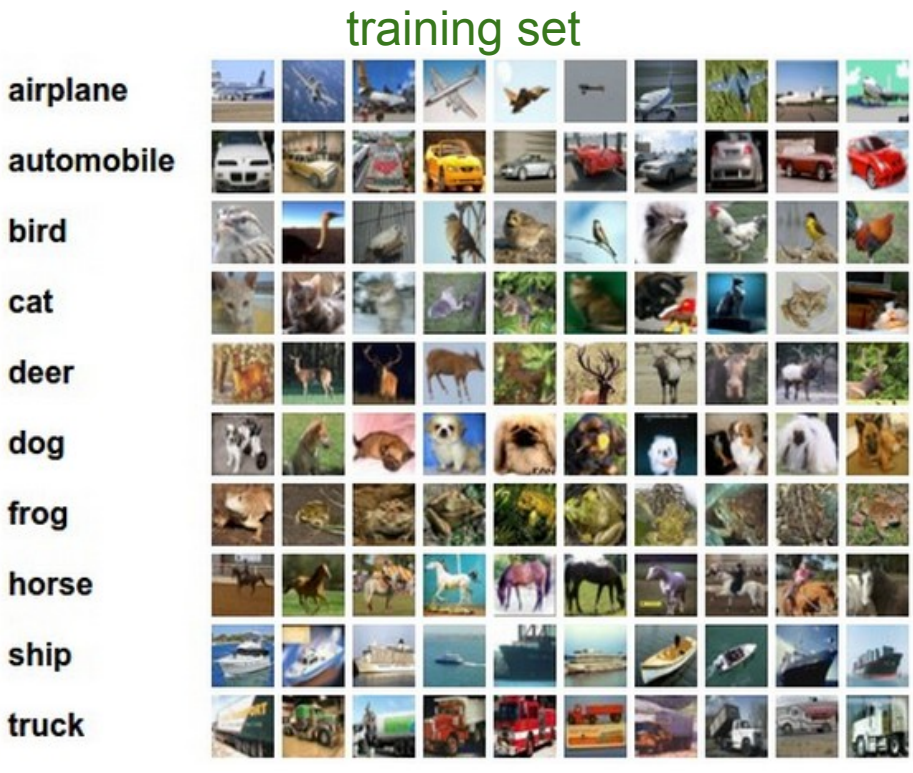
Lecture 3:
Linear Classification

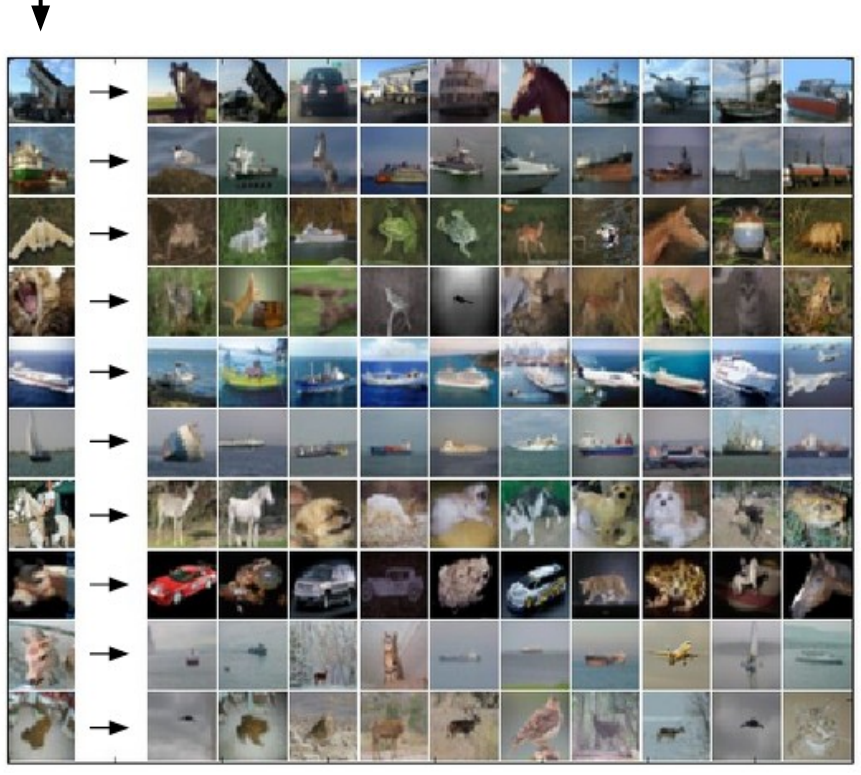
Last time: Image Classification

assume given set of discrete labels
{dog, cat, truck, plane, ...}

cat

k-Nearest Neighbor





Linear Classification

1. define a **score function**

class scores

Linear Classification

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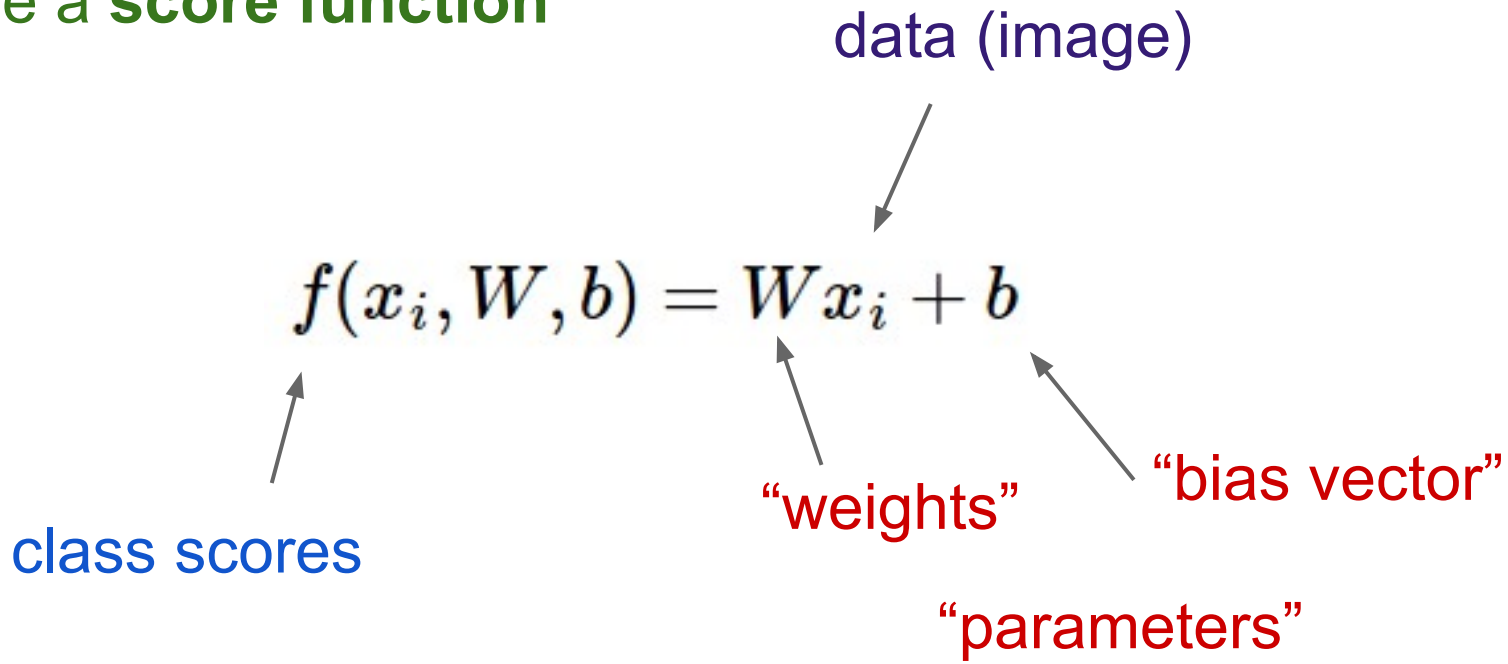

