Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. Given a general quadratic equation in x, **[7]**

$$a\,x^2 + b\,x + c = 0\,,$$

where a, b and c are constants, complete the square to derive the quadratic-solutions formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}\,.$$

State how the number of real roots depends on the value of $b^2 - 4\,a\,c$.

2. The parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the forms **[7]**

$$x = a(\theta - \sin\theta)\,, \qquad y = a(1 - \cos\theta)\,,$$

where a is a constant. Show that the tangent to the curve has a slope, i.e. $\frac{dy}{dx}$, of $\cot\frac{\theta}{2}$.

3. A classical wave propagating in the x-direction has a dependency **[6]**

$$y(x,t) = A\sin(kx - \omega t)$$

where y may be e.g. a linear displacement; the amplitude, A, is a constant; ω, the angular frequency, is related to the frequency by $\omega = 2\pi f$; and k, the wavenumber, is related to the wavelength by $k = 2\pi/\lambda$. Show that this satisfies the differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

where v is the velocity of the wave.

[Part marks]

4. For vectors, $\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $\mathbf{b} = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$, write down the vector product, $\mathbf{a} \times \mathbf{b}$, in terms of the components. **[6]**

   For a particle moving in a circular orbit

   $$\mathbf{r} = \mathbf{i}\,r\cos\,\omega t + \mathbf{j}\,r\sin\,\omega t\,,$$

   (a) evaluate $\mathbf{r} \times \dot{\mathbf{r}}$;

   (b) evaluate $\ddot{\mathbf{r}} + \omega^2\mathbf{r}$.

   The radius, r, and angular velocity, ω, are constant.

5. (a) Given two complex numbers, $z_1 = 3 + 7\,i$ and $z_2 = -6\,i$, determine **[6]**

   $$\text{(i)}\ \ z_1 + z_2 \qquad \text{(ii)}\ \ z_1 - z_2 \qquad \text{(iii)}\ \ z_1\,z_2 \qquad \text{(iv)}\ \ z_1/z_2\,.$$

   (b) Determine the real and imaginary parts of the following :

   $$\text{(i)}\ \ \frac{1}{i^5} \qquad\qquad \text{(ii)}\ \ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)^2 .$$

6. Write down the general form of the Taylor series for a function $f(x)$. **[8]**

   Determine the Taylor series up to the cubic power of the following :

   (a) $\ln x$, about $x = 1$,

   (b) $\tan x$, about $x = \pi$.

[Part marks]

Section B

7. (a) Write down the product rule of differentiation. **[1]**

(b) Use the product rule to derive the equation for integration by parts. **[2]**

(c) Hence or otherwise, integrate the following : **[6]**

$$\text{(i)} \quad \int x^2\, e^{ax}\, dx\,, \qquad \text{(ii)} \quad \int x^n \, \ln x \, dx\,.$$

(d) Set up a reduction formula for **[7]**

$$I_n = \int_0^{\pi/2} \cos^n x \, dx$$

in order to find a relationship between I_n and I_{n-2}.

(e) Hence evaluate : **[4]**

$$\text{(i)} \quad \int_0^{\pi/2} \cos^4 x \, dx\,, \qquad \text{(ii)} \quad \int_0^{\pi/2} \cos^3 x \, dx\,.$$

[Part marks]

8. (a) From **[3]**

$$e^{i\theta} = \cos\theta + i\,\sin\theta\,,$$

express $\cos\theta$ and $\sin\theta$ in terms of exponentials.

(b) Write down the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials. Using these and the relationships derived in (a), express $\sinh ix$ in terms of $\sin x$ and $\cosh ix$ in terms of $\cos x$. **[4]**

(c) Express $\sinh(x + i\,y)$ in the form $u + i\,v$, where x, y, u and v are all real, and show that **[7]**

$$|\sinh(x + i\,y)|^2 = \frac{1}{2}(\cosh 2x - \cos 2y)\,.$$

(d) Show that $y = (\sinh^{-1} x)^2$ satisfies the equation **[6]**

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2\,.$$

[Part marks]

9. (a) For two vectors, $\mathbf{a}$ and $\mathbf{b}$, define the scalar and vector products in terms of the magnitudes of the vectors and angle between the vectors. **[3]**

(b) For two vectors, $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, determine the angle between them using both scalar and vector products. **[3]**

(c) Find the distance, d, from the point $P = (1, 1, 1)$ to the line, L, which passes through the points, $P_1 = (-3, 1 - 4)$ and $P_2 = (4, 4, -6)$. **[6]**

(d) The magnetic induction, $\mathbf{B}$, is defined by the Lorentz force equation **[8]**

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) .$$

Carrying out two experiments, we find

$$\mathbf{v} = \mathbf{i}, \qquad \frac{\mathbf{F}}{q} = 2\mathbf{k} - 4\mathbf{j} ,$$
$$\mathbf{v} = \mathbf{j}, \qquad \frac{\mathbf{F}}{q} = 4\mathbf{i} - \mathbf{k} .$$

From the results of these two experiments, calculate the magnetic field induction, $\mathbf{B}$, and verify that it agrees with a third experiment where

$$\mathbf{v} = \mathbf{k}, \qquad \frac{\mathbf{F}}{q} = \mathbf{j} - 2\mathbf{i} .$$

10. (a) Write down the general expression for the Maclaurin series expansion of a function $f(x)$ (i.e. the Taylor series expansion about $x = 0$). **[4]**

    Determine the Maclaurin series for $f(x) = \ln(1 + x)$.

  (b) Write down the binomial expansion for $(1 + x)^n$. **[2]**

  (c) The Klein–Nishina formula for the scattering of photons by electrons contains a term of the form **[6]**

$$f(\epsilon) = \frac{1 + \epsilon}{\epsilon^3} \left[\frac{2\epsilon(1 + \epsilon)}{1 + 2\epsilon} - \ln(1 + 2\epsilon) \right] ,$$

    where $\epsilon = h\nu/mc^2$ is the ratio of the photon's energy to the electron's rest mass energy. Using the series from (a) and (b), determine

$$\lim_{\epsilon \to 0} f(\epsilon) .$$

  (d) In a head-on proton-proton collision, the ratio of the kinetic energy in the centre-of-mass system to the incident kinetic energy is **[8]**

$$R = \frac{\sqrt{2mc^2(E_k + 2mc^2)} - 2mc^2}{E_k} ,$$

    where m and c are the mass of the proton and speed of light. Find the value of this ratio for :

(i) $E_k \ll mc^2$ (ii) $E_k \gg mc^2$.

[Part marks]

11. (a) We have a function $f(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$. Given **[7]**

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

show that we can change variables to get :

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\,.$$

(b) We have a function $u = f(x,y)$, where x and y are rectangular Cartesian coordinates which can also be expressed in polar coordinates (r,θ) as $x = r\cos\theta$ and $y = r\sin\theta$. Determine $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ and hence show that solving for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ gives **[6]**

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\,,$$
$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\,.$$

Hence in Cartesian coordinates given that $u = f(x,y)$ satisfies Laplace's equation **[7]**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,,$$

show that the form of this equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0\,.$$

 END OF PAPER