PHAS1247 Classical Mechanics 2016–2017

Full Notes with Additional Worked Examples

1 Introduction & Mathematical Preliminaries

1.1 Why study it?

Classical mechanics is important as it gives the foundation for most of physics. The theory, based on **Newton's laws of motion**, provides essentially an **exact** description of almost all **macroscopic** phenomena.

We study classical mechanics at the start of a programme in Physics and Astronomy because

- First and foremost it gives an essentially complete description of a huge range of phenomena. It requires modification only for
  - *microscopic systems*, e.g. atoms, molecules, nuclei—for which we need to use **quantum mechanics**
  - *particles travelling at speeds close to the speed of light*— for which we need to use **relativistic mechanics**
- It is both a logical foundation for these more exotic theories, and a limit to which they reduce for large particles moving at speeds small compared with the speed of light.
- It is an excellent example of how physics and mathematics are tied together—indeed, Newton developed calculus in order to solve problems in mechanics. So it is the perfect setting to learn to apply maths to physical problems.
- It emphasises the importance of symmetry in physical theories.

The subject is usually divided into

1. **statics** - systems at rest and in *equilibrium*,
2. **kinematics** - systems in motion, often accelerating. Concerned here with general relationships, e.g. $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, (Newton's second law, without specifying the details of the force.)
3. **dynamics** - details of the force law are specified, e.g. gravitational force, force due to a stretched spring.

1.2 Problem solving

Classical mechanics, like all of physics, is also a great way to teach students how to solve real-life problems using mathematical tools. The best way to learn how to solve problems is of course working on a lot of them, but also reading them and their solution will help; the textbook has several problems with worked out solutions. There are several strategies to deal with a new problem. The most common are:

- Draw a diagram
- write down what you know, and where you are aiming at
- solve it symbolically first
- always check units and dimensions
- check limits and special cases
- if computing numerical answers, check orders of magnitude

Often solving a problem requires writing the relation between physical expressions like the equation of motion, the speed and the acceleration as a function of time, and solve a differential equation. The general solution of this equation will depend on parameters that have to be fixed using initial or boundary conditions.

1.3 Mathematical preliminaries

1.4 Units and dimensions

A physical quantity consists of a numerical *value* multiplied by a *dimension.* When we use algebraic symbols to refer to the quantities they represent *both* parts of the expression.

Example 1 *A particle is moving with speed*

$$\begin{array}{lll} v & = 5 & \mathrm{m\,s^{-1}} \\ & \textit{numerical part}\times & \textit{dimensional part.} \end{array}$$

The dimensional part is expressed in terms of **units** (in this case $\mathrm{m\,s^{-1}}$). We could change the units, while keeping the same fundamental *dimensions*—this means that a speed always has to correspond to travelling such-and-such a distance in such-and-such a time.

Example 2 *The speed above can be re-written*

$$v = 6.21 \times 10^9 \, \textit{inch}\, \textit{year}^{-1}.$$

(but in the scientific literature and in this course only metric units are allowed!). We refer to the way in which the dimensional part of a quantity relates to those of other quantities as the **dimensions** of that quantity: in this case we say speed has the **dimensions** of distance divided by time. We denote dimensions by square brackets, so in this case we would write

$$[v] = [L]\,[T]^{-1}.$$

Note this is a statement only about the *dimensions* of v; it says nothing about its magnitude.

The units (and magnitude) of a dimensional quantity will change according to the system of units adopted, but the dimensions will remain the same.

In mechanics these units are always combinations of those for three fundamental variables: conventionally these are chosen to be mass $[M]$, length $[L]$ and time $[T]$. The most widely used set of units for these variables is the Système International (S.I.). The base units of the S.I. system are

variable	**unit name**	**abbreviation**
mass $[M]$	kilogram	kg
length $[L]$	metre	m
time $[T]$	second	s

From these base units we can obtain derived units for other variables, e.g. speed (as we saw above).

Example 3 *For force = mass × acceleration the dimensions are* $[F] = [m]\,[L]\,[T]^{-2}$ *and the unit is* $\mathrm{kgms^{-2}}$ *= Newton* (N).

Units have dimensions and **all** equations in physics must be **dimensionally homogeneous**; i.e. both sides and each term of the equation must have the same dimensions (put another way—any two quantities that are added together or equated to one another must have the same dimensions).

A corollary is that dimensionless variables (pure numbers) can depend only on other dimensionless variables.

These requirements often enables one to determine

1. distance travelled in time t at constant speed v is $s = vt$. The dimension are
$$[L] = [L]\,[T]^{-1}\ [T] = [L]\,. \tag{1}$$

2. distance travelled in time t under a constant acceleration a from an initial speed u is
$$s = ut + \frac{1}{2}at^2. \tag{2}$$
The dimensions are
$$\begin{aligned} [L] &= [L]\,[T]^{-1}\ [T] + [L]\,[T]^{-2}\ [T]^2 \quad (3)\\ [L] &= [L] + [L]\,. \quad (4) \end{aligned}$$

Dimensional analysis of equations provides a very useful check on the correctness of algebraic expressions. However it does not give information about dimensionless constants, such as the $\frac{1}{2}$ in equation (2) above.

Failure to make correct unit conversions is a frequent source of mistakes—not just by physics students. It caused the loss of the Mars Climate Orbiter in September 1999 when a data file written in Imperial units (pound-force) was assumed to be in metric units (newtons) when read by other programs.

You can guarantee to perform unit conversions correctly by systematically manipulating the expression:

$$\begin{aligned} v &= 6.21 \times 10^9 \text{inch year}^{-1} \\ &= 6.21 \times 10^9 \frac{\text{inch}}{\text{m}} \left(\frac{\text{year}}{\text{s}}\right)^{-1} \text{m s}^{-1} \\ &= 6.21 \times 10^9 \times 0.0254 \times (365 \times 24 \times 3600)^{-1} \text{ m s}^{-1} \\ &= 5.00 \text{ m s}^{-1}. \end{aligned}$$

1.5 Vectors

There are two main types of variables in mechanics.

Scalar – has only magnitude, e.g. mass, energy, speed.

Vector – has magnitude and direction, e.g. position, velocity, acceleration, force.

A vector may be represented graphically by a directed line segment. The length of the line represents the magnitude of the vector, the direction of the line shows the direction of the vector; the line represents the displacement through space necessary to pass from its starting point to its end point.

In this course printed text vectors are often written in **bold** type (though you will also see arrows used thus $\vec{v}$); since boldface is hard to see in in hand-written text we will use underlines instead.

In Cartesian form we can write a vector in terms of multiples of three **Cartesian unit vectors**: vectors $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ of unit length oriented along the $\{x, y, z\}$ axes. (I use hats for unit vectors throughout the course.) We would write a vector $\mathbf{r}$ with components x, y and z as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{5}$$

We can also represent the same vector in the form of a column vector:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{6}$$

Example 4 *Consider Newton's second law, (coming later):*

$$\mathbf{F} = m\mathbf{a}. \tag{7}$$

This is a vector equation: not only the magnitudes on both sides of the equation equal, but also their directions. Equivalently, every component has to be equal on both sides of the equation: if

$$\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} \tag{8}$$

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}, \tag{9}$$

then

$$F_x = ma_x \tag{10}$$

$$F_y = ma_y \tag{11}$$

$$F_z = ma_z. \tag{12}$$

We will generally use subscripts like this to indicate the components of a vector; the main exception is the position vector $\mathbf{r}$, which we will write as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{13}$$

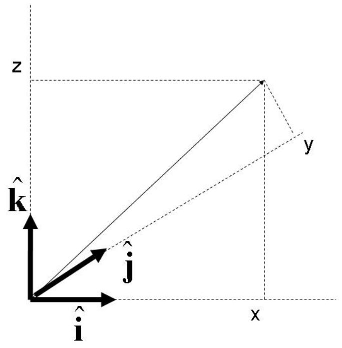

Figure 1: Representation of a vector $\mathbf{r}$ in terms of the Cartesian basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

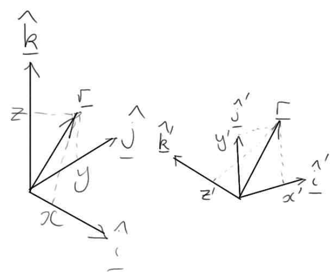

Figure 2: Representation of the same vector $\mathbf{r}$ in terms of two different basis sets.

1.5.1 Rotational transformations on vectors

We could choose to express the same vector in terms of two different sets of basis vectors. The most important example would be a rotation of the axes, with a corresponding change to a new set of basis vectors $\{\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'\}$. Then we can express the same vector in terms either of the two different basis sets (Figure 2):

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}'. \tag{14}$$

The *components* of the vector change depending on the basis vectors; on the other hand, a scalar quantity will be unaffected by the change.

Example 5 *Consider the vector*

$$\mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}.$$

We perform a rotation of our basis vectors through 45° *about the z-axis so that the new basis vectors (still orthonormal) are*

$$\hat{\mathbf{i}}' = \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}}); \quad \hat{\mathbf{j}}' = \frac{1}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}); \quad \hat{\mathbf{k}}' = \hat{\mathbf{k}}.$$

Then the same vector $\mathbf{r}$ *can be expressed in terms of the new set as*

$$\mathbf{r} = \sqrt{2}\hat{\mathbf{i}}' - 3\hat{\mathbf{k}}'.$$

If a vector equation $\mathbf{a} = \mathbf{b}$ is true in one basis set (i.e. the vectors on the LHS and RHS are equal when expressed in one basis) they will also be equal in every other basis:

$$\left.\begin{array}{l} a_x = b_x \\ a_y = b_y \\ a_z = b_z \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} a'_x = b'_x \\ a'_y = b'_y \\ a'_z = b'_z \end{array}\right. \tag{15}$$

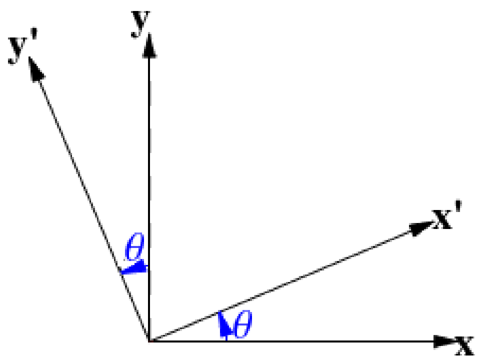

Figure 3: Rotation of a vector basis by an angle θ around the origin.

In two dimensions, the rotation of an orthonormal basis around the origin by an angle θ, shown in figure 5 has an easy expression. From trigonometry, we see that

$$\mathbf{x}' = \mathbf{x}\cos\theta + \mathbf{y}\sin\theta$$

$$\mathbf{y}' = -\mathbf{x}\sin\theta + \mathbf{y}\cos\theta$$

In other words,

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

(refer to next chapter about matrix algebra if not familiar with the notation). Notice that transforming the coordinates of a vector requires performing the inverse rotation. To express vector $\mathbf{r} = r_x\mathbf{x} + r_y\mathbf{y}$ in terms of the new basis, we need to express the initial basis in terms of the new one using the opposite rotation:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix}$$

so the initial vector becomes

$$\mathbf{r} = r_x(\cos\theta\mathbf{x}' - \sin\theta\mathbf{y}') + r_y(\sin\theta\mathbf{x}' + \cos\theta\ \mathbf{y}') = r'_x\mathbf{x}' + r'_y\mathbf{y}'$$

So

$$\begin{pmatrix} r'_x \\ r'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

1.5.2 Scalar products

The scalar ('dot') product of two vectors is a scalar equal to the product of the magnitudes of the two vectors times the cosine of the angle between them.

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta. \tag{16}$$

You can think of it as 'magnitude of **a** times component of **b** in direction of **a**'.

Note that it can be positive (for parallel vectors), negative (for anti-parallel vectors) or zero (for perpendicular vectors). Note also that the cosine function is even ($\cos(-\theta) = \cos\theta$) so the dot product is *commutative*:

$$\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}.$$

Applied to the Cartesian unit vectors the definition gives

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1; \quad \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0.$$

We sum up this set of properties by saying that the set of vectors is **orthonormal** (i.e. of unit length or 'normalised to 1', and orthogonal to one another).

This means we can multiply out the brackets to express the dot product in terms of the vectors' components (since the operation is 'distributive'):

$$\mathbf{a} \cdot \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \cdot (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) = a_xb_x + a_yb_y + a_zb_z. \tag{17}$$

The *magnitude* of a vector is given by the square root of its dot product with itself; this is obvious from taking $\mathbf{a} = \mathbf{b}$ either in the original definition (16) or in writing out (17) and using Pythagoras' theorem:

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_x^2 + a_y^2 + a_z^2. \tag{18}$$

Example 6 *The speed of a particle is the magnitude of its velocity vector:*

$$v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

We shall use the simple, non-underlined letter as a shorthand for the magnitude of the corresponding vector: for example, the magnitude of the acceleration vector is

$$a = |\mathbf{a}|$$

and the speed (magnitude of the velocity vector) is

$$v = |\mathbf{v}|.$$

Since the dot product is a scalar, its value is independent of the basis in which we write the vectors. In particular the formulae (17) and (18) are true for the components of the vector(s) with respect to any orthonormal basis.

Example 7 *Consider the vector* $\mathbf{r}$ *that we previously expressed in two different bases:*

$$\mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}} = \sqrt{2}\hat{\mathbf{i}}' - 3\hat{\mathbf{k}}'.$$

Its magnitude is the same computed in either basis: in the first,

$$|\mathbf{r}|^2 = 1^2 + 1^2 + (-3)^2 = 11,$$

while in the second

$$|\mathbf{r}|^2 = (\sqrt{2})^2 + (-3)^2 = 11.$$

Example 8 *To find the* $\hat{\mathbf{i}}'$ *and* $\mathbf{j}'$ *components of the vector* $\mathbf{r}$*, we can use use the dot product:*

$$\mathbf{r} \cdot \hat{\mathbf{i}}' = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) \cdot \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \frac{1 \times 1 + 1 \times 1}{\sqrt{2}} = \sqrt{2},$$

whereas

$$\mathbf{r} \cdot \hat{\mathbf{j}}' = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) \cdot \frac{1}{\sqrt{2}}(\hat{\mathbf{j}} - \hat{\mathbf{i}}) = \frac{1 \times 1 - 1 \times 1}{\sqrt{2}} = 0,$$

consistent with what we wrote earlier.

Thus the scalar product tells us about the product of one vector with the component of another in the same direction. We will find it is particularly useful in the discussion of *work* and *energy* in mechanics.

1.5.3 Vector (cross) products

A second useful product is the vector ('cross') product. The cross product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is defined as a vector of magnitude

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\sin\theta|, \tag{19}$$

whose direction is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ and in the direction you would obtain by rotating a right-handed screw from $\mathbf{a}$ to $\mathbf{b}$.

You will also see the notation $\mathbf{a} \wedge \mathbf{b}$ used in some books, but we will avoid it in this course because it has a different meaning in more advanced geometry. We will also continue to use $\times$ sometimes to multiply ordinary numbers—the type of product intended will always be clear from the context.

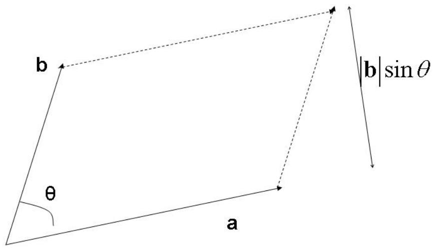

Figure 4: Area of the parallelogram enclosed between two vectors.

Example 9 *The angle between the unit vectors* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *is* $90°$ *(or* $\pi/2$ *radians). Therefore the magnitude of their cross product is*

$$|\hat{\mathbf{i}} \times \hat{\mathbf{j}}| = 1 \times 1 \times \sin(\pi/2) = 1,$$

and its direction is along the positive z-axis (right-hand screw rule). Hence

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}.$$

The cross product is anti-symmetric:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}. \tag{20}$$

This follows from the definition of the screw rule for the direction—reversing the order of the vectors means we have to reverse the screw. You can also think of it as being consistent with the fact that the sine function is odd: $\sin(-\theta) = -\sin\theta$.

Hence the cross product of any vector with itself is zero:

$$\mathbf{a} \times \mathbf{a} = -\mathbf{a} \times \mathbf{a} \quad \Rightarrow \quad \mathbf{a} \times \mathbf{a} = 0. \tag{21}$$

This can also be seen by noticing that the 'angle between the two vectors' is zero in this case, so $\sin\theta = 0$.

Example 10 *Find an expression for the area of the parallelogram formed by the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in terms of the cross product.*

The area is the base of the parallelogram times its perpendicular height. Take the base to be along $\boldsymbol{a}$ *so its length is* $|\mathbf{a}|$*; the height is then* $|\mathbf{b}||\sin\theta|$ *(see Figure 4). The area is therefore*

$$|\mathbf{a}||\mathbf{b}||\sin\theta| = |\mathbf{a} \times \mathbf{b}|.$$

Can also think of the magnitude of $\mathbf{a} \times \mathbf{b}$ as $|\mathbf{b}|$ multiplied by the perpendicular distance of its line of action from the origin when it is displaced through $\mathbf{a}$. This reminds us of the definition of a **moment**.

The cross product can be evaluated in terms of the components by using the distributive law (as we did for the dot product), our result for $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ (see above), and the fact that the cross product of each unit vector with itself gives zero ($\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$ etc):

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\
&= a_x b_y \hat{\mathbf{i}} \times \hat{\mathbf{j}} + a_x b_z \hat{\mathbf{i}} \times \hat{\mathbf{k}} + a_y b_x \hat{\mathbf{j}} \times \hat{\mathbf{i}} + a_y b_z \hat{\mathbf{j}} \times \hat{\mathbf{k}} + a_z b_x \hat{\mathbf{k}} \times \hat{\mathbf{i}} + a_z b_y \hat{\mathbf{k}} \times \hat{\mathbf{j}} \\
&= (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}}.
\end{aligned} \tag{22}$$

or

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}. \tag{23}$$

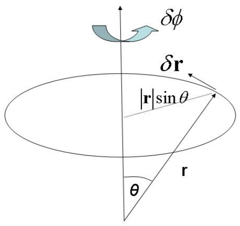

Figure 5: Effect of a small rotation on an arbitrary vector.

Like the dot product, the cross product is defined independent of any particular basis set. Similarly the formulae (22) and (23) hold for any 'right-handed' set of orthonormal basis vectors—right-handed in this case means that the cross product of the first vector with the second gives the third.

Example 11 *Find the Cartesian components of the cross product of the vectors*

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \times 2 - 2 \times 1 \\ 2 \times 3 - 1 \times 2 \\ 1 \times 1 - 2 \times 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}.$$

Many people find it helpful to remember equations (22) and (23) as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Interchanging the order of the two vectors swaps the order of two rows in the determinant and therefore changes the sign, as we would expect.

Example 12 *Suppose a body's position vector is* $\mathbf{r}$*. The body is rotated through a small angle* $\delta\phi$ *(all angles will be in radians!) around an axis* $\hat{\mathbf{n}}$ *passing through the origin of the coordinates, such that the direction of* $\hat{\mathbf{n}}$ *is given by a right-hand grip rule (i.e. it's in the direction that a right-handed screw would move when rotated). What is the change* $\delta\mathbf{r}$ *in the position vector?*

Let θ *be the angle between the vector and the rotation axis. The length of* $\delta\mathbf{r}$ *will be* $\delta\phi$ *multiplied by the object's perpendicular distance from the rotation axis. The perpendicular distance is* $|\mathbf{r}|\sin\theta$*, in other words*

$$|\delta\mathbf{r}| = \delta\phi|\mathbf{r}|\sin\theta. \tag{24}$$

Since $\delta\phi$ *is small, the direction of* $\delta\mathbf{r}$ *will be perpendicular to* $\mathbf{r}$*, and also in a plane perpendicular to the rotation axis* $\hat{\mathbf{n}}$*. We can therefore write the vector* $\delta\mathbf{r}$ *as*

$$\delta\mathbf{r} = \delta\phi\hat{\mathbf{n}} \times \mathbf{r}. \tag{25}$$

Let us define a vector $\delta\theta = \delta\phi\hat{\mathbf{n}}$*, where the direction is again defined so that the rotation is given by a right-hand grip rule. Then the change in position can also be written*

$$\delta\mathbf{r} = \delta\theta \times \mathbf{r}.$$

The cross product therefore expresses the turning effect of one vector on another; it is particularly useful for situations involving rotational motion.

1.6 Matrices

Matrices can be seen as vectors or vectors, or as two-dimensional representations ofsets of numbers. A matrix is represented as a capital letter, with its elements being lowercase letters with running indexes. For instance, a matrix A with $(n \times m)$ elements will be:

$$A = \begin{pmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & a_{mn} \end{pmatrix}$$

A transposed matrix A^T will have raws and columns swapped wrt the original one. If the elements of matrix A are indicated as a_{ij}, those of A^T will be a_{ji}.

Matrices are multiplied by raws and columns, It is only possible to multiply two matrices if the number of columns in the first matrix matches the numnber of rows of the second one. The product between a matrix A with dimensions $(n \times m)$ and elements a_{ni} and a matrix B with dimensions $(p \times n)$ and elements b_{jn} will be another matrix with dimensions $(p \times m)$, whose values are:

$$c_{ij} = \Sigma_n a_{ni} b_{jn}$$

Notice that in this convention, the first letter in parenthesis represents the number of columns in the matrix and the second letter the number of rows; students are allowed to use the opposite convention as long as it is indicated. Commonly used special cases of matrices are square matrices (with the same number of rows and columns), and vectors (whose number of rows or of columns is one).

A diagonal matrix is a square matrix whose only non-zero elements are along the diagonal, namely have the same row and column number. The identity matrix is a diagonal matrix whose diagonal elements are all equal to 1. The inverse of a matrix A^{-1} is a matrix such tath the product $A^{-1}A$ is equal to the identity matrix.

1.7 Calculus

Mechanics is to with the rate of change of things: we therefore need the concepts of calculus to describe them.

1.7.1 Differentiation

The **derivative** with respect to time of some quantity y is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}. \tag{26}$$

It corresponds to the *rate of change* of the quantity f with time; equivalently it is the slope of the graph of f against t at time t.

