UCL DEPARTMENT OF PHYSICS AND ASTRONOMY
Gower Street Teaching Laboratories

First Year Laboratory: Course PHAS1240

Study of an Oscillating Mechanical System Driven into Resonance (NX3)

4 Sessions
Autumn 2016

Relevant Lecture Courses:
PHAS1245 Mathematical Methods I
PHAS1246 Mathematical Methods II
PHAS1247 Classical Mechanics

Revisions:
September 2014: PAB, JG
August 2015: JG, KD
July 2016: JG
November 2016: JG

Risk Assessment (as submitted to UCL's `Risknet')

Department: Physics and Astronomy **Risk Assessment Form** Torsion Pendulum

WORK/PROJECT TITLE : Study of an oscillating mechanical system driven into resonance

LOCATION(S): First year teaching laboratory, Gower Street

DESCRIPTION OF WORK: The measurement of the nature of the resonant peak associated with a torsion pendulum for different values of damping

PERSONS INVOLVED: Undergraduate students and academic staff

Hazard	Risk	Level	Control
Laser	may damage eyes	low	Laboratory Health and Safety procedures and laser safety protocols to be followed
Oil spills	may cause accidents	low	clean up immediately
Free end of wire	may damage eyes	low	End cap to be applied
Unshielded moving parts	crushing of fingers in mechanism	low	Fingers to be kept away from motor mechanism when operating.
	hair caught in mechanism	low	Long hair to be tied back
Bags and coats in walkways	trip hazard	low	stow bags and clothing safely in a designated place

DECLARATION
I, the undersigned, have assessed the work titled above, and declare that there is no significant risk/the risks will be controlled by the methods stated on this form (delete as applicable) and that the work will be carried out in accordance with Departmental codes of practice.

Name P Bartlett

Signed

Date

Laser Safety Notes

1. Do not attempt to look along the direct laser beam - permanent eye damage can result.

2. Avoid looking at the laser beam directly reflected from a metal surface (specular reflection) or transmitted through an optical system without attenuation.

3. Under no circumstances should the beam or a reflected beam be viewed through a focusing element.

4. Avoid prolonged staring at the beam directly scattered from a matte surface.

5. The lasers used in this experiment will not cause damage if the direct laser beam falls upon your skin or clothing, but this is not the case with all lasers, so it is good practice to avoid this situation.

6. Turn off the laser when not needed.

7. If you are in doubt about any aspect of the safe use of a laser, consult a member of the academic or technical staff.

Introduction:

Many objects or systems of objects will oscillate at a characteristic frequency – their *natural* frequency – if disturbed from rest. Such oscillations will die away due to *damping* – the dissipation of the energy of oscillation – again, at a characteristic rate. If such a system is *forced* to oscillate by driving it with a periodic driving force with frequency close to the natural frequency of the free oscillation, it exhibits the phenomenon known as *resonance*, where the amplitude of the oscillation is a maximum. The motion depends on various parameters of the system, including the damping level.

Aims

In this experiment, the response of a system, forced into oscillation, is studied as a function of drive frequency and damping value, and compared with theoretical predictions.

Objectives

You will be using a particular type of oscillating system in this experiment. **In a few words**, say what it is, how you will investigate its motion, and how you will vary the parameters of interest. Most details of the particular experimental set-up should be left to the "Experiment" section.

Theory

The first part of the theory is for a linear system and may already be familiar to you. If it is, you may skip it, and you do not need to record it in your lab book. The second part is for the particular case of a rotating mass (torsion pendulum). For more information about periodic motion with driving and damping, see *e.g.* chapter 13 of [1]; for information about rotational motion of solid objects, refer to *e.g.* chapters 9 and 10 of [1].

Linear system

Consider a mass, m, on a spring with periodic driving. The equation of motion is:

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = A_0 \cos \omega t \tag{1}$$

where x is the displacement, t the time, λ the damping constant, k the spring constant (constant of the restoring force), A_0 the (maximum) driving force, ω its (angular) frequency ($\omega = 2\pi f$).

