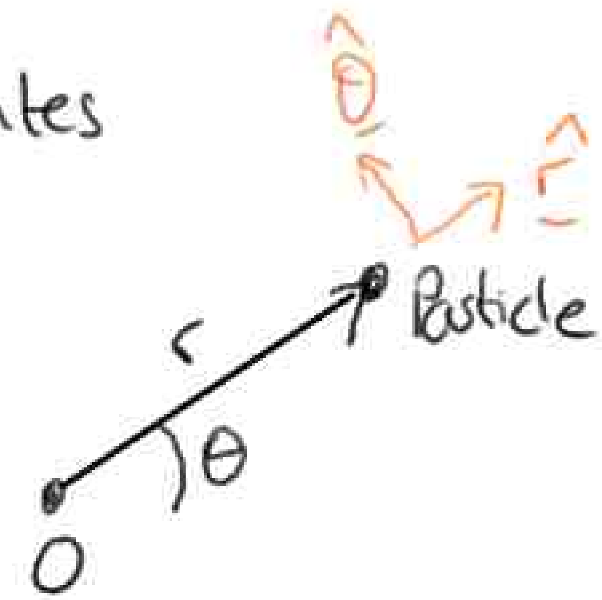
PHAS1247 Session 11

Admin – Problem Sheet 2 scripts for non-PoA students available from me
– Problem Sheet 3 will be handed out on Thursday
– Any outstanding Problem Sheet 1 marks, please come and see me (or ask your academic tutor to mail me)

Review – Velocity and acceleration in plane polar coordinates



$$\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\underline{\theta}}$$

↑ Centripetal acceleration

Menu – Angular momentum
– Central forces

[Lecture questionnaires]

Angular Momentum

Suppose an object has position vector $\underline{r}$ relative to some origin. The angular momentum $\underline{L}$ is defined as

$$\underline{L} = \underline{r} \times \underline{p}$$

where $\underline{p} = m\underline{v}$ is the ordinary (linear) momentum.

We say that the angular momentum is measured "about" the origin of coordinates. It's useful because it plays a similar role in <u>rotational</u> motion to linear momentum in <u>translational</u> motion.

Consider rate of change of angular momentum:

$$\frac{d\underline{L}}{dt} = \frac{d}{dt}(\underline{r} \times m\underline{v})$$

$$= \underbrace{\underline{v} \times m\underline{v}}_{=0} + \underline{r} \times m\underline{a}$$

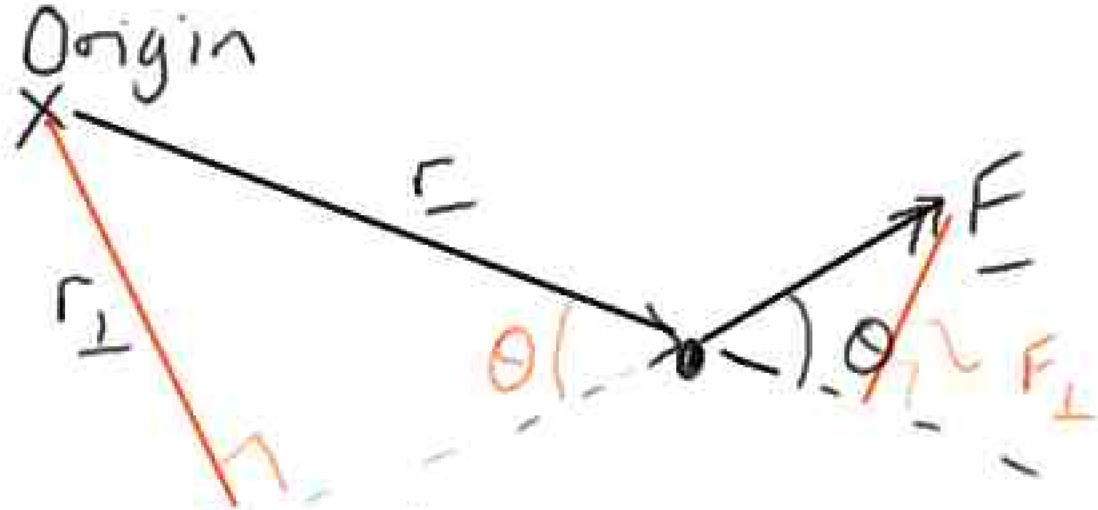
$$= \underline{r} \times \underline{F}$$