Linear Classification

1. define a **score function**

(assume CIFAR-10 example so
32 x 32 x 3 images, 10 classes)

data (image)
[3072 x 1]

$$f(x_i, W, b) = Wx_i + b$$

class scores

weights

bias vector

Linear Classification

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data (image)
[3072 x 1]

$$f(x_i, W, b) = Wx_i + b$$

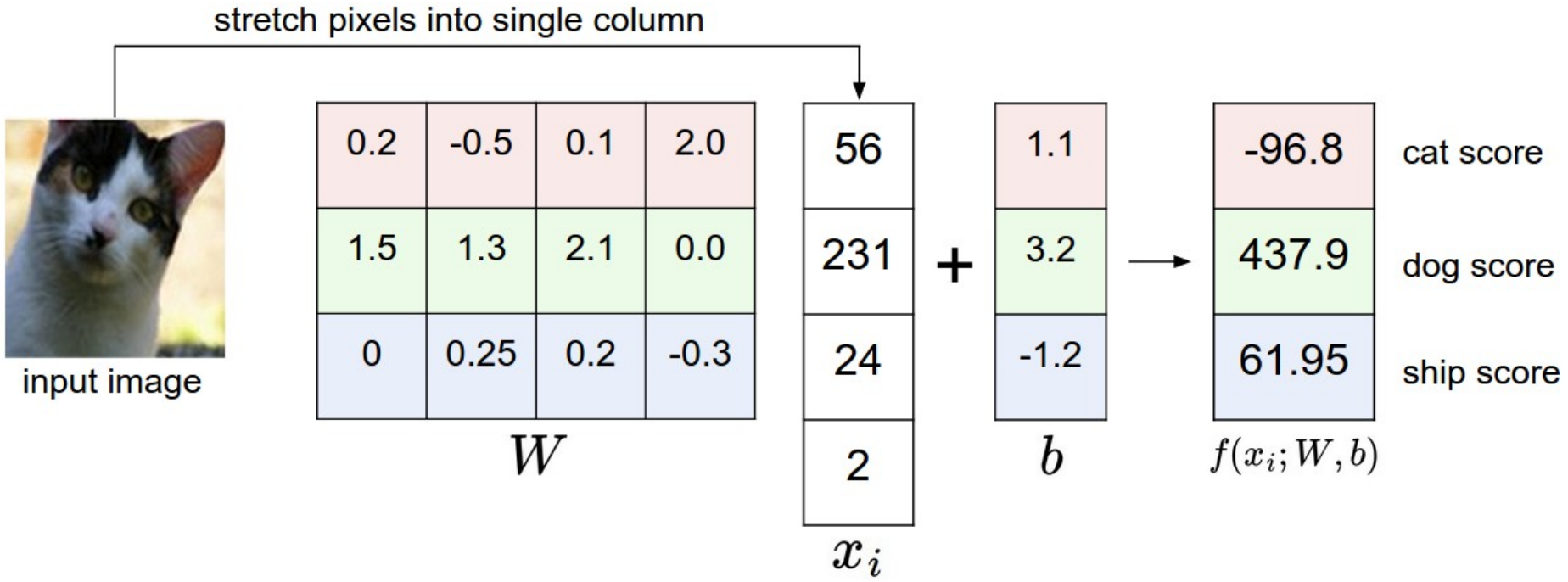
class scores
[10 x 1]

weights
[10 x 3072]

bias vector
[10 x 1]

