IV Integration

21. Definite and Indefinite Integrals (Riley 2.2.1+2)

21.1 Area under a Curve

The starting problem to be solved with integrals is to calculate the area between a function $y = f(x)$ and the x-axis, within an interval from $x = a$ to $x = b$. Apart from a constant or linear function, this is highly non-trivial and to solve it we use a limiting approach, similar to differentiation: We subdivide and approximate the area with N rectangular strips with widths $\Delta x = (a-b)/N$ and heights determined by the function value at the position of the strip, $f(x_k) = f(a + k\Delta x)$ with $k = 0, 1, \ldots, N-1$. Clearly, for a finite number of such strips, the sum of the rectangular areas is only an approximation of the area to be calculated. The approximation becomes better, the more strips are used, i.e. the larger N is. We now make the mathematical limit with N approaching infinity while $\Delta x = (a-b)/N$ approaches zero at the same time.

Definition 21.1 — Definite Integral. We thus write for the area

$$A_{ab} = \int_a^b f(x)dx \equiv \lim_{N\to\infty} \sum_{k=0}^{N-1} f(x_k)\Delta x, \tag{21.1}$$

where $x_k = a + k\Delta x$ and $\Delta x = (a-b)/N$. This is the definition of the **definite integral** of $f(x)$ between the limits $x = a$ and $x = b$. The definite integral results in a number corresponding to the area between the function curve, the x-axis and the limits. The function $f(x)$ inside the integral is called the **integrand**.

R The definite integral is negative for regions where the function value is negative. Also, if the limits are in the 'wrong' order, i.e. $b < a$, another negative sign is introduced.

21.2 The Indefinite Integral Function

From the definite integral, we can define a function $F(x)$, called the **indefinite integral**, where x is now the upper limit.

Definition 21.2 — Indefinite Integral. The indefinite integral is defined by

$$F(x) = \int_a^x f(u)du, \tag{21.2}$$

where a is an arbitrary value and u is a **dummy variable**. The indefinite integral is a function of x, the upper limit of the integration.

R The variable u is called a dummy variable because it is 'integrated over', i.e. the result on the left-hand side does not depend on it. One may thus choose any symbol for it that does not clash with anything else. In this case, we cannot use x as this now denotes the upper limit of integration.

Formula 21.3 — Basic Properties of Integrals. A couple of basic properties of definite and indefinite integrals can be easily derived from the definitions.

- Multiplying a function by a constant factor or adding/subtracting two functions simply changes the height and one has

$$\int_a^b (c_1 f(x) \pm c_2 g(x))dx = c_1 \int_a^b f(x)dx \pm c_2 \int_a^b g(x)dx, \quad c_1, c_2 \text{ are constants.} \tag{21.3}$$

- Splitting an integration interval simply divides the area and one has

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{21.4}$$

  Note that this relation holds for any values of a, b and c, i.e. it is not necessary that $a < b < c$.
- If the upper and lower integration limits are equal, the definite integral vanishes,

$$\int_a^a f(x)dx = 0. \tag{21.5}$$

- Reversing the order of the limits yields

$$\int_b^a f(x)dx = -\int_a^b f(x)dx. \tag{21.6}$$

- Integrating an **even function**, $f_{\text{even}}(x) = f_{\text{even}}(-x)$, over an interval symmetric with $x = 0$ yields

$$\int_{-a}^a f_{\text{even}}(x)dx = 2\int_0^a f_{\text{even}}(x)dx. \tag{21.7}$$

- Integrating an **odd function**, $f_{\text{odd}}(x) = -f_{\text{odd}}(-x)$, over an interval symmetric with $x = 0$ yields zero,

$$\int_{-a}^a f_{\text{odd}}(x)dx = 0, \tag{21.8}$$

  because the areas to the left and right of $x = 0$ are equal but of opposite sign to each other.

R It is worth mentioning that integration is much more 'forgiving' than differentiation. While the latter cannot be defined at points where the function is discontinuous (has a finite gap) or not smooth (has a kink), integration generally works fine. In practice, this is usually done by splitting up the integration interval at such special points as in Equation 21.4

■ **Example 21.1** Calculate the integral

$$\int_{-\pi}^{\pi} x\cos x dx.$$

The integrand is an odd function, $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$, and thus the integral vanishes,

$$\int_{-\pi}^{\pi} x\cos x dx = 0.$$

■

21.3 Integration as the Reverse of Differentiation

While differentiation and integration seem to be very different things, the crucial realization in calculus is that the two can be regarded as reverse operations from each other. To show what this means, we differentiate the above indefinite integral with respect to x. Using the basic definition of the derivative we find

$$\begin{aligned} \frac{dF(x)}{dx} = \frac{d}{dx}\left[\int_a^x f(u)du\right] &= \lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(u)du - \int_a^x f(u)du}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\int_a^x f(u)du + \int_x^{x+\Delta x} f(u)du - \int_a^x f(u)du}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(u)du}{\Delta x}, \end{aligned} \tag{21.9}$$

where we have split up the integration interval from a to $x+\Delta x$ into two parts: a to x and x to Δx. The key point now is that we can simplify the remaining integral in the numerator as

$$\int_x^{x+\Delta x} f(u)du \to f(x)\Delta x, \tag{21.10}$$

as Δx approaches zero, because the integral can be approximated better and better by a single rectangular slice with the area $f(x)\Delta x$ when $\Delta x \to 0$.

Formula 21.4 — Fundamental Theorem of Calculus. We can then easily calculate the limit which leads to the 'Fundamental Theorem of Calculus',

$$\frac{dF(x)}{dx} \equiv \frac{d}{dx}\left[\int_a^x f(u)du\right] = f(x). \tag{21.11}$$

Equivalently, we can also write

$$\int f(x)dx = F(x) + c. \tag{21.12}$$

In this schematic notation, we do not bother to write down upper and lower integration limits, and we do not worry about a dummy integration variable. Furthermore, the arbitrary constant

c indicates that the indefinite integral $F(x)$ can only be determined up to a constant because $\frac{d}{dx}(F(x)+c) = f(x)$ for any c. It is called the **constant of integration**.

