II Complex Numbers

10. Real vs Complex Numbers (Riley 3.1, 3.3)

Complex numbers are a generalisation of real numbers. They occur in many branches of mathematics and have numerous applications in physics.

10.1 The Imaginary Unit i

Definition 10.1 — The Imaginary Unit. The imaginary unit is defined by

$$i = \sqrt{-1} \Leftrightarrow i^2 = -1. \tag{10.1}$$

The obvious place to see where we have already needed this is in the solution to quadratic equation. For example, find the roots of

$$\begin{aligned} z^2 + 4z + 5 &= 0 \\ \Rightarrow (z+2)^2 + 1 &= 0 \\ \Rightarrow (z+2)^2 &= -1 \end{aligned} \tag{10.2}$$

with two solutions

$$z_{1,2} = -2 \pm \sqrt{-1}. \tag{10.3}$$

So in this case we can use the imaginary unit and write the solutions as

$$z_{1,2} = -2 \pm i. \tag{10.4}$$

which is called a complex number.

10.2 Cartesian Representation

Formula 10.2 — Cartesian Representation of a Complex Number. The general form of a complex number is

$$z = x + iy \tag{10.5}$$

in the Cartesian representation given by the sum of the **real part** x and i times the **imaginary part** y: These are also denoted as

$$\mathrm{Re}(z) = x, \tag{10.6}$$
$$\mathrm{Im}(z) = y, \tag{10.7}$$

respectively.

The real and/or imaginary part can be zero; if the imaginary part is zero, the number is real and hence real numbers are just a subset of complex numbers.

As seen above, when using the quadratic solutions formula, we can have situations where there are no real roots, i.e when the discriminant is $b^2 - 4ac < 0$ for the equation $ax^2 + bx + c = 0$. We could have solved the above quadratic equation using the explicit formula:

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2\sqrt{-1}}{2} = -2 \pm i. \tag{10.8}$$

A complex number may also be written more compactly as $z = (x, y)$ where x and y are two real numbers which define the complex number and may be thought of as Cartesian coordinates.

10.3 Polar Representation

Recall that Cartesian and polar coordinates are related by

$$x = r\cos\theta, \tag{10.9}$$
$$y = r\sin\theta. \tag{10.10}$$

Formula 10.3 — Polar Representation of a Complex Number. Therefore we can represent z in polar coordinates as (see Figure 10.3)

$$z = x + iy = r(\cos\theta + i\sin\theta). \tag{10.11}$$

The number r is called the **modulus** of z, written as $|z|$ or mod z. In terms of x and y it is given by

$$|z| = \sqrt{x^2 + y^2}. \tag{10.12}$$

The angle θ is called the **argument** of z, written as arg(z) (or arg z) and it is defined as

$$\tan(\arg z) = \frac{y}{x}, \tag{10.13}$$

so $\arg z$ is the angle that the line joining the origin to z on an **Argand diagram** makes with the positive x axis. The anti-clockwise direction is taken to be positive by convention.

However, θ is not unique since $\theta + 2n\pi$ (n is zero or any integer) are also arguments for the same complex number. We therefore define a **principal value** of a complex number as that value

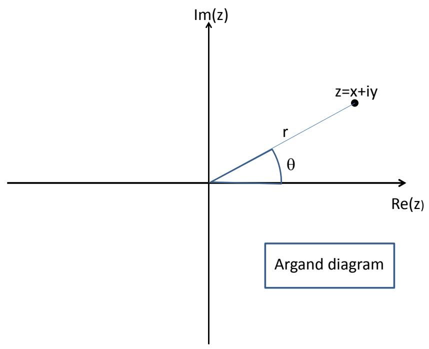

Figure 10.1: Polar representation of a complex number

of θ which satisfies $-\pi < \theta \leq \pi$ (this is a convention; it could instead also be $0 < \theta \leq 2\pi$). Also, account must be taken of the signs of x and y when determining in which quadrants $\arg z$ lies. E.g. if x and y are both negative then $-\pi < \arg z < -\pi/2$ rather than $0 \leq \arg z < \pi/2$ even though the ratios of x and y will be the same when both are negative or both are positive.

■ **Example 10.1** Find the modulus and argument of $z = -3 + 5i$.

$$|z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34},$$
$$\arg z = \tan^{-1} \frac{5}{-3} \approx -1.03 \quad \text{or} \ -1.03 + \pi = 2.11$$

Given that z is in the 2nd (upper left) quadrant on the Argand diagram, the argument is

$$\arg z = 2.11,$$

i.e. positive and $< \pi$. (Note: For $z = 3 - 5i$ we would have had $\arg z \approx -1.03$). ■

11. Operations with Complex Numbers (Riley 3.2)

11.1 Addition and Subtraction

The addition or subtraction of two complex numbers leads to another complex number where the real and imaginary components are added separately.