When the drive is first turned on, there is a time lag before the mass shows regular periodic motion (reaches its steady state) - this is the *transient state*, which can be modelled mathematically by the *homogeneous* differential equation constructed from (1) but with zero on the right hand side. The transient state dies away quite rapidly, and we shall not be concerned with it in this experiment.

Once the *steady* state (represented by a solution of the full *in*homogeneous equation (1)) is reached, the mass oscillates at the same frequency as the driving frequency, but there is a phase difference between the drive and response. The response of the mass can then be expressed as:

$$x = x_0 \cos(\omega t + \phi) \tag{2}$$

where ϕ is the phase angle between drive and response, and x_0 is the maximum displacement.

If this is substituted into the equation of motion, eq. (1), ϕ and x_0 can be determined analytically:

$$\tan\phi = \frac{\lambda}{\omega m - k/\omega} \tag{3}$$

$$x_0 = \frac{A_0}{\sqrt{\omega^2\lambda^2 + m^2(\omega^2 - \omega_0^2)^2}} \tag{4}$$

where ω_0 is the natural frequency of the mass if there is *no* driving or damping ($A_0 = 0$ and $\lambda = 0$).

Rotating system

In rotational motion, the position of the rotating body is described in terms of its *angular* position *(θ)* rather than its linear displacement *x*. We also use a rotational version of Newton's Second Law, with linear acceleration replaced by angular acceleration. We do not speak of applying a *force* to a rotating body, because what counts here is not just the magnitude of the force but how far from the pivot the force is applied – in other words it is the *moment* of the force about the pivot that is important, and we give it a special name, *torque* (*T*). Thus the Second Law becomes

$$T = I\frac{d^2\theta}{dt^2} \tag{5}$$

where, in turn, the mass has been replaced by the moment of inertia (*I*), a property of a body about a given axis which incorporates information about the distribution of the mass about the axis. Moment of inertia has dimensions ML^2.

In the case of a torsional pendulum – a mass, hanging on a wire, which is capable of rotation – any displacement of the mass through an angle θ will twist the wire so that there is now a *restoring torque*, tending to turn the mass back to its equilibrium position, which we assume is proportional to the angle turned: $T = -s\theta$ (where s is a constant). Substituting this into (5) we obtain

$$I\frac{d^2\theta}{dt^2} = -s\theta \tag{6}$$

whose solution is a simple harmonic oscillation at the *natural frequency* ω_0 where

$$\omega_0 = \sqrt{\frac{s}{I}} \tag{7}$$

If the pendulum is moving through a viscous medium, the motion will also be opposed by a *damping torque* which we assume to be proportional to the angular velocity of the pendulum, and hence equal to $-\lambda \frac{d\theta}{dt}$, where λ is a constant. Including this term in (6) gives us

$$I\frac{d^2\theta}{dt^2} = -\lambda\frac{d\theta}{dt} - s\theta \tag{8}$$

the solution of which is an oscillation which is *damped*, and dies away. (This is the *homogeneous* differential equation, as referred to on page 4). If, however, we *force* the system to oscillate by applying a sinusoidal driving torque $T_0 \cos \omega t$, the full equation looks like (9), the right hand side of which gives the *resultant* torque:

$$I\frac{d^2\theta}{dt^2} = T_0 \cos \omega t - \lambda\frac{d\theta}{dt} - s\theta \tag{9}$$

In standard mathematical format, this equation becomes

$$I\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + s\theta = T_0 \cos \omega t \tag{10}$$

By analogy with (2), the response of the cylinder is

$$\theta = \theta_0 \cos(\omega t + \phi) \tag{11}$$

Substituting (11) into (10) then gives the following expressions for the phase difference ϕ and maxmum angular displacement θ_0:

$$\tan \phi = \frac{\lambda}{\omega I - s/\omega} \tag{12}$$

$$\theta_0 = \frac{T_0}{\sqrt{\omega^2\lambda^2 + I^2(\omega^2 - \omega_0^2)^2}} \tag{13}$$

Differentiating (13) with respect to ω gives us the following expression for the peak or *resonant* angular velocity, ω_λ:

$$\omega_\lambda^2 = \omega_0^2 - \frac{\lambda^2}{2I^2} \tag{14}$$

Q: From (12), describe what happens to the phase angle as $\omega \rightarrow \omega_0$ from below.
Q: What happens when $\omega = \omega_0$?
Q: Is this dependent on the damping factor?

