter

### 2.2. Oscillating Circuit

The circuit seen in figure (2), is assembled and the output voltage is measured by an oscilloscope at the ohmic resistor. This time the following components are used:

- Resistance with magnitude 100Ω ,
- Capacitor of magnitude 5,6 nF ± 10% ,
- Inductance of magnitude 1,8 mH ± 5% ,

so that resonance frequency is expected to be in the range 40 kHz to 70 kHz. We set the function generator to a sinusoidal input signal and tweaked the output frequency to determine the resonant value. Afterwards frequencies in range of ±30 % around the resonance frequency are measured. Additionally the damping coefficient is measured via fit through the transfer curve and phase-shift curve.
The damping coefficient is also measured by changing the signal form of the frequency generator to a (low-frequency) rectangular shape. With this we can describe the LRC-circuit with the homogenous differential equation (4). By tracking the amplitude of as many (half-)oscillation at time t we can fit the data points with a theory function to obtain the damping constant.

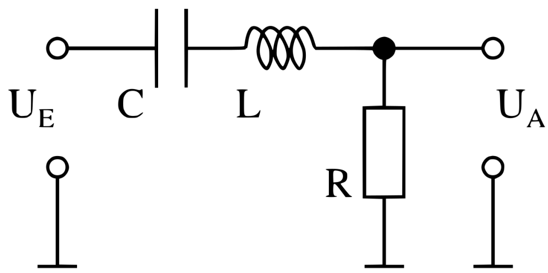


FIG. 2: Basic circuit of a serial oscillator; here $U_{\mathrm{E}} = V_{\mathrm{in}}$ and $U_{\mathrm{A}} = V_{\mathrm{out}}$.

### 2.3. Differentiating and integrating effect of high- and low-Pass Filter

For this experiment only two specific input frequencies are used, as opposed to an entire range of values. We observe the signalform of the output voltages for

- a frequency (50Hz) much lower than the cut-off frequency on the high-pass filter, and
- a frequency much higher than the cut-off frequency on the low-pass filter

for different input signal forms (sinusoidal,rectangular and triangle). The components are the same as in the high-pass circuit measurements.

### 2.4. Capacity of co-axial cable

The circuit is assembled as displayed in (3). The component properties are as follows:

- Ohmic resistor of magnitude 10 kΩ ± 10 %,
- Inductor of magnitude 1,8 mH ± 5 %,
- Capacitor of magnitude 5,6 nF ± 10 %,
- Capacity per unit length—of co-axial cable)—of magnitude 100 pF.

First, the output voltage is measured using only the calibrated probe on the circuit and the resonance frequency is determined by tweaking the input frequency. Then, an additional cable of length 10m is connected between the probe and the circuit. Again, we determine the new resonance frequency by tweaking the input.

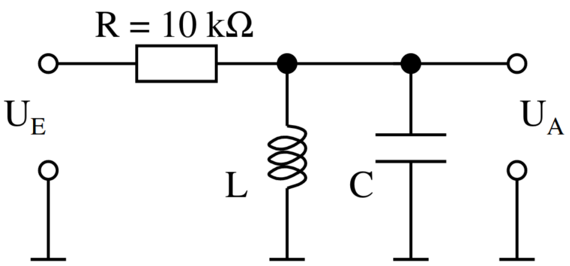


FIG. 3: Basic circuit of a parallel oscillator; here $U_{\mathrm{E}} = V_{\mathrm{in}}$ and $U_{\mathrm{A}} = V_{\mathrm{out}}$.

## 3. RESULTS AND DISCUSSION

### 3.1. High-Pass Filter

Figures (4) and (5) show the transfer function curve and phase-shift curve.

