6 Geometrical Optics

In this section we will consider mirrors, lenses, and the instruments which can be made using them. In order to study this we only ever really need to deal with one equation. If we define p as the object distance, q as the image distance and f as a focal length then both mirrors and lenses obey the same common equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}. \tag{127}$$

However, in various situations we have to consider the definitions of what p, q and f are, and also look at the general consequences.

6.1 Mirrors

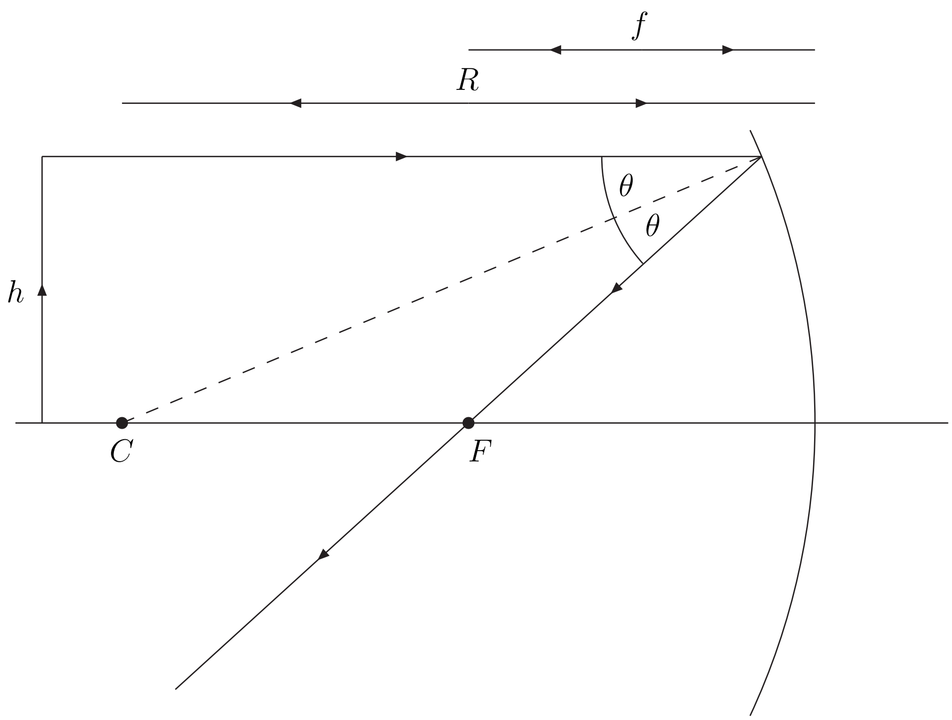

Figure 42: The relationship of centre of curvature to focal point.

We will always consider a mirror to be a section of a spherical shell with a given centre of curvature C, and a radius of curvature R. The reflecting surface may be on the inner surface of the spherical shell, in which case it is a concave mirror, or on the outer surface of the spherical shell, in which case it is convex. A concave mirror will focus light rays towards a point and a convex one will cause them to diverge from a point. More precisely, we define the principal axis of the mirror as the line which is the normal to the centre of the mirror, and then the focal point is that point at which all light rays parallel to the to the principal axis will be focused. For a convex mirror this point is implicitly behind the mirror.

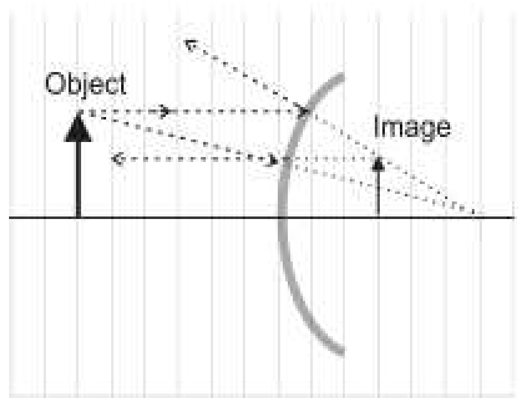

Figure 43: The focus of light rays for a convex mirror.

We can determine the position of the focal point by considering the geometry shown in figure 42. Using the fact that when a light ray strikes the surface at any point the angle of incidence relative to the line from C, which is normal to the mirror, is equal to the angle of reflection, we see we have the results

$$\frac{h}{R} = \tan\theta, \qquad \frac{h}{f} = \tan 2\theta, \tag{128}$$

where f is the distance from the focal point to the mirror. If θ is small this can be approximated by

$$\frac{h}{R} = \theta, \qquad \frac{h}{f} = 2\theta, \tag{129}$$

and clearly in this limit the focal length f is defined by $f = R/2$. This is taken as a general definition, but it should be remembered that it assumes that all rays striking the mirror which are parallel to the principal axis are making a small angle to the normal of the mirror. This assumes the mirror is only a very small part of the full spherical shell.