We thus immediately have a means to calculate certain integrals. If we know the derivative of a function $F(x)$, $\frac{dF}{dx} = f(x)$, then $F(x)$ is the indefinite integral (or anti-derivative) of $f(x)$. For example, $\frac{d(x^2)}{dx} = 2x$, thus $\int(2x)dx = x^2 + c$.

Formula 21.5 — Definite from Indefinite Integrals. Having determined the indefinite integral, any definite integral, i.e. with arbitrary but fixed limits $x = a$ and $x = b$, can be easily calculated,

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx = \int_{x_0}^b f(x)dx - \int_{x_0}^a f(x)dx \\ &= F(b) - F(a) \equiv [F(x)]_a^b,\end{aligned} \tag{21.13}$$

where x_0 is an arbitrary fixed point used in the definition of the indefinite integral over $f(x)$.

■ **Example 21.2** Calculate the integral

$$I = \int_1^3 x\,dx.$$

From the above example, we know the indefinite integral $\int x dx = \frac{1}{2}x^2 + c$. Thus the definite integral is

$$I = \left[\frac{1}{2}x^2\right]_1^3 = \frac{1}{2}(3^2 - 1^2) = 4.$$

■

22. Integrals of Basic Functions (Riley 2.2.3+4)

In this section we obtain integrals by regarding integration as the reverse of differentiation. This method may be extended slightly by differentiating the result of an integration, and adjusting the result, so as to obtain the integrand that we started with.

Formula 22.1 — Integral of $\mathbf{x^n}$ $(\mathbf{n \neq -1})$. Since we have

$$\frac{d(x^n)}{dx} = nx^{n-1} \Leftrightarrow x^n = \int nx^{n-1} dx + c, \tag{22.1}$$

$$\Rightarrow \int x^n dx = \frac{1}{n+1} x^{n+1} + c, \tag{22.2}$$

therefore by redefining the power we have

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c. \tag{22.3}$$

This result is valid for any real n except $n = -1$, where the above fails.

Formula 22.2 — Integral of $\mathbf{1/x}$. Although the curve of $f(x) = 1/x$ has clearly an area between it and the x-axis, we have seen that the formula above fails for $n = -1$. Instead, we know that the natural logarithm $\ln x$ has the derivative

$$\frac{d \ln x}{dx} = \frac{1}{x}, \tag{22.4}$$

and thus the integral of $1/x$ is

$$\int \frac{1}{x} dx = \ln|x| + c. \tag{22.5}$$

R The absolute value of x in the above result is introduced to generalise the integral to apply to negative x as well. While the natural logarithm is not defined for $x < 0$, $1/x$ is and there is a corresponding (negative) area as in the positive x case. Care must only be taken if the integration interval contains $x = 0$ where the integrand goes to infinity. Such **improper integrals** will be covered briefly later.

Formula 22.3 — Integral of e^x and a^x. The natural exponential is its own derivative,

$$\frac{d(e^x)}{dx} = e^x, \tag{22.6}$$

therefore

$$\int e^x dx = e^x + c. \tag{22.7}$$

Consider now the general exponential a^x. We first calculate its derivative by expressing it in terms of the natural exponential using $a = e^{\ln a}$,

$$a^x = (e^{\ln a})^x = e^{x \ln a}, \tag{22.8}$$

and therefore via the chain rule,

$$\frac{d(a^x)}{dx} = \frac{d(e^{x \ln a})}{dx} = e^{x \ln a} \ln a = a^x \ln a. \tag{22.9}$$

Therefore the integral of a^x is

$$\int a^x dx = \frac{a^x}{\ln a} + c. \tag{22.10}$$

Formula 22.4 — Integrals of $\sin x$ and $\cos x$. We have seen that

$$\frac{d(\sin x)}{dx} = \cos x, \text{ and} \tag{22.11}$$

$$\frac{d(\cos x)}{dx} = -\sin x. \tag{22.12}$$

Therefore the integrals of sine and cosine are

$$\int \sin x \, dx = -\cos x + c, \text{ and} \tag{22.13}$$

$$\int \cos x \, dx = \sin x + c. \tag{22.14}$$

By taking derivatives it is also immediate to verify that:

$$\int \tan x \, dx = -\ln|\cos x| + c, \tag{22.15}$$

$$\int \cos x \sin^n x \, dx = \frac{\sin^{n+1} x}{n+1} + c, \tag{22.16}$$

$$\int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1} + c. \tag{22.17}$$

■ **Example 22.1** To calculate the integral $\int \sin^5 x dx$, we can re-write the integrand to apply the rules above,

$$
\begin{aligned}
\int \sin^5 x\, dx &= \int \sin^4 x \sin x\, dx \\
&= \int (1-\cos^2 x)^2 \sin x\, dx \\
&= \int (1-2\cos^2 x+\cos^4 x) \sin x\, dx \\
&= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x) dx \\
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.
\end{aligned}
$$

■

Formula 22.5 — Integration of Hyperbolic Functions. Integrals of hyperbolic functions are given by

$$\int \sinh x\, dx = \cosh x + c, \tag{22.18}$$

$$\int \cosh x\, dx = \sinh x + c, \tag{22.19}$$

$$\int \tanh x\, dx = \ln|\cosh x| + c. \tag{22.20}$$

Formula 22.6 — Integrals producing Inverse Trigonometric and Hyperbolic Functions. Using the anti-derivative argument, we can immediately verify the following integrals resulting in inverse trigonometric or hyperbolic functions:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(x/a) + c = -\arccos(x/a) + c', \tag{22.21}$$

$$\int \frac{a\, dx}{a^2+x^2} = \arctan(x/a) + c, \tag{22.22}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{arcsinh}(x/a) + c, \tag{22.23}$$

$$\int \frac{a\, dx}{a^2-x^2} = \operatorname{arctanh}(x/a) + c. \tag{22.24}$$

For the first integral, note the relation $\arcsin x = -(\arccos x - \pi/2)$.