Q: What does this tell you about the phase angle (and frequency) at which the phase curves intersect?
Q: What does (14) tell you about how the resonant frequency varies with damping?

The conversion between the terms and properties of the two systems is summarised in table 1.

Linear Motion (mass on a spring)		**Angular motion (torsion pendulum)**	
mass	m	moment of inertia	I
linear displacement	x	angular displacement	θ
maximum displacement	x_0	maximum angular displacement	θ_0
spring constant	k	torsion constant of wire	s
maximum driving force	A_0	maximum driving torque	T_0

Table 1: the 1-1 correspondence between a linear system with periodic driving and damping and the torsion pendulum system with angular motion. The driving angular velocity (sometimes referred to as the *frequency*), damping constant, phase difference and time have the same symbols and meaning in both systems.

The theoretical expressions derived above can be plotted against ω for various damping levels:

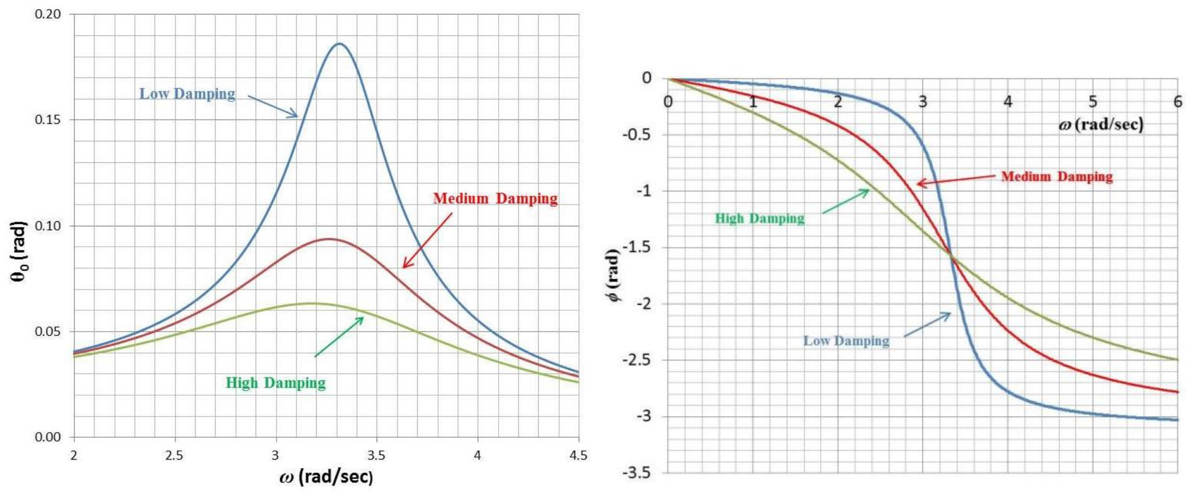
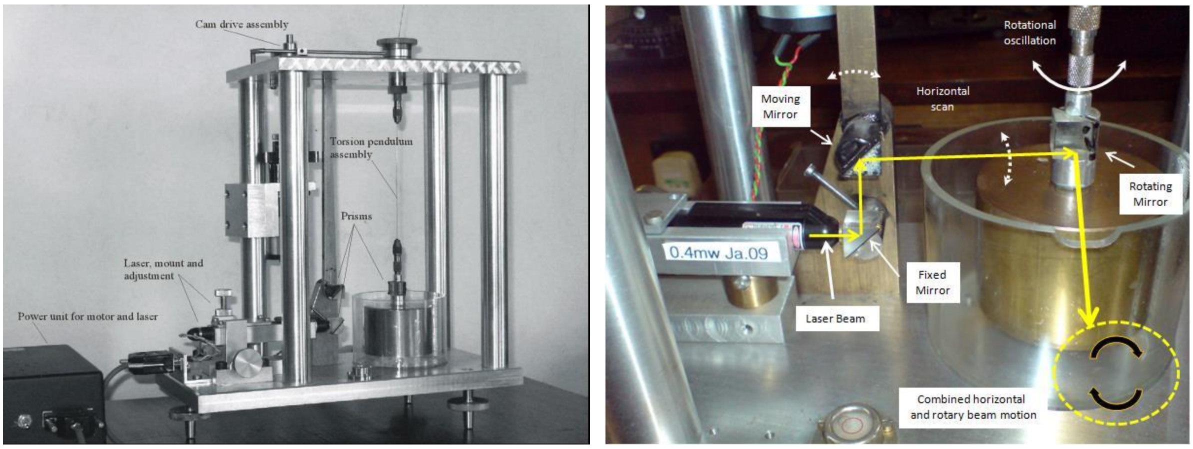