The situation for the convex mirror is shown in figure 43. It is the same as for the concave mirror, but parallel rays are reflected outwards and it is only the continuation of these reflected rays behind the mirror which meet at the focal point. This means that f is negative for a convex mirror.

6.1.1 Images from Concave Mirrors

We begin the study of the image formed by a concave mirror by making an explicit derivation of the mirror equation. This can be done in a similar manner for any situation, but this is perhaps the most straightforward to present. We consider the case in figure 44, where we have an object O of height h at a

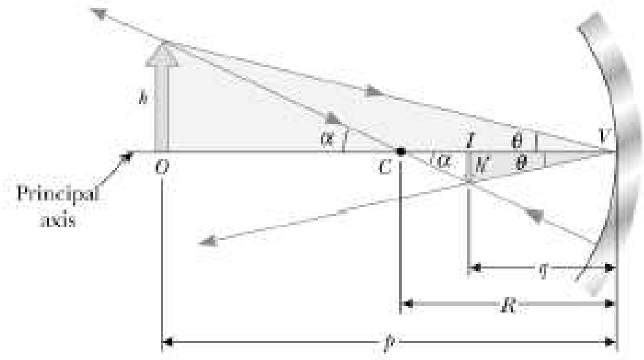

Figure 44: The image formed by a spherical concave mirror when the object O is placed outside the centre of curvature C.

distance $p > R$ from the vertex of the principal axis with the mirror at point V. We look at two separate rays from the top of the object. One goes through the centre of curvature C, and striking the mirror at right angles, is reflected back along the same line. There is another ray which strikes the mirror at V, and hence has angle of incidence θ equal to angle of reflection. The two rays meet at one point, and this forms the "top" of the image I at distance q from the mirror. This has height h', but it is inverted, so h' is negative.

To find the relationship between p and q we can consider the two triangles which have their vertex at C. O and I form the opposite sides of these triangles. The angle at the vertex is the same, let us call it α, it each case. This means we can define $\tan\alpha$ for each triangle and equate the results, i.e.

$$\tan\alpha = \frac{h}{p-R} = \frac{-h'}{R-q}. \tag{130}$$

But, from consideration of the two triangles with angles θ at the vertex, and I and O as the opposite sides we see that

$$\frac{-h'}{h} = \frac{q}{p}. \tag{131}$$

This is a general result which is true for all mirrors and lenses. Putting this in the expression for $\tan\alpha$ we obtain

$$\frac{-h'}{h} = \frac{q}{p} = \frac{R-q}{p-R}. \tag{132}$$

A little rearrangement gives

$$\frac{R-q}{q} = \frac{p-R}{p} \quad \rightarrow \quad \frac{R}{q} - 1 = 1 - \frac{R}{p} \quad \rightarrow \quad R\left(\frac{1}{p} + \frac{1}{q}\right) = 2. \tag{133}$$

This immediately becomes

$$\left(\frac{1}{p} + \frac{1}{q}\right) = \frac{2}{R} = \frac{1}{f}, \tag{134}$$

i.e. we obtain the mirror equation.

We now consider the type of image formed by a concave mirror. For all image distances $p > f$ the general situation is as illustrated in figure 44. The light rays reflected from a mirror do indeed really focus at a point, so the image formed is real. The image is in front of the mirror, which corresponds to the image distance q being positive. The image is also inverted, and in this case reduced in size.

Let us consider an explicit example, $f = 5$cm and $p = 20$cm. Using the lens equation

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{fp} \quad \rightarrow \quad q = \frac{fp}{p-f}. \tag{135}$$

In this case $q = 100/15 = 20/3$cm. We now introduce one more quantity, the lateral magnification M. This is defined by

$$M = \frac{h'}{h} = -\frac{q}{p}. \tag{136}$$

M will be positive if the image is upright and negative if it is inverted. In this example

$$M = -(20/3)/20 = -1/3, \tag{137}$$

and the image is inverted and reduced.

All these features in the example are common to the situation $p > f$ for a concave mirror, except the image is not always reduced (equivalently q is not always less than p. It is easy to see that if $2f > p > f$ then $q > p$, and the image will be enlarged. Indeed as p approaches f from above the image distance $q \rightarrow \infty$.