■ **Example 22.2**

$$\int_a^b \frac{dx}{1+x^2} = [\arctan x]_a^b = \arctan b - \arctan a.$$

For example, as $a \to -\infty$ and $b \to \infty$, $\arctan x$ approaches $-\pi/2$ and $\pi/2$, respectively. Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Thus the area under the whole curve $1/(1+x^2)$ is exactly π. ■

The above is an example of an **improper** integral, where the integration limits are not finite numbers but instead approach infinity. Strictly speaking such integrals need to be calculated as a limit. For example consider

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2} &\equiv \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} \\
&= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b \\
&= \lim_{b\to\infty}\left(1-\frac{1}{b}\right) \\
&= 1
\end{aligned} \tag{22.25}$$

Thus the area approaches the finite value $A = 1$ as the upper limits becomes larger and larger.

A second class of improper integrals occurs when an integration is calculated with a limit where the integrand is not defined (or more generally if the integration interval contains such points). For example, $f(x) = 1/x$ is not defined at $x = 0$ and thus

$$\begin{aligned}
\int_0^1 \frac{dx}{x} &\equiv \lim_{a\to 0}\int_a^1 \frac{dx}{x} \\
&= \lim_{a\to 0}\left[\ln x\right]_a^1 \\
&= \lim_{a\to 0}(-\ln a) \to +\infty.
\end{aligned} \tag{22.26}$$

The integral in this case is not defined; it approaches infinity. In other words, the area under the curve $1/x$ between $x = 0$ and $x = 1$ is infinitely large.

23. Integration Techniques

As opposed to differentiation, where the derivative can usually be calculated following a straightforward set of rules, integration often requires an inspired way of transforming the integrand or a rather roundabout approach to the problem. There are numerous techniques to calculate integrals analytically and we here discuss only a few of the most fundamental ones.

It is also important to mention that not every integral can actually be calculated analytically; this means that while the integral $\int f(x)dx$ exists (there is a well-defined area under the curve $y = f(x)$), it is not possible to write $F(x)$ in terms of other elementary mathematical functions. For example, the integral

$$\int \frac{2}{\sqrt{\pi}} e^{-x^2} dx, \tag{23.1}$$

while being deceptively simple and certainly well-defined (the integrand describes a Gaussian bell curve), cannot be expressed in terms of (combinations of) exponential, trigonometric, etc. functions. Instead, the integral **defines** a new, special function, namely the error function $\mathrm{erf}(x)$,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du. \tag{23.2}$$

Note that for some such integrals, specific definite integrals may still be found analytically, e.g. in the above case one has

$$\int_0^\infty \frac{2}{\sqrt{\pi}} e^{-x^2} dx = 1. \tag{23.3}$$

23.1 Substitution (Riley 2.2.7)

The first technique we discuss is related to the chain rule of differentiation.

Formula 23.1 — Integration by Substitution. One can attempt to view the integrand as a nested function $f(u(x))$ and transform the integral over x as an integral over u,

$$\int f(u(x))dx = \int \frac{f(u)}{u'}du. \tag{23.4}$$

The appearance of the derivative $u'(x)$ in the denominator is due to the transformation of the **differential** $du/dx = u'(x) \Rightarrow dx = du/u'(x)$ in the process. Note that the resulting integrand must be expressed in terms of u only.

When calculating a definite integral, it is also necessary to transform the integration limits accordingly,

$$\int_a^b f(u(x))dx = \int_{u(a)}^{u(b)} \frac{f(u)}{u'}du. \tag{23.5}$$

■ **Example 23.1** A straightforward simplification is achieved for integrands of the form $f(ax+b)$. For example,

$$I = \int \frac{1}{3x+2}dx.$$

We substitute

$$u = 3x+2 \Rightarrow du = 3dx,$$

and determine the integral

$$I = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u|+c = \frac{1}{3}\ln|3x+2|+c.$$

In the last step, we re-inserted $u \to 3x+2$, as the final result needs to be a function of x. ■

■ **Example 23.2**

$$I = \int \sin x \; \cos x \; dx.$$

We could use the standard formula for $\int \sin x \cos^n x dx$. However, it can also be easily solved directly. By substituting

$$u = \sin x \Rightarrow du = \cos x dx,$$

we obtain

$$I = \int u du = \frac{u^2}{2}+c = \frac{1}{2}\sin^2 x+c.$$

Alternatively, we could have substituted

$$u = \cos x \Rightarrow du = -\sin x dx,$$

giving

$$I = -\int u du = -\frac{u^2}{2}+c = -\frac{1}{2}\cos^2 x+c'.$$

While seemingly different, the two results differ only by a constant,

$$\frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 x+\frac{1}{2},$$

which is irrelevant when calculating an indefinite integral. ■

■ Example 23.3

$$I = \int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

We substitute

$$u = \cos x \Rightarrow du = -\sin x dx,$$

and obtain

$$I = \int \frac{-1}{u} du = -\ln |u| + c = -\ln |\cos x| + c.$$

■

Trigonometric substitutions
Integrals of the form

$$I = \int \frac{dx}{a + b\cos x} \quad \text{and} \quad I = \int \frac{dx}{a + b\sin x}, \tag{23.6}$$

can be solved by making the less obvious substitution

$$t = \tan\left(\frac{x}{2}\right).$$

Thus,

$$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1 + t^2), \tag{23.7}$$

and

$$dx = \frac{2dt}{1 + t^2}. \tag{23.8}$$

Sine and cosine can now be expressed in terms of t as

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2t}{1 + t^2}, \tag{23.9}$$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - t^2}{1 + t^2}. \tag{23.10}$$