Linear Classification

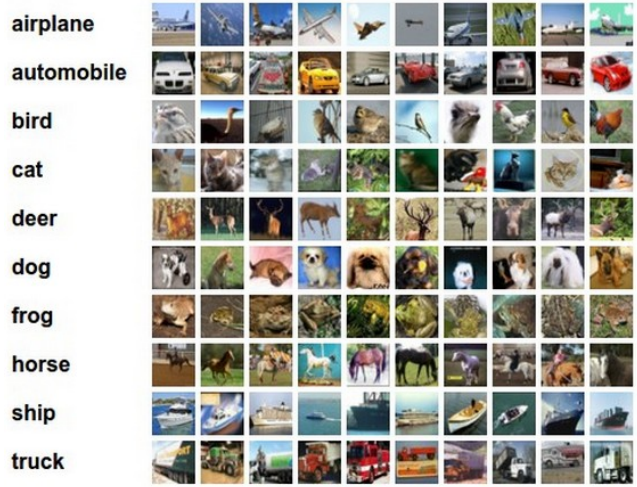
$$f(x_i, W, b) = Wx_i + b$$



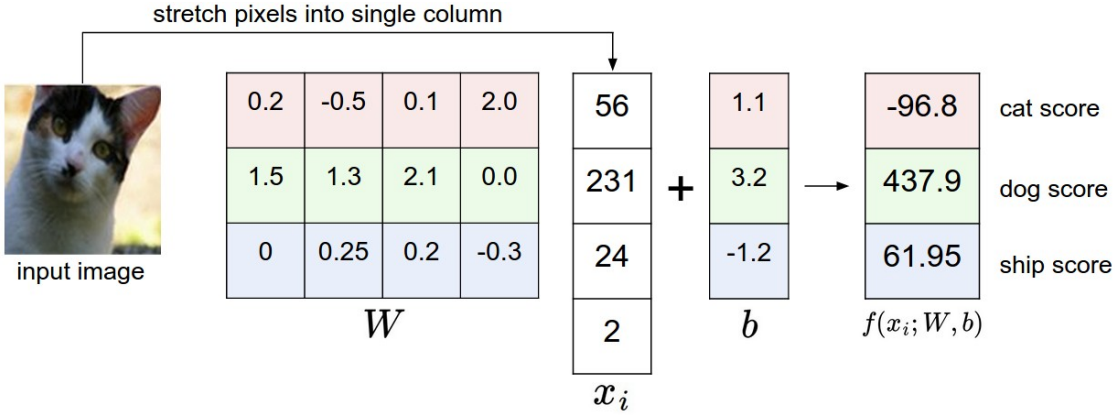
Interpreting a Linear Classifier

Question:

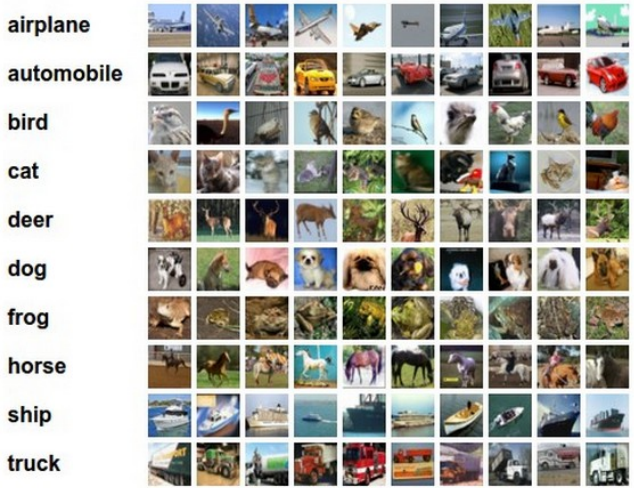
what can a linear classifier do?