Figure 1: (a) The amplitude response of a resonant system over a range of driving frequencies for low, medium and high damping. (b) The phase of a resonant system (relative to driving) over a range of driving frequencies for low, medium and high damping.

Experimental System

The torsional pendulum forms the mechanical system being studied in this experiment. It consists of a brass cylinder suspended from a steel wire, the upper end of which is held in a chuck (see figure 2(a)). The chuck is made to execute small-amplitude oscillations by means of a lever and cam mechanism. The cam is driven by a variable speed motor. The restoring torque of the wire is assumed proportional to the angular displacement.



(a) (b)

Figure 2: (a) Photo of the torsion pendulum assembly showing key moving parts. (b) Photo of the central section of the pendulum assembly showing the laser path and the motion of the mirrors.

The motion of the cylinder is damped by immersion in silicone oil, which produces a viscous damping torque that is approximately proportional to the angular velocity of the cylinder. A laser beam reflected from prism mirrors fixed to the driving mechanism and the cylinder is used to project the driving and response motions: see figure 2(b). Analysis of the laser path allows the calculation of the amplitude of oscillation of the cylinder and the phase of this oscillation relative to the driving to be made (see page 11).

Examine the apparatus to ensure that you understand how the torsion cylinder is driven into oscillation and how, as a result of the arrangement of light beam and mirrors, the driving oscillation and the oscillation of the cylinder are represented by mutually perpendicular components of the motion of a single light beam, producing an elliptical motion of the laser spot.

Determine which direction on the plotting table displays the motion of the driving system (i.e. the motor) and which direction displays that of the driven system (i.e. the cylinder). Figure 4 may help explain the details.

Diagrams

Make a sketch of the apparatus in your lab book; this should show the essential parts of the entire system (i.e. not just the parts shown in fig 2(b)) but can omit such things as structural components. You will probably need at least two separate sketches due to the complicated layout of the apparatus.

The diagrams in figures 3 and 4 are solely to demonstrate the details of the light path and the formation of the ellipses observed.

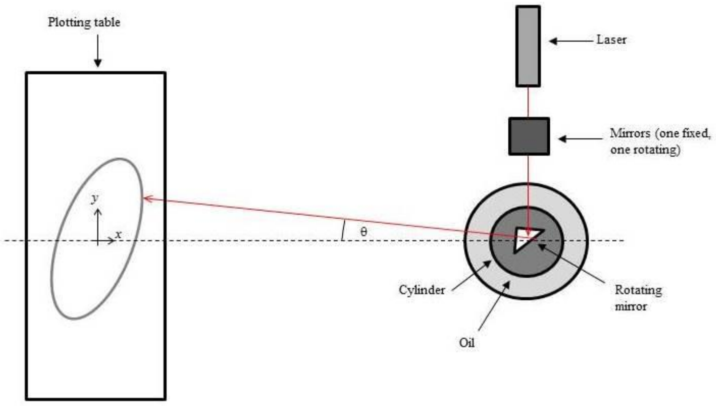
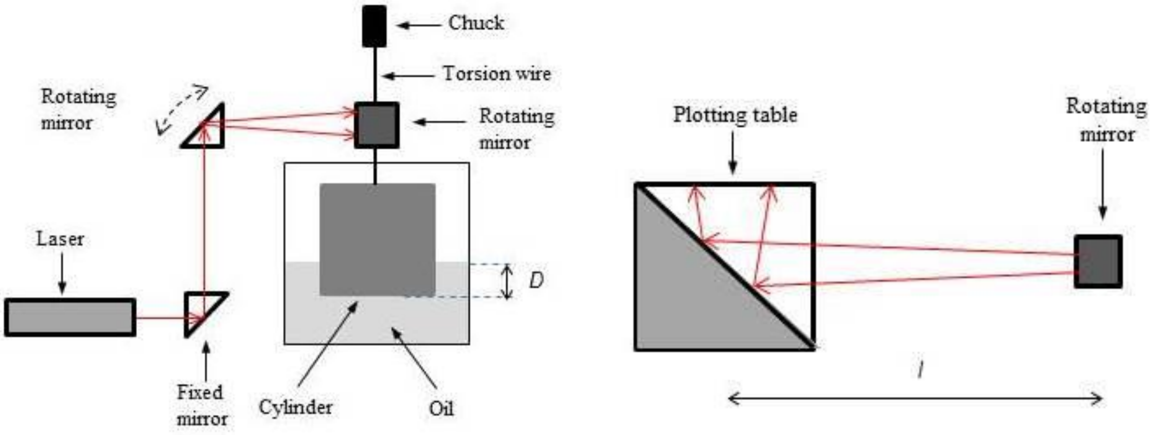


