Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. (a) State the formal definition of the derivative of a function $f(x)$. **[2]**

   (b) Using the formal definition of the derivative, calculate from first principles the derivative of $f(x) = 2\sqrt{x}$ at $x = 1$. **[4]**

(a) See lecture notes.

(b) $f'(x)|_{x=1} = 1$.

[Part marks]

2. (a) Use partial fractions to calculate the following indefinite integrals: **[4]**

$$(i) \quad \int \frac{x}{x^2 - 3x - 4} dx \ ;$$
$$(ii) \quad \int \frac{3x - 1}{(x-1)(x^2+1)} dx \ .$$

(b) Use integration by parts to calculate the following indefinite integrals: **[4]**

$$(i) \quad \int x \sin x \, dx \ ;$$
$$(ii) \quad \int x^3 \ln x \, dx \ .$$

(a) (i) $\int \frac{x}{x^2-3x-4} = \frac{4}{5} \ln|x-4| + \frac{1}{5} \ln|x+1| + const.$

(ii) $\int \frac{3x-1}{(x-1)(x^2+1)} = \ln|x-1| - \frac{1}{2} \ln(x^2+1) + 2\tan^{-1} x + const.$

(b) (i) $\int x \sin x dx = -x \cos x + \sin x + const$.

(ii) $\int x^3 \ln x dx = \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + const$.

[Part marks]

3. (a) Calculate the first derivative of the following functions: **[4]**

$$(i) \quad f(x) = \frac{\sqrt{x}}{\sqrt{x}-1} \; ;$$
$$(ii) \quad g(x) = e^{\ln(x^2)} - 3x^{-7} \; .$$

(b) Given a function $y = f(x)$, state the condition for a point x_0 to be stationary. **[2]**

(c) Given a function $y = f(x)$, state the criteria to determine the nature of a stationary point. **[2]**

(a) (i) $f'(x) = -\frac{1}{2}\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}$.

(ii) $g'(x) = 2x + 21x^{-8}$.

(b) See lecture notes.

(c) See lecture notes.

[Part marks]

4. (a) Let $\underline{v} = 5\underline{i} + \underline{j} - 2\underline{k}$ and $\underline{w} = 4\underline{i} - 4\underline{j} + 3\underline{k}$. Calculate $\underline{v} \cdot \underline{w}$. **[2]**

   (b) Let $\underline{v} = 8\underline{i} + 4\underline{j} + 3\underline{k}$ and $\underline{w} = 2\underline{i} + \underline{j} + 4\underline{k}$. Is $\underline{v}$ perpendicular to $\underline{w}$? Justify your answer. **[2]**

   (c) Calculate $\underline{u} \times (\underline{v} \times \underline{w})$ for $\underline{u} = \underline{i} + 2\underline{j} + 4\underline{k}$, $\underline{v} = 2\underline{i} + 2\underline{j}$, $\underline{w} = \underline{i} + 3\underline{j}$. **[2]**

(a) $\underline{v} \cdot \underline{w} = 10$.

(b) $\underline{v}$ and $\underline{w}$ are not perpendicular.

(c) $\underline{v} \times \underline{w} = 4\underline{k}$,

$\underline{u} \times (\underline{v} \times \underline{w}) = 8\underline{i} - 4\underline{j}$.

[Part marks]

5. (a) Write an expression for the absolute value $|z|$ of the complex number $z = a+ib$, with a and b real. **[1]**

   (b) Write an expression for the complex conjugate z^* of the complex number $z = a + ib$, with a and b real. **[1]**

   (c) Compute the absolute value and the complex conjugate of the complex number **[2]**

$$w = i^{17} \ .$$

   (d) Determine the real and imaginary parts of the complex number **[2]**

$$z = \frac{1 + 4i}{3 + 2i} \ .$$

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5. (a) $|z| = \sqrt{a^2 + b^2}$.

   (b) $z^* = a - ib$.

   (c) $|w| = 1, w^* = -i$.

   (d) $Re(z) = \frac{11}{13},\ Im(z) = \frac{10}{13}$.

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6. Calculate the following limits: **[6]**

$$(i) \quad \lim_{x\to+\infty} \frac{6x+1}{2x+5} \ ;$$

$$(ii) \quad \lim_{x\to 1} \frac{x^5-1}{x-1} \ ;$$

$$(iii) \quad \lim_{x\to 0} \frac{2\sin x - \sin(2x)}{x - \sin x} \ .$$

(a) $\lim_{x\to\infty} \frac{6x+1}{2x+5} = 3$.

(b) $\lim_{x\to 1} \frac{x^5-1}{x-1} = 5$.

(c) $\lim_{x\to 0} \frac{2\sin x-\sin(2x)}{x-\sin x} = 6$

Section B

[Part marks]

7. (a) Write down the general form of the Maclaurin series for a function $f(x)$. **[4]**

(b) Determine the Maclaurin series for the function **[2]**

$$f(x) = \ln(1+x) \ .$$

Hence, or otherwise, determine the Maclaurin series for the function **[4]**

$$g(x) = \ln\left(\frac{1+x}{1-x}\right) \ .$$

(c) Determine the first three non-zero terms in the Maclaurin series for the function **[2]**

$$f(x) = \cos(\sin x) \ .$$

Hence, or otherwise, determine the limit **[2]**

$$\lim_{x\to 0} \frac{1-\cos(\sin x)}{x^2} \ .$$

(d) Determine the limit **[6]**

$$\lim_{x\to +\infty} x(\sqrt{x^2+1} - \sqrt[3]{x^3+1}) \ .$$

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7. (a) See lecture notes.