Formula 11.1 — Sum and Difference of Complex Numbers. Therefore for two complex numbers z_1 and z_2,

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2). \tag{11.1}$$

Note that complex numbers, as real numbers, satisfy the commutative and associative laws of addition,

$$z_1 + z_2 = z_2 + z_1, \qquad \text{[commutative]}, \tag{11.2}$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3, \qquad \text{[associative]}. \tag{11.3}$$

11.2 Multiplication

Multiplication of two complex numbers gives another complex number.

Formula 11.2 — Product of Complex Numbers. The product is calculated by multiplying out in full and using the property of the imaginary unit $i\,i = i^2 = -1$,

$$\begin{aligned} z_1\,z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned} \tag{11.4}$$

Multiplication is both commutative and associative

$$z_1\,z_2 = z_2\,z_1, \qquad \text{[commutative]}, \tag{11.5}$$

$$(z_1\,z_2)z_3 = z_1(z_2\,z_3), \qquad \text{[associative]}, \tag{11.6}$$

and also has the simple properties

$$|z_1\, z_2| = |z_1|\, |z_2|, \tag{11.7}$$
$$\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2). \tag{11.8}$$

■ **Example 11.1** For $z_1 = 5-3i$ and $z_2 = 1+2i$ one has

$$|z_1| = \sqrt{5^2+(-3)^2} = \sqrt{34}, \qquad \arg z_1 = -\tan^{-1}(3/5) \approx -0.54,$$
$$|z_2| = \sqrt{1^2+2^2} = \sqrt{5}, \qquad \arg z_2 = \tan^{-1} 2 \approx 1.11,$$
$$z_1\, z_2 = (5-(-6)) + i(10+(-3)) = 11+7i,$$
$$|z_1\, z_2| = \sqrt{11^2+7^2} = \sqrt{170} = \sqrt{34\; 5} = |z_1|\, |z_2|, \quad \arg(z_1\, z_2) = \tan^{-1}(7/11) \approx 0.57.$$

Thus the relation $\arg(z_1\, z_2) = \arg z_1 + \arg z_2$ also holds. ■

11.3 Complex Conjugation

Definition 11.3 — Complex Conjugate. For a complex number $z = x+iy$, the complex conjugate is defined by

$$z^* = x - iy. \tag{11.9}$$

On an Argand diagram, this corresponds to 'reflecting' the given point for z about the real axis.

So the complex conjugate has the same magnitude as z and when multiplied by z gives a real positive result:

$$z\, z^* = (x+iy)(x-iy) \tag{11.10}$$
$$= x^2 - ixy + ixy - i^2 y^2 \tag{11.11}$$
$$= x^2 + y^2 = |z|^2$$

Likewise for any two complex numbers

$$|z_1\, z_2|^2 = z_1\, z_2\, z_1^*\, z_2^* = z_1\, z_1^*\, z_2\, z_2^* = |z_1|^2\, |z_2|^2, \tag{11.12}$$

and since all moduli are positive,

$$|z_1\, z_2| = |z_1|\, |z_2| \tag{11.13}$$

as stated before. Also

$$z + z^* = (x+iy) + (x-iy) = 2x = 2\mathrm{Re}(z), \tag{11.14}$$
$$z - z^* = (x+iy) - (x-iy) = 2iy = 2i\mathrm{Im}(z). \tag{11.15}$$

Note that no matter how complicated the expression, we can always form the conjugate by replacing every i by $-i$.

■ **Example 11.2** Consider

$$z = w^{(3y+2ix)}, \qquad \text{where} \qquad w = x+5i, \text{ and } x, y \text{ real},$$

so

$$z = (x+5i)^{3y+2ix}, \text{ and}$$
$$z^* = (x-5i)^{3y-2ix}$$

■

11.4 Division

What is the ratio of two complex numbers?

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}. \tag{11.16}$$

To evaluate this, we multiply both the numerator and denominator by the complex conjugate of the denominator, z_2^*,

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \end{aligned} \tag{11.17}$$

This multiplication thus allowed us to make the denominator a real number and ultimately separate the real and imaginary components.