$$f(x_i, W, b) = Wx_i + b$$



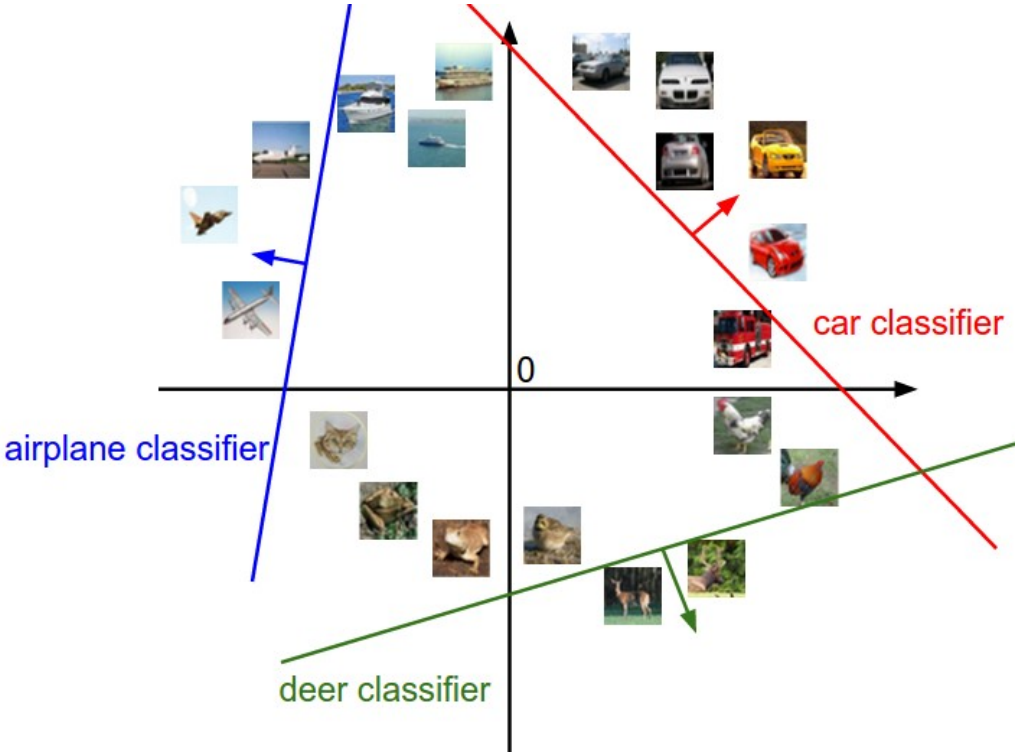
Interpreting a Linear Classifier



$$f(x_i, W, b) = Wx_i + b$$

Example training classifiers on CIFAR-10:

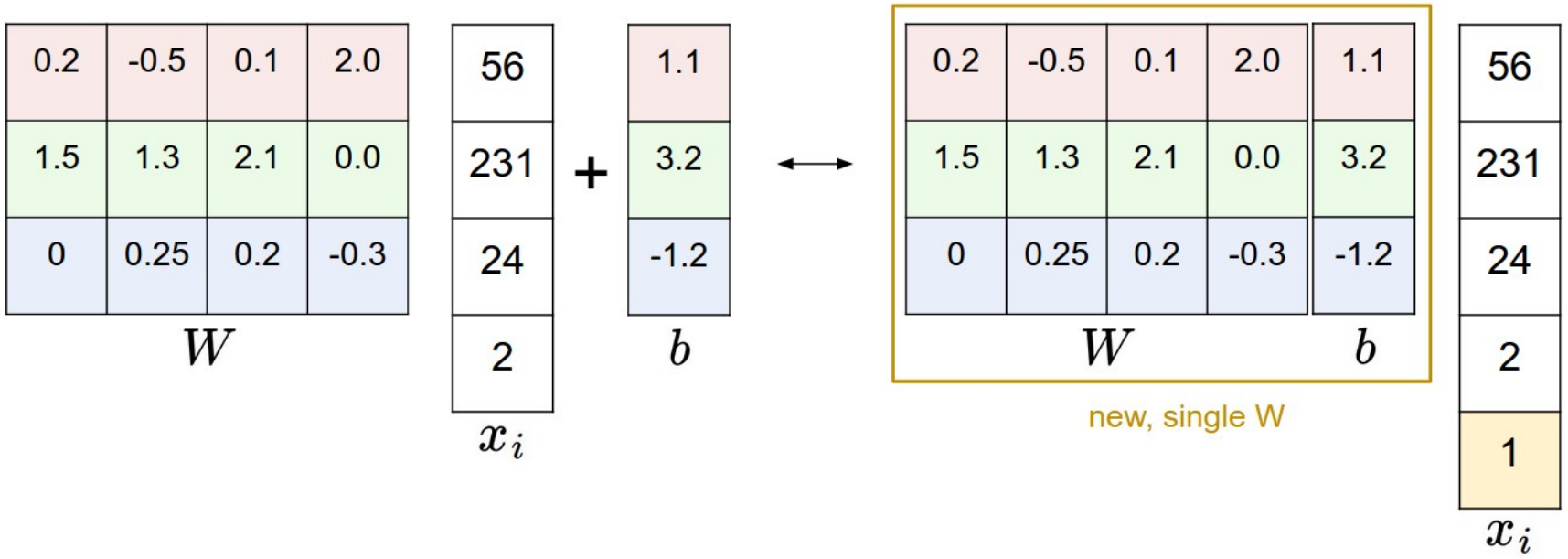
Interpreting a Linear Classifier



$$f(x_i, W, b) = Wx_i + b$$

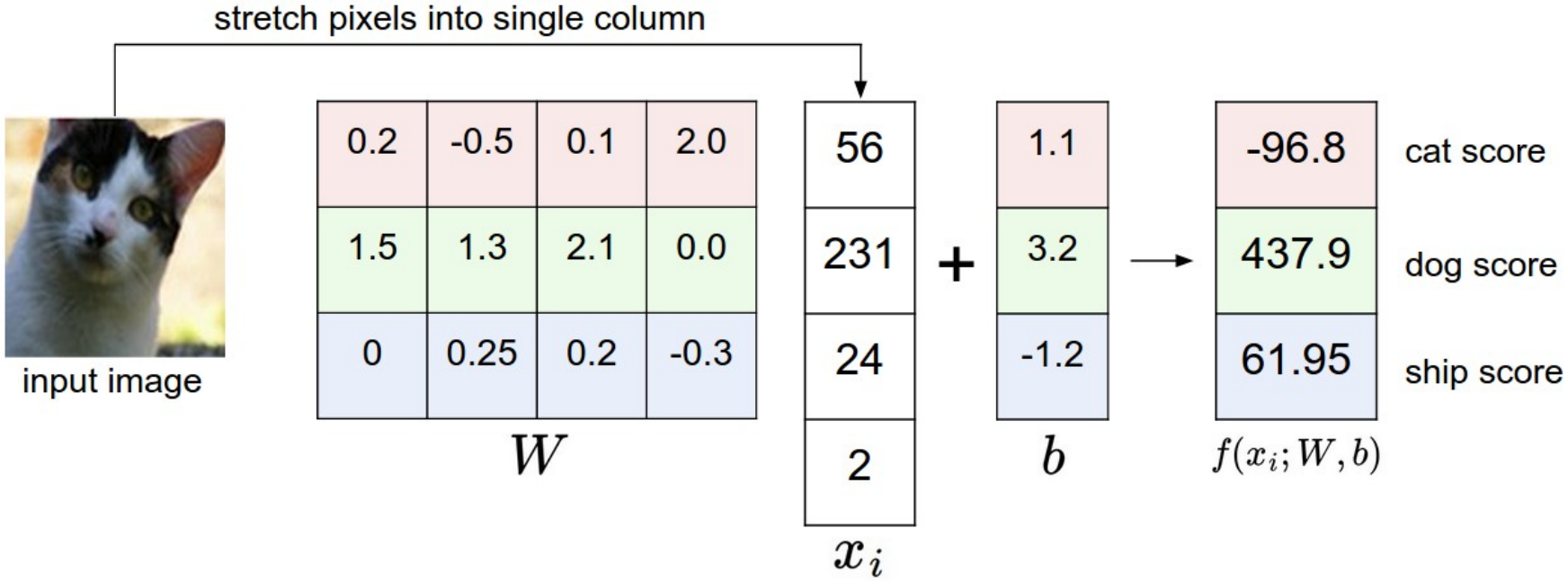
Bias trick

$$f(x_i, W, b) = Wx_i + b \longrightarrow f(x_i, W) = Wx_i$$



So far:

We defined a (linear) **score function**: $f(x_i, W, b) = Wx_i + b$



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Example:

$$f(x_i, W) = [13, -7, 11]$$

$$y_i = 0$$

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- scores, label ⟶ *loss*.

$f(x_i, W)$ y_i L_i

Example:

$$f(x_i, W) = [13, -7, 11]$$

$$y_i = 0$$

Question: if you were to assign a single number to how "unhappy" you are with these scores, what would you do?

2. Define a **loss function** (or cost function, or objective)
One (of many ways) to do it: **Multiclass SVM Loss**

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

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(One possible generalization of Binary Support Vector Machine to multiple classes)

$$L_i = C \max(0, 1 - y_i w^T x_i) + R(W)$$

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loss due to example i

sum over all incorrect labels

difference between the correct class score and incorrect class score

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e.g. 10

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$$y_i = 0$$

---

loss = ?

Example: $$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

e.g. 10

$$f(x_i, W) = [13, -7, 11]$$

$$y_i = 0$$

$$L_i = \max(0, -7 - 13 + 10) + \max(0, 11 - 13 + 10)$$

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

```
def L_i_vectorized(x, y, W):
  scores = W.dot(x)
  margins = np.maximum(0, scores - scores[y] + 1)
  margins[y] = 0
  loss_i = np.sum(margins)
  return loss_i
```

There is a bug with the objective…

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right]$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$

Regularization strength

L2 regularization: motivation

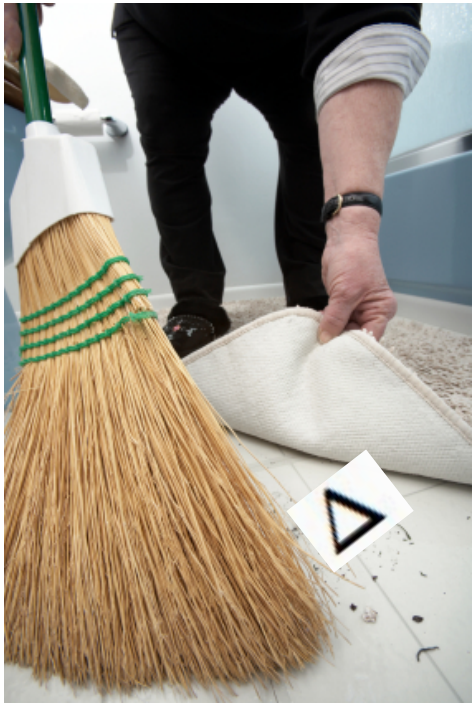
$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$



$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$

Do we have to cross-validate both Δ and λ ?

