清華大學數學系數學組
碩士論文

完備流形上的四頂點定理

A note on four-vertex theorem on complete manifolds

指導教授：宋瓊珠（Chiung-Jue Sung）

研 究 生：蔡柏聖（Bo-Sheng Tsai）

學　　號：103021501

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摘要

我們證明了平面上的四頂點定理能夠被推廣到任意簡單連通的二維非負常曲率流形上。另一方面，在特定形式的非簡單連通流形上，我們給出四頂點定理無法被推廣的反例。

致謝

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學生 蔡柏聖 敬上

A NOTE ON FOUR-VERTEX THEOREM ON COMPLETE MANIFOLDS

BO-SHENG TSAI

Abstract. We show that the Four-Vertex Theorem can be extended on any two-dimensional simply connected space form with non-negative sectional curvature. On the other hand, we give counterexamples to show that the Four-Vertex Theorem can not be extended on a type of non-simply connected space form.

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1. Introduction

Let M be a two-dimensional complete manifold, V an orientable open set of M, $\gamma : I \to M$ a smooth regular curve in V parametrized by arc-length, and k the signed geodesic curvature function of γ. We say γ has a vertex at s_0 if $k'(s_0) = 0$. In particular, if k achieves its local extremum at s_0, we say γ has an honest vertex at s_0.

The Four-Vertex Theorem, first proved by S. Mukhopadhaya in 1909, states that any smooth simple closed convex plane curve has at least four honest vertices. In 1912, A. Kneser extended the Four-Vertex Theorem to a non-convex case which states that any smooth simple closed plane curve has at least four honest vertices.

Later, many results about vertices of curves have been extended. In 1987, U. Pinkall [8] proved that any smooth closed plane curve which bounds a compact immersed surface with genus $g \geq 0$ has at least four honest vertices. Also, he conjectured that for each $g \geq 1$, the minimal number of the vertices is $4g + 2$. However, in 1992, Cairs, Ozdemir, and Tjaden [1] gave counterexamples for the conjecture which states that for each $g \geq 1$, there exists a curve with only six honest vertices. Not long after, in 1994, M. Umehara [10] proved that if a smooth closed plane curve bounds a compact immersed surface with genus $g \geq 1$, the minimal number of honest vertices of the curve is six.

It will be an interesting topic if the Four-Vertex Theorem holds on any two-dimensional complete manifold. In 1945, Jackson [6] proved that if M is a Riemannian surface with non-constant sectional curvature K, for any non-stationary point $p \in M$ of the sectional curvature K, that is, $dK_p \neq 0$, there exists a geodesic circle centered at p with only two vertices under the geodesic polar coordinate of a neighborhood of p. In other words, for any non-constant curvature Riemannian surface, there exists a local orthogonal parametrization such that the Four-Vertex Theorem fails. Hence we turn our study to the Four-Vertex Theorem on conformal parametrizations and constant curvature manifolds.

In Section 3, we show that there is a relation between vertices of curves and conformal parametrizations.

Theorem 1.1. *Let M be a complete two-manifold and ϕ an orientation preserving local conformal parametrization of an open set V of M. Then the vertices of any smooth regular curve on V correspond to the vertices of its pre-image curve β on $\mathbb{R}^2$ if and only if the related metric $g = Gdx^2 + Gdy^2$ of ϕ satisfies*

$$\frac{1}{\sqrt{G(x,y)}} = A(x^2 + y^2) + Bx + Cy + D, \tag{1}$$

for some constants A, B, C, and D. Moreover, if the condition (1) holds, the isolated honest vertices of α correspond to the isolated honest vertices of β.

The equality (1) implies that the manifold M has constant sectional curvature on V. Hence, if M is a two-dimensional space form which is covered by a family of conformal parametrizations that all satisfy (1), we can study vertices of any curves on M without any concern. In other words, under this special class of conformal parametrizations, the number of the vertices of curves becomes a geometric invariant.

As a result, we show in Section 4 that the Four-Vertex Theorem can be extended on any simply connected space forms with non-negative sectional curvature under conformal parametrizations which satisfy (1).

On the other hand, one wonders whether the Four-Vertex Theorem can be extended on non-simply connected two-dimensional space forms. However, the answer is no. In Section 5, we will give examples which show that the Four-Vertex Theorem of Kneser and Pinkall can not be extended on any two-dimensional space form which is isometric to a cylinder.

2. Preliminaries

In this section, we introduce definitions and theorems which play important roles in this note.

We will introduce the geodesic curvature function of curves on two-dimensional manifolds. Let us start with the oriented base fields.

Recall that an orientation for a finite-dimensional vector space is an equivalence class of ordered bases as follows: Let X be an n-dimensional vector space. The ordered basis $\{b_1, ..., b_n\}$ determines the same orientation as the basis $\{b'_1, ..., b'_n\}$ if

$$b'_i = \sum_j a_{ij} b_j \text{ with } \det(a_{ij}) > 0.$$

It determines the opposite orientation if $\det(a_{ij}) < 0$. For any complete two-manifold M, we say V is an orientable open set of M if V is open in M and there exists a pair of ordered vector fields $\{X_1, X_2\}$ on V such that $\{X_1(p), X_2(p)\}$ determines the same orientation on the tangent plane T_pM for any point p in V.

Here we define the oriented base fields along a smooth curve.

Definition 2.1. Let M be a complete two-manifold with metric g and γ a smooth curve on an open set V of M. Let $X_1 = a_{11}\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}$, $X_2 = a_{21}\frac{\partial}{\partial x} + a_{22}\frac{\partial}{\partial y}$ be two smooth vector fields along γ. We say $\{X_1, X_2\}$ is a positively(negatively) orthonormal oriented base frame along γ on V (or simply called an oriented base frame along γ) if $g(X_i(p), X_j(p)) = \delta_{ij}$ for any $i = 1, 2$ and the determinant of the related matrix $(a_{ij}(p))$ satisfies $\det(a_{ij}(p)) > 0 (< 0)$ for any $p \in V$. That is, $\{X_1(p), X_2(p)\}$ forms an orthonormal basis of the tangent plane T_pM and determines the same orientation on T_pM for any point p in V.

The definition of the geodesic curvature function of a curve is given by the following.

Definition 2.2. Let M be a complete two-manifold with metric g and $\gamma : I \to M$ a smooth regular curve in an open set V of M, which is parametrized by arc-length s. If there exists a vector field $n(s)$ along γ such that $\{\gamma'(s), n(s)\}$ forms an oriented base frame along γ on V, we define the (signed)geodesic curvature function $k : I \to \mathbb{R}$ of γ by

$$k(s) = \left\langle \frac{D\gamma'(s)}{ds}, n(s) \right\rangle,$$

where $\frac{D\gamma'(s)}{ds}$ is the covariant derivative of γ and $\langle \cdot, \cdot \rangle$ is the inner product associated to the metric g.

Remark.

(i) If we choose opposite oriented base field along γ, the sign of the geodesic curvature changes.

(ii) If V is an orientable open set of M, the geodesic curvature function of any smooth regular curve in V is well-defined. Therefore, if the manifold M is orientable, the geodesic curvature function of any smooth regular curve in M can be globally defined.

(iii) The regularity of curves is necessary for the well-definedness of the curvature function. For example,

$$\gamma(t) = (1 + 2\cos t + \cos 2t, 2\sin t + \sin 2t), \quad t \in [0, 2\pi].$$

Its curvature function

$$k(t) = \frac{3}{8\sqrt{2}} \frac{1 + \cos t}{(1 + \cos t)^{\frac{3}{2}}}$$

blows up as t approaches π (Figure 1).

In this note we study vertices of smooth regular curves on orientable open sets of complete two-manifolds since the geodesic curvature function is well-defined.

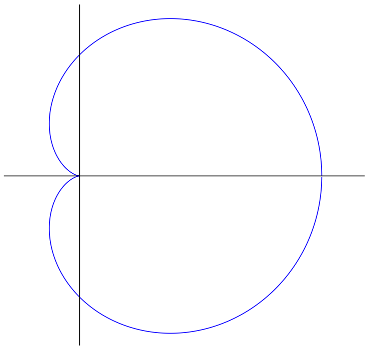

FIGURE 1.

Definition 2.3. Let M be a complete two-manifold, V an orientable open set of M, and $\gamma : I \to M$ a smooth regular curve on V with geodesic curvature k, which is parametrized by arc-length. If $k'(s_0) = 0$, we say γ has a vertex at s_0, $\gamma(s_0)$ is called a vertex of γ. In particular, if k achieves its local extremum at s_0, the vertex at s_0 is called an honest vertex.

Remark. It follows directly from compactness and the well-definedness of the geodesic curvature function that any smooth regular closed curve on any complete orientable two-manifolds has at least two honest vertices.

We introduce two results of the Four-Vertex Theorem in planar case which are given by Kneser and Pinkall [8]. These two results have played important roles for studying the planer case of the Four-Vertex Theorem.

Theorem 2.1 (Kneser)**.** *Any smooth regular simple closed plane curve has at least four honest vertices.*

Theorem 2.2 (Pinkall)**.** *Any smooth regular closed plane curve, which bounds a compact immersed surface on* $\mathbb{R}^2$*, has at least four honest vertices.*

Remark. A plane curve γ bounding a compact immersed surface with genus g means that there exists a regular map $f : C \to \mathbb{R}^2$, where C is a connected boundary of a compact regular surface S with genus g, such that $f(C) = \gamma$ and f can be extended to an immersion $F : S \to \mathbb{R}^2$.

3. The correspondence theorem of vertices

In this note, we try to extend the Four-Vertex Theorem of Kneser and Pinkall on complete two-manifolds. Recall that in 1945, Jackson [6] proved the following theorem, which gives us a direction of the study on the Four-Vertex Theorem.