Figure 3: Schematic of the light path from above



(a) (b)

Figure 4: Details of the light path (side views): (a) view along x-axis (b) view along y-axis

The ellipse is projected onto the plotting table by reflection at the angled mirror (see figure 4(b)). This can then be recorded by tracing the motion of the observed spot.

Equipment list

Although the majority of the experimental apparatus is contained within a fixed unit it is good practice to record the details of the individual components. If there are serial numbers (or any other unique identifiers) on any parts of the equipment it is a very good idea to record these too. The motor is driven by a variable power supply; the knob labelled “SPEED” actually varies the voltage output of the power supply, and is calibrated in **arbitrary units**. Do **not** assume a linear relationship between this knob position and the motor speed.

The following information about some of the components of the system may be relevant (but note that this is NOT a complete equipment list!):

Damping fluid: silicone oil, viscosity 100cP .
Torsion wire: steel; diameter 0:56mm; free length 11cm.
Torsion bob: mass = 1:165kg; height = 44:6mm; diameter = 63:1mm.
Drive motor: The period can be varied from 1.5 to 5 seconds. The resonant period is about 2s.

Safety

The formal risk assessment for this experiment is included on page 2, with additional notes about laser safety on page 3. Try to construct your own risk assessment, then check it against the script. Are there other hazards associated with this experiment?

Procedure

Construct a procedure for this experiment; a general outline is given below. Leave space (spare lines) so you can add details/change method as you go along. *Note:* you must have some damping present in the system to obtain a stable ellipse; without damping the system is unstable.

Setting up

Add oil: What depth? Over the course of the full experiment, you should consider several different dampings. We suggest using $d \approx 5$mm, 25mm to start with (do the smaller level first; it will give you a clearer resonance peak). Take care that bubbles of air do not become trapped under the cylinder. The damping constant is not equal to the depth - you may wish to consider how it could be determined.

Q: If there are bubbles in the oil, will they adversely affect your experiment if they stay where they are?

Position of plotting table: ensure that the plotting table is aligned as shown in figure 3. Record the distance to the plotting table (see figure 4(b)). The distance l is to the centre (y-axis) of the ellipse which is undetermined at this stage; how can you make sure that you can obtain l?

Q: How precisely can you determine *l*, and how precisely do you need to determine it?
Q: What will you do if the ellipse becomes too large for the plotting table?

Tracing paper: always have a sheet of paper on the plotting table before you turn on the laser and set the pendulum into motion.

Control and range of drive speeds: before you trace any ellipses, quickly run through full range of drive speeds (both inner and outer dials) and note down the approximate location of resonance and where you are definitely well above and well below resonance.

Driving amplitude: is this constant?

**Sampling `rate':** Do you want to take data at evenly spaced drive speeds (numbers on dials) or take more data in particular regions? What is your `experimental strategy' and why?

Data collection

The raw data for this experiment consists of various properties of the ellipses traced by the laser spot, and the period of the drive. From these you will be able to obtain the amplitude and phase response and the driving angular frequency (see page 12).

