Basic waves - Transverse

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Waves are seen in lots of areas of physics, but are quite hard to describe

Some do not require a medium, such as EM waves, some only propagate through a material, such as sound waves.

Waves also do not need to be periodic, but often are

Overall definition - the involve the transfer of energy through a disturbance which propagates in time and space.

1 dimensional wave equation

Propagation can be described by a wave equation.

Linear second-order partial differential which describes propagation of oscillation at a fixed speed, c

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{dx^2}$$
$$\psi = f(x,t)$$

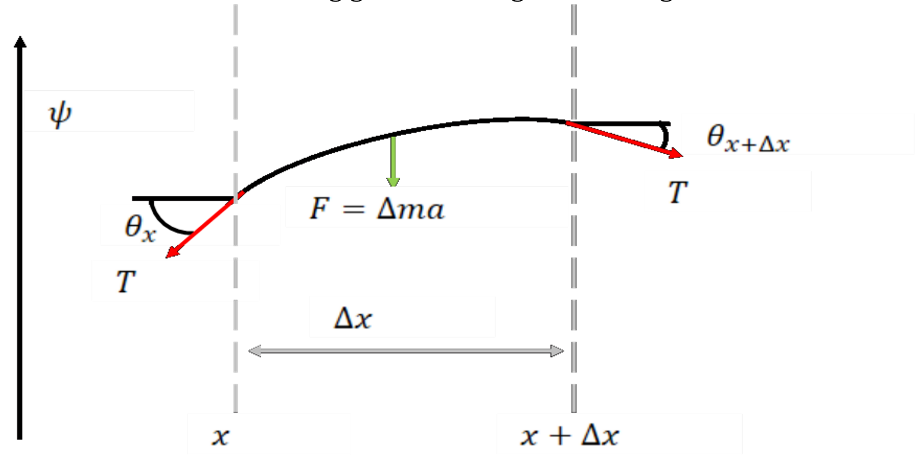
Deriving the wave equation using a string segment:

We wish to derive:

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi}{dx^2}$$

This is the wave equation in a string, where T is the tension in the string and μ is the mass per unit length

The wave acts to make the string go from straight to having a curve in it:



If we resolve the force F:

$$F = ma$$
$$dm = \mu\Delta x$$
$$a = \frac{\partial^2\psi}{\partial t^2}$$
$$F = \mu\Delta x\frac{\partial^2\psi}{\partial t^2}$$

If we resolve the force through the tensions:

$$F = Tsin(\theta_{x+\Delta x}) - Tsin(\theta_x)$$

However, we can write this as:

$$F = T\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - T\left(\frac{\partial\psi}{\partial x}\right)_x$$

As the gradient on the line is equal to sin(θ) for each point

We can now equate the force:

$$\mu\Delta x\frac{\partial^2\psi}{\partial t^2}=T\left(\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial\psi}{\partial t}\right)_x\right)$$

$$\frac{\partial^2\psi}{\partial t^2}=\frac{T}{\mu}\frac{\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial\psi}{\partial t}\right)_x}{\Delta x}$$

The term divided by Δx by definition gives the derivative, and so we can write:

$$\frac{\partial^2\psi}{\partial t^2}=\frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2}$$

As required

This gives $c=\sqrt{\frac{T}{\mu}}$

From this result we can also see:

$$\frac{\partial\psi}{\partial t}=\pm c\frac{\partial\psi}{\partial x}$$

This is only true for small displacements of the string.

