2015 first attempt

28 March 2019 14:40

1. State Newton's second and third laws. **[2]**
   In what type of frame is Newton's second law valid? **[1]**
   Show, using these laws, that the total momentum of a system of particles is conserved in the absence of external forces. **[3]**

2nd Law - $F = ma$

3rd Law $F_a = -F_b$

Lab frame

$$F = \frac{dp}{dt}$$

$$p = p_1 + p_2$$

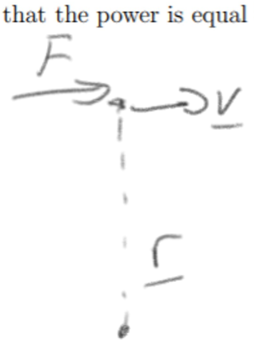
$$= F_1 t + F_2 t$$

$$\frac{dp}{dt} = F_1 + F_2$$

$$F_1 = -F_2$$

$$\rightarrow \frac{dp}{dt} = 0$$

2. A force $\underline{\mathbf{F}}$ acts on a particle with velocity $\underline{\mathbf{v}}$ at position $\underline{\mathbf{r}}$ for a time dt causing the particle to move a distance $\underline{\mathbf{dr}}$. Write down the expressions for the torque, power, impulse and work. **[4]**
   Show that the power is equal to the rate of change of kinetic energy with time. **[2]**



$$\tau = \underline{r} \times \underline{F}$$

$$P = Fv$$

$$I = F dt$$

$$I = F\,dt$$
$$W = \frac{FV}{dt}$$
$$KE = \frac{1}{2}mv^2$$
$$\frac{d}{dt}(KE) = maV$$
$$= FV$$
$$= P$$

3. Define the centre of mass position for two particles of mass m_1 and m_2 and positions $\underline{\mathbf{r}}_1$ and $\underline{\mathbf{r}}_2$ respectively and use your definition to determine the formula for the velocity of the centre of mass in terms of the velocities of the particles: $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$. [2]

   Show that the momentum of mass m_1 in the centre of mass frame, $\underline{\mathbf{p}}_1'$, is given by:

$$\underline{\mathbf{p}}_1' = \mu\underline{\mathbf{v}}$$

   where μ is the reduced mass of the two particles and $\underline{\mathbf{v}} = \underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_2$. [3]

   What is the total momentum of the particles in the CM frame? [1]

   Show that the total angular momentum of the two particles in the centre of mass frame, $\underline{\mathbf{L}}'$, is given by:

$$\underline{\mathbf{L}}' = \mu\underline{\mathbf{r}} \times \underline{\mathbf{v}}$$

   where $\underline{\mathbf{r}} = \underline{\mathbf{r}}_1 - \underline{\mathbf{r}}_2$. [2]

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

$$V_{cm} = \frac{d}{dt}(R_{cm}) = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2}$$

$$V_1' = \frac{m_2}{M}\underline{V}$$
$$\underline{P_1'} = m_1 V_1'$$
$$= \frac{m_1 m_2}{m_1 + m_2}\underline{V}$$
$$= \mu V$$

$m_1 + \dots$

$= \mu \underline{V}$

total momentum $= 0$

$L = mr^2\dot{\theta}$?

4. Sketch the plane polar coordinate system, showing the coordinates r and θ and the unit vectors $\hat{\underline{\mathbf{r}}}$ and $\hat{\underline{\boldsymbol{\theta}}}$. Express $\hat{\underline{\mathbf{r}}}$ and $\hat{\underline{\boldsymbol{\theta}}}$ in terms of the Cartesian unit vectors $\hat{\underline{\mathbf{i}}}$ and $\hat{\underline{\mathbf{j}}}$. [4]

Show that:

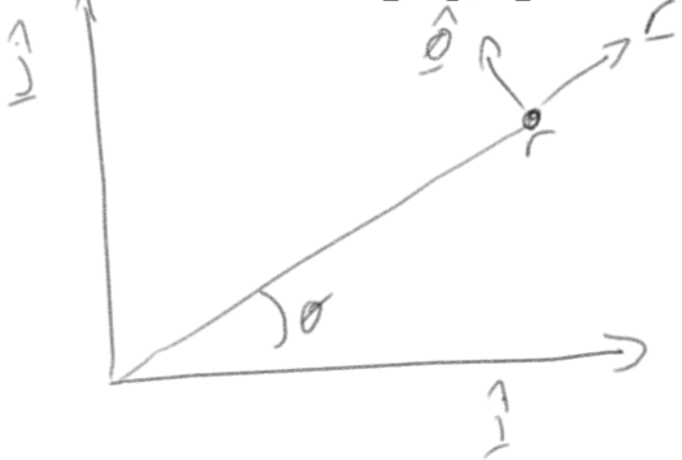
$$\dot{\hat{\underline{\mathbf{r}}}} = \dot{\theta}\,\hat{\underline{\boldsymbol{\theta}}}.$$

[2]

And hence that:

$$\underline{\mathbf{v}} = \dot{r}\,\hat{\underline{\mathbf{r}}} + r\dot{\theta}\,\hat{\underline{\boldsymbol{\theta}}}.$$

[2]



$\hat{\underline{r}} = \cos\theta\,\hat{\underline{i}} + \sin\theta\,\hat{\underline{j}}$

$\hat{\underline{\theta}} = -\sin\theta\,\hat{\underline{i}} + \cos\theta\,\hat{\underline{j}}$

$\dot{\hat{\underline{r}}} = \frac{d}{dt}\left(\cos\theta\,\hat{\underline{i}} + \sin\theta\,\hat{\underline{j}}\right)$

$= -\dot{\theta}\sin\theta\,\hat{\underline{i}} + \dot{\theta}\cos\theta\,\hat{\underline{j}}$

$= \dot{\theta}\,\hat{\underline{\theta}}$

$$v = \frac{dr}{dt} = \frac{d}{dt}\left(r\hat{\underline{r}}\right)$$
$$= r\dot{\hat{\underline{r}}} + \dot{r}\hat{\underline{r}}$$
$$= \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$

5. For a rotating rigid body, under what circumstances is the angular momentum conserved? **[1]**

   For such a body, how is the angular momentum and rotational kinetic energy related to the moment of inertia, I? **[2]**

   For a rotating rigid body, what condition is satisfied when the body rolls without sliding? **[1]**

   If the coefficient of sliding friction is μ, what is the frictional force on an object in vertical equilibrium sliding on a horizontal plane? **[2]**

Only radial force: $F(r)\hat{\underline{\theta}} = 0$

$L = I\omega$

R~~k~~e $= \frac{1}{2}I\omega^2$

?

$F = \mu mg$

6. What two conditions define a conservative, central force? **[2]**

   Show that angular momentum is conserved for a central force. **[3]**

   In one dimension, how is the magnitude of a conservative force, F, related to the potential energy, V? **[1]**

Conservative - work done independent of path.
Central - only acts in $\hat{\underline{r}}$ direction

Central force: $m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}} = 0$

$\therefore 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$L = mr^2\dot{\theta}$

$\dot{L} = \quad L$

$L = mr^2\dot{\theta}$

$\dot{\theta} = \frac{L}{mr^2}$

$\ddot{\theta} = \frac{-2\dot{r}L}{mr^3}$

$\frac{2\dot{r}L}{mr^2} + \frac{r \times -2\dot{r}L}{mr^3} = 0$

$2\dot{r}L - 2\dot{r}L = 0$

?

