V Partial Differentiation

25. The Partial Derivative

So far we have considered functions of a single variable i.e. $f = f(x)$, and the slope or gradient at x has been defined by

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{25.1}$$

We now consider a function of two (or more) variables $f = f(x,y)$, a so called **multivariate function**. For two variables, such a function can be represented by a 2D surface within 3D space, the z-axis denoting the function value $z = f(x,y)$. Many of the results in this chapter can be generalized to any number of dimensions but we will largely focus on the case of two variables, $f = f(x,y)$ in the following.

25.1 Definition (Riley 5.1)

It is clear that a function $f = f(x,y)$ of two variables can in general have a different gradient depending on the direction considered for the variable change in the xy-plane. The starting point to quantify these rates of change (slopes or gradients) is through the gradients along the direction of the x-axis and y-axis, respectively.

Definition 25.1 — Partial Derivative. We can define the partial derivative with respect to x (holding y constant),

$$\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \tag{25.2}$$

and the partial derivative with respect to y (holding x constant),

$$\left(\frac{\partial f}{\partial y}\right)_x = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \tag{25.3}$$

The partial derivatives are (in general) functions of both x and y.

In practice, calculating partial derivatives is no different from a 'normal' derivative. We simply treat the variable(s) not differentiated over as constants. It is important though to interpret the different ways of differentiating correctly.

R We will use the following equivalent notations to denote partial derivatives,

$$\left(\frac{\partial f}{\partial x}\right)_y \equiv \frac{\partial f}{\partial x} \equiv f_x,$$

and similar for the partial derivative with respect to y and higher order derivatives (see below). The first form emphasizes that the variable y is kept constant when differentiating. Sometimes we will also emphasize if the partial derivative is to be evaluated at a specific point (x_0, y_0) by

$$\left(\frac{\partial f(x_0,y_0)}{\partial x}\right)_y \equiv \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} \equiv f_x(x_0,y_0).$$

Formula 25.2 — Higher-order Partial Derivatives. As the partial derivatives are again functions of x and y, we can calculate second and higher-order partial derivatives,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx}, \tag{25.4}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy}. \tag{25.5}$$

Here it is important to note that mixed derivatives can occur as well,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy}, \tag{25.6}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv f_{yx}. \tag{25.7}$$

Provided that the second-order derivatives are sufficiently well-behaved, specifically that the second-order derivatives are continuous, **mixed partial derivatives are symmetric**,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \text{or} \quad f_{xy} = f_{yx}, \tag{25.8}$$

due to **Schwarz's Theorem**.

■ **Example 25.1** Consider the multivariate function $s = s(t,u) = t^u$. Its partial derivatives are given by

$$\frac{\partial s}{\partial t} = \frac{\partial (t^u)}{\partial t} = ut^{u-1},$$

$$\frac{\partial s}{\partial u} = \frac{\partial (t^u)}{\partial u} = \frac{\partial}{\partial u}(e^{u\ln t}) = e^{u\ln t}\ln t = t^u \ln t.$$

The second-order partial derivatives are

$$\begin{aligned}
\frac{\partial^2 s}{\partial t^2} &= \frac{\partial}{\partial t}(ut^{u-1}) = u(u-1)t^{u-2},\\
\frac{\partial^2 s}{\partial u^2} &= \frac{\partial}{\partial u}(t^u \ln t) = t^u \ln^2 t,\\
\frac{\partial^2 s}{\partial u \partial t} &= \frac{\partial}{\partial u}(ut^{u-1}) = t^{u-1}(1+u\ln t),\\
\frac{\partial^2 s}{\partial t \partial u} &= \frac{\partial}{\partial t}(t^u \ln t) = t^{u-1}(1+u\ln t).
\end{aligned}$$

Thus the symmetry of the mixed partial derivatives can be seen to hold. ■

25.2 The Gradient Vector (Riley 10.7.1)

Definition 25.3 — Gradient Vector. The partial derivatives can be interpreted as (Cartesian) components of a vector called the **gradient vector**,

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \equiv \begin{pmatrix} f_x \\ f_y \end{pmatrix}. \tag{25.9}$$

The notation for the gradient vector uses the so called **Nabla operator** ∇, applied to a function $f(x,y)$. ∇ combines the properties of a partial derivative and a vector.

R Note that the gradient vector of a function $f(x,y)$ 'lives' in the 2D space of the xy-plane of variables, not in the 3D space of the function surface $(x,y,f(x,y))$.

The gradient vector will in general depend both in magnitude and direction on the position (x,y), as the partial derivatives are functions of the independent variables x and y. This is our first example of a so called **vector field**, which we will discuss below in more detail in the context of line integrals. The gradient has a very useful interpretation: its direction at a given point in the xy-plane coincides with the direction where the function $f(x,y)$ increases most rapidly (maximal change of rate/slope). The magnitude of the gradient vector at a given point then gives this maximal slope.

■ **Example 25.2** Determine the gradient vector field for the function $f(x,y) = x^2 + y^2$. Evaluate and describe the gradient at the point $x = 1, y = 1$.

The partial derivatives of $f(x,y)$ are given by

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

The gradient vector field is therefore

$$\nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

At the point $x = 1$, $y = 1$, the gradient vector is given by

$$\mathbf{v} = \nabla f(1,1) = 2\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

The maximum slope of the function f at $(1,1)$ is the magnitude of this vector, $|\mathbf{v}| = 2\sqrt{2}$. The direction of steepest ascent can for example be determined from the unit vector in the direction of $\mathbf{v}$,

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

It points radially outward from the origin with an angle of $\pi/4$ to the positive x-axis. ■