Period of drive: Look at the cam mechanism; is there a convenient way of observing a single drive period? How can uncertainties be reduced?

Tracing the ellipses: Use a sharp pencil on the paper on top of the plotting table to mark points on the ellipses (join the dots later). Make sure you know which ellipse corresponds to each drive.

Record the **direction** of laser spot: why?

Multiple ellipses: try to avoid overlapping ellipses (there is plenty of paper available).

Any other notes/observations: write them down, but make sure that you know which set of parameters they relate to.

If you update your method make a note of what you did and why.

Results

Make sure that you and your partner(s) have copies of the raw data stuck into your lab books at the end of each session (photocopy as necessary). Construct tables before you fill them in.

Data Analysis

You will plot the amplitude of the response and the phase angle against the driving (angular) frequency. To do this you will need to calculate the following from your raw data (ellipses and periods): $\omega = 2\pi f$, lengths: AB and CD from figure 5. You will also need the distance l from the mirror on the cylinder to the centre (y-axis) of the ellipse.

Record these in a systematic fashion. (What equations do you need and what are your uncertainties?)

Figure 5 shows the formation of an ellipse from a driving displacement

$$x(t) = x_0 \cos \omega t \qquad (15)$$

and a response displacement, which is out of phase by an angle ϕ,

$$y(t) = y_0 \cos(\omega t + \phi) \qquad (16)$$

Calculating amplitude

From figure 5 it is easily seen that the maximum displacement is the distance CD/2 of each ellipse. Since the displacement of the pendulum is *angular*, refer to figure 3, and see that:

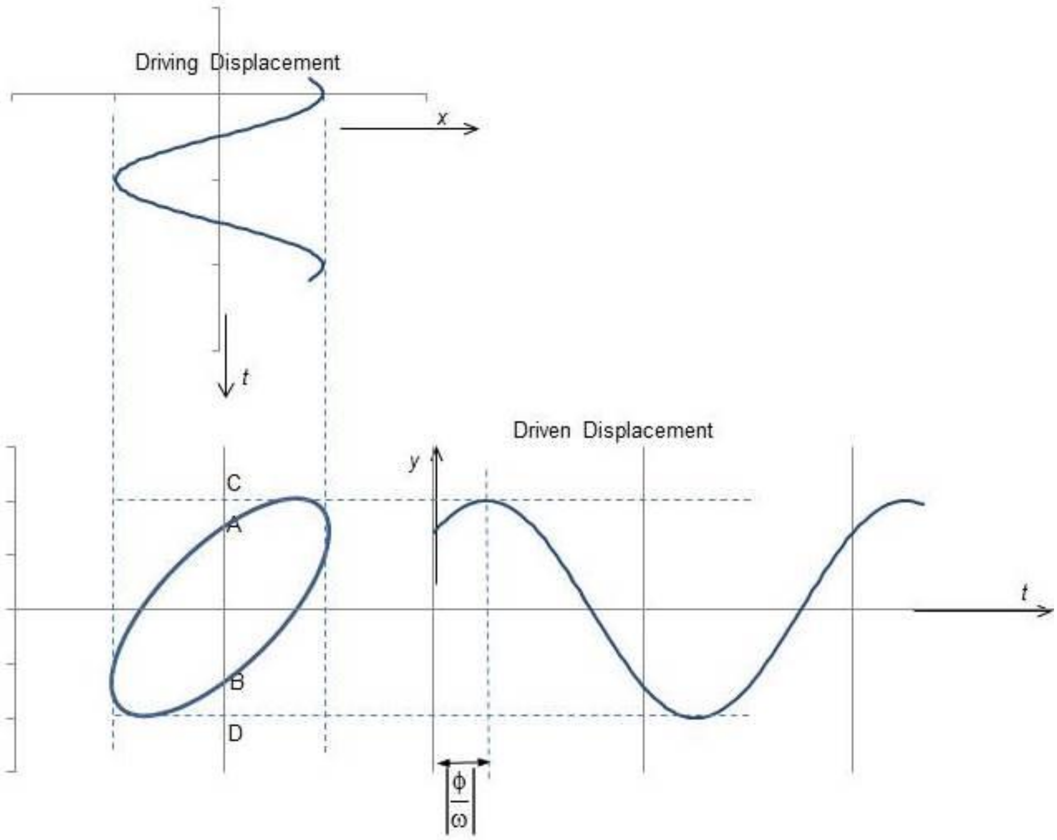


Figure 5: An ellipse (generalised circle) is the superposition of two sinusoidal waves travelling at right angles to each other with a relative phase ϕ. From figure 4, the motion in the y-direction comes from the response of the cylinder and the motion in the x-direction is the driving.