$F = -\frac{dU}{dx}$

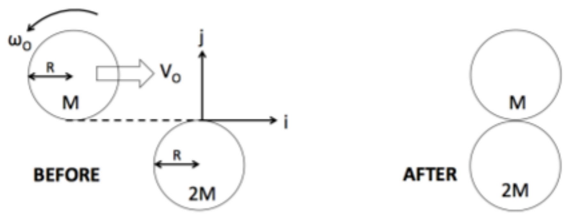
7. Write down in integral form the equation defining the moment of inertia, I, of a rigid body. [2]

Show, that the moment of inertia of a uniform circular disk of radius R and mass M about an axis perpendicular to the disk and passing through its centre is:

$$I = \frac{1}{2}MR^2.$$

[5]

This disk moves on a frictionless horizontal surface towards a stationary circular disk of mass 2M and radius R. The moving disk has a velocity $v_0\,\hat{\mathbf{i}}$ and is rotating counter clockwise with an angular speed of ω_o as shown in the figure below which depicts the disks before and after the collision as viewed from above the surface. The moving disk collides with the stationary disk and they instantly stick to each other and subsequently move as a single combined object.



What two quantities are conserved in the collision? [2]

What is the velocity of the combined disks after the collision? [2]

What is the position of the centre of mass of the disks at the time of collision? [2]

Show that the moment of inertia of the final system with respect to an axis through the centre of mass and perpendicular to the disks is: $\frac{25}{6}MR^2$. [3]

The initial angular momentum of the disks has a contribution from the rotation and the linear momentum with respect to this centre of mass. Using this, determine the angular velocity of the final system and show that it is zero if:

$$\omega_o = \frac{8v_o}{3R}.$$

[4]

$$I = \int_{vol} r^2 \, dm$$

Circle: $A = \pi r^2$

$$\rho = \frac{M}{A}$$

$$dA = 2\pi R$$

$$\rho \, dA = dm$$

$$\frac{M}{2\pi r^2} \times R = dm$$

$$dm = \frac{M}{2r}$$

$$I = \frac{MR}{2\pi} \int_0^{2\pi} d\theta \int_0^R dr \mid dr$$

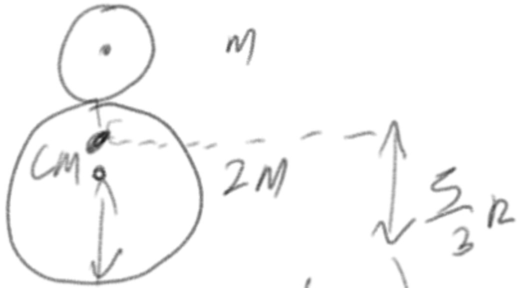
$$= \frac{1}{2} MR^2$$

Angular momentum ?

linear momentum.

$$v_0 M = V_1 (M + 2M)$$

$$V_1 = \frac{1}{3} V_0$$



$$CM = \frac{2MR + M(3R)}{3M}$$

$$= \frac{5}{3}R$$

$$I_0 = \frac{1}{2}(2M)R^2$$

$$I_{cm} = I_0 + 2M\left(\frac{2}{3}R\right)^2$$

$$= MR^2 + \frac{4}{9}\times 2MR^2$$

$$= \frac{17}{9}MR^2$$

$$I_0 = \frac{1}{2}MR^2$$

$$I_{cm} = I_0 + M\left(\frac{4}{3}R\right)^2$$

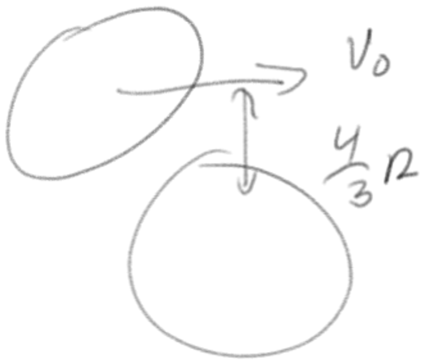
$$= \frac{1}{2}MR^2 + \frac{16}{9}MR^2$$

$$= 41\ MR^2$$

$$= \frac{41}{18} MR^2$$

&

$$\frac{41}{18} + \frac{17}{9} = \frac{75}{18} = \frac{25}{6} MR^2$$



$\rightarrow F = \frac{dp}{dt}$

$p = V_0 M$

$\tau = \underline{F} \times \underline{d}$

$$\tau = \frac{d}{dt}(V_0 M) \times \frac{4}{3}R$$

$$\tau = \frac{dL}{dt}$$

$L = V_0 M \times \frac{4}{3} R$

(from linear)