■ Example 23.4

$$I = \int_{-\pi/2}^{\pi/2} \frac{2}{1 + 3\cos x} dx.$$

We substitute

$$t = \tan\frac{x}{2} \Rightarrow dx = \frac{2}{1 + t^2} dt,$$

and thus as above,

$$\cos x = \frac{1 - t^2}{1 + t^2}.$$

Also, the integration limits are substituted as $-\pi/2 \to \tan(-\pi/4) = -1$ and $\pi/2 \to \tan(\pi/4) = 1$. Altogether we get the integral

$$
\begin{aligned}
I &= \int_{-1}^{1} \frac{2}{1+3\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt \\
&= \int_{-1}^{1} \frac{4}{1+t^2+3(1-t^2)} dt \\
&= \int_{-1}^{1} \frac{2}{2-t^2} dt \\
&= \int_{0}^{1} \frac{4}{2-t^2} dt,
\end{aligned}
$$

where the last simplification is due to the integrand being even. This can be cast in the form of the standard integral

$$
\begin{aligned}
I &= 2\sqrt{2} \int_{0}^{1} \frac{\sqrt{2}}{\sqrt{2}^2 - t^2} dt \\
&= 2\sqrt{2} \left[\operatorname{arctanh}\left(\frac{t}{\sqrt{2}} \right) \right]_0^1 \\
&= 2\sqrt{2} \operatorname{arctanh}\left(\frac{1}{\sqrt{2}} \right).
\end{aligned}
$$

Alternatively, we can also factor the denominator and write

$$
\begin{aligned}
I &= \int_{0}^{1} \frac{4}{(\sqrt{2}-t)(\sqrt{2}+t)} dt \\
&= \sqrt{2} \int_{0}^{1} \left(\frac{1}{\sqrt{2}-t} + \frac{1}{\sqrt{2}+t} \right) dt \\
&= \sqrt{2} \left[-\ln(\sqrt{2}-t) + \ln(\sqrt{2}+t) \right]_0^1 \\
&= \sqrt{2} \left[\ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t} \right) \right]_0^1 \\
&= \sqrt{2} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right).
\end{aligned}
$$

The two results are in fact identical due to the relation

$$
\operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x} \right).
$$

■

■ Example 23.5

$$
I = \int \frac{1}{3+\cos^2 x} dx.
$$

We change variable,

$$
\begin{aligned}
t = \tan x \Rightarrow \frac{dt}{dx} = 1 + \tan^2 x \Rightarrow dx = \frac{dt}{1+t^2}, \\
\Rightarrow \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2},
\end{aligned}
$$

and thus

$$
\begin{aligned}
I &= \int \frac{1}{3+\frac{1}{1+t^2}} \frac{dt}{1+t^2} \\
&= \int \frac{dt}{4+3t^2} \\
&= \frac{1}{3} \int \frac{dt}{4/3+t^2} \\
&= \frac{1}{3} \frac{\sqrt{3}}{2} \arctan\left(\frac{t}{\frac{2}{\sqrt{3}}}\right) + c \\
&= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{2}\tan x\right) + c.
\end{aligned}
$$

■

23.2 Integration by Parts (Riley 2.2.8)

The method of integration by parts is the analogue of the product rule for differentiation and indeed we start from there,

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x). \tag{23.11}$$

We take the indefinite integral of this (note the need to replace x with a different, dummy variable t inside the integrals),

$$\int_a^x (u(t)v(t))'dt = \int_a^x u'(t)v(t)dt + \int_a^x u(t)v'(t)dt. \tag{23.12}$$

Due to the fundamental theorem of calculus, the left-hand side is

$$\int_a^x (u(t)v(t))'dt = u(x)v(x) + c. \tag{23.13}$$

Thus rearranging we get

$$\int_a^x u(t)v'(t)dt = u(x)v(x) + c - \int_a^x u'(t)v(t)dt. \tag{23.14}$$

The integration constant c is irrelevant as the above relates two indefinite integrals.

Formula 23.2 — Integration by Parts. In short-hand notation the above reads

$$\int uv'dx = uv - \int u'vdx. \tag{23.15}$$

For a definite integral, the integration by parts relation reads

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx. \tag{23.16}$$

Therefore for a product we assign the first term as u and the second as v' where the former should be easy to differentiate and the latter easy to integrate. For the method to be useful, the remaining integral over $u'v$ should be easier to determine.

■ **Example 23.6**

$$I = \int xe^x dx.$$

We identify u and v as follows:

$$\begin{aligned} u &= x \quad &\Leftrightarrow \quad u' &= 1, \\ v &= e^x \quad &\Leftrightarrow \quad v' &= e^x. \end{aligned}$$

Thus

$$I = xe^x - \int 1 \times e^x dx = (x-1)e^x + c.$$

■

■ **Example 23.7**

$$I = \int \arctan x \, dx.$$

The integrand is not a product but we can intepret it as such with 1 as a factor! We thus identify u and v as follows:

$$\begin{aligned} u &= \arctan x \quad &\Leftrightarrow \quad u' &= \frac{1}{1+x^2}, \\ v &= x \quad &\Leftrightarrow \quad v' &= 1. \end{aligned}$$

Thus

$$I = x \arctan x - \int \frac{x}{1+x^2} dx.$$

To determine the second integral, we substitute

$$t = 1 + x^2 \Rightarrow dt = 2x dx,$$

to get

$$\int \frac{x}{1+x^2} dx = \int \frac{1}{t} \frac{dt}{2} = \frac{1}{2} \ln|t| + c = \frac{1}{2} \ln(1+x^2) + c.$$

Note that the absolute value around the argument of the logarithm can be dropped as $1 + x^2 > 0$ for all x. We finally get

$$I = x \arctan x - \frac{1}{2} \ln(1+x^2) + c.$$

■

■ **Example 23.8**

$$I = \int \ln x \, dx.$$

We apply the same trick and insert a factor of 1. We then identify

$$\begin{aligned} u &= \ln x \quad &\Leftrightarrow \quad u' &= \frac{1}{x}, \\ v &= x \quad &\Leftrightarrow \quad v' &= 1. \end{aligned}$$

This gives

$$I = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x(\ln x - 1) + c.$$

■

23.3 Partial Fractions (Riley 2.2.6)

First let us consider expressing proper fractional functions in terms of partial fractions[1]. The procedure we adopt is justified by considering that denominators like $(x+1)^2$ could result from partial fractions with a denominator of the form $(x+1)$ as well as its square. Also, quadratic denominators will in general have terms linear in x in the numerator. We do a series of illustrative examples.