Theorem 3.1. *Let M be a complete Riemannian surface with non-constant sectional curvature K. Then for any non-stationary point p of K, that is, $dK_p \neq 0$, there exists $R > 0$ such that under the geodesic polar coordinate of a neighborhood of p, there are only two vertices on the geodesic circle centered at p with radius R .*

Remark. For the proof of the theorem of Jackson, one may refer to Appendix for details.

Since Jackson's Theorem point out that for any Riemannian surface with non-constant sectional curvature, there exists an orthogonal parametrization such that the Four-Vertex Theorem fails. Therefore, we turn our study on conformal parametrizations and two-dimensional space forms. In the following we first focus on complete two-manifolds with a local conformal parametrization.

To extend the Four-Vertex Theorem, we start with an idea: Let M be a complete two-manifold. If there exists a local parametrization ϕ of an open set V of M such that the honest vertices of any smooth regular curve in V correspond to the honest vertices of its pre-image curve on $\mathbb{R}^2$, we may extend the Four-Vertex Theorem directly on V by the result of Kneser and Pinkall.

By considering complete two-manifold with a local conformal parametrization, we obtain a result.

Theorem 3.2. *Let M be a complete two-manifold and ϕ an orientation preserving local conformal parametrization of an open set V of M. Then the vertices of any smooth regular curve on V correspond to the vertices of its pre-image curve β on $\mathbb{R}^2$ if and only if the related metric $g = Gdx^2 + Gdy^2$ of ϕ satisfies*

$$\frac{1}{\sqrt{G(x,y)}} = A(x^2 + y^2) + Bx + Cy + D, \tag{2}$$

for some constants A, B, C, and D. Moreover, if the above condition holds, the isolated honest vertices of α correspond to the isolated honest vertices of β.

Remark. We say $\alpha(s_0)$ is an isolated vertex of a smooth regular curve α if there exists an $\epsilon > 0$ such that there is no other critical point of k_g in $(s_0 - \epsilon, s_0 + \epsilon)$. In particular, if $\alpha(s_0)$ is also an honest vertex of α, we say $\alpha(s_0)$ is an isolated honest vertex of α.

From the result, we can extend a part of the Four-Vertex Theorem of Kneser and Pinkall locally under orientation preserving conformal parametrization which satisfies equality (2).

In the following, we will prove the main result. For simplicity, we use the following settings in lemmas and theorems:

Let M be a complete two-manifold. Assume there exists an orientation preserving local conformal parametrization ϕ of an open set V of M with the related metric $g(x, y) = G(x, y)dx^2 + G(x, y)dy^2$, $G > 0$, where (x, y) represents the local coordinate of V.

We denote the standard basis of the tangent bundle TV by $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$.

Let $\alpha : I \to V$ be an arbitrary smooth regular curve on V parametrized by arc-length s and $\beta(s) = (u(s), v(s))$ the pre-image curve of α under ϕ.

Let s_e be the arc-length of $\beta(s)$, that is, $\beta_2(s_e) := \beta(s(s_e))$ is a smooth regular curve with $|\beta_2'(s_e)| = 1$. Also, we denote the geodesic curvature of α, β_2 by k_g, k_e respectively, where k_g depends on the metric g. Last, we define the smooth directional angle function $\theta(s)$ to be the angle between the first coordinate vector $\left(\frac{\partial}{\partial x}\right)_{\alpha(s)}$ and $\alpha'(s)$.

We expect that there is a correspondence between the vertices of curves α and β under ϕ. Therefore we start with the geodesic curvature function k_g and expect there is a relation between k_g and k_e under ϕ.

Theorem 3.3. *The geodesic curvature function k_g of α is given by*

$$k_g(s) = \theta'(s) - \left[\left(\frac{1}{\sqrt{G}}\right)_x (\beta(s))\right] \sin(\theta(s)) + \left[\left(\frac{1}{\sqrt{G}}\right)_y (\beta(s))\right] \cos(\theta(s)).$$

Remark. This formula only depends on the metric of M.

proof of Theorem 3.3. For simplicity, in this proof we use notations $\alpha = \alpha(s)$, $\theta = \theta(s)$, $u = u(s)$, $v = v(s)$, $G = G(u(s), v(s))$. Also, we write $\left(\frac{\partial}{\partial x}\right)_{\alpha(s)}$ and $\left(\frac{\partial}{\partial y}\right)_{\alpha(s)}$ by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively.

Before proving the theorem, we state several useful substitution formulas about the directional angle θ:

$$\cos\theta = \sqrt{G}u', \quad \sin\theta = \sqrt{G}v', \tag{3}$$

$$
\begin{aligned}
-(\sin\theta)\theta' &= [(\sqrt{G})_x u' + (\sqrt{G})_y v']u' + \sqrt{G}u'', \\
(\cos\theta)\theta' &= [(\sqrt{G})_x u' + (\sqrt{G})_y v']v' + \sqrt{G}v''.
\end{aligned} \tag{4}
$$

Here we give the proofs of these formulas. Let θ_2 be the directional angle between the second coordinate vector $\frac{\partial}{\partial y}$ and α'. Then $\theta - \theta_2 = \pi/2$. This implies $\cos\theta_2 = \cos(\theta - \pi/2) = \sin\theta$.

Since $\alpha' = u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$, we have

$$
\cos\theta = \frac{\left\langle \frac{\partial}{\partial x}, \alpha' \right\rangle}{|\frac{\partial}{\partial x}|\,|\alpha'|} = \frac{Gu'}{\sqrt{G}} = \sqrt{G}u' \tag{5}
$$

and

$$
\sin\theta = \cos\theta_2 = \frac{\left\langle \frac{\partial}{\partial y}, \alpha' \right\rangle}{|\frac{\partial}{\partial y}|\,|\alpha'|} = \frac{Gv'}{\sqrt{G}} = \sqrt{G}v'. \tag{6}
$$

This proves (3).

For 4, taking the derivative of (5) and (6) with respect to s, we directly obtain

$$
\begin{aligned}
-(\sin\theta)\theta' &= [(\sqrt{G})_x u' + (\sqrt{G})_y v']u' + \sqrt{G}u'', \\
(\cos\theta)\theta' &= [(\sqrt{G})_x u' + (\sqrt{G})_y v']v' + \sqrt{G}v'',
\end{aligned}
$$

where $G = G(u(s), v(s))$. This proves (4).

Here we start calculating the geodesic curvature function of α. The covariant derivative of α' is

$$
\begin{aligned}
\frac{D\alpha'(s)}{ds} &= \left[u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1\right]\frac{\partial}{\partial x} \\
&\quad + \left[v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2\right]\frac{\partial}{\partial y} \\
&= \left[u'' + (u')^2\left(\frac{G_x}{2G}\right) + u'v'\left(\frac{G_y}{G}\right) + (v')^2\left(-\frac{G_x}{2G}\right)\right]\frac{\partial}{\partial x} \\
&\quad + \left[v'' + (u')^2\left(-\frac{G_y}{2G}\right) + u'v'\left(\frac{G_x}{G}\right) + (v')^2\left(\frac{G_y}{2G}\right)\right]\frac{\partial}{\partial y},
\end{aligned}
$$

where $u = u(s)$, $v = v(s)$ and Γ_{ij}^k are the Christoffel symbols of M in the parametrization ϕ.

Since ϕ is a conformal parametrization, we may choose $\left\{\alpha', -v'\frac{\partial}{\partial x} + u'\frac{\partial}{\partial y}\right\}$ to be the oriented base frame along α. By definition, the sectional curvature function of α is

$$\begin{aligned}
k_g(s) &= \left\langle \frac{D\alpha'(s)}{ds}, -v'\frac{\partial}{\partial x} + u'\frac{\partial}{\partial y} \right\rangle_g \\
&= -Gv'\left[u'' + (u')^2\left(\frac{G_x}{2G}\right) + u'v'\left(\frac{G_y}{G}\right) + (v')^2\left(-\frac{G_x}{2G}\right)\right] \\
&\quad +Gu'\left[v'' + (u')^2\left(-\frac{G_y}{2G}\right) + u'v'\left(\frac{G_x}{G}\right) + (v')^2\left(\frac{G_y}{2G}\right)\right] \\
&= -\sqrt{G}v'\left[\sqrt{G}u'' + (u')^2\left(\frac{G_x}{2\sqrt{G}}\right) + u'v'\left(\frac{G_y}{\sqrt{G}}\right) + (v')^2\left(-\frac{G_x}{2\sqrt{G}}\right)\right] \\
&\quad +\sqrt{G}u'\left[\sqrt{G}v'' + (u')^2\left(-\frac{G_y}{2\sqrt{G}}\right) + u'v'\left(\frac{G_x}{\sqrt{G}}\right) + (v')^2\left(\frac{G_y}{2\sqrt{G}}\right)\right] \\
&= -\sqrt{G}v'\left[\sqrt{G}u'' + (u')^2\left(\sqrt{G}\right)_x + 2u'v'\left(\sqrt{G}\right)_y - (v')^2\left(\sqrt{G}\right)_x\right] \\
&\quad +\sqrt{G}u'\left[\sqrt{G}v'' - (u')^2\left(\sqrt{G}\right)_y + 2u'v'\left(\sqrt{G}\right)_x + (v')^2\left(\sqrt{G}\right)_y\right].
\end{aligned}$$