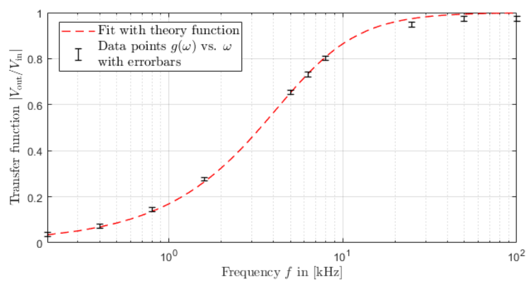


FIG. 4: Transfer-function of a high-pass-filter circuit

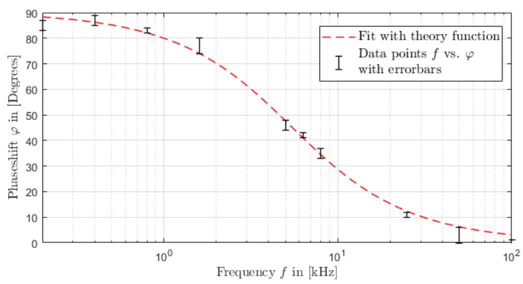


FIG. 5: Phase-shift curve of a high-pass-filter circuit

These fits were produced using the theory functions in equations (1) and (3), where $f_{\mathrm{G}}$ was chosen as a fitparameter. These fits with Matlab also yield the values of both parameters, $f_{\mathrm{G},1}$ and $f_{\mathrm{G},2}$, with uncertainty, whose weighted mean value, $\overline{f_{\mathrm{G}}}$, is then determined according to equations (A3) and (A5). These measurements and the theoretical value, $f_{\mathrm{G,Th}}$—which is calculated with equations (2) and (A1) —can be found in table (I).
We see that the measurement result and theoretical value match within their confidence range and conclude that measurements predict the theory accurately.

TABLE I: Measurement and theoretical value of the cut-off frequency in comparison.

| | frequency in [kHz] with absolute uncertainty |
|---|---|
| $f_{\mathrm{G},1}$ | $(5.8 \pm 0.3)$ |
| $f_{\mathrm{G},2}$ | $(5.4 \pm 0.7)$ |
| $\overline{f_{\mathrm{G}}}$ | $(5.7 \pm 0.3)$ |
| $f_{\mathrm{G,Th}}$ | $(6.0 \pm 0.6)$ |

### 3.2. Oscillating Circuit

Figures (6) and (7) show the transfer function curve and phase-shift curve of the LRC-circuit.

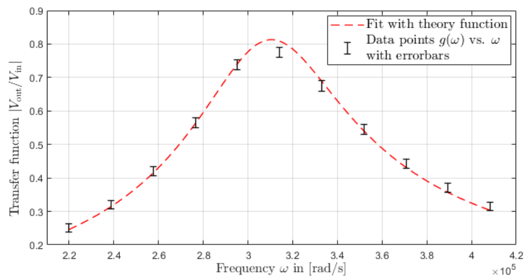


FIG. 6: Transfer function of a serial LRC-circuit

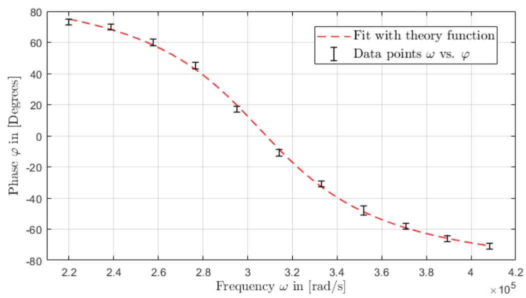


FIG. 7: Phase-shift curve of a serial LRC-circuit

These fits were obtained using the theory functions (11) and (12), where $R_{\mathrm{m}}/L$, $\delta$ and $\omega_0$ are chosen as the three fit parameters for equation (11), and $\delta$ and $\omega_0$ are chosen as the two fit parameters for equation (12). These two Matlab fits yield a value for the parameters, $\omega_0^{(1)}$, $\omega_0^{(2)}$,

$\delta^{(1)}$, $\delta^{(2)}$, whose weighted average is taken according to equations (A3) and (A5).
We can also obtain values for $\delta$ by fitting the data points for amplitudes of (half-)oscillations against time, see figure (8), with the theory function

$$f(t) = A\exp(-\delta t) \quad ,$$

where we denote our fit of the damping constant as $\delta^{(3)}$. All the measurements and theoretical values—which are

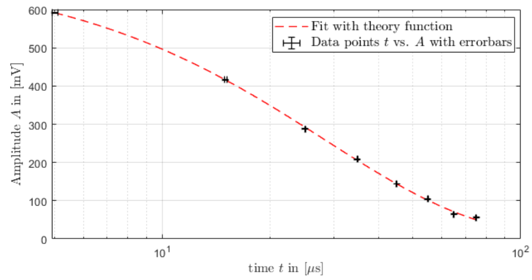


FIG. 8: Plot of the (half-)oscillations

calculated with equations (13), (7), (15), (16) and (A1)—can be found in table (II).
With the previous results found, we can then calculate the bandwidth and quality of the LRC-circuit, by using the equations (15) and (16), which can be found in table (III).
First lets compare the resonance frequencies to each other: We see that measurement result is in confidence range with the theoretical value and conclude that theory predicts the measurements accurately.Note that the uncertainty of the theoretical value is high; this comes mainly from the high uncertainty of the capacitor.
Now lets compare the damping constants, bandwidth and quality to each other: We see that that the fit results $\delta^{(3)}$ and $\overline{\delta}$ are in confidence range with each other, but are not in confidence range with the theoretical value. This is also true for the bandwidth and quality, because they are determined with $\delta$ as a variable. We can only guess what went wrong here; we either used a false approach calculating the theoretical value of $\delta$ or we made a measurement error.