Example 13 *Consider a particle moving in one dimension with a displacement $x(t)$. The velocity of the particle is the rate of change of its displacement:*

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Its acceleration is the rate of change of its velocity:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t},$$

and hence the second derivative of its displacement:

$$a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

We will sometimes use Newton's 'dot' notation for time derivatives:

$$v = \dot{x}; \quad a = \dot{v} = \ddot{x}.$$

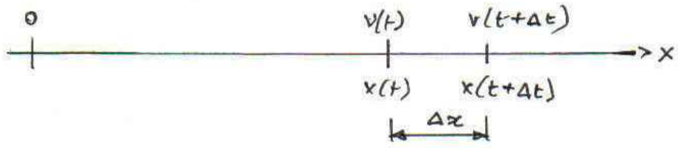

Figure 6: One-dimensional motion along the x-axis.

1.7.2 Integration

Integration corresponds to finding the area under the graph of the function $f(t)$. The **definite integral** from a to b is the area under the curve between $t = a$ and $t = b$. For example, we can write

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{\delta t \to 0} \sum_i f(t_i)\delta t, \tag{27}$$

where the function f is evaluated at a set of equally spaced points t_i separated by intervals δt.

The **indefinite integral** (or primitive) is the function F such that

$$\int_a^b f(t)\,\mathrm{d}t = [F(t)]_{t=a}^{t=b} = F(b) - F(a). \tag{28}$$

We write

$$\int f(t)\,\mathrm{d}t = F(t) + C$$

(where C is the arbitrary constant of integration).

The Fundamental Theorem of Calculus asserts that differentiation and integration are inverse operations, so that

$$\frac{\mathrm{d}F}{\mathrm{d}t} = f(t).$$

Example 14 *The distance moved in one dimension between times t_1 and t_2 by a particle whose velocity is $v(t)$ is*

$$s = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t.$$

Example 15 *Suppose instead we know the velocity of the particle as a function of position x, and we know that the particle always moves to the right (i.e. the velocity is positive). We want to know the time taken to travel between two given points, $x = a$ and $x = b$ (with $a < b$). We can find this by writing the total time taken T as*

$$T = \sum_i \delta t_i,$$

where each time interval δt_i is the time taken to move a small distance δx_i. Then we know

$$\delta t_i = \frac{\delta x_i}{v},$$

so

$$T = \sum_i \frac{\delta x_i}{v} = \int_a^b \frac{\mathrm{d}x}{v(x)}$$

We will not go over the mechanics of how to differentiate and integrate in PHAS1247; this will be covered in course PHAS1245.

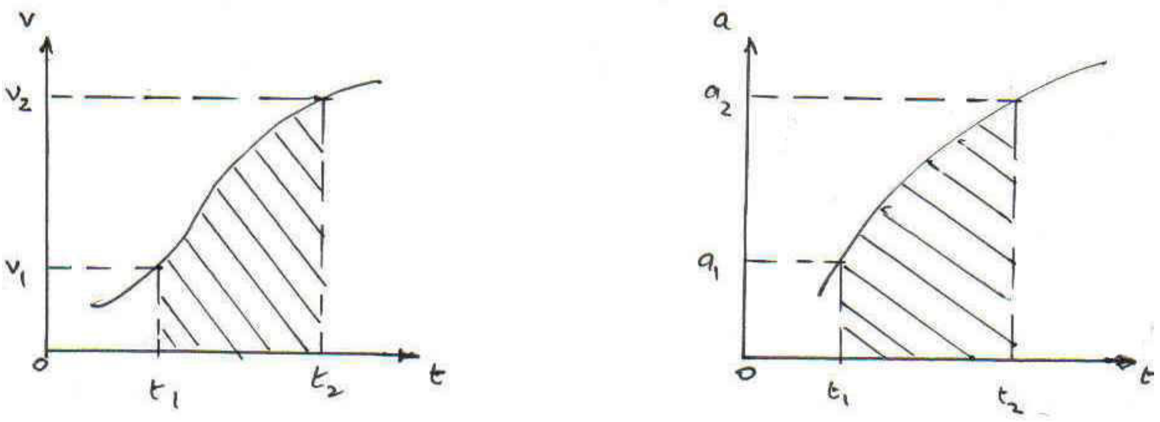

Figure 7: Distance and velocity for a path of a particle moving in one dimension with constant acceleration.

Example 16 *Consider a particle of mass, m, moving along the positive x-axis as in Fig. 6.*

The velocity is positive for motion in sense of x increasing and negative for x decreasing.

In time $\mathrm{d}t$ distance travelled by particle is $\mathrm{d}x = v\mathrm{d}t$. In the finite time interval between t_1, when position of particle is x_1, and time t_2 when position is x_2, distance travelled is

$$s = (x_2 - x_1) = \int_{t_1}^{t_2} v\,\mathrm{d}t. \tag{29}$$

This is represented by the shaded area in fig 7(a). We would need to know how v varies with t in order to calculate s.

Similarly for acceleration, a, the change in velocity in time interval between time t_1 and t_2 is

$$(v_2 - v_1) = \int_{t_1}^{t_2} a\,\mathrm{d}t, \tag{30}$$

and is represented by the shaded area in fig 7(b). As before we need to know how a varies as a function of time in order to calculate v.

1.7.3 Time rate of change of vectors

The same principles can be applied to vectors: we can define the derivative of a vector $\mathbf{v}$ with respect to time by

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t} = \lim_{\delta t\to 0}\frac{\delta\mathbf{v}}{\delta t}. \tag{31}$$

In terms of components, since the basis vectors are time-independent, we have

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \lim_{\delta t\to 0}\left[\frac{v_x(t+\delta t) - v_x(t)}{\delta t}\hat{\mathbf{i}} + \frac{v_y(t+\delta t) - v_y(t)}{\delta t}\hat{\mathbf{j}} + \frac{v_z(t+\delta t) - v_z(t)}{\delta t}\hat{\mathbf{k}}\right] \\ &= \frac{\mathrm{d}v_x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\hat{\mathbf{k}},\end{aligned} \tag{32}$$

i.e. the Cartesian components of the derivative are simply the derivatives of the Cartesian components of the original vector. Note we could also get the same result by applying the normal rules of differentiation, taking the basic vectors as constants.

Newton's second law, $\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$, involves the time rate of change of the velocity vector, $\mathbf{v}$. Consider

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}. \tag{33}$$

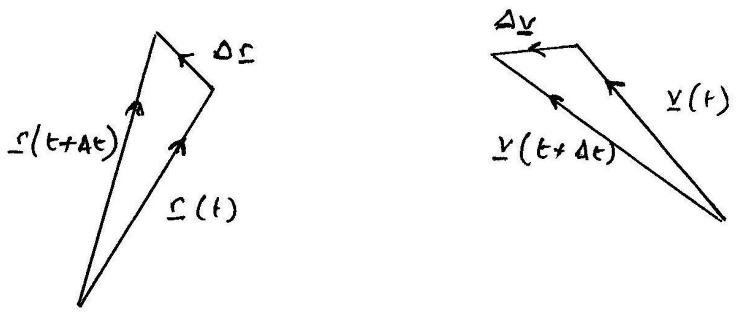

Figure 8: Change of a particle's position $\delta\mathbf{r}$ and of velocity $\delta\mathbf{r}$ in time interval δt.

In Cartesian coordinates

$$\begin{aligned}
\mathbf{r} &= x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}, &(34)\\
\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\widehat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\widehat{\mathbf{k}}, &(35)\\
&= v_x\widehat{\mathbf{i}} + v_y\widehat{\mathbf{j}} + v_z\widehat{\mathbf{k}}. &(36)
\end{aligned}$$

For acceleration, replace $\mathbf{r}$ by $\mathbf{v}$ in the above expressions,

$$\mathbf{a} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{37}$$

In Cartesian coordinates

$$\begin{aligned}
\mathbf{v} &= v_x\widehat{\mathbf{i}} + v_y\widehat{\mathbf{j}} + v_z\widehat{\mathbf{k}}, &(38)\\
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}v_x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\widehat{\mathbf{j}} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\widehat{\mathbf{k}}, &(39)\\
&= \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\widehat{\mathbf{i}} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\widehat{\mathbf{j}} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\widehat{\mathbf{k}}. &(40)
\end{aligned}$$

1.7.4 Integration with vectors

The fundamental theorem of calculus also works for vectors: for example, if the velocity vector is the rate of change of the position vector with time (a scalar),

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

then the change in position from an initial time t_1 to a final time t_2 is the integral of the velocity vector with respect to time between t_1 and t_2:

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = \lim_{\delta t\to 0}\sum_i \mathbf{v}(t_i)\delta t = \int_{t_1}^{t_2}\mathbf{v}(t)\,\mathrm{d}t. \tag{41}$$

We will also sometimes need a different kind of integral involving vectors (particularly when we look at the work done by a force): this involves integrating 'along a path' and taking at each point the dot product of the vector (say $\mathbf{v}$) to be integrated with the small displacement vector $\delta\mathbf{r}$ corresponding to a motion along the path. The result is written

$$\int_{\text{path}}\mathbf{v}\cdot\mathrm{d}\mathbf{r} = \sum_{\text{path}}\mathbf{v}\cdot\delta\mathbf{r}. \tag{42}$$

In practice these integrals can be calculated by expanding out the dot product and turning them into ordinary integrals.

Example 17 *Consider the integral of the variable vector* $\mathbf{v} = x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}$ *along a straight-line path from* $(x, y, z) = (0, 0, 0)$ *to* $(x, y, z) = (1, 1, 1)$.

The integral can be expanded as

$$\int_{\text{path}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_{\text{path}} [x^2 \,\mathrm{d}x + xy \,\mathrm{d}y + y^2 \,\mathrm{d}z]. \tag{43}$$

Now we need to put in the information about the path. Everywhere along this particular path the values of x, y *and* z *are equal, as are the changes* δx, δy *and* δz *when moving along any path segment. We can choose any of the variables to work in terms of—let's choose* x, *for the sake of argument. Then we can write the integral as*

$$\int_{\text{path}} [x^2 \,\mathrm{d}x + xy \,\mathrm{d}y + y^2 \,\mathrm{d}z] = 3 \int_{x=0}^{1} x^2 \,\mathrm{d}x = 1. \tag{44}$$

(We would have got the same result by working in terms of any of the other variables.)

1.7.5 Differential equations

Are relations between a function and its derivatives (usually time derivatives); the solution is the function satisfying the relation. Usually the solution depends on some free parameters, that have to be determined by boundary conditions (for instance, the position and velocity of the system at time $t = 0$). In this course we will only deal with linear differential equations, where the function is the linear combination of its derivatives. Since the only function tat maintain its form after derivation, so can be a linear combination of its dervatives, is the exponential, the solution of linear differential equations will be an exponential, with complex arguments. The simplest differential equation is a relation between a function and its first derivative:

$$\frac{dx(t)}{dt} = -bx(t)$$

A formally incorrect (but easy to remember) way to solve this equation is to divide both terms of the equation by $x(t)$ and multiply by dt, then integrate:

$$\frac{dx(t)}{x(t)} = -bdt$$

$$ln(x(t)) = -bt + c$$

Taking the exponential of both terms

$$x(t) = e^{-bt}e^{c} = Ae^{-bt}$$

where the exponential of integration constant C has been incorporated in the multiplicative constant A, whose value has to be derived from the boundary conditions (only one is needed in this case, since only one variable is to be determined).

A second order differential equation has the form:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Let us assume that the function $x(t) = e^{rt}$ is a solution, with r in general a complex number. Taking the first and second derivative, and substituting into the differential equation, we obtain:

$$e^{rt}(ar^2 + br + c) = 0$$

so the possible values of r are the roots of the quadratic equation:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and the solution of the differential equation will be a linear combination of the exponentials of the two roots:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

where the constants A and B have to be determined from the boundary conditions.

Example 18 *A non-linear (simple) case. Consider some tape coiled around a spinning axis, with initial radius r_0. The tape , with thickness σ, is pulled with constant speed v_0. What is the radius of the tape coil as a function of time?*

The radius of the coil is

$$r(t) = n(t)\sigma$$

and in the unit time the amount of tape that is pulled away is

$$dl = v_0 dt$$

As it happens, the number of turns of the tape around the axis is reduced in an infinitesimal amount:

$$dn = -\frac{v_0 dt}{2\pi r(t)}$$

That in turns changes the radius by an amount

$$dr = dn\sigma = -\frac{v_0 dt\sigma}{2\pi r}$$

so

$$rdr = -\frac{\sigma v_0 dt}{2\pi}$$

$$\int_{r_0}^{r(t)} = -\int_0^t \frac{\sigma v_0}{2\pi} dt$$

$$\frac{1}{2}(r_0^2 - r^2(t)) = \frac{\sigma v_0}{2\pi} t$$

$$r = \sqrt{r_0^2 - \frac{\sigma v_0 t}{\pi}}$$

2 Statics

Reading:

- Morin, Chapter 2

The branch of mechanics dealing with balancing forces, and non-moving objects. A force is a push or a pull opon an object resulting from an interaction with another object, and that can set the object in motion. A torque, or momentum of a force, is a force applied to a point around a pivot of an extended object, aiming at setting the object in motion.

2.1 Passive forces: normal force, tension

In statics, where objects do not move, the sum of all forces and torques has to be zero. Since a body standing on a floor will not move despite the force of gravity trying to pull it downwards, it means that the floor is producing a force that balances exactly gravity. This force, called "normal force" since it is normal to the surface of the floor, is dynamic, since it will perfectly compensate gravity: if an empty glass is filled with water, the normal force will increaase to compensate for the increased weight of the glass.

Similarly, a body attached to a wire in the earth's gravitational field will not fall: the wire will produce a tension force that exactly compensates the gravitationsl pull (until it breaks). Both the normal force and the tension of a cable are electrical forces at microscopic level, due to the external electrons of atoms moving from their equilibrium position.

2.2 Friction

2.3 Force of friction and other dissipative forces

Reading:

- Serway and Beichner §5.8;
- Kibble and Berkshire not covered;
- Kleppner and Kolenkow §2.5.

2.3.1 Sliding friction between solid surfaces

Consider a block of mass m sliding on a **rough** surface. The frictional force, $\mathbf{F}_f$, **as it acts on the block**, is in a direction **opposite** to the relative motion of the two surfaces (i.e., opposite to the motion of the block). It acts at the surface of the block in contact with the rough surface. The **normal reaction**, $\mathbf{N}$, of the surface **on the block** is perpendicular to the surface.

Sliding Friction is not a conservative force, since its direction depends on the relative velocity of the two surfaces, not on the positions of the objects involved.

To a good approximation, for most surfaces, the force of friction is

- Independent of the apparent contact area;
- Independent of the relative speed of the surfaces once they are in motion;
- Proportional to the normal force between the surfaces. The constant of proportionality is called the coefficient of friction, often denoted by μ. Hence we write $|\mathbf{F}_f = \mu|\mathbf{N}|$ for the magnitude.

But friction is still poorly understood. It is known that the frictional force is proportional to the true contact area between the surfaces (usually much less than the apparent contact area), so smooth surfaces can have more friction than rough ones. It is also known that friction can occur without wear, i.e. is not necessarily a result of damage to the surfaces, and that the independence of friction of the relative velocity is approximate, not exact.

The interactions of atoms and molecules near the surface are dominated by (conservative) electrostatic interactions among the nuclei and electrons; it is not understood in detail how these conservative interactions give rise to a non-conservative force.

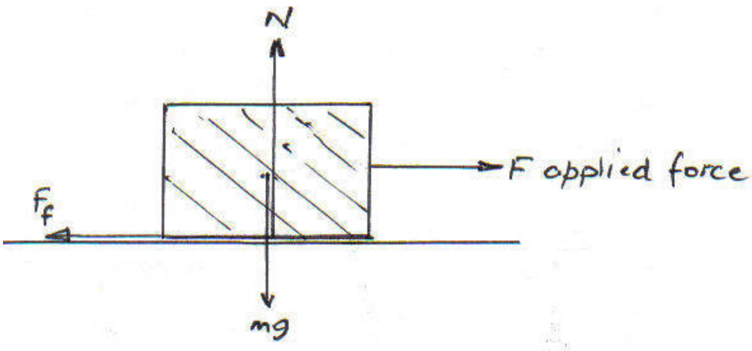

Figure 9: A block sliding over a rough horizontal surface.

As there is no resultant vertical force on the block, (see fig 9) then the magnitude

$$N = mg. \tag{45}$$

Some typical values of μ are

surfaces	μ
steel on steel	0.4
teflon on teflon	0.04
lead on steel	1

2.3.2 Static friction

If a force is applied to an object at rest on a rough surface, a frictional force develops on the object which is equal and opposite to component of the the total applied force parallel to the surface: $\mathbf{F}_f = -\mathbf{F}_{\text{applied},\|}$, i.e. it opposes the motion that would be produced by the applied forces if there was no friction.

This state of affairs continues until the magnitude of the applied force parallel to the surface exceeds $\mu|\mathbf{N}|$, where $\mathbf{N}$ is the normal force. Once this happens the object starts to slide and we have the situation described above, i.e. the direction of the frictional force opposes the direction of motion and its magnitude is $\mu|\mathbf{N}|$.

If the body is **just about to slide** we shall assume that $F_f = \mu N$. In practice the coefficient of **sliding friction** is often slightly less than the coefficient of **static friction**.

Example 19 *It is necessary to ease the pressure on car brakes just before stopping in order to keep the frictional force constant and avoid a 'jolt' to passengers.*

2.3.3 Equation of motion for a block on a rough horizontal surface

Consider the motion of a block sliding on a rough horizontal surface. Resolving the forces vertically,

$$mg - N = 0 \tag{46}$$

so

$$F_f = \mu N = \mu mg \tag{47}$$

and the resultant horizontal force on the body is

$$(F - F_f) = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{48}$$

This is the **equation of motion** of the body.

Example 20 *Suppose body is initially sliding to the right and the applied force $F = 0$. The body slides to rest under the influence of the frictional force. Equation of motion is*

$$-F_f = -\mu mg = ma, \tag{49}$$

so the acceleration

$$a = -\mu g \tag{50}$$

is constant. Hence previous results for motion under a constant acceleration can be used. In particular the time to come to rest if initial velocity is **u** *is given by*

$$\begin{aligned} 0 &= u - \mu gt, & (51)\\ t &= u/(\mu g). & (52) \end{aligned}$$

2.3.4 Motion of body on rough inclined plane

Consider the body being pulled/pushed **up** the plane at a constant speed. The forces are as in the diagram, Figure 10.

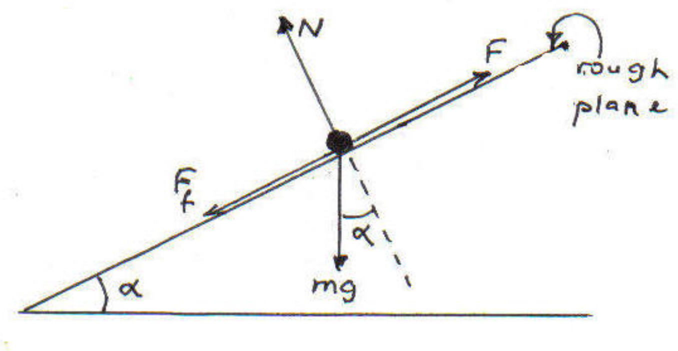

Figure 10: Balance of forces on an inclined plane.

The body is sliding so we take frictional force $F_f = \mu N$. There is no resultant force perpendicular to the plane (it remains in contact) so

$$N = mg\cos\alpha. \tag{53}$$

Resolving forces parallel to plane surface,

$$\begin{aligned} F - F_f - mg\sin\alpha &= ma = m\frac{\mathrm{d}v}{\mathrm{d}t} & (54)\\ F - \mu mg\cos\alpha - mg\sin\alpha &= m\frac{\mathrm{d}v}{\mathrm{d}t}. & (55) \end{aligned}$$

Example 21 *What is largest value for the angle of inclination of the plane for which body remains at rest on the plane* ***without sliding down****. In this case the body wants to slide down, so the frictional force F_f acts up the plane, as in fig 11.*

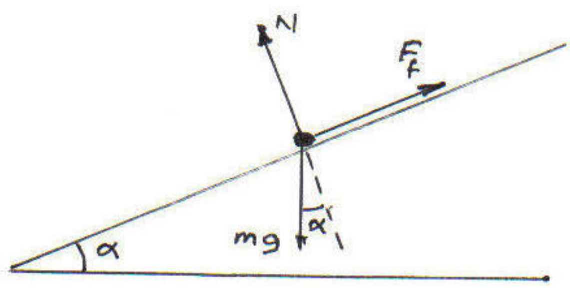

Figure 11: Balance of forces on an inclined plane, to find the maximum angle before slipping occurs.

For equilibrium there is no resultant force on the particle in any direction. Resolving forces perpendicular to the plane

$$N = mg\cos\alpha \tag{56}$$

and parallel to the plane

$$F_f = mg\sin\alpha. \tag{57}$$

The maximum value of F_f is μN, so

$$F_f = \mu N = \mu mg\cos\alpha \tag{58}$$

so the maximum value of α is when

$$\mu mg\cos\alpha = mg\sin\alpha \tag{59}$$

$$\tan\alpha = \mu. \tag{60}$$

2.4 Torque

Torque is the vector product between a force and its centre of application about an axis:

$$\tau = \mathbf{r} \times \mathbf{F} \tag{61}$$

where $\mathbf{r}$ is the vector connecting the origin to the point of application of the forcem and $\mathbf{F}$ is the force vector. The torque represents the capacity of a force to perform a rotation around the origin. To make calculations easier, it is convenient to have a smart choice of the reference frame. The vector product between the force and the "arm" $\mathbf{r}$ can be intuitively understood from our deily experience that the application of a force using a longer lever arm, and perpendicular to the rotation, will result in a strong rotational effect than smaller lever arms or different directions.

Torques will be discussed in more detail later on in the course, but they are important in statics, because equilibrium requires not only the sum of the forces to be zero, but also the sum of the torques.