Substituting $\sqrt{G}u'$, $\sqrt{G}v'$, $\sqrt{G}u''$, $\sqrt{G}v''$ by (5) and (6), we obtain

$$\begin{aligned}
k_g(s) &= -\sin\theta\left[-\theta'\sin\theta - \frac{\cos\theta}{G}\left(\left(\sqrt{G}\right)_x\cos\theta + \left(\sqrt{G}\right)_y\sin\theta\right)\right] \\
&\quad -\sin\theta\left[\frac{\cos^2\theta}{G}\left(\sqrt{G}\right)_x + \frac{2\sin\theta\cos\theta}{G}\left(\sqrt{G}\right)_y - \frac{\sin^2\theta}{G}\left(\sqrt{G}\right)_x\right] \\
&\quad +\cos\theta\left[\theta'\cos\theta - \frac{\sin\theta}{G}\left(\left(\sqrt{G}\right)_x\cos\theta + \left(\sqrt{G}\right)_y\sin\theta\right)\right] \\
&\quad +\cos\theta\left[\frac{\cos^2\theta}{G}\left(\sqrt{G}\right)_y + \frac{2\sin\theta\cos\theta}{G}\left(\sqrt{G}\right)_x + \frac{\sin^2\theta}{G}\left(\sqrt{G}\right)_y\right] \\
&= \theta' - \frac{\sin\theta}{G}\left[\left(\sqrt{G}\right)_x\cos 2\theta + \left(\sqrt{G}\right)_y\sin 2\theta\right] + \frac{\cos\theta}{G}\left[\left(\sqrt{G}\right)_x\sin 2\theta + \left(\sqrt{G}\right)_y\cos 2\theta\right] \\
&= \theta' - \frac{\sin\theta}{G}\left[\left(\sqrt{G}\right)_x(-\sin\theta\cos 2\theta + \cos\theta\sin 2\theta) + \left(\sqrt{G}\right)_y(-\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)\right] \\
&= \theta' + \frac{\left(\sqrt{G}\right)_x}{G}\sin(\theta) - \frac{\left(\sqrt{G}\right)_y}{G}\cos(\theta) \\
&= \theta'(s) - \left(\frac{1}{\sqrt{G}}\right)_x\sin(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_y\cos(\theta(s)).
\end{aligned}$$

This completes the proof. □

In the next lemma, we show that there is a relation between $k_e(s_e)$ and $\theta'(s(s_e))$:

$$\theta'(s(s_e)) = \frac{k_e(s_e)}{\sqrt{G(s(s_e))}}.$$

As a result, we can obtain a relation between k_g and k_e by Theorem 3.3:

(7)

$$k_g(s(s_e)) = \frac{k_e(s_e)}{\sqrt{G(s(s_e))}} - \left[\left(\frac{1}{\sqrt{G}}\right)_x (\beta(s(s_e)))\right] \sin(\theta(s(s_e))) + \left[\left(\frac{1}{\sqrt{G}}\right)_y (\beta(s(s_e)))\right] \cos(\theta(s(s_e))).$$

Then taking the derivative on both sides, we can study under what condition the vertices of α will correspond to the vertices of β.

For simplicity, in the following we use these notations for any differentiable function f since we will reparametrize s by s_e:

$$\begin{aligned} s &:= s(s_e), \quad f(s) := f(s(s_e)), \\ \frac{ds}{ds_e} &:= \frac{ds(s_e)}{ds_e}, \quad \frac{ds_e}{ds} := \frac{ds_e(s)}{ds}, \\ f'(s) &= \frac{df(s)}{ds} := f'(s(s_e)) \equiv \left.\frac{df(s)}{ds}\right|_{s=s(s_e)}, \\ \frac{df(s)}{ds_e} &:= \frac{df(s(s_e))}{ds_e} \left(= \left.\frac{df(s)}{ds}\right|_{s=s(s_e)} \frac{ds}{ds_e} \right). \end{aligned}$$

Lemma 3.4. *Same assumption as the beginning of this section, we have the following substitution formulas.*

First, the correlations between the arc-length functions are

$$\frac{ds(s_e)}{ds_e} = \sqrt{G(s(s_e))}, \ \textit{and} \ \frac{ds_e(s)}{ds} = \frac{1}{\sqrt{G(s)}}. \tag{8}$$

Second, the correlations between $\beta(s)$ and $\theta(s)$ are

$$\frac{du(s)}{ds_e} = \cos\theta(s), \ \textit{and} \ \frac{dv(s)}{ds_e} = \sin\theta(s). \tag{9}$$

Third, the correlations between $\theta(s)$ and $k_e(s_e)$ are

$$\frac{d\theta(s)}{ds} = \frac{k_e(s_e)}{\sqrt{G(s)}}, \ \textit{and} \ \frac{d\theta(s)}{ds_e} = k_e(s_e), \tag{10}$$

where $s = s(s_e)$ in the second and third formulas.

Proof. In this proof we also denote $G(u(s), v(s))$ by $G(s)$.

For the first, since α is regular and parametrized by arc-length s, we have

$$|\alpha'(s)|^2 = G(s)\left((u'(s))^2 + (v'(s))^2\right) \equiv 1. \tag{11}$$

On the other hand, since β_2 is regular and parametrized by arc-length s_e, we have

$$|\beta_2'(s_e)|^2 = \left(u'(s)\frac{ds}{ds_e}\right)^2 + \left(u'(s)\frac{ds}{ds_e}\right)^2 = \left((u'(s))^2 + (v'(s))^2\right)\left(\frac{ds}{ds_e}\right)^2 \equiv 1. \tag{12}$$

Therefore from (11) and (12) we obtain

$$\frac{ds}{ds_e} = \sqrt{G(s)}. \tag{13}$$

And then by inverse function theorem,

$$\frac{ds_e}{ds} = \frac{1}{\sqrt{G(s)}}. \tag{14}$$

This proves (8).

For the second one, from (13) we have

$$\begin{aligned}
\frac{du(s)}{ds_e} &= u'(s)\frac{ds}{ds_e} = u'(s)\sqrt{G(s)},\\
\frac{dv(s)}{ds_e} &= v'(s)\frac{ds}{ds_e} = v'(s)\sqrt{G(s)}.
\end{aligned}$$

Substituting $u'(s)\sqrt{G(s)}$ and $v'(s)\sqrt{G(s)}$ by (3), we obtain

$$\begin{aligned}
\frac{du(s)}{ds_e} &= \cos\theta(s),\\
\frac{dv(s)}{ds_e} &= \sin\theta(s),
\end{aligned}$$

This proves (9).

For the third one, recall that $\beta_2(s_e) = \beta(s(s_e))$.

Let $\theta_\beta(s)$ be the angle between $\beta'(s)$ and the first coordinate vector $(1,0)$ on its tangent plane and $\theta_{\beta_2}(s_e)$ the angle between $\beta_2'(s_e)$ and $(1,0)$ on its tangent plane. From Theorem 3.3, by choosing our manifold $\mathbb{R}^2$ we directly obtain

$$k_e(s_e) = \frac{d\theta_{\beta_2}(s_e)}{ds_e}. \tag{15}$$

Since ϕ is a conformal parametrization, we have $\theta(s) = \theta_\beta(s)$.

On the other hand, since $\beta_2(s_e)$ and $\beta(s) := \beta(s(s_e))$ represent the same curve, the tangent vector $\beta_2'(s_e)$ has same direction with $\beta'(s)$. Hence

$$\theta_{\beta_2}(s_e) = \theta_\beta(s) = \theta(s). \tag{16}$$

Note that

$$\frac{d\theta(s)}{ds_e} = \frac{d\theta(s)}{ds}\frac{ds}{ds_e} = \frac{d\theta(s)}{ds}\sqrt{G(s)}. \tag{17}$$

Combining (15), (16), and (17), we obtain

$$\frac{d\theta(s)}{ds} = \frac{1}{\sqrt{G(s)}}\frac{d\theta(s)}{ds_e} = \frac{1}{\sqrt{G(s)}}\frac{d\theta_{\beta_2}(s_e)}{ds_e} = \frac{k_e(s_e)}{\sqrt{G(s)}}. \tag{18}$$

This proves the first one of (10).

Plugging (18) into (17), then we obtain

$$\frac{d\theta(s)}{ds_e} = \sqrt{G(s)}\frac{d\theta(s)}{ds} = k_e(s_e).$$

This completes the proof. $\square$

Here we start proving the main theorem. For simplicity, we use the following notations:

$$\left(\frac{1}{\sqrt{G}}\right) := \left(\frac{1}{\sqrt{G}}\right)(u(s), v(s)), \quad \left(\frac{1}{\sqrt{G}}\right)_{x_i} := \frac{\partial\left(\frac{1}{\sqrt{G}}\right)}{\partial x_i}(u(s), v(s)),$$

and

$$\left(\frac{1}{\sqrt{G}}\right)_{x_i x_j} := \frac{\partial^2\left(\frac{1}{\sqrt{G}}\right)}{\partial x_i \partial x_j}(u(s), v(s)),$$

where $s = s(s_e)$.