Define the <u>torque</u> $\underline{\tau} = \underline{r} \times \underline{F}$

$$\Rightarrow \quad \frac{d\underline{L}}{dt} = \underline{\tau}$$

This has a structure similar to Newton's 2^{nd} law.

$\Rightarrow$ <u>torque</u> plays a role analogous to <u>force</u>.



$\underline{\tau} = \underline{r} \times \underline{F}$

In this case, magnitude of torque is

$$|\underline{\tau}| = |\underline{r}|\,|\underline{F}| \sin\theta$$

$$= |\underline{F}|\, r_\perp \quad \text{or} \quad |\underline{r}|\, F_\perp$$

⇒ the magnitude of the torque is equal to the moment, or couple, of the force about the origin.

If $\underline{F}$ and $\underline{r}$ are parallel (or antiparallel)

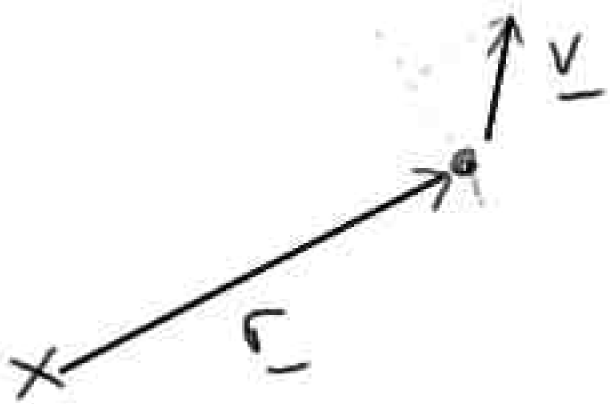
$$\underline{\tau} = 0.$$

Similarly, the angular momentum

$$\underline{L} = \underline{r} \times \underline{p}$$

is the "moment" of the linear momentum about the origin.

For a particle moving in the x-y plane:



$$\underline{v} = \dot{r}\,\hat{\underline{r}} + \underbrace{r\dot{\theta}\,\hat{\underline{\theta}}}_{\text{Component perpendicular to } \underline{r}}$$

Angular momentum lies in z direction, magnitude is

$$L_z = m r \, r\dot{\theta} = m r^2 \dot{\theta}$$

For planar motion, angular momentum is always along z

$\Rightarrow$ just write $L_z \to L$

Central forces

A central force is one that is always directed towards or away from some central point (the 'centre of force'). We will assume further that it only depends on the distance from the centre of force:

$$\underline{F} = F(r)\,\hat{\underline{r}}$$

If we choose the centre of force as our origin, then the torque (about the origin) is zero:

$$\underline{\tau} = \underline{r} \times \underline{F} = 0 \quad \text{for a central force}$$

$\Rightarrow$ angular momentum obeys $\dfrac{d\underline{L}}{dt} = \underline{\tau} = 0$

$\Rightarrow$ angular momentum is conserved.

Examples of central forces:

(a) Electrostatic force between charges q_1, q_2

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{\underline{r}}$$

Always along line separating the two charges; attractive if $q_1 q_2 < 0$ (i.e. unlike charges attract), repulsive if $q_1 q_2 > 0$ (i.e. like charges repel).

(b) Gravitational force between masses m_1, m_2;

$$\underline{F} = -G \frac{m_1 m_2}{r^2} \hat{\underline{r}}$$

Always attractive.