Example 22 *Consider a ladder of length l and mass m, leaning against a frictionless wall with an angle θ with respect to the floor. There is a static friction coefficient μ between the bottom of the ladder and the floor. What is the smallest angle the ladder can make without starting to move?*

There are two vertical and two horizontal forces acting on the ladder. The vertical ones are gravity, pointing downwards with magnitude mg, and the normal reaction force of the floor, that we call N_1. The horizontal forces will be the one from the vertical, frictionless wall, that we call N_2, perpendicular to the wall and pointing outwards, and friction force of the floor, that we call F. Since forces have to be in equilibrium, we deduce that $N_1 = mg$ and $N_2 = F$. But force equilibrium cannot hel us in finding the values of N_2 nor F. For that we need to impose the condition on torques. The smartest choice for the origin is usually the point where most of the forces act, since there the "arm" $\mathbf{r}$ will be zero, and so the force. So we chose the origin (pivot point) as the foot of the ladder, the point where it touches ground.

There the torque from N_1 and F will be zero, and the equilibrium condition is that the torque produced by N_2 and gravity will be equal. So

$$N_2 l \sin\theta = mg(l/2)\cos\theta \tag{62}$$

$$N_2 = \frac{mg}{2\tan\theta} \tag{63}$$

Let us remember that $F = N_2$; so the condition on the friction coefficient is that $F < \mu N_1 = \mu mg$, so

$$\frac{mg}{2\tan\theta} \leq \mu mg \tag{64}$$

$$\tan\theta \geq \frac{1}{2\mu} \tag{65}$$

3 Newton's laws of motion and simple examples

Reading:

- Morin chapter 3

These were formulated in his book *Principia Mathematica*, written over several years but published in 1686–7. They are the basis of all classical mechanics.

1. A body remains at rest or in a state of uniform motion (non-accelerating) unless acted on by an **external** force.
2. Force = time rate of change of momentum, i.e.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{66}$$

where $\mathbf{p} = m\mathbf{v}$ = momentum of body of mass m moving with velocity $\mathbf{v}$. If m is constant then

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a}, \tag{67}$$

with acceleration, $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$.

3. To every **force** (**action**) there is an equal but opposite **reaction**.

As discussed in §3.1.7, these laws are only true in an **inertial** (non-accelerating) **frame of reference**. We shall discuss later how we can treat motion relative to an accelerating frame of reference.

Notes:

- From the second law, if $\mathbf{F} = 0$, then the acceleration $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0$, so the velocity, $\mathbf{v}$, is constant. Thus the first law is special case of the second law.
- We can also derive the third law from the second law as follows. Apply a force $\mathbf{F}$ to body 1, which is in contact with a second body 2. Body 1 pushes on body 2 with force $\mathbf{F}_2$ and body 2 pushes back on body 1 with force $\mathbf{F}_1$ as shown in fig 12.

  Applying Newton's second law, for the combined system,

$$\mathbf{F} = (m_1 + m_2)\,\mathbf{a}. \tag{68}$$

For body 1,

$$\mathbf{F} + \mathbf{F}_1 = m_1\mathbf{a}, \tag{69}$$

for body 2,

$$\mathbf{F}_2 = m_2\mathbf{a}. \tag{70}$$

Adding

$$\begin{aligned} \mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 &= (m_1 + m_2)\,\mathbf{a} = \mathbf{F} && (71)\\ \mathbf{F}_1 + \mathbf{F}_2 &= 0 && (72) \end{aligned}$$

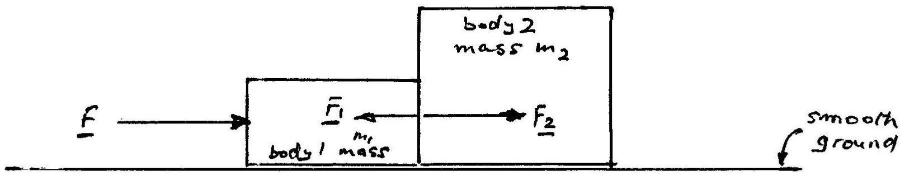

Figure 12: Deriving Newton's third law from the consistency of forces on a composite system.

and hence

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{73}$$

So that forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude but opposite in direction.

- Note the nature of the second law: it simultaneously defines a quantity (force) and says something about it (that it causes a change in momentum).
- Mathematically, Newton's second law is a **differential equation**: assuming we know how the force on a body is determined by the position (and perhaps velocity) of that body, it tells us how the velocity changes. It is a **second-order** differential equation for the position vector, and therefore its solution will contain two arbitrary constants. We will see that you can think of these constants as determining the initial position and velocity of the particle.

3.1 Symmetries

There are a number of symmetries that we often take for granted in physics (and indeed in the whole of the scientific method).

3.1.1 Invariance under spatial translations

We expect physical laws to be the same if we translate the position vectors $\mathbf{r}_i$ of all particles i by a constant vector $\mathbf{R}$ (leaving time unchanged):

$$\mathbf{r}_i \to \mathbf{r}_i + \mathbf{R} \qquad t \to t. \tag{74}$$

As far as we know, this is an exact symmetry of nature.

3.1.2 Time translations

Similarly we expect the results of out experiments do not depend on *when* we perform them. We therefore expect the laws are invariant under the replacements

$$t \to t + T, \qquad \mathbf{r}_i \to \mathbf{r}_i \tag{75}$$

for any fixed time T (leaving positions unchanged). As far as we know, this is an exact symmetry of nature though there have been some claims that the fundamental constants change as a function of time.

3.1.3 Invariance under rotations

We also do not expect there to be a preferred direction in space, so we would expect physical laws to be invariant under a rotation of the spatial coordinate system (leaving time unchanged):

$$\mathbf{r}_i \to R[\mathbf{r}_i], \qquad t \to t \tag{76}$$

where R denotes an arbitrary rotation. As far as we know, this is an exact symmetry of nature.

3.1.4 Invariance under uniform boosts

We might expect to see no difference if we compare the results of an experiments done with two sets of apparatus moving, one with respect to another, at a uniform velocity v; this corresponds to physical laws being left unchanged by a transformation

$$\mathbf{r}_i \to \mathbf{r}_i + \mathbf{v}t, \qquad t \to t.$$

This is known as a *Galilean transformation.*

In informal language: there is no experiment we can perform purely inside a uniformly moving aeroplane by which we can determine its velocity.

It is an exact symmetry of Newton's laws of motion as we study them in this course, but in nature it is approximate and only applies at low velocities (much less than the velocity of light). The equations of electricity and magnetism (Maxwell's equations) do not have symmetry under Galilean transformations—instead they have a different symmetry under the *Lorentz transformations* (see PHAS1246 next term).

3.1.5 Time reversal and spatial inversion

The above symmetries are all *continuous symmetries*—the time/space translations and velocity boosts can be varied continuously. There are also *discrete symmetries* which cannot be varied arbitrarily; two of the most important are *inversion symmetry* where all position vectors are inverted through the origin

$$\mathbf{r}_i \to -\mathbf{r}_i \qquad t \to t \tag{77}$$

and *time reversal*

$$\mathbf{r}_i \to \mathbf{r}_i \qquad t \to -t. \tag{78}$$

These are *almost* exact symmetries of nature: spatial inversion is respected by all forces of nature except one (the weak nuclear force, which is responsible for nuclear β-decay but otherwise has few macroscopically observable consequences) and time reversal is respected everywhere except in some rather exotic decays of particular elementary particles (neutral kaons and neutral B-mesons—see Nobel Prize for Physics 2008 to Kobayashi and Maskawa).

3.1.6 Broken (or hidden) symmetries

Even when the underlying symmetry is exact, we do not necessarily observe the full consequences of the symmetry because the arrangement of particles in our vicinity may not respect the underlying symmetry.

Example 23 *For us, living near the Earth's surface, all directions are not equivalent: the existence of the Earth's gravitational field means that things fall down, not up. The true underlying symmetry would only be restored if we rotated the whole system (us plus our surroundings plus the Earth).*

Example 24 *A car attempting to drive along the Euston Road in the rush hour cannot increase its speed to 70mph without experiencing different physical consequences (a crash). The underlying symmetry with respect to velocity boosts can only be restored by applying the boost to the whole environment: the car, the other traffic, and the Earth (hence the road).*

Example 25 *Under what conditions will Newton's second law be invariant under a translation of the coordinates*

$$\mathbf{r}'_i = \mathbf{r} + \mathbf{R}?$$

Writing the second law as

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i,$$

we see that the left-hand side is left the same since $\ddot{\mathbf{r}}'_i = \ddot{\mathbf{r}}_i$. Hence Newton's law is invariant so long as the force is left unchanged by a uniform translation of all the particles. This will be true so long as the particles interact only with one another and not with some external object which is not translated.

3.1.7 Accelerating frames of reference

Empirically, we find that physical laws are *not* invariant if we look in an accelerating frame of reference: we can tell when an aeroplane is taking off without looking our of the window. Hence Newton's laws of physics are not the same under acceleration; we will see later, however, that they can be made invariant by the introduction of **fictitious forces**.

3.2 Motion under a constant force

Reading:

- Jewett and Serway §2.6 and §4.2.
- Kibble and Berkshire §3.2.
- Kleppner and Kolenkow §1.7.

If the force is constant, then by Newton's second law, $\mathbf{F} = m\mathbf{a}$, and therefore so is the acceleration.

3.2.1 General theory

We can think of the change in velocity from the area under a graph of constant acceleration:

$$\mathbf{v}(t_2) = \mathbf{v}(t_1) + \mathbf{a}\,(t_2 - t_1)\,, \tag{79}$$

or by integrating the very simple differential equation

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a}; \tag{80}$$

$$\mathbf{v} = \mathbf{a}t + \mathbf{C} \tag{81}$$

where $\mathbf{C}$ is an arbitrary (vector) constant of integration.

Suppose $\mathbf{v} = \mathbf{u}$ at $t = 0$. This is a **boundary** or **initial condition**, so

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t \tag{82}$$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u} + \mathbf{a}t \tag{83}$$

and integrating again gives

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 + \mathbf{C}_2, \tag{84}$$

where $\mathbf{C}_2$ is another vector constant of integration determined by another boundary (initial) condition. Suppose $\mathbf{r} = \mathbf{r}_0$ at $t = 0$, then

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 \tag{85}$$

and the displacement up to time t is

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 = \frac{1}{2}(\mathbf{u} + \mathbf{v})t. \tag{86}$$

Since $\mathbf{a} = (\mathbf{v} - \mathbf{u})/t$, we can dot both sides with $\mathbf{a}$ to get

$$\begin{aligned}
\mathbf{a}\cdot\Delta\mathbf{r} &= \mathbf{u}\cdot(\mathbf{v} - \mathbf{u}) + \frac{1}{2}(\mathbf{v} - \mathbf{u})^2 && (87)\\
&= \mathbf{u}\cdot\mathbf{v} - u^2 + \frac{1}{2}v^2 - \mathbf{v}\cdot\mathbf{u} + \frac{1}{2}u^2 && (88)\\
&= \frac{1}{2}(v^2 - u^2), && (89)\\
\Rightarrow\quad v^2 &= u^2 + 2\mathbf{a}\cdot\Delta r. && (90)
\end{aligned}$$

Collecting together the equations of motion we have for linear motion under a constant acceleration (or force)

$$\begin{aligned}
\mathbf{v} &= \mathbf{u} + \mathbf{a}t, && (91)\\
\Delta\mathbf{r} &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2, && (92)\\
v^2 &= u^2 + 2\mathbf{a}\cdot\Delta\mathbf{r}, && (93)\\
\Delta\mathbf{r} &= \frac{1}{2}(\mathbf{u} + \mathbf{v})\,t. && (94)
\end{aligned}$$

3.2.2 One dimension

In the special case of one-dimensional motion (i.e. where the velocity, acceleration and displacement are all co-linear) we define s as the distance travelled and then have

$$v = u + at, \tag{95}$$
$$s = ut + \frac{1}{2}at^2, \tag{96}$$
$$v^2 = u^2 + 2as, \tag{97}$$
$$s = \frac{1}{2}(u+v)\,t. \tag{98}$$

These results should be familiar from school.

Example 26 *The Department of Transport recommended safe stopping distances for cars as a function of speed contain two components: a thinking distance s_{think} and a braking distance s_{brake}.*

Let the initial speed of the car be u.

The distance covered while thinking is

$$s_{\text{think}} = ut_{\text{think}}.$$

The distance covered while braking can be found from $v^2 = u^2 + 2as$; putting $v = 0$ gives

$$s_{\text{brake}} = \frac{u^2}{2(-a)}$$

(note the acceleration will be negative since we are braking). Hence the total distance is

$$s = ut_{\text{think}} + \frac{u^2}{2(-a)}.$$

The published distances fit this curve with $t_{\text{think}} = 0.68\,\text{s}$ and $a = -13\,\text{ms}^{-2}$.

Example 27 *A force is acting on a body with mass m, at rest at the origin of a reference system, according to the time dependence*

$$F = \frac{m}{(t+c)^2}$$

where c is an arbitrary constant. Notice that c has to be positive and non-zero, otherwise the acceleration at time -c would be infinite. Calculate velocity and position as a function of time.

$$a = \frac{1}{(t+c)^2}$$

$$v = \int_0^t \frac{dt'}{(t'+c)^2} = \frac{-1}{t+c}|_0^t = \frac{1}{c} - \frac{1}{t+c}$$

So the velocity starts at zero, but will tend to a value $1/c$ at infinite time (it is expected that since acceleration tends to zero, the velocity becomes a constant)

$$s = \frac{t}{c} - ln(t+c)|_0^t = \frac{t}{c} + ln\frac{c}{t+c}$$

the position has a component linear with the terminal velocity, minus another that accounts for the fact that the velocity is actually always smaller than that asymptotic value. however, the linear infinity of the first term is stronger than the logaritmic one of the second term.

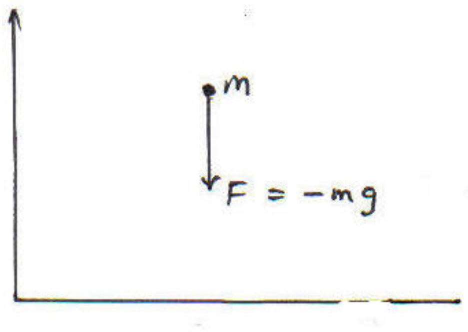

Figure 13: Mass falling freely under gravity.

3.2.3 Free fall under gravity—one dimension

Reading:

- Jewett and Serway §2.7.

In this case the constant acceleration is downwards, with $a = -g$, as height z, velocity v and acceleration a are positive when measured upwards.

Consider a body thrown upwards with an initial speed u, along the positive vertical direction, taken as axis y. Forces are as shown in fig 13. The initial conditions are, $y = 0$, $v = u$ at $t = 0$. To find the maximum height reached, we find the distance s where $v = 0$. Thus from eq(93), $0 = u^2 - 2gs$ giving $s = u^2/(2g)$. The time to the maximum height is found using eq(95) giving $t_{\max} = u/g$.

To find the time to reach some specified height, y_1, then use eq(96),

$$y_1 = ut - \frac{1}{2}gt^2 \tag{99}$$

$$gt^2 - 2ut + 2y_1 = 0. \tag{100}$$

This is a quadratic equation in t, with two roots,

$$t = \frac{u \pm \sqrt{u^2 - 2gy_1}}{g}. \tag{101}$$

If $u^2 > 2gy_1$, the body can reach a height greater than z_1 and the two roots are real, with

$$t_1 = \frac{u - \sqrt{u^2 - 2gy_1}}{g} \tag{102}$$

giving the time to reach z_1 on the way up and

$$t_2 = \frac{u + \sqrt{u^2 - 2gy_1}}{g} \tag{103}$$

giving the time to reach height z_1 going down after reaching the maximum height. Note that t_1 and t_2 are symmetric about the time to maximum height $t_{\max} = u/g$ (taking the special case $z_1 = 0$ we see that the time taken to fall back to the original starting point is $2t_{\max} = 2u/g$). If $x_1 > u^2/g$ no particle can reach this height (the quadratic has no real solution).

3.2.4 Trajectories of particle in a uniform gravitational field (ballistic trajectories)

Reading:

- Jewett and Serway §4.3

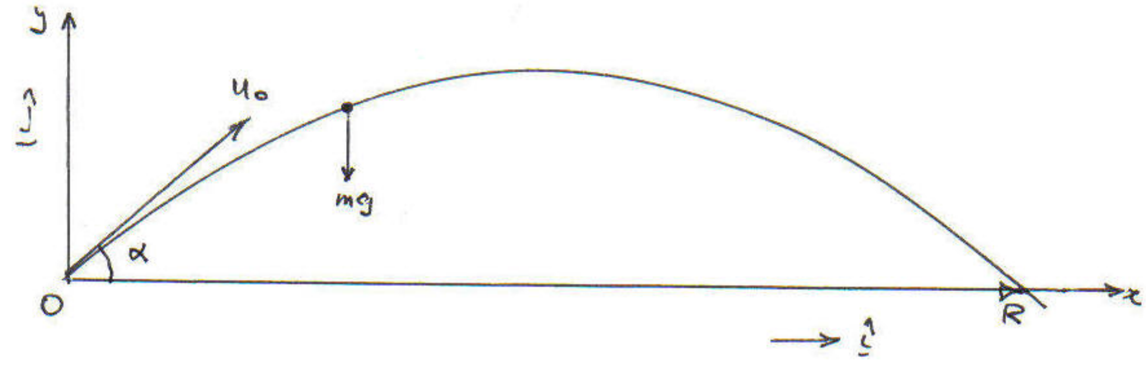

Figure 14: Ballistic trajectory

Now we consider the corresponding problem in three dimensions. Suppose the initial velocity $\mathbf{u}$ makes an angle α with the horizontal (sometimes called the angle of elevation) as in Fig. 14. Take this velocity to lie in the xz plane, so

$$\mathbf{u} = u(\cos\alpha\hat{\mathbf{i}} + \sin\alpha\hat{\mathbf{j}}).$$

The equation of motion is

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\left(\frac{\mathrm{d}v_x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\hat{\mathbf{j}}\right) = -mg\hat{\mathbf{j}}. \tag{104}$$

or

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a} = -g\hat{\mathbf{j}}.$$

Hence immediately we see that

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t = u\cos\alpha\hat{\mathbf{i}} + (u\sin\alpha - gt)\hat{\mathbf{j}}, \tag{105}$$

and so v_x is constant, equal to its initial value

$$v_x\,(t) = u\cos\alpha, \tag{106}$$

while

$$v_y = u\sin\alpha - gt. \tag{107}$$

The displacement is

$$\begin{aligned} \mathbf{r} - \mathbf{r}_0 &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 & (108)\\ &= \mathbf{u}t - \frac{1}{2}gt^2\hat{\mathbf{k}} & (109)\\ &= ut\cos\alpha\hat{\mathbf{i}} + (ut\sin\alpha - \frac{1}{2}gt^2)\hat{\mathbf{j}}, & (110) \end{aligned}$$

so

$$\begin{aligned} x\,(t) &= ut\cos\alpha, & (111)\\ y(t) &= ut\sin\alpha - \frac{1}{2}gt^2. & (112) \end{aligned}$$

Finally we have

$$v^2 = v_x^2 + v_y^2 = u^2 + 2\mathbf{a}\cdot\Delta\mathbf{r} = u_x^2 + u_y^2 - 2gy. \tag{113}$$

Since $u_x^2 = v_x^2$ this implies that

$$v_y^2 = u^2\sin^2\alpha - 2gy. \tag{114}$$

Note we could also have obtained all these results by applying the equations for one-dimensional motion to the individual components.

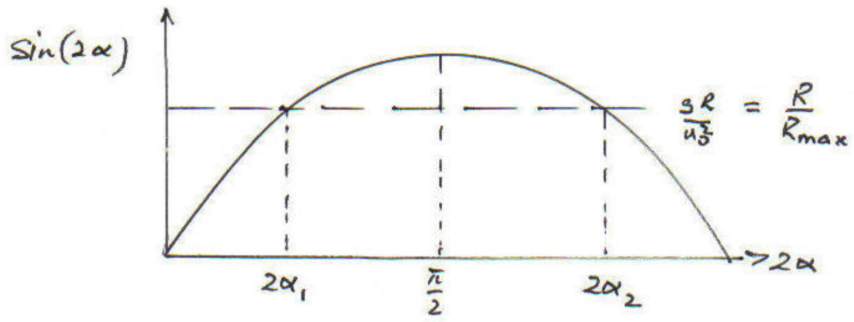

Figure 15: Angles for different ranges

3.2.5 Shape of trajectory

Since $x(t) = ut\cos\alpha$ then $t = x/(u\cos\alpha)$. Hence using the z position, eq(112)

$$\begin{aligned} y &= u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2 && (115)\\ &= x\tan\alpha - \left(\frac{gx^2}{2u^2\cos^2\alpha}\right). && (116)\end{aligned}$$

This is the equation of a parabola.

The range R on the horizontal plane is found from setting $y = 0$. Then

$$0 = x\tan\alpha - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\alpha}\right) = x\left(\tan\alpha - \frac{gx}{2u^2\cos^2\alpha}\right) \tag{117}$$

which has one solution, $x = 0$, and the other

$$x = R = \frac{2u^2}{g}\sin\alpha\cos\alpha = \frac{u^2}{g}\sin(2\alpha). \tag{118}$$

Time to reach $y = 0$ is found from

$$0 = ut\sin\alpha - \frac{1}{2}gt^2 \tag{119}$$

giving either $t = 0$ (the initial position) or for the whole range

$$T = \frac{2u\sin\alpha}{g}. \tag{120}$$

Once again this is twice the time t_{max} taken to reach the maximum height.

Maximum range occurs when $\sin 2\alpha = 1$, so when $\alpha = \pi/4 = 45°$, and $R_{\max} = u^2/g$.

For a range $R < R_{\max}$ there are two values of α which give rise to the same range, as illustrated in fig 15 and fig 16.