$\rightarrow w_1 = \frac{L}{I_{cm}}$

$$= \frac{\frac{4}{3}MR}{\frac{25}{6}MR^2} V_0$$

$\frac{8}{} V_0$

Re consume:

Befor:

$w_1 I_{cm} = W_0 I_0$

$$\omega_1 I_{cm} = \omega_0 I_0 \qquad = \frac{8}{25R} v_0$$

$$\omega_2 = \frac{\omega_0 I_0}{I_{cm}}$$

$$= \frac{\frac{1}{2}MR^2}{\frac{25}{6}MR^2}\omega_0$$

$$= \frac{3}{25}\omega_0$$

$$\omega_t = \omega_1 + \omega_2$$

$$\omega_t = \frac{8}{25R}v_0 - \frac{3}{25}\omega_0$$

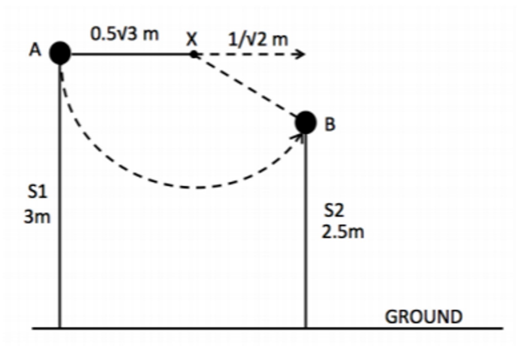
$$\omega_t = 0 \rightarrow \frac{3}{25}\omega_0 = \frac{8}{25R}v_0$$

$$\omega_0 = \frac{8}{3R}v_0$$

8. Under what circumstances is it convenient to introduce fictitious forces? **[2]**

A ball of mass m moves along the earth's surface at a latitude of 45° with a uniform velocity $\underline{\mathbf{v}}$. Its trajectory is measured precisely by a stationary laser system. If the earth rotates with a uniform angular velocity $\underline{\boldsymbol{\omega}}$, write down the formula for the two fictitious forces which cause the trajectory to be measured as a curve. **[3]**

As shown in the figure below a mass (A) of 5 kg is fixed to point X via an inextensible string of negligible mass and length $\frac{\sqrt{3}}{2}$ m. The mass is stationary on a support (S1) that is is 3 m above the ground and $-\frac{\sqrt{3}}{2}$ m horizontally from X. S1 is removed and the mass falls freely under gravity with the string remaining under tension. A second mass (B) of 2 kg is stationary on a support (S2) at a height of 2.5 m above the ground and a horizontal distance of $+\frac{1}{\sqrt{2}}$ m from X.



At the instant A collides with B the string detaches from A and the masses coalesce to form a single object that falls freely under gravity. Neglecting the size of the masses and any forces due to the earth's rotation:

Determine the angle $A\hat{X}B$ at which the balls collide. **[2]**

What is the speed of A and the tension of the string at the instant just before the collision? **[4]**

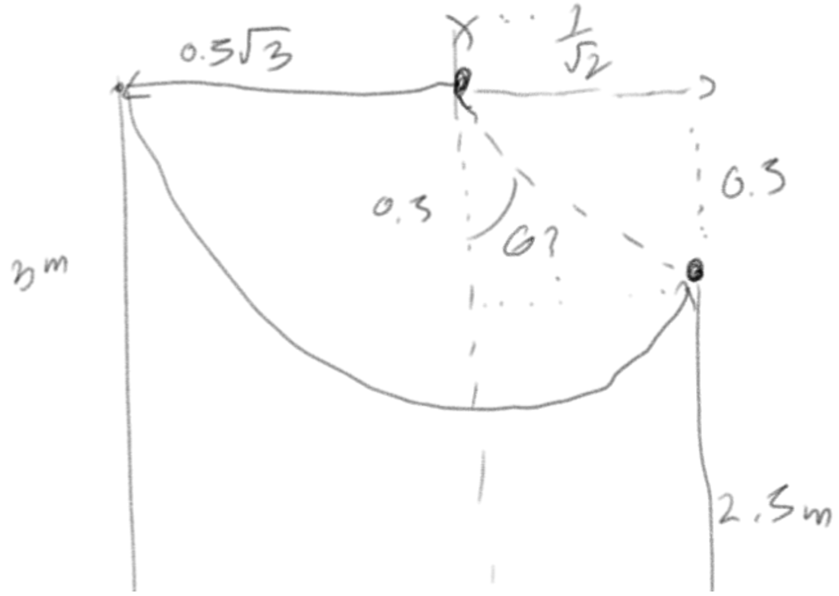
Determine the maximum height above the ground attained by the coalesced masses after the collision. **[5]**

