PHAS1247 Classical Mechanics
Problems for Week 9/10 of Lectures (2016)

1. A *compound pendulum* consists of a spherical mass of mass 0.02 kg and radius 0.5 cm fixed to a wire of negligible mass such that the centre of mass of the system is at the centre of the spherical mass. The centre of mass is 5 cm from a rigid pivot to which the wire is attached and around which the mass rotates. Using the formula for the angular frequency of a compound pendulum *and* the parallel axes theorem, determine the pendulum's period of oscillation if its motion is undamped.

   Now suppose the spherical mass is in fact moving through a fluid while attached to the pendulum rod. According to the laws of fluid dynamics, a sphere of radius a experiences a resistive force
   $$\mathbf{F}_{\mathrm{res}} = -6\pi a\eta\mathbf{v}$$
   when it moves through such a fluid at a velocity $\mathbf{v}$, where η is a constant for any given fluid and is known as the *viscosity*. What value of η would be required to give critical damping in this situation? What will be the pendulum's period of oscillation if the spherical mass is instead immersed in glycerol? (Assume you can neglect the resistive force on the rod.)

   [Acceleration due to gravity $g = 9.81\,\mathrm{ms}^{-2}$; viscosity of glycerol $\eta = 1.2\,\mathrm{N\,s\,m}^{-2}$. You may quote results derived in the lectures without proof.]

2. A mass m moves along a line on a rough table and is attached to a spring of spring constant k. Suppose the coefficients of static and sliding friction between the mass and the table are equal and have the value μ. The frictional force is defined as the normal force multiplied by μ.

   (a) Show that in the absence of friction (i.e. if $\mu = 0$), the particle executes simple harmonic motion with angular frequency $\omega = \sqrt{\frac{k}{m}}$.

   (b) Now we include the effect of friction. Suppose the particle is released *from rest* at time $t = 0$ with a positive displacement x_0 from equilibrium. Describe its initial motion, distinguishing between the cases (i) $kx_0 > \mu mg$ and (ii) $kx_0 \leq \mu mg$.

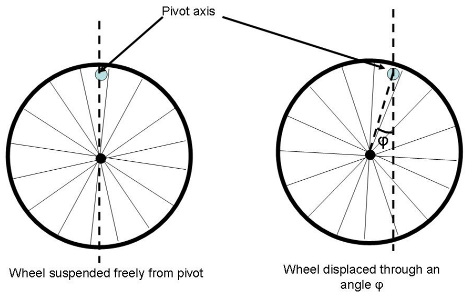
   (c) For case (i), write down the differential equation satisfied by the displacement x of the mass as long as it remains moving. Verify that it is satisfied by a solution of the form
   $$x(t) = A\cos(\omega t) + B\sin(\omega t) + C,$$
   and find the values of the constants A, B and C for the data given.

   (d) Find the time t_1 and position x_1 at which the particle next comes to rest.

   (e) Find the condition for the particle to move again after it has stopped at position x_1. What happens subsequently? Assuming $kx_0 \gg \mu mg$, outline how the amplitude of the oscillation varies with time, and how the frequency of the mass's oscillation is affected by the friction.

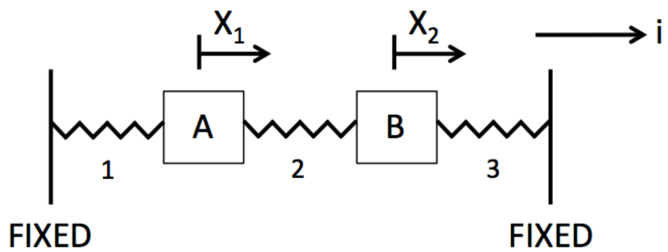
3. Show that the moment of inertia of a rectangular plane of mass M, having length l and width w, about an axis passing through its centre and at right angles to the plane is
   $$I = \frac{M(l^2 + w^2)}{12}.$$

4. A bicycle wheel of mass m and radius a is suspended from just inside its rim so that it is free to rotate about an axis perpendicular to the wheel. Assuming all the mass of the wheel is concentrated uniformly around the rim and that the rim is very thin, find the moment of inertia of the wheel about the axis of rotation. [HINT—use the Parallel Axes Theorem.]



With the wheel pivoted at the point of suspension, its centre is displaced to one side (see diagram). Find the torque about the axis of suspension when the wheel is displaced through an angle ϕ. Hence show that for small displacements the wheel executes simple harmonic motion, and find its period in terms of a and the gravitational acceleration g.

5. Two masses A and B each of mass m are connected on a horizontal frictionless surface by a spring of spring-constant k and to two fixed supports by springs also of spring-constant k as shown below.



Initially the springs are at their equilibrium lengths, B is at rest and A is given a velocity $\mathrm{v}\,\hat{\mathbf{i}}$. Neglecting gravity, write down the equations of motion for A and B if their displacements from equilibrium are x_1 and x_2 respectively.

If $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ show that both y_1 and y_2 obey the equation describing undamped, undriven simple harmonic motion and that the ratio of the frequencies of oscillation of y_1 and y_2 is $\frac{1}{\sqrt{3}}$.

Using the given boundary conditions, show that a solution to the equation for y_1 is:

$$y_1 = v\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right).$$

A mechanical damping system is now attached to A and B that applies a force opposing the motion of A and B. The magnitude of this force is βv_r where v_r is the relative velocity of A and B.

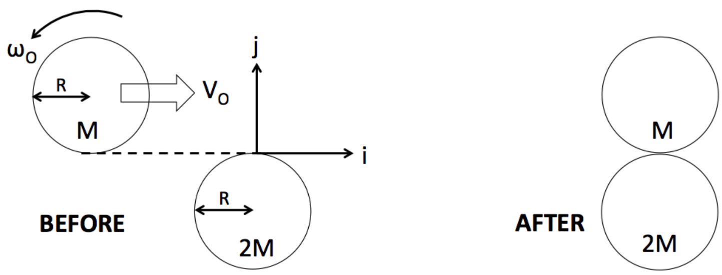
Write down the new equations of motion for A and B in terms of x_1 and x_2 that include this damping force.

By considering these equations of motion in terms of y_1 and y_2 and a trial solution of the form Ae^{qt} for y_2, show that $x_1 = x_2$ as $t \to \infty$ and hence determine the maximum amplitude of A or B as $t \to \infty$.

6. The moment of inertia of a uniform circular disk of radius R and mass M about an axis perpendicular to the disk and passing through its centre is:

$$\mathrm{I} = \frac{1}{2}\mathrm{MR}^2.$$

This disk moves on a frictionless horizontal surface towards a stationary circular disk of mass 2M and radius R. The moving disk has a velocity $\mathrm{v}_0\,\hat{\mathbf{i}}$ and is rotating counter clockwise with an angular speed of ω_o as shown in the figure below which depicts the disks before and after the collision as viewed from above the surface. The moving disk collides with the stationary disk and they instantly stick to each other and subsequently move as a single combined object.



What is the velocity of the combined disks after the collision?

What is the position of the centre of mass of the disks at the time of collision?

Show that the moment of inertia of the final system with respect to an axis through the centre of mass and perpendicular to the disks is: $\frac{25}{6}\mathrm{MR}^2$.

The initial angular momentum of the disks has a contribution from the rotation and the linear momentum with respect to this centre of mass. Using this, determine the angular velocity of the final system and show that it is zero if:

$$\omega_o = \frac{8\mathrm{v_o}}{3\mathrm{R}}.$$