Solving the wave equation:

We know the wave equation takes the form:

$$\frac{\partial^2\psi}{\partial t^2}=c^2\frac{\partial^2\psi}{\partial x^2}$$

The second time derivative must be equal to the second displacement derivative multiplied by c^2

We can let:

$$u=x-ct\ or\ u=x+ct$$

Such that:

$$\psi(x,t)=f(u)$$

We can now confirm this:

$$u=x-ct\rightarrow\frac{\partial u}{\partial x}=1,\frac{\partial u}{\partial t}=(\pm)c$$

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}=\frac{\partial\psi}{\partial u}$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u}\right)=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}\left(\frac{\partial\psi}{\partial u}\right)=\frac{\partial^2\psi}{\partial u^2}$$

$$\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial t}=(\pm)c\frac{\partial\psi}{\partial u}$$

$$\frac{\partial^2\psi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right)=\frac{\partial}{\partial t}\left(\pm c\frac{\partial\psi}{\partial u}\right)=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial t}\left(\pm c\frac{\partial\psi}{\partial u}\right)=c^2\frac{\partial^2\psi}{\partial u^2}$$

Which gives us the final result:

$$\frac{\partial^2\psi}{\partial t^2}=c^2\frac{\partial^2\psi}{\partial u^2},\frac{\partial^2\psi}{\partial x^2}=\frac{\partial^2\psi}{\partial u^2}$$

$$\therefore\frac{\partial^2\psi}{\partial t^2}=c^2\frac{\partial^2\psi}{\partial x^2}$$

As both solutions with $-ct$ and $+ct$ are valid, we write:

$$\psi(x,t)=f(x-ct)+g(x+ct)$$

Where $g(u)$ represents a wave travelling in the opposite direction to $f(u)$

As the waves follow a sinusoidal pattern, we can incorporate 2π into the equations:

$$let\ k=\frac{2\pi}{\lambda},\omega=\frac{2\pi c}{\lambda}$$

$$\psi(x,t)=f(kx-\omega t)+g(kx+\omega t)$$

Energy in waves can be stored as kinetic energy and potential energy

If looking at our string example from earlier:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{1}{2}\mu A^2\omega^2\sin^2(kx-\omega t)$$

The potential energy in the string comes as a result of the displacement causing extra tension from the increased length of the string.

The length increase, ΔL can be found by:

$$\Delta L = \sqrt{\Delta x^2 + \Delta\psi^2}$$

$$\Delta\psi = \Delta x\frac{\partial\psi}{\partial x}$$

$$\Delta L = \sqrt{\Delta x^2 + \Delta x^2\left(\frac{\partial\psi}{\partial x}\right)^2} = \Delta\mathrm{x}\left(1+\left(\frac{\partial\psi}{\partial\mathrm{x}}\right)^2\right)^{\frac{1}{2}} = \Delta\mathrm{x}\left(1+\frac{1}{2}\left(\frac{\partial\psi}{\partial\mathrm{x}}\right)^2\right)$$

The total change in length of the segment is $\Delta L - \Delta x = \frac{1}{2}\left(\frac{\partial\psi}{\partial\mathrm{x}}\right)^2\Delta\mathrm{x}$

The potential energy is the work done against tension per unit length. Unit length $= \Delta x$

$$\therefore U = T * \frac{\Delta L - \Delta x}{\Delta x} = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 = \frac{1}{2}TA^2k^2\sin^2(kx-\omega t)$$

This means total energy:

$$E = KE + U$$

$$E(x,t) = \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2$$

$$\therefore E(x,t) = \frac{1}{2}\frac{Z_0}{c}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 + c^2\left(\frac{\partial\psi}{\partial x}\right)^2\right]$$

We now introduce impedance, Z_0, which is the resistance towards a wave. For a stretch string $Z_0 = \sqrt{T\mu}$

We also already know $c = \sqrt{\frac{T}{\mu}}$

The rate of energy flow through a wave, i.e. the power, is found by the force times the velocity:

$$P(x,t) = F\frac{\partial\psi}{\partial t}$$

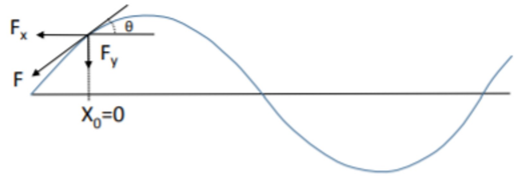
$$F \approx -T\frac{\partial\psi}{\partial x}$$

$$P(x,t) \approx -T\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}$$

$$Z_0 = \sqrt{T\mu}, c = \sqrt{\frac{T}{\mu}} \rightarrow T = Z_0 c$$

$$P(x,t) \approx Z_0 c\,\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}$$