Theorem 3.5. *Same assumption as the beginning of this section, the vertices of any smooth regular curve α on V correspond to the vertices of its pre-image curve β on $\mathbb{R}^2$ under ϕ if and only if the related metric $g = G(x,y)dx^2 + G(x,y)dy^2$ of ϕ satisfies*

$$\left(\frac{1}{\sqrt{G}}\right)_{xy} \equiv 0, \text{ and } \left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx} \equiv 0. \tag{19}$$

Proof. Recall that from Theorem 3.3 and (10) we have

$$k_g(s) = \frac{k_e(s_e)}{\sqrt{G(s)}} - \left[\left(\frac{1}{\sqrt{G}}\right)_x\right]\sin(\theta(s)) + \left[\left(\frac{1}{\sqrt{G}}\right)_y\right]\cos(\theta(s)), \tag{20}$$

where $s = s(s_e)$. Then taking the derivative on both sides with respect to s_e:

$$\begin{aligned}
\frac{dk_g(s)}{ds_e} &= \left(\frac{1}{\sqrt{G}}\right)\frac{dk_e(s_e)}{ds_e} + k_e(s_e)\left[\left(\frac{1}{\sqrt{G}}\right)_x \frac{du(s)}{ds_e} + \left(\frac{1}{\sqrt{G}}\right)_y \frac{dv(s)}{ds_e}\right] \\
&\quad - \left[\left(\frac{1}{\sqrt{G}}\right)_{xx} \frac{du(s)}{ds_e} + \left(\frac{1}{\sqrt{G}}\right)_{yx} \frac{dv(s)}{ds_e}\right]\sin(\theta(s)) - \left[\left(\frac{1}{\sqrt{G}}\right)_x\right]\cos(\theta(s))\frac{d\theta(s)}{ds_e} \\
&\quad + \left[\left(\frac{1}{\sqrt{G}}\right)_{xy} \frac{du(s)}{ds_e} + \left(\frac{1}{\sqrt{G}}\right)_{yy} \frac{dv(s)}{ds_e}\right]\cos(\theta(s)) - \left[\left(\frac{1}{\sqrt{G}}\right)_y\right]\sin(\theta(s))\frac{d\theta(s)}{ds_e}.
\end{aligned}$$

Using the formulas (9) and (10), we have

$$\begin{aligned}
\frac{dk_g(s)}{ds_e} &= \left(\frac{1}{\sqrt{G}}\right)\frac{dk_e(s_e)}{ds_e} + k_e(s)\left[\left(\frac{1}{\sqrt{G}}\right)_x \cos(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_y \sin(\theta(s))\right] \\
&\quad - \left[\left(\frac{1}{\sqrt{G}}\right)_{xx} \cos(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_{yx} \sin(\theta(s))\right]\sin(\theta(s)) - \left[\left(\frac{1}{\sqrt{G}}\right)_x\right]\cos(\theta(s))k_e(s_e) \\
&\quad + \left[\left(\frac{1}{\sqrt{G}}\right)_{xy} \cos(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_{yy} \sin(\theta(s))\right]\cos(\theta(s)) - \left[\left(\frac{1}{\sqrt{G}}\right)_y\right]\sin(\theta(s))k_e(s_e) \\
&= \left(\frac{1}{\sqrt{G}}\right)\frac{dk_e(s_e)}{ds_e} - \left[\left(\frac{1}{\sqrt{G}}\right)_{xx} \cos(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_{yx} \sin(\theta(s))\right]\sin(\theta(s)) \\
&\quad + \left[\left(\frac{1}{\sqrt{G}}\right)_{xy} \cos(\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_{yy} \sin(\theta(s))\right]\cos(\theta(s)) \\
&= \left(\frac{1}{\sqrt{G}}\right)\frac{dk_e(s_e)}{ds_e} + \sin(\theta(s))\cos(\theta(s))\left[\left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx}\right] \\
&\quad + \left(\cos^2(\theta(s)) - \sin^2(\theta(s))\right)\left(\frac{1}{\sqrt{G}}\right)_{yx}.
\end{aligned}$$

Hence

(21)

$$\frac{dk_g(s)}{ds_e} = \frac{1}{\sqrt{G}}\frac{dk_e(s_e)}{ds_e} + \frac{1}{2}\left[\left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx}\right]\sin(2\theta(s)) + \left(\frac{1}{\sqrt{G}}\right)_{yx}\cos(2\theta(s)),$$

where $s = s(s_e)$.

If the vertices of α correspond to vertices of β for any smooth curve α on the open set V which contains p, we may choose α_0 and $\alpha_{\pi/4}$ to be the geodesic curves on V passing through p, whose tangent vectors at p have directional angles 0 and $\pi/4$ with respect to the first coordinate $\left(\frac{\partial}{\partial x}\right)_p$.

Since α_0 and $\alpha_{\pi/4}$ are geodesic curves, their geodesic curvature functions are both zero and, by the assumption, so do their pre-image curves on $\mathbb{R}^2$. Therefore from (21), by setting $\theta = 0$ and $\pi/4$ respectively, we have

$$\left\{\left(\frac{1}{\sqrt{G}}\right)_{yx}\right\} \equiv 0, \quad \text{and} \quad \left\{\frac{1}{2}\left[\left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx}\right]\right\} \equiv 0.$$

This proves one direction of the statement of the theorem.

On the other hand, if

$$\left(\frac{1}{\sqrt{G}}\right)_{yx} \equiv 0 \text{ and } \left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx} \equiv 0,$$

from (21) we have

$$\frac{dk_g(s)}{ds_e} = \frac{1}{\sqrt{G(s)}}\frac{dk_e}{ds_e}. \tag{22}$$

From the chain rule

$$\frac{dk_g(s)}{ds_e} = \frac{dk_g(s)}{ds}\frac{ds}{ds_e} = \frac{dk_g(s)}{ds}\sqrt{G(s)},$$

and combining with (22) and we obtain

$$\frac{dk_g(s)}{ds} = \frac{1}{G(s)}\frac{dk_e}{ds_e}, \tag{23}$$

where $s = s(s_e)$. Therefore, the zeros of $k'_g(s(s_e))$ correspond to the zeros of $k'_e(s_e)$. That is, all vertices of α correspond to all vertices of β and this completes the proof of the theorem. □

Moreover, we observe that we can assure the correspondence between isolated honest vertices of α and isolated honest vertices of its pre-image curve β.

Theorem 3.6. *Let M be a complete two-manifold and ϕ an orientation preserving local conformal parametrization of an open set V of M which satisfies the condition (19). Then the isolated honest vertices of α correspond to the isolated honest vertices of β.*

Proof. Again we denote the geodesic curvature function of α, β by k_g, k_e respectively.

First, let $\beta(a_0)$ be an isolated honest vertex of $\beta(s_e)$ and $k_e(s_e)$ achieves its local maximum(minimum) at a_0. We want to show that $\alpha(s(a_0))$ is an isolated honest vertex of

$\alpha(s(s_e))$ and $k_g(s(s_e))$ also achieves its local maximum(minimum) at a_0.

Recall that in the proof of Theorem 3.5, the condition (19) implies (23) which states that

$$\frac{dk_g(s)}{ds} = \frac{1}{G(s)}\frac{dk_e}{ds_e},$$

where $s = s(s_e)$. Here we denote $k_g'(s(s_e)) = \frac{dk_g(s)}{ds}$ and $k_e'(s_e) = \frac{dk_e}{ds_e}$.

Without loss of generality we may assume $k_e(s_e)$ achieves its local minimum at a_0. Since $\beta(a_0)$ is a vertex of $\beta(s_e)$, $k_e'(a_0) = 0$. From (23) we know that $k_g'(s(a_0)) = 0$.

Because now $\beta(a_0)$ is an isolated honest vertex of β, by definition of isolated point we know that there exists an $\epsilon > 0$ such that

$$k_e'(s_e) \neq 0 \text{ on } (a_0 - \epsilon, a_0 + \epsilon) - \{0\}\,. \tag{24}$$

By (24) and Intermediate Value Theorem we know that on $(a_0 - \epsilon, a_0)$, $k_e'(s_e)$ must has the same sign(otherwise there exists another vertex in $(a_0 - \epsilon, a_0)$ and yields a contradiction). Similarly, on $(a_0, a_0 + \epsilon)$, $k_e'(s_e)$ also has the same sign.

As a result, since now $k_e(s_e)$ achieves its local minimum at a_0, we must have $k_e'(s_e) < 0$ on $(a_0 - \epsilon, a_0)$ and $k_e'(s_e) > 0$ on $(a_0, a_0 + \epsilon)$. Otherwise k_e is increasing on $(a_0 - \epsilon, a_0)$ or decreasing on $(a_0, a_0 + \epsilon)$ and $k_e(a_0)$ fails to be a local minimum of k_e. Therefore from (23), we have:

$$k_g'(s(s_e)) < 0 \text{ on } (a_0 - \epsilon, a_0) \text{ and } k_g'(s(s_e)) > 0 \text{ on } (a_0, a_0 + \epsilon). \tag{25}$$

This implies that $k_g(s(s_e))$ has local minimum at a_0. Then again from (25), $\alpha(s(s_e))$ has an isolated honest vertex at a_0.

Conversely, assume $\alpha(s(s_e))$ has an isolated honest vertex at a_0, and $\alpha(s(s_e))$ achieves its local minimum at a_0. Using the same argument as above, one may obtain

$$k_e'(s_e) < 0 \text{ on } (a_0 - \epsilon, a_0) \text{ and } k_e'(s_e) > 0 \text{ on } (a_0, a_0 + \epsilon).$$

This implies that $k_e(s_e)$ has local minimum at a_0. Also, from (25) we know that $\beta(s_e)$ has an isolated honest vertex at a_0.

Therefore, from the above arguments, the points where $k_e(s_e)$ achieves local minimum correspond to the points where $k_g(s(s_e))$ achieves local minimum.

Similarly, we obtain from the same method that the points where $k_e(s_e)$ achieves local maximum correspond to the points where $k_g(s(s_e))$ achieves local maximum and this completes the proof. $\square$

Notice that the conditions

$$\left(\frac{1}{\sqrt{G}}\right)_{xy} = \left(\frac{1}{\sqrt{G}}\right)_{yx} \equiv 0,$$

$$\left(\frac{1}{\sqrt{G}}\right)_{yy} - \left(\frac{1}{\sqrt{G}}\right)_{xx} \equiv 0$$

form a system of partial differential equation of the function $\frac{1}{\sqrt{G}}$.

Solving the equations, then we obtain

$$\frac{1}{\sqrt{G(x,y)}} = f(x,y) = A(x^2+y^2) + Bx + Cy + D,$$

for some constants A, B, C, and D. Therefore this condition is equivalent to the condition (19) in Theorem 3.5. As a result, together with Theorem 3.6, we have the following.