(b) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{x^3}{3} + ...,$

$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}.$

(c) $\cos(\sin x) = 1 - \frac{x^2}{2} + \frac{5x^4}{24} + ...$

$\lim_{x\to 0} \frac{1-\cos(\sin x)}{x^2} = \frac{1}{2}.$

(d) $\lim_{x\to\infty} x(\sqrt{x^2+1} - \sqrt[3]{x^3+1}) = \frac{1}{2}.$

[Part marks]

8. (a) State and derive de Moivre's theorem. **[4]**

   (b) Determine all the solutions of the equation **[4]**

$$z^5 + 32 = 0 \ ,$$

   and plot them in an Argand diagram.

   (c) Let $z = 1 - i$. Calculate the real and imaginary parts of z^{10}. **[4]**

   (d) For each of the following equations, find, describe and plot in an Argand diagram the set of solutions z: **[8]**

$$\begin{aligned} (i) &\quad |z| = 1; \\ (ii) &\quad z - |z| = z^*; \\ (iii) &\quad \arg z = \frac{\pi}{4}; \\ (iv) &\quad |z|^2 - 2|z| - 3 = 0. \end{aligned}$$

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8. (a) See lecture notes.

   (b) The roots are:

$$\begin{aligned} z_0 &= 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right), \\ z_1 &= 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right), \\ z_2 &= -2, \\ z_3 &= 2\left(\cos\frac{-3\pi}{5} + i\sin\frac{-3\pi}{5}\right), \\ z_4 &= 2\left(\cos\frac{-\pi}{5} + i\sin\frac{-\pi}{5}\right). \end{aligned}$$

   (c) $Re(z^{10}) = 0, \ Im(z^{10}) = -32$.

   (d) (i) Circle of radius 1 centered at the origin.

   (ii) The point at the origin.

   (iii) Line from the origin (excluding the origin), forming an angle of $\pi/4$ with the positive horizontal axis.

   (vi) Circle of radius 3 centered at the origin.

[Part marks]

9. (a) Determine the sums of the following infinite series: **[8]**

$$(i) \quad \sum_{n=1}^{\infty} \frac{2}{n^2+2n} \ ;$$

$$(ii) \quad \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} \ .$$

(b) Show, by any means, whether the following sums converge or diverge: **[6]**

$$(i) \quad \sum_{n=0}^{\infty} \frac{n^2}{3^n} \ ;$$

$$(ii) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \ ;$$

$$(iii) \quad \sum_{n=0}^{\infty} n\, e^{-n^2} \ .$$

(c) For each of the following series, determine the value for the real parameter α so that the given sum is equal to 1/3: **[6]**

$$(i) \quad \sum_{n=0}^{\infty} (\ln \alpha)^n \ ;$$

$$(ii) \quad \sum_{n=1}^{\infty} \frac{1}{(1+\alpha)^n} \ .$$

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9. (a) (i) $S = \frac{3}{2}$.
   (ii) $S = 1$.
   (b) (i) Series converges.
   (ii) Series diverges.
   (iii) Series converges.
   (c) (i) No solution for α.
   (ii) $\alpha = 3$.

[Part marks]

10. The position vector of a point in 2D Cartesian coordinates is given by

$$\underline{r} = x\,\underline{i} + y\,\underline{j}\,.$$

Consider now polar coordinates, defined by

$$\begin{aligned} x &= r\cos\theta\ , \\ y &= r\sin\theta\ . \end{aligned}$$

(a) Derive an expression for the unit vectors of polar coordinates $\underline{\hat{r}}$ and $\underline{\hat{\theta}}$ in terms of the unit vectors in Cartesian coordinates $\underline{i}$ and $\underline{j}$. **[4]**

(b) (*i*) Show that **[4]**

$$\frac{\partial}{\partial r} = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}\ ,$$

and **[4]**

$$\frac{\partial}{\partial \theta} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}\ .$$

(*ii*) Determine an expression for the Laplace operator **[8]**

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in polar coordinates [Hint: Calculate first $\frac{\partial^2}{\partial r^2}$ and $\frac{\partial^2}{\partial \theta^2}$].

10.

(a) $\underline{\hat{r}} = \cos\theta\underline{i} + \sin\theta\underline{j}$,

$\underline{\hat{\theta}} = -\sin\theta\underline{i} + \cos\theta\underline{j}$.

(b) (i) Result as shown.

(ii) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

[Part marks]

11. (a) Write the general expression for the Maclaurin expansion up to the second order of a function of two variables $z = f(x, y)$. **[2]**

(b) Determine the Maclaurin expansion up to the second order of the following functions of two variables: **[9]**

$$\begin{aligned} (i) \quad & f(x,y) = \sin x \, \sin y \ ; \\ (ii) \quad & g(x,y) = x \, e^{x\,y} \ ; \\ (iii) \quad & h(x,y) = x^2 \sin(y^2) \ . \end{aligned}$$

(c) Find the stationary point(s) of the following functions and discuss its (their) nature: **[9]**

$$\begin{aligned} (i) \quad & f(x,y) = e^{-(x^2+y^2)} \ ; \\ (ii) \quad & g(x,y) = x^2 y + x^2 - 2y \ ; \\ (iii) \quad & h(x,y) = \ln(1 + x^2 y^2) \ . \end{aligned}$$

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11. (a) See lecture notes.

(b) (i) $\sin x \sin y = xy +$ higher than 2nd order.

(ii) $xe^{xy} = x +$ higher than 2nd order.

(iii) $x^2(\sin(y^2)) = 0 +$ higher than 2nd order.

(c) (i) One maximum at $(0, 0)$.

(ii) Two saddle points at $(\pm\sqrt{2}, -1)$.

(iii) Infinite number of stationary points defined by either $x = 0$ or $y = 0$. Their nature cannot be determined using the methods presented in the course.