26. Total Differential and Derivative (Riley 5.2)

26.1 The Total Differential (Riley 5.2)

For a function of one variable, $f = f(x)$, the change in the function value when x is changed **infinitesimally** by dx is given by

$$df = f(x+dx) - f(x) = \frac{df}{dx}dx. \tag{26.1}$$

It is called the **differential** (change) in f corresponding to the infinitesimal change dx of the independent variable. It is an abstract concept that should be understand as the limit $\Delta x \to 0$ of a finite change Δf where the corresponding relation holds approximately,

$$\Delta f = f(x+\Delta x) - f(x) \approx \frac{df}{dx}\Delta x. \tag{26.2}$$

Having defined the partial derivatives of multivariate function, we now ask what is the corresponding differential change df in $f = f(x,y)$ when the coordinates (x,y) are changed to $(x+dx, y+dy)$,

$$\begin{aligned} df &= f(x+dx, y+dy) - f(x,y) \\ &= f(x+dx, y+dy) - f(x, y+dy) + f(x, y+dy) - f(x,y), \end{aligned} \tag{26.3}$$

where we added and subtracted $f(x, y+dy)$. We multiply and divide appropriate terms by dx and dy,

$$df = \left(\frac{f(x+dx, y+dy) - f(x, y+dy)}{dx}\right)dx + \left(\frac{f(x, y+dy) - f(x,y)}{dy}\right)dy. \tag{26.4}$$

Formula 26.1 — Total Differential. The terms in the brackets correspond to the partial derivatives in the limit $dx \to 0$ and $dy \to 0$, respectively, and we can write for the differential df of a function $f = f(x,y)$,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy. \tag{26.5}$$

It should be interpreted as the infinitesimal limit of the approximate relation for a finite change. For example, the function f can be approximated close to a point (x_0, y_0) by

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0), \tag{26.6}$$

with the correspondence $df \to \Delta f = f(x,y) - f(x_0,y_0)$, $dx \to \Delta x = x - x_0$, $dy \to \Delta y = y - y_0$, and the partial derivatives are evaluated at (x_0, y_0).

Using the gradient vector and defining the infinitesimal displacement vector

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} = \begin{pmatrix} dx \\ dy \end{pmatrix}, \tag{26.7}$$

the total differential can also be written as the vector scalar product

$$df = (\nabla f) \cdot d\mathbf{r}. \tag{26.8}$$

■ **Example 26.1** Find the total differential of the function $f(x,y) = ye^{x+y}$.
We have

$$\left(\frac{\partial f}{\partial x}\right)_y = ye^{x+y}, \qquad \left(\frac{\partial f}{\partial y}\right)_x = ye^{x+y} + e^{x+y}.$$

Thus the total differential is

$$df = ye^{x+y}dx + (1+y)e^{x+y}dy.$$

For example, close to the point $(x,y) = (0,0)$, the function may be simply approximated as

$$f(x,y) \approx f(0,0) + f_x(0,0)(x-x_0) + f_y(0,0)(y-y_0) \approx y,$$

i.e. it is approximately constant for changes in x and rises linearly in y with unit slope. ■

26.2 The Total Derivative (Riley 5.2)

The total differential gives the change in the function value df when making infinitesimal steps dx and dy from a point (x,y) in the variable plane. To determine the rate of change, we need to specify a certain direction, as the total differential clearly depends on the values of dx and dy and thus for example on the direction of $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$.

A general problem is to find the rate of change df/dt of a function $f = f(x,y)$ as $x = x(t)$ and $y = y(t)$ follow a parametrized path. Then $f(x,y)$ becomes essentially a function of one variable, the parameter t, only: $f(x(t), y(t))$.

Formula 26.2 — The Total Derivative. To calculate the **total derivative** df/dt, one could substitute $x(t)$ and $y(t)$ but it is often more straightforward to formally divide the total differential by dt to obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (\nabla f) \cdot \frac{d\mathbf{r}}{dt}, \tag{26.9}$$

where the second form employs the gradient vector and the vector derivative $\frac{d\mathbf{r}}{dt}$ of $\mathbf{r}(t) =$

$x(t)\mathbf{i}+y(t)\mathbf{j}$ describing the path in terms of a parameter t.

R In some applications (see the example below where t is time), f may depend on the parameter t as well, $f = f(x,y,t)$. In this case, the total derivative is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t} = (\nabla f)\cdot\frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial t}.$$

This describes the general change of a function that both **explicitly** and **implicitly** depends on t through $f(x(t),y(t),t)$.

■ **Example 26.2** The temperature on a 2D plate depends on the position (x,y) and the time t as $T(x,y,t) = \ln t + xe^{-y}$. A particle/object moving along the path $x(t) = 1+at$, $y(t) = bt^3$ (a, b are constants) on the plate, will feel the temperature change dT/dt.

We first calculate the partial derivatives of T,

$$\frac{\partial T}{\partial t} = \frac{1}{t}, \quad \frac{\partial T}{\partial x} = e^{-y}, \quad \frac{\partial T}{\partial y} = -xe^{-y},$$

and the (usual) derivatives of $x(t)$ and $y(t)$ with respect to t,

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = 3bt^2.$$

Thus, the total derivative is given by

$$\frac{dT}{dt} = \frac{1}{t} + e^{-y}a - xe^{-y}3bt^2 = \frac{1}{t} + e^{-bt^3}\left(a - (1+at)3bt^2\right),$$

where we substituted $x(t)$ and $y(t)$ to make the total derivative a function of t only. The time-dependence of the temperature consists of two parts: the explicit dependence of T on time, and the implicit dependence through the change in position $(x(t),y(t))$ along the path. ■

26.3 Derivative in a Direction

If one is simply interested in the rate of change at a specific point (x_0,y_0) and in a specific direction, it is not necessary to describe a complicated path in full.

Formula 26.3 — Derivative in a Direction. We can instead consider the special case of calculating the total derivative along the simple, linear path

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(y) \end{pmatrix} = \begin{pmatrix} x_0 + n_x t \\ y_0 + n_y t \end{pmatrix}, \tag{26.10}$$

originating at the point (x_0,y_0) and with the components n_x and n_y of a **unit direction vector** $\hat{\mathbf{e}} = n_x\mathbf{i} + n_y\mathbf{j}$, $n_x^2 + n_y^2 = 1$. The total derivative at the point (x_0,y_0), i.e. the rate of change of the function f in this direction is then given by

$$\left.\frac{df}{dt}\right|_{0,\hat{\mathbf{e}}} = \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} n_x + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} n_y = (\nabla f)_0 \cdot \hat{\mathbf{e}}. \tag{26.11}$$

The notation with zero in the subscript is to indicate that the partial derivatives are evaluated at the point (x_0,y_0) of interest.