Theorem 3.7. *Let M be a complete two-manifold and ϕ an orientation preserving local conformal parametrization of an open set V of M. Then the vertices of any smooth regular curve on V correspond to the vertices of its pre-image curve β on $\mathbb{R}^2$ if and only if the relating metric $g = Gdx^2 + Gdy^2$ of ϕ satisfies*

$$\frac{1}{\sqrt{G(x,y)}} = A(x^2+y^2) + Bx + Cy + D, \tag{26}$$

for some constants A, B, C, and D. Moreover, if the condition (26) holds, the isolated honest vertices of α correspond to the isolated honest vertices of β.

Moreover, by computation we observe that condition (26) implies that the sectional curvature on V is constant $K = 4AD - B^2 - C^2$. This means if a complete constant curvature two-manifold M is covered by a family of conformal parametrizations which all satisfy the condition (26), we can study the number of vertices of any curve on M without any concern under these parametrizations.

Next we consider another case. If M is $\mathbb{R}^2$ and we introduce a metric $g = G(x,y)dx^2 + G(x,y)dy^2$ on an open set V of $\mathbb{R}^2$, under what condition the vertices of curves in $\mathbb{R}^2$ will correspond to the vertices of the same curve in the manifold $(\mathbb{R}^2, g)$?

Let $\alpha(s) = (u(s), v(s))$ be a smooth regular curve in V of $(\mathbb{R}^2, g)$, which is parametrized by arc-length s. Define $\beta(s) = (u(s), v(s))$ to be the image curve of α under the inclusion map $\eta : (\mathbb{R}^2, g) \to \mathbb{R}^2$ and let s_e be the arc length of β.

Notice that α, β represent the same curves on $\mathbb{R}^2$, the metric g preserves angles under the view point of $(\mathbb{R}^2, g)$ and $\mathbb{R}^2$, and we may choose the orientations of $\mathbb{R}^2$ and $(\mathbb{R}^2, g)$ to be the same.

Although there is no parametrization in this case, we get the similar result as those previous lemmas and theorems simply because the computations only depend on the metric, angle functions, and the orientation choosing. Our theorem is in the following.

Theorem 3.8. *Let V be an open set in $\mathbb{R}^2$ and $g = G(x,y)dx^2 + G(x,y)dy^2$ a well-defined metric on V. Then the vertices of any smooth regular curve in V correspond to the vertices of its image curve β on $\mathbb{R}^2$ under the inclusion map $\eta : (\mathbb{R}^2, g) \to \mathbb{R}^2$ if and only if the metric $g = Gdx^2 + Gdy^2$ satisfies*

$$\frac{1}{\sqrt{G(x,y)}} = A(x^2+y^2) + Bx + Cy + D, \text{ for some constants } A, B, C \text{ and } D.$$

Moreover, if the condition (26) holds, the isolated honest vertices of any smooth regular curve in $(\mathbb{R}^2, g)$ correspond to the isolated honest vertices of such curve in $\mathbb{R}^2$.

4. Examples for extension of the Four-Vertex Theorem

In this section we give three applications of the correspondence theorem of vertices.

From Theorem 3.6 and Theorem 3.8, the first extension of the Four-Vertex Theorem is given as follows.

Theorem 4.1. *Let M be a two-dimensional simply connected space form with non-negative sectional curvature. Then we have*

(A) If α is a simple closed curve on M with only isolated vertices, it has at least four honest vertices.

(B) If α is a closed curve on M which bounds a compact immersed surface and with only isolated vertices, it has at least four vertices.

Here the curves are assumed to be smooth and regular.

Proof. If $M = \mathbb{R}^2$, (A) and (B) are special cases of Four-Vertex Theorem of Kneser and Pinkall. Hence both of the statements hold.

If $M = \mathbb{S}^2$, let α be a smooth regular closed curve on $\mathbb{S}^2$ with only isolated vertices. Then we can always take a point p on $\mathbb{S}^2$ such that γ does not pass through p. Without loss of generality, we may assume p is the north pole of $\mathbb{S}^2$(up to a proper linear orthogonal transformation ρ, which is an isometry of $\mathbb{S}^2$).

We define a conformal parametrization $\phi : \mathbb{R}^2 \to \mathbb{S}^2 - (0,0,1)$ by

$$\phi(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}),$$

which is the inverse of the stereographic projection and its related metric is given by

$$g_{ij}(x,y) = \frac{4\delta_{ij}}{(1+x^2+y^2)^2}.$$

Clearly that ϕ satisfies condition (26).

Since isometries preserve metrics, the related metric of $\rho \circ \phi$ is also a conformal parametrization that satisfies condition (26). That is, $\mathbb{S}^2$ is covered by a family of stereographic projection, which all satisfy the condition (26). Hence, from Theorem 3.6 we know that isolated honest vertices of α on $\mathbb{R}^2$ correspond to isolated honest vertices of its image curve on $\mathbb{S}^2 - (0,0,1)$ under ϕ. Therefore, again by the Four-Vertex Theorem of Knrser and Pinkall(Theorem 2.1 and 2.2), (A) and (B) hold.

Finally, it is known in [3] that any simply connected space form with non-negative constant curvature is isometric to $\mathbb{S}^2$ or $\mathbb{R}^2$. Therefore by Theorem 3.6 and 3.8 and the Four-Vertex Theorem of Knerser and Pinkall, (A) and (B) hold for any two-dimensional simply connected space form M. This completes the proof. □

Remark. If M is a two-dimensional space form but not simply connected, we can also extend the Four-Vertex Theorem on any simply connected open set of M. The main idea is that the universal covering of space form is isometric to $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$(this is also known in [3]). One may refer to [7] for details.

The following theorem is another extension of the Four-Vertex Theorem, which is a special case of Theorem 3.8.

Theorem 4.2. *Consider the Poincaré half-plane model* $(\mathbb{R}^2, g)$ *([3], page 160) of the hyperbolic plane* $\mathbb{H}^2$*. The metric on the upper half-plane* $\{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ *is given by*

$$g(x,y) = \frac{dx^2+dy^2}{y^2}.$$

Then any smooth regular closed curve on $\mathbb{H}^2$*, which has only isolated vertices, contains at least four honest vertices.*

Proof. Clearly that the metric g satisfies the condition (26). From Theorem 3.8 we know that the isolated honest vertices of any smooth regular curve on $(\mathbb{R}^2, g)$ correspond to the isolated honest vertices of such curve in $\mathbb{R}^2$. Again, combining with the Four-Vertex Theorem of Knrser and Pinkall we know that if α is a smooth regular closed curve with only isolated vertices on $\mathbb{H}^2 = (\mathbb{R}^2, g)$, it has at least four honest vertices.

□

Next, the following is the third application of the correspondence theorem of vertices.

Theorem 4.3. *Let γ be a smooth regular curve on $\mathbb{R}^2$, which does not pass through the origin. Define the inversion map about the unit circle $R_0 : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$ by*

$$R_0(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \quad (x,y) \neq (0,0).$$

Then the vertices of γ correspond to the vertices of its image curve $R_0(\gamma)$ in $\mathbb{R}^2$. That is, the inversion map on $\mathbb{R}^2 - \{(0,0)\}$ preserves vertices of curves.

Proof. Notice that R_0 is an orientation reversing map. We consider a reflection map $F : (x,y) \mapsto (-x,y)$. Then $R = F \circ R_0$ becomes an orientation preserving map on $\mathbb{R}^2 - \{(0,0)\}$.

Let $M = \mathbb{R}^2 - \{(0,0)\}$ which is parametrized by R. Then by computation, the metric of M is given by

$$g(x,y) = \frac{dx^2+dy^2}{(x^2+y^2)^2},$$

where (x,y) is the local coordinate of M. Since g satisfies the condition (26), from Theorem 3.7 we know that the vertices of γ correspond to the vertices of its image curve $R(\gamma)$.

On the other hand, since R maps (x,y) to $\left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ in $\mathbb{R}^2 - \{(0,0)\}$, we may see the manifold (M,g) to be $(\mathbb{R}^2 - \{(0,0)\}, g_2)$, where

$$g_2\left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = \frac{dx^2+dy^2}{(x^2+y^2)^2}.$$

Using the change of parameter $(u,v) = \left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$, then $R\left(\frac{-u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) = (u,v)$ and we have

$$g_2(u,v) = \frac{du^2+dv^2}{(u^2+v^2)^2}, \text{ for any } (u,v) \in \mathbb{R}^2 - \{(0,0)\}.$$

The metric g_2 also satisfies the condition (26). Then from Theorem 3.8, the vertices of $R(\gamma)$ in $(M,g) = (\mathbb{R}^2 - \{(0,0)\}, g_2)$ correspond to the vertices of the same curve in $\mathbb{R}^2 - \{(0,0)\}$.

Finally, it is clearly that the reflection map F preserves vertices of curves. Therefore $R_0 = F \circ R$ also preserves vertices of curves. This completes the proof. □

Remark. One can also obtain that the translations $(x,y) \mapsto (x+c,y)$ and dilations $(x,y) \mapsto (ax,ay)$ also preserve vertices of curves in $\mathbb{R}^2$, for any $c, a > 0$. Actually, it is known in [8] that any orientation preserving Möbius transformation preserves vertices of curves.

5. Counterexamples on non-simply connected space forms

In this section we give two examples on the cylinder. One is a smooth regular simple closed curve on cylinder with only two vertices. The other is a smooth regular closed curve on cylinder, which bounds a compact immersed surface, also with only two vertices. These two examples show that the Four-Vertex Theorem of Kneser and Pinkall can not be extended on cylinder.

We consider here a flat cylinder $S = \mathbb{R}^2/\langle T_L\rangle$ for some translation $T_L : (x, y) \mapsto (x + L, y)$, $L \neq 0$. The universal covering of the cylinder is $\mathbb{R}^2$. Since $\langle T_L\rangle$ acts in a totally discontinuous manner on $\mathbb{R}^2$, the quotient space $\mathbb{R}^2/\langle T_L\rangle$ is a zero curvature two-manifold, whose metric is induced from the covering map $\eta : \mathbb{R}^2 \to \mathbb{R}^2/\langle T_L\rangle$ (see [3], p.165).