We need to be able to describe a wave during its motion



A wave needs a driving mechanism to continue

The force must cause the string to move up and down

This force must balance the transverse component of tension

$$F_D = F_y = -T\sin(\theta) = -T\left(\frac{\partial\psi}{\partial x}\right)_{x=0}$$

$$\left(\frac{\partial\psi}{\partial t}\right) = \pm c\left(\frac{\partial\psi}{\partial x}\right) \therefore$$
$$F_D = -T\left(\frac{\partial\psi}{\partial x}\right)_{x=o} = -\frac{T}{c}\left(\frac{\partial\psi}{\partial t}\right)_{x=0}$$
This is linked with impedance:
$$\frac{T}{c} = Z_0$$
$$F_D = -Z_0\left(\frac{\partial\psi}{\partial t}\right)_{x=0}$$
The force at any instant must be proportional to the transverse velocity of string
This drag comes as a result of energy being transported by the wave

Terminating a wave:

We want to look at what occurs at the end of the string.
We can model a finite string as an infinite one,
The force at the end, at point $x = L$, the force must balance the tranverse component (as with the beginning):
$$F_L = T\left(\frac{\partial\psi}{\partial x}\right)_{x=L} = Z_0\left(\frac{\partial\psi}{\partial t}\right)_{x=L}$$
This basically means all the energy is absorbed at the end, and so no reflection occurs
This then models an infinitely long piece of string
This idea of the force at the end balancing the force of the wave is impedance matching

Different things can occur at the end of a wave:

Impedance matching (all energy absorbed, wave terminates)
Wave meets a boundary:
Reflection/Transmission

If a wave meets a solid wall, a reflection will occur. The total wave equation can be written as:
$$\psi = \psi_i + \psi_r = f(kx - \omega t) + g(kx + \omega t)$$
Where ψ_i, ψ_r are initial wave and reflected wave

We shall look at 3 boundary conditions:

Fixed end (wall)
String with free end
Joining another string

Fixed wall:

Wall has infinite impedance (will not move)
At the point at the wall:
$\psi = 0$ (there is no wave movement due to fixed position)
$\therefore \psi_i + \psi_r = 0 \rightarrow \psi_r = -\psi_i$
Reflection occurs and 180 degree phase difference

Free end:

There is 0 impedance at the end of the wall.
$$T\left(\frac{\partial\psi}{\partial x}\right) = 0 \therefore \frac{\partial\psi}{\partial x} = 0$$
$$\psi = \psi_i + \psi_r = 2\psi_i \rightarrow \psi_r = \psi_i$$

Joining another string:

Let string one have Length=L_1, $Z_0 = Z_1, \mu = \mu_1$

Let string two have length=$L_2, Z_0 = Z_2, \mu = \mu_2$

Strings connect at x=0

As tension is equal, and $Z_0 = \sqrt{T\mu}$

$$Z_1 = \sqrt{T\mu_1}, Z_2 = \sqrt{T\mu_2}$$

We will also assume there is perfect termination at the end of string 2

$$-T\left(\frac{\partial\psi}{\partial x}\right) = Z\left(\frac{\partial\psi}{\partial t}\right)$$

Some wave is reflected, and some is transmitted.

