Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. (a) State the formal definition of a derivative of a function $f(x)$. **[3]**

   (b) Using the formal definition of a derivative, calculate from first principles the derivative of $y = 3x^2 - 7$. **[3]**

2. (a) Determine the following indefinite integrals: **[4]**

$$(i) \int 3x^{2/3} dx \ ; \qquad (ii) \int \frac{x}{(x-2)^2} dx \ .$$

   (b) Calculate the following definite integrals: **[4]**

$$(i) \int_{-1}^{+1} x \exp(-|x|) dx \ ; \qquad (ii) \int_{-1}^{+1} |x| \exp(-|x|) dx \ .$$

3. (a) Given a function $y = f(x)$, state the condition for a point x_0 to be stationary. **[2]**

   (b) Given a function $y = f(x)$, state the criteria to determine the nature of a stationary point. **[2]**

   (c) Find the stationary point(s) of $f(x) = x^4 + 4x^3 - 6$ and discuss its/their nature. **[4]**

4. (a) Calculate $|\underline{a} \times \underline{b}|$ when $|\underline{a}| = 2$, $|\underline{b}| = 4$ and the angle between the vectors $\underline{a}$ and $\underline{b}$ is $\theta = \pi/4$. **[2]**

   (b) Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} + 3\underline{j} + \underline{k}$. **[2]**

   (c) Determine the value of λ such that $\underline{a} = 4\underline{i} + 2\underline{j} - \lambda\underline{k}$ is perpendicular to $\underline{b} = 2\underline{i} - 6\underline{j} - 3\underline{k}$. **[2]**

[Part marks]

5. (a) Write **[3]**

$$z = i^5 + i + 1$$

in the form $(a + ib)$, with a and b real.

(b) Determine the real and imaginary parts of **[3]**

$$z = \frac{i - 4}{2i - 3} \ .$$

6. Determine the following limits **[6]**

$$\lim_{x \to 0} \frac{\sin(2x)}{x} \ ,$$

$$\lim_{x \to 0} \frac{\sin(x^4)}{\sin^2(x^2)} \ .$$

Section B

7. (a) Write down the general form of the Maclaurin series for a function $f(x)$. **[4]**

   (b) Determine the Maclaurin series, up to and including the fourth order term $\propto x^4$, of the following two functions : **[3]**

      i. $f(x) = \ln(1 - 8x^2)$. **[3]**

      ii. $g(x) = -\dfrac{4}{1+2x^2}$.

      Then determine the limit **[4]**

$$\lim_{x\to 0} \frac{f(x) + g(x) + 4}{x^6 - x^4} \ .$$

[6]

   (c) Determine the limit

$$\lim_{x\to 0} \frac{\int_0^x t\sqrt[3]{1+t^3}dt}{x^2} \ .$$

8. (a) State and derive de Moivre's theorem. **[4]**

   (b) Determine the cubic roots of $z = -8$. **[4]**

   (c) Show that **[4]**

$$\frac{\cos 2\theta + i \sin 2\theta}{\cos 3\theta + i \sin 3\theta} = \frac{1}{\cos\theta + i\sin\theta} \ .$$

[8]

   (d) Show that

$$1 + \frac{\cos\theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + ... = \frac{4 - 2\cos\theta}{5 - 4\cos\theta} \ .$$

[Part marks]

9. (a) Derive the sum of the arithmetic series **[3]**

$$S_N = a + (a + d) + (a + 2d) + ... + (a + (N - 1)d) \ .$$

(b) Derive the sum of the geometric series **[3]**

$$S_N = a + ar + ar^2 + ... + ar^{N-1} \ .$$

(c) Determine the sum of the series given by **[4]**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} \ .$$

(d) Show, by any means, whether the following sum converges or diverges : **[4]**

$$\sum_{n=1}^{\infty} \frac{n^2}{4(n+1)(n+2)} \ .$$

(e) Determine the radius of convergence of the sum of the power series **[6]**

$$\sum_{n=0}^{\infty} \left(\frac{3n+7}{4n+2} \right)^n z^n \ ,$$

where z is a complex number.

[Part marks]

10. (a) The position vector of a particle in 2D cartesian coordinates is given by **[4]**

$$\underline{r} = x\,\underline{i} + y\,\underline{j}\,.$$

Consider now polar coordinates, defined by

$$\begin{aligned} x &= r\cos\theta\ , \\ y &= r\sin\theta\ . \end{aligned}$$

Show that the unit vectors in 2D polar coordinates are given by :

$$\begin{aligned} \underline{\hat{r}} &= \cos\theta\,\underline{i} + \sin\theta\,\underline{j}\ , \\ \underline{\hat{\theta}} &= -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\ . \end{aligned}$$

(b) Show that the velocity of the particle **[4]**

$$\underline{v} = \frac{d\underline{r}}{dt}$$

is given by

$$\underline{v} = \frac{dr}{dt}\,\underline{\hat{r}} + r\,\frac{d\theta}{dt}\,\underline{\hat{\theta}}\,.$$

(c) Show that the acceleration of the particle **[4]**

$$\underline{a} = \frac{d\underline{v}}{dt}$$

is given by

$$\underline{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\underline{\hat{r}} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\underline{\hat{\theta}}\,.$$

(d) Find the points at which the curve given by the function

$$r = 1 + \cos\theta,$$

expressed in terms of the polar coordinates r and θ, has a vertical or horizontal tangent line. **[8]**

[Part marks]

11. (a) Given a function $z = f(x, y)$, state the condition for a point (x_0, y_0) to be stationary. **[3]**

(b) Given a function $z = f(x, y)$, state the criteria to determine the nature of a stationary point. **[5]**

(c) Find the stationary point(s) of $f(x, y) = x^3 + 3y - y^3 - 3x$ and discuss its/their nature. **[6]**

(d) Use the method of Lagrange Multipliers to find the stationary points of $f(x, y, z) = x + 2y - 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$. **[6]**

END OF PAPER