http://www.star.ucl.ac.uk/~idh/PHAS1102

Problem Paper 4: Notes for Answers

Question 1

The velocities of individual galaxies don't obey Hubble's Law exactly – galaxies have 'peculiar velocities' (largely resulting from gravitational interactions with neighbouring galaxies) in addition to 'cosmological velocities' resulting from the Hubble flow.
Suppose that a typical peculiar velocity is 500 km s^{-1}. How far away would a galaxy have to be if you wanted to use it to determine the Hubble constant with an uncertainty of 10 per cent? (Assume $H_0 = 72\ \mathrm{km\ s^{-1} Mpc^{-1}}$, and that the distance and velocity of the galaxy can be measured perfectly accurately; express your answer in Mpc.)

The Hubble constant is basically velocity divided by distance, in suitable units (normally km/s and Mpc). This question therefore essentially asks "at what distance does 500 km/s represent 10 per cent of the (cosmological) recession velocity", i.e., to what distance does 5000 km/s correspond? For $H_0 = 72$ km s^{-1} the answer is simply (5000/72) Mpc, or 69 Mpc.

[3 marks; lose a mark if the answer isn't to 2 sig figs, or if it isn't in Mpc]

For the given value of H_0, calculate a value for the Hubble time. (Express your answer in years, to 2 significant figures.)

This question really requires no more than attention to units (plus knowing that there are 3.08568×10^{19} km Mpc^{-1}, and remembering that the Hubble time, an approximate estimate of the age of the universe, is just $1/H_0$):

$$\begin{aligned}
\tau_0 &= \frac{1}{H_0} \\
&= \frac{3.08568 \times 10^{19}\ \text{km Mpc}^{-1}}{72\ \text{km s}^{-1}\ \text{Mpc}^{-1}} \\
&= 4.29 \times 10^{17}\ \text{s} = 1.4 \times 10^{10}\ \text{yr}
\end{aligned}$$

[3 marks; lose a mark if the answer isn't in years.]

Question 2

The velocity of recession of a galaxy is very often expressed in terms of its redshift, z. If $v \ll c$ we can convert from redshift to velocity by using the usual Doppler formula,

$$1 + z \equiv \frac{\lambda}{\lambda_o} \simeq 1 + \frac{v}{c} \quad \text{(i.e., } v \simeq cz\text{);}$$

otherwise the relativistic form must be used:

$$1 + z \equiv \frac{\lambda}{\lambda_o} = \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}.$$

To show that v and cz are significantly different even for quite small redshifts, rearrange the special-relativistic formula for z into an equation for v/c.

*[*Hint: *to obtain an expression for v/c, try writing $(1 - v^2/c^2)$ as $(1 - v/c)(1 + v/c)$.]*

Following the 'hint', we write

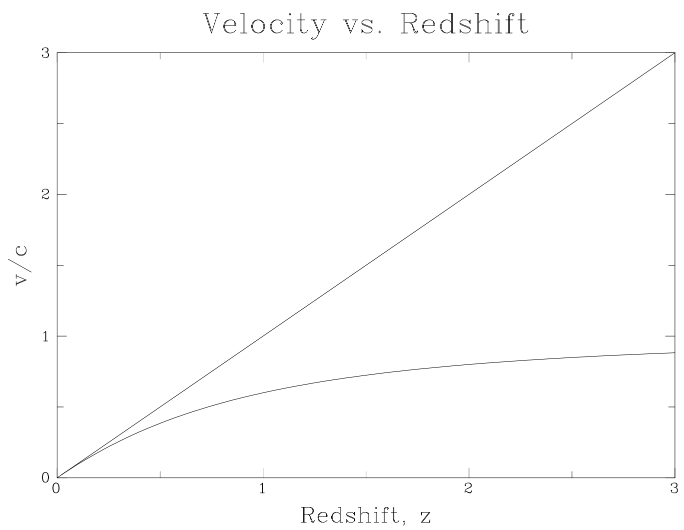
$$1 + z \equiv \frac{\lambda}{\lambda_o} = \frac{1 + v/c}{\sqrt{(1 - v/c)(1 + v/c)}} = \sqrt{\frac{(1 + v/c)(1 + v/c)}{(1 - v/c)(1 + v/c)}}$$

$$(1 + z)^2(1 - v/c) = (1 + v/c)$$

$$\frac{v}{c} = \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1}$$ [3 marks]

Plot a graph of v/c vs. z for z in the range 0 to 3; for comparison, also plot the line for the (unrealistic, hypothetical, non-relativistic) case $v = cz$. Put redshift on the x axis.