$\psi = \psi_i + \psi_r$

At join, transmitted wave form must equal initial wave form:

$$\psi(0,t) = \psi_{t(0,t)} \rightarrow \psi_i(0,t) + \psi_r(0,t) = \psi_t(0,t)$$

At join, drag on first string from the second string comes from the impedance multiplied by transverse velocity:

$$F_{drag} = Z_2\left(\frac{\partial\psi_t}{\partial t}\right) = Z_2\frac{\partial}{\partial t}(\psi_i + \psi_r) = Z_2\left(\frac{\partial\psi_i}{\partial t} + \frac{\partial\psi_r}{\partial t}\right)$$

This will be balanced by transverse force on first string:

$$-T\left(\frac{\partial\psi}{\partial x}\right) = -T\left(\frac{\partial\psi_i}{\partial x} + \frac{\partial\psi_r}{\partial x}\right) = Z_1\left(\frac{\partial\psi_i}{\partial t} + \frac{\partial\psi_r}{\partial t}\right)$$

We can now equate:

$$Z_1\left(\frac{\partial\psi_i}{\partial t} + \frac{\partial\psi_r}{\partial t}\right) = Z_2\left(\frac{\partial\psi_i}{\partial t} + \frac{\partial\psi_r}{\partial t}\right)$$

If we rearrange to find $\frac{\partial\psi_r}{\partial t}$:

$$\frac{\partial\psi_r}{\partial t} = \frac{Z_1 - Z_2}{Z_1 + Z_2}\left(\frac{\partial\psi_i}{\partial t}\right)$$

$$\rightarrow \psi_r(0,t) = \frac{Z_1 - Z_2}{Z_1 + Z_2}\psi_i(0,t)$$

We can related values at a given time to a different time:

$$\psi_i\left(-l, t - \frac{l}{c}\right) = \psi_i(0,t)$$

$$\psi_r\left(-l, t + \frac{l}{c}\right) = \psi_r(0,t) = R\psi_i(0,t) = R\psi_i\left(-l, t - \frac{l}{c}\right)$$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

This means reflected wave is same as incident but at a time $2\frac{l}{c}$ in the past and scaled by R

The transmitted wave:

$$\psi_t(0,t) = \psi_i(0,t) + \psi_r(0,t)$$

$$\psi_t(0,t) = (1 + R)\psi_i(0,t) = T\psi_i(0,t)$$

$$T = 1 + R = \frac{2Z_1}{Z_1 + Z_2}$$

If $Z_2 \gg Z_1$ (ie, solid wall)

$R = -1, T = 0$

No transmission and phase change

If $Z_1 \gg Z_2$ (ie, free end)

$R = 1, T = 2$

Reflection with same phase occurs, amplitude at turning point is 2*A

The frequencies of the transmitted wave and reflected wave must be equal, but the phase velocity and wavelengths will differ

$$\omega_1 = \omega_2 = \omega$$
$$v_1 = v_2 = v$$
$$c_1 = \sqrt{\frac{T}{\mu_1}}, c_2 = \sqrt{\frac{T}{\mu_2}}$$

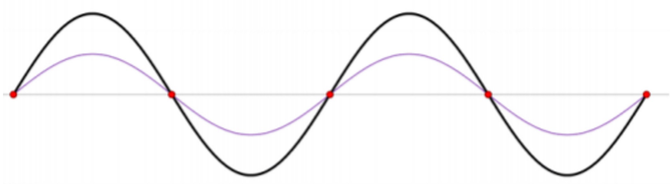
If we write our equations as:

$$\psi(x,t) = f(kx - \omega t) + g(kx + \omega t) = f\left(\frac{2\pi}{\lambda}(x - ct)\right) + g\left(\frac{2\pi}{\lambda}(x + ct)\right)$$

We can find:

$$k_1 = \frac{\omega}{c_1}, k_2 = \frac{\omega}{c_2}$$
$$\lambda_1 = \frac{c_1}{v}, \lambda_2 = \frac{c_2}{v}$$

A standing wave is one that occurs when 2 waves of the wave wavelength, frequency, and amplitude travel in opposite directions and interfere with each other.



This can occur when a wave reflects back on its self

We can find a solution by looking at boundary conditions.