(c) Tension in a string or spring joining particle to the centre of force

Motion of a particle in a plane subject to a central force:

$$F(r)\,\hat{\underline{r}} = m\underline{a}$$

(taking centre of force as origin)

$$\Rightarrow F(r)\,\hat{\underline{r}} = m\left[(\ddot{r} - r\dot{\theta}^2)\,\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\underline{\theta}}\right]$$

Radial component: $F(r) = m(\ddot{r} - r\dot{\theta}^2)$

Angular component: $0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Consider $L = mr^2\dot{\theta}$

$$\Rightarrow \frac{dL}{dt} = m\frac{d}{dt}(r^2\dot{\theta}) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

$\Rightarrow$ angular equation expresses conservation of angular momentum, as expected for a central force.

Example: consider a planet of mass m moving in a circular orbit about a star of mass M (assumed fixed) with angular velocity ω and radius r.

r is constant $\Rightarrow$ radial equation becomes

$$F(r)\,\hat{\underline{r}} = -mr\dot{\theta}^2\,\hat{\underline{r}}$$

$$\Rightarrow \quad \underbrace{-\frac{GMm}{r^2}\,\hat{\underline{r}}}_{\text{Gravitational attraction}} = \underbrace{-mr\omega^2}_{\text{Centripetal force}}\,\hat{\underline{r}}$$

$$\Rightarrow \quad \omega^2 = \frac{GM}{r^3} \quad \text{or, for period } T, \quad \left(\frac{2\pi}{T}\right)^2 = \frac{GM}{r^3}$$

$$\Rightarrow \quad T^2 \propto r^3 \qquad \text{(Kepler's third law)}$$

What if the motion is not circular?

In general, have $\quad m(\ddot{r} - r\dot{\theta}^2) = F(r)$

Remember also $\quad L = mr^2\dot{\theta} = \text{constant}$

$$\Rightarrow \quad \dot{\theta} = \frac{L}{mr^2}$$

$$\Rightarrow \quad r\dot{\theta}^2 = \frac{L^2}{m^2r^3}$$

$\Rightarrow$ equation of motion becomes

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}$$

Central force

Extra "fictitious" force representing rotation, known as centrifugal force,

Like 1D motion, but with a modified force.

Central forces and potential energy

As we have defined them, central forces are conservative.

Can define a potential energy (depending only on distance r from centre of force)

$$V(r) = -\int F(r)\,dr \quad \Rightarrow \quad F(r) = -\frac{dV}{dr}$$

Example: for a planet of mass m at distance r from star of mass M, have

$$V(r) = -\int F(r)\,dr = \int \frac{GMm}{r^2}\,dr = -\frac{GMm}{r} + C$$

Choose $C = 0$ so that $V(r) \to 0$ as $r \to \infty$.

PHAS1247 Session 12

Admin:
- Problem Sheet 3 available today (hand in Thu 26 Nov)
- Remaining non-P&A Problem Sheet 2 scripts for collection
- Any more missing marks for Problem Sheet 1 please see me
- Office Hour today 1300 - 1400
- Non-PoA students please see me at half time to arrange tutorials next week

Review

Angular momentum $\underline{L} = \underline{r} \times \underline{p}$ Torque $\underline{\tau} = \underline{r} \times \underline{F}$ (couple, or moment)

$$\frac{d\underline{L}}{dt} = \underline{\tau}$$

Central force: $\underline{F} = F(r)\,\hat{\underline{r}}$ $\Rightarrow$ angular momentum conserved

Equations of motion in a plane:

Angular: $\frac{d}{dt}(mr^2\dot{\theta}) = \text{constant}$ $\Rightarrow$ L conserved

Radial: $m\ddot{r} = F(r) + \frac{L^2}{mr^3}$

Menu: More about potential energy
"Nearly circular" orbits
Motion in an inverse-square law force
[Institute of Physics presentation]

Last time: potential energy $V(r) = -\int F(r)\,dr$

$$\Rightarrow \quad F(r) = -\frac{dV}{dr}$$

Often convenient to think about a "centrifugal potential" V_c, such that the centrifugal force can be written

$$F_c = \frac{L^2}{mr^3} = -\frac{dV_c}{dr}$$

Integrating gives $V_c = \dfrac{L^2}{2mr^2} + C$

Choose $C=0$ so $V_c \to 0$ as $r \to \infty$.