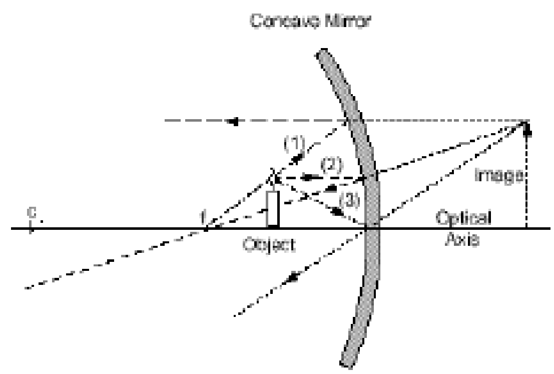

Figure 45: The image formed by a concave mirror when $f > p$.

The features change qualitatively if $f > p$. The general situation is shown in the diagram in figure 45, where this time a light ray incident parallel to the principal axis and focused at F, and one incident from F and reflected parallel to the principal axis are drawn, as well as one which is reflected at equal angle at the vertex. We now see that the general feature is that the light rays from the mirror do not meet, but will appear to do so by an observer looking at the mirror, because they appear to originate from a point behind the mirror. Hence, a virtual image is obtained which is behind the mirror. This image is upright and enlarged.

As a specific example we take $f = 10$cm and $p = 6$cm. This time

$$q = \frac{fp}{p-f} = 60/(-4)\text{cm} = -15\text{cm}. \tag{138}$$

The lateral magnification $h'/h = -q/p = 15/6 = 2.5$, and the image is indeed enlarged.

6.1.2 Images from Convex Mirrors

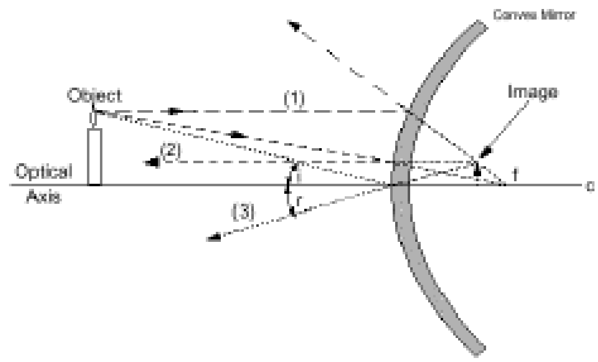

Figure 46: The image formed by a convex mirror.

In this case the equation is the same, but light is focused (in a virtual sense) behind the mirror, so $f < 0$. The situation is shown in figure 46, where again the lines either incident or reflected parallel to the principal axis are shown. The rays diverge away from the mirror and to an observer appear to meet at a point between the mirror and the focal point. The image is upright, virtual and reduced. As a specific example we take $p = 15$cm and $f = -5$cm. This gives

$$q = \frac{fp}{p-f} = -75/20\text{cm} = -15/4\text{cm}. \tag{139}$$

The magnification is $M = -(-15/4)/15 = 1/4$. The magnitude of the image distance q is always less than either p or f, and the magnification is always < 1.

6.1.3 Image in a Plane Mirror

The focal length of a plane mirror is $f \to \infty$, so this is the limit of infinite focal length for either the concave or convex mirror. The mirror equation gives

$$1/p + 1/q = 0, \tag{140}$$

i.e. $p = -q$, and the image is upright, virtual, and has magnification $M = 1$. The ray diagram consistent with this is shown in figure 47.

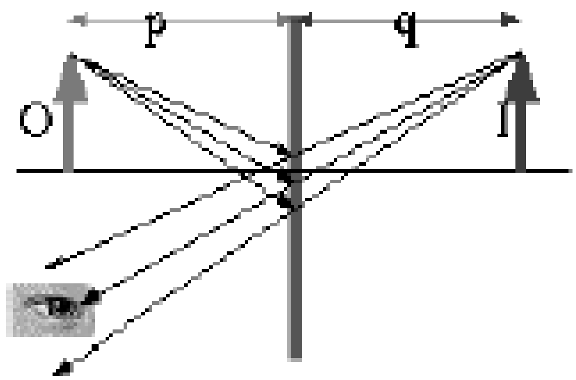

Figure 47: The image in a plane mirror.