If the end occurs at $x = L$

$$\psi(0,t) = \psi(L,t) = 0$$

As there is no movement at end points

$$\left(\frac{\partial\psi}{\partial t}\right)_{x=0} = \left(\frac{\partial\psi}{\partial t}\right)_{x=L} = 0$$

As the transverse speed at both ends is 0

We can the find the wave equations in complex form:

$$\psi(x,t) = Ae^{i(\omega t + kx)} + Be^{i(\omega t - kx)}$$
$$\psi(0,t) = Ae^{i\omega t} + Be^{i\omega t} = 0$$
$$\rightarrow A = -B$$
$$\therefore \psi(x,t) = Ae^{i\omega t}\left(e^{ikx} - e^{-ikx}\right)$$
$$= 2iAe^{i\omega t}\sin(kx)$$

We take the real part:

$$\psi(x,t) = -2A\sin(\omega t)\sin(kx)$$

We can now look at the boundary at $x = L$,

$$\psi(x,t) = 2iAe^{i\omega t}\sin(kx)$$
$$\psi(L,t) = 2iAe^{i\omega t}\sin(kL) = 0$$
$$\therefore kL = n\pi$$

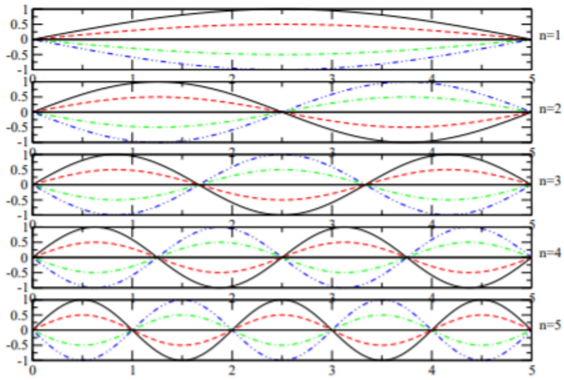
This means that we have multiple values of k and λ:

$$k_n = \frac{n\pi}{L}, \lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$$

Because

$$\omega = \frac{2\pi c}{\lambda} \rightarrow \omega_n = ck_n = \frac{n\pi c}{L}$$
$$f = \frac{\omega}{2\pi} \rightarrow f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L}$$

This gives the following pattern for varying values of n:



The wave consists of points which have no movement at any time (Nodes) and points which fluctuate between maximum amplitudes (Antinodes)

There are a total of n maximum points (n antinodes) and n-1 nodes (excluding start and end points)

The frequency $f_{n=1}$ (1 antinode, 0 nodes) is called the fundamental frequency, all following frequences (f_n $n = 2,3,4 \dots$) are called the harmonics

Overall, when $n = 1$, the wave length is the largest allowed, and the frequency the lowest.

$$\lambda_1 = 2L, f_1 = \frac{c}{2L}$$

Most natural systems will have a resonant frequency.

In a damped force driven harmonic oscillator, the amplitude will reach a maximum when the driving frequency is equal to the resonant frequency of the object.

ASK LECTURER ABOUT RESONANCE

Nodes and Antinodes require a different wave equation to the simple one found earlier:

$$\psi(x,t) = Re\left[2iAe^{i\omega t}\sin(kx)\right] = -2Asin(\omega t)\sin(kx)$$

From this equation it is easy to see how points where $kx = n\pi$ will have 0 amplitude, and how points where $kx = \left(n + \frac{1}{2}\pi\right)$ will oscillate over time with an amplitude of $2A$

We can find the positions of the nodes and antinodes by substitution:

$$k_n = \frac{n\pi}{L}, \lambda_n = \frac{2L}{n} \rightarrow L = \frac{n\lambda_n}{2} \therefore k_n = \frac{2\pi}{\lambda_n}$$

$$k_n x_{node} = n\pi \rightarrow x_{node} = \frac{n\pi}{k_n} = \frac{n\pi}{\frac{2\pi}{\lambda_n}} = n\frac{\lambda}{2}$$

$$k_x x_{antinode} = \left(n + \frac{1}{2}\right)\pi = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$$