Since $R = R_{\max}\sin 2\alpha$, and $\sin 2\alpha = \sin(\pi - 2\alpha)$ then

$$2\alpha_1 = \sin^{-1}\left(\frac{R}{R_{\max}}\right) = \sin^{-1}\left(\frac{gR}{u^2}\right), \tag{121}$$

$$(\pi - 2\alpha_2) = \sin^{-1}\left(\frac{R}{R_{\max}}\right) = 2\alpha_1. \tag{122}$$

Hence

$$\alpha_2 = \frac{\pi}{2} - \alpha_1 \tag{123}$$

and then

$$\alpha_2 - \frac{\pi}{4} = \frac{\pi}{4} - \alpha_1 \tag{124}$$

so we see that the two angles for the same range are symmetric about $\pi/4$, see fig 15. Maximum range occurs when $\alpha_1 = \alpha_2 = \pi/4$.

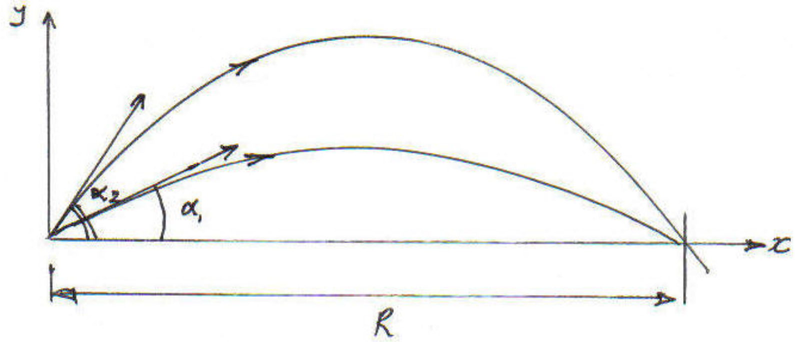

Figure 16: Two trajectories giving the same range.

3.3 Projectile motion accounting for air friction

The friction of a fluid like air can be epproximated as being proportional to the speed of the body moving in the fluid, and will be opposite to the direction of the velocity vector:

$$\mathbf{F}_{air_friction} = -\beta\mathbf{v}$$

Let us consider a body with an initial velociy v_0 moving in air, with no influence from external forces (for instance, moving horizontally on a frictionless plane). The only force acting on the system will be the air friction, so

$$m\frac{dv(t)}{dt} = -\beta v(t)$$

This is a first order differential equation, that has as solution

$$v(t) = Ae^{-\beta t/m}$$

where the parameter A has to be determined by the initial condition, namely the speed at time zero: $v(0) = A = v_0$. The position as a function of time can be found integratin the velocity function:

$$x(t) = \int v_0 e^{-\beta t/m} = -\frac{v_0 m}{\beta}e^{-\beta t/m} + C$$

We can set as initial condition that the position at time $t = 0$ is $x_0 = 0$, so $C = \frac{v_0 m}{\beta}$, and the solution is

$$x(t) = \frac{v_0 m}{\beta}(1 - e^{-\beta t/m})$$

Let us consider now the case when the body feels the earth's gravitational field, so for instance it is left falling at time $t = 0$ from position $y = 0$ and with initial speed 0. Assuming the direction of the vertical axis to point downwards, the relation between acceleration and force becomes:

$$m\frac{dv(t)}{dt} = -(\beta v(t) - mg)$$

If we divide both sides by $m(v(t) - mg/\beta)$ and multiply by dt, we obtain:

$$\frac{dv(t)}{v(t) - mg/\beta} = -\beta/mdt$$

Integrating

$$\ln(v(t) - mg/\beta) = -\beta/mt + C$$

exponentiating

$$v(t) = Ae^{-\beta/mt} + mg/\beta$$

Imposing the condition that the speed at $t = 0$ is zero, $A = -mg/\beta$, so

$$v(t) = mg/\beta(1 - e^{-\beta/mt})$$

So the speed will tend to a constant value $v_f = mg/\beta$ for very long values of t; this is the speed at which the air friction perfectly equates the gravitational pull.

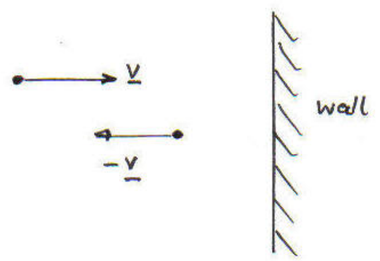

Figure 17: Particle bouncing elastically from a wall (very large mass)

3.4 Momentum and Impulse

Consider a force $\mathbf{F}$ which acts on a body of mass m during a time interval from t_1 to t_2. The impulse (a vector quantity) of $\mathbf{F}$ during this time interval is defined as

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, \mathrm{d}t \tag{125}$$

But from Newton's second law the force at each instant is the rate of change of momentum:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{126}$$

so

$$\mathbf{I} = \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\, \mathrm{d}t = \int_{\mathbf{p}_1}^{\mathbf{p}_2} \mathrm{d}\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1) \tag{127}$$

where $\mathbf{p}_1 = m\mathbf{v}_1$, the momentum of particle at time t, and similarly $\mathbf{p}_2 = m\mathbf{v}_2$ at time t_2. Hence Newton's second law can also be stated as Impulse = change in momentum,

$$\mathbf{I} = \Delta\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)\,. \tag{128}$$

If force, $\mathbf{F}$, is constant (in direction and magnitude) throughout the time interval $\Delta t = (t_2 - t_1)$, then

$$\mathbf{I} = \mathbf{F}\,(t_2 - t_1) = \mathbf{F}\Delta t. \tag{129}$$

An important special case is when force, $\mathbf{F}$, is very large and Δt is very small, so that $\mathbf{I} = \Delta\mathbf{p}$ is finite. Note as $\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, \mathrm{d}t$ then

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} \tag{130}$$

Example 28 *Particle of mass m bouncing off a wall elastically (i.e. with no loss of kinetic energy) as in fig 17.*

Change in momentum of particle is $\Delta\mathbf{p} = (-m\mathbf{v}) - m\mathbf{v} = -2m\mathbf{v}$ is the impulse on the particle. Therefore by Newton's third law the impulse on the wall is $+2m\mathbf{v}$.

Note that for this type of very short collision we can calculate the impulse, but not the force (since we do not know exactly how long the collision lasts).

3.5 Conservation of momentum for isolated systems

From Newton's second law for a single body, $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$, if $\mathbf{F} = 0$, then $\mathbf{p}$ is constant.

The same applies to a collection of arbitrarily many bodies: let body i have mass m_i. Let it experience an external force $\mathbf{F}_{\mathrm{ext},i}$, and an internal force $\mathbf{F}_{i,j}$ from each other particle j. Then Newton's second law gives

$$\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} = \mathbf{F}_{\mathrm{ext},i} + \sum_j \mathbf{F}_{i,j}. \tag{131}$$

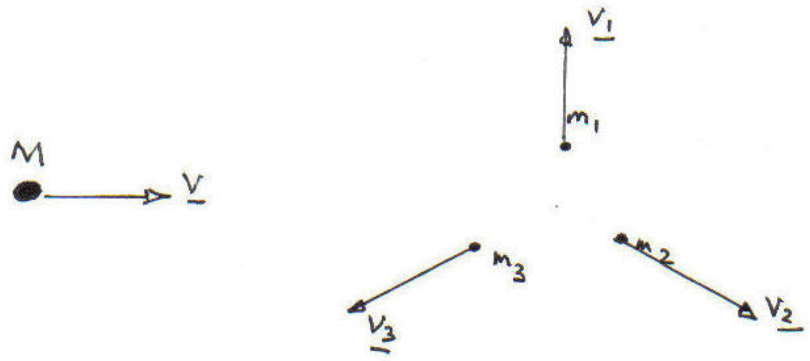

Figure 18: An exploding mass.

But from Newton's third law,

$$\mathbf{F}_{j,i} = -\mathbf{F}_{i,j} \tag{132}$$

so on adding the equations together,

$$\sum_i \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} = \sum_i \mathbf{F}_{\mathrm{ext},i} + \sum_{ij} \mathbf{F}_{i,j} = \sum_i \mathbf{F}_{\mathrm{ext},i}, \tag{133}$$

or

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{ext,tot}} \tag{134}$$

where

$$\mathbf{P} = \sum_i \mathbf{p}_i; \quad \mathbf{F}_{\mathrm{ext,tot}} = \sum_i \mathbf{F}_{\mathrm{ext},i}. \tag{135}$$

The rate of change of the total momentum is equal to the total external force.

If there is no external force (i.e. if only internal forces act) then

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0 \tag{136}$$

and the total momentum $\mathbf{P}$ is conserved.

In terms of impulses, the change ΔP in the total momentum from time t_1 to t_2 is equal to the total impulse $\mathbf{I}_{\mathrm{ext}}$ from the external force:

$$\Delta\mathbf{P} = \mathbf{I}_{\mathrm{ext}} \quad \text{where} \quad \mathbf{I}_{\mathrm{ext}} = \int_{t_1}^{t_2} F_{\mathrm{ext},i}\,\mathrm{d}t.$$

Example 29 *Consider a body of mass M moving with velocity $\mathbf{V}$ and not subject to any forces. Its momentum remains constant, i.e. $\mathbf{p} = M\mathbf{V} =$ constant. The body then explodes into several pieces (say at least three) with masses m_i, $i = 1, 2, 3\ldots$ travelling with velocities $\mathbf{v}_i$, $i = 1, 2, 3\ldots$ as illustrated in fig 18.*

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots. \tag{137}$$

The explosion gives impulses $\mathbf{I}_i$ to each of the three pieces. Because of Newton's third law (action and reaction are equal an opposite). If impulse on particle 1 due to particle 2 is $\mathbf{I}_{12}$ and that due to particle 2 on particle 1 is $\mathbf{I}_{21}$, then

$$\mathbf{I}_{12} = -\mathbf{I}_{21}. \tag{138}$$

For particle 1 the total impulse on it due to the other particles is

$$\mathbf{I}_1 = \mathbf{I}_{12} + \mathbf{I}_{13} \tag{139}$$

and for the whole system we have

$$
\begin{aligned}
\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 &= (\mathbf{I}_{12} + \mathbf{I}_{13}) + (\mathbf{I}_{21} + \mathbf{I}_{23}) + (\mathbf{I}_{31} + \mathbf{I}_{32}) && (140)\\
&= (\mathbf{I}_{12} + \mathbf{I}_{21}) + (\mathbf{I}_{23} + \mathbf{I}_{21}) + (\mathbf{I}_{31} + \mathbf{I}_{32}) && (141)\\
&= 0 + 0 + 0 && (142)
\end{aligned}
$$

Thus total impulse is zero and so the total momentum is the ***same before and after*** *the explosion,*

$$
\begin{aligned}
\mathbf{p} &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots && (143)\\
M\mathbf{V} &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots && (144)
\end{aligned}
$$

that is, ***total momentum is conserved****. In this case (non-relativistic) mass is conserved,*

$$M = m_1 + m_2 + m_3 + \ldots. \tag{145}$$

On the other hand ***kinetic energy is not conserved****,*

$$\frac{1}{2}MV^2 \neq \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots \tag{146}$$

but the extra kinetic energy comes from chemical potential energy or energy stored in springs.

Example 30 *Two particles, one initially at rest, collide and stick together (opposite of the explosion). Initial momentum is*

$$\mathbf{p}_i = m_1\mathbf{v}_1 + 0. \tag{147}$$

Final momentum is

$$\mathbf{p}_f = M\mathbf{V} = (m_1 + m_2)\,\mathbf{V}. \tag{148}$$

Since total momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f$*, so* $m_1\mathbf{v}_1 = (m_1 + m_2)\,\mathbf{V}$ *and*

$$\mathbf{V} = \frac{m_1}{m_1 + m_2}\mathbf{v}_1 \tag{149}$$

and combined particle moves in the same direction as the incoming one, but at a reduced speed. The initial kinetic energy is

$$K_i = \frac{1}{2}m_1v_1^2 + 0. \tag{150}$$

The final kinetic energy

$$K_f = \frac{1}{2}(m_1 + m_2)\,V^2 = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}v_1\right)^2 = \frac{1}{2}\frac{m_1^2}{(m_1 + m_2)}v_1^2 = \frac{m_1}{m_1 + m_2}K_i. \tag{151}$$

Thus there is a ***loss*** *of kinetic energy in a 'sticking' collision. The energy is converted into heat and/or sound.*

4 Kinematic relations—work, power, kinetic and potential energy

Reading:

- Jewett and Serway Chapter 7.
- Kibble and Berkshire §2.1 and §3.1.
- Kleppner and Kolenkow Chapter 4.

Before considering other problems we need to develop some other general aspects of mechanics.

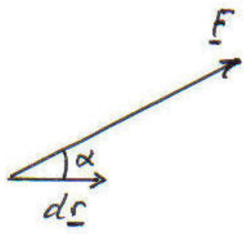

Figure 19: Work is the scalar product of force and displacement.

4.1 Work

A force **F** is applied to a body. If point of application of the force moves, i.e. body moves a distance $\delta\mathbf{r}$ as in fig 19, the **work done by the force** δW is defined to be

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} = (F\cos\theta)\ |\delta\mathbf{r}|. \tag{152}$$

where $F\cos\theta$ is component of force in direction of motion of the body.

Hence **work is a scalar quantity**. For finite displacements of a particle from position $\mathbf{r}_1$ to $\mathbf{r}_2$ as in fig 19 the work done **by** the force is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} [F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z]. \tag{153}$$

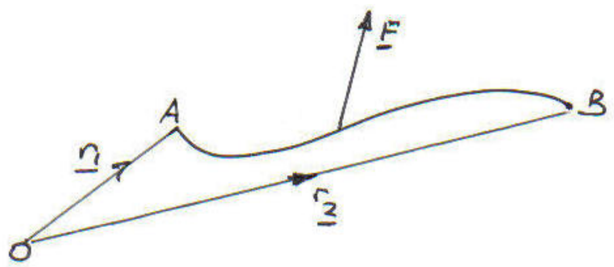

Figure 20: Work done as an integral for a body moving over a curved path.

This formula is needed in the general case where the force **F** varies along the path. For a **constant force** (in both **magnitude** and **direction**), work done is

$$W = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1)\,. \tag{154}$$

Work is **energy** and the S.I. unit is the joule (J) or newton metre (Nm).

For one-dimensional motion (e.g. if force acts in the x-direction and the particle can only move in the x-direction) we can drop the scalar product in eq(153) and write

$$W = \int_{x_1}^{x_2} F\,\mathrm{d}x \tag{155}$$

as the work done by force F in moving particle from position x_1 to x_2.

Example 31 *The work that must be done to stretch a spring through a distance x, as illustrated in fig 21.*

The tension in the spring when extended is proportional to the (small) extension, (Hooke's law), so $F_{\text{spring}} = -T = -kx$ where k is the stiffness constant. To stretch the spring we must apply an opposing external force $F_{\text{ext}} = +kx$. The work done by force F_{ext} (i.e. by us) to stretch spring to an extension x_1 is

$$W = \int_0^{x_1} F_{\text{ext}}\,\mathrm{d}x = \int_0^{x_1} kx\,\mathrm{d}x = \frac{1}{2}kx_1^2. \tag{156}$$

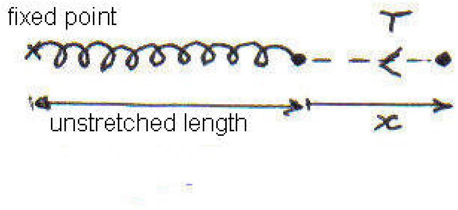

Figure 21: Work done to stretch a spring.

Note the work done by the force from the spring is equal and opposite:

$$\int_0^{x_1} F_{\text{spring}}\,\mathrm{d}x = -\frac{1}{2}kx_1^2.$$

This quantity is negative: we say the spring has work done on it.

Example 32 *The work done by gravity when a mass m falls through a vertical distance h.*

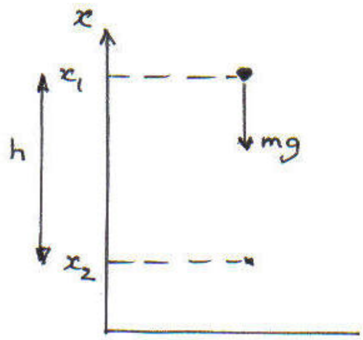

Figure 22: A body falling through distance h under gravity.

Let the vertical direction be the z direction. The force $F_z = -mg$ and the work done by it is

$$W = \int_{z_1}^{z_2} F\,\mathrm{d}z = \int_{z_1}^{z_2} (-mg)\,\mathrm{d}z = (-mgz)\,|_{z_1}^{z_2} = -mg\,(z_2 - z_1) = mgh. \tag{157}$$

Example 33 *The work done when a particle is constrained to fall under gravity inside a **smooth** tube, as in the fig 23.*

*Force of tube **on** the particle is N and is normal to the tube. So resultant force on particle is $\mathbf{N} + \mathbf{F}$, where $\mathbf{F} = -mg\hat{\mathbf{k}}$.*

Work done by these forces as particle moves a distance $\mathrm{d}\mathbf{r}$ along the tube is

$$\mathrm{d}W = (\mathbf{N} + \mathbf{F}) \cdot \mathrm{d}\mathbf{r} = \mathbf{N} \cdot \mathrm{d}\mathbf{r} + \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{158}$$

But $\mathbf{N}$ is orthogonal (perpendicular) to $\mathrm{d}\mathbf{r}$ at every point along tube, so $\mathbf{N} \cdot \mathrm{d}\mathbf{r} = 0$. Hence

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{159}$$

This is an example of the fact that purely constraining forces do no work.

Let the motion lie in the xz plane, so $\mathrm{d}\mathbf{r} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}z\hat{\mathbf{k}}$, so

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\,mg\hat{\mathbf{k}} \cdot \left(\mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}z\hat{\mathbf{k}}\right) = -mg\mathrm{d}z. \tag{160}$$

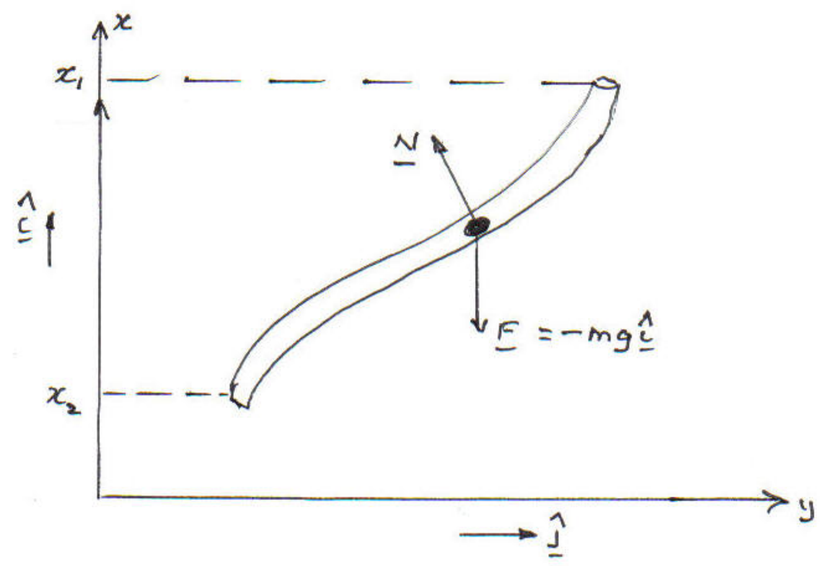

Figure 23: Constrained motion of particle in a tube under gravity.

and hence the work done by gravity when particle falls from height z_1 *to* z_2 *is*

$$W = \int_{z_1}^{z_2} (-mg)\,\mathrm{d}z = mgh \tag{161}$$

as before.

4.2 Power

Power, P, is defined as the rate of doing work,

$$P = \lim_{\delta t \to 0} \frac{\delta W}{\delta t} = \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{162}$$

The S.I. unit is the joule/second $\left(\mathrm{Js}^{-1}\right)$ or watt (W). Since $\delta W = \mathbf{F} \cdot \delta \mathbf{r}$,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v} = F_x v_x + F_y v_y + F_z v_z. \tag{163}$$

Average power over extended time interval t is $P = W/t$, where W is work done in time t.

4.3 Kinetic energy

The displacement in a short interval δt is $\delta \mathbf{r} = \mathbf{v}\delta t$. Hence the work done in moving the particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = m \int_{t_1}^{t_2} \mathbf{a} \cdot \mathbf{v}\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v}\mathrm{d}t. \tag{164}$$

But we see that

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} = \frac{\mathrm{d}v_x}{\mathrm{d}t} v_x + \frac{\mathrm{d}v_y}{\mathrm{d}t} v_y + \frac{\mathrm{d}v_z}{\mathrm{d}t} v_z = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2), \tag{165}$$

so

$$W = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} m v^2\right). \tag{166}$$

Define the **kinetic energy**

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}. \tag{167}$$

Then

$$W = K(t_2) - K(t_1), \tag{168}$$

i.e. the work done is equal to the change in kinetic energy. This is sometimes referred to (e.g. in Jewett and Serway) as the 'work-kinetic-energy theorem'.

Note that kinetic energy, like work, is a scalar.

The existence of the relationship (168) between work and kinetic energy is the reason we define both quantities the way we do.

Example 34 *For motion with a constant acceleration* $\mathbf{a}$ *we derived the equation*

$$v^2 = u^2 + 2\mathbf{a} \cdot \Delta r.$$

Multiplying by $\frac{1}{2}m$ *and remembering that* $\mathbf{F} = m\mathbf{a}$*, we find*

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + m\mathbf{a} \cdot \Delta r \quad \Rightarrow \quad \mathbf{F} \cdot \Delta r = W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.$$

In other words this equation tells us that the change in kinetic energy is equal to the work done by the force, as we expect.