### 3.3. Differentiating and integrating effect of high- and low-Pass Filter

With input frequencies in ranges opposite of the ones passed by the respective filters, they attenuate most of the signals amplitude, leaving only a fraction of the original Voltage to be observed. Running a low frequency input signal of different shapes into the high-pass-filter, we can observe the differentiating effect. Here, the majority of the input Voltage declines at the condensator. However, because the output signal is measured at the resistor, we obtain the following equation for the output current:

$$V_{\text{out}} = R \cdot I = R \cdot C \cdot \frac{\mathbf{d}V}{\mathbf{d}t} \approx R \cdot C \cdot \frac{\mathbf{d}V_{\text{in}}}{\mathbf{d}t} \tag{17}$$

This shows that $V_{\text{out}} \propto \frac{\mathbf{d}V_{\text{in}}}{\mathbf{d}t}$ , hence the name "differentiating effect".
Viewing the input signal on the top as the initial function, the output signal on the bottom shows the derivative of this function. See Figures (9), (10) and (11).

Similarly, inputting a high frequency signal into the low-pass-filter, one can see the integrating effect of the circuit. This time, most of the input voltage declines at the resistor and the output is measured at the capacitor. We find:

$$V_{\text{in}} = R \cdot I = R \cdot C \cdot \frac{\mathbf{d}V}{\mathbf{d}t} \tag{18}$$

TABLE II: Measurement and theoretical value of the resonance frequency and damping constant of the LRC-circuit in comparison.

| | value in $[10^3 \cdot 1/\mathrm{s}]$ with absolute uncertainty |
|---|---|
| $\omega_0^{(1)}$ | $(310.5 \pm 1.8)$ |
| $\omega_0^{(2)}$ | $(308.2 \pm 1.6)$ |
| $\overline{\omega_0}$ | $(309.5 \pm 1.2)$ |
| $\omega_{0,\mathrm{Th}}$ | $(315.0 \pm 17.6)$ |
| $\delta^{(1)}$ | $(34.6 \pm 2.2)$ |
| $\delta^{(2)}$ | $(41.2 \pm 6.8)$ |
| $\overline{\delta}$ | $(35.2 \pm 2.1)$ |
| $\delta^{(3)}$ | $(35.3 \pm 1.0)$ |
| $\delta_{Th}$ | $(27.8 \pm 1.4)$ |

TABLE III: Measurement and theoretical value of the bandwidth and quality of the LRC-circuit in comparison.

| | $\overline{\delta}$ | $\delta_{Th}$ |
|---|---|---|
| $B_\omega$ in $[10^3 1/\mathrm{s}]$ | $(70.4 \pm 4.2)$ | $(55.6 \pm 2.8)$ |
| $Q$ | $(4.39 \pm 0.26)$ | $(5.66 \pm 0.43)$ |

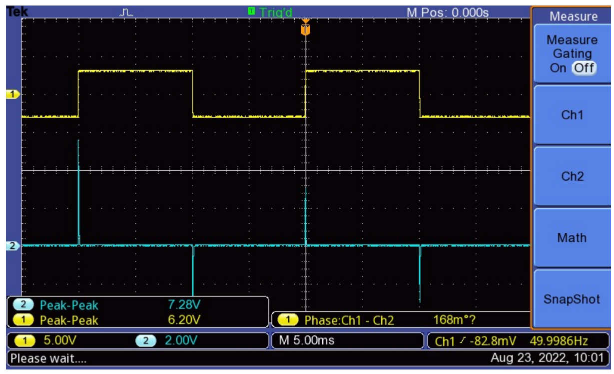


FIG. 9: Differentiating effect acting on a rectangular input signal

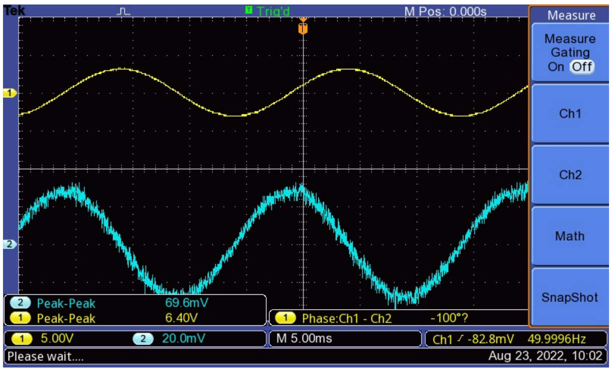


FIG. 10: Differentiating effect acting on a sinusoidal input signal.