How far from the base of S2 does the coalesced mass hit the ground? **[4]**

During Non-linear reference frames

Coreolis: $F = -2m\,\underline{\omega} \times \underline{V}$

centrifugal: $F = m r \dot{\theta}^2$



2.5m

$$\tan\theta = \frac{\frac{1}{\sqrt{2}}}{0.5}$$

$$\theta = 54.74^\circ$$

$$mgh = \frac{1}{2}mv^2$$

$$h = 0.5$$

$$v = \sqrt{g}$$

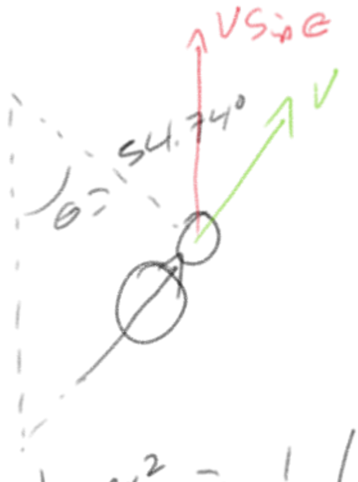
$$= 3.13$$

$$F = T = \frac{mv^2}{r}$$

$$r = 0.5\sqrt{3}$$

$$T = 56.58N + mg\cos(\theta)$$

$$= 84.87\ N$$



$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m_1+m_2)v^2$$

$$v_0^2 = g$$

$v_0 \quad -g$

$$v^2 = \frac{5 \times g}{5+2}$$

$= 7$

$v = 2.645$

$v\sin\theta = 2.16 \text{ m/s}$

$\frac{1}{2}mv^2 = mgh$

$$h = \frac{\frac{1}{2}v^2}{g}$$

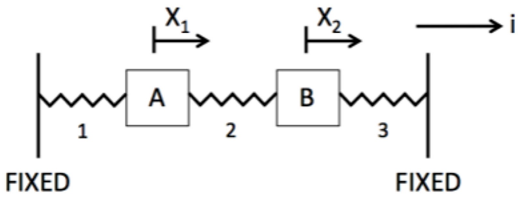
$= 0.238\text{m}$

$0.238 + 2.5 = 2.738$?

distance is trivial

9. Write down the differential equation for the displacement, x, from equilibrium for a mass undergoing undamped, undriven simple harmonic motion of period T. [2]

Two masses A and B each of mass m are connected on a horizontal frictionless surface by a spring of spring-constant k and to two fixed supports by springs also of spring-constant k as shown below.



Initially the springs are at their equilibrium lengths, B is at rest and A is given a velocity $v\hat{\mathbf{i}}$. Neglecting gravity, write down the equations of motion for A and B if their displacements from equilibrium are x_1 and x_2 respectively. [4]

If $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ show that both y_1 and y_2 obey the equation describing undamped, undriven simple harmonic motion and that the ratio of the frequencies of oscillation of y_1 and y_2 is $\frac{1}{\sqrt{3}}$. [3]

Using the given boundary conditions, show that a solution to the equation for y_1 is:

$$y_1 = v\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right).$$

[3]

A mechanical damping system is now attached to A and B that applies a force opposing the motion of A and B. The magnitude of this force is βv_r where v_r is the relative velocity of A and B.