Formula 11.4 — Division of Complex Numbers. So in brief, division of two complex numbers z_1 by z_2 can be expressed as

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}. \tag{11.18}$$

■ **Example 11.3**

$$\begin{aligned}\frac{-7+3i}{4+i} &= \frac{(-7+3i)(4-i)}{(4+i)(4-i)} \\ &= \frac{-28+7i+12i+3}{16+1} = -\frac{25}{17} + \frac{19}{17}i,\end{aligned}$$

where the final result is explicitly expressed in terms of the real and imaginary component. ■

Division also has some simple properties with regard to the modulus and argument,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{11.19}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2). \tag{11.20}$$

12. Euler's Equation (Riley 3.3)

We have already defined the Cartesian form

$$z = x + iy \tag{12.1}$$

of a complex number, and seen that this can be written in polar form as

$$z = r(\cos\theta + i\sin\theta). \tag{12.2}$$

Formula 12.1 — Euler's Equation and Exponential Form of a Complex Number. Another way to express a complex number, which will allow various operations to be performed far more easily, uses **Euler's Equation**,

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{12.3}$$

Using this we can express a complex number in **exponential form**,

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{12.4}$$

R We simply state Euler's equation without proof at this point. We will revisit it and outline a derivation in Chapter 6 on "Series and Limits" once we have the necessary tools.

From this simply follows

$$e^{-i\theta} = \frac{1}{e^{i\theta}} = \cos\theta - i\sin\theta, \tag{12.5}$$

and thus

$$z^* = re^{-i\theta}. \tag{12.6}$$

We again associate r with $|z|$ and θ with $\arg z$ and note that rotation by θ is the same as rotation by $\theta + 2n\pi$ where n is any integer,

$$re^{i\theta} = re^{i(\theta+2n\pi)}. \tag{12.7}$$

Complex numbers z with **unit modulus** are simply given by $z = e^{i\theta}$, i.e. $r = |z| = 1$ and arbitrary argument $-\pi < \theta \leq \pi$. Such numbers form a unit circle around the origin in the Argand diagram. Simple relations hold for such numbers,

$$z = e^{i\theta} = \cos\theta + i\sin\theta, \quad z\, z^* = |z|^2 = 1$$

$$\Rightarrow \quad z^* = e^{-i\theta} = z^{-1} = \frac{1}{z}, \tag{12.8}$$

$$z + z^* = z + \frac{1}{z} = 2 \text{ Re } z = 2\cos\theta, \tag{12.9}$$

$$z - z^* = z - \frac{1}{z} = 2i \text{ Im } z = 2i\sin\theta. \tag{12.10}$$

■ **Example 12.1** Write

$$z = (4+3i)e^{i\pi/3}$$

in the Cartesian form $x + iy$ (with x, y real).

We first expand the exponent,

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and then multiply,

$$z = (4+3i)\frac{1}{2}(1+i\sqrt{3}) = \frac{4-3\sqrt{3}}{2} + i\frac{3+4\sqrt{3}}{2},$$

yielding the desired result. ■

Multiplication and division become much more simple when using this exponential form. For

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}, \tag{12.11}$$

one has

$$z_1\, z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}, \tag{12.12}$$

by applying the usual rules for an exponential expression. Thus

$$|z_1 z_2| = |z_1|\,|z_2|, \tag{12.13}$$

$$\arg(z_1\, z_2) = \arg z_1 + \arg z_2, \tag{12.14}$$

follow immediately. An analogous result holds for division,

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_1 e^{i\theta_1}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}, \tag{12.15}$$

where

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{12.16}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2, \tag{12.17}$$

follow immediately.

■ **Example 12.2** Considering the real and imaginary parts of the product

$$e^{i\theta}e^{i\phi},$$

prove the trigonometric formulae for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.
Collecting the exponentials in one yields

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi),$$

whereas expressing the exponentials in polar form first gives

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi). \end{aligned}$$

As the two above expressions for $e^{i\theta}e^{i\phi}$ must be equal, we can equate the real and imaginary part, respectively, giving

$$\begin{aligned} \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi, \\ \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi, \end{aligned}$$

i.e. the well known trigonometric relations for the sine and cosine of a sum of angles. ■

13. de Moivre's Theorem (Riley 3.4)

Formula 13.1 — de Moivre's Theorem. The exponential relation

$$\left(e^{i\theta}\right)^n = e^{in\theta}, \tag{13.1}$$

is simply written in polar form as

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{13.2}$$

This is **de Moivre's Theorem**. It is in fact valid for all n: real, imaginary or complex.

13.1 Trigonometric Identities

A useful application of de Moivre's theorem is to express trigonometric identities for multiple angle functions in terms of a polynomial of single angles, e.g. for $\cos(2x)$, the above gives

$$\cos(2x) + i\sin(2x) = (\cos x + i\sin x)^2 = \cos^2 x - \sin^2 x + 2i\sin x\cos x. \tag{13.3}$$

Equating real and imaginary parts yields

$$\cos(2x) = \cos^2 x - \sin^2 x, \tag{13.4}$$

$$\sin(2x) = 2\sin x\ \cos x. \tag{13.5}$$

Note: This may also be seen as a special case of the angle sum considered in Example 12.2.