Evaluating this over the required range gives a plot like this:



[3 marks; lose a mark for plotting z as a function of v/c instead of v/c as a function of z, as requested.]

Note that the relativistic and non-relativistic lines diverge substantially for redshifts in excess of just a few tenths; and that by redshift 3, galaxies are receding at about 88% the speed of light (and not $3c$!).

Question 3 [SI units required; lose 1 mark for each answer in any other units]
The energy density of black-body radiation at temperature T is given by $(4\sigma/c)T^4$ where σ is the Stefan-Boltzmann constant and c is the speed of light. Calculate the energy density of the Cosmic Microwave Background ($T = 2.728$ K).

A simple sum: $(4\sigma/c)T^4 = 4.19 \times 10^{-14}$ J m^{-3} [2 marks]

The typical energy of a photon in a black-body distribution of temperature T is roughly $3kT$, where k is Boltzmann's constant. What is the number density of CMB photons in the Universe today?

For 2.728 K, the average photon energy is $3kT = 1.13 \times 10^{-22}$ J;
so the photon number density is $(4.19 \times 10^{-14})/(1.13 \times 10^{-22})$ m^{-3}, i.e., 3.71×10^8 m^{-3} [3 marks]

Suppose that the average number density of hydrogen atoms in the local universe is one per cubic metre. Express this as an energy density ($E = mc^2$). Compare the average number densities and energy densities of atoms and photons in the universe.

If there's 1 atom m^{-3} and 3.71×10^8 CMB photons m^{-3} then there are $\sim 10^8$–10^9 times as many photons as atoms; and we've just seen that the energy density of photons is 4.19×10^{-14} J m^{-3}. [2 marks]

The equivalent energy density of matter is

$$E = mc^2 = 1 \times 1.66 \times 10^{-27} \times (3.00 \times 10^8)^2 = 1.5 \times 10^{-10} \text{ J m}^{-3};$$

that is, 1 hydrogen atom represents about 12 orders of magnitude more energy than a 3K photon; and even though the photons outnumber the atoms by 8–9 orders of magnitude, their *total* energy is still much less (by 3–4 orders of magnitude). [2 marks]

Question 4
In Problem-Solving Tutorial 3, you were invited to consider a spiral galaxy viewed from 'above'. You should've been able to show that the total luminosity of the disk, integrated from the centre out to some radius R, is given by

$$L(R) = 2\pi L_0 r_0 \left[r_0 - (R + r_0) \exp\left(\frac{-R}{r_0} \right) \right] \quad (1)$$

where L_0 is the central surface brightness and r_0 is the scale length for the particular galaxy. Here we will explore this same topic in more detail than was possible in the PST environment.

For a particular galaxy, we suppose the central surface brightness is

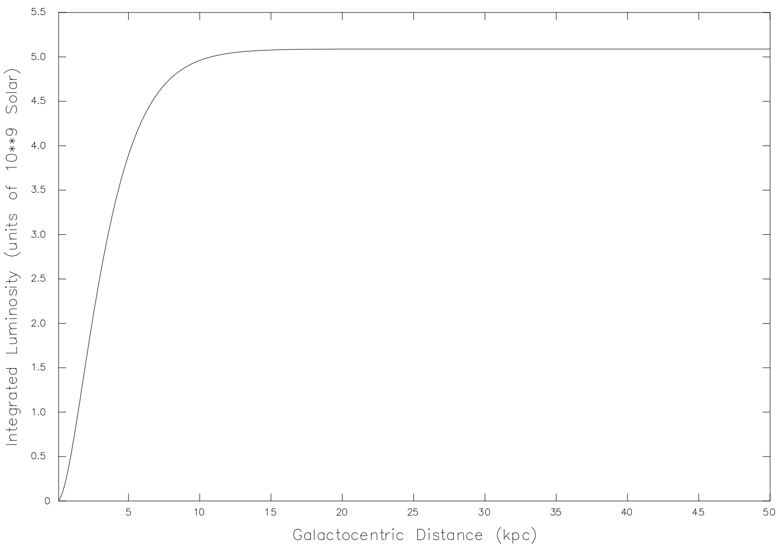
$$L_0 = 250 L_\odot \text{ pc}^{-2}$$

and the scale length is

$$r_0 = 1.8 \text{ kpc}.$$

By using a spreadsheet (or other technique of your choice), evaluate and plot the total luminosity (in units of $L_\odot$) for R =0–50 kpc at steps of 0.5 kpc.