$$\frac{CD}{2} = l \tan \theta_0$$

If we can use the `small angle' approximation, then $\tan \theta_0 \approx \theta_0$ (in radians) and:

$$\theta_0 = \frac{CD}{2l} \qquad (17)$$

Q: Is the small angle approximation valid here, considering uncertainties?

Plot θ_0 against ω (remember uncertainties) on a single graph for all damping considered.

Calculating phase angle

Ensure you know how to calculate ϕ and its uncertainty: remember to write out at least the final formulae you will be using. If you are struggling, ask a demonstrator. Remember that this is not a maths course!

The line AB in figure 5 corresponds to the driving amplitude being zero (i.e. cos $\omega t = 0$). The response amplitude on this line is then (by properties of cosines):

$$y = -y_0 \sin \omega t \sin \phi,$$

with $\sin \omega t = \pm 1$ when $\cos \omega t = 0$. To determine the sign, note that if we differentiate eq. (15) we get:

$$\frac{dx}{dt} = -\omega x_0 \sin \omega t \tag{18}$$

Therefore increasing x requires $\sin \omega t < 0$ while decreasing x requires $\sin \omega t > 0$. These two possibilities are related to the direction of motion of the light spot.

1. Light spot moving clockwise: x is increasing at A and decreasing at B, so we have

$$y(A) = y_0 \sin \phi; \qquad y(B) = -y_0 \sin \phi$$

Here y_0 is the maximum displacement, CD/2, so that

$$AB = y(A) - y(B) = CD \sin \phi \tag{19a}$$

whence $\sin \phi > 0$, so that ϕ must be in the **1st or 2nd quadrant** ($0 < \phi < \pi$).

2. Light spot moving anticlockwise: x is decreasing at A and increasing at B, so we have

$$y(A) = -y_0 \sin \phi; \qquad y(B) = y_0 \sin \phi$$

and hence

$$AB = -CD \sin \phi \tag{19b}$$

so that ϕ is in the **3rd or 4th quadrant** ($-\pi < \phi < 0$).

To determine the quadrant we also consider which way the ellipse is *leaning*.

Differentiating (16),

$$\frac{dy}{dt} = -\omega y_0 \sin(\omega t + \phi)$$

so that when x=0,

$$\frac{dy}{dt} = -\omega y_0 \sin \omega t \cos \phi \tag{20}$$

Comparing (18) and (20) we see that dx/dt and dy/dt have the same sign if $\cos \phi$ is positive, and opposite signs if $\cos \phi$ is negative. If the ellipse is leaning to the right (like the one in Fig. 5) clearly they have the same sign at A and B whichever way round the beam is going, so we infer that $\cos \phi$ is positive, and hence ϕ must be in either the first or the fourth quadrant (i.e. $-\pi/2 < \phi < \pi/2$) whereas if the ellipse is leaning to the left, the signs will be opposite, again independent of beam direction, so that $\cos \phi$ must be negative, and hence ϕ is in the 2nd or 3rd quadrant ($-\pi < \phi < -\pi/2$ or $\pi/2 < \phi < \pi$).

Hence these two pieces of information (the direction of the laser spot and the lean of the ellipse) enable us to identify the correct quadrant.

Equations (19a) and (19b) together tell us that **we will need to compute $\sin^{-1}\left(\frac{AB}{CD}\right)$.**

Calculating the inverse sine on a calculator or computer usually returns an angle between zero and 90° (in the first quadrant; $0 < \theta < \pi/2$ in radians). If you have decided that ϕ is in a different quadrant, carry out the adjustment shown in the last column of this table.