■ Example 23.9

$$\frac{x+3}{(x-2)(x+4)} \equiv \frac{A}{x-2} + \frac{B}{x+4} = \frac{A(x+4)+B(x-2)}{(x-2)(x+4)},$$

which means

$$x+3 = (A+B)x + (4A-2B).$$

Choosing $x=2$ to eliminate the B term we find $A = \frac{5}{6}$. Similarly choosing $x=-4$ to eliminate A, we find $B = \frac{1}{6}$. ■

■ Example 23.10

$$\frac{x^2-3}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

which means

$$x^2-3 = A(x^2+1)+(Bx+C)(x-1) = (A+B)x^2+(C-B)x+(A-C).$$

Substituting $x=1$ so as to eliminate B and C gives $A=-1$ and substituting $x=0$ will eliminate B. Since $A=-1$, we find $C=2$ and substituting any other value for x, a small value is sensible, will determine $B=2$. ■

■ Example 23.11

$$\frac{x-1}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{D}{(x-2)^2},$$

which means

$$x-1 = A(x-2)^2+B(x+1)(x-2)+D(x-1).$$

Choosing the obvious values for x determines D to be $\frac{1}{3}$ and A to be $-\frac{2}{9}$. Then comparing the coefficients of x^2 determines B to be $\frac{2}{9}$. ■

We now consider the use of partial fractions for integration. We express a fractional functions in terms of two or more simpler fractional functions on which integration can be done more easily. The is best demonstrated in a specific case, and we continue with Example 23.9:

$$\begin{aligned} I &= \int \frac{x+3}{(x-2)(x+4)} dx \\ &= \int \left(\frac{5}{6(x-2)} + \frac{1}{6(x+4)} \right) dx \\ &= \frac{5}{6} \int \frac{1}{x-2} dx + \frac{1}{6} \int \frac{1}{x+4} dx \\ &= \frac{5}{6} \ln|x-2| + \frac{1}{6} \ln|x+4| + c. \end{aligned} \tag{23.17}$$

[1] In a **proper** fraction the degree of the numerator is less than that of the denominator. For an **improper** fraction the degree of the numerator is greater or equal to that of the denominator.

We can now also perform a complete classification of the integration of fractions with quadratic polynomials as the denominator,

$$I = \int \frac{dx}{ax^2+bx+c}. \tag{23.18}$$

The result depends on how many solutions the equation $ax^2+bx+c=0$ has, i.e. it depends on the **discriminant** b^2-4ac:

a) If $b^2-4ac>0$, there are two solutions $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and the polynomial factorizes,

$$ax^2+bx+c = a(x-x_1)(x-x_2). \tag{23.19}$$

The resulting fraction is then decomposed into partial fractions, and the integration proceeds as outlined above.

b) If $b^2-4ac=0$, there is one solution and we can write

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2, \tag{23.20}$$

and thus

$$I = \int \frac{dx}{a(x+b/2a)^2} = -\frac{1}{ax+b/2}+\text{const.} \tag{23.21}$$

c) If $b^2-4ac<0$, no solution exists but we can 'complete the square', i.e. write the polynomial in the form

$$ax^2+bx+c = a(x-A)^2+B^2, \tag{23.22}$$

which can be integrated yielding arctan after substitution.

■ **Example 23.12**

$$I = \int \frac{dx}{x^2-2x+5}.$$

The discriminant is <0 in this case so we can complete the square and write the integral as

$$I = \int \frac{dx}{(x-1)^2+4}.$$

We substitute $u=x-1$, thus $du=dx$ and the integral becomes

$$I = \int \frac{du}{u^2+4}.$$

We recall the standard integral

$$\int \frac{a\,dx}{a^2+x^2} = \arctan\left(\frac{x}{a}\right)+c,$$

from which we can derive

$$I = \frac{1}{2}\arctan\frac{u}{2}+c = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right)+c.$$

■

23.4 Differentiation with Respect to a Parameter

Integrands that contain one or more parameters, in addition to the integration variable x, may be simplified first by taking the derivative with respect to such a parameter a, or treating the integrand as the derivative already. The resulting function can then be integrated with respect to x, which then is integrated/differentiated with respect to a to yield the desired result. This procedure is valid as the x and a can be treated as independent from each other. This is best demonstrated in an example:

$$I = \int xe^{-ax}dx. \tag{23.23}$$

We notice that

$$xe^{-ax} = -\frac{d}{da}\left(e^{-ax}\right), \tag{23.24}$$

and hence

$$I = -\int \frac{d}{da}e^{-ax}dx. \tag{23.25}$$

As the differentiation with respect to a is an independent operation from the integration with respect to x, we can do whichever we prefer first and then the other. Therefore we have

$$\begin{aligned} I &= -\frac{d}{da}\int e^{-ax}dx \\ &= -\frac{d}{da}\left(\frac{e^{-ax}}{-a}\right) + c(a) \\ &= \frac{d}{da}\left(\frac{e^{-ax}}{a} + c(a)\right) \\ &= -\frac{xe^{-ax}}{a} - \frac{e^{-ax}}{a^2} + c'(a) \\ &= -e^{-ax}\frac{1+ax}{a^2} + c. \end{aligned} \tag{23.26}$$

Note the appearance of the integration constant c in this context. While a constant with regard to x it will in general be a function of the parameter a, $c = c(a)$. We include it throughout but simply write it as a constant in the final result as per usual for an indefinite integral, $c = c'(a)$. It may depend on a but it is still just an arbitrary expression which e.g. will drop out when calculating definite integrals from this.

