PHAS1202 - Atoms, Stars and The Universe
Problem Solving Tutorial Sheet 2 - 2015

All questions (or variations of them) may appear in the In-Course-Assessment test. Questions are made available approximately one week before the PST. Please attempt the problem sheet in advance of the PST class. A solution sheet will be made available after all PSTs have taken place. **Please print this question sheet and bring it to the PST.**

Objectives:

1. Perform a dimensional analysis on Planck's constant.

2. Gain practise with normalising wave-functions and calculating their properties, including expectation values.

3. Practise using the TISE to calculate energies for an unfamiliar potential.

4. Get practise with the integration and differentiation which arises very often in problems of this kind.

Useful definitions

Planck's constant h is 6.6×10^{-34} Js (2 s.f.).

The time-independent Schrödinger equation (TISE) for a particle in one-dimensional potential $V(x)$ with mass m, energy E with wave-function $\psi(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

1: The dimensions of h

In Physics, studying the dimensions of quantities can be a very important way of discovering relationships (or potential relationships) between them. Here we will consider the dimensions arising in Planck-Einstein's photon energy law and Bohr's atomic model.

1.a) Write down the dimensions of Planck's constant h in terms of the fundamental quantities of mass M, length L and time T, taking the definition of h to be through Planck and Einsteins relationship $E = hf$. Why does $\hbar$ have the same dimensions as h?

1.b) Find the dimensions (in terms of mass M, length L and time T) of angular momentum and show that these are the same as those of Plancks constant h. Hence show that Bohrs quantisation assumption $L = n\hbar$ (where L is the angular momentum and n is an integer) is dimensionally consistent.

2: Calculating properties of wave-functions

Normalise the following wave-functions; sketch the normalised wave-function and the corresponding probability density; use the normalised function to calculate the probability of finding a particle in region $0 \leq x \leq 1$; and compute the expectation value of position $\langle x \rangle$.

2a)

$$\psi_1(x) = \begin{cases} \cos(\pi x) & \text{if } -1.5 \leq x \leq 1.5 \\ 0 & \text{otherwise} \end{cases}$$

2b)

$$\psi_2(x) = \begin{cases} \sin(2\pi x) & \text{if } -2 \leq x \leq 0 \\ \sin(\pi x) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

3: Which functions are valid wave-functions?

To represent a physical particle, wave-functions must be continuous and normalisable. Consider the following functions. Can they represent a wave-function for a physical system? If not, explain why.

3a)

$$f_1(x) = \sin(px/\hbar)$$

3b)

$$f_2(x) = \begin{cases} \cos(2\pi x) - \cos(\pi x) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

3c)

$$f_3(x) = \begin{cases} \cos(\pi x) & x \leq 0 \\ \sin(\pi x) & x > 0 \end{cases}$$

3d)

$$f_f(x) = e^x$$

4: Ground state of a chemical bond

A chemical bond can be approximated by a simple spring between two masses (which represent the atoms), the potential energy of which depends on their separation x, according to:

$$V(x) = \frac{1}{2}kx^2$$

where k is the spring constant. If we solve the TISE for such a potential, the lowest energy state has a wave function of the form:

$$\psi(x) = A\exp[-\alpha^2 x^2/2]$$

where $\alpha = \sqrt{m\omega/\hbar}$ and $\omega = \sqrt{k/m}$.

4a) Use the TISE to compute the energy of wavefunction $\psi(x)$.

4b) Use the symmetry of the wave-function to determine the expectation value for separation x.

4c) Using the method of integration by substitution, and the definite integral given below, calculate the value of constant A which ensures that the wavefunction is normalised.

You may find the following integral useful

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

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