$\Rightarrow$ radial motion obeys

$$m\ddot{r} = -\frac{d}{dr}\Big(\underbrace{V(r) + V_c(r)}_{\text{Effective potential } V_{eff}(r)}\Big)$$

A circular orbit ($\ddot{r}=0$) corresponds to a stationary point of $V_{eff}(r)$

A <u>stable</u> circular orbit corresponds to a <u>minimum</u> of $V_{eff}(r)$.

Nearly circular motion.

Suppose central force is an attractive power law:

$$F(r) = -Kr^n \qquad (n, K \text{ constants}; K > 0)$$

Radial equation of motion:

$$m\ddot{r} = -Kr^n + \frac{L^2}{mr^3}$$

Suppose there is a circular orbit of radius r_0. Can find its angular velocity (and hence period) by solving

$$0 = -Kr_0^n + \frac{L^2}{mr^3} \Rightarrow Kr_0^n = mr_0\dot{\theta}^2$$

$$\Rightarrow \text{angular velocity } \omega^2 = \dot{\theta}^2 = \frac{Kr_0^{n-1}}{m}$$

$$\Rightarrow \text{period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{Kr_0^{n-1}}}$$

Make a small charge in $r \Rightarrow$ write $r = r_0 + x$ (with x small)

Central force changes to $F(r) \simeq F(r_0) + x \left.\frac{dF}{dr}\right|_{r=r_0} + \dots$

Remember $F(r) = -Kr^n \Rightarrow \frac{dF}{dr} = -nKr^{n-1}$

$$\Rightarrow F(r) = -Kr_0^n - xnKr_0^{n-1}$$

Centrifugal force also changes, to

$$\frac{L^2}{mr_0^3} + x \left.\frac{d}{dr}\left(\frac{L^2}{mr^3}\right)\right|_{r=r_0} = \frac{L^2}{mr_0^3} - \frac{3xL^2}{mr_0^4}$$

Equation of motion is

$$m\ddot{r} = m\ddot{x} = \underbrace{-Kr_0^n + \frac{L^2}{mr_0^3}}_{=0} - x\left[nKr_0^{n-1} + \frac{3L^2}{mr_0^4}\right]$$

Also, $\frac{L^2}{mr_0^4} = Kr_0^{n-1} \Rightarrow m\ddot{x} = -xKr_0^{n-1}(n+3)$

$\Rightarrow$ change x in the radius executes simple harmonic motion.

Can write $\ddot{x} = -\Omega^2 x$

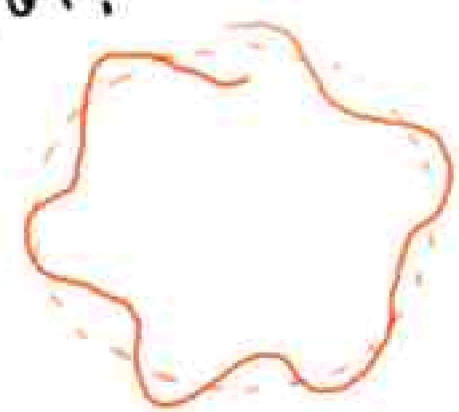
with $\Omega = \sqrt{\dfrac{K r_0^{n-1}(n+3)}{m}}$ $\Rightarrow$ $x(t) = A\cos(\Omega t) + B\sin(\Omega t)$

$\Rightarrow$ radius will oscillate with a period

$$T_{osc} = \frac{2\pi}{\Omega}$$

Compare with angular velocity of original circular motion:

$$\omega = \sqrt{\frac{K r_0^{n-1}}{m}}, \qquad T = \frac{2\pi}{\omega}$$



$\Rightarrow$ find $\Omega = \sqrt{n+3}\,\omega$ $\qquad T_{osc} = \dfrac{T}{\sqrt{n+3}}$

In general, $\sqrt{n+3}$ is irrational $\Rightarrow$ no simple ratio between orbital periods $\Rightarrow$ orbit will not close on itself.