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$

$$\begin{aligned}
&\max(0, \delta f + \Delta) \\
\rightarrow &\max(0, \delta f + \Delta_2) && \text{if } \Delta \rightarrow \Delta_2 \\
\rightarrow &\max(0, \frac{\Delta_2}{\Delta} \delta f + \Delta_2) && \text{then scale the weights by } \frac{\Delta_2}{\Delta} \\
= &\max(0, \frac{\Delta_2}{\Delta} (\delta f + \Delta)) \\
= &\frac{\Delta_2}{\Delta} \max(0, \delta f + \Delta)
\end{aligned}$$

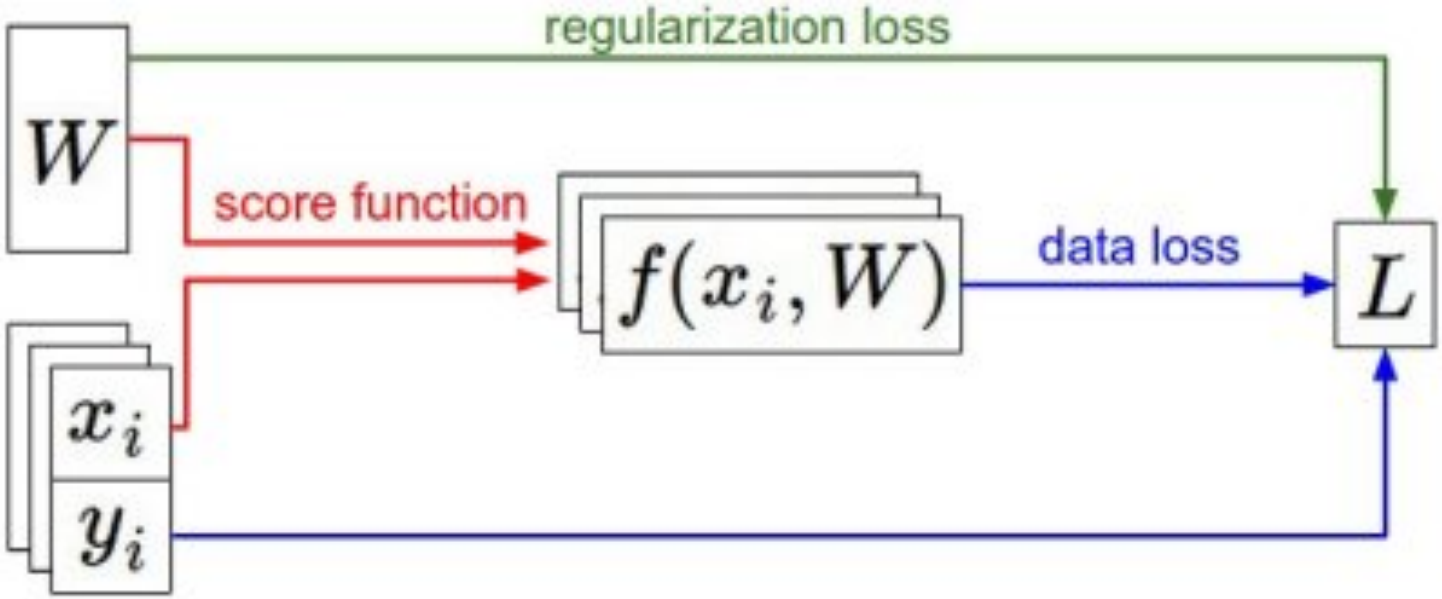
So far…

1. Score function

$$f(x_i, W, b) = Wx_i + b$$

2. Loss function

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$


regularization loss
W
score function
$f(x_i, W)$
data loss
L
x_i
y_i

Softmax Classifier

$$f(x_i, W) = Wx_i$$

score function is the same

(extension of Logistic Regression to multiple classes)

Softmax Classifier

$$f(x_i, W) = Wx_i$$

score function is the same

$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

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softmax function

$$P(y_i \mid x_i; W)$$

i.e. we're minimizing the negative log likelihood.

