PHAS1245
Mathematical Methods 1
Exam 2018

[Dr Frank Deppisch]

Answer ALL SIX questions from Section A and ALL THREE questions from Section B

The numbers in square brackets in the right-hand margin indicate a provisional allocation of maximum possible marks for different parts of each question.

Section A

(Answer ALL SIX questions from this section)

1. (a) Determine the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, in terms of their components a_x, a_y, a_z and b_x, b_y, b_z. **[2]**

   (b) A plane is defined by a point A (position vector $\mathbf{a}$) on it and a unit vector $\hat{\mathbf{n}}$ perpendicular to it. Write down the equation of the plane, satisfied by any point R (position vector $\mathbf{r}$) on that plane. **[1]**

   (c) Determine x and y such that the vector $\mathbf{a} = x\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ has unit magnitude and is perpendicular to the vector $\mathbf{b} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$. **[3]**

2. (a) Given two complex numbers, $z_1 = 3 + 7i$ and $z_2 = 6e^{-i\pi/2}$, determine **[3]**

   i. $z_1 - z_2$,

   ii. $z_1 z_2$,

   iii. z_1/z_2.

   Express each result in the form $x + iy$, where x, y are real numbers.

   (b) Find all roots of **[3]**

   $$z^3 = -4\sqrt{2}(1 + i),$$

   and express them in exponential form using the convention that $-\pi < \arg z \leq \pi$.

   (c) Evaluate $\mathrm{Re}(e^{3iz})$, where $z = x + iy$ (x, y are real numbers). **[2]**

3. (a) State the formal definition of the derivative of a function $f(x)$. **[1]**

   (b) Using the formal definition, calculate the derivative of **[3]**

   $$f(x) = \frac{1}{x^2}.$$

   (c) Find all stationary points of **[3]**

   $$f(x) = x^4 + 6x^3 - 6,$$

   and determine their nature.

4. (a) Determine the following indefinite integrals: **[4]**

   i. $\int x^{5/2} dx$,

   ii. $\int x^n \ln x \; dx$, $\quad$ ($n > 0$ is a positive integer).

   (b) Determine the definite integral **[2]**

$$\int_{-1}^{1} \frac{\sin x}{1+x^2} dx,$$

   and justify your answer.

5. (a) Determine the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function **[3]**

$$f(x,y) = \ln\left(1 + xy^2\right),$$

   and calculate the total derivative $\frac{df}{dt}$ for the parametrized path defined by $x(t) = t$, $y(t) = \sqrt{t}$.

   (b) Show that the above function $f(x,y)$ in (a) satisfies the equation **[3]**

$$2\frac{\partial^2 f}{\partial x^2} + y^3 \frac{\partial^2 f}{\partial x \partial y} = 0.$$

6. (a) Write down the general form of the Maclaurin series of a function $f(x)$. **[2]**

   (b) Determine the first three non-zero terms in the Maclaurin series of the following functions: **[5]**

   i. $f(x) = \sqrt{1+2x}$,

   ii. $f(x) = \sin(2x^2)$.

Section B

(Answer ALL THREE questions from this section)

7. (a) Find the minimal distance d between the point $P = (1,1,1)$, and the line passing through the points $A = (2,1,5)$ and $B = (3,4,3)$. **[5]**

(b) Find the equation of the line formed by the intersection of the two planes **[5]**

$$3x + y - z = 3,$$
$$2y + 4z = -4.$$

Express the equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of a point on the line, $\mathbf{b}$ is a vector in the direction of the line and λ is a real parameter.

(c) Calculate the scalar and vector product between the vectors **[5]**

$$\mathbf{a} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j},$$
$$\mathbf{b} = \cos\phi\ \mathbf{i} + \sin\phi\ \mathbf{j},$$

and hence prove that

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi,$$
$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

(d) Sketch (in separate Argand diagrams) and describe the regions on the complex z plane, defined by the following inequalities: **[5]**

i. $|z + 2 - 3i| \leq 2$,

ii. $\mathrm{Re}(z^2) > 0$.

8. (a) Calculate the derivative $\frac{df}{dx}$ of the following functions: **[6]**

   i. $f(x) = \arctan x$,

   ii. $f(x) = x^{(x^2)}$,

   iii. $f(x) = e^{-x^2} + \int_0^x e^{-t^2} dt$.

(b) Calculate the volume of revolution formed by rotating the curve **[3]**

$$f(x) = \frac{1}{\sqrt{1+x^2}},$$

around the x-axis in a full circle. The volume extends over the range $-\infty < x < \infty$.

(c) i. Given a function $f(x, y)$, state the condition for a point (x_0, y_0) to be stationary, and the criteria to determine its nature. **[3]**

   ii. Find all stationary points of the function **[4]**

$$f(x, y) = x^3 - yx^2 + y^2,$$

and determine their nature.

(d) A tilted ellipse in the x-y plane is described by the implicit relation

$$x^2 + xy + y^2 = 12.$$

Find the location (x, y) of the **[4]**

   i. top-most (largest y value),

   ii. bottom-most (smallest y value),

   iii. right-most (largest x value), and

   iv. left-most (smallest x value)

point on the ellipse.

9. (a) i. Given a general differential of the form

$$A(x,y)dx + B(x,y)dy,$$

state the condition that means that the differential is exact. **[1]**

ii. Hence determine whether the following differentials are exact or not: **[5]**

1) $\left(2x + y^2 + \frac{1}{x}\right) dx + \left(2xy - \frac{1}{y}\right) dy$,

2) $\frac{x}{x^2+y^2} dy - \frac{xy}{x^2+y^2} dx$.

In case a differential is exact, determine the corresponding function $f(x,y)$ such that $df = A(x,y)dx + B(x,y)dy$.

(b) A vector field in two-dimensional Cartesian coordinates is given by

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2+y^2)^{3/2}}.$$

Calculate the line integral $W = \int_C \mathbf{F} \cdot d\mathbf{r}$ for the path defined by a clockwise half-circle around the origin, from $\mathbf{r}_A = -2\mathbf{i}$ to $\mathbf{r}_B = 2\mathbf{i}$. **[3]**

(c) Show that the sum of squared integers, $\sum_{k=1}^{N} k^2$, is given by **[4]**

$$\sum_{k=1}^{N} k^2 = \frac{1}{6} N(N+1)(2N+1).$$

Hint: Use the identity $(k+1)^3 - k^3 = 3k^2 + (3k+1)$ to express the given series in terms of two other, explicitly summable, series.

(d) Evaluate the following limits: **[7]**

i.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}},$$

ii.

$$\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x\right)^x.$$