Answer ALL questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

The questions are worth the following total marks :

Q1 : 10 marks, Q2 : 15 marks, Q3 : 15 marks, Q4 : 15 marks, Q5 : 5 marks.

[Part marks]

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1. (a) Factorise **[3]**

$$x^2 - 5x - 36 \quad \text{and}$$

$$6x^2 + 23x - 55$$

(b) From the general quadratic equation **[4]**

$$ax^2 + bx + c = 0\,,$$

derive the quadratic solutions formula by completing the square.

(c) Given $x = 2$ is one root of **[3]**

$$f(x) = 2x^4 + 4x^3 - 9x^2 - 11x - 6 = 0\,,$$

determine all the real roots.

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1. (a) $x^2 - 5x - 36$. What factors of -36 add up to -5 ? Solution is

$$x^2 - 5x - 36 = (x-9)(x+4)\,.$$

$6x^2 + 23x - 55$. What factors of -330 add up to 23 ? So

$$\begin{aligned} 6x^2 + 23x - 55 &= 6x^2 - 10x + 33x - 55 \\ &= 2x(3x-5) + 11(3x-5) \\ &= (3x-5)(2x+11)\,. \end{aligned}$$

(b)

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
\Rightarrow x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

(c) Given $x = 2$ is a root, the student should first factorise, by any means, e.g. long division, the equation in terms of $x - 2$ and a cubic equation :

$$
\begin{aligned}
f(x) &= 2x^4 + 4x^3 - 9x^2 - 11x - 6 \\
&= (x - 2)(2x^3 + 8x^2 + 7x + 3)\,.
\end{aligned}
$$

To factorise the cubic, it is obvious that a positive value of x is not a solution to $f(x) = 0$. Try some negative values or plot a graph. After a little trial and error, $f(-3) = 0$. So $f(x)$ factorises further to :

$$f(x) = (x - 2)(x + 3)(2x^2 + 2x + 1)$$

Then by considering $\Delta = \sqrt{b^2 - 4ac}$ in the factor formula, it can be seen that $\Delta = 4 - 4 \times 2 \times 1 = -4$ and so the quadratic has no real roots. So $x = 2$ and $x = -3$ are the real roots.

[Part marks]

2. You are given the function

$$f(x) = x \ln x \qquad\qquad x > 0\,.$$

(a) Find the solution(s) of the equation $f(x) = 0$. **[2]**

(b) Find any minima/maxima that $f(x)$ has. **[4]**

(c) Sketch the function based on the features derived in the previous subquestions. **[4]**

(d) Find the first derivative $\left(\frac{d}{dx}\right)$ of $\ln(x^a + x^{-a})$. **[2]**

(e) Find the first derivative $\left(\frac{d}{dx}\right)$ of x^x. **[3]**

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2. (a) Since the function defined only for $x > 0$, $f(x) = 0$ is only possible if

$$\ln x = 0 \;\Rightarrow\; x = 1\,.$$

(b)

$$\begin{aligned} \frac{df}{dx} &= \ln x + x\frac{1}{x} = \ln x + 1 = 0 \\ \Rightarrow \ln x &= -1 \\ \Rightarrow x &= e^{-1}\,. \end{aligned}$$

To determine whether this is a minimum or maximum, we need the second derivative:

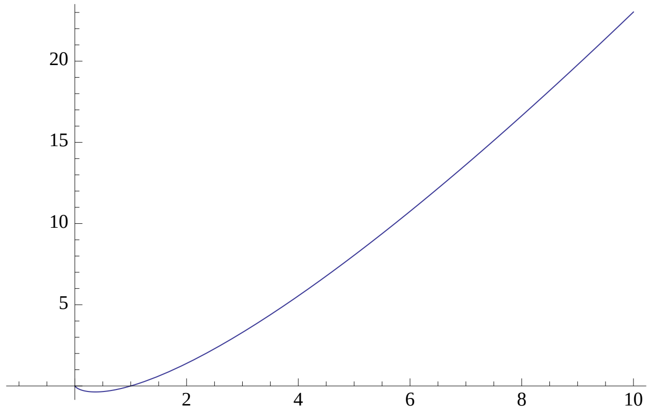
$$\frac{d^2 f}{dx^2} = \frac{1}{x}\,,$$

and for $x = 1/e$ the second derivative is positive, so this point is a minimum.

(c) We have $f(1/e) = -1/e$, so the point of minimum is $(1/e, -1/e)$. The function is negative for $x < 1$ and since there are no other stationary points it goes smoothly to 0, as $x \to 0$ and goes to $+\infty$ when $x \to +\infty$. The student should show all these points by drawing a diagram. A quick sketch is given below.

(d)

$$\begin{aligned} \frac{d}{dx}\left(\ln(x^a + x^{-a})\right) &= \frac{1}{x^a + x^{-a}}\frac{d}{dx}\left(x^a + x^{-a}\right) \\ &= \frac{1}{x^a + x^{-a}}(ax^{a-1} - ax^{-a-1}) \\ &= \frac{a(x^a - x^{-a})}{x(x^a + x^{-a})} \end{aligned}$$



(e)

$$
\begin{aligned}
\text{Let} \quad y &= x^x \\
\Rightarrow \ln y &= x \ln x \\
\text{Therefore, differentiating} \quad \frac{1}{y}\frac{dy}{dx} &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\
&= 1 + \ln x \\
\Rightarrow \frac{dy}{dx} &= y(1 + \ln x) \\
\frac{dy}{dx} &= x^x(1 + \ln x)
\end{aligned}
$$