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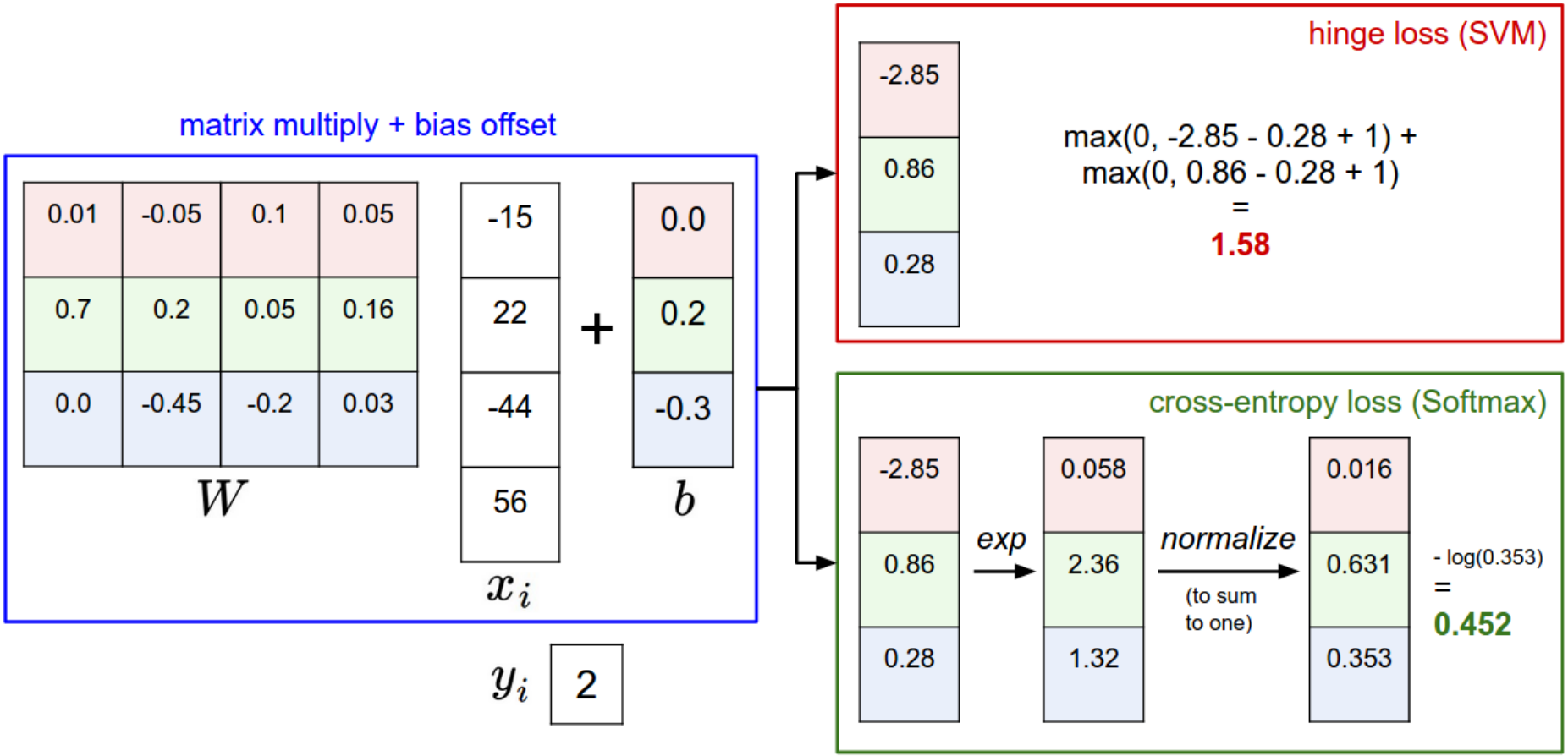
$$P(y_i \mid x_i; W)$$

i.e. we're minimizing the negative log likelihood.

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$


matrix multiply + bias offset
0.01
-0.05
0.1
0.05
0.7
0.2
0.05
0.16
0.0
-0.45
-0.2
0.03
W
-15
22
-44
56
x_i
+
0.0
0.2
-0.3
b
y_i
2
hinge loss (SVM)
-2.85
0.86
0.28
max(0, -2.85 - 0.28 + 1) +
max(0, 0.86 - 0.28 + 1)
=
1.58
cross-entropy loss (Softmax)
-2.85
0.86
0.28
exp
0.058
2.36
1.32
normalize
(to sum
to one)
0.016
0.631
0.353
- log(0.353)
=
0.452

Softmax vs. SVM

- Interpreting the probabilities from the Softmax

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$

Softmax vs. SVM

- Interpreting the probabilities from the Softmax

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$

suppose the weights W were only half as large

(we use a higher regularization strength)

Softmax vs. SVM

- Interpreting the probabilities from the Softmax

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$

suppose the weights W were only half as large:

$$[0.5, -1, 0] \rightarrow [e^{0.5}, e^{-1}, e^0] = [1.65, 0.37, 1] \rightarrow [0.55, 0.12, 0.33]$$

Softmax vs. SVM

- Interpreting the probabilities from the Softmax

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$

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What happens in the limit, as the regularization strength goes to infinity?

Softmax vs. SVM

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + 1)$$

$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

scores:

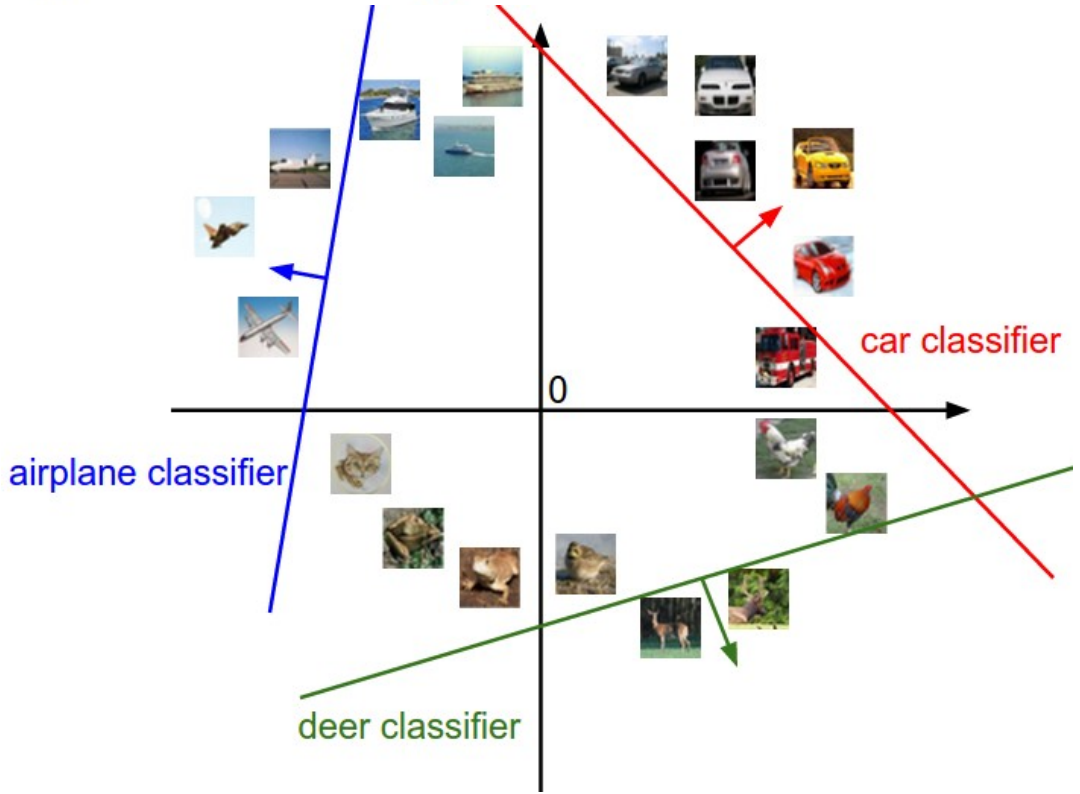
[10, -2, 3]