Since the covering map η is a local isometry, the number of the vertices of any smooth regular curve $\alpha : \mathbb{R}/l \to S$ in S is the same as the number of the vertices of any lifting curve of α in $[0, l)$. We will use this fact to count the number of vertices of the given smooth regular curve in S.

Here we give the first example. We construct a smooth regular closed curve on S as follows (Figure 2 shows its lifting curve in $\mathbb{R}^2$).

$$\alpha : \mathbb{R}/L\mathbb{Z} \to S, \quad \alpha(t) = \left(t, \sin\left(\frac{2\pi t}{L}\right)\right).$$

This is a closed curve on S since $T_L(\alpha(t)) = \alpha(t)$ for any $t \in [0, 2\pi)$.

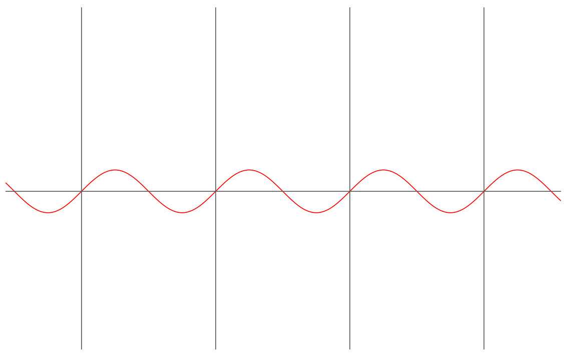

Figure 2.

By computation, the derivative of the curvature function k of α is given by

$$k'(t) = \frac{\left(\frac{2\pi}{L}\right)^3 \cos\left(\frac{2\pi t}{L}\right)\left[\left(\frac{2\pi}{L}\right)^2 \cos\left(\frac{4\pi t}{L}\right) - \left(1 + 2\left(\frac{2\pi}{L}\right)^2\right)\right]}{\left(1 + \left(\frac{2\pi}{L}\right)^2 \cos^2\left(\frac{2\pi t}{L}\right)\right)^{\frac{5}{2}}},$$

which has only two zeros $t = \frac{L}{4}$ and $t = \frac{3L}{4}$ in $[0, L)$. Hence, α has only two vertices. This example shows that Kneser's Four-Vertex Theorem can not be extended on any two-dimensional space form which is isometric to any cylinder.

Here we give the second example. This time we show that Pinkall's Four-Vertex Theorem can not be extended on any cylinder $S = \mathbb{R}^2/\langle T_L\rangle$, $L \neq 0$. We will construct a smooth regular closed curve γ in $\mathbb{R}^2$ with only two vertices and see it as one of the lifting curve of $\eta(\gamma)$. Then $\eta(\gamma)$ is also a smooth regular closed curve with only two vertices since the covering map η is a local isometry. Moreover, all the lifting curves of $\eta(\gamma)$ are $(T_L)^k(\gamma)$, $k \in \mathbb{Z}$. We will show that the curve $\eta(\gamma)$ bounds a compact immersed surface on the cylinder S by observing its lifting curves in the fundamental region $[0, L] \times \mathbb{R}$.

Let us start with construction of the curve. One of the lifting curve γ on $\mathbb{R}^2$ is given by

$$\gamma(t) = \frac{L}{5.5}\left[\frac{\tau(t)}{|\tau(t)|^2} - (2, 0)\right], \ t \in \mathbb{R}/(5\pi)\mathbb{Z},$$

where $\tau(t) = (\cos(t/5)\cos(t) + 0.08, \cos(t/5)\sin(t))$.

Here we show that γ has only two vertices in $[0, 5\pi)$. First, by computation, the derivative of the curvature function κ of τ is given by

$$\kappa'(t) = \frac{24\left(8 + 6\cos\left(\frac{2t}{5}\right)\right)\sin\left(\frac{2t}{5}\right)}{\left(13 + 12\cos\left(\frac{2t}{5}\right)\right)^{\frac{5}{2}}},$$

which has only two zeros $t = 0$ and $t = \frac{5\pi}{2}$ in $[0, 5\pi)$. Hence the smooth regular closed curve τ has only two vertices (Figure 3).

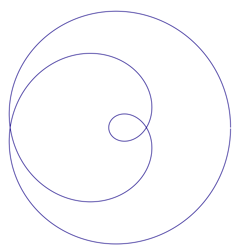

FIGURE 3.

Next, we construct γ by taking the inversion of τ, then a translation $(x, y) \mapsto (x-2, y)$, and then finally a dilation $(x, y) \mapsto (\frac{L}{5.5}x, \frac{L}{5.5}y)$. Since from Theorem 4.3 we know that the inversion map, translations, and dilations all preserve vertices of curves, γ has only two vertices in $[0, 5\pi)$. Figure 4 shows the curve γ and other lifting curves of $\eta(\gamma)$.

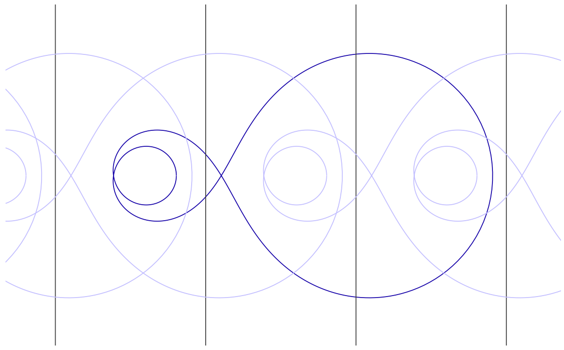

FIGURE 4.

The curve $\eta(\gamma)$ actually bounds a compact immersed surface with genus one. In the fundamental region $D = [0, L] \times \mathbb{R}$, the immersed surface T bounded by $\eta(\gamma)$ is shown in Figure 5, where the vertical lines at right and left hand sides of D are seen to be identical.

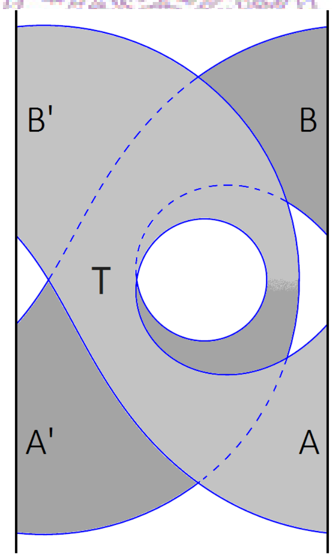

FIGURE 5.

Here we explain why $\eta(\gamma)$ bounds a compact immersed surface with genus one. In Figure 6, picture (a) shows an orientable compact surface M with genus one and a connected boundary C. Picture (b), (c), (d), (e) show continuous deformations of M. We can immerse the surface in picture (e), denoted by N, into a plane by an immersion $F : N \to \mathbb{R}^2$, $F(N) = K$.

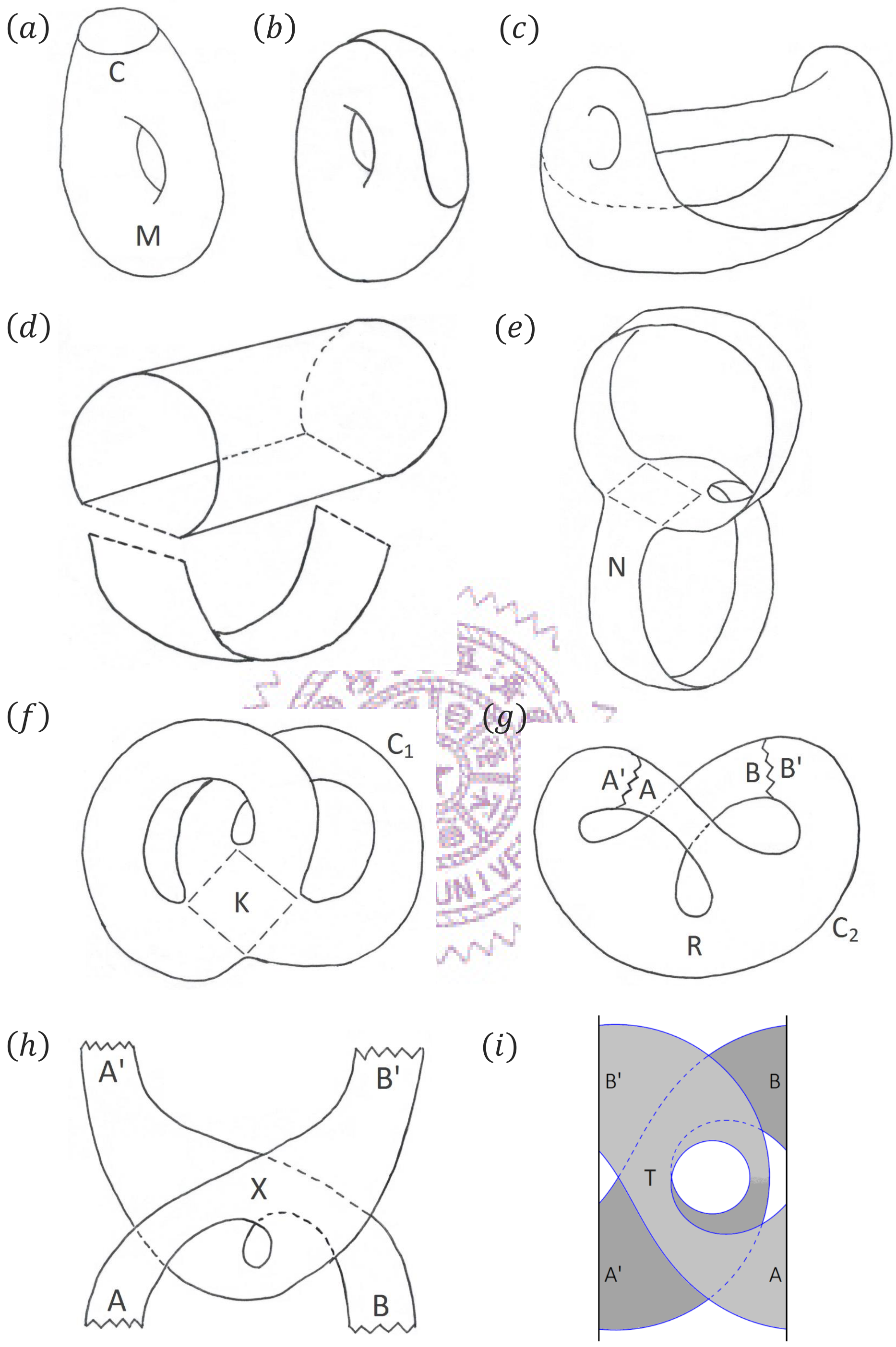

FIGURE 6.