Using the properties of the vector scalar product, several useful geometric interpretations can be inferred from this result:

- The derivative in a direction, $\frac{df}{dt}|_{0,\hat{\mathbf{e}}}$, gives the slope of the function f at the point (x_0, y_0) in the direction of $\underline{\hat{e}}$.
- If $\hat{\mathbf{e}}$ is pointing in the same direction as the gradient vector $(\nabla f)_0$, the slope is maximal with value $|(\nabla f)_0|$ (maximal ascent, this coincides with our earlier interpretation of the gradient vector).
- If $\hat{\mathbf{e}}$ is pointing in the opposite direction of the gradient vector, the slope is negatively maximal and given by $-|(\nabla f)_0|$ (maximal descent).
- If $\hat{\mathbf{e}}$ is perpendicular to the gradient vector, the slope is zero, i.e. the function f remains approximately constant when (slightly) changing x, y in this direction.
- If the gradient vector is zero, the slope is zero in any direction $\hat{\mathbf{e}}$, thus the point (x_0, y_0) is a stationary point (will be discussed in more detail in Section 27).

26.4 Exact and Inexact Differentials (Riley 5.3)

In the previous section we obtained the total differential df by determining the partial derivatives from $f(x,y)$. We now address the inverse problem. Consider the general differential

$$df = A(x,y)dx + B(x,y)dy. \tag{26.12}$$

Can we go back to find the function $f(x,y)$? If we can, this is called an **exact differential**, otherwise it is an **inexact differential**.

Assuming the above expression represents an exact differential, we can identify the partial derivatives as

$$\frac{\partial f}{\partial x} = A(x,y), \qquad \frac{\partial f}{\partial y} = B(x,y). \tag{26.13}$$

For a well-behaved function f, the second-order derivatives would need to be symmetric,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{26.14}$$

Formula 26.4 — Exactness Condition. This means that the differential $df = A(x,y)dx + B(x,y)dy$ is exact if

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}. \tag{26.15}$$

Otherwise the differential is not exact, i.e. it does not describe the infinitesimal change of a function $f(x,y)$.

R An inexact differential is sometimes denoted as $đf$. In physics, inexact differentials for example appear in thermodynamic quantities whose change depends on the path and which cannot be represented as a function $f(x,y)$.

On the other hand, exact differentials correspond to changes of state functions $f(x,y)$ that do not depend on the path taken. For example, the potential energy $U(x,y,z)$ of a mass in a 3D gravitational force field is a function of the position (x,y,z) only. The change in the potential energy when moving from a point (x_1,y_1,z_1) to another point (x_2,y_2,z_2) is simply $\Delta U = U_2 - U_1 = U(x_2,y_2,z_2) - U(x_1,y_1,z_1)$, independent of how the mass is moved between the points.

■ **Example 26.3** Determine whether $x^2dy - (y^2+xy)dx$ is an exact or inexact differential.

The functions $A(x,y)$ and $B(x,y)$ are (as per standard convention, A is associated with dx and B with dy)

$$A(x,y) = -(y^2+xy), \qquad B(x,y) = x^2.$$

Their partial derivatives are

$$\frac{\partial A}{\partial y} = -2y - x, \qquad \frac{\partial B}{\partial x} = 2x,$$

which shows that the given differential is inexact. ■

Given an exact differential $df = A(x,y)dx + B(x,y)dy$, a natural question to ask is: What is the corresponding function $f(x,y)$? This can be done by 'partially' integrating $\frac{\partial f}{\partial x} = A(x,y)$ over x, $\frac{\partial f}{\partial y} = B(x,y)$ over y, and matching the results. This is best demonstrated in an example.

■ **Example 26.4** Consider the differential $df = \frac{x}{y^2}dy - \frac{x+y}{xy}dx$. The functions A and B are

$$A(x,y) = -\frac{x+y}{xy} = -\frac{1}{x} - \frac{1}{y}, \qquad B(x,y) = \frac{x}{y^2},$$

and thus

$$\frac{\partial A}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial B}{\partial x} = \frac{1}{y^2}.$$

The exactness condition therefore holds and there exists a function (or rather functions) $f(x,y)$ for which the total differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is given as above. To find f, we identify

$$\frac{\partial f}{\partial x} = A(x,y) = -\frac{1}{x} - \frac{1}{y}, \qquad \frac{\partial f}{\partial y} = B(x,y) = \frac{x}{y^2},$$

and 'partially' integrate, i.e. integrate over x and y, respectively while treating the other variable as a constant,

$$f(x,y) = \int \frac{\partial f}{\partial x}dx = \int \left(-\frac{1}{x} - \frac{1}{y}\right)dx = -\ln x - \frac{x}{y} + c_1(y),$$
$$f(x,y) = \int \frac{\partial f}{\partial y}dy = \int \frac{x}{y^2}dy = -\frac{x}{y} + c_2(x).$$

Important to note here is the appearance of the integration 'constants' $c_1(y)$ and $c_2(x)$; while they cannot depend on the integration variable x and y, respectively, they can be (at this point) arbitrary well-behaved functions of the other variable. Both results for $f(x,y)$ have to match in the end, thus we require

$$-\ln x - \frac{x}{y} + c_1(y) = -\frac{x}{y} + c_2(x).$$

This equality has to hold for any values of x and y which is only possible if

$$c_1(y) = c_1, \qquad \text{[i.e. an overall constant in both } x \text{ and } y\text{]},$$
$$c_2(x) = -\ln x + c_2,$$

to match the dependence of each other, apart from overall constants. This yields

$$f(x,y) = -\ln x - \frac{x}{y} + c,$$

with an arbitrary overall constant of integration c, i.e. independent of both x and y. ■

27. Multivariate Function Analysis

27.1 Maxima, Minima and Saddle Points (Riley 5.8)

For a multivariate function $f = f(x,y)$, special points of interest are stationary points, i.e. local maxima and minima but also so-called saddle points which emerge for functions of more than one variable. At a stationary point (x_0, y_0), the function value remains approximately constant when making small steps in any direction (dx, dy), i.e. the total differential vanishes,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \tag{27.1}$$

for any dx and dy.