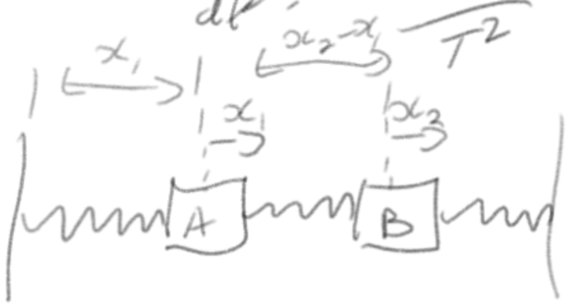
Write down the new equations of motion for A and B in terms of x_1 and x_2 that include this damping force. [2]

By considering these equations of motion in terms of y_1 and y_2 and a trial solution of the form Ae^{qt} for y_2, show that $x_1 = x_2$ as $t \to \infty$ and hence determine the maximum amplitude of A or B as $t \to \infty$. [6]

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{d^2x}{dt^2} = -\frac{4\pi^2 x}{T^2}$$



$$A: \quad \frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$
$$= k(x_2 - 2x_1)$$

B: $\frac{d^2x_2}{dt^2} = kx_2 - k(x_1 - x_2)$

$= k(2x_2 - x_1)$

$y_1 = x_1 + x_2$

$\frac{d^2y_1}{dt^2} = 3k(x_2 - x_1)$

$\frac{d^2y_2}{dt^2} = k(x_2 - x_1)$

$\omega^2 = 3k$

$\omega = \sqrt{3k}$

$\frac{y_2}{y_1} = \frac{1}{\sqrt{3}}$

$\frac{d^2y_1}{dt^2} = 3k(x_2 - x_1)$

$y = v\sqrt{\frac{m}{n}} \sin\left(t\sqrt{\frac{k}{m}}\right)$

$\frac{dy}{dt} = v\cos\left(t\sqrt{\frac{k}{m}}\right)$

$\frac{d^2y}{dt^2} = -v\sqrt{\frac{k}{m}} \sin\left(t\sqrt{\frac{k}{m}}\right)$

$\frac{d^2y_1}{dt^2} + y = 0$ ✓ ?

$\frac{d^2 y}{dt^2} + y = 0$ ✓ ?...

?

10. What quantities are conserved in an elastic collision with no external forces? **[2]**

In a one dimensional elastic collision of two objects how does the relative velocity of the objects change as a result of the collision? **[2]**

Two elastic balls are placed vertically above each other with a small gap between and released to fall from a height h under gravity. The top ball has mass m_1 and the bottom m_2. Neglecting air resistance and the size of the balls relative to h, determine the velocity, u, of the balls when the bottom one reaches the ground. **[2]**

Assuming the collision with the ground is elastic and the subsequent collision between the balls is also elastic, determine the value of m_1/m_2 which maximises the kinetic energy of the upper ball after the collision of the two balls and for this value of m_1/m_2 determine the maximum height the upper ball reaches after the collision. **[7]**

Determine the velocity of m_2 in the centre of mass frame after it has bounced from the ground and just before it collides with m_1. **[3]**

The maximum possible height that can be attained by the upper ball occurs when $m_2 \gg m_1$. Show in this case that the upper ball reaches a height of $9h$. **[4]**

KE & momentum.

separation speed = approach speed
i.e. rel velocity is constant.

