PHAS1245—Problem Sheet 2, Solutions

1.

$$\frac{d}{dx}\left(x^2e^x\right) = 2xe^x + x^2e^x = xe^x(x+2)\ .$$

2.

$$\frac{d}{dx}\ln\left(a^x + a^{-x}\right) = \frac{1}{(a^x + a^{-x})}\left\{a^x\ln a + a^{-x}(-1)\ln a\right\}$$

$$= \ln a\frac{a^x - a^{-x}}{a^x + a^{-x}}\ .$$

3.

$$\frac{d}{dx}\ln\left(x^a + x^{-a}\right) = \frac{1}{x^a + x^{-a}}\left\{ax^{a-1} + (-a)x^{-a-1}\right\}$$

$$= \frac{\left(ax^{(a-1)} - ax^{-a-1}\right)}{x^a + x^{-a}} = \frac{a\left(x^a - x^{-a}\right)}{x\left(x^a + x^{-a}\right)}\ .$$

4. If $y = x^x$ then $\ln y = x\ln x$. Now differentiate both sides wrt x :

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\frac{1}{x}$$

$$\text{Thus}\quad \frac{dy}{dx} = y\left(\ln x + 1\right)$$

$$\text{and}\quad \frac{dy}{dx} = x^x(\ln x + 1)\ .$$

Or you can write $y = x^x = e^{x\ln x}$ and take it from there.

5. $y = \cot^{-1}x$ means $x = \cot y$. Then use inverse function differentiation :

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{d}{dy}\left(\frac{\cos y}{\sin y}\right)\\
&= -\cos y\frac{\cos y}{\sin^2 y} - \frac{\sin y}{\sin y}\\
&= -\cot^2 y - 1\\
\Rightarrow \frac{dy}{dx} &= -\frac{1}{1+x^2}
\end{aligned}$$

6. The shape is a rectangle with the top side replaced by half a circle. If r is the radius of the circle and a is the vertical side of the rectangle, then the area A of the tunnel is

$$A = 2ra + \frac{1}{2}\pi r^2$$

and its perimeter S is

$$S = 2a + 2r + \pi r\,.$$

Solving the first for a and substituting in the second, we get

$$S = \frac{A}{r} + \left(2 + \frac{\pi}{2}\right) r$$

We want to minimize S wrt r, so

$$\frac{dS}{dr} = -\frac{A}{r^2} + 2 + \frac{\pi}{2} = 0 \Rightarrow A = r^2\left(2 + \frac{\pi}{2}\right)\,.$$

The second derivative is

$$\frac{d^2S}{dr^2} = \frac{2A}{r^3}\,,$$

greater than 0 for all r, so the stationary point is a minimum (as we required). Hence, substituting A in the very first equation we find $a = r$.

7. (a) (i) Differentiating the equation of the curve implicitly :

$$12y^2\frac{dy}{dx} = a^2 + 3a^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}\,.$$

(ii) In parametrised form :

$$\frac{dy}{d\theta} = -a\sin\theta, \qquad \frac{dx}{d\theta} = -3a\sin 3\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sin\theta}{-3a\sin 3\theta}\,.$$

Using trigonemtry relations :

$$\begin{aligned}\sin 3\theta &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\ &= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta \\ &= \sin\theta(4\cos^2\theta - 1)\end{aligned}$$

this gives

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} = \frac{a^2}{12a^2\cos^2\theta - 3a^2}$$

with $a\cos\theta = y$, (i) and (ii) can be seen to be equivalent.

(b) At a point of inflection, $y'' = 0$. So :

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \cdot \frac{dy}{dx} = -\frac{a^2}{(12y^2 - 3a^2)^2} \cdot 24y \cdot \frac{a^2}{12y^2 - 3a^2}$$

This can only be zero at $y = 0$ (and x is also 0). But, when $y = 0$, $\frac{dy}{dx} = -\frac{1}{3}$. As this is non-zero, the point of inflection is not a staionary point.

(c) See figure :