Formula 27.1 — Local Maxima, Minima and Saddle Points. As the total differential is required to vanish for any direction (dx, dy), both partial derivatives must vanish simultaneously at a stationary point,

$$\left(\frac{\partial f}{\partial x}\right)_y = 0, \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)_x = 0. \tag{27.2}$$

Analogously, we can say that the gradient vector vanishes at a stationary point, $\nabla f = \mathbf{0}$. To determine the nature of a stationary point, we must consider the second-order partial derivatives and the following conditions apply. The stationary point is

1. A **local minimum** if the following three conditions are satisfied:

$$f_{xx} > 0, \quad f_{yy} > 0, \quad f_{xx}f_{yy} > f_{xy}^2. \tag{27.3}$$

2. A **local maximum** if the following three conditions are satisfied:

$$f_{xx} < 0, \quad f_{yy} < 0, \quad f_{xx}f_{yy} > f_{xy}^2. \tag{27.4}$$

The last condition is the same as for a minimum.

3. A **saddle point** if the following condition holds:

$$f_{xx}f_{yy} < f_{xy}^2. \tag{27.5}$$

Note that this includes the case that f_{xx} and f_{yy} have opposite signs to each other and thus $f_{xx}f_{yy} < 0$.

4. **Undetermined** in any other case, i.e. specifically if $f_{xx}f_{yy} = f_{xy}^2$, including the case $f_{xx} = f_{yy} = f_{xy} = 0$. Further investigation is required, e.g. by Taylor-expanding the function to higher orders (see Chapter 6).

In the above we have used the usual short-hand notation,

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y}, \tag{27.6}$$

and all partial derivatives are to be evaluated at the stationary point (x_0, y_0).

R We will delay the proof of these conditions until Chapter 6 when we determine the Taylor expansion of a multivariate function.

■ **Example 27.1** Find all stationary points of the function

$$f(x,y) = x^2 - 2xy + 2y^2 - 2y + 2,$$

and determine their nature.

The first partial derivatives are

$$f_x = 2x - 2y,$$
$$f_y = -2x + 4y - 2.$$

Setting these to zero we find

$$\begin{aligned} 2x - 2y = 0 \quad &\Rightarrow \quad x = y \\ -2x + 4y - 2 = 0 \quad &\Rightarrow \quad 2x - 2 = 0 \quad \Rightarrow \quad x = 1. \end{aligned}$$

Thus $x = 1, y = 1$ is the only stationary point. Now, we calculate the second-order derivatives,

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = -2.$$

Since $f_{xx} > 0$, $f_{yy} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$, $(1,1)$ is a local minimum of f. ■

For functions of more than one variable, the structure of stationary points can be more complicated than a set of isolated stationary points as the following example demonstrates.

■ **Example 27.2** Find all stationary points of the function

$$f(x,y) = x^4 + y^4 - 4x^2 - 4y^2 + 2x^2y^2,$$

and determine their nature.

Setting the first partial derivatives to zero,

$$f_x = 4x^3 - 8x + 4xy^2 = 4x(x^2 + y^2 - 2) = 0,$$
$$f_y = 4y^3 - 8y + 4x^2y = 4y(x^2 + y^2 - 2) = 0.$$

Both equations are simultaneously satisfied if

$$x = 0, \ y = 0 \quad \text{or} \quad x^2 + y^2 = 2.$$

This means that there is a solitary stationary point at the origin $(0,0)$ but there is also a continuous set of stationary points forming a circle with radius of $\sqrt{2}$ around the origin.

We can calculate the second-order derivatives

$$f_{xx} = 12x^2 - 8 + 4y^2, \quad f_{yy} = 12y^2 - 8 + 4x^2, \quad f_{xy} = 8xy.$$

From this we find that $(0,0)$ is a local maximum. On the other hand, it can be shown that for the stationary points on the circle one has $f_{xx}f_{yy} = f_{xy}^2$, i.e. their nature is undetermined according to our classification. By inspection of the function one can see that they form a collective minimum in the shape of circle. ■

27.2 Stationary Points under a Constraint (Riley 5.9)

In the previous section we have determined stationary points of a function $f(x,y)$ where the variables x and y are fully independent from each other. It is often useful to consider instead that there is a constraint on x and y. Specifically, we consider determining stationary points of a function $f(x,y)$ under the presence of an implicit relation between x and y,

$$\phi(x,y) = c. \tag{27.7}$$

Here, c is a constant number. As discussed in Section 19.1, this constraint can be visualized as a curve in the xy-plane. Thus we are interested to find stationary points of a function $f = f(x,y)$, but only among those points (x,y) which satisfy $\phi(x,y) = c$. We could in fact attempt to use $\phi(x,y) = c$ to eliminate y from f, but as mentioned before, this can be difficult or even impossible. The method of the **Lagrange multiplier** outlined below is an elegant way of handling this problem.

R Such problems occur often in practice, i.e. when one wants to find values of (x,y) that maximise/minimise a certain quantity (e.g. maximal volume of an object, maximal profit from a product in economics) while being under a constraint (surface area to be kept constant, amount of money to put into production/marketing of product is given).

To determine the solution, we calculate the total differentials of the function $f(x,y)$ and the constraint function $\phi(x,y)$,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \tag{27.8}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy. \tag{27.9}$$

Now, $df = 0$ at a stationary point but note that dx and dy are not independent of each other any more as they need to satisfy the constraint $\phi(x,y,) = c$ at the same time. On the other hand, $d\phi = 0$ as $\phi(x,y) =$ constant by definition. Thus we can multiply $d\phi$ by a yet arbitrary factor λ, called the Lagrange multiplier, and add it to df,

$$df + \lambda d\phi = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy = 0. \tag{27.10}$$

As this automatically incorporates the constraint, we can now treat dx and dy as independent.