Other identities related to de Moivre's identity can be derived from Equations (12.9) and (12.10) as a starting point (where $z = e^{i\theta}$),

$$\left(z + \frac{1}{z}\right)^n = 2^n\cos^n\theta, \tag{13.6}$$

$$\left(z - \frac{1}{z}\right)^n = (2i)^n\sin^n\theta. \tag{13.7}$$

On the other hand, Equations (12.9) and (12.10) equally apply for $e^{in\theta} = (e^{i\theta})^n = z^n$, and thus

$$z^n + \frac{1}{z^n} = 2\cos(n\theta), \tag{13.8}$$

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta). \tag{13.9}$$

13.2 Finding Roots

The **complex roots** of the equation

$$z^n - 1 = 0, \tag{13.10}$$

are the complex numbers z for which the above equation is true. Here, n is a positive integer. In order to find the roots, we formally solve the above for z as

$$z = 1^{1/n}. \tag{13.11}$$

We then use a rather convoluted way of writing the number 1,

$$1 = e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k), \tag{13.12}$$

where k can be any integer including zero.

Formula 13.2 — Complex Roots of One. Thus the roots of 1, i.e. the complex solutions z_k of the equation $z^n = 1$, are given by

$$z_k = 1^{1/n} = (e^{2\pi ki})^{1/n} = e^{2\pi ki/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right). \tag{13.13}$$

By letting k take the values $k = 0, 1, 2, 3, \ldots, (n-1)$, one can construct exactly n distinct roots (i.e. solutions) z_0, z_1, $\ldots$, z_{n-1}. By choosing k beyond this range, the same solutions z_i are recovered, i.e. it is only necessary to select $k = 0, 1, 2, 3, \ldots, (n-1)$.

In an Argand diagram, the roots form a regular polygon with n corners.

■ **Example 13.1** Find the roots $(-8)^{1/3}$, i.e. find all complex numbers z that satisfy the equation

$$z^3 = -8$$

There is one obvious value $z = -2$, on the real axis, but in fact there are two other complex roots. To find them we write -8 in exponential form, including rotations by multiples of 2π,

$$-8 = 8e^{i(\pi+2k\pi)},$$

and thus

$$z = (-8)^{1/3} = 8^{1/3}(e^{i(\pi+2k\pi)})^{1/3} = 2e^{i\pi(2k+1)/3}.$$

The three different roots z_k are then derived by inserting $k = 0, 1, 2$ in the above equation,

$$\begin{aligned}
z_0 &= 2e^{i\pi/3} && = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) && = 1 + i\sqrt{3},\\
z_1 &= 2e^{i\pi} && = 2\left(\cos\pi + i\sin\pi\right) && = -2,\\
z_2 &= 2e^{i5\pi/3} = 2e^{-i\pi/3} && = 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right) && = 1 - i\sqrt{3}.
\end{aligned}$$

Note that for z_2, the argument $\arg z_2 = 5\pi/3$ was initially outside the conventional range $-\pi < \arg z \leq \pi$, and was brought inside the range $\arg z_2 = 5\pi/3 \to -\pi/3$ to express the solution in polar form. ■

R The above is generalized in the "Fundamental Theorem of Algebra" which states that any complex polynomial of degree n with generally complex coefficients a_i, $i = 0, 1, 2, \ldots, n$,

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0 = 0,$$

has exactly n complex roots. Here, a root may be counted multiple times, e.g. consider

$$z^4 + 2z^2 + 1 = (z^2 + 1)(z^2 + 1) = 0,$$

which has two doubly-degenerate roots $z_0 = z_1 = i$, $z_2 = z_3 = -i$.

14. Hyperbolic & Trigonometric Functions (R. 3.7)

Given the exponential representation of a complex number, we can find new expressions for the cosine and sine of a quantity. We have

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{14.1}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta, \tag{14.2}$$

from which we simply derive

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \tag{14.3}$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{14.4}$$

Definition 14.1 — Hyperbolic Functions. The hyperbolic functions cosh and sinh are defined in a similar fashion,

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x}), \tag{14.5}$$

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}). \tag{14.6}$$

So there are simple relationships between the trigonometric and hyperbolic functions,

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x, \tag{14.7}$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x, \tag{14.8}$$

and in the same way

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i\sin x, \tag{14.9}$$

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\sinh x. \tag{14.10}$$