6 Series

6.1 Definitions

Given a sequence of numbers $a_1, a_2, a_3, ...a_n$, the sum of the first n numbers is called the nth partial sum S_n:

$$S_n = a_1 + a_2 + ... + a_n = \sum_{r=1}^{n} a_r \ .$$

If the partial sums S_1, S_2, S_3,... converge to a finite limit S

$$S = \lim_{n\to\infty} S_n \ ,$$

then S is defined as the sum of the *infinite* series

$$S = a_1 + a_2 + ... = \sum_{r=1}^{\infty} a_r \ ,$$

and the series is said to be *convergent*.

6.2 Different types of series and summation [see Riley 4.1]

<u>Arithmetic series</u>

In an arithmetic series there is a common difference d between successive terms in the series.
The sum S_n of an arithmetic series of n terms can be written as

$$S_n = \sum_{k=0}^{n-1}(a + kd) = a + (a + d) + (a + 2d)......... + (a + (n-2)d) + (a + (n-1)d) \ .$$

If we add the first term to the last, the second to the next to last and so on, we find that each term in the addition is the same and equal to $(2a + (n-1)d)$. Thus

$$S_n = \frac{n}{2}(2a + (n-1)d) \ .$$

For $n \to \infty$ the series will increase or decrease indefinitely, it diverges.
Example: find the sum of integer from 1 to N

$$S_n = 1 + 2 + 3 + 4 + ... + n = n + (n-1) + (n-2) + (n-3) + (n-4) + ... + 1 = \sum_{r=1}^{n} r$$

where we have written the series backwards as well. Adding the two series we see that each term adds to $n + 1$ and since there are n such terms which add to $2S_n$ we have the result

$$S_n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2} \ .$$

Or from the general formula, by replacing $a = 1$ and $d = 1$ we have:

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(2 + (n-1)) = \frac{n(n+1)}{2} \ .$$

<u>Geometric Series</u>

In a geometric series there is a common ratio r between successive terms in the series.
The sum S_n of a geometric series can be written term by term as

$$S_n = \sum_{k=0}^{n-1} ar^k = a + ar + ar^2 + ... + ar^{n-1}.$$

The above series when multiplied throughout by r yields

$$rS_n = \qquad ar + ar^2 + ... + ar^{n-1} + ar^n \ .$$

On subtraction of the above series we find

$$(1-r)S_n = (a - ar^n) = a(1-r^n) \ \text{ and } \ S_n = a\frac{(1-r^n)}{1-r} \ .$$

For $|r| < 1$ and $n \to \infty$ the series converges to the value

$$S = a\frac{1}{1-r}.$$

For $|r| \geq 1$ it diverges or oscillates.

Example: Consider a ball that drops from a height of 27 m and on each bounce retains only a third of its kinetic energy; thus after one bounce it will return to a height of 9 m, after two bounces to 3m, etc. Find the total distance travelled between the first bounce and the n^{th} bounce.

Given by the sum of $n-1$ terms (or distances):

$$S_{n-1} = 2(9 + 3 + 1 + ...) = 2\sum_{k=0}^{n-2} \frac{9}{3^k} \ ,$$

where the factor of 2 takes into account the up and down travel. The common ratio is 1/3.

$$S_{n-1} = 18\sum_{k=0}^{n-2} \frac{1}{3^k} = 18\frac{(1 - \left(\frac{1}{3}\right)^{n-1})}{1 - \frac{1}{3}} = 27[1 - \left(\frac{1}{3}\right)^{n-1}]$$

<u>Other series (Arithmetic-geometric series)</u>

Essentially a combination of the above has the form:

$$S_n = a + (a+d)r + (a+2d)r^2 + ... + [a + (n-1)d]r^{n-1} = \sum_{k=0}^{n-1}(a + kd)r^k \ .$$

To find the sum, we do as with a geometric series, and multiply by r.

$$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + ... + [a + (n-1)d]r^n \ .$$

Subtracting the two equations:

$$(1-r)S_n = a + dr + dr^2 + dr^3 + ... + dr^{n-1} - [a + (n-1)d]r^n \ .$$

The terms

$$dr + dr^2 + dr^3 + ... + dr^{n-1}$$

are just the geometric series we have had before and is equal to

$$d\frac{(1-r^n)}{1-r} - d = rd\frac{(1-r^{n-1})}{1-r}$$

thus

$$(1-r)S_n = a + d\frac{(1-r^n)}{1-r} - d - [a + (n-1)d]r^n \ .$$

and

$$\begin{aligned} S_n &= d\frac{(1-r^n)}{(1-r)^2} + \frac{a - d - [a + (n-1)d]r^n}{1-r} && (1) \\ &= rd\frac{(1-r^{n-1})}{(1-r)^2} + \frac{a - [a + (n-1)d]r^n}{1-r} \ . && (2) \end{aligned}$$

For an infinite series with $|r| < 1$, $n \to \infty$ ($r^n \to 0$):

$$S = \frac{a}{1-r} + \frac{rd}{(1-r)^2} \ .$$

If $|r| \geq 1$, the series diverges or oscillates.

The difference method for summation

Given a general series

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + ... + u_n$$

if u_k can be expressed as

$$u_k = f(k) - f(k-1) \ ,$$

then

$$S_n = [f(1) - f(0)] + [f(2) - f(1)] + ... + [f(n) - f(n-1)]$$

and cancelling terms

$$S_n = f(n) - f(0) \ .$$

Example:

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+2)}$$

First, we use partial fractions:

$$\frac{1}{k(k+2)} = \frac{1}{2k} - \frac{1}{2(k+2)} \ .$$

Hence

$$u_k = f(k) - f(k-2)$$

with

$$f(k) = -\frac{1}{2(k+2)} \ .$$

Thus

$$\begin{aligned}
S_n &= u_1 + u_2 + ... + u_n = \\
&= [f(1) - f(-1)] + [f(2) - f(0)] + [f(3) - f(1)] + ...[f(n-1) - f(n-3)] + [f(n) - f(n-2)] \\
&= f(n) + f(n-1) - f(0) - f(-1)
\end{aligned}$$

as all other terms cancel. Replacing for f we obtain:

$$S_n = -\frac{1}{2(n+2)} - \frac{1}{2(n+1)} + \frac{1}{4} + \frac{1}{2} = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{n+1}\right) \ .$$

Therefore the series converges, with sum

$$S = \lim_{n\to\infty} S_n = \frac{3}{4} \ .$$

Example
Consider the series:

$$\sum_{k=1}^{\infty} \frac{4}{(k+1)(k+3)} \ .$$

Using partial fractions we can write

$$\frac{4}{(k+1)(k+3)} = \frac{2}{(k+1)} - \frac{2}{(k+3)} = f(k) - f(k+2)$$

with

$$f(k) = \frac{2}{k+1} \ .$$

This allows us to evaluate the partial sum

$$S_n = \sum_{k=1}^{n} \frac{4}{(k+1)(k+3)}$$

as

$$S_n = f(1)-f(3)+f(2)-f(4)+f(3)-f(5)+...+f(n-2)-f(n)+f(n-1)-f(n+1)+f(n)-f(n+2)$$

from which

$$S_n = f(1) + f(2) - f(n+1) - f(n+2) = 1 + \frac{2}{3} - \frac{2}{n+2} - \frac{2}{n+3}$$

which shows that the series converges, with a sum

$$S = \frac{5}{3} \ .$$

Example
Determine for which values of x the series :

$$\sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)} \ .$$

converges. For these values, determine the sum of the series.
We can write

$$u_k = \frac{1}{(x+k)(x+k-1)} = \frac{1}{(x+k-1)} - \frac{1}{(x+k)} = f(k-1) - f(k) \ .$$

Thus for $x \neq 0, -1, -2, -3...., -k$ we have

$$S_n = u_1 + ... + u_n = f(0) - f(1) + f(1) - f(2) + ... + f(n-1) - f(n) = f(0) - f(n) = \frac{1}{x} - \frac{1}{x+n}$$

which tends to $1/x$. Thus, the series is convergent for $x \neq 0, -1, -2, -3....$ and has sum $\frac{1}{x}$.

6.3 Convergence of infinite series [see Riley 4.3]

We have touched on the convergence of infinite series and although we can sometimes give a value, it is not always easy. There are a number of tests we can use to assess convergence or not.

Preliminary test

If

$$\lim_{k\to\infty} u_k \neq 0 \ ,$$

then S_n does not converge. We stress that the condition $\lim_{k\to\infty} u_k = 0$ is necessary but not sufficient for convergence. There are many series which satisfy such a condition but which are divergent.

Example

$$S = \sum_{k=0}^{\infty} \frac{k^2}{4(k+1)(k+2)} \ .$$

As

$$\lim_{k\to\infty} \frac{k^2}{4(k+1)(k+2)} = \frac{1}{4} \ ,$$

S diverges.

Example

$$S = \sum_{k=1}^{\infty} \frac{1}{k} \ .$$

As

$$\lim_{k\to\infty} \frac{1}{k} = 0 \ ,$$

we cannot conclude whether the series converges or diverges on the basis of the preliminary test. We will actually show, using a different method, that it diverges.

Comparison test $(u_k, v_k > 0)$

If

$$\sum v_n \quad \text{converges}$$

and

$$u_n \leq v_n \ \ \text{for} \ \ n > N,$$

where N is some fixed number dependent on the series, then

$$\sum u_n \quad \text{converges} \ .$$

Similarly, if

$$\sum v_n \quad \text{diverges}$$

and

$$u_n \geq v_n \ \ \text{for} \ \ n > N,$$

where N is some fixed number dependent on the series, then

$$\sum u_n \quad \text{diverges} \; .$$

Example Consider the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....=\sum_{r=1}^{\infty}\frac{1}{r}\; .$$

It can be written as

$$1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+...$$

where every term in the brackets is larger than 1/2. Thus by comparison with the divergent series

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+....$$

we can conclude that the harmonic series is divergent.

Example Does

$$S=\sum_{k=0}^{\infty}\frac{1}{k!+1}=\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{25}+...$$

converge?

We use the fact (it will be demonstrated later) that

$$\sum_{k=0}^{\infty}\frac{1}{k!}$$

converges. Comparing:

$$\frac{1}{n!+1}<\frac{1}{n!}$$

we can conclude that S converge.

<u>D'Alembert ratio test</u>

If

$$\sum_{k=0}^{\infty}u_k\; ,$$

we define

$$\rho=\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|\; .$$

Then:

- If $\rho < 1$, the series converges
- If $\rho > 1$, the series diverges
- If $\rho = 1$, the behaviour is undetermined by this test.

Example

Determine if

$$S = \sum_{n=0}^{\infty} \frac{1}{n!}$$

converges. We calculate

$$\rho = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0 \ .$$

So $\rho < 1$ and therefore S converges.

Example

Determine if

$$S = \sum_{k=1}^{\infty} \frac{k!}{10^k}$$

converges. We calculate

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{(k+1)!}{10^{k+1}} \frac{10^k}{k!} = \lim_{k\to\infty} \frac{k+1}{10} = \infty \ .$$

So $\rho > 1$ and therefore S diverges.

Example

Determine if

$$S = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

converges. We calculate

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}} \frac{2^k}{k} = \lim_{k\to\infty} \frac{1}{2} \frac{k+1}{k} = \frac{1}{2} \ .$$

So $\rho < 1$ and therefore S converges.

6.4 Power series [see Riley 4.5]

A power series has the form

$$P(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + ...$$

To test the convergence of the series we use d'Alembert ratio test:

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1}x^{k+1}}{a_k x^k} \right| = |x| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \ .$$

$P(x)$ converges if

$$\rho < 1 \Rightarrow |x| < \frac{1}{\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|} \ .$$

This range of x is called the interval of convergence.

Example:

Does

$$P(x) = \sum_{k=1}^{\infty} (2x)^k = 1 + 2x + 4x^2 + 8x^3 + ...$$

converge?

We first calculate ρ:

$$\rho = \lim_{k\to\infty} \left| \frac{(2x)^{k+1}}{(2x)^k} \right| = |2x| \ .$$

Thus $P(x)$ converges for

$$|x| < \frac{1}{2} \ .$$

Example: Study the convergence of the series

$$\sum_{r=1}^{\infty} \frac{x^r}{r} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...$$

We first calculate ρ:

$$\rho = \lim_{r\to\infty} \left| \frac{x^{r+1}}{(r+1)} \frac{r}{x^r} \right| = \lim_{r\to\infty} \left| x \frac{r}{r+1} \right| = |x| \ .$$

Thus the power series converges for

$$|x| < 1 \ .$$

Example: Study the convergence of the series

$$\sum_{r=1}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

We first calculate ρ:

$$\rho = \lim_{r\to\infty} \left| \frac{x^{r+1}}{(r+1)!} \frac{r!}{x^r} \right| = \lim_{r\to\infty} \left| \frac{x}{(r+1)} \right| = 0$$

irrespective of the value of $|x|$. The series converges for all values of x, i.e. the interval of convergence is $-\infty < x < +\infty$.

Power series of complex numbers

The above may be extended to power series of complex numbers. For:

$$P(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + ...$$

$P(z)$ converges if

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| = |z| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1 \ .$$

Example

Determine the radius of convergence of the series

$$P(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n = 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + ...$$

$P(z)$ converges if

$$\rho = \frac{|z|}{2} \Rightarrow \quad |z| < 2 \ .$$

So $P(z)$ converges if z lies within a circle on an Argand diagram with radius equal to 2.

6.5 Taylor (and Maclaurin) series [see Riley 4.6]

The Taylor series is very useful and it is used to define a given function in terms of a power series. It can be applied to functions that are continuous and differentiable within the x range of interest.

Suppose we want to write $f(x)$ in the form:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ...$$

where we have expanded $f(x)$ about $x = x_0$. We now need to identify the constants.

For

$$x = x_0 \ , \qquad f(x_0) = a_0 \ .$$

Now we differentiate and set $x = x_0$:

$$\frac{df}{dx} = 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + ...$$

from which

$$\left| \frac{df}{dx} \right|_{x=x_0} = a_1 \ .$$

And again:

$$\frac{d^2 f}{dx^2} = 2a_2 + 6a_3(x - x_0) + ...$$

from which

$$\left|\frac{d^2 f}{dx^2}\right|_{x=x_0} = 2a_2 \ .$$

Continuing like this we get:

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... \\ &= f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=x_0} (x - x_0)^n \end{aligned}$$

For the case $x_0 = 0$, this is called the Maclaurin series:

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... \\ &= f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=0} x^n \end{aligned}$$

<u>Example</u>

Determine the Maclaurin series for $f(x) = \sqrt{1 + x}$.

By taking derivatives:

$$\begin{aligned} f(x) &= (1 + x)^{1/2} \implies f(0) = 1 \\ f'(x) &= \frac{1}{2}(1 + x)^{-1/2} \implies f'(0) = 1/2 \\ f''(x) &= -\frac{1}{4}(1 + x)^{-3/2} \implies f''(0) = -1/4 \\ ... \end{aligned}$$

from which

$$f(x) = 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{3}{48}x^3 + ...$$

<u>Example</u>

Determine the Maclaurin series for $f(x) = \ln(1 + x)$, and the interval of convergence.

We have

$$
\begin{array}{rcl}
f(x) & = & \ln(1+x) \implies f(0)=0 \\
f'(x) & = & (1+x)^{-1} \implies f'(0)=1 \\
f''(x) & = & -(1+x)^{-2} \implies f''(0)=-1 \\
f'''(x) & = & 2(1+x)^{-3} \implies f'''(0)=2 \\
f^{(4)}(x) & = & -6(1+x)^{-4} \implies f^{(4)}(0)=-6 \\
\ldots\ldots\ldots & .. & \ldots\ldots\ldots. \\
f^{(n)}(x) & = & (-1)^{n-1}(n-1)!(1+x)^{-n} \implies f^{(n)}(0)=(-1)^{n-1}(n-1)!
\end{array}
$$

from which

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

which can also written as ($k=n-1$, i.e. $n=k+1$)

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}$$

We determine now the radius of convergence.

$$\rho=\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|=\lim_{k\to\infty}\left|\frac{k+1}{k+2}x\right|=|x|\ .$$

Thus the interval of convergence is $|x|<1$.

<u>Example</u>

Determine the McLaurin expansion and the interval of convergence of

$$f(x)=\frac{1}{1-x}\ .$$

We have

$$
\begin{array}{rcl}
f(x) & = & (1-x)^{-1} \implies f(0)=1 \\
f'(x) & = & (1-x)^{-2} \implies f'(0)=1 \\
f''(x) & = & 2(1-x)^{-3} \implies f''(0)=2 \\
f'''(x) & = & 6(1-x)^{-4} \implies f'''(0)=6 \\
\ldots\ldots\ldots & .. & \ldots\ldots\ldots. \\
f^{(n)}(x) & = & n!(1-x)^{-n-1} \implies f^{(n)}(0)=n!
\end{array}
$$

from which

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

which is not surprising, as the initial expression is the sum of a geometric series.

We determine now the interval of convergence.

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \right| = |x| \ .$$

Thus the interval of convergence is $|x| < 1$.

<u>Example</u>

Determine the Maclaurin series of

$$\frac{4+5x}{(2+x)(1-x)}$$

up to the third order.

We can relate this series to the one previously examined. We decompose the fraction into partial fractions:

$$\frac{4+5x}{(2+x)(1-x)} = \frac{A}{(2+x)} + \frac{B}{(1-x)}$$

which implies

$$4 + 5x = A(1-x) + B(2+x) \ .$$

This gives $A = -2$ and $B = 3$, i.e.

$$\frac{4+5x}{(2+x)(1-x)} = \frac{-2}{(2+x)} + \frac{3}{(1-x)}$$

Using the previous result:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3....$$

with an interval of convergence $|x| < 1$. And

$$\frac{2}{2+x} = \frac{1}{1-\left(-\frac{x}{2}\right)} = 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} +,,,$$

with an interval of convergence $|x/2| < 1$, i.e. $|x| < 2$.
Thus

$$\frac{4+5x}{(2+x)(1-x)} = -(1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} +,,,) + 3(1 + x + x^2 + x^3 + ...) = 2 + \frac{7}{2}x + \frac{11}{4}x^2 + \frac{25}{8}x^3 + ...$$

which converges for $|x| < 1$, i.e. for the interval for which both series converge.

<u>Example</u>

Determine the Maclaurin series for e^x, and the interval of convergence.

As
$$\frac{d^n e^x}{dx^n} = e^x$$
so
$$\left.\frac{d^n e^x}{dx^n}\right|_{x=0} = 1 \ .$$
Thus the Maclaurin series is
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ .$$
We saw in a previous example that the series converges for all values of x, i.e. the interval of convergence is $-\infty < x < +\infty$.

Example

Determine the Maclaurin series of $\sin\theta$ and $\cos\theta$.

For $\sin\theta$:
$$\begin{aligned}
\frac{d}{d\theta}(\sin\theta) &= \cos\theta = 1 \quad \text{for} \quad \theta = 0 \\
\frac{d^2}{d\theta^2}(\sin\theta) &= -\sin\theta = 0 \quad \text{for} \quad \theta = 0 \\
\frac{d^3}{d\theta^3}(\sin\theta) &= -\cos\theta = -1 \quad \text{for} \quad \theta = 0 \\
\frac{d^4}{d\theta^4}(\sin\theta) &= \sin\theta = 0 \quad \text{for} \quad \theta = 0 \ .
\end{aligned}$$
Thus the Maclaurin series of $\sin\theta$ is:
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} ...$$
For $\cos\theta$ we can proceed in the same way:
$$\begin{aligned}
\frac{d}{d\theta}(\cos\theta) &= -\sin\theta = 0 \quad \text{for} \quad \theta = 0 \\
\frac{d^2}{d\theta^2}(\cos\theta) &= -\cos\theta = -1 \quad \text{for} \quad \theta = 0 \\
\frac{d^3}{d\theta^3}(\cos\theta) &= \sin\theta = 0 \quad \text{for} \quad \theta = 0 \\
\frac{d^4}{d\theta^4}(\cos\theta) &= \cos\theta = 1 \quad \text{for} \quad \theta = 0 \ .
\end{aligned}$$
and find
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} ...$$
We notice that the same result could have been found by differentiating the series for $\sin\theta$ as $(\sin\theta)' = \cos\theta$.

Example

Determine the Maclaurin series of
$$f(x) = e^{-3x}\cos(2x)$$
up to the third order.
Using previous results:
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$
with convergence for all values of x. Thus
$$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3!} + ...$$
Also, we saw that:
$$\cos x = 1 - \frac{x^2}{2} + ...$$
with convergence for all values of x. Thus
$$cos(2x) = 1 - \frac{(2x)^2}{2} + ... = 1 - 2x^2 + ...$$
Thus
$$e^{-3x}\cos(2x) = (1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3)(1 - 2x^2) + ... = 1 - 3x + \frac{5}{2}x^2 + \frac{3}{2}x^3 + ...$$
which converges for all x.

Example

Determine the Maclaurin series of
$$f(x) = (\sin x)\ln(1 - 2x)$$
up to the fourth order.

Using previous results:
$$\sin(x) = x - \frac{x^3}{3!} + ... = x - \frac{x^3}{6} + ...$$
which converges for all x. We also saw that
$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}...$$
Therefore
$$\ln(1 - 2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4}... = -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - ...$$
Combining the two
$$(\sin x)\ln(1 - 2x) = (x - \frac{x^3}{6})(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4) = -2x^2 - 2x^3 - \frac{7}{3}x^4 + ...$$

Proof of Euler's equation

We are now in a position to prove Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We use the series of the exponential we derived earlier:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + ... \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

Taylor expansion for $f(x, y)$ about (x_0, y_0)

$$\begin{aligned}
f(x,y) &= f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \Delta y \\
&+ \frac{1}{2!}\left[\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} (\Delta x)^2 + 2\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} \Delta x \Delta y + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} (\Delta y)^2\right] + ...
\end{aligned}$$

where $\Delta x = x - x_0$ and $\Delta y = y - y_0$.

Example

Find a quadratic approximation to the function $f(x, y) = \sin x \sin y$ near the origin.

We calculate the derivatives:

$$\begin{aligned}
f(0,0) &= 0 \\
f_x(0,0) &= \cos x \sin y|_{(0,0)} = 0 \\
f_y(0,0) &= \sin x \cos y|_{(0,0)} = 0 \\
f_{xx}(0,0) &= -\sin x \sin y|_{(0,0)} = 0 \\
f_{xy}(0,0) &= \cos x \cos y|_{(0,0)} = 1 \\
f_{yy}(0,0) &= -\sin x \sin y|_{(0,0)} = 0
\end{aligned}$$

Thus up to the second order

$$\sin x \sin y = xy$$

Example

Find the second order Maclaurin expansion of $f(x,y) = e^{2x}\sin(3y)$.

This can be done in two different ways. The first way is to compute all partial derivatives up to the second order:

$$\begin{aligned}
f(0,0) &= e^{2x}\sin(3y)|_{(0,0)} = 0 \\
f_x(0,0) &= 2e^{2x}\sin(3y)|_{(0,0)} = 0 \\
f_y(0,0) &= 3e^{2x}\cos 3y|_{(0,0)} = 3 \\
f_{xx}(0,0) &= 4e^{2x}\sin 3y|_{(0,0)} = 0 \\
f_{xy}(0,0) &= 6e^{2x}\cos 3y|_{(0,0)} = 6 \\
f_{yy}(0,0) &= -9e^{2x}\sin 3y|_{(0,0)} = 0
\end{aligned}$$

Thus up to the second order

$$f(x,y) = 3y + 6xy$$

A second way exploits the single variable Maclaurin expansions:

$$e^x = 1 + x + \frac{x^2}{2} + ..$$

and

$$\sin y = y + ...$$

Replacing x by $2x$ in the first and y by $3y$ in the second we obtain after multiplication

$$f(x,y) = (1 + 2x + \frac{(2x)^2}{2})3y = 3y + 6xy$$

where we retain terms up to the second order.

We can use the result for the Taylor expansion to determine the nature of stationary points in multivariate functions which we previously just stated in the partial differentiation section.

For a stationary point:

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

thus using the Taylor expansion we have

$$f(x,y) - f(x_0,y_0) = \frac{1}{2!}\left[f_{xx}(\Delta x)^2 + 2f_{xy}\Delta x\Delta y + f_{yy}(\Delta y)^2\right]$$

where we neglected higher order terms.

We can rewrite this as:

$$f(x,y) - f(x_0,y_0) = \frac{1}{2f_{xx}}\frac{1}{2!}\left[\left(f_{xx}\Delta x + f_{xy}\Delta y\right)^2 - (\Delta y)^2\left(f_{xy}^2 - f_{xx}f_{yy}\right)\right]$$

or

$$f(x,y) - f(x_0,y_0) = \frac{1}{2f_{yy}}\frac{1}{2!}\left[(f_{xy}\Delta x + f_{yy}\Delta y)^2 - (\Delta x)^2\left(f_{xy}^2 - f_{xx}f_{yy}\right)\right]$$

If

$$f_{xy}^2 - f_{xx}f_{yy} < 0 \ ,$$

then the terms in the square brackets are always positive and so the sign of $f(x,y) - f(x_0,y_0)$ is completely determined by the sign of f_{xx} and f_{yy}.

We have a maximum if

$$f_{xy}^2 < f_{xx}f_{yy} \quad \text{and} \quad f_{xx} < 0 \quad f_{yy} < 0$$

and have a minimum if

$$f_{xy}^2 < f_{xx}f_{yy} \quad \text{and} \quad f_{xx} > 0 \quad f_{yy} > 0$$

Finally we have a saddle point if

$$f_{xy}^2 > f_{xx}f_{yy}$$

6.6 Limits [see Riley 4.7]

Definition of limit

Consider the function $f(x)$. If we can make $f(x)$ as near as we want to a given number l by choosing x sufficiently near to a number a, then l is said to be the limit of $f(x)$ as $x \to a$ and it is written as

$$\lim_{x\to a} f(x) = l \ .$$

Properties of limits

If $f(x)$ and $g(x)$ are two functions of x such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:

- Limit of a sum
$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
- Limit of a product
$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$
- Limit of a quotient
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$
provided that $\lim_{x\to a} g(x) \neq 0$.

The limit of a function $f(x)$ as x tends to a being $f(a)$ seems reasonable. However $f(a)$ may be undefined. E.g.

$$\lim_{x\to 0}\frac{\sin x}{x}$$

which can be shown to be equal to 1 (see below) although $\sin x/x$ is undefined for $x = 0$.

Evaluating limits using Taylor expansion

Example I

Consider the limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Now we know that the Maclaurin expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

Thus for small values of x we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

From which we easily derive that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \ .$$

Example II

Consider the limit

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

Now we know that the Maclaurin expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Thus for small values of x

$$\frac{e^x - 1}{x} = \frac{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - ...) - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + ...$$

From which we easily derive that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \ .$$

Evaluating limits using l'Hôpital's rule

To evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

when

$$f(a) = g(a) = 0$$

take a Taylor expansion:

$$\frac{f(x)}{g(x)} = \frac{f(a) + (x-a)f'(a) + [(x-a)^2/2!]f''(a) + ...}{g(a) + (x-a)g'(a) + [(x-a)^2/2!]g''(a) + ...}$$

but $f(a) = g(a) = 0$, and dividing by $(x-a)$

$$\frac{f(x)}{g(x)} = \frac{f'(a) + [(x-a)/2!]f''(a) + ...}{g'(a) + [(x-a)/2!]g''(a) + ...} \ .$$

Hence:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}}$$

assuming $g'(a) \neq 0$.

If $f'(a) = g'(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}$$

Provided the limit exists it is usually possible to find a value of n such that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

Note that the rule of de l'Hôpital also applies for $a = \infty$ and also for $f(a) = g(a) = \infty$. In fact if $f(x)$ and $g(x)$ both tend to infinity as $x \to a$, one can show that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

still holds.

Similarly, if $f(x)/g(x)$ becomes $0/0$ or ∞/∞ as $x \to \infty$, then we can go back to the standard form to de l'Hôpital rule by introducing the new variable y

$$x = \frac{1}{y}$$

so that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0} \frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)} = \lim_{y\to 0} \frac{-\frac{1}{y^2} f'\left(\frac{1}{y}\right)}{-\frac{1}{y^2} g'\left(\frac{1}{y}\right)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

<u>Example I</u>

Determine the limit:

$$\lim_{x\to 0} \frac{\sin(3x)}{\sinh(x)}$$

This is an indefinite form of the type 0/0. From de l'Hôpital rule:

$$\lim_{x\to 0}\frac{\sin(3x)}{\sinh(x)} = \lim_{x\to 0}\frac{3\cos(3x)}{\cosh(x)} = 3\ .$$

Example II

Determine the limit:

$$\lim_{x\to 0}[x\ln x]$$

This is an indefinite form of the type $0\cdot\infty$. In order to apply the de l'Hôpital rule, we re-arrange as follows:

$$\lim_{x\to 0}[x\ln x] = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -x = 0\ .$$

Example III

Determine the limit:

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{x\to 0}\frac{2e^{2x}}{1} = 2$$

Example IV

Determine the limit:

$$\lim_{x\to 1}\frac{1+\cos(\pi x)}{x^2-2x+1} = \lim_{x\to 1}\frac{-\pi\sin(\pi x)}{2x-2} = \lim_{x\to 1}\frac{-\pi^2\cos(\pi x)}{2} = \frac{\pi^2}{2}$$

where we applied the rule of de l'Hôpital twice.

Example V

Determine the limit:

$$\lim_{x\to\infty}\left(x^2e^{-x}\right)$$

his is an indefinite form of the type $\infty\cdot 0$. In order to apply the de l'Hôpital rule, we re-arrange as follows:

$$\lim_{x\to\infty}\left(x^2e^{-x}\right) = \lim_{x\to\infty}\frac{x^2}{e^x}$$

which is an indefinite form of the type ∞/∞. And applying the rule of de l'Hôpital twice.

$$\lim_{x\to\infty}\left(x^2e^{-x}\right) = \lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

Example VI

Determine the limit:

$$\lim_{x\to\infty} x^{1/x}$$

This is an indefinite form of the type ∞^0. We can re-rarrange it in a form appropriate for the application of de l'Hôpital rule by introducing the new variable

$$y = x^{1/x}$$

from which

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

whose limit for $x \to \infty$ is an indefinite form of the type ∞/∞. We can therefore apply de l'Hôpital rule to $\ln y$:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

from which

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1 \ .$$

Example VII

Determine the limit:

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$$

This is an indefinite form of the type $\infty - \infty$. We can re-arrange it in a form appropriate for the application of de l'Hôpital:

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$$

which is of the form $0/0$. And applying de l'Hôpital rule repeatedly:

$$\lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x\to 0}\frac{2x - \sin 2x}{2x\sin^2 x + x^2 \sin 2x} = \lim_{x\to 0}\frac{2 - 2\cos 2x}{2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x} =$$

$$\lim_{x\to 0}\frac{4\sin 2x}{6\sin 2x + 12x\cos 2x - 4x^2\sin 2x} = \lim_{x\to 0}\frac{8\cos 2x}{24\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 2x} = \frac{1}{3}$$