Formula 27.2 — Stationary Point under Constraint. Thus we find stationary points of $f = f(x,y)$ satisfying the constraint $\phi(x,y) = c$ by solving the following three equations simultaneously for the three unknowns x, y and λ,

$$\begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} &= 0, \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} &= 0, \\ \phi(x,y) &= c. \end{aligned} \tag{27.11}$$

(R) We do not discuss criteria to determine the nature of stationary points under a constraint. In many practical examples, the nature can be inferred by inspecting the function.

■ **Example 27.3** Find the values of x and y that maximise the function

$$f(x,y) = xy^{3/2}, \qquad x,y \geq 0$$

subject to the constraint

$$x + 2y = 100.$$

The conditions are

$$y^{3/2} + \lambda = 0, \quad \frac{3}{2}xy^{1/2} + 2\lambda = 0,$$

from which follows

$$\frac{3}{2}xy^{1/2} - 2y^{3/2} = 0 \quad \Rightarrow \quad y = \frac{3}{4}x.$$

Substituting the constraint,

$$x + 2y = 100 \quad \Rightarrow \quad x + 2\frac{3}{4}x = 100 \quad \Rightarrow \quad \frac{5}{2}x = 100 \quad \Rightarrow \quad x = 40$$

from which follows $x = 40$ and $y = \frac{3}{4}x = 30$ as the only stationary point. One can easily see that $f(40,30) > 0$ whereas $f(0,50) = f(100,0) = 0$ (two special points also satisfying the constraint $x + 2y = 100$). As $(40,30)$ is the only stationary point, it must therefore be a local (and global for $x,y > 0$ maximum. ■

■ **Example 27.4** Find the rectangle of maximum area which can be placed with its sides parallel to the x and y axes inside an ellipse with the equation $x^2 + 4y^2 = 1$.

We consider the upper-right corner of the rectangle at (x,y) to be situated on the ellipse. Thus we want to maximize the area of the rectangle,

$$A = 2x \cdot 2y = 4xy,$$

under the constraint that (x,y) satisfies the equation of the ellipse,

$$x^2 + 4y^2 = 1.$$

We identify $f(x,y)$ and $\phi(x,y)$ as

$$f = 4xy, \quad \phi = x^2 + 4y^2,$$

and calculate

$$\begin{aligned}
\frac{\partial A}{\partial x}+\lambda\frac{\partial \phi}{\partial x} &= 4y+2\lambda x=0 &&\Rightarrow\quad 2y+\lambda x=0 &&\Rightarrow\quad \lambda=-\frac{2y}{x},\\
\frac{\partial A}{\partial y}+\lambda\frac{\partial \phi}{\partial y} &= 4x+8\lambda y=0 &&\Rightarrow\quad x+2\lambda y=0 &&\Rightarrow\quad x+2\left(-\frac{2y}{x}\right)y=0,\\
& &&\Rightarrow\quad x^2-4y^2=0 &&\Rightarrow\quad x=\pm 2y.
\end{aligned}$$

As we take the upper-right corner, $x \geq 0$, $y \geq 0$ by convention and therefore the above condition reads $x=2y$. To determine x and y we now consider the constraint, i.e. the equation of the ellipse,

$$x^2+4y^2=1 \quad\Rightarrow\quad 4y^2+4y^2=1 \quad\Rightarrow\quad y=\frac{1}{2\sqrt{2}},\quad x=\frac{1}{\sqrt{2}}.$$

Therefore, $(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}})$ is a stationary point of f under the constraint. We can interpret it as the point where the area of the rectangle,

$$A=4xy=4\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{2}}=1,$$

is maximal. To infer the nature of the stationary point, imagine sliding the upper-right corner (x,y) along the ellipse in the first quadrant: the area will necessarily decrease to zero when sliding to $x=0$ or $y=0$, thus the stationary point must be a maximum. ∎

28. Line Integrals (Riley 11.1)

We have seen that the slope/rate of change of a multivariate function is not uniquely defined as it depends on the direction in the xy-plane of variables. Likewise, how to define the reverse operation through integration is not immediately obvious and must necessarily include a description of a path in the xy-plane. It will turn out that the such an integration procedure can be best understood in terms of the integration of a vector field over a path. We first introduce the meaning of such quantities and then mathematically define their integration.

Scalar and Vector Fields

A **field** assigns a quantity to each point in a region of space.

A **scalar field** assigns a single value (usually a real number) to a point, so it is nothing but a multivariate function $f = f(x,y)$ we have been discussing so far in this chapter. Examples of scalar fields in physics are height of land above sea level, air pressure or temperature (in 3D space), potential energy of a particle in a gravitational field.

A **vector field** assigns a vector, i.e. a quantity with magnitude and direction, to each point, $\mathbf{v} = \mathbf{v}(x,y)$. Examples of vector fields in physics are: velocity of air in the atmosphere, electric and magnetic field strengths, the gravitational force field. We have already encountered an important example of a vector field: the gradient of a function, $\nabla f(x,y)$. A physical application of the gradient is the determination of a 3D force field $\mathbf{F}(x,y,z)$ (the force a particle feels as a function of x, y and z) from the potential energy $U(x,y,z)$, $\mathbf{F} = -\nabla U(x,y,z)$. The minus sign applies here as the force will be in a direction such that the potential energy *decreases*.

Instead of writing the position as x,y,z, we can also express the position dependence in terms of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, i.e. we use the notation $f(\mathbf{r}) \equiv f(x,y,z)$, etc..