Direction of Lean	**Direction of Laser Spot Motion**	**Quad-Rant**	**Phase angle, ϕ (radians)**	**Adjustment**
left	anti-clockwise	3	$-\pi < \phi < -\pi/2$	Subtract π
right	anti-clockwise	4	$-\pi/2 < \phi < 0$	Multiply by (-1)
right	clockwise	1	$0 < \phi < \pi/2$	None
left	clockwise	2	$\pi/2 < \phi < \pi$	Subtract from π

Table 2. Quadrants for calculating ϕ.

Note that the table gives only the *principal* quadrants (between $-\pi$ and π). You can add or subtract 2π radians, and you may wish to do this in order to avoid an unphysical discontinuity in your ϕ graph. Note that the theory (equation (12) and fig. 1b) suggests that ϕ must be in the interval $-\pi < \phi < 0$; however, you may find that some of your values are outside this range.

Making the Experiment Quantitative

This experiment can be completed satisfactorily by making only *qualitative* comparisons – e.g. "the low damping resonance peak is higher or lower than the high damping peak". But it can be made more *quantitative* by estimating actual values (with uncertainties) for the resonant frequency, the maximum amplitude, the phase angle and frequency at the crossing point, and the sharpness of the resonance peak.

When estimating peak amplitudes and frequencies, remember that the position of the peak is usually more uncertain than the individual points, unless you have managed to take plenty of points around resonance; hence you should quote uncertainties that reflect this.

To compare the sharpness of the resonance curves, one considers the width ($\delta\omega$) of the resonant peak where the amplitude is $1/\sqrt{2}$ times the maximum value (power transferred is half that transferred at resonance; this concept has an analogue in electronic tuned circuits). The quality factor Q is then defined as:

$$Q = \frac{\omega_\lambda}{\delta\omega} \tag{21}$$

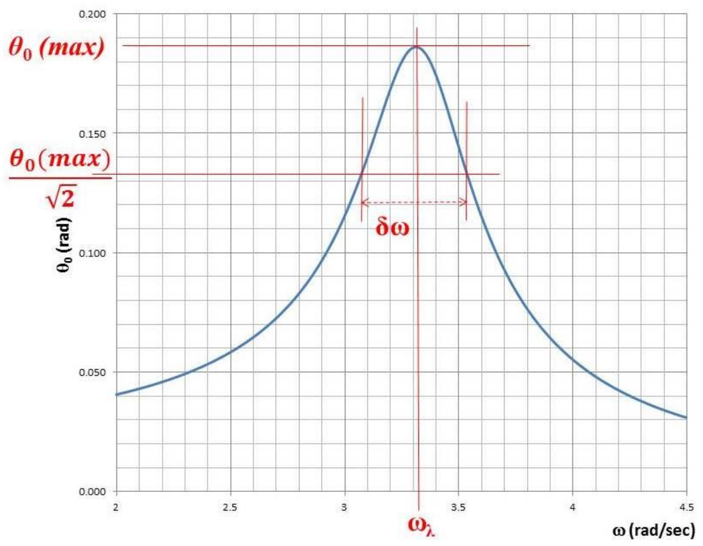


Figure 6: Measurement of Q

Finally, if you know how to fit a line to a set of data points, you could consider fitting equations (12) and (13) to your data. This may give you estimates of the parameters, T_0, I, s, λ. Do **not** attempt to fit a polynomial – these functions are not polynomials.

Conclusions and Discussion

You should now have two graphs with (ideally) three curves on each corresponding to three levels of damping. You may wish to consider the following aspects of what you have done.

How do the resonant frequency, sharpness and size of the resonance peak vary with damping?

What about the phase angle? What can you say about the point where the phase angle curves intersect?

Are your graphs in agreement with the expected results (figure 1)?

(*If you have made quantitative estimates of the paramete*rs) How precise is the determination of Q?

Are there any flaws in your method/approach or anything you would change?

References

[1] H. D. Young and R. A. Freedman. University Physics. Addison Wesley, 12th edition, 2008.

[2] C. Isenberg and S. Chomet. Physics experiments and projects for students, Vol.1. Newman-Hemisphere, London, 1991.