6.2 Lenses

6.2.1 Definitions

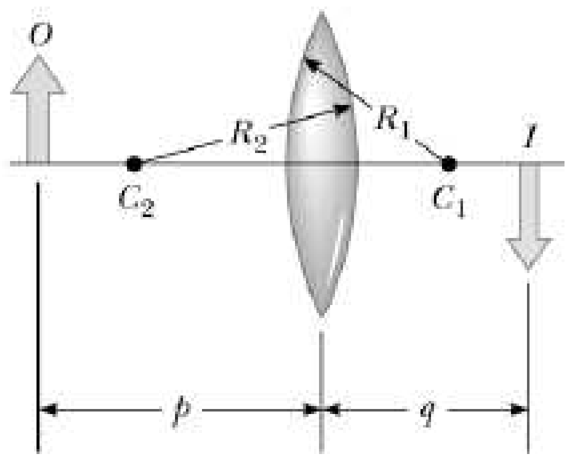

Figure 48: The geometry for a thin lens.

The general situation for a lens is shown in figure 48, where a biconvex lens is shown, but either, or both surfaces could be convex. As with the mirror the surface of the lens is assumed to be a section of the surface of a sphere, but the lens has two distinct surfaces, each with a centre of curvature, C_1 and C_2. Hence, the situation for the lens is similar to the mirror, but there are more degrees of freedom. We make the following definitions.

The object is conventionally in front (in diagrams usually left) of the lens, so this is classified as $p > 0$. If the object for a lens was behind the lens we would have $p < 0$. This can be the case in practice if we consider a system of lenses, as discussed later.

If the image is behind (right of) the lens then $q > 0$. This corresponds to a real image. An image in front of the lens has $q < 0$, and is virtual

The focal length f is defined as $f > 0$ for a converging lens and $f < 0$ for a diverging lens.

The radii of curvature R_1, R_2 are each > 0 if C_1, C_2 are behind the lens and are < 0 if C_1, C_2 are in front of the lens.

As with the mirror the lateral magnification $M = h'/h = -q/p$, and $M > 0$ is an upright image and $M < 0$ is an inverted image.

Figure 48 shows a biconvex lens with $R_1 > 0$ and $R_2 < 0$. A biconcave lens would have $R_1 < 0$ and $R_2 > 0$. It is possible to have one convex and one concave surface, but we will not consider an example of this. Unlike a mirror the lens has a focal point either side of the lens, as illustrated in figure 49.

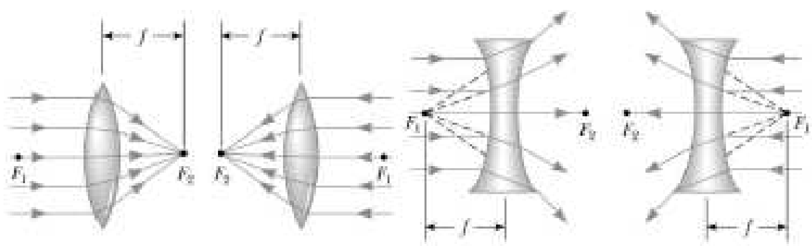

Figure 49: The focal points of a converging (left) and diverging (right) lens.

Having made clear our definitions it would be possible to derive the lens equation using the sort of geometrical arguments we used for the mirror in combination with Snell's law for refraction. This is rather complicated and we will not go through details. We will merely note that the result, in the limit that the lens is thin (equivalent to the statement that the surface of the lens is only a very small section of a spherical surface), is the **lens maker's formula**

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{141}$$

where n is the refractive index of the lens and that of the surrounding medium is implicitly equal to 1. So a bi-convex lens is converging, a bi-concave lens diverging, and a mixed lens depends on the relative radii of curvature. This result for the focal length of the lens can then be used in the lens equation,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}. \tag{142}$$

This results in the images for lenses being much the same as for mirrors, but where light was reflected back in the previous case, it is now refracted through.

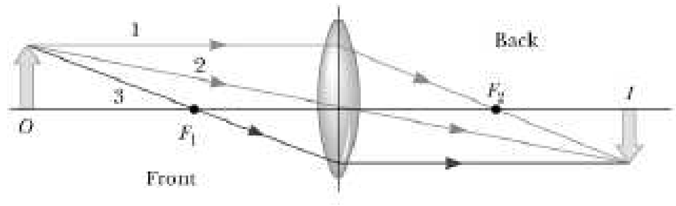

Figure 50: The image for a converging lens with $p > f$.