Definition of a Line Integral

A line integral takes a position-dependent vector field $\mathbf{F}(\mathbf{r})$ and a path C, defined by a parametrization $C : \mathbf{r} = \mathbf{r}(t)$, and 'integrates' over their scalar product,

$$I = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{28.1}$$

The notation used in the above equation reflects how the line integral is mathematically defined: The path is subdivided into infinitesimally small segments $d\mathbf{r}$ and the scalar product $dI = \mathbf{F} \cdot d\mathbf{r}$ is calculated for each segment. Finally, one sums over all pieces and the integral is achieved by taking the limit such that the lengths of the segments go to zero, $\lim_{|d\mathbf{r}|\to 0} \sum_C dI$. As we are taking the scalar product between the vector field (at a given point of the path) and the path vector segment, the end result of the integration is a number.

Definition 28.1 — Line Integral. The above procedure may sound complicated but if we have a parametrization of the path, i.e. functions $x = x(t)$, $y = y(t)$ that determine the path $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$, we can write the line integral as

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} \equiv \int_{t_A}^{t_B} \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \right] dt. \tag{28.2}$$

Here we have written the vector differential in terms of the vector derivative, $d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt$. The right-hand side of the above equation is nothing but a normal integral over t with the integrand given in the square brackets. The integral limits t_A and t_B represent the start and end points of the path C in the given parametrization: $\mathbf{r}_A = \mathbf{r}(t_A)$ (start point), $\mathbf{r}_B = \mathbf{r}(t_B)$ (end point).

In general, a line integral will depend on both the vector field and the choice of path. Even if the vector field and the start/end point are fixed, the line integral can depend on the choice of path from $\mathbf{r}_A$ to $\mathbf{r}_B$. (Note: the line integral will be independent of the specific *parametrization* of a given path, though).

■ **Example 28.1** A vector field $\mathbf{F}$ is defined by $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$. Evaluate the line integral for the path defined by a straight line from the start point $\mathbf{r}_A = (0,0)$ to the end point $\mathbf{r}_B = (2,1)$.

We first need a parametrization of the described path. As it is a straight line, we can write

$$x(t) = 2t, \quad y(t) = t \quad \Rightarrow \quad \mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j},$$

with the values $t_A = 0$ and $t_B = 1$ of the parameter t corresponding to the start and end point, respectively. The line integral can then be calculated as

$$\begin{aligned} I &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{t_A}^{t_B} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^1 \left(x(t)y(t)\mathbf{i} - y^2(t)\mathbf{j} \right) \cdot \frac{d(2t\mathbf{i} + t\mathbf{j})}{dt} dt \\ &= \int_0^1 \left(2t^2\mathbf{i} - t^2\mathbf{j} \right) \cdot (2\mathbf{i} + \mathbf{j}) dt \\ &= \int_0^1 3t^2 dt = \left[t^3 \right]_0^1 = 1. \end{aligned}$$

Note that after substituting the parametrization $x = 2t$, $y = t$, and performing the vector scalar product, we have an ordinary integral over the parameter t. ■

For sufficiently simple paths, it is often possible to use one of the coordinates as the parameter. This speeds up the calculation as we can take a few short cuts. In the above example, we could have simply chosen y as parameter and expressed x in terms of y: $x = 2y$.

■ **Example 28.2** For the same vector field $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$, calculate the line integral over the parabolic path $y = x^2/4$ from $(0,0)$ to $(2,1)$.

We choose x as the path parameter ranging from 0 to 2. Therefore $dy = \frac{1}{2}xdx$ and the line integral is given by

$$\begin{aligned} I &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C (xy\mathbf{i} - y^2\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) \\ &= \int_C (\frac{1}{4}x^3\mathbf{i} - \frac{1}{16}x^4\mathbf{j}) \cdot (dx\mathbf{i} + \frac{1}{2}xdx\mathbf{j}) \\ &= \int_0^2 (\frac{1}{4}x^3 - \frac{1}{32}x^5)dx = \left[\frac{1}{16}x^4 - \frac{1}{192}x^6\right]_0^2 = 1 - \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

It can be seen that the line integrals in the two examples differ as a consequence of the different paths chosen. ■

Physics Application: Work done in a Force Field

Consider a particle in a force field $\mathbf{F}$, e.g. a satellite in the gravitational force field of the earth. At a given point $\mathbf{r}$, the particle will feel the force $\mathbf{F}(\mathbf{r})$ and moving it to an infinitesimally near point $\mathbf{r} + d\mathbf{r}$ will require the mechanical work

$$dW = -\mathbf{F} \cdot d\mathbf{r}.$$

The scalar product applies to ensure that only the component of the force along the displacement vector takes effect. Summing over such infinitesimal steps along a given path C will therefore correspond to taking the line integral

$$W_C = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Loop Integrals

A loop integral is a line integral for which the path forms a closed loop, i.e. the start and end points coincide, $\mathbf{r}_A = \mathbf{r}_B$. Such loop integrals are sometimes denoted using the special $\oint$ notation,

$$I = \oint_C \mathbf{F} \cdot d\mathbf{r}. \tag{28.3}$$

In order to practically calculate a loop integral one needs an appropriate parametrization such that t_A is different from t_B.

■ **Example 28.3** Consider the vector field $\mathbf{F} = (y + 2xy)\mathbf{i} + (x + x^2)\mathbf{j}$. Calculate the loop integral formed by the following path: Starting at A: $(1,1)$, go straight to B: $(2,1)$, then straight to C: $(2,2)$, then straight to D: $(1,2)$ and finally straight back to A.