Example 23.13

$$I = \int_0^1 \frac{x^2-1}{\ln x}dx.$$

While this definite integral does not contain a parameter, we can treat is as a special case of the more general integral

$$I(b) = \int_0^1 \frac{x^b-1}{\ln x}dx, \quad b > -1.$$

We proceed by first calculating the derivative with respect to b,

$$\begin{aligned}
I'(b) &= \frac{d}{db}\left[\int_0^1 \frac{x^b-1}{\ln x}dx\right] \\
&= \int_0^1 \left[\frac{d}{db}\left(\frac{x^b-1}{\ln x}\right)\right] dx \\
&= \int_0^1 \left[\frac{1}{\ln x}\frac{d(x^b)}{db}\right] dx \\
&= \int_0^1 \left[\frac{1}{\ln x}(x^b \ln x)\right] dx \\
&= \int_0^1 x^b dx = \left[\frac{x^{b+1}}{b+1}\right]_0^1 \\
&= \frac{1}{b+1}.
\end{aligned}$$

We integrate this over b to calculate $I(b)$,

$$I(b) = \int \frac{db}{b+1} = \ln(b+1) + c.$$

The integration constant can in this case be determined by considering a special case: when $b = 0$, the integral $I(b = 0)$ vanishes as the numerator $x^0 - 1$ is identically zero. This fixes $c = 0$. Finally, the solution to the original integral is

$$I = I(2) = \ln(3).$$

■

23.5 Reduction Formulae (Riley 2.2.9)

The determination of a reduction formula to solve an integral is not an integration technique per se; instead, it describes a process in which a given integral can be iteratively determined through successive simplification. Again, this is best demonstrated in an example.

Consider the class of definite integrals described by

$$I_n = \int_0^1 (1-x^3)^n dx, \tag{23.27}$$

where n is a positive integer or zero. We can attempt to solve it as follows:

$$\begin{aligned}
I_n &= \int_0^1 (1-x^3)(1-x^3)^{n-1} dx \\
&= \int_0^1 (1-x^3)^{n-1} dx - \int_0^1 x^3(1-x^3)^{n-1} dx \\
&= I_{n-1} - \int_0^1 xx^2(1-x^3)^{n-1} dx, \qquad (23.28)
\end{aligned}$$

where we have identified the first integral on the second line with the definition of I_{n-1}. We can evaluate the remaining integral by parts, identifying

$$\begin{aligned}
u &= x & &\Leftrightarrow & u' &= 1, \\
v &= \frac{(1-x^3)^n}{-3n} & &\Leftrightarrow & v' &= x^2(1-x^3)^{n-1}.
\end{aligned}$$

This gives

$$\begin{aligned} I_n &= I_{n-1} + \left[\frac{x}{3n}(1-x^3)^n\right]_0^1 - \frac{1}{3n}\int_0^1 (1-x^3)^n dx \\ &= I_{n-1} - \frac{1}{3n} I_n, \end{aligned} \tag{23.29}$$

where we again identified the remaining integral with the definition of I_n. While we have not really found a closed result, we can express the integral I_n in terms of the integral I_{n-1},

$$I_n = \frac{3n}{3n+1} I_{n-1}. \tag{23.30}$$

The strategy now is to determine I_n iteratively, i.e. in terms of I_{n-1} which can be determined from I_{n-2}, etc. The lowest integral, I_0, is easy to calculate,

$$I_0 = \int_0^1 (1-x^3)^0 dx = \int_0^1 dx = [x]_0^1 = 1. \tag{23.31}$$

All the other integrals then follow from this,

$$\begin{aligned} I_1 &= \frac{3\times 1}{3\times 1+1} \times 1 = \frac{3}{4}, \\ I_2 &= \frac{3\times 2}{3\times 2+1} \times \frac{3}{4} = \frac{9}{14}, \\ I_3 &= \frac{3\times 3}{3\times 3+1} \times \frac{9}{14} = \frac{81}{140}, \\ &\vdots \\ I_n &= \frac{3n}{3n+1}\frac{3(n-1)}{3(n-1)+1} \cdots \frac{6}{7}\frac{3}{4}. \end{aligned}$$

23.6 Extension to Complex Numbers (Riley 3.6)

Some functions can be considered as the real or imaginary part of a more general complex number expression. Performing integration (or differentiation) on the 'complexified' can be considerably easier. For example, consider the integral

$$I = \int e^{ax} \cos(bx) dx. \tag{23.32}$$

This could be solved by doing integration by parts twice. Instead, we remember Euler's Equation,

$$e^{ibx} = \cos(bx) + i\sin(bx), \tag{23.33}$$

which allows us to express the integrand as the real part of

$$e^{ax}(\cos(bx) + i\sin(bx)) = e^{ax}e^{ibx} = e^{(a+ib)x}. \tag{23.34}$$

We therefore have to integrate over a single exponential only, and we bring the resulting complex expression in a form separating real and imaginary parts,

$$\begin{aligned} \int e^{(a+ib)x} dx &= \frac{e^{(a+ib)x}}{a+ib} + c \\ &= \frac{e^{(a+ib)x}}{(a+ib)}\frac{a-ib}{a-ib} + c \\ &= \frac{e^{ax}}{a^2+b^2}(\cos(bx) + i\sin(bx))(a-ib) + c \\ &= \frac{e^{ax}}{a^2+b^2}[(a\cos(bx) + b\sin(bx)) + i(a\sin(bx) - b\cos(bx))] + c. \end{aligned} \tag{23.35}$$

The original integral is now simply the real part of this,

$$\int e^{ax}\cos(bx)dx = \mathrm{Re}\left[\int e^{(a+ib)x}dx\right] = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx)+b\sin(bx)\right)+c. \tag{23.36}$$

Analogously, the following integral corresponds to the imaginary part of the complex expression,

$$\int e^{ax}\sin(bx)dx = \mathrm{Im}\left[\int e^{(a+ib)x}dx\right] = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx)-b\cos(bx)\right)+c. \tag{23.37}$$

Exercise 23.1 Calculate the integral $\int x^i dx$, by treating the imaginary unit i as any other constant. Use the result to determine the integrals $\int \cos(\ln x)dx$ and $\int \sin(\ln x)dx$. ■