6.2.2 Images using a Converging Lens.

As for the concave (focusing) mirror there are two qualitatively different cases – $p > f$ and $p < f$. The situation for the former is shown in figure 50. The light rays converge at a point behind the mirror to form a real, inverted image which may be reduced or enlarged, depending on the position of the object compared to the focal length. As an explicit example we consider $p = 11$cm and $f = 5$cm. This leads to

$$q = \frac{pf}{p-f} = 55/6\,\text{cm}, \tag{143}$$

and q is indeed positive. The lateral magnification $M = -q/p = -(55/6)/11 = -5/6$, so the image is reduced and inverted. The image will be enlarged if $f < p < 2f$, e.g for the same lens if $p = 8$cm, $q = 40/3$cm and $M = -5/3$.

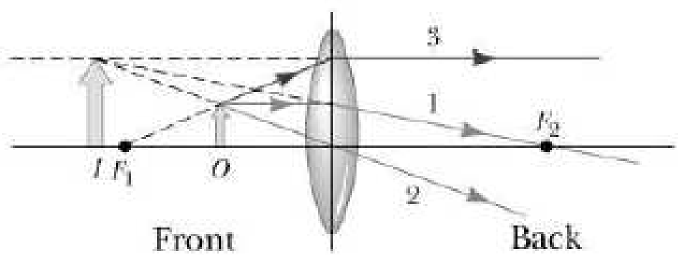

Figure 51: The image for a converging lens with $p < f$.

For $p < f$ the image formation is illustrated in figure 51. The light rays emergent from the lens are divergent and an observer sees them appear to converge in front of the lens. The image is virtual, upright and enlarged. For example, if $f = 10$cm and $p = 6$cm then

$$q = \frac{pf}{p-f} = 60/(-4)\,\text{cm} = -15\,\text{cm}. \tag{144}$$

The magnification is $M = -(-15)/6 = 2.5$, and the image is indeed enlarged.

6.2.3 Images using a Diverging Lens.

As for the convex (diverging) mirror the qualitative situation is the same whatever the value of p, and is shown in figure 52 for a bi-concave lens. The emergent

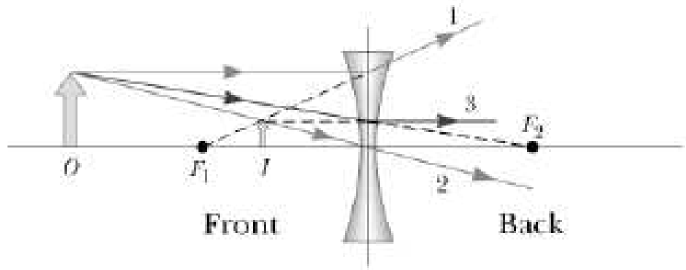

Figure 52: The image for a diverging lens.

rays diverge, so again an observer sees a virtual image appearing to originate from in front of the lens. It is upright, reduced and between the lens and the focal point. For example if $f = -5$cm and $p = 9$cm then

$$q = \frac{pf}{p-f} = -45/14\,\text{cm}. \tag{145}$$

The lateral magnification $M = -(45/14)/9 = 5/14$, and the image is reduced.

6.2.4 Systems of Lenses

It is rather easier to set up a system of more than one lens than it is a system of mirrors. This can be used to obtain enhanced magnification in optical instruments such as microscopes and telescopes, and we will soon consider these specific examples. However, in principle any system is possible. We will only explicitly consider the case of two lenses.

The generalisation from one lens to a system of two with a common principal axis is straightforward. If we have the first lens with focal length f_1 and an object at distance p_1, and the second lens has focal length f_2, the only other parameter to be specified is the separation of the lenses L. The image for the first lens is given by

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1}, \tag{146}$$

and the image from the first lens then becomes the object for the second. The object distance p_2, i.e. the distance of the first image from the second lens is just $p_2 = L - q_1$, where it is possible that $p_2 < 0$. The resultant image from the second lens is then at position q_2, given by

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}. \tag{147}$$

The overall lateral magnification is the ratio of the final image height to initial object height, which is the product of the individual magnifications, since the first image height is identical to the second object height. So overall

$$M = M_1 \times M_2. \tag{148}$$

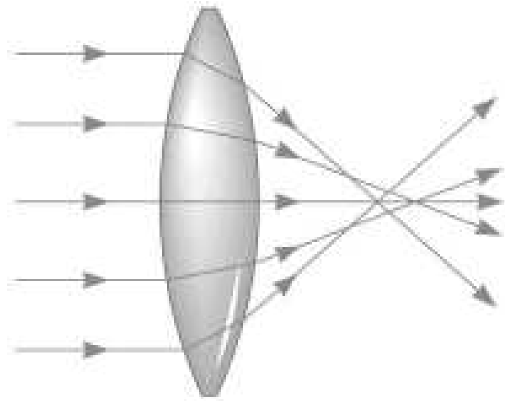

Figure 53: Spherical aberration.