We calculate the loop integral by splitting it into four parts:

$$I = \oint_C \mathbf{F} \cdot d\mathbf{r} = I_{A\to B} + I_{B\to C} + I_{C\to D} + I_{D\to A}.$$

The individual parts are calculated as conventional line integrals where we use either x or y as parameter, as appropriate:

$$\begin{aligned}
I_{A\to B} &= \int \mathbf{F}\cdot dx\mathbf{i} && \text{with } dy = 0 \text{ as } y = 1 = \text{const} \\
&= \int_1^2 (1+2x1)dx = [x+x^2]_1^2 = 4, \\
I_{B\to C} &= \int \mathbf{F}\cdot dy\mathbf{j} && \text{with } dx = 0 \text{ as } x = 2 = \text{const} \\
&= \int_1^2 (2+2^2)dy = 6, \\
I_{C\to D} &= \int \mathbf{F}\cdot dx\mathbf{i} && \text{with } dy = 0 \text{ as } y = 2 = \text{const} \\
&= \int_2^1 (2+4x)dx && \text{[note the order of the limits]} \\
&= \left[2x+2x^2\right]_2^1 = 4-12 = -8, \\
I_{D\to A} &= \int \mathbf{F}\cdot dy\mathbf{j} && \text{with } dx = 0 \text{ as } x = 1 = \text{const} \\
&= \int_2^1 (1+1^2)dy = -2.
\end{aligned}$$

Summing all parts yields

$$I = \oint_C \mathbf{F}\cdot d\mathbf{r} = 4+6-8-2 = 0.$$

The loop integral is in fact zero as $\mathbf{F}$ is the gradient vector field of the function $f(x,y) = x^2y+xy$. The example illustrates the technique of splitting up a path into two or more parts and summing the individual results afterwards. ■

An important special case of a vector field is the gradient vector field of a scalar function, $\mathbf{F} = \nabla f$. The line integral in this case reads

$$I = \int_C (\nabla f)\cdot d\mathbf{r} = \int_{t_A}^{t_B} \left[(\nabla f)\cdot \frac{d\mathbf{r}}{dt}\right] dt. \tag{28.4}$$

Note that the integrand is nothing but the total derivative, Equation 26.9. Therefore,

$$I = \int_C (\nabla f)\cdot d\mathbf{r} = \int_{t_A}^{t_B} \frac{df}{dt}dt = f(x(t_B),y(t_B)) - f(x(t_A),y(t_A)), \tag{28.5}$$

according to the fundamental theorem of calculus. The line integral over a vector gradient simply yields the difference of function values at the start and end points of the path. The result is therefore also independent of the shape of the path. Using the total differential as a short-hand this is also easy to see,

$$I = \int_C (\nabla f)\cdot d\mathbf{r} = \int_C df = \Delta f = f(\mathbf{r}_B) - f(\mathbf{r}_A). \tag{28.6}$$

This can be seen as the generalization of the fundamental theorem of calculus to multivariate functions. As another consequence, loop integrals over vector gradient fields vanish,

$$\oint_C (\nabla f)\cdot d\mathbf{r} = 0. \tag{28.7}$$

R There is a technical caveat to the above: The gradient vector field needs to be well-behaved (differentiable, no poles) on a surface inside the loop.

29. Non-Cartesian Coordinate Systems

29.1 Change of Variables (Riley 5.6)

Functions of more than one variable appear frequently in physics when describing a position-dependent quantity, e.g. temperature of air as the function of the 3D Cartesian coordinates, $T = T(x,y,z)$.

As we have already mentioned, Cartesian coordinates are not the only possibility to specify a position. In 2D, we can for example use **polar coordinates** $(x,y) = (r\cos\theta, r\sin\theta)$, whereas in 3D we will learn the **spherical** and **cylindrical coordinate systems** below.

We are interested how the functional dependence changes as we make the transformation from one coordinate system to another, i.e. as we **change variables**. In general we can formulate the following problem: Given a function $f(x,y)$ and a transformation of variables from (x,y) to (s,t) with $x = x(s,t)$ and $y = y(s,t)$, we want to determine the partial fractions $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

In terms of the x,y variables the total differential of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{29.1}$$

On the other hand we can also determine the differentials of x,y when they are expressed as functions of s,t,

$$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt, \tag{29.2}$$

$$dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt. \tag{29.3}$$

Plugging these into df,

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right) \\ &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)dt. \end{aligned} \tag{29.4}$$

On the other hand, we can treat f as a function of s,t after the change of variables and the correspondin total differential is

$$df = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt. \tag{29.5}$$

Formula 29.1 — Change of Variables. We now compare the last two equations, to determine the partial derivatives with respect to s and t,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \tag{29.6}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{29.7}$$

R Above we neglected the detail that we have to change the function name when making the change of variables, $f(x,y) \to f(x(s,t),y(s,t)) = \bar{f}(s,t)$, because the functional dependence in terms of s,t is different from that of x,y. The crucial point, though, is that the differentials will need to be the same $df = d\bar{f}$ as both functions still describe the same quantity.

■ **Example 29.1** Given the function $z(x,y) = xy$ and the transformation of the variables as $x(s,t) = s-t$ and $y(s,t) = \sin(s+t)$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

We first calculate

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= y \cdot 1 + x \cdot \cos(s+t) \\ &= \sin(s+t) + (s-t)\cos(s+t),\end{aligned}$$

and similarly for $\frac{\partial z}{\partial t}$. We can check the above result by first expressing z in terms of s and t,

$$z = xy = (s-t)\sin(s+t),$$

from which we can calculate the partial derivatives directly, e.g.

$$\frac{\partial z}{\partial s} = \sin(s+t) + (s-t)\cos(s+t),$$

and similarly for $\frac{\partial z}{\partial t}$. ■

Coordinates come with their associated natural basis vectors. In 2D Cartesian coordinates, the basis vectors $\mathbf{i}$ and $\mathbf{j}$ can be expressed as partial derivatives of the position vector,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \tag{29.8}$$

with respect to the coordinates x and y,

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial x}, \quad \mathbf{j} = \frac{\partial \mathbf{r}}{\partial y}. \tag{29.9}$$

Formula 29.2 — Basis Vectors in General Coordinate System. We generalize this to define basis vectors associated with arbitrary coordinates s,t,

$$\mathbf{e}_s = \frac{\partial \mathbf{r}}{\partial s}, \quad \mathbf{e}_t = \frac{\partial \mathbf{r}}{\partial t}. \tag{29.10}$$

It is not guaranteed that these are unit vectors, i.e. are normalized to have a unit magnitude. To determine the unit basis vectors we thus have to divide by the magnitude, $\hat{\mathbf{e}}_s = \mathbf{e}_s/|\mathbf{e}_s|$ and $\hat{\mathbf{e}}_t = \mathbf{e}_t/|\mathbf{e}_t|$.