It follows from equations (168) and (162) that power is the rate of change of kinetic energy:

$$P = \mathbf{F} \cdot \mathbf{v} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv^2\right] = \frac{\mathrm{d}K}{\mathrm{d}t}. \tag{169}$$

4.4 Potential energy

4.4.1 One dimension

Consider a one-dimensional problem where a particle is subject to a force $\mathbf{F}$ in the x-direction. Then the work done by the force as the particle moves from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F \,\mathrm{d}x.$$

Suppose the force depends only on the position x (and not, e.g., on the velocity). Then we define a quantity V, the **potential energy**, in such a way that the change in V is the work that would need to be done by an **external force** F_{ext}, acting from outside the system and opposing the force F, in order to move the particle from x_1 to x_2. It follows that the change in V is minus the integral of the force with respect to x:

$$V(x_2) - V(x_1) = \int_{x_1}^{x_2} F_{\text{ext}} \,\mathrm{d}x = -\int_{x_1}^{x_2} F \,\mathrm{d}x. \tag{170}$$

It follows that the potential energy V is minus the **indefinite integral** of the force F:

$$V(x) = -\int F(x) \,\mathrm{d}x. \tag{171}$$

There will be an arbitrary constant of integration in this expression; we fix it by setting the potential energy equal to zero at some arbitrarily chosen reference position x_0.

We can then write the work done by the physical force as the difference in potential energy between points x_1 and x_2:

$$W = -[V(x_2) - V(x_1)] = V(x_1) - V(x_2). \tag{172}$$

Note the signs: equation (172) tells us that if the potential energy goes up ($V(x_2) > V(x_1)$) then the work done by the internal force F is negative, i.e. the force has work done on it by the external force. This is consistent with the interpretation that the potential energy is energy stored in the configuration of the system.

The fundamental theorem of calculus then tells us that the force can be written as the (minus) the derivative of the potential energy wrt x:

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{173}$$

Again, note the sign: the physical (internal) force F acts in the direction of *decreasing* V.

Why the minus signs? Combining (168) and (172) for a particle moving from position x_1 at time t_1 to position x_2 at time t_2, we get

$$W = K(t_2) - K(t_1) = V(x_1) - V(x_2) \quad \Rightarrow \quad K(t_2) + V(x_2) = K(t_1) + V(x_1) \tag{174}$$

i.e. the sum of the kinetic energy and the potential energy is conserved. We refer to this sum as the **total energy** E:

$$E = K + V. \tag{175}$$

Example 35 *For the stretched string discussed earlier, taking the zero of potential energy at the unextended spring ($x = 0$),*

$$V(x) = \frac{1}{2}kx^2.$$

Note that the force from the spring is

$$F_{\text{spring}} = -kx = -\frac{\mathrm{d}V}{\mathrm{d}x},$$

as we expect.

Example 36 *Gravitational force $F = -mg$, height z (defined positive upwards).*

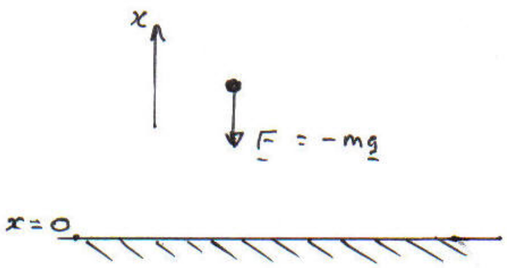

Figure 24: Particle falling under gravity

Then

$$V(z_1) = \int_{z_1}^{z_0} F\,\mathrm{d}z = \int_{z_1}^{z_0} (-mg)\,\mathrm{d}z = (-mgz)\,\big|_{z_1}^{z_0} = mgz_1 - mgz_0. \tag{176}$$

We can choose $z_0 = 0$ as the ground, giving

$$V(z) = mgz \tag{177}$$

and

$$F = -\frac{\mathrm{d}V}{\mathrm{d}z} = -mg \tag{178}$$

as above.

Example 37 *The "Gauss gun" involves firing a ball bearing into a well of low potential energy as it is attracted towards one side of a strong magnet. At the same time a second bearing is ejected from a shallower well on the other side of the magnet. The difference between the large potential energy decrease of the first ball bearing, and the smaller increase of the second, appears as a much higher kinetic energy of the second ball.*

4.4.2 Three dimensions

In three-dimensions we should like to be able to write, by analogy,

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) = \text{change of P.E.} \tag{179}$$

and

$$V(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{180}$$

$$V(\mathbf{r}_0) = 0. \tag{181}$$

But if the potential is to exist, the integrals must depend only on the end points and be independent of the path taken between them. If this condition is satisfied the force is said to be **conservative**.

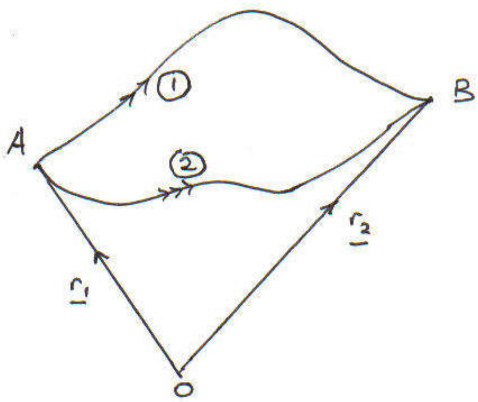

Figure 25: Work done over different paths by a conservative force

Example 38 *This is not necessarily true! Previously we considered the example* $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, *where* $\mathbf{F} = x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}$, *along a straight-line path from* $(0,0,0)$ *to* $(1,1,1)$. *We found that the integral had the value 1. Now let's evaluate it instead along a different path:*

- *First the line from* $(0,0,0)$ *to* $(1,0,0)$. *Along here* $y = z = 0$ *and the displacement is purely in the x-direction (i.e.* $\mathrm{d}\mathbf{r} = \mathrm{d}x\hat{\mathbf{i}}$*) so the contribution to the integral is*

$$\int_{x=0}^{x=1} x^2 \,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}.$$

- *Next take the line from* $(1,0,0)$ *to* $(1,1,0)$. *Along this line* $x = 1$ *and* $z = 0$, *and the displacement is in the y-direction, i.e.* $\mathrm{d}\mathbf{r} = \mathrm{d}y\hat{\mathbf{j}}$, *so the contribution is*

$$\int_{y=0}^{y=1} y \,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{y=0}^{y=1} = \frac{1}{2}.$$

- *Finally consider the line from* $(1,1,0)$ *to* $(1,1,1)$. *Along this line* $x = y = 1$ *and the displacement is in the z-direction, so* $\mathrm{d}\mathbf{r} = \mathrm{d}z\hat{\mathbf{k}}$. *We therefore get*

$$\int_{z=0}^{z=1} 1 \,\mathrm{d}z = 1.$$

The total integral is then $\frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$, *which is different from the value of 1 we found earlier. Such a force, where the work done depends on the path and not only on the end points, is called* ***non-conservative*** *and does not have an associated potential energy function.*

Put another way, the work done must be zero going around any closed loop:

$$\int_1 \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_2 \mathbf{F}\cdot \mathrm{d}\mathbf{r} \quad \Rightarrow \quad \int_1 \mathbf{F}\cdot \mathrm{d}\mathbf{r} - \int_2 \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathrm{loop}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0. \tag{182}$$

A force satisfying this condition is said to be **conservative**.

If this condition holds, then for a small displacement $\delta\mathbf{r}$

$$\delta V = -\mathbf{F}\cdot\delta\mathbf{r} = -[F_x\delta x + F_y\delta y + F_z\delta z]. \tag{183}$$

Considering a small displacement in x with y, z held constant we obtain

$$F_x = -\frac{\partial V}{\partial x}, \tag{184}$$

and similarly

$$F_y = -\frac{\partial V}{\partial y}; \quad F_z = -\frac{\partial V}{\partial z} \tag{185}$$

The curly-d symbols are 'partial derivatives': they are defined as the derivative with respect to one of the variables (x, say) with the others (y and z) held constant.

All three equations can be summed up as

$$\mathbf{F} = -\nabla V \tag{186}$$

where the gradient operator is defined as

$$\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{187}$$

We can now find a mathematical requirement for the force to be conservative: since

$$\frac{\partial^2 V}{\partial x\partial y} = \frac{\partial^2 V}{\partial y\partial x},$$

we must have

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

and so on.

Examples of conservative forces are: electrostatics, gravitation, force due to a stretched spring. However there are many examples of forces which, even though they may depend only on position, are not conservative.

Example 39 *Consider an electrical circuit around which there is an electromotive force (emf) Φ (arising, for example, from a variation in the magnetic field through the circuit). There is an electric field $\mathcal{E}$ everywhere in the circuit which produces a force $\mathbf{F} = q\mathcal{E}$ on a charge q. Thus*

$$\int_{circuit} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = q\Phi \neq 0,$$

so the force is not conservative.

Example 40 *The force*

$$\mathbf{F} = -2xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

is conservative, because it can be written as

$$\mathbf{F} = -\nabla(x^2y - 4z).$$

Example 41 *The force used in our previous integration example, $\mathbf{F} = x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}$, is not conservative because*

$$\frac{\partial F_x}{\partial y} = 0 \quad \textit{but} \quad \frac{\partial F_y}{\partial x} = y.$$

5 Simple Harmonic motion

Reading:

- Morin chapter 4
- Jewett and Serway Chapter 15
- Kibble and Berkshire §2.2, SS2.5–2.6
- Kleppner and Kolenkow Chapter 10.

Consider a particle of mass m attached to a light (massless) spring with a displacement x from the equilibrium (unstretched) position $x = 0$ as shown in fig 26. Positive x means extension, negative x means compression of the spring. The restoring force is given by Hooke's law, $F = -kx$.

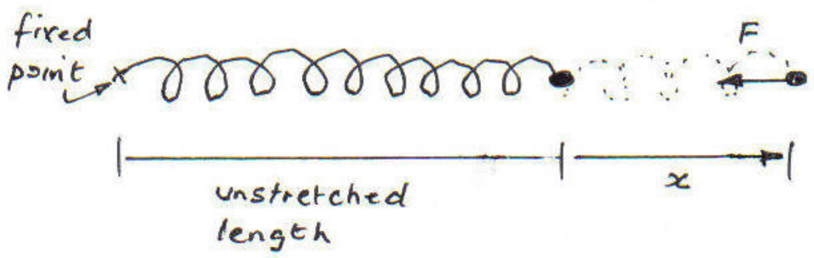

Figure 26: Force from a stretched spring.

The equation of motion is

$$F = -kx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{188}$$

or, writing $\omega = \sqrt{k/m}$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0. \tag{189}$$

This equation, in one form or another, is one of the most common equations in physics.

5.1 Undamped motion—three ways of writing the solution

The general solution to (189) is

$$x\,(t) = A\cos(\omega t) + B\sin(\omega t), \tag{190}$$

where A and B are two arbitrary constants of integration of the second-order differential equation eq(189). The functions $\sin\omega t$ and $\cos\omega t$ are shown in figs 28 and 27.

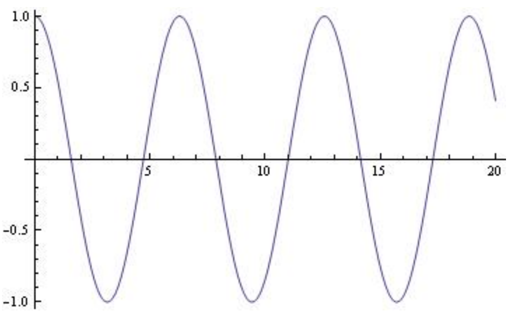

Figure 27: Cosine curve

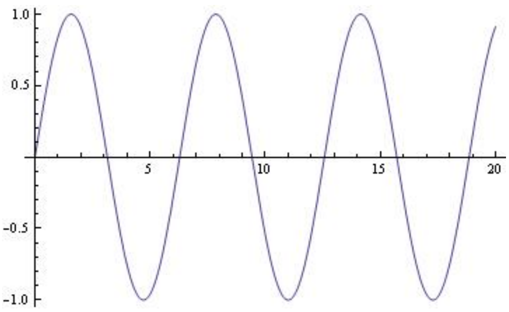

Figure 28: Sine curve

Where does this solution come from? We can confirm it by differentiating. But another way to find it is by making the guess (or *ansatz*)

$$x(t) = \alpha e^{qt},$$

where a and q are constants (such a guess always works in solving a linear differential equation with constant coefficients). Substituting in the equation gives

$$\begin{aligned} \dot{x}(t) &= q\alpha e^{qt} = qx; \qquad \ddot{x}(t) = q^2 \alpha e^{qt} = q^2 x; \\ q^2 + \omega^2 &= 0 \qquad q = \pm i\omega, \end{aligned} \tag{191}$$

where i is the square root of minus one, as usual ($i^2 = -1$). What does this mean? A general solution will be a linear combination of terms with the two possible values of q:

$$x(t) = Ce^{i\omega t} + De^{-i\omega t}, \tag{192}$$

where C and D are two more arbitrary constants. We can relate C and D to A and B by using **de Moivre's theorem**:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \tag{193}$$

so the solution involving complex exponentials can be rewritten

$$x(t) = C[\cos(\omega t) + i\sin(\omega t)] + D[\cos(\omega t) - i\sin(\omega t)] = (C + D)\cos(\omega t) + i(C - D)\sin(\omega t). \tag{194}$$

Thus $A = C + D$ and $B = i(C - D)$.

Although it's convenient to introduce complex numbers in order to solve the problem we have to remember that $x(t)$ is still a real physical quantity; we therefore require A and B to be real, and hence $C + D$ to be real and $C - D$ to be imaginary. This means that C and D have to be complex conjugates of one another, so $D = C^*$ (in fact $C = (A + iB)/2$ and $D = (A - iB)/2$) and therefore

$$x(t) = Ce^{i\omega t} + (Ce^{i\omega t})^* = 2\Re[Ce^{i\omega t}],$$

since $2\Re(z) = z + z*$ for any complex number z. Hence we can also write

$$x(t) = R\cos(\omega t + \phi) \tag{195}$$

where

$$C = \frac{1}{2}Re^{i\phi} = \frac{1}{2}R(\cos\phi + i\sin\phi), \tag{196}$$

with R and ϕ both real. We can think of R as the distance from the origin of a point in the complex plane (Argand diagram) rotating in a circle around the origin at angular velocity ω; ϕ is then the angle of the point to the real axis at time $t = 0$. The quantities R and ϕ are known as the *amplitude* and *phase angle* of the oscillation respectively. The physical solution is the real part of this rotating complex number (i.e. the projection onto the horizontal axis).

In any case there are two arbitrary constants (A and B, or C and D, or R and ϕ); their values are determined by the **initial** or **boundary conditions**.

Any solution to the equation of motion is **periodic**, the time for one complete period/oscillation being

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{197}$$

The frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{Hz.} \tag{198}$$

and the angular frequency $\omega = \sqrt{k/m}$ rad/s.

Example 42 *Suppose $x = a$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$ are the initial conditions. Then from*

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{199}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\omega A\sin\omega t + \omega B\cos\omega t \tag{200}$$

substituting the initial values gives,

$$a = A, \tag{201}$$

$$0 = \omega B. \tag{202}$$

Thus a particular solution satisfying the initial conditions is

$$x(t) = a\cos\omega t \tag{203}$$

$$v(t) = -a\omega\sin\omega t. \tag{204}$$

The amplitude of the oscillation is therefore a in this case.

5.2 Potential and kinetic energy in simple harmonic motion

We now consider the variations of the potential energy and kinetic energy of the particle as it undergoes simple harmonic motion. The force $F = -kx$ is a conservative force so the potential energy

$$V(x) = \frac{1}{2}kx^2 + V_0. \tag{205}$$

We can choose $V(x = 0) = 0$ giving $V_0 = 0$ and the potential energy is

$$V(x) = \frac{1}{2}ka^2\cos^2(\omega t) \tag{206}$$

The kinetic energy K_E is

$$K_E = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\omega^2\sin^2(\omega t). \tag{207}$$

The total energy $E = K_E + V$ is

$$E = \frac{1}{2}ma^2\omega^2\sin^2(\omega t) + \frac{1}{2}ka^2\cos^2(\omega t). \tag{208}$$

But $\omega^2 = k/m$ so

$$E = \frac{1}{2}ma^2\frac{k}{m}\sin^2(\omega t) + \frac{1}{2}ka^2\cos^2(\omega t), \tag{209}$$

$$= \frac{1}{2}ka^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right], \tag{210}$$

$$E = \frac{1}{2}ka^2 = K_E + V. \tag{211}$$

Fig 29 shows the potential energy curve for a stretched spring. If total energy is E, then the motion is restricted to $|x| < a$. At $x = a$, $V = \frac{1}{2}ka^2 = E$, and $K_E = 0$; at $x = 0$, $V = 0$, and $K_E = \frac{1}{2}ka^2 = E$.

Whenever the potential energy function of a system has a rounded minimum, the vibrational motion about the equilibrium position of minimum potential energy approximates to simple harmonic motion. An example is two atoms bound together to form a molecule, such as in fig 30,. If r_0 is the separation for minimum potential energy, (the equilibrium separation), the two atoms vibrate about this value r_0.

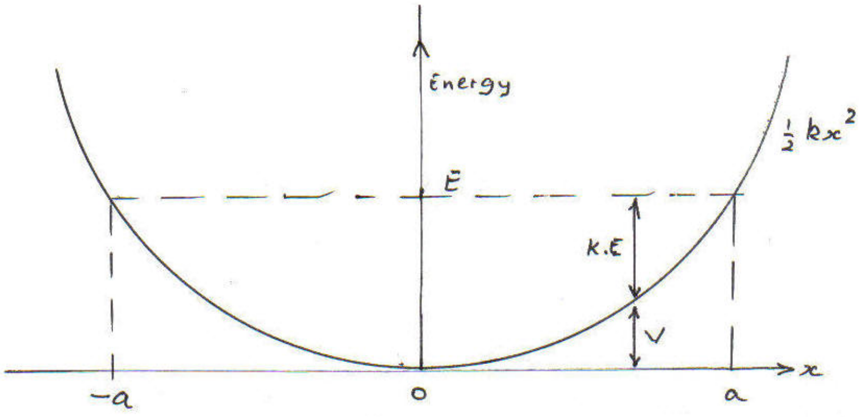

Figure 29: Potential energy curve for simple harmonic motion

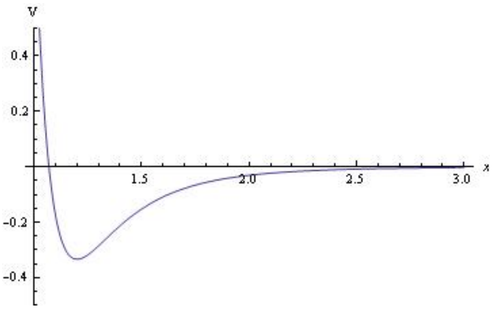

Figure 30: Potential energy as a function of atomic separation.

This can be shown by taking a Taylor's series expansion about the equilibrium position. If x_0 is the equilibrium separation, then

$$V(x) - V(x_0) = \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0}(x-x_0) + \frac{1}{2}\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0}(x-x_0)^2 + \dots. \tag{212}$$

But at $x = x_0$ is a minimum, so $\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0} = 0$ and $\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0} > 0$. Since the force for small displacements $(x - x_0)$ is

$$F(x) = -\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right), \tag{213}$$

$$= -\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0}(x-x_0). \tag{214}$$

Hence force is of the form $F = -kx$ where x is displacement from equilibrium, with $k > 0$, i.e.S.H.M.

5.3 Simple pendulum

Simple pendulum: a body of mass m on light inextensible string of length l, making an angle θ with the vertical, in downward gravitational field g, see figure 31. There are many ways of solving the pendulum, and if oscillations are not small the problem can become quite complicated; here we will limit ourselves to small oscillations.

The plot shows both the x coordinate, and the "curvilinear coordinate" s, the distance from the lowest point along the circle, that by definition of a radiant angle is $s = l\theta$. As a first approach, let's look at the force diagram. The external forces acting on the system are gravity, $F = -mg$, and the string tension. So we can separate gravity into two components, parallel and perpendicular to the wire. The

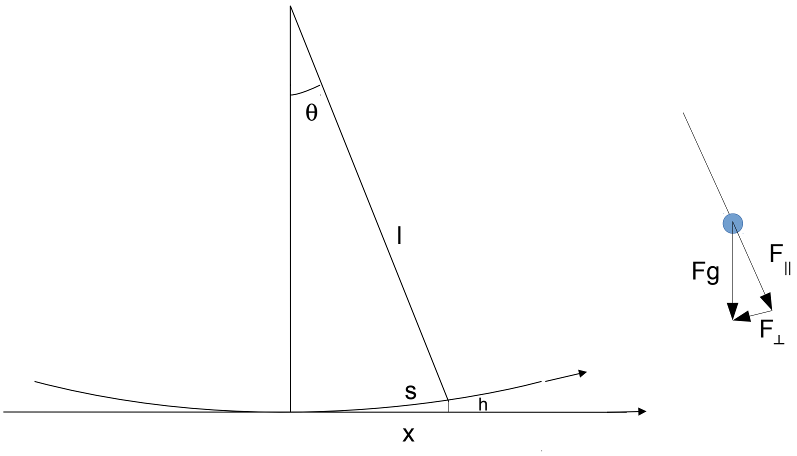

Figure 31: A simple pendulum, showing both the curvilinear distance s and the x axis.

parallel component of the force, $F_{||} = -mg\cos\theta$, is completely compensated by the string tension. The perpendicular component, $F_{\perp} = -mg\sin\theta$, is acting along the s direction. So the equation of motion along s is

$$m\ddot{s} = ml\ddot{\theta} = -mg\sin\theta \approx -mg\theta \tag{215}$$

The mass cancels, and this equation corresponds to a harmonic oscillator in the variable θ, with $\omega = \sqrt{l/g}$.