[Part marks]

3. (a) Find the first derivative $\left(\frac{d}{dx}\right)$ of $\arccos x$. **[4]**

(b) Write down the product rule of differentiation. **[2]**

(c) Using the above rule explain the method of integration by parts. **[4]**

(d) Using integration by parts, evaluate the indefinite integral and then the definite integral **[5]**

$$\int \arccos x \, dx \,, \qquad \int_0^1 \arccos x \, dx \,.$$

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3. (a)

$$y = \text{arccos } x \Leftrightarrow x = \cos\, y$$

$$\begin{aligned} \frac{dx}{dy} &= -\sin\, y \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{-1}{\sin\, y} \\ &= \frac{-1}{\sqrt{1-\cos^2\, y}} \\ &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

(b)

$$\frac{d}{dx}\,(uv) = \frac{du}{dx}\,v + u\,\frac{dv}{dx}$$

(c) Integrate the product rule :

$$\int \frac{d}{dx}\,(uv)\ dx = \int \frac{du}{dx}\,v\,dx + \int u\,\frac{dv}{dx}\,dx$$

But

$$\int \frac{d}{dx}\,(uv)\ dx = uv$$

Therefore

$$\int u\,\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}\,v\,dx\,.$$

(d)

$$\begin{aligned}
\text{Let} \quad u &= \arccos x \qquad \frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} \\
\text{and} \quad \frac{dv}{dx} &= 1 \qquad\qquad v = x
\end{aligned}$$

$$\begin{aligned}
I &= \int \arccos x \, dx \\
&= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx \\
\text{Substitute} \quad t &= 1 - x^2, \quad dt = -2x \, dx \\
\Rightarrow I &= x \arccos x + \int \frac{-dt}{2\sqrt{t}} \\
&= x \arccos x - \sqrt{t} + c \\
&= x \arccos x - \sqrt{1-x^2} + c \, .
\end{aligned}$$

$$\begin{aligned}
I &= \int_0^1 \arccos x \, dx \\
&= \Big[x \arccos x - \sqrt{1-x^2} \Big]_0^1 \\
&= \arccos 1 - 0 - 0 + \sqrt{1} \\
&= 1 \, .
\end{aligned}$$

[Part marks]

4. The hyperbolic functions are defined as

$$\cosh\, y = \frac{1}{2}\left(e^{y} + e^{-y}\right)\,, \qquad \sinh\, y = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

(a) Show that **[2]**

$$\cosh\, 2y = \cosh^2\, y + \sinh^2\, y$$

(b) Evaluate $y = \sinh^{-1}\, x$ in logarithmic form. **[4]**

(c) Evaluate $\frac{dy}{dx}$ using two methods, firstly using the relationship $dy/dx = 1/(dx/dy)$ and secondly by differentiating the logaritmic form of $\sinh^{-1}\, x$. **[6]**

(d) Hence show that **[3]**

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0\,.$$

[You may use $\cosh^2\, y - \sinh^2\, y = 1$ without proof.]

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4. (a)

$$\begin{aligned}
\cosh^2\, y &= \frac{1}{4}\left(e^{2y} + e^{-2y} + 2\right) \\
\sinh^2\, y &= \frac{1}{4}\left(e^{2y} + e^{-2y} - 2\right) \\
\Rightarrow \cosh^2\, y + \sinh^2\, y &= \frac{1}{2}\left(e^{2y} + e^{-2y}\right) \\
&= \cosh\, 2y
\end{aligned}$$

(b) Given the above exponential forms for $\cosh y$ and $\sinh y$ and $x = \sinh\, y$,

$$\begin{aligned}
e^{y} &= \cosh\, y + \sinh\, y \\
&= \sqrt{1+\sinh^2\, y} + \sinh\, y \\
&= \sqrt{1+x^2} + x \\
\Rightarrow y &= \ln\left(\sqrt{1+x^2} + x\right)\,.
\end{aligned}$$

(c)

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$$

$$\begin{aligned}
\frac{dx}{dy} &= \cosh y \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{1}{\cosh y} \\
&= \frac{1}{\sqrt{1+\sinh^2 y}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\ln\left(\sqrt{1+x^2}+x\right)\right) &= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{x+\sqrt{1+x^2}}\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{1}{2}(1+x^2)^{-3/2}\cdot 2x \\
&= \frac{-x}{(1+x^2)^{3/2}} \\
\Rightarrow (x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx} &= (x^2+1)\frac{-x}{(1+x^2)^{3/2}}+x\frac{1}{\sqrt{1+x^2}} \\
&= \frac{-x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}} \\
&= 0\,.
\end{aligned}$$

[Part marks]

5. You are given the function

$$f(x, y, z) = x^2 \, e^{z^2} \, \sin y.$$

(a) Find the first and second derivatives, **[4]**

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}.$$

(b) Show that **[1]**

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

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5. (a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x \, e^{z^2} \, \sin y \\
\frac{\partial^2 f}{\partial x^2} &= 2 \, e^{z^2} \, \sin y \\
\frac{\partial f}{\partial y} &= x^2 \, e^{z^2} \, \cos y \\
\frac{\partial^2 f}{\partial y^2} &= -x^2 \, e^{z^2} \, \sin y
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(x^2 \, e^{z^2} \, \cos y\right) = 2x \, e^{z^2} \, \cos y \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(2x \, e^{z^2} \, \sin y\right) = 2x \, e^{z^2} \, \cos y \\
\Rightarrow \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x}.
\end{aligned}$$