Important special cases:

(i) $n=1 \Rightarrow$ "Hooke's law" force $\quad F(r) = -Kr$

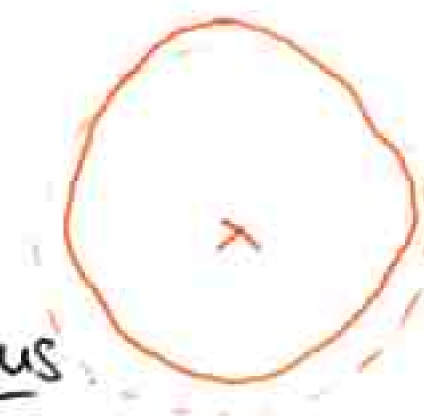
$\Rightarrow \sqrt{n+3} = 2 \quad \Rightarrow \quad T_{osc} = \frac{T}{2}$

$\Rightarrow$ 2 oscillations per rotation

Elliptical orbit, with its centre at the centre of force.

(ii) $n=-2 \Rightarrow$ inverse square law (e.g. gravity)

$\Rightarrow \sqrt{n+3} = 1 \quad \Rightarrow \quad T_{osc} = T$



$\Rightarrow$ 1 oscillation per rotation

Also an elliptical orbit, centre of force at focus

When $n<-3$, circular orbit is unstable

$\Rightarrow$ orbit "pushed away" from circular motion.

Motion in an inverse-square-law force

This is a central force with

$$F(r) = \frac{K}{r^2}$$

$K > 0$: repulsive force

$K < 0$: attractive force

Potential energy $V(r) = -\int F(r)\,dr = \frac{K}{r} + C$

Take $C = 0$ so $V(r) \to 0$ as $r \to \infty$.

Angular momentum will be conserved, radial equation of motion is

$$m\ddot{r} = \frac{K}{r^2} + \frac{L^2}{mr^3}$$

$\boxed{L = mr^2\dot{\theta}}$

Change variable to $u = \frac{1}{r}$, try to find $u(\theta)$

$$\Rightarrow \quad \dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{1}{u^2}\frac{L}{mr^2}\frac{du}{d\theta} = -\frac{L}{m}\frac{du}{d\theta}$$

and $\ddot{r} = \frac{d}{dt}\left(-\frac{L}{m}\frac{du}{d\theta}\right) = -\frac{L}{m}\frac{d^2u}{d\theta^2}\cdot\underbrace{\frac{L}{mr^2}}_{=\dot{\theta}} = -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}$

$\Rightarrow$ radial equation of motion becomes

$$m\ddot{r} = -\frac{L^2u^2}{m}\frac{d^2u}{d\theta^2} = Ku^2 + \frac{L^2}{m}u^3$$

Multiply by $\frac{-m}{L^2u^2}$ $\Rightarrow$ $\frac{d^2u}{d\theta^2} = -\frac{mK}{L^2} - u$

or $\frac{d^2u}{d\theta^2} + u = \frac{-mK}{L^2}$.

Final substitution: $y = u + \frac{mK}{L^2}$ $\Rightarrow$ $\frac{d^2y}{d\theta^2} + y = 0$

$\Rightarrow$ general solution $y = A\cos(\theta-\theta_0)$

$\Rightarrow$ $u = \frac{1}{r} = A\cos(\theta-\theta_0) - \frac{mK}{L^2}$.

