ECE 250 Homework 1 Solutions

Prob 1

$$E[Y^n] = E[e^{-nX}] = \int_{-\infty}^{\infty} e^{-nx} f_X(x)\,dx$$

$$= \alpha \int_0^{\infty} e^{-(\alpha+n)x}\,dx$$

$$\boxed{E[Y^n] = \frac{\alpha}{n+\alpha}}$$

Prob. 2

Clearly $G(0) = 1$ and $|G(u)| \leq 1$

Examine $G(u) = \dfrac{1+iu}{1+4u^2} = \dfrac{1+iu}{(1+2iu)(1-2iu)}$

Partial Fraction $\longrightarrow$ $= \dfrac{1}{4}\dfrac{1}{1+2iu} + \dfrac{3}{4}\dfrac{1}{1-2iu}$

From Transform Pairs

$$\frac{1}{4}\frac{1}{1+2iu} \longleftrightarrow \frac{1}{8} e^{(x/2)} u(-x)$$

$$\frac{3}{4}\frac{1}{1-2iu} \longleftrightarrow \frac{3}{8} e^{-(x/2)} u(x)$$

$\therefore$

$$G(u) \longleftrightarrow \underbrace{\frac{1}{8} e^{(x/2)} u(-x) + \frac{3}{8} e^{-(x/2)} u(x)}_{\text{These functions are non-negative and the sum integrates to } 1}$$

$\therefore$ $\boxed{\text{G(u) can be the ch.f. of a prob. density}}$

Prob. 3 Use characteristic function

$$\Phi_X(u) = \sum_{n=1}^{\infty} e^{iun} P(X=n)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}e^{iu}\right)^n = \frac{e^{iu}}{2-e^{iu}}$$

From Useful Formulas

$$E[X] = \frac{1}{i}\frac{d}{du}\Phi_X(u)\Big|_{u=0} = \frac{2e^{iu}}{(2-e^{iu})^2}\Big|_{u=0}$$

$$\boxed{E[X] = 2}$$

$$E[X^2] = \left(\frac{1}{i}\right)^2 \frac{d^2}{du^2}\Phi_X(u)\Big|_{u=0} = \frac{2e^{iu}(2+e^{iu})}{(2-e^{iu})^3}\Big|_{u=0}$$

$$E[X^2] = 6$$

$\therefore$

$$Var[X] = E[X^2] - (E[X])^2$$

$$\boxed{Var[X] = 2}$$

Prob. 4 $E[X^n] = \mu^n$

Expand ch.f. $\Phi_X(u) = E[e^{iuX}] = E\left[\sum_{n=0}^{\infty} \frac{(iu)^n}{n!} X^n\right]$

$$= \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} \mu^n$$

$$= \sum_{n=0}^{\infty} \frac{(iu\mu)^n}{n!} = e^{iu\mu}$$

The density is obtained via the inverse Fourier Transform

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(u)\, e^{-iux}\, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu(x-\mu)}\, du$$

$$\boxed{f_X(x) = \delta(x-\mu)}$$

Prob. 5 Use Chebyshev inequality

$$P(|Z - E[Z]| \geq \mu) \leq \frac{Var[Z]}{\mu^2}$$

$Z = X - Y$ $\qquad E[Z] = E[X] - E[Y] = 0$

X and Y are indep. $\qquad Var[Z] = Var[X] + Var[Y]$

$Var[X] = Var[Y] = 1$

$\therefore\ Var[Z] = 2$

$\boxed{\text{Prob. 5 Cont.}}$ The Chebyshev inequality now appears as

$$P(|X-Y| \geq \nu) \leq \frac{2}{\nu^2}$$

let $\nu = 10$ and this becomes

$$\boxed{P(|X-Y| \geq 10) \leq \frac{1}{50}}$$

$\boxed{\text{Prob. 6}}$ $$P(X=n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \cdots$$

$$P(Y=1) = P(X=\text{even}) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} P(X=n)$$

Aside

$$\frac{1}{2}(e^{A}+e^{-A}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{A^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{A^n}{n!}(-1)^n = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty}\frac{A^n}{n!}$$

apply to with $\lambda = A$

$$= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= \frac{1}{2}(e^{\lambda}+e^{-\lambda})e^{-\lambda}$$

$$= \frac{1}{2}(1+e^{-2\lambda})$$

$\therefore$

$$\boxed{\begin{aligned} P(Y=1) &= \frac{1}{2}(1+e^{-2\lambda}) \\ P(Y=-1) &= 1-P(Y=1) = \frac{1}{2}(1-e^{-2\lambda}) \end{aligned}}$$