This just requires evaluating equation 1. Most students will probably do this with Excel; I wrote a computer program. Here's what I get:



[3 marks for correctly labelled graph with a reasonable curve (marks for numerical accuracy follow later).]

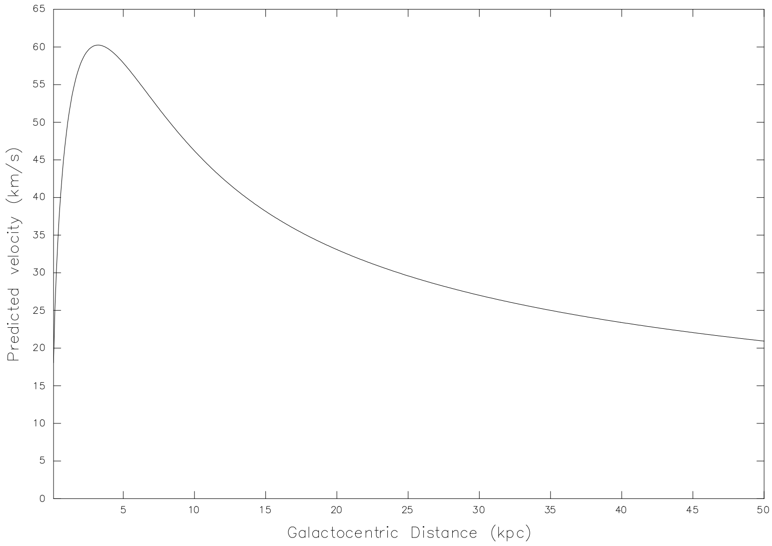
Again suppose, as a crude but reasonable approximation, that $M(R)/M_\odot$, the mass within radius R, equals $L(R)/L_\odot$, the luminosity within radius R, where both are measured in solar units. Plot the predicted rotation curve (the orbital velocities of stars, in km/s, as a function of galactocentric distance, R, in kpc).

For example, if the galaxy emits $\sim 5 \times 10^9$ solar luminosities within a radius of 10 kpc, we suppose that this luminosity is generated by $\sim 5 \times 10^9$ solar masses of stars (i.e., by $\sim 5 \times 10^9$ stars if each star weighed one solar mass). So we can simply switch the luminosity data already calculated to mass data (in effect, change the y-axis label on the above diagram from 'Luminosity' to 'Mass'). If the mass contained within some radius R is $M(R)$, then we can estimate the orbital velocity by requiring that the centrifugal and gravitational forces balance (this is, in effect, the definition of a stable orbit); that is, for some star of mass m,

$$\frac{GM(R)m}{R^2} = \frac{mv^2(R)}{R}; \text{ i.e.,}$$

$$v(R) = \sqrt{\frac{GM(R)}{R}}.$$

Again, I wrote a computer program to compute $v(R)$, the orbital velocity at radius R. My results are shown overleaf.



[3 marks for correctly labelled graph with a reasonable curve (marks for numerical accuracy follow later).]

Your answer should consist of two plots, plus a table listing $L(R)$, $M(R)$, and $v(R)$ for $R = 0, 10, 20, 30, 40$, and 50 kpc.

[You may find it helpful to look at the notes on PST 3, available on the PHAS 1102 web page.]

The plots have already been given. Here's a selection of my numerical results (more numbers than are required from students):

R (kpc)	$L(R)/L_\odot \equiv M(R)/M_\odot$	$v(R)$ (km s^{-1})
0.00000D+00	0.00000D+00	0.00000D+00
1.00000D+00	5.47081D+08	4.85087D+01
2.00000D+00	1.55245D+09	5.77813D+01
3.00000D+00	2.52602D+09	6.01799D+01
4.00000D+00	3.31224D+09	5.96794D+01
5.00000D+00	3.89394D+09	5.78767D+01
1.00000D+01	4.96040D+09	4.61904D+01
1.50000D+01	5.07796D+09	3.81586D+01
2.00000D+01	5.08846D+09	3.30805D+01
2.50000D+01	5.08931D+09	2.95906D+01
3.00000D+01	5.08937D+09	2.70125D+01
3.50000D+01	5.08938D+09	2.50088D+01
4.00000D+01	5.08938D+09	2.33936D+01
4.50000D+01	5.08938D+09	2.20557D+01
5.00000D+01	5.08938D+09	2.09238D+01

[5 marks for numerical accuracy and correct units.]