24. Applications of Integration (Riley 2.2.13)

24.1 Average of a Function

The average of a set of N numbers $y_1, y_2, y_3, \ldots, y_N$ (e.g. experimental measurements of a quantity) is given by

$$\langle y \rangle = \frac{1}{N}\sum_{k=1}^{N} y_k. \tag{24.1}$$

We can extend this to the average of a continuous function $f = f(x)$ over an interval from $x = a$ to $x = b$. Imagine sampling function values $y_k = f(x_k)$ at equally spaced points within the interval, x_1, $x_2, \ldots, x_n$ with spacing $\Delta x = (b-a)/N$, and taking the average,

$$\langle f \rangle = \frac{1}{N}\sum_{k=1}^{n} f(x_k). \tag{24.2}$$

We notice that $1/N = \Delta x/(b-a)$ and write the average $\langle f \rangle$ as

$$\langle f \rangle = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x. \tag{24.3}$$

Formula 24.1 — Average Function Value over an Interval. As $N \to \infty$, the function average over the interval $[a,b]$ can be expressed in terms of its integral as

$$\langle f \rangle = \frac{1}{b-a}\int_a^b f(x)dx. \tag{24.4}$$

The average function value $\langle f \rangle$ can be geometrically interpreted as the height of a rectangle with width $(b-a)$ that has the the same area as the area under the function curve over the same interval,

$$A = \langle f \rangle (b-a) = \int_a^b f(x)dx. \tag{24.5}$$

■ **Example 24.1** Determine the time-averaged electrical power generated by an alternating current $I(t) = I_0 \cos(2\pi f t)$ and a voltage $V(t) = V_0 \cos(2\pi f t + \delta)$ over one cycle, i.e. from $t = 0$ to $t = T = 1/f$. Here, I_0 and V_0 are the peak current and voltage, respectively, f is the frequency and $0 \leq \delta \leq \pi/2$ is the phase-shift of the voltage with respect to the current.

The electrical power is given $P(t) = I(t)V(t)$ and thus the time-averaged power over a cycle from $t = 0$ to $t = T = 1/f$ is

$$\begin{aligned}
\langle P \rangle &= \frac{1}{T}\int_0^T I_0 V_0 \cos(2\pi f t)\cos(2\pi f t + \delta) dt \\
&= \frac{I_0 V_0}{T}\int_0^T \cos(2\pi f t)(\cos(2\pi f t)\cos\delta - \sin(2\pi f t)\sin\delta) dt \\
&= \frac{I_0 V_0}{T}\int_0^T \left(\cos^2(2\pi f t)\cos\delta - \cos(2\pi f t)\sin(2\pi f t)\sin\delta\right) dt \\
&= \frac{I_0 V_0}{T}\int_0^T \left(\frac{1+\cos(4\pi f t)}{2}\cos\delta - \frac{\sin(4\pi f t)}{2}\sin\delta\right) dt \\
&= \frac{I_0 V_0}{T}\left\{\left[\frac{\cos\delta}{2}t\right]_0^T + \underbrace{\left[\frac{\sin(4\pi f t)}{2\times 4\pi f}\cos\delta\right]_0^T}_{=0} + \underbrace{\left[\frac{\cos(4\pi f t)}{2\times 4\pi f}\sin\delta\right]_0^T}_{=0}\right\} \\
&= \frac{I_0 V_0}{2}\cos\delta.
\end{aligned}$$

The average power thus is maximal if the current and voltage are in phase, $\delta = 0$, and zero if $\delta = \pi/2$. ■

24.2 Average over a Distribution (Riley 30.4.1+2)

Average over a discrete probability distribution

Let x be a discrete random variable taking the N possible values $x_1, x_2, \ldots, x_N$, each with respective probability $P(x_1), P(x_2), \ldots, P(x_N)$. As the variable must take one of the possible values, the probabilities have to sum to 1,

$$\sum_{k=1}^{N} P(x_k) = 1. \tag{24.6}$$

The mean, or expectation (or expected) value of x is given by

$$\langle x \rangle = \sum_{k=1}^{N} x_k P(x_k). \tag{24.7}$$

■ **Example 24.2** Let x be the sum of two fair, six-sided dice. It can therefore take the following values with associated probabilities

x_k	2	3	4	5	6	7	8	9	10	11	12
$P(x_k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

For example, the sum $x = 2$ is only possible if both dice give a one, thus the probability is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. The sum $x = 3$ results from one die giving one and the other two, and vice versa; thus the probability is $2 \times \frac{1}{6} \times \frac{1}{6} = \frac{2}{36} = \frac{1}{18}$, etc. As is easy to check, all probabilities add to one, and the expectation value, i.e. the average sum of two dice is given by

$$\langle x \rangle = \frac{1}{36}(2 \times 1 + 3 \times 2 + \cdots + 11 \times 2 + 12 \times 1) = 7.$$

This can also be understood from the symmetry of the probabilities around $x = 7$. ∎

Average over a continuous probability distribution

The above considerations can be generalized to a **continuous random variable** x that can take any value within an interval $[a,b]$, with probabilities determined by a **probability density function** $p = p(x)$. As the probability for any outcome of x needs to be one, $p(x)$ is 'normalized' as

$$\int_a^b p(x)dx = 1. \tag{24.8}$$

The probability of certain outcomes can be calculated as integrals over $p(x)$. For example, the probability of x taking a value within the interval $x_1 \leq x \leq x_2$ ($a \leq x_1, x_2 \leq b$) is given by

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} p(x)dx. \tag{24.9}$$

R For a continuous random variable, the probability to take one specific value is strictly speaking zero, as there is an infinite number of outcomes. Only integrals over intervals give meaningful probabilities. This is also the reason why $p(x)$ is called a probability *density* function. It does not give the probability directly, but rather the probability per unit interval of x.

Formula 24.2 — Expectation Value of Continous Random Variable. The mean, or expectation value of a random variable with probability density $p(x)$ is defined analogously to the discrete case,

$$\langle x \rangle = \int_a^b x\, p(x)dx. \tag{24.10}$$

Here, $[a,b]$ is the full allowed range of values of x.