Let C_1 be the boundary curve of K. Then from the above process, we know that after the immersion, the closed curve α bounds a compact immersed surface with genus one because any regular map $f : \partial N \to \mathbb{R}^2$ can be extended to the immersion F with $f(\partial N) = C_1$.

The closed curve C_2 in picture (g) is homotopic to the closed curve C_1 in picture (f). Hence C_2 also bounds a compact immersed surface with genus one. Let R be the immersed surface bounded by C_2. We cut R along the serrated lines which are shown in picture (g), where A, A', B, and B' represent the different sides near the cutting lines. Now we drag the side A to the bottom left, and side B to the bottom right, side A to the top left, and side B to the top right, then R becomes the immersed surface in picture (h), called X.

Finally, we put X on the cylinder and contact the side A with A', B with B'. Then X becomes the immersed surface T we have shown in Figure 5 (we show it again in picture (i)). The process from (g) to (i) shows that the immersed surface R is homotopic to T. Therefore, the boundary curve $\eta(\gamma)$ of T also bounds a compact immersed surface with genus one on the cylinder S.

Since $\eta(\gamma)$ has only two vertices, from this example we know that Pinkall's Four-Vertex Theorem can not be extended on any two-dimensional space form which is isometric to any cylinder.

Appendix: A Theorem of Jackson

In this section, we show the following theorem, which is first proved by Jackson [6]. As we state in Section 3, this result gives us a direction of the study on the Four-Vertex Theorem.

Theorem 5.1. *Let M be a complete Riemannian surface with non-constant sectional curvature K. Then for any non-stationary point p of K, that is , $dK_p \neq 0$, there exists $R > 0$ such that there are only two vertices on the geodesic circle centered at p with radius R.*

Before proving Theorem 5.1, we first compute the geodesic curvature of geodesic circles.

Lemma 5.2. *Given a two-dimensional complete manifold M. Let U be a neighborhood of a point $p \in M$ and γ_R a geodesic circle in U centered at p with radius R. Consider the geodesic polar coordinate (ρ, θ) on U, and ϕ the local parametrization on U with $\phi(0,0) = p$. Then the geodesic curvature k_R of γ_R is given by*

$$k_R(t) = \frac{1}{\sqrt{G}}(R,t)\frac{\partial\sqrt{G}}{\partial\rho}(R,t), \quad t \in [0, 2\pi].$$

Proof. Note that under the geodesic polar coordinate, the metric g is given by $g(\rho,\theta) = d\rho^2 + G(\rho,\theta)d\theta^2$ (see [2], page 287), where $G \geq 0$ and $G(\rho,\theta) = 0$ only at $\rho = 0$.

Let $\gamma_R(t) = \phi(R,t)$ be the geodesic circle centered at p with radius R, where $t \in [0, 2\pi]$.

Then the arc-length s of γ is

$$s(t) = \int_0^t |\gamma_R'(u)|\, du = \int_0^t \sqrt{G(R,u)}du$$

Hence $\gamma'(s) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta}$.

The covariant derivative of γ_R is

$$\begin{aligned}
\frac{D\gamma_R'(s)}{ds} &= D_{\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta}}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta}\right) = \frac{1}{\sqrt{G}}D_{\frac{\partial}{\partial\theta}}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta}\right)\\
&= \frac{1}{\sqrt{G}}\left[\left(\frac{\partial}{\partial\theta}\frac{1}{\sqrt{G}}\right)\frac{\partial}{\partial\theta} + \frac{1}{\sqrt{G}}\left(\Gamma_{22}^1\frac{\partial}{\partial\rho} + \Gamma_{22}^2\frac{\partial}{\partial\theta}\right)\right]\\
&= \frac{1}{\sqrt{G}}\left[-\frac{1}{2}G^{-\frac{3}{2}}G_\theta\frac{\partial}{\partial\theta} + \frac{1}{\sqrt{G}}\left(-\frac{G_\rho}{2}\frac{\partial}{\partial\rho} + \frac{G_\theta}{2G}\frac{\partial}{\partial\theta}\right)\right]\\
&= -\frac{G_\rho}{2G}\frac{\partial}{\partial\rho}.
\end{aligned}$$

Let $n(s) = -\frac{\partial}{\partial\rho}$, then we have

$$|n(s)| = 1, \quad \langle\gamma'(s), n(s)\rangle = 0, \quad \text{and the related matrix is } \begin{pmatrix} 0 & -1 \\ \frac{1}{\sqrt{G}} & 0 \end{pmatrix},$$

whose determinant is positive for any s. Therefore we may choose $\left\{\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta}, -\frac{\partial}{\partial\rho}\right\}$ to be the oriented base field along γ. Then by definition, the geodesic curvature of $\gamma_R(s)$ is

$$k_R = \left\langle -\frac{G_\rho}{2G}\frac{\partial}{\partial\rho}, -\frac{\partial}{\partial\rho}\right\rangle_g = \frac{G_\rho}{2G} = \frac{\left(\sqrt{G}\right)_\rho}{\sqrt{G}},$$

where $G = G(R,s)$. Reparametrizing γ_R by t, then we complete the proof. $\square$

Under the same assumptions in Lemma 5.2, we give the proof of Theorem 5.1.

Proof of Theorem 5.1. Let K be the sectional curvature of M.

For $R > 0$, we want to find the vertices of γ_R, that is, the zeros of $\frac{\partial k_R}{\partial\theta}$. From Lemma 5.2 we have

$$\frac{\partial k_R}{\partial\theta} = \frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial\rho\partial\theta} - \frac{1}{G}\frac{\partial\sqrt{G}}{\partial\rho}\frac{\partial\sqrt{G}}{\partial\theta}.$$

Since $G(\rho,\theta) = 0$ only at $\rho = 0$, it suffices to find the zeros of

$$\mu(R,\theta) := -G\frac{\partial k_R}{\partial\theta} = -\sqrt{G}\frac{\partial^2\sqrt{G}}{\partial\rho\partial\theta} + \frac{\partial\sqrt{G}}{\partial\rho}\frac{\partial\sqrt{G}}{\partial\theta}, \tag{27}$$

where R can be seen as a variable.

We may expand $\sqrt{G}$ in a power series in ρ. Recall that under the geodesic polar coordinate, it is known in [3] that

$$\sqrt{G}(0,\theta)=0,\quad \left(\sqrt{G}\right)_\rho(0,\theta)=1,\quad \text{for any } \theta\in[0,2\pi].$$

The other coefficients in the expansion may be computed from the above two identities by the Gauss equation(in the limit sense $r\to 0$, see [3]):

$$\left(\sqrt{G}\right)_{\rho\rho}(r,\theta)=-\sqrt{G}(r,\theta)K(r,\theta).$$

For examples,

$$\begin{aligned}
\frac{\partial^2\sqrt{G}}{\partial\rho^2}(0,\theta) &= -\sqrt{G}(0,\theta)K(0,\theta)=0,\\
\frac{\partial^3\sqrt{G}}{\partial\rho^3}(0,\theta) &= -\left(\sqrt{G}\right)_\rho(0,\theta)K(0,\theta)-\sqrt{G}(0,\theta)K_\rho(0,\theta)\\
&= -K(0,\theta),\\
\frac{\partial^4\sqrt{G}}{\partial\rho^4}(0,\theta) &= -\left(\sqrt{G}\right)_{\rho\rho}(0,\theta)K(0,\theta)-2\left(\sqrt{G}\right)_\rho(0,\theta)K_\rho(0,\theta)-\sqrt{G}(0,\theta)K_{\rho\rho}(0,\theta)\\
&= \sqrt{G}(0,\theta)K^2(0,\theta)-2\left(\sqrt{G}\right)_\rho(0,\theta)K_\rho(0,\theta)-\sqrt{G}(0,\theta)K_{\rho\rho}(0,\theta)\\
&= -2K_\rho(0,\theta),
\end{aligned}$$

and so on.