6.2.5 Lens Aberrations

All the above has assumed lenses behave perfectly. This is not always the case, so we quickly review the reasons for this. The departures from ideal behaviour are called aberrations. One type is called **spherical aberration**, and is shown in figure 53. If a lens surface really is part of a section of a perfect sphere then for large sections the focal point is not constant (remember that for the mirror $f = R/2$ was derived in the small angle limit). Hence, light refracted near the edges does not go through the normal focal point. For large lenses this is countered by altering the shape at distances far from the lens centre in the appropriate manner.

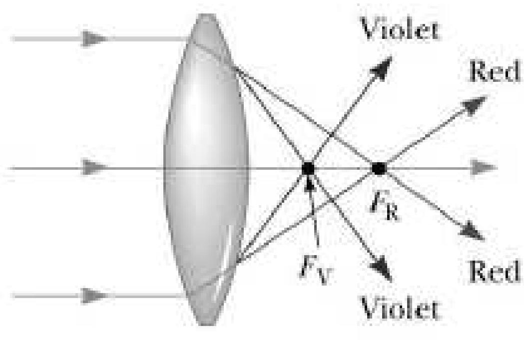

Figure 54: Chromatic aberration.

There is also a feature called **chromatic aberration**. For many materials the refractive index n is slightly higher for lower wavelength light, causing lower wavelengths to be focused slightly nearer to the lens. This is shown in figure 54. It can be minimised by making the complete lens from a combination of materials of different refractive index.

6.3 Optical Instruments

6.3.1 The Eye and Vision Correction

Very simply speaking the eye is a system where a single converging lens focuses light and forms an image on the retina at the back of the eye. The muscle surrounding the eye is able to change its shape to some extent, altering the focal point depending on what is being viewed. However, it does not have unlimited flexibility so the eye has two limiting points. The **near point** is the smallest object distance at which the eye can effectively focus properly. For a normal eye it is about 25cm. Hence, at about 25cm one can see fine detail most precisely, but if the object is closer than this its image will start to blur. The **far point** is the furthest distance at which the lens can focus onto the retina. For a normal eye it is effectively at infinity. However, both the near and far point can change from these values due to imperfections of the eye. There are two standard problems.

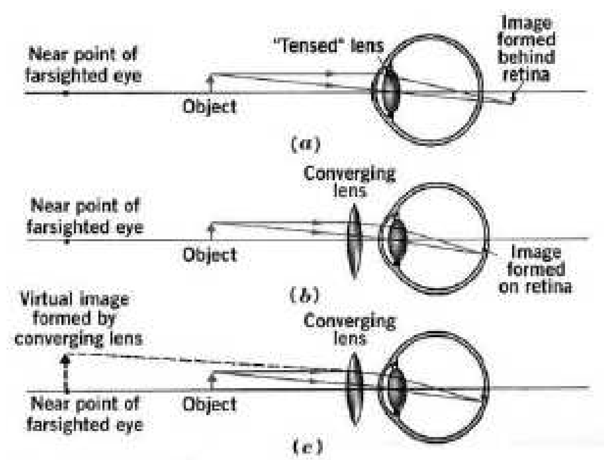

Figure 55: An illustration of farsightedness.

Farsightedness or hyperopia occurs when the light from relatively nearby objects focuses behind the retina. It can be due to the distance between the lens and retina being small, or the curvature of the lens not being able to become great enough. It can be corrected by placing an appropriate converging lens in front of the eye. The object, whose image is now focused onto the retina, then appears to be at the real near point of the eye. This is illustrated in figure 55.

Nearsightedness or myopia occurs when the light from distant objects focuses in front of the retina. It can be due to the distance between the lens and retina being too large, or the cornea – the outer surface of the eye – giving too much of a focusing effect. It can be corrected by placing an appropriate diverging lens in front of the eye. The object, whose image is now focused onto the retina, then appears to be at the real far point of the eye. This is illustrated in figure 56.