Let us look now at the problem from the point of view of the potential. As long as the string remains taut, the potential energy is

$$V(\theta) = -mgh = mgl(1 - \cos\theta) \tag{216}$$

If we take the zero of the potential at $\theta = \pi/2$, the constant term will be zero and the potential will be

$$V(\theta) = -mgl\cos\theta \tag{217}$$

The force acting on the system will be minus the space derivative of this potential

$$F = -\frac{dV}{dx} = \frac{dV}{d\theta}\frac{d\theta}{dx} \tag{218}$$

We know that $x = l\sin\theta$, so

$$\frac{dx}{d\theta} = l\cos\theta \tag{219}$$

From the chain rule we can write

$$\frac{d\theta}{d\theta} = 1 = \frac{d\theta}{dx}\frac{dx}{d\theta} \tag{220}$$

so

$$\frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{l\cos\theta} \tag{221}$$

(notice that this relation is no longer valid close to discontinuity points like $\theta = \pi/2$, but our pendulum will never get that far). So:

$$F_x = mgl \sin\theta \frac{1}{l\cos\theta} \tag{222}$$

This is actually not exact, since the proper treatment would involve a Lagrangiam-mechanics equation with constraints. But it is a good approximation for small θ, for which $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and so

$$l\ddot{\theta} \approx -g\theta. \tag{223}$$

identical to the one found using the forces, and we have again simple harmonic motion with angular frequency $\omega = \sqrt{\frac{g}{l}}$. Instead of taking the complete expression of the potential and then making the approximation later, we can directly take the second-order Taylor expansion directly on the potential. For small θ, the potential energy can be approximated by

$$V(\theta) = -mgl\cos\theta \approx -mgl\frac{\theta^2}{2}, \tag{224}$$

again yielding the same result.

A yet different way to get to the same equation is using energy conservation. The total energy of the system is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{225}$$

Taking the time derivative

$$0 = ml^2\ddot{\theta}\dot{\theta} + mgl\sin\theta\dot{\theta} \tag{226}$$

Dividing by $ml\dot{(\theta)}$, and again taking the small-angle approximation

$$l\ddot{\theta} + g\theta = 0 \tag{227}$$

5.4 Damped oscillations

In all **real** mechanical oscillators there is some **damping** (or friction). The damping force opposes the motion of the particle. Consider a damping force that is proportional to the velocity of the particle, i.e. $F_f = -\lambda\frac{\mathrm{d}x}{\mathrm{d}t}$ with λ a positive constant. Suppose the velocity is positive, i.e. motion in direction of increasing x, then the damping force is in the opposite direction (see fig 32).

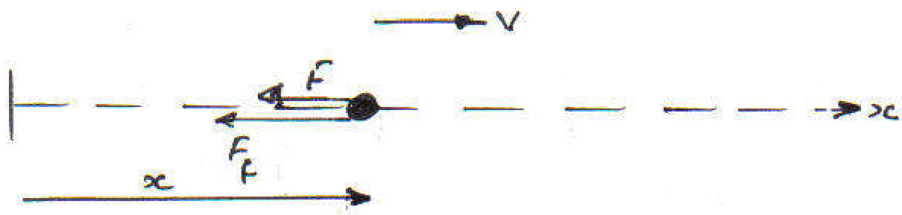

Figure 32: Displacement and forces in a damped harmonic oscillator.

The equation of motion of the particle is

$$F + F_f = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{228}$$

$$-kx - \lambda\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{229}$$

or

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{230}$$

We introduce the quantities

$$\gamma = \frac{\lambda}{2m}. \tag{231}$$

and the natural (undamped) angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$ Note that γ has the dimensions of $[T]^{-1}$ and so can be directly compared with ω_0.

Now we can write

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0. \tag{232}$$

This is a second-order, linear, homogeneous differential equation with constant coefficients 1, 2γ and ω_0^2. The general solution is

$$x(t) = A\mathrm{e}^{q_1 t} + B\mathrm{e}^{q_2 t}, \tag{233}$$

where q_1 and q_2 are the roots of the quadratic equation (the auxiliary equation)

$$q^2 + 2\gamma q + \omega_0^2 = 0, \tag{234}$$

$$q = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{235}$$

In the absence of damping, $\gamma = 0$, we would have

$$q = \pm\sqrt{-\omega_0^2} = \pm\mathrm{i}\omega_0 \tag{236}$$

so the solutions would be $\mathrm{e}^{\pm\mathrm{i}\omega_0 t}$ as before.

In the general case we write

$$\begin{aligned} q &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} && (237)\\ &\equiv -\gamma \pm i\omega, && (238) \end{aligned}$$

where the angular frequency

$$\omega = \sqrt{\omega_0^2 - \gamma^2}. \tag{239}$$

There are three possibilities;

(a) $\gamma < \omega_0$, so ω is real and positive with $0 < \omega < \omega_0$

(b) $\gamma = \omega_0$, so $\omega = 0$,

(c) $\gamma > \omega_0$, so ω is imaginary, q is wholly real, and there is no oscillation.

We will consider these cases in turn.

(a) $\gamma < \omega_0$, known as light damping ($0 < \omega < \omega_0$).

The roots are complex quantities, $q_1 = -\gamma + i\omega$ and $q_2 = -\gamma - i\omega$, so

$$\begin{aligned} x(t) &= C\mathrm{e}^{q_1 t} + D\mathrm{e}^{q_2 t}, && (240)\\ &= \mathrm{e}^{-\gamma t}\left[C\mathrm{e}^{i\omega t} + D\mathrm{e}^{-i\omega t}\right]. && (241) \end{aligned}$$

From de Moivre's theorem $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, $x(t)$ can be re-written as

$$x(t) = \mathrm{e}^{-\gamma t}\left[A\cos\omega t + B\sin\omega t\right] \tag{242}$$

which is a product of a function $\mathrm{e}^{-\gamma t}$ exponentially decaying with time and an oscillating function $[C\cos\omega t + D\sin\omega t]$ of time with angular frequency ω as shown in fig 33.

Energy is continually being lost due to the damping force but at any given time the energy of the oscillator is

$$E = K_E + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{243}$$

The rate of energy loss is the power dissipated against the damping force

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\lambda v^2 = -\frac{2\lambda}{m}K = -4\gamma K. \tag{244}$$

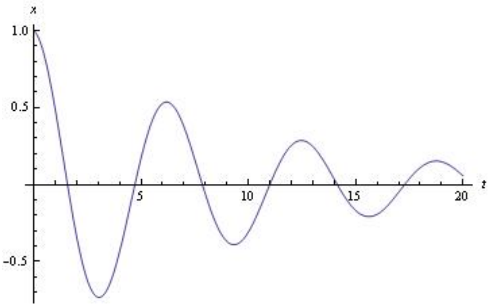

Figure 33: Displacement as function of time in an under damped oscillator.

For very light damping where the average kinetic and potential energies are still roughly equal we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \approx -2\gamma E. \tag{245}$$

Hence in time interval δt the fractional energy loss

$$\frac{\delta E}{E} \approx -2\gamma \delta t. \tag{246}$$

We define the Q**-factor** or **quality factor** such that the fraction of energy lost in a time interval $\delta t = 1/\omega$ (the time taken for the oscillation to move through a phase angle of 1 radian) is $1/Q$, or equivalently the fractional energy loss in one period $T = 2\pi/\omega$ is $2\pi/Q$. Therefore

$$\frac{1}{Q} = \frac{2\gamma}{\omega} \quad \text{or} \quad Q = \frac{\omega}{2\gamma} \approx \frac{\omega_0}{2\gamma}, \tag{247}$$

for light damping since $\omega \approx \omega_0$. For a lightly damped oscillator the Q-factor is large.

(b) $\lambda^2 = 4mk$, $(\gamma = \omega_0, \omega = 0)$ known as critical damping
The two roots are equal to $-\lambda/(2m)$. In this case the general solution to the equation of motion is

$$x(t) = (A + Bt)\, e^{-\gamma t}. \tag{248}$$

If $x = a$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$, $x(t)$ is never negative - there is no oscillation and the system returns to the equilibrium position is the shortest time (see fig 34).

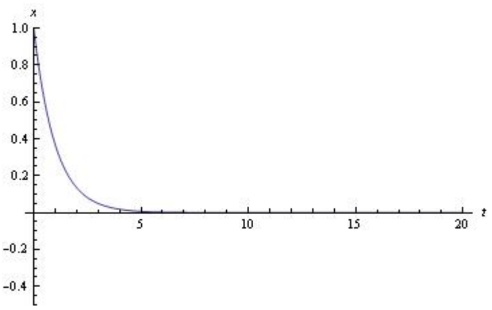

Figure 34: Displacement as function of time for critical damping.

Critical damping is required for swing-doors and car suspensions.

(c) $\lambda^2 > 4mk$ $(\gamma > \omega_0)$, known as heavy damping, ω is imaginary and q is wholly real. The two roots are

$$q_1 = -\frac{\lambda}{2m} + \sqrt{\frac{\lambda^2}{4m^2} - \omega_0^2} < 0, \tag{249}$$

$$q_2 = -\frac{\lambda}{2m} - \sqrt{\frac{\lambda^2}{4m^2} - \omega_0^2} < 0. \tag{250}$$

Both roots are negative but q_2 is more negative then q_1, i.e. $|q_2| > |q_1|$. The displacement

$$x(t) = A\mathrm{e}^{-|q_1|t} + B\mathrm{e}^{-|q_2|t} \tag{251}$$

is the addition of two term, the first $A\mathrm{e}^{-|q_1|t}$ dies away slowly (middle curve in fig 35), the second $B\mathrm{e}^{-|q_2|t}$ dies away more quickly (lower curve in fig 35).

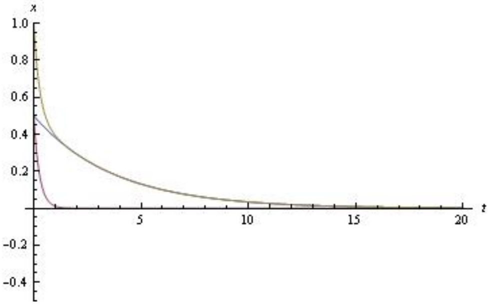

Figure 35: Displacement as function of time for overdamped (heavy) oscillator

As damping increases $|q_1|$ becomes smaller and the displacement decreases more slowly with time.

5.5 Forced damped oscillator

Consider applying a periodic force $F = F_0 \cos \omega_f t = \Re[F_0 \mathrm{e}^{\mathrm{i}\omega_f t}]$ to a damped oscillator in order to keep it oscillating. The equation of motion of the particle is now

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0 \cos \omega_f t \tag{252}$$

where the term on the R.H.S. is the driving or forcing term. To be noticed that the convention is different to that used in the Morin book, where the angular frequence of the external force is ω_0, and the proper oscillation frequency of the oscillator is ω. The general solution to this equation is the sum of the general solution to the homogeneous equation (zero on R.H.S., eq(232)) plus a **particular solution** to the full equation. It was shown above that the solution to the homogeneous equation eq(232) dies away exponentially (this is called the transient solution) so that after a sufficiently long time the motion of the particle is described by the particular solution alone.

Let's take as an ansatz a particular solution of the form

$$x(t) = R\cos(\omega_f t + \phi) = \Re[z(t)] \quad \text{with} \quad z(t) = R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)}. \tag{253}$$

That is one having the same frequency ω_f as the driving frequency but with an unknown phase difference ϕ and amplitude R. To determine R and ϕ substitute the assumed solution into the differential equation of motion. It's easiest to use the complex form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}\omega_f R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)}, \tag{254}$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\omega_f^2 R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)}, \tag{255}$$

so we have

$$-\omega_f^2 R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)} + 2\mathrm{i}\gamma\omega_f R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)} + \omega_0^2 R\mathrm{e}^{\mathrm{i}(\omega_f t+\phi)} = \frac{F_0}{m}\mathrm{e}^{\mathrm{i}\omega_f t} = f_0\mathrm{e}^{\mathrm{i}\omega_f t}, \tag{256}$$

where $f_0 = F_0/m$. Therefore

$$R\mathrm{e}^{\mathrm{i}\phi}[-\omega_f^2 + 2\mathrm{i}\gamma\omega_f + \omega_0^2] = f_0 \quad \Rightarrow \quad R\mathrm{e}^{\mathrm{i}\phi} = \frac{f_0}{(\omega_0^2 - \omega_f^2) + 2\mathrm{i}\gamma\omega_f} \tag{257}$$

Taking the modulus squared of this equation gives us (since R is real)

$$R^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2} \tag{258}$$

or

$$R = \frac{f_0}{\left[\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2\right]^{1/2}} = \frac{F_0}{m\left[\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2\right]^{1/2}}. \tag{259}$$

We can now investigate the variation of R with ω_f. It reaches its maximum value $R_{\max} = \frac{f_0}{2\gamma\omega}$ when $\omega_f^2 = \omega_0^2 - 2\gamma^2 = \omega^2 - \gamma^2$, i.e. near $\omega_f = \omega_o$. This is the phenomena of **resonance**.

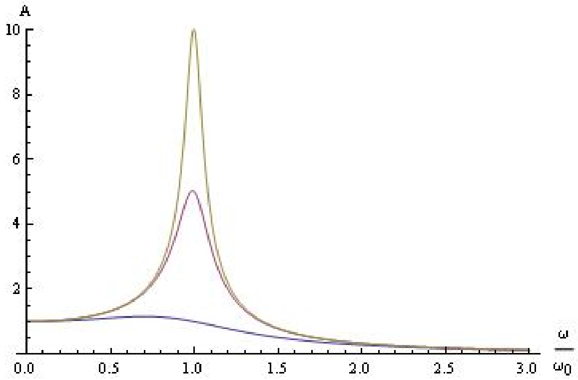

Figure 36: Amplitude of forced oscillator; Lower curve—heavy damping; Upper curve—light damping

Amplitude of the oscillation becomes much larger when the angular frequency of the driving force is equal to the natural frequency of the undamped oscillator as shown in fig 36. The resonance amplitude becomes larger and more sharply peaked in frequency as the damping is reduced. The amplitude is reduced to $1/\sqrt{2}$ times its peak value when $\omega_f = \omega_0 \pm \frac{1}{2}\Delta\omega$, where

$$|(\omega_0 \pm \frac{1}{2}\Delta\omega)^2 - \omega_0^2| \approx 2\gamma\omega_0 \quad \Rightarrow \quad |\Delta\omega|\omega_0 \approx 2\gamma\omega_0. \tag{260}$$

Hence the width $\Delta\omega$ of the resonance peak between the two points where the amplitude response drops by $1/\sqrt{2}$ (also between the points where $\tan\phi \approx 1$) is

$$\Delta\omega = 2\gamma = \frac{\omega_0}{Q}. \tag{261}$$

In other words the Q-factor directly tells us the width of resonance (as measured by the width of the central peak or the range over which the phase switches) relative to the position of its centre. Low damping corresponds to high Q and a high, sharp resonance.

Dramatic example - Tacoma Narrows bridge.

To find an expression for ϕ, we need to separate real and imaginary parts of equation (257).

$$R(\cos\phi + i\sin\phi) = f_0\frac{(\omega_0^2 - \omega_f^2) - 2\mathrm{i}\gamma\omega_f}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2} \tag{262}$$

$$R\cos\phi = \frac{f_0(\omega_0^2 - \omega_f^2)}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2} \tag{263}$$

$$R\sin\phi = \frac{-2f_0\gamma\omega_f}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2} \tag{264}$$

$$\tag{265}$$

and taking the ratio of the two expressions

$$\tan\phi = \frac{2\gamma\omega_f}{\left(\omega_f^2 - \omega_0^2\right)}. \tag{266}$$

Below the resonance frequency, $\omega_f < \omega_0$ the phase angle ϕ is fairly small as shown in fig 37. Therefore the displacement x is approximately **in phase** with the driving force. As ω_f increases through ω_0 the

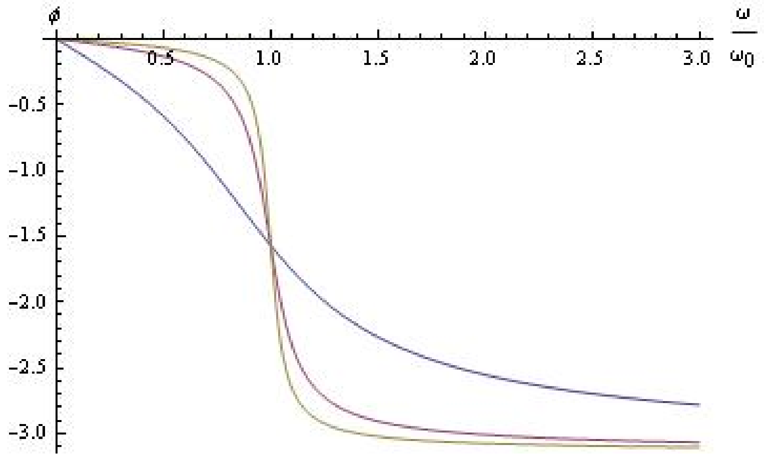

Figure 37: Phase of a forced oscillator as a function of the driving frequency.

phase angle drops rapidly (becomes more negative) and drops through $-\pi/2$ to $-\pi$, at which stage x is then out of phase with the driving force for $\omega_f \gg \omega_0$. At resonance, the displacement lags the applied force by $\pi/2$ and the velocity is therefore in phase with the driving force.

The work done by the driving force over one period $T_f = 2\pi/\omega_f$ of the forced oscillation is the integral of the power with respect to time:

$$W = \int_0^{T_f} F\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = -F_0 R\omega_f \int_0^{T_f} \cos(\omega_f t)\sin(\omega_f t + \phi)\,\mathrm{d}t.$$

Substitute $\omega_f t = \theta$, so $\mathrm{d}t = \mathrm{d}\theta/\omega_f$ and

$$W = -F_0 R \int_0^{2\pi} \cos\theta \sin(\theta + \phi)\,\mathrm{d}\theta = -F_0 R \int_0^{2\pi} \cos\theta[\sin\theta\cos\phi + \cos\theta\sin\phi]\,d\theta.$$

But

$$\int_0^{2\pi} \sin\theta\cos\theta\,\mathrm{d}\theta = 0, \qquad \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = \pi,$$

so

$$W = -\pi F_0 R \sin\phi$$

and the average power is

$$\overline{P} = \frac{W}{T_f} = \frac{\omega_f}{2\pi} W = -\frac{1}{2} F_0 R\omega_f \sin\phi = -\frac{1}{2} F_0 v_{\max}\sin\phi,$$

where $v_{\max} = R\omega_f$ is the maximum speed reached. Since

$$\sin\phi = \frac{-2R\gamma\omega_f}{f_0} = \frac{-2mv_{\max}\gamma}{F_0}$$

we can also write this as

$$\overline{P} = m\gamma v_{\max}^2 = 2K_{\max}\gamma,$$

where $K_{\max}$ is the maximum kinetic energy. This too is sharply peaked around $\omega_f = \omega_0$ for weak damping (see Figure 38)—in fact it is evident from equation (266) that both $\sin\phi$ and $v_{\max}$ peak when $\phi = -\pi/2$, which occurs exactly at $\omega_f = \omega_0$.

5.6 Oscillations in 2d

Let us consider now the case of a potential $V(x, y)$ that depends on two variables. Consider this potential to have a minimum in position (x_0, y_0). It will be possible to take the second derivative of this potential

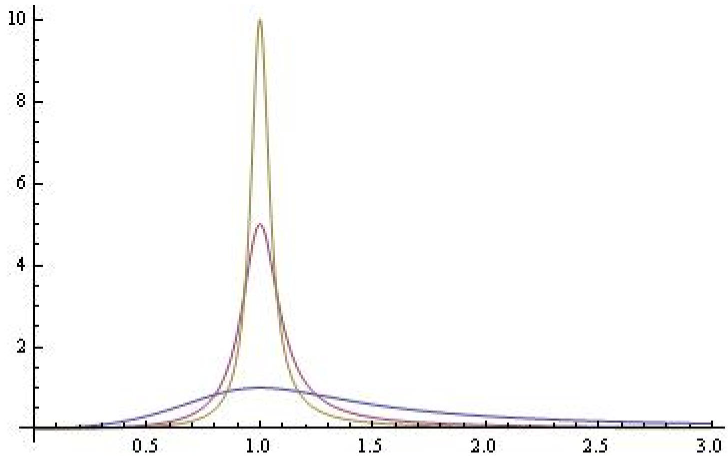

Figure 38: Average power dissipated in a forced oscillator as a function of the driving frequency.

with respect to x and y computed at that position, that we will call (k_x, k_y). The system can be decoupled in the two dimensions, with the potential near the equilibrium position being

$$V(x, y) \approx \frac{1}{2}k_x(x - x_0)^2 + \frac{1}{2}k_y(y - y_0)^2 \tag{267}$$

Notice that in the most general case, also a term proportional to $(x - x_0)(y - y_0)$ is present, but a proper rotation of the axes can be chosen to set this term to zero. The motion governed by the two equations

$$m\ddot{x} = -k_x(x - x_0) \tag{268}$$

$$m\ddot{y} = -k_y(y - y_0) \tag{269}$$

with solutions

$$x(t) = A_x \cos(\sqrt{k_x/m}t + \phi_x) \tag{270}$$

$$y(t) = A_y \cos(\sqrt{k_y/m}t + \phi_y) \tag{271}$$

with the constants A and ϕ determined from the initial conditions. The resulting motion will be a 2D figure (Lissajou figure) in the (x,y) plane, closed for if the $\sqrt{k_x/k_y}$ ratio is a rational number, very complex and cahotic otherwise. The special case $k_x = k_y$ leads to an ellipse, that collepses to a straight line for $\phi_x = \phi_y$, and to a circle for $A_x = A_y$.

Notice that if you have a mass attached to a spring in 2 dimensions with constant k, in a position with respect to the spring rest position (assumed for semplicity to be the origin of the coordinate system) described by the polar coordinates (r, θ), the two components of the elastic force in cartesian coordinates will be $-kr(\cos\theta, \sin\theta)$, so the two components of the force will be

$$F_x = -kr\cos\theta = -kx \tag{272}$$

$$F_y = -kr\sin\theta = -ky \tag{273}$$

so the system is decomposed in a 2d motion, and each component will have the same value of $\omega = \sqrt{k/m}$.

5.7 Coupled oscillators

Let us consider the case of two oscillators, coupled among them with another elastic force. An example are two masses connected to each other and to two walls via springs (figure 39.