∎ **Example 24.3** The Maxwell-Boltzmann function describes the distribution of the speed v of particles in an ideal gas,

$$p(v) = Av^2 e^{-B^2 v^2},$$

where

$$A = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2}, \quad B^2 = \frac{m}{2kT}.$$

Here, m is the mass of a particle in the gas, T is the temperature of the gas, and k is the Boltzmann constant. The speed v of a particle can be in the range $[0,\infty]$. One may verify that the Maxwell-Boltzmann function is properly normalized,

$$\int_0^\infty p(v)dv = 1,$$

and we interpret it as the probability distribution of the speeds of particles. Thus the average speed of a particle in the gas is calculated as

$$\langle v \rangle = \int_0^\infty v\, A v^2 e^{-B^2 v^2} dv,$$

We can integrate this by parts, identifying

$$\begin{aligned} u &= v^2 & &\Leftrightarrow & u' &= 2v, \\ w &= -\frac{1}{2B^2} e^{-B^2 v^2} & &\Leftrightarrow & w' &= v e^{-B^2 v^2}. \end{aligned}$$

This gives

$$\begin{aligned} \langle v \rangle &= A \left[-v^2 \frac{e^{-B^2 v^2}}{2B^2} \right]_0^\infty + \frac{A}{B^2} \int_0^\infty v e^{-B^2 v^2} dv \\ &= 0 + \frac{A}{B^2} \left[\frac{e^{-B^2 v^2}}{-2B^2} \right]_0^\infty \\ &= \frac{A}{2(B^2)^2}, \end{aligned}$$

noting that the functions in the square brackets vanish when taking the limit $x \to \infty$. Replacing A and B^2 with the physical parameters, we obtain

$$\langle v \rangle = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \times \frac{1}{2} \times \left(\frac{m}{2kT} \right)^{-2} = \sqrt{\frac{8kT}{\pi m}}.$$

The average velocity thus increases with the square-root of the temperature of the gas. ■

24.3 Length of Curve, Surface and Volume of Revolution

The integration of a function yields the area under a curve, but we may also use it to calculate other geometrical quantities related to it.

Length of Curve

One can use integration to find the length of a curve $f = f(x)$ between two points a and b. Consider a small element of the curve with a change of dx in x-direction and $dy = df$ in the y-direction. The length of the curve element ds is calculated via Pythagoras,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{df}{dx} \right)^2} dx, \tag{24.11}$$

where we have written the expression using the usual derivative $\frac{df}{dx}$ of the curve.

Formula 24.3 — Length of Curve. To determine the length S of the curve between a and b, we sum over all small curve elements and take the continuum limit yielding the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{df}{dx} \right)^2} dx. \tag{24.12}$$

■ **Example 24.4** We can calculate the length of the upper half of a circle with radius r using this method. The function curve in this case is given by $f(x) = \sqrt{r^2 - x^2}$ and thus

$$\frac{df}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}.$$

Integrating the curve length from $-r$ to r gives

$$\begin{aligned}
S &= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx \\
&= r \int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}} \\
&= r \left[\arcsin\left(\frac{x}{r}\right) \right]_{-r}^{r} = r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi r,
\end{aligned}$$

as expected ■

Surface of Revolution

One can also use integration to find the surface area of a solid generated when the curve $y = f(x)$ is rotated in a full circle about the $x-$axis. We can think of the surface as being composed of rings with radius $f(x)$ and thickness given by the length of the curve element ds described above. Hence the surface area of one such ring is

$$dA = 2\pi f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx. \tag{24.13}$$

Formula 24.4 — Surface Area of Revolution. To find the total area, we sum all rings between the limits $x = a$ and $x = b$. In the continuum limit this gives

$$A = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx. \tag{24.14}$$

Volume of Revolution

Analogously, we can find the volume of a solid generated when a curve $y = f(x)$ is rotated about the $x-$axis. We can think of the volume as being composed of disks with radius $f(x)$ and thickness dx, and hence the volume for one such disk is

$$dV = \pi f^2(x) dx. \tag{24.15}$$

Formula 24.5 — Volume of Revolution. To find the total volume, we sum up all disks between the limits $x = a$ and $x = b$. In the continuum limit this gives

$$V = \pi \int_a^b f^2(x) dx. \tag{24.16}$$

■ **Example 24.5** Derive the volume of a sphere of radius r.

A sphere is formed by rotating a circle around the x axis, which has the equation

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad f^2(x) = r^2 - x^2.$$

Putting this into the volume of revolution formula, we have

$$
\begin{aligned}
V_{\text{sphere}} &= \pi \int_{-r}^{r} (r^2 - x^2)dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left(r^3 - \frac{r^3}{3} \right) - \pi \left((-r)^3 - \frac{(-r)^3}{3} \right) \\
&= \frac{4}{3}\pi r^3,
\end{aligned}
$$

as expected. ■

24.4 Numerical Integration (Riley 27.4.1)

In practical applications, many integrals encountered are not solvable analytically and numerical methods to determine definite integrals have to be used.

The simplest method, albeit not the most efficient one, is based on the basic definition of the integral; we divide the area under a curve between the limits $[a,b]$ into N strips of equal width $\Delta x = (b-a)/N$. We can somewhat improve the approximation over that used in the basic definition (where we used rectangular strips), by spanning a trapezium between neighbouring function values $f(x_k)$ and $f(x_{k+1})$ with the area

$$
\Delta A = \frac{1}{2}(f(x_{k+1}) + f(x_k))\frac{b-a}{N}. \tag{24.17}
$$

Formula 24.6 — Simple Numerical Integration. We can then approximate the integral by summing up all trapezia,

$$
\begin{aligned}
\int_a^b f(x)dx &\approx \frac{b-a}{N}\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1}) + \frac{1}{2}f(x_N)\right] \\
&\approx \frac{b-a}{N}\left[\frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{N-1} f(x_k)\right],
\end{aligned} \tag{24.18}
$$

with $x_k = a + k(b-a)/N$.

The quality of the approximation improves as N becomes larger but this also requires the evaluation of the function at many points which can be computationally costly.