[10, 9, 9]

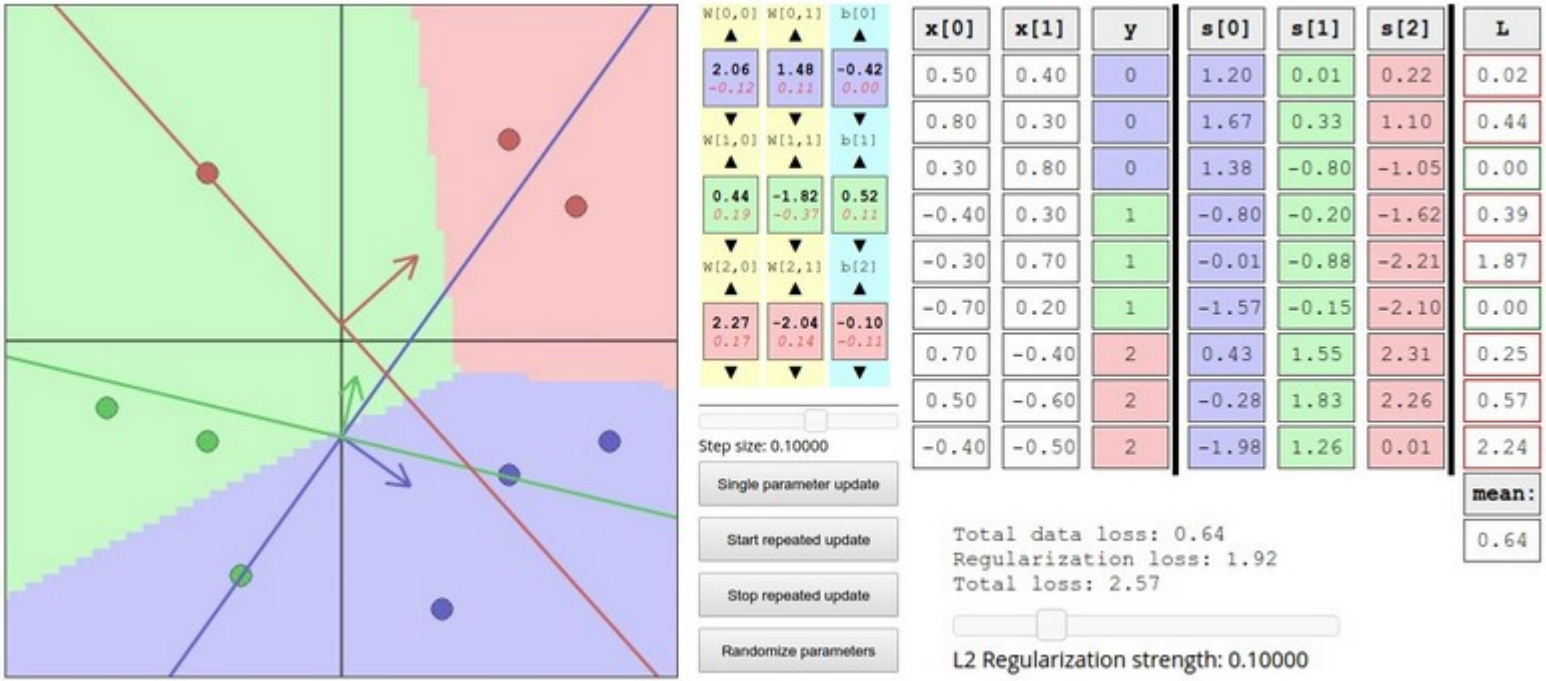
$y_i = 0$ [10, -100, -100]

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + 1)$$

$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$



Interactive Web Demo time....



http://vision.stanford.edu/teaching/cs231n/linear-classify-demo/

Summary

- We introduced a **parametric approach** to image classification
- We defined a **score function** (linear map)
- We defined a **loss function** (SVM / Softmax)

One problem remains: How to find W,b ?

Next class:
Optimization, Backpropagation

Find the W,b that minimizes the loss function.