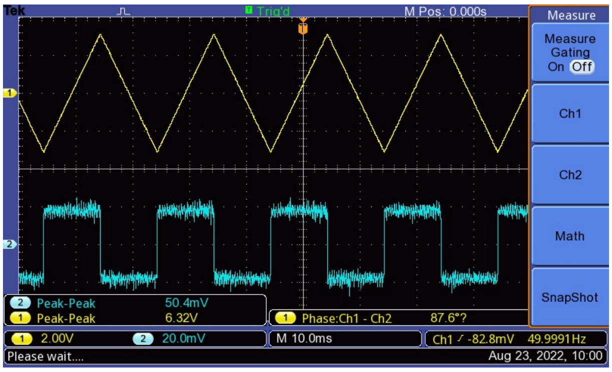


FIG. 11: Differentiating effect acting on a triangular input signal

With $V = V_{\text{in}}$ this equation can be rearranged and integrated and one can see:

$$V_{\text{out}} = \frac{1}{R \cdot C} \int V_{\text{in}} \mathbf{d}t \propto \int V_{\text{in}} \mathbf{d}t \tag{19}$$

Hence "integrating effect". This is approved by the measurement. The input signal on the top being the base function, the output signal on the bottom shows an antiderivative of that function. See Figures (12), (13) and (14).

### 3.4. Capacity of co-axial cable

For the resonance frequency using only the probe, connected directly to the oscilloscope, we obtain $f_{\text{res}} = (50,0 \pm 0,1)$ kHz . Adding the cable to the contraption by connecting it between circuit and probe, the new value is observed to be $f'_{\text{res}} = (49,5 \pm 0,1)$ kHz . With this information, we can derive the capacitance of the cable by using equation (14), which applies to both situations, with and without the cable between probe and circuit. When adding the cable, the capacitance changes like $C \to C + C_{\text{cab}}$. This is where we observe the altered resonance frequency $f'_{\text{res}}$. This results in a system of two

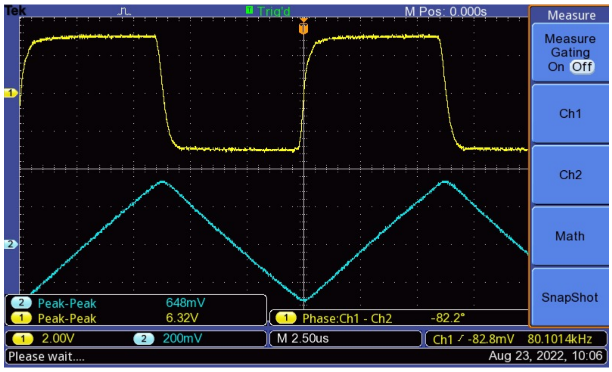


FIG. 12: Integrating effect acting on a rectangular input signal

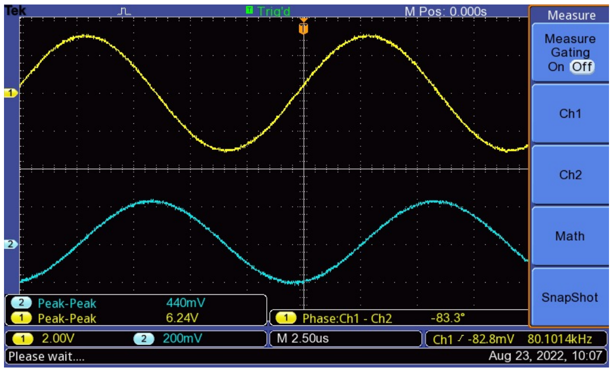


FIG. 13: Integrating effect acting on a sinusoidal input signal

equations, namely (14) and

$$f'_{\text{res}} = \frac{1}{2\pi} \sqrt{\frac{1}{L\,(C + C_{\text{cab}})}} \tag{20}$$

Rearranging these equations and eliminating $C$ , the least precisely defined variable of the system, we find for the