$mgh = \frac{1}{2}mv^2$

$v = \sqrt{2gh}$

max KE for m_1 ∴ KE for $m_2 = 0$

$V_1 - V_2 = 2\sqrt{2gh}$

$$V_{m_1} - V_{m_2} = 2\sqrt{2gh}$$

$$v_1 = 0$$
$$v_2 = ?$$

$$m_2 u_2 - m_1 u_1 = m_1 v$$

$$v = 2\sqrt{2gh}$$

$$m_2 - m_1 = 2m_1$$

$$3m_1 = m_2$$

$$\frac{m_1}{m_2} = \frac{1}{3}$$

$$v = 2\sqrt{2gh}$$

$$\frac{1}{2}mv^2 = mgh$$

$$h = \frac{v^2}{2g} = \frac{8gh}{2g}$$
$$= 4h$$

$$v = \frac{m_2 - m_1}{m_1 + m_2} v$$

$m_2 >> m_1$

approach $= 2u$

sep $= 2u \; (e = 1)$

$\therefore \; v - u = 2u$

$v = 3u$

$u = \sqrt{2gh}$

$V_{m_1} = 3\sqrt{2gh}$

$mgh = \frac{1}{2}mv^2$

$h = \frac{18gh}{2g}$

$= 9gh$

11. What condition must the eccentricity of a planetary orbit satisfy for the orbit to be closed and what are the possible shapes of closed, planetary orbits? [3]

A spherical, rotating neutron star has a density of $6 \times 10^{17}\,\mathrm{kg\,m^{-3}}$. Determine the minimum period of rotation for the star to remain intact. Assume the centripetal acceleration is provided by the gravitational interaction and this stops the star from disintegrating. [3]

What quantities are conserved for a planet orbiting a star in a closed orbit? [2]

A planet of mass m is in a circular orbit of radius r_0 around a massive star of mass M ($M \gg m$). Show that the kinetic energy of the planet, KE_P, is given by:

$$KE_P = \frac{GMm}{2r_0}.$$

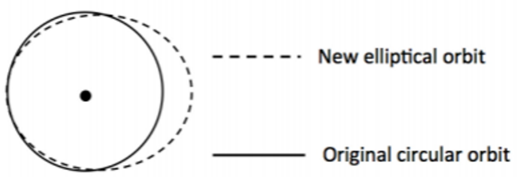
[2]

The star suddenly explodes and ejects 1% of its mass to a distance well beyond the planet's orbit such that the ejected mass has no gravitational effect on the planet.
If α is the ratio of the mass of the star after the explosion to the mass before the explosion and β is the ratio of the planet's farthest and closest distance to the star in its subsequent orbit, show, by considering the appropriate conserved quantities, that:

$$1 - 2\alpha = \frac{1}{\beta^2} - \frac{2\alpha}{\beta}.$$

[6]

Determine the eccentricity, e, of the new elliptical orbit of the planet. [4]



(An ellipse of eccentricity e is described by the equation of a conic section:

$\frac{1}{r} = \frac{1}{h}(e\cos\theta + 1)$ *where* $h = a(1 - e^2)$ *and a is the length of the semi major axis)*

total E (KE+PE) ≤ 0

Parabolic? (maybe not)

eliptical

circular ...

$$mr^2\dot{\theta} = \frac{GMm}{r^2} \quad ?$$

Angular momentum
total energy (KE + PE)

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2}$$

$$\ddot{r} = 0$$

$$r\dot{\theta}^2 = \frac{GM}{r^2}$$

$\dot{\theta} = \omega$

$\omega r = v$

$$(r\omega)^2 = \frac{GM}{r_0}$$

$= v^2$

$$KE = \frac{1}{2}mv^2$$

$$KE = \frac{1}{2}\frac{GMm}{r_0}$$

$$= \frac{GMm}{2r_0}$$

Angular momentum conserved.

$$m_0 r_0^2 \dot{\theta}_0 = m_1 r_1^2 \dot{\theta}_1^2$$

Circle:

$KE + PE = 0$

$\therefore PE = -KE$

$$PE_0 = -\frac{GMm}{2r}$$

$$\alpha = \frac{0.99M}{M}$$

Li conserved:

$$mr_0^2\dot{\theta}_0 = mr_1^2\dot{\theta}_1^2$$

$$v = r\omega$$

$$mv_0r_0 = mv_1r_1$$

$$v_0 = \frac{v_1r_1}{r_0}$$

$$KE_0 + PE_0 = KE_1 + PE_1$$

$$\frac{1}{2}m\left(\frac{v_1r_1}{r_0}\right)^2 +$$