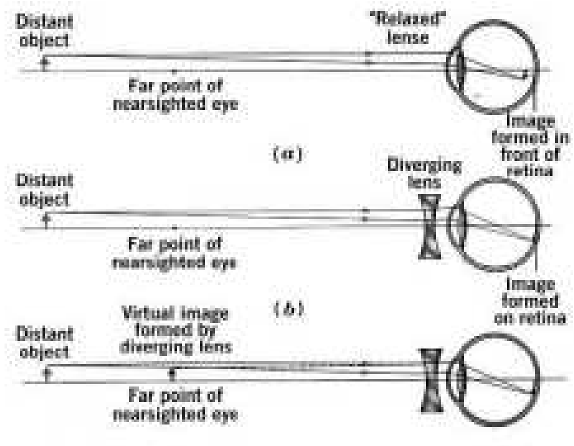

Figure 56: An illustration of nearsightedness.

6.3.2 The Compound Microscope

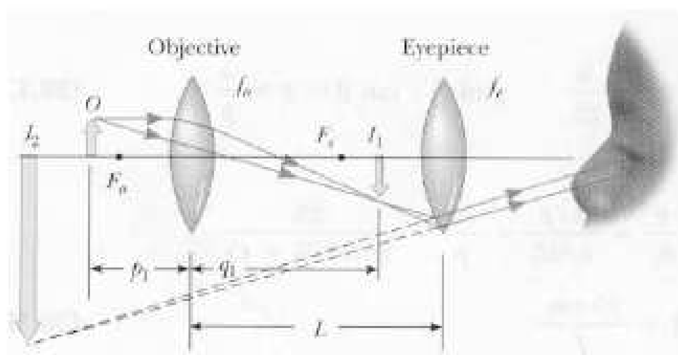

Figure 57: The arrangement for a compound microscope.

A single lens can only provide so much magnification. This can be improved by combining two lenses in a device known as a compound microscope. The arrangement is shown in figure 57. It consists of two focusing lenses. The *objective*, which is nearer to the object, has a very short focal length $f_o < 1$cm, and the *eyepiece*, which is used to view the image, has a focal length f_e of a few cm. The two lenses are separated by a distance $L \gg f_o$ or f_e, and which in practice will be a few tens of cm.

The object is placed slightly outside the focal point of the objective. The first lens forms a real, enlarged, inverted image I_1 just inside the focal point of the eyepiece. The eyepiece then forms the image I_2, which is viewed by the observer, and is virtual, enlarged, and the same orientation as I_1, i.e. inverted compared to the object.

The lateral magnification M_1 of the first image is $-q_1/p_1$. But from the positions described above, $p_1 \approx f_o$ and $q_1 \approx L$. Thus, the lateral magnification by the objective is

$$M_1 \approx -\frac{L}{f_o}. \tag{149}$$

The overall magnification of the microscope is sometimes defined using the angular magnification of the eyepiece, but we will only introduce this concept for the telescope. For the microscope it makes little difference to the answer. Instead we will consider the lateral magnification of the eyepiece. It is natural to take the image distance q_2 to be at the near point of the eye, i.e. $q_2 = -25$cm. The object distance for the lens is the distance of I_1 from the lens, which is given by $p_2 \approx f_e$. Hence, the lateral magnification of the eyepiece is

$$M_2 = -\frac{-25\text{cm}}{f_e} = \frac{25\text{cm}}{f_e}. \tag{150}$$

The overall lateral magnification of the compound microscope is just the product of the magnifications of the objective and of the eyepiece, i.e.

$$M = M_1 \times M_2 \approx -\frac{25L}{f_o f_e}\,\text{cm}, \tag{151}$$

Where the negative sign indicates that the image is inverted. Using, for example, $f_o = 1$cm, $f_e = 5$cm and $L = 50$cm, $M = -250$, which is a typical value.

6.3.3 The Telescope

For the telescope we are viewing objects at very large distances, e.g. planets in the solar system, so the objects are very large, but extremely far away, i.e. the image distance is tending to infinity compared to the focal length. In this case it is not the lateral magnification which is important, but the angular magnification, which is defined by the ratio of the angle subtended by the image, e.g. some region of the moon, compared to the angle subtended by the object viewed without a telescope. There are two different types of telescope, refracting, which uses two lenses, and reflecting, where one mirror is used along with a lens. We will consider both, but the former in more detail.