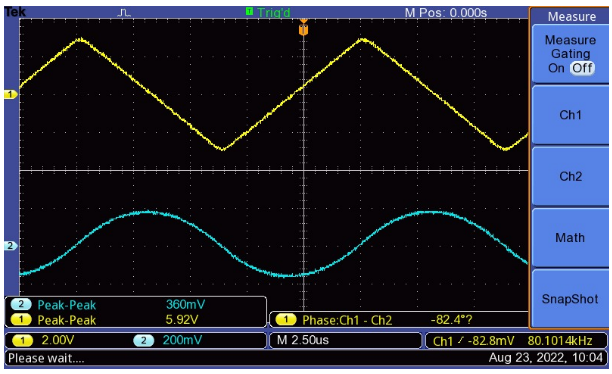


FIG. 14: Integrating effect acting on a triangular input signal

capacitance of the cable:

$$C_{\text{cab}} = \frac{1}{L\left(2\pi f'_{\text{res}}\right)^2} - \frac{1}{L\left(2\pi f_{\text{res}}\right)^2} \tag{21}$$

Pluggins in the values and using the gaussian error propagation(A1) yields: $C_{\text{cab}} = (0.11 \pm 3.17)$ nF
This very high ratio of uncertainty to value shows that the measurement was probably flawed.

## 4. SHORTQUESTIONS

### 4.1. Differences and Applications of an LRC oscillating circuit and a low- and high-pass filter

A low-pass filter filters out higher frequencies, a high-pass filter filters out lower frequencies, and an oscillating circuit filters out frequencies around a desired frequency—which is called a band-pass filter.
Filters have applications in sound design in so called Digital Audio Workstations.


FIG. 15: Example of an equalizer with a high-, two band- and a low-pass filter (left to right).

These equalizers, see figure (15), are used to filter out or amplify frequencies in an instrument or sample sound, for example it is common to filter out the high frequencies in a bass guitar with a low-pass filter to give more "room" for instruments with higher frequencies, which results in a much clearer sound.
Oscillating circuits can be also used to generate electromagnetic waves that are widely used in communication via cellphone or radio broadcasting and Wi-Fi networks. They can also be used as a band-pass filter.

### 4.2. Analogies between a LRC oscillating circuit and a spring-mass system

Let us compare the differential equation of an LRC-Circuit,

$$0 = L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q \quad , \tag{22}$$

with the differential equation of a spring-mass system

$$0 = m\ddot{x} + b\dot{x} + kx \quad , \tag{23}$$

where $x$ is the displacement of the masspoint from its rest position, $m$ the mass, $b$ the damping constant and $k$ the spring constant. We see that both these differential equations have the same structure, but different variables, which gives us many interesting analogies, see Table (IV).

TABLE IV: Analogies between an LRC-Circuit and a spring-mass system.

| LRC-Circuit | Spring-Mass system |
|---|---|
| Inductance $L$ | Mass $m$ |
| Ohmic resistor $R$ | Damping constant $b$ |
| Inverse of capacitance $C^{-1}$ | Spring constant $k$ |
| Charge $Q$ | Displacement $x$ |
| Current $\dot{Q} = I$ | Velocity $\dot{x} = v$ |
| Electrical energy $\frac{Q^2}{2C}$ | Potential energy $\frac{kx^2}{2}$ |
| Magnetic energy $\frac{LI^2}{2}$ | Kinetic energy $\frac{mv^2}{2}$ |

[1] Technische Universität München. Hinweise zur Beurteilung von Messungen, Messergebnissen und Messunsicherheiten (ABW). Year 2021, last updated.

## Appendix A: Auxillary material

The Uncertainty calculations are done using this reference material [1].

### 1. Gaussian Error Propagation

We can use the Gaussian error propagation,

$$u(g) = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial g}{\partial x_i} u(x_i) \right)^2} \quad , \tag{A1}$$

assuming the variables $x_i$ , $i \in \{1, ..., n\}$ of the function g are independent; $u(x_i)$ is the uncertainty of the variable.

### 2. Consideration of multiple uncertainties

Individual can have uncertainty of type A and multiple uncertainties of type B. The overall uncertainty is then given by

$$u(x_i) = \sqrt{u_A(x_i)^2 + u_{B1}(x_i)^2 + u_{B2}(x_i)^2 + ...} \quad . \tag{A2}$$

### 3. Weighted mean value

The weighted mean value can be calculated using

$$\overline{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \tag{A3}$$

whereby

$$w_i = \frac{1}{u(x_i)^2} \quad . \tag{A4}$$

The uncertainty is given by

$$\begin{aligned} u_{int}(\overline{x}) &= \sqrt{\frac{1}{\sum_{i=1}^{n} w_i}} \\ u_{ext}(\overline{x}) &= \sqrt{\frac{\sum_{i=1}^{n} w_i (x_i - \overline{x})^2}{(n-1) \sum_{i=1}^{n} w_i}} \\ u(\overline{x}) &= \max(u_{int}, u_{ext}) \end{aligned} \tag{A5}$$