For sake of simplicity, we assume the masses and elastic constants of the springs to be equal: masses are m and elastic constants k. Let $x_1(t)$ and $x_2(t)$ be the positions of the masses with respect to their equilibrium position. The net force acting on the first mass is $F_1 = -kx_1 + k(x_2 - x_1)$ and that acting on the second mass is $F_2 = -kx_2 - k(x_2 - x_1)$. Defining $\omega^2 = k/m$ and using Newton's law, we can write the differential equations for the two systems:

$$\frac{d^2x_1}{dt^2} + 2\omega^2 x_1 - \omega^2 x_2 = 0$$

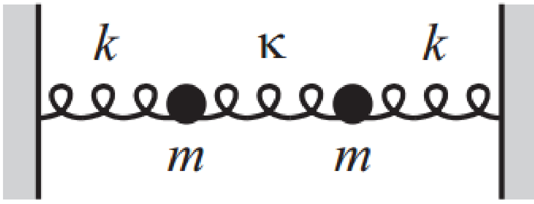

Figure 39: A coupled oscillator

$$\frac{d^2x_2}{dt^2} + 2\omega^2 x_2 - \omega^2 x_1 = 0$$

Taking the sums and differences of the two equations we get:

$$\frac{d^2(x_1+x_2)}{dt^2} + \omega^2(x_1+x_2) = 0$$
$$\frac{d^2(x_1-x_2)}{dt^2} + 3\omega^2(x_1-x_2) = 0$$

The solutions for the sums and differences are

$$x_1 + x_2 = A_+\cos(\omega t + \Phi_+)$$
$$x_1 - x_2 = A_-\cos(\sqrt{3}\omega t + \Phi_-)$$

leading to the solutions

$$x_1(t) = \frac{A_+}{2}\cos(\omega t + \Phi_+) + \frac{A_-}{2}\cos(\sqrt{3}\omega t + \Phi_-)$$
$$x_2(t) = \frac{A_+}{2}\cos(\omega t + \Phi_+) - \frac{A_-}{2}\cos(\sqrt{3}\omega t + \Phi_-)$$

Where A_+ and A_- have to be determined from the initial conditions; it is anyway interesting to notice that the motion of the two balls is a combination of two modes: one where the central sprint does not move, and the two masses move coherently with frequency ω, as independent oscillators, and one where the two masses move in a symmetrical way around their centre, each feeling the strength of the three springs and therefore having a frequency $\sqrt{3}\omega$.

Consider now the case where the elastic coefficient of the central spring is $2k$. The equations will become

$$\frac{d^2x_1}{dt^2} + 3\omega^2 x_1 - \omega^2 x_2 = 0$$
$$\frac{d^2x_2}{dt^2} + 3\omega^2 x_2 - \omega^2 x_1 = 0$$

taking again the difference we get

$$\frac{d^2(x_1+x_2)}{dt^2} + 2\omega^2(x_1+x_2) = 0$$

$$\frac{d^2(x_1-x_2)}{dt^2} + 4\omega^2(x_1-x_2) = 0$$

so the modes of oscillation would still be the sum and differnce of the positions, but with different frequences.

Let us treat the problem a bit more formally. The positions of the two masses can be described by a vector $\mathbf{x}$, and the elastic forces like a matrix.

$$m\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{K} = \begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}$$

We can find aigenvalue sand eigenvectors of the matrix

$$s_{1,2} = \frac{1}{\sqrt{2}}(x_1 \pm x_2)$$

$$\mathbf{D} = \begin{pmatrix} k & 0 \\ 0 & k+2k' \end{pmatrix}$$

so the original equation will be

$$m\ddot{\mathbf{s}} = -\mathbf{D}\mathbf{s}$$

so the normal vibrations $\mathbf{s}$ are harmonic oscillators, with frequencies respectively $\omega = \sqrt{k/m}$ and $\sqrt{(k+2k')/m}$. In the case where $k' = k$, the bottom-right term of the matrix is $3k$, and the second oscillations frequency will be $\sqrt{3k/m}$, as previously found.

A slightly more complicated case is when the system is asymmetric, due to different masses or different values of the elastic constant. Let us consider here the case where the elastic constants of the springs attached to the wall are different: k_1 different from k_2. Let us define

$$\bar{k} = (k_1 + k_2)/2, \Delta k = (k_1 - k_2)/2$$

such that

$$k_1 = \bar{k} + \Delta k; k_2 = \bar{k} - \Delta k$$

So the matrix $\mathbf{K}$ will be

$$\mathbf{K} = \begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix} = \begin{pmatrix} \bar{k} + \Delta k + k'' & -k' \\ -k' & \bar{k} - \Delta k + k' \end{pmatrix}$$

As before, the frequencies of the normal modes are defined by the eigenvalues of this matrix. We can define the eigenvalues as $e_\pm = m\omega_\pm^2$, and they can be found from

$$det(\mathbf{k} - \mathbf{ei}) = \begin{vmatrix} \bar{k} - k' - e + \Delta k & -k' \\ -k' & \bar{k} - \Delta K + K' \end{vmatrix}$$

so

$$(\bar{k} + k' - e)^2 - \Delta k^2 - k'^2 = e^2 - 2(\bar{k} + k')e + \bar{k}^2 + 2\bar{k}k' - \Delta k^2 = 0$$

so the aigenvalues (frequences of the normal vibration modes) are

$$m\omega_\pm^2 = e_\pm = \bar{k} = k' \pm \sqrt{k'^2 + \Delta k^2}$$

and eigenvectors can be found after them.

6 Angular momentum, and motion in a central force

6.1 Motion in a plane expressed in plane polar coordinates

Particle of mass m is at position P defined by Cartesian coordinates (x, y) or polar coordinates (r, θ) or vector $\mathbf{r}$ as shown in Figure 40.

Let us introduce a unit radial vector $\hat{\mathbf{r}}$ and unit transverse vector $\hat{\theta}$, which are orthogonal (perpendicular) to each other. The position vector of P is $\mathbf{r} = r\hat{\mathbf{r}}$.

In terms of unit vectors along the fixed Cartesian axes

$$\widehat{\mathbf{r}} = \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}} \quad (274)$$

$$\widehat{\theta} = -\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}. \quad (275)$$

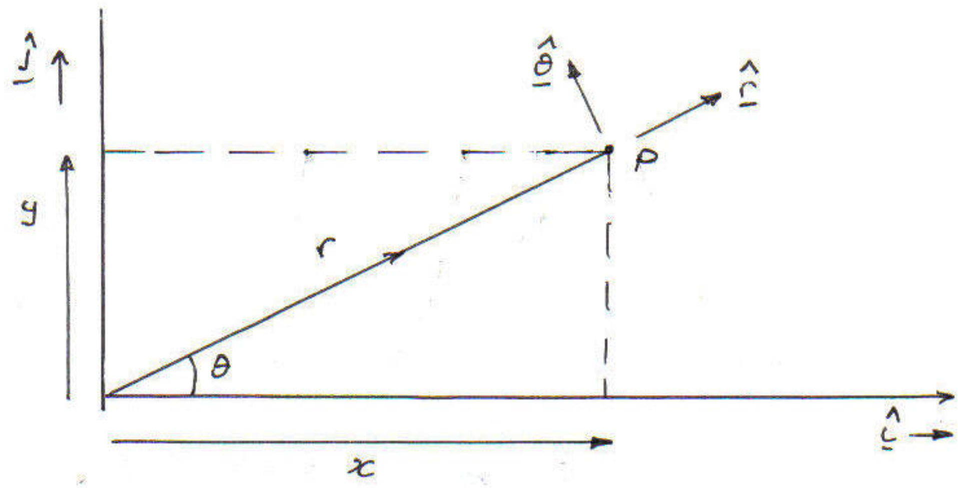

Figure 40: Unit vectors for 2D motion in polar coordinates.

As the particle moves the **directions** of $\widehat{\mathbf{r}}$ and $\widehat{\theta}$ **vary**.

Consider the velocity of the particle: we now have to account for the changes in the unit vectors with time.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\widehat{\mathbf{r}}) = \frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\mathbf{r}} + r\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t}. \tag{276}$$

An expression for $\mathrm{d}\widehat{\mathbf{r}}$ can be found algebraically or graphically as shown in fig 41.

Algebraically we have (using the 'dot' notation for derivatives for brevity):

$$\begin{aligned} \widehat{\mathbf{r}} &= \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}, & (277)\\ \dot{\widehat{\mathbf{r}}} &= -\sin\theta\dot{\theta}\widehat{\mathbf{i}} + \cos\theta\dot{\theta}\widehat{\mathbf{j}} & (278)\\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\left(-\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}\right) & (279)\\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}. & (280) \end{aligned}$$

So

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\widehat{\mathbf{r}} + r\dot{\theta}\hat{\theta}. \tag{281}$$

In other words, the radial component of $\mathbf{v}$ is $\frac{\mathrm{d}r}{\mathrm{d}t}$; transverse component is $r\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

The speed of the particle is

$$v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \tag{282}$$

Consider acceleration of the particle

$$\begin{aligned} \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\mathbf{r}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\theta}\right) & (283)\\ &= \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\widehat{\mathbf{r}} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\theta} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\widehat{\theta} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t}. & (284) \end{aligned}$$

The time derivative $\frac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t}$ can be found also algebraically or graphically, as illustrated in fig 41.

$$\begin{aligned} \widehat{\theta} &= -\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}, & (285)\\ \frac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t} &= -\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{i}} - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{j}}, & (286)\\ \frac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t} &= -\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}\right) = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{r}}. & (287) \end{aligned}$$

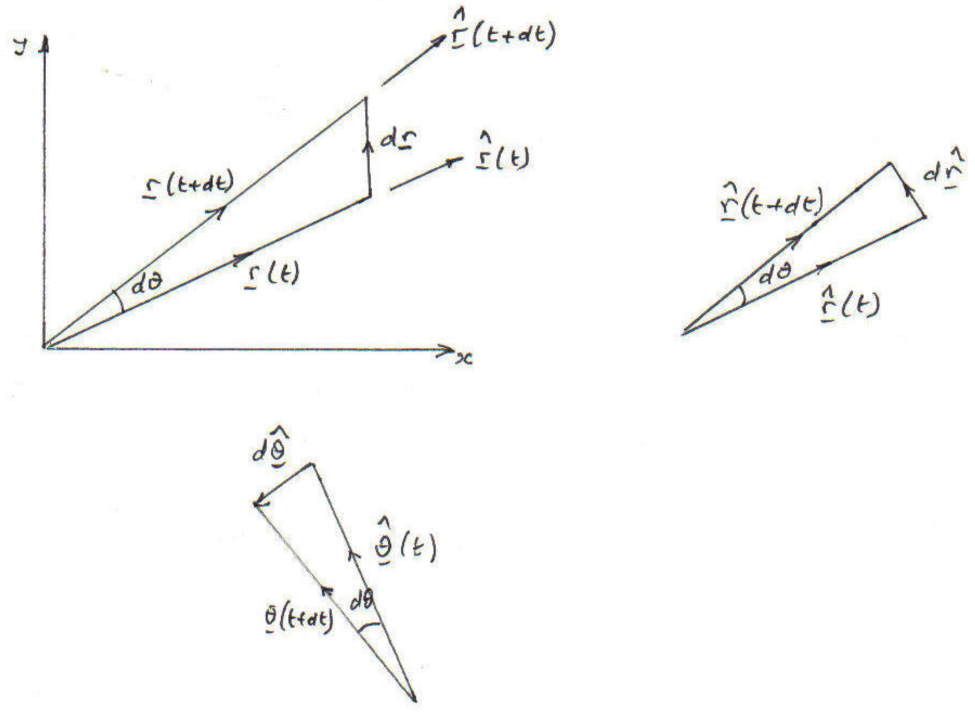

Figure 41: Time dependence of polar vectors.

Hence the acceleration

$$\begin{aligned}
\mathbf{a} &= \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\widehat{\mathbf{r}} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\theta} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\theta} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\widehat{\theta} - r\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{r}}, && (288)\\
&= \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\widehat{\mathbf{r}} + \left[2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right]\widehat{\theta}, && (289)\\
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\theta}. && (290)
\end{aligned}$$

The radial component of the acceleration is $\left(\ddot{r} - r\dot{\theta}^2\right)$, the transverse component is $\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$.

6.2 Circular motion

Consider a particle moving in a circle of radius r as shown in Figure 42.

Then r is constant, so $\dot{r} = 0$, $\ddot{r} = 0$. Hence

$$\mathbf{v} = r\dot{\theta}\widehat{\theta} \tag{291}$$

and the velocity is purely transverse.

We often write $\dot{\theta}$ as the **angular velocity** $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$, in which case

$$\mathbf{v} = r\omega\widehat{\theta}. \tag{292}$$

The acceleration

$$\mathbf{a} = \left(-r\dot{\theta}^2\right)\widehat{\mathbf{r}} + r\ddot{\theta}\widehat{\theta}. \tag{293}$$

If the angular velocity is constant, $\dot{\omega} = \ddot{\theta} = 0$, and

$$\mathbf{a} = -r\dot{\theta}^2\widehat{\mathbf{r}} = -r\omega^2\widehat{\mathbf{r}}. \tag{294}$$

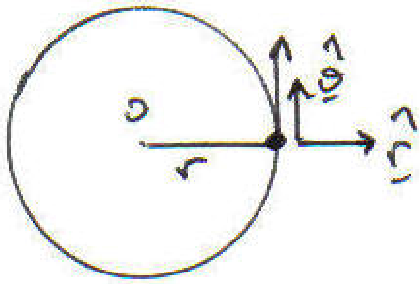

Figure 42: Motion in a circle.

Since $\mathbf{v} = r\omega\widehat{\theta}$ then for the magnitudes $v = \omega r$ and so

$$\mathbf{a} = -\frac{v^2}{r}\widehat{\mathbf{r}} \tag{295}$$

and the acceleration is directed towards the centre of the circle. This acceleration is called the **centripetal acceleration**. It follows that there **must** be a force $\mathbf{F} = m\mathbf{a} = -\, m\omega^2 r\widehat{\mathbf{r}} = -m\frac{v^2}{r}\widehat{\mathbf{r}}$ acting radially. This is the **centripetal force**.

6.3 Angular momentum and torques

Particle of mass m has velocity $\mathbf{v}$ at position vector $\mathbf{r}$ as in fig 43.

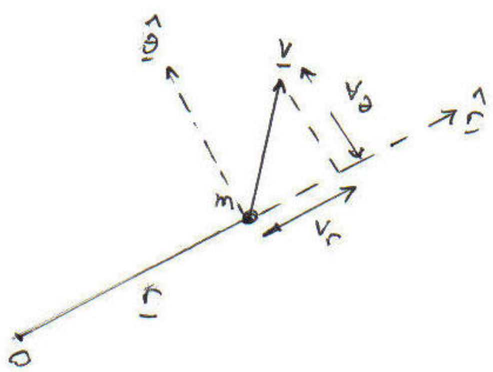

Figure 43: Position and velocity vectors for rotational motion

The **angular momentum is defined** by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} \tag{296}$$

about point O. The direction of $\mathbf{L}$ is given by the usual right-hand rule for a vector product. Note that this definition means that $\mathbf{L}$ is a vector; it has a direction associated with it, which is related to the direction of rotation (see below).

Consider a force $\mathbf{F}$ acting on the particle. The **torque** or **moment of force** about point O is **defined** by

$$\tau = \mathbf{r} \times \mathbf{F}. \tag{297}$$

But

$$\mathbf{F} = \mathbf{m}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{298}$$

so

$$\tau = m\mathbf{r} \times \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{299}$$

Consider

$$\begin{aligned}
\frac{\mathrm{d}(\mathbf{r}\times\mathbf{v})}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{v}+\mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &(300)\\
&= \mathbf{v}\times\mathbf{v}+\mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0+\mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &(301)\\
\frac{\mathrm{d}(\mathbf{r}\times\mathbf{v})}{\mathrm{d}t} &= \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &(302)
\end{aligned}$$

and therefore

$$\tau = m\mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\frac{\mathrm{d}(\mathbf{r}\times\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{r}\times\mathbf{v})}{\mathrm{d}t} \tag{303}$$

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}. \tag{304}$$

Hence **torque is equal to the rate of change of angular momentum**. This is analogous to the linear case of force equal to rate of change of linear momentum, i.e. Newton's second law. If $\tau = 0$ then $\mathbf{L}$ is constant in magnitude and direction. If $\tau = 0$ then either force $\mathbf{F} = 0$ or $\mathbf{r}\times\mathbf{F} = 0$ in which case force $\mathbf{F}$ and $\mathbf{r}$ are parallel and the force is a central force. Thus $\mathbf{L}$ is constant for any central force, $\mathbf{F}(\mathbf{r}) = F(r)\,\widehat{\mathbf{r}}$.

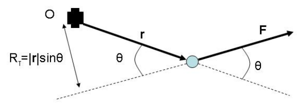

Figure 44: Torque (cross product of force and displacement) as the moment of the force.

Note that both torque and angular momentum can be expressed in terms of the **moments** of the force and linear momentum respectively about the rotation axis. For torque:

$$|\tau| = |\mathbf{F}||\mathbf{r}|\sin\theta = |\mathbf{F}||\mathbf{r}_\perp|, \tag{305}$$

where $|\mathbf{r}_\perp|$ is the perpendicular distance from the origin to the line of action of the force.

6.3.1 Angular momentum in polar coordinates

If the motion is known to be in a plane (i.e. both $\mathbf{r}$ and $\mathbf{p}$ lie in the plane) then the angular momentum vector is perpendicular to that plane. If the motion is in the xy-plane, then the angular momentum along the z direction is equal to the distance from the origin multiplied by the angular component of the momentum:

$$L_z = rv_\theta = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{306}$$

Note the direction: for anticlockwise rotation in the xy-plane (positive $\frac{\mathrm{d}\theta}{\mathrm{d}t}$), L_z is positive, consistent with what we would expect from the right-hand rule for vector products.

Since L_z is the only non-zero component of $\mathbf{L}$ for motion in a plane, we will write $L_z = L$ for the rest of this section.

6.3.2 Angular momentum and reduced mass

Consider the total angular momentum of two objects about their common centre of mass, viewed in the centre-of-mass frame. In the notation we used when discussing collision problems, it is

$$\mathbf{L} = \mathbf{r}'_1\times\mathbf{p}'_1+\mathbf{r}'_2\times\mathbf{p}'_2. \tag{307}$$

But we saw previously that

$$\mathbf{p}_1' = -\mathbf{p}_2' = \mu \mathbf{v}, \tag{308}$$

where μ is the reduced mass and $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity. It follows that

$$\mathbf{L} = \mu(\mathbf{r}_1' - \mathbf{r}_2') \times \mathbf{v} = \mu \mathbf{r} \times \mathbf{v}, \tag{309}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In other words the angular momentum (as with the linear momentum and kinetic energy discussed earlier) is the same as that of a single particle, whose mass is the reduced mass μ, whose position is $\mathbf{r}$ and whose velocity is $\mathbf{v}$.

6.4 Central forces

A central force is one which is directed directly towards or away from a central point (the **center of force**), which we usually choose as the origin of coordinates. If we further suppose that the force depends only on the radius (distance from the centre of force), we have

$$\mathbf{F}(\mathbf{r}) = F(r)\,\widehat{\mathbf{r}}. \tag{310}$$

If $F(r)$ is **positive** the force is **repulsive** or **centrifugal**; if $F(r)$ is **negative** the force is **attractive** or **centripetal**. Examples of central forces are

(a) electrostatic force between point or spherically symmetric charges:

$$\mathbf{F} = \frac{q_1 q_2}{2\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{311}$$

Like charges repel and unlike charges attract.

(b) gravitational force

$$\mathbf{F}(\mathbf{r}) = -G\frac{M_1 M_2}{r^2}\hat{\mathbf{r}} \tag{312}$$

between point or spherically symmetric masses; this is always attractive.

(c) tension in a string or spring.

The expressions above for velocity (see fig 45) and acceleration in plane polar coordinates are particularly important in the case of central forces.

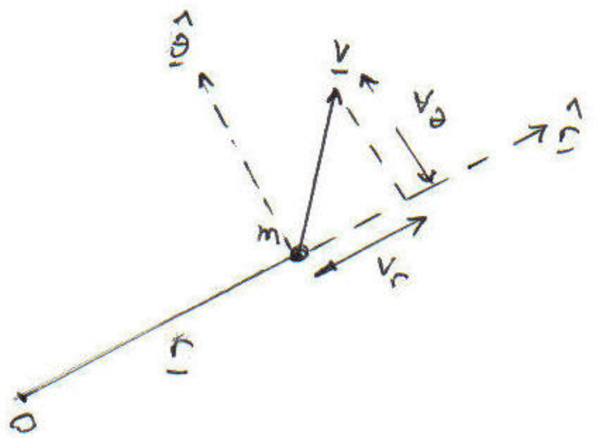

Figure 45: Vectors for rotational motion.

The equation of motion is

$$m\mathbf{a} = \mathbf{F} = F(r)\,\widehat{\mathbf{r}}, \tag{313}$$

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\theta}\right] = F(r)\,\widehat{\mathbf{r}} \tag{314}$$

We see that that the force **only** enters into the radial component,

$$m\left(\ddot{r}-r\dot{\theta}^{2}\right)=F\left(r\right) \tag{315}$$

and the transverse component

$$m\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=0. \tag{316}$$

is independent of the nature of the force. One may therefore make significant deductions about the motion regardless of the force. The radial component of velocity is $v_r=\dot{r}$ and the transverse component is $v_\theta=r\dot{\theta}$. Recall that the z-component (out of plane) of the angular momentum is

$$L=\left(mv_\theta\right)r=mr^2\dot{\theta}. \tag{317}$$

$$L = mr^2\dot{\theta} \tag{318}$$

$$\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right)=m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right) \tag{319}$$

$$= mr\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=mra_\theta \tag{320}$$

where we have used the expression for the transverse acceleration $a_\theta=2\dot{r}\dot{\theta}+r\ddot{\theta}$. But for a central force the transverse acceleration must be zero, so

$$\frac{\mathrm{d}L}{\mathrm{d}t}=0, \tag{321}$$

and hence the angular momentum L is **constant** during the motion for **any central force**.