6.3.4 Refracting Telescope

The basic arrangement is shown in figure 58. As with the compound microscope there is an objective and eyepiece. The separation between the two focusing lenses is very close to the sum of the focal lengths, i.e. $L \approx f_o + f_e$. The focal length of the objective is much larger than that of the eyepiece, i.e. $f_o/f_e \gg 1$. The object is at a very large distance, i.e. $p_1 \to \infty$, so the image distance is given by $q_1 \approx f_o$. The image is real and inverted. The distance between the top and bottom of the image is taken to subtend an angle θ_o from the objective. The image I_1 is formed very near to the focal point of both lenses. The image I_1 subtends an angle θ_0 from the objective, so we have the identity

$$\tan\theta_0 \approx \theta_0 = \frac{-h'}{f_o}. \tag{152}$$

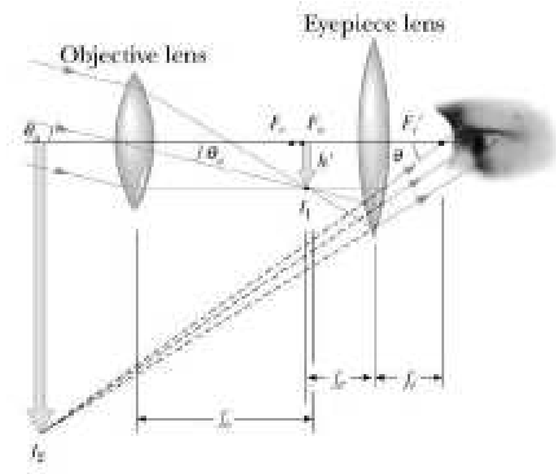

Figure 58: The arrangement for a refracting telescope.

The eyepiece forms an enlarged, virtual image I_2 of the initial object, which is inverted because its object, I_1 is inverted. Since the object distance $p_2 \approx f_e$ the image I_2 has image distance $q_2 \to \infty$. However, the angle θ subtended by the final image at the eye is the same as the angle a ray coming from the tip of I_2 and travelling parallel to the principal axis makes with the axis after it passes through the lens. This ray is focused through the focal point of the eyepiece, so

$$\tan\theta \approx \theta = \frac{h'}{f_e}. \tag{153}$$

The total angular magnification is then expressed as

$$m = \frac{\theta}{\theta_o} = \frac{h'/f_e}{-h'/f_o} = -\frac{f_o}{f_e}, \tag{154}$$

where the negative sign shows that the image is inverted.

In a terrestrial rather than astronomical telescope one generally wants an upright image. This can be achieved by using a diverging lens as eyepiece. This can be understood by noticing in figure 57 that a ray travelling along the line parallel to the principal axis would then diverge out from the focal point left of the eyepiece, rather than towards the focal point right of the eyepiece. It can be shown that again the angular magnification is $m = -f_0/f_e$, which is now a positive number because f_e is negative. Similarly $L = f_o + f_e = f_o - |f_e|$.

Finally we note that whatever the angular magnification the resolution of the telescope is limited by $\theta_R = 1.22\lambda/D$, where D is the objective diameter due to image spreading from diffraction. This leads to reflecting telescopes.

6.3.5 Reflecting Telescope

In order to have the greatest resolution, as well as greatest intensity, the objective of a telescope needs to be as large as possible. It is difficult to make very

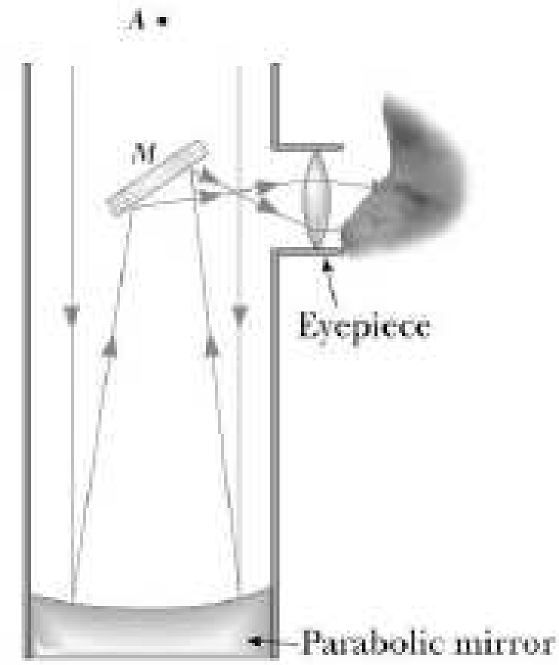

Figure 59: The arrangement for a reflecting telescope.

large lenses since they are very heavy, so can sag causing spatial aberrations, and increased diameter also means thickness which leads to more chromatic aberration due to increased path lengths in the refracting material. Large mirrors can be supported since they do not require light to enter one side and leave the other, and chromatic aberration is not an issue. An example of a reflecting telescope is shown in figure 59, where there is a converging mirror and lens. Reflecting telescopes can be 10m, whereas refracting are limited at about 1m.