By this method, we obtain

(28)
$$\sqrt{G}(\rho,\theta)=\rho-\frac{1}{6}K(0,\theta)\rho^3-\frac{1}{12}(\frac{\partial K}{\partial\rho}(0,\theta))\rho^4+\frac{1}{120}(K^2(0,\theta)-3\frac{\partial^2K}{\partial\rho^2}(0,\theta))\rho^5+....$$

In order to use (28) to express the formula μ in (27), we take the partial derivatives of (28):

$$\begin{aligned}
\left(\sqrt{G}\right)_\rho &= 1-\frac{1}{2}K(0,\theta)\rho^2-\frac{1}{3}k_\rho(0,\theta)\rho^3+\frac{1}{24}(K^2(0,\theta)-3\frac{\partial^2K}{\partial\rho^2}(0,\theta))\rho^4+...\\
\left(\sqrt{G}\right)_\theta &= -\frac{1}{6}\frac{\partial}{\partial\theta}(K(0,\theta))\rho^3-\frac{1}{12}\frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right)\rho^4\\
&\quad +\frac{1}{120}\left(2K(0,\theta)\frac{\partial}{\partial\theta}\left(K(0,\theta)\right)-3\frac{\partial}{\partial\theta}\left(\frac{\partial^2K}{\partial\rho^2}(0,\theta)\right)\right)\rho^5+...\\
&= -\frac{1}{12}\frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right)\rho^4+\frac{1}{120}\left(2K(0,\theta)\frac{\partial}{\partial\theta}\left(K(0,\theta)\right)-3\frac{\partial}{\partial\theta}\left(\frac{\partial^2K}{\partial\rho^2}(0,\theta)\right)\right)\rho^5+...
\end{aligned}$$

$$\begin{aligned}
\left(\sqrt{G}\right)_{\rho\theta} &= -\frac{1}{2}\frac{\partial}{\partial\theta}(K(0,\theta))\rho^2 - \frac{1}{3}\frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right)\rho^3 \\
&\quad +\frac{1}{24}\left(2K(0,\theta)\frac{\partial}{\partial\theta}\left(K(0,\theta)\right) - 3\frac{\partial}{\partial\theta}\left(\frac{\partial^2 K}{\partial\rho^2}(0,\theta)\right)\right)\rho^4 + \dots \\
&= -\frac{1}{3}\frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right)\rho^3 + \frac{1}{24}\left(2K(0,\theta)\frac{\partial}{\partial\theta}\left(K(0,\theta)\right) - 3\frac{\partial}{\partial\theta}\left(\frac{\partial^2 K}{\partial\rho^2}(0,\theta)\right)\right)\rho^4 + \dots
\end{aligned}$$

where $\frac{\partial}{\partial\theta}(K(0,\theta)) = 0$ since $K(0,\theta) = K(p)$ is a constant function.

Substituting (27) by (28) and the above three, we obtain

$$\mu(R,\theta) = \frac{1}{4}\frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right)R^4 + \frac{1}{12}\frac{\partial}{\partial\theta}\left(\frac{\partial^2 K}{\partial\rho^2}(0,\theta)\right)R^5 + \dots.$$

Since we consider $R > 0$, to find the zeros of μ, it is sufficient to find the zeros of

$$\tilde{\mu}(R,\theta) := \frac{\partial}{\partial\theta}\left(\frac{\partial K}{\partial\rho}(0,\theta)\right) + \frac{1}{3}\frac{\partial}{\partial\theta}\left(\frac{\partial^2 K}{\partial\rho^2}(0,\theta)\right)R + \dots. \tag{29}$$

We now focus on the first term of $\tilde{\mu}$ in (29).

Consider the normal coordinate system (x, y) with change of variables:

$$W(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta) = (x, y),$$

and let $\psi = \phi \circ W^{-1}$ be the normal coordinate parametrization of the neighborhood U of p.

Moreover, we let C_1, C_2 and C be the curves

$$\begin{aligned}
C_1(s_1) &= \phi(s_1, 0) = \psi(s_1, 0), \\
C_2(s_2) &= \phi(s_2, \pi/2) = \psi(0, s_2), \\
C(s) &= \phi(s, \theta_0) = \psi(s\cos\theta_0, s\sin\theta_0),
\end{aligned}$$

where $\theta_0 \in [0, 2\pi]$ and s_1, s_2, s are the arc-length of C_1, C_2 and C respectively.

Then we have the following:

$$\begin{aligned}
\frac{dK(C_1(s_1))}{ds_1} &= \frac{\partial K}{\partial\rho}(C_1(s_1)) = \frac{\partial K}{\partial x}(C_1(s_1)), \\
\frac{dK(C_2(s_2))}{ds_2} &= \frac{\partial K}{\partial\rho}(C_2(s_2)) = \frac{\partial K}{\partial y}(C_2(s_2)), \\
\frac{dK(C(s))}{ds} &= \frac{\partial K}{\partial\rho}(C(s)) = \frac{\partial K}{\partial x}(C(s))\cos\theta_0 + \frac{\partial K}{\partial y}(C(s))\sin\theta_0.
\end{aligned}$$

This implies

$$\frac{\partial K}{\partial \rho}(s_1, 0) = \frac{\partial K}{\partial x}(s_1, 0), \tag{30}$$

$$\frac{\partial K}{\partial \rho}(s_2, \pi/2)) = \frac{\partial K}{\partial y}(0, s_2), \tag{31}$$

and

$$\frac{\partial K}{\partial \rho}(s, \theta_0) = \frac{\partial K}{\partial x}(C(s)) \cos\theta_0 + \frac{\partial K}{\partial y}(C(s)) \sin\theta_0. \tag{32}$$

From (30) and (31) we have

$$\frac{\partial K}{\partial \rho}(0,0) = \frac{\partial K}{\partial x}(0,0), \quad \text{and } \frac{\partial K}{\partial \rho}(0, \pi/2)) = \frac{\partial K}{\partial y}(0,0).$$

Apply on (32) by choosing $s = 0$, then we obtain

$$\begin{aligned} \frac{\partial K}{\partial \rho}(0, \theta_0) = & \frac{\partial K}{\partial x}(0,0) \cos\theta_0 + \frac{\partial K}{\partial y}(0,0) \sin\theta_0. \\ = & \frac{\partial K}{\partial \rho}(0,0) \cos\theta_0 + \frac{\partial K}{\partial \rho}(0, \frac{\pi}{2}) \sin\theta_0. \end{aligned} \tag{33}$$

Now we see θ_0 as a variable θ. Then by taking partial derivative with respect to θ in both sides, it becomes

$$\frac{\partial}{\partial \theta}\left(\frac{\partial K}{\partial \rho}(0, \theta)\right) = -\frac{\partial K}{\partial \rho}(0,0) \sin\theta + \frac{\partial K}{\partial \rho}(0, \frac{\pi}{2}) \cos\theta. \tag{34}$$

Therefore by (29) we have

$$\tilde{\mu}(0, \theta) = -\frac{\partial K}{\partial \rho}(0,0) \sin\theta + \frac{\partial K}{\partial \rho}(0, \frac{\pi}{2}) \cos\theta. \tag{35}$$

Note that if $\frac{\partial K}{\partial \rho}(0,0)$ and $\frac{\partial K}{\partial \rho}(0, \frac{\pi}{2})$ are both zero, by (34) we know that $\frac{\partial K}{\partial \rho}(0,0) \sin\theta = \frac{\partial K}{\partial \rho}(0, \frac{\pi}{2}) \cos\theta = 0$. And then by (33), $\frac{\partial K}{\partial \rho}(s, \theta) \equiv 0$. This implies that p is a stationary point of K and yields a contradiction. Therefore $\frac{\partial K}{\partial \rho}(0,0)$ and $\frac{\partial K}{\partial \rho}(0, \frac{\pi}{2})$ are not both zero. And then from (33) and (35), $\tilde{\mu}(0, \theta) = 0$ is equivalent to

$$\tan\theta = \frac{\frac{\partial K}{\partial \rho}(0, \frac{\pi}{2})}{\frac{\partial K}{\partial \rho}(0,0)}, \quad \theta \in [0, 2\pi). \tag{36}$$

From (36) we know that there are exactly two solutions, say θ_1 and $\theta_2 = \theta_1 + \pi$.

Consider another point $(0, \theta_3)$, $\theta_3 \neq \theta_1, \theta_2$. Then $\tilde{\mu}(0, \theta_3) \neq 0$. Since K is continuous, there exists a neighborhood V of $(0, \theta_3)$ such that $\tilde{\mu}(\rho, \theta) \neq 0$ on V.

On the other hand, for points $(0, \theta_i)$, $i = 1, 2$, by (36) we may write

$$\begin{aligned} \sin\theta_i &= \frac{\frac{\partial K}{\partial \rho}(0, \frac{\pi}{2})}{m}, \\ \cos\theta_i &= \frac{\frac{\partial K}{\partial \rho}(0, 0)}{m}, \end{aligned} \tag{37}$$

where m can be set as any non-zero value. Since p is not a stationary point of K, we may set $m^2 = \left(\frac{\partial K}{\partial \rho}_{(0,\frac{\pi}{2})}\right)^2 + \left(\frac{\partial K}{\partial \rho}_{(0,0)}\right)^2 \neq 0$.

Plugging (37) into the derivative of (34), we have

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial K}{\partial \rho}(0, \theta)\right) &= -\frac{\partial K}{\partial \rho}(0, 0)\cos\theta - \frac{\partial K}{\partial \rho}(0, \frac{\pi}{2})\sin\theta \\ &= \mp m\left(\cos\theta\cos\theta_i + \sin\theta\sin\theta_i\right) \\ &= \mp m\cos(\theta - \theta_i), \end{aligned}$$

where $-$, $+$ are with respect to $i = 1, 2$.

This implies $\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial K}{\partial \rho}(0, \theta_i)\right) = \mp m \neq 0$ and therefore

$$\frac{\partial \tilde{\mu}}{\partial \rho}(0, \theta_i) \neq 0, \text{ for any } i = 1, 2. \tag{38}$$

Since now θ_1 and θ_2 are zeros of $\tilde{\mu}$, from (38) and the implicit function theorem we know that for each $i = 1, 2$, there exists a neighborhood V_i of $(0, \theta_i)$ such that $\tilde{\mu}(\rho, \theta) = 0$ can be solved for θ uniquely. Therefore we can choose R small enough such that $\{(R, \theta) \,|\, \theta \in [0, 2\pi]\} \cap (V_1 \cup V_2) \neq \varnothing$. And then for any $r \leq R$, there exist only two zeros of $\mu(r, \theta) = 0$. That is, there exists r such that for any $0 < r \leq R$, the geodesic circle γ_R has only two vertices. $\square$

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