6.4.1 Example—motion in a circle

Example 43 *Consider a planet, mass m, moving in a circular orbit about a star of mass M as in fig 46.*

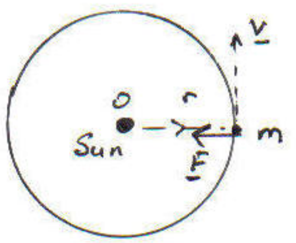

Figure 46: Planet in circular motion about Sun

The centripetal force is due to gravity,

$$\mathbf{F}\left(\mathbf{r}\right) = -G\frac{Mm}{r^2}\widehat{\mathbf{r}} \tag{322}$$

$$-G\frac{Mm}{r^2}\widehat{\mathbf{r}} = m\left(-r\dot{\theta}^{2}\right)\widehat{\mathbf{r}} \tag{323}$$

$$r\dot{\theta}^{2} = \frac{GM}{r^2} \tag{324}$$

$$\dot{\theta}^{2} = \omega^2=\frac{GM}{r^3}. \tag{325}$$

The period of rotation $T = 2\pi/\omega$, so

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{r^3} \tag{326}$$

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{327}$$

$$T^2 \propto r^3, \tag{328}$$

as expressed in Kepler's third law of planetary motion.

Speed of the planet is $v = r\omega$, so

$$v = \sqrt{\frac{GM}{r}}. \tag{329}$$

Example 44 *Consider a car of mass m moving in a circular path of radius r on a horizontal surface with coefficient of friction μ as in fig 47. What is the maximum speed without skidding sideways (ignore toppling over for the moment)?*

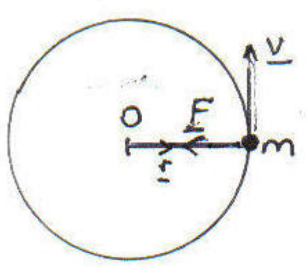

Figure 47: Car in circular motion on horizontal track

The centripetal force necessary for it to travel round the circular path is

$$\mathbf{F} = m\left(-\frac{v^2}{r}\right)\widehat{\mathbf{r}} \tag{330}$$

and this must be provided by the frictional force between the wheels and the ground. So as the normal reaction $N = mg$, then

$$m\left(-\frac{v^2}{r}\right)\widehat{\mathbf{r}} = -\mathbf{F}_{friction} \tag{331}$$

$$m\left(\frac{v^2}{r}\right) = \mu mg \tag{332}$$

$$v_{\max} = \sqrt{\mu g r}. \tag{333}$$

If $\mu = 1$, $g \simeq 10\mathrm{ms}^{-2}$ then $v_{\max} \simeq \sqrt{10r}\mathrm{ms}^{-1}$. Thus for $r = 20\mathrm{m}$, $v_{\max} \simeq 14\mathrm{ms}^{-1} \simeq 31$ mph.

Example 45 *(From 2000 exam paper, question 11.)*

Particle of mass m moves on a smooth horizontal table subjected to a force

$$\mathbf{F} = -\frac{K}{r^3}\widehat{\mathbf{r}}. \tag{334}$$

If particle is initially moving in a circle of radius r as in fig 48we will determine the speed of the particle and its angular momentum.

For this motion,

$$\mathbf{F} = -\frac{K}{r^3}\widehat{\mathbf{r}} = m\left(-r\dot{\theta}^2\right)\widehat{\mathbf{r}}, \tag{335}$$

so

$$\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}. \tag{336}$$

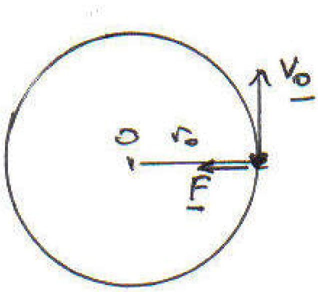

Figure 48: Motion in a circle

The initial value of the angular momentum at $r = r_0$ *is*

$$L = mr^2\dot{\theta} = \sqrt{mK}, \tag{337}$$

is independent of r_0 *and so is the only possible value for the angular momentum for motion in a circle under an inverse cube force law.*

Suppose particle is given a radially outward impulse. The impulse does not change the angular momentum so it remains at a value $\sqrt{mK}$ *for any subsequent motion. Radial equation of motion after the impulse is*

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{K}{r^3}. \tag{338}$$

But $\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}$ *so*

$$m\left[\ddot{r} - r\left(\frac{1}{r^2}\sqrt{\frac{K}{m}}\right)^2\right] = -\frac{K}{r^3}, \tag{339}$$

$$m\ddot{r} = 0. \tag{340}$$

Hence $\dot{r}$ *is constant. So particle moves in a spiral with constant radial component of velocity as illustrated in fig 49.*

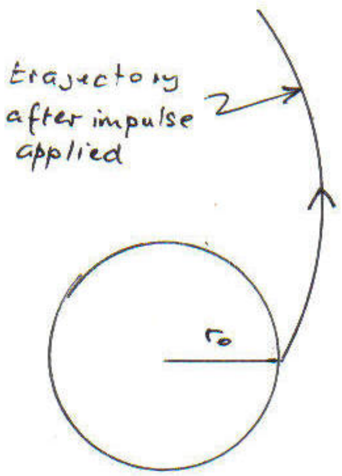

Figure 49: Trajectory of particle

Example 46 *Motion of a particle attached to a string and moving in a vertical circle, radius R, under gravity, see fig 50. Assume string remains taut throughout the motion.*

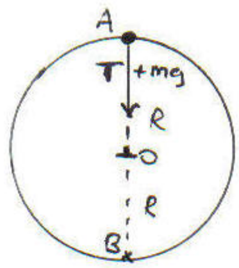

Figure 50: Motion of mass on string swung in circle in vertical plane

At highest point, A, force is

$$\mathbf{F} = -\left(T_A + mg\right)\widehat{\mathbf{r}} = m\left(-R\dot{\theta}_A^2\right)\widehat{\mathbf{r}} \tag{341}$$

giving$_A$

$$\dot{\theta}_A = \sqrt{\frac{(T_A + mg)}{mR}}. \tag{342}$$

The velocity at A

$$v_A = R\dot{\theta}_A = \sqrt{\frac{(T_A + mg)\,R}{m}}. \tag{343}$$

The minimum speed at A is when the tension $T_A = 0$*, i.e.*

$$v_{A\,\mathrm{min}} = \sqrt{gR}. \tag{344}$$

By conservation of energy at lowest point B

$$\frac{1}{2}mv_B^2 = \frac{1}{2}mv_A^2 + mg\,(2R) \tag{345}$$

$$v_B^2 = v_A^2 + 4mgR. \tag{346}$$

At lowest point B the tension in the string T_B*, is given by*

$$\left(-T_B + mg\right)\widehat{\mathbf{r}} = m\left(-R\dot{\theta}_B^2\right)\widehat{\mathbf{r}} \tag{347}$$

$$T_B - mg = m\left(R\dot{\theta}_B^2\right) = m\frac{v_B^2}{R} \tag{348}$$

$$= \frac{m}{R}\left(v_A^2 + 4mgR\right). \tag{349}$$

If $v_A = v_{A\,\mathrm{min}} = \sqrt{gR}$ *then*

$$T_B = mg + \frac{m}{R}\left(gR + 4mgR\right) = 6mg. \tag{350}$$

6.5 Potential energy for a central force

A central force whose magnitude depends only on the distance from the origin

$$\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$$

is always conservative. To see this, consider the work done around the loop 12341 shown in Figure 51. The total work done is

$$\int_1^2 \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_2^2 \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_3^4 \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_4^1 \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{r_1}^{r_2} F(r)\,\mathrm{d}r + 0 - \int_{r_1}^{r_2} F(r)\,\mathrm{d}r + 0 = 0. \tag{351}$$

Any path can be decomposed into a series of such loops, so the total work done is always zero.

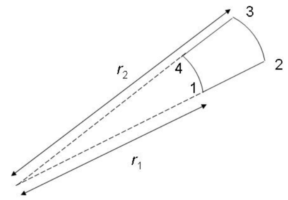

Figure 51: The work done around the loop 12341 is always zero for a central force depending only on the distance from the origin.

The potential energy also depends only on the distance from the origin, and can be written as

$$V(r) = -\int F(r)\,\mathrm{d}r$$

so

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}.$$

Example 47 *The potential energy for particle of mass m interacting gravitationally with a fixed mass M is*

$$V(r) = -\int F(r)\,\mathrm{d}r = -\int \frac{GMm}{r^2}\,\mathrm{d}r = -\frac{GMm}{r} + C. \tag{352}$$

As usual we can choose the constant of integration C; this is usually done so the potential energy goes to zero as $r \to \infty$, i.e. as the two masses become very far apart. In that case we have $C = 0$ and

$$V(r) = -\frac{GMm}{r}. \tag{353}$$

Note once again the connection between a symmetry and a conservation law: for a central force the potential depends on r alone and is independent of angle, so the potential energy is left the same by **rotating** the system. It's in exactly this situation that we saw angular momentum is conserved.

7 Centre of mass and collision problems

Reading:

- Jewett and Serway Chapter 9;
- Kibble and Berkshire Chapter 7 (though the approach is a little different);
- Kleppner and Kolenkow SS3.2, 3.3 and 4.14.

7.1 Centre of mass

The *centre of mass* of a set of masses is defined as the average of the positions of the individual masses, weighted by the magnitude of the masses:

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \tag{354}$$

where $M = \sum_i m_i$.

The total momentum can be easily written in terms of the velocity of the centre of mass:

$$M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \sum_i m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_i \mathbf{p}_i = \mathbf{P}. \tag{355}$$

Hence the centre of mass accelerates only in response to the external force on the system:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \mathbf{F}_{\mathrm{ext}} \tag{356}$$

and its motion is as if the whole mass M of the system of particles was concentrated at that point.

In particular in the absence of external forces (i.e. when $\mathbf{P}$ is conserved) the velocity of the centre of mass is constant:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = 0 \quad \Rightarrow \quad \mathbf{V} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \text{constant if} \quad \mathbf{F}_{\mathrm{ext}} = 0. \tag{357}$$

It is often convenient to measure positions and velocities *relative to the centre of mass*, especially in collision problems. We often refer to this as working 'in the centre-of-mass frame' or 'in the centre-of-momentum frame'; the original positions and velocities relative to an external laboratory observer are sometimes said to be 'in the laboratory frame'. The transformation from the lab to the centre-of-mass frame is an example of a Galilean transformation (uniform velocity boost) and Newton's laws are exactly the same in the CM frame provided only internal forces act between the particles that depend only on their relative positions and velocities.

The new positions and velocities are then

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R} \tag{358}$$

and

$$\mathbf{v}'_i = \frac{\mathrm{d}\mathbf{r}'_i}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{v}_i - \mathbf{V}. \tag{359}$$

The corresponding momenta in the centre of mass (or centre of momentum) frame are

$$\mathbf{p}'_i = m_i\mathbf{v}'_i = m_i(\mathbf{v}_i - \mathbf{V}). \tag{360}$$

In the centre of mass frame, the total momentum is zero:

$$\sum_i m_i\mathbf{v}'_i = \sum_i m_i(\mathbf{v}_i - \mathbf{V}) = \sum_i m_i\mathbf{v}_i - M\mathbf{V} = 0, \tag{361}$$

i.e. the total momentum of the system is entirely 'carried by the centre of mass'. This means that for a two-body collision, the velocities of the component particles in the centre-of-mass frame must always be in opposite directions.

The kinetic energy can also be nicely written in terms of the centre of mass:

$$\begin{aligned} K = \sum_i \frac{m_i v_i^2}{2} &= \sum_i \frac{m_i(\mathbf{v}_i - \mathbf{V} + \mathbf{V})^2}{2} \\ &= \sum_i \frac{m_i(\mathbf{v}'_i + \mathbf{V})^2}{2} \\ &= \sum_i \frac{m_i v_i'^2}{2} + \sum_i m_i(\mathbf{v}'_i)\cdot\mathbf{V} + \sum_i \frac{m_i V^2}{2}. \end{aligned}$$

But from equation (361)

$$\sum_i m_i\mathbf{v}'_i = 0,$$

so

$$K = \sum_i \frac{m_i v_i'^2}{2} + \sum_i \frac{m_i V^2}{2} = K_{\mathrm{rel}} + K_{\mathrm{cm}}. \tag{362}$$

The kinetic energy is equal to the sum of K_{cm} (the kinetic energy that the system would have if the total mass M were moving at the centre-of-mass velocity $\mathbf{V}$) and K_{rel} (the kinetic energy coming from the motion of the component bodies relative to the centre of mass).

7.2 Relative displacement and reduced mass

For collisions or other problems involving just two objects it is often convenient to define a *relative position*, the position of particle 1 relative to particle 2:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{363}$$

and a corresponding relative velocity

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2. \tag{364}$$

This is the same in the centre-of-mass frame:

$$\mathbf{r} = \mathbf{r}'_1 - \mathbf{r}'_2.$$

But since $m_1\mathbf{r}'_1 + m_2\mathbf{r}'_2 = 0$, we can recover $\mathbf{r}'_1$ and $\mathbf{r}'_2$ knowing just $\mathbf{r}$:

$$\mathbf{r}'_1 = \frac{m_2}{M}\mathbf{r}; \qquad \mathbf{r}'_2 = \frac{-m_1}{M}\mathbf{r}. \tag{365}$$

and similarly

$$\mathbf{v}'_1 = \frac{m_2}{M}\mathbf{v}; \qquad \mathbf{v}'_2 = \frac{-m_1}{M}\mathbf{v}. \tag{366}$$

The momentum in the CM frame is then

$$\mathbf{p}'_1 = m_1\mathbf{v}'_1 = -m_2\mathbf{v}'_2 = \frac{m_1 m_2}{M}\mathbf{v} = \mu\mathbf{v}, \tag{367}$$

where μ is the **reduced mass**

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{368}$$

Suppose the force on particle 1 from particle 2 is $\mathbf{F}$; we suppose this is the only force, since no external forces are acting and therefore $\mathbf{P}$ is conserved. Then the equation of motion will be

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}'_1}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{p}_1}{\mathrm{d}t} = m_1\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = \mu\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mu\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}. \tag{369}$$

In other words the motion is equivalent to that of a single particle of the reduced mass μ, whose position vector is equal to the relative displacement of the true physical particles 1 and 2.

Similarly the relative part of the kinetic energy is

$$K_{\mathrm{rel}} = \frac{1}{2}(m_1 {v'_1}^2 + m_2 {v'_2}^2) = \frac{v^2}{2M^2}(m_1 m_2^2 + m_2 m_1^2) = \frac{v^2}{2M} m_1 m_2 = \frac{1}{2}\mu v^2. \tag{370}$$

7.3 Single-body collision with a rigid wall

If wall is smooth, i.e. no friction, then impulse given to the particle by the wall is perpendicular to the wall, there is no component parallel to the wall's surface. as in fig 52 Hence tangential component of velocity is unaltered so

$$v_i \sin\alpha = v_f \sin\beta. \tag{371}$$

If the collision is perfectly elastic, i.e. no loss of kinetic energy,

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 \tag{372}$$

and so

$$v_i = v_f, \tag{373}$$
$$\alpha = \beta. \tag{374}$$

Thus the normal component of velocity is simply reversed in the collision.

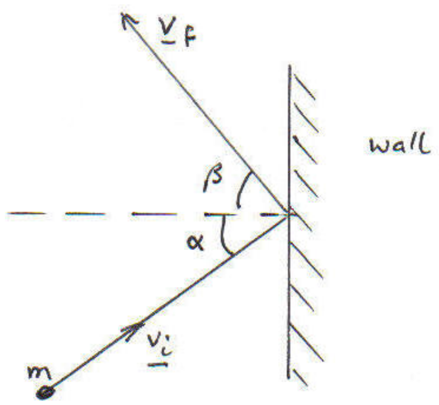

Figure 52: Collision of a small mass with a rigid wall.

7.4 Coefficient of restitution

In practice the normal component of velocity is reduced by a factor $e < 1$ (called coefficient of restitution), such that

$$v_f \cos\beta = e v_i \cos\alpha. \tag{375}$$

Since

$$v_f \sin\beta = v_i \sin\alpha \tag{376}$$

and

$$\tan\beta = \frac{\tan\alpha}{e}, \tag{377}$$

so $v_f < v_i$ and $\beta > \alpha$. There is a loss of kinetic energy since

$$v_f^2 = v_i^2(\sin^2\alpha + e^2\cos^2\alpha) = v_i^2 + v_i^2\left(e^2 - 1\right)\cos^2\alpha \tag{378}$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}mv_i^2\left(e^2 - 1\right)\cos^2\alpha < 0. \tag{379}$$

For an elastic collision, $e = 1$, and kinetic energy is conserved

7.5 Collision between two bodies of finite mass, one initially at rest

No external forces act during the collision so the total momentum is conserved (always true).

7.5.1 Head-on collision

Figure 53: Head-on elastic collision

Consider this collision in the centre-of-mass frame, assume all velocities lie in the same direction. Take just components of vectors along direction of motion. Initial velocity of centre of mass is

$$V = \frac{m_1}{M}u_1.$$

So relative to centre of mass, initial velocities are

$$u_1' = (1 - \frac{m_1}{M})u_1 = \frac{m_2}{M}u_1; \quad u_2' = -V = -\frac{m_1}{M}u_1 = -\frac{m_1}{m_2}u_1'.$$

(note initial relative velocity is equal to u_1), and initial relative kinetic energy is

$$\frac{1}{2}(m_1 {u_1'}^2 + m_2 {u_2'}^2) = \frac{\mu {u_1}^2}{2}.$$

Final velocities along same line; let final relative velocity be v, then

$$v_1' = \frac{m_2}{M}v, \qquad v_2' = -\frac{m_1}{M}v.$$

Final relative kinetic energy is

$$\frac{1}{2}(m_1 {v_1'}^2 + m_2 {v_2'}^2) = \frac{\mu v^2}{2}.$$

For an elastic collision, kinetic energy is conserved, only solution is $v = \pm u_1$. Positive sign corresponds to situation before collision, so we have

$$v = -u = -u_1 \quad \Rightarrow \quad v_1' = \frac{m_2}{M}(-u_1) = -\frac{m_2}{M}u_1 \text{ and } \quad v_2' = -\frac{m_1}{M}(-u_1) = \frac{m_1}{M}u_1$$

and therefore

$$v_1 = v_1' + V = -\frac{m_2}{M}u_1 + \frac{m_1}{M}u_1 = \frac{m_1 - m_2}{M}u_1; \;\; v_2 = v_2' + V = \frac{m_1}{M}u_1 + \frac{m_1}{M}u_1 = \frac{2m_1}{M}u_1.$$

Special cases:

- If the masses are equal, $m_1 = m_2$, we get

  $$v_1 = 0, \quad v_2 = u_1,$$

  so all the momentum is transferred from the first particle to the second.

  Example 48 *Newton's cradle.*

- If $m_2 \gg m_1$ then

  $$v_1 \approx -u_1; \qquad v_2 \approx 0,$$

  and the problem becomes like a collision with a rigid 'wall'.

- If $m_1 \gg m_2$, then

  $$v_1 \approx u_1; \qquad v_2 \approx 2u_1.$$

Example 49 *Two objects are dropped from rest at a height h_0, object 2 directly above object 1. Object 1 collides elastically with the ground, then immediately collides (also elastically) with m_2. If $m_1 \gg m_2$, what heights do the two balls reach as they rebound?*

Both balls acquire a velocity $v = -\sqrt{2gh_0}$ (taking velocities positive upwards) immediately before impact with the ground. Immediately after rebound object 1 has velocity $u_1 = +v$, while its relative velocity to the second balls is $v - (-v) = 2v$. Since the collision is elastic the relative velocity is reversed after the collision, but since $M \gg m$ the velocity of the first ball is almost unaffected. Hence after the collision with the second ball the velocities are $v_1 \approx u_1 = v$ and $u_2 \approx= 3v$. Hence the heights reached are

$$h_1 = \frac{v_1^2}{2g} = h_0; \qquad h_2 = \frac{v_2^2}{2g} = 9h_0.$$

(Ask about conservation of energy!)

For inelastic collisions with coefficient of restitution e we get

$$v_1' = -eu_1' \quad v_2' = -eu_2',$$

and so

$$v_1 = v_1' + V = -\frac{em_2}{M}u_1 + \frac{m_1}{M}u_1 = \frac{m_1 - em_2}{M}u_1; v_2 = v_2' + V = \frac{em_1}{M}u_1 + \frac{m_1}{M}u_1 = \frac{(1+e)m_1}{M}u_1.$$

The case $e = 0$ gives the 'sticking collision' discussed above.

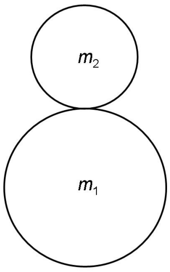

Figure 54: Two falling balls with $m_1 \gg m_2$.

7.5.2 Glancing collision—general case

In the CM frame the objects approach each other with relative speed u and recede with relative speed v. For elastic collisions

$$\frac{\mu u^2}{2} = \frac{\mu v^2}{2} \quad \Rightarrow \quad v = \pm u \tag{380}$$

and the relative speed of recession is equal to the relative speed of approach (but in a different direction); for inelastic collisions with coefficient of restitution e we have

$$v = eu \quad \Rightarrow \quad \frac{\mu v^2}{2} = e^2 \frac{\mu u^2}{2} \tag{381}$$

so the relative kinetic energy is reduced by a factor of e^2 (though the KE of the centre of mass is unaffected) and the speed of recession is less than the speed of approach. In either case, if we consider collisions with all possible outcomes (i.e. all possible directions for the final velocity $\mathbf{v}$) the final velocities of the balls in the CM frame

$$\mathbf{v}_1' = \frac{m_2}{M}\mathbf{v}; \qquad \mathbf{v}_2' = -\frac{m_1}{M}\mathbf{v} \tag{382}$$

lie on two circles, of radii $\frac{m_2}{M}eu$ and $\frac{m_1}{M}eu$ respectively.

- In the case $m_1 = m_2$ the two circles coincide;
- In the case $e = 1$ (elastic collision) they have radius $u/2$.

Now consider adding back the CM velocity to recover the final velocities in the lab frame

$$\mathbf{v}_1 = \mathbf{v}_1' + \mathbf{V}; \qquad \mathