Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. (a) Given the vectors $\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$, $\underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$, state expressions for the following:

   i. the scalar product $\underline{a} \cdot \underline{b}$, **[1]**

   ii. the vector product $\underline{a} \times \underline{b}$, **[2]**

   iii. the modulus $|\underline{a}|$, **[1]**

   in terms of the coordinates a_x, a_y, a_z and b_x, b_y, b_z .

   Answer:

   (i) $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

   (ii) $\underline{a} \times \underline{b} = (a_y b_z - a_z b_y)\underline{i} + (a_z b_x - a_x b_z)\underline{j} + (a_x b_y - a_y b_x)\underline{k}$

   (iii) $|\underline{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

   (b) For the vectors $\underline{a} = 3\underline{i}+6\underline{j}+3\underline{k}$, $\underline{b} = \underline{i}+2\underline{j}-5\underline{k}$, determine the angle between them. **[2]**

   Answer: Angle is $\pi/2$

2. (a) Reduce to the standard form $x + iy$ (x, y real) the following expressions

   i. $1 + i^2 + i^3 + \frac{1}{i^3}$, **[1]**

   ii. $\frac{3+5i}{4-7i}$, **[1]**

   iii. $\frac{|3+2i|}{(2-3i)^2}$. **[2]**

   Answer:

   (i) 0 (zero)

   (ii) $-\frac{23}{65} + \frac{41}{65}i$

   (iii) $-\frac{5}{\sqrt{13^3}} + \frac{12}{\sqrt{13^3}}i$

   (b) Solve $z^4 + 16 = 0$ for complex z . Give the solution(s) in the standard form $x + iy$ (x, y real). **[2]**

   Answer:

   $z_0 = \sqrt{2} + i\sqrt{2},\ z_1 = -\sqrt{2} + i\sqrt{2},\ z_2 = -\sqrt{2} - i\sqrt{2},\ z_0 = \sqrt{2} - i\sqrt{2}$

(c) Given two complex numbers z_1 and z_2 , prove that **[2]**

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)\,.$$

Answer: Derivations not shown as answers

3. (a) State the formal definition of the derivative of a function $f(x)$, and provide its graphical interpretation. **[2]**

   Answer: See lecture notes

   (b) Derive the rule for the derivative of the product $u(x)v(x)$ of the two functions $u(x)$ and $v(x)$. **[3]**

   Answer: See lecture notes

   (c) Calculate the derivative of $y = \cosh^{-1} x$. **[2]**

   Answer: $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$

[Part marks]

4. (a) Write down the general form of the Maclaurin series for a function $f(x)$. **[2]**
   Answer: See lecture notes
   (b) Write down or derive the Maclaurin series for the functions e^x, $\sin x$ and $\cos x$. **[2]**
   Answer: See lecture notes
   (c) Hence show that **[2]**
$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$
   Answer: See lecture notes

5. (a) By resolving the integrand into two terms, evaluate the following indefinite integral: **[3]**
$$I = \int \frac{x^3}{x^2-1}dx\,.$$
   Answer: $\frac{1}{2}x^2 + \frac{1}{2}\ln|x^2-1| + c$
   (b) Determine the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function **[2]**
$$f(x,y) = \ln(x^2+y^2) - 2\arctan\left(\frac{y}{x}\right)\,.$$
   Answer: $\frac{\partial f}{\partial x} = \frac{2(x+y)}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{2(y-x)}{x^2+y^2}$
   (c) Given a general differential df, of the form **[2]**
$$df = A(x,y)dx + B(x,y)dy\,,$$
   state the condition on $A(x,y)$ and $B(x,y)$ such that df is exact.
   Answer: $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

6. (a) Consider a ball that drops from a height of 27 m and on each bounce retains only a third of its height. Thus after one bounce it will return to a height of 9 m, after two bounces, 3 m, etc..
      i. Find the total distance travelled between the *first bounce* and the N^{th} bounce, in terms of N. **[2]**
      ii. As $N \to \infty$, what is the total distance travelled? **[1]**

   Answer:
   (i) $27\left[1 - \left(\frac{1}{3}\right)^{N-1}\right]$
   (ii) 27
   (b) Determine the limit **[3]**
$$\lim_{x\to 0}\frac{\sin(2x)}{5x^2+7x}\,.$$
   Answer: $\frac{2}{7}$

Section B

Answer ANY THREE questions from this Section.
Note: Only three Section B answers will be marked.

7. (a) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, prove **[4]**

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,.$$

Answer: Derivations not shown as answers

(b) Show that **[4]**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\,.$$

Answer: Derivations not shown as answers

(c) By substituting $z = x+iy$ or $z = re^{i\theta}$ in the following equations and inequalities, sketch and describe the following regions on the complex plane in separate Argand diagrams:

i. $|z-3-4i| < 5\,,$ **[3]**

ii. $\arg(z) = \frac{\pi}{3}\,,$ **[3]**

iii. $e^z = 1\,,$ **[3]**

iv. $\mathrm{Im}(z^2) < 0\,.$ **[3]**

Answer:

(i) Circular disk (excluding bordering circle) around (3,4) with radius 5

(ii) Line from (but excluding) origin with angle of $\pi/3$ against positive x-axis (in first quadrant only)

(iii) Set of points $(x,y) = (0, 2\pi n)$ with $n = \ldots, -2, -1, 0, 1, 2, \ldots$

(iv) Region covering 2nd and 4th quadrants completely but excluding the axes

8. (a) Determine the following integrals:

   i. $\int \ln x \, dx$, **[2]**

   ii. $\int_0^{\pi/2} \sin^3 x \, dx$. **[2]**

   Answer:

   (i) $x(\ln x - 1) + c$

   (ii) $\frac{2}{3}$

(b) Find the derivative of each of the following functions $y(x)$:

   i. $y(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$, **[2]**

   ii. $y(x) = x^x - 3x^2$, **[2]**

   iii. $\frac{y(x)}{x - y(x)} = x^2 + 1$. **[2]**

   Answer:

   (i) $\frac{1+x}{2x^{3/2}}$

   (ii) $x^x(\ln x + 1) - 6x$

   (iii) $\frac{x^4+5x^2+2}{(x^2+2)^2}$

(c) Let

$$f(x) = \frac{x - \frac{3}{2}}{x^2 + 2}$$

and

$$g(x) = \frac{x^2 + 1}{x^2 + 2} \; .$$

At what value(s) of x do the curves $y = f(x)$ and $y = g(x)$ have parallel tangent lines? **[5]**

Answer: Tangents are parallel at the points $x_1 = -1$, $x_2 = 2$

(d) Suppose that $g(x)$ is a differentiable function and that $f(x) = g(x + 5)$ for all x . If the derivative of g is $g'(1) = 3$, determine a value x_0 such that the derivative of f at this point is $f'(x_0) = 3$. **[5]**

Answer: $x_0 = -4$

[Part marks]

9. (a) The position vector of a particle in 2D polar coordinates is given by $\underline{r} = x\underline{i} + y\underline{j}$. **[4]**
Briefly explain, with the aid of a diagram, the relationships between the unit vectors in 2D polar coordinates ($\underline{\hat{r}}$, $\underline{\hat{\theta}}$) and the Cartesian unit vectors ($\underline{i}$, $\underline{j}$):

$$\begin{aligned} \underline{\hat{r}} &= \quad \cos\theta\,\underline{i} + \sin\theta\,\underline{j}\,, \\ \underline{\hat{\theta}} &= -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\,. \end{aligned}$$

Answer: See lecture notes

(b) Show that the velocity vector of the particle is given by **[5]**

$$\underline{v} = \frac{dr}{dt}\underline{\hat{r}} + r\frac{d\theta}{dt}\underline{\hat{\theta}}\,.$$

Answer: See lecture notes

(c) Show that that the acceleration vector of the particle is given by **[6]**

$$\underline{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\underline{\hat{r}} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\underline{\hat{\theta}}\,.$$

Answer: See lecture notes

(d) A particle moves with $\frac{d\theta}{dt} = \omega$ and $r = r_0 e^{\beta t}$, where ω , r_0 and β are constants. **[5]**
For what values of β does the radial part of the acceleration vector $\underline{a}$ vanish?
Answer: $\beta = \pm\omega$

10. (a) A plane P contains three points A, B and C with position vectors $\underline{a} = 5\underline{i}$, $\underline{b} = 2\underline{j}$ and $\underline{c} = 3\underline{k}$, respectively.
Write the equation of the plane P in the form $n_x x + n_y y + n_z z = d$, where n_x, n_y and n_z are the components of a vector $\underline{n}$ perpendicular to the plane. **[4]**
Answer: $\frac{6}{19}x + \frac{15}{19}y + \frac{10}{19}z = \frac{30}{19}$

(b) A particle moves from the origin $\underline{r}_D = 0$ to $\underline{r}_E = 6\underline{i} + 4\underline{j} + 3\underline{k}$, in a straight line with constant velocity such that the total trip requires 10 seconds.

i. Write down a parametrisation of the path C taken by the particle. **[3]**
ii. Show that the path C intersects with the plane P at the time $t_I = 50/21$ seconds. Determine the location $\underline{r}_I$ of this point of intersection. **[6]**

Answer:
(i) $\underline{r}(t) = t\left(\frac{3}{5}\underline{i} + \frac{2}{5}\underline{j} + \frac{3}{10}\underline{k}\right)$ with $t_D = 0$, $t_E = 10$
(ii) Derivation not shown; point of intersection is $\frac{10}{7}\underline{i} + \frac{20}{21}\underline{j} + \frac{5}{7}\underline{k}$

(c) The plane P separates the space into two parts, each with different force vector fields present:
In the part containing the origin $\underline{r} = 0$, the field is $\underline{F}_1 = y\underline{i} + \frac{1}{2}x\underline{j}$.
In the other part, the field is $\underline{F}_2 = \frac{1}{\sqrt{x^2+y^2+z^2}}(3\underline{j} - 4\underline{k})$.
Determine the work done by the particle as it travels along the path C, i.e. calculate the line integral **[7]**

$$W = \int_C \underline{F} \cdot d\underline{r}\,.$$

Here, $\underline{F}$ is the force experienced by the particle, i.e. either $\underline{F}_1$ or $\underline{F}_2$, depending on the position of the particle.
Answer: $W = \frac{50}{49}$

[Part marks]

11. (a) Given a function $z = f(x, y)$, state the condition for a point (x_0, y_0) to be stationary. **[2]**

    Answer: See lecture notes

    (b) You are asked to manufacture cylindrical drink cans of height h and radius r. The volume of a can is required to be $V = 10$ (in arbitrary units) and its total surface area A should be minimised. Determine the optimal values for h and r to satisfy these requirements. **[4]**

    Answer: $r = \left(\frac{5}{\pi}\right)^{1/3}$, $h = 2\left(\frac{5}{\pi}\right)^{1/3}$

    (c) i. State and derive l'Hôpital's rule. **[3]**

    ii. Evaluate, using l'Hôpital's rule or otherwise, the limit **[3]**

$$\lim_{x \to 0} \frac{\tan x - x}{x - \sin x} \, .$$

    Answer:

    (i) See lecture notes

    (ii) 2

    (d) Evaluate the limit **[4]**

$$\lim_{x \to +\infty} \sqrt{x} \left(\sqrt{x+3} - \sqrt{x-2} \right) \, .$$

    Answer: $\frac{5}{2}$

    (e) Evaluate the limit **[4]**

$$\lim_{x \to 0} \frac{\sqrt{|x|} \cos(\pi^{\frac{1}{x^2}})}{2 + \sqrt{x^2 + 3}} \, .$$

    Answer: 0 (zero)