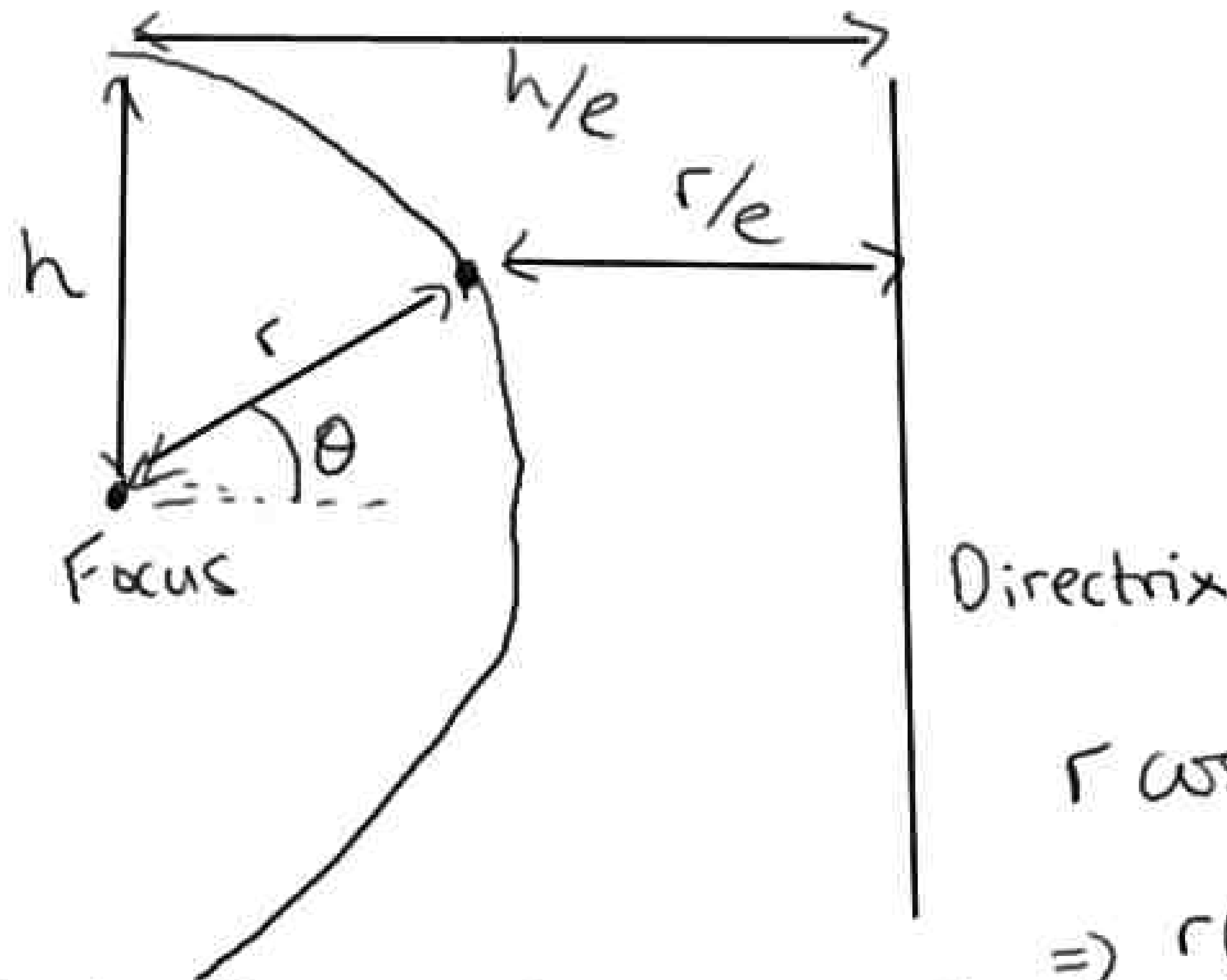
Choose $\theta_0 = 0$ and $A > 0$ $\Rightarrow$ $\theta = 0$ corresponds to closest approach.

We can now write this in the form

$$u = \frac{1}{r} = \frac{1}{h}(1 + e\cos\theta)$$

where h and e are constants.

This describes a curve called a 'conic section'.



Ratio of distances from point to focus and point to directrix is e, the "eccentricity".

$$r\cos\theta + \frac{r}{e} = \frac{h}{e}$$

$$\Rightarrow r(1 + e\cos\theta) = h$$

$$\Rightarrow u = \frac{1}{r} = \frac{1}{h}(1 + e\cos\theta)$$

h is called the "semi-latus rectum"