The basis vectors generally point in a direction determined by an infinitesimal change of the associated coordinate variable.

29.2 Polar Coordinates in 2D

We apply the above general considerations to the case of 2D polar coordinates. The transformation of variables from 2D Cartesian to polar coordinates is given by

$$x = r\cos\theta, \tag{29.11}$$
$$y = r\sin\theta. \tag{29.12}$$

As already discussed, r is the distance from the origin, $0 < r \leq \infty$ and θ is the angle counter-clockwise from the positive x-axis, $0 \leq \theta < 2\pi$.

The unit basis vectors in polar coordinates are written as $\hat{\mathbf{r}} \equiv \hat{\mathbf{e}}_r$ and $\hat{\theta} \equiv \hat{\mathbf{e}}_\theta$. As opposed to the Cartesian basis vectors, they are not constant because their directions change depending on the position.

We can calculate the polar basis vectors in terms of $\mathbf{i}$ and $\mathbf{j}$ as

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \quad \Rightarrow \quad \hat{\mathbf{e}}_r \equiv \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \tag{29.13}$$
$$\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} \quad \Rightarrow \quad \hat{\mathbf{e}}_\theta \equiv \hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \tag{29.14}$$

As expected, $\hat{\mathbf{r}}$ points radially outwards from the origin (increase in r) and $\hat{\theta}$ points laterally in a counter-clockwise fashion (increase in θ). They form an orthonormal basis system, i.e. in addition to being unit vectors, they are orthogonal to each other, $\hat{\mathbf{r}} \cdot \hat{\theta} = 0$.

We can calculate the following useful derivatives

$$\frac{d\hat{\mathbf{r}}}{d\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} = \hat{\theta}, \tag{29.15}$$
$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} = -\hat{\mathbf{r}}, \tag{29.16}$$

where we use an ordinary derivative with respect to θ as the vectors are functions of θ only.

■ **Example 29.2** The motion of a particle in 2D can be described as follows. The position vector $\mathbf{r} = \mathbf{r}(t)$ is a function of time t. In polar coordinates, the position vector can be best decomposed as

$$\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(t),$$

i.e. expressed in terms of the radial distance $r(t)$ and the radial unit vector $\hat{\mathbf{r}}(t)$. In general, both are functions of time. Then, the velocity is

$$\begin{aligned}\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(r\hat{\mathbf{r}})}{dt} &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} \\ &= v_r\hat{\mathbf{r}} + v_\theta\hat{\theta},\end{aligned}$$

where we followed the usual rules of vector differentiation and expressed the derivative of $\hat{\mathbf{r}}$ with respect to time in terms of the derivative with respect to θ using the chain rule. The final result shows the decomposition of the velocity in a radial part v_r and a transverse part v_θ.

In a similar fashion, one can calculate the acceleration,

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}\right) \\ &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta},\end{aligned}$$

or using the dot notation with $\dot{r} \equiv dr/dt$, etc.,

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}.$$

Here, the first term describes a genuine radial acceleration, the second term the centripetal acceleration induced by the angular motion, the third term a genuine angular acceleration and the fourth term the Coriolis acceleration. As a special case, circular motion with constant speed is described by $r = \text{const}$ and $\dot{\theta} = \text{const}$, i.e. the acceleration reduces to the centripetal acceleration, $\mathbf{a} = -r\dot{\theta}^2\hat{\mathbf{r}} = -\frac{v_\theta^2}{r}\hat{\mathbf{r}}$. ■

29.3 Cylindrical Coordinates (Riley 10.9)

While we do not discuss curved coordinate systems in 3D space in great detail, we here present cylindrical and spherical coordinates for reference.

Formula 29.3 — Cylindrical Coordinates. The transformation of variables from 3D Cartesian to cylindrical coordinates is given by

$$\begin{aligned}x &= \rho\cos\phi, \\ y &= \rho\sin\phi, \\ z &= z,\end{aligned} \tag{29.17}$$

where ρ is the polar distance in the xy-plane, $0 \le \rho < \infty$, θ is the angle counter-clockwise from the positive x-axis, $0 \le \theta < 2\pi$ (these two coordinates are simply polar coordinates of the projection onto the xy-plane). The z coordinate remains unchanged.

The position vector $\mathbf{r}$ can thus be expressed as

$$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k}. \tag{29.18}$$

The unit basis vectors $\hat{\rho}$, $\hat{\phi}$ and $\hat{\mathbf{k}}$ point in the directions of increasing ρ, ϕ and z, respectively:

$$\begin{aligned}\hat{\rho} &= \cos\phi\ \mathbf{i} + \sin\phi\ \mathbf{j},\\ \hat{\phi} &= -\sin\phi\ \mathbf{i} + \cos\phi\ \mathbf{j},\\ \hat{\mathbf{k}} &= \mathbf{k}.\end{aligned} \tag{29.19}$$

29.4 Spherical Coordinates (Riley 10.9)

Formula 29.4 — Spherical Coordinates. The transformation of variables from 3D Cartesian to spherical coordinates is given by

$$\begin{aligned}x &= r\cos\phi\sin\theta,\\ y &= r\sin\phi\sin\theta,\\ z &= r\cos\theta,\end{aligned} \tag{29.20}$$

where r is the radial distance from the origin, $0 \le r < \infty$, θ is the angle with respect to the positive z-axis, $0 \le \theta \le \pi$, and ϕ is the polar angle of the projection onto the xy-plane from the x-axis in a counter-clockwise fashion, $0 \le \phi < 2\pi$.

The position vector $\mathbf{r}$ can thus be expressed as

$$\mathbf{r} = r\cos\phi\sin\theta\mathbf{i} + r\sin\phi\sin\theta\mathbf{j} + r\cos\theta\mathbf{k}, \tag{29.21}$$

and the unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$ point in the directions of increasing r, θ and ϕ, respectively:

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k},\\ \hat{\theta} &= \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k},\\ \hat{\phi} &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}.\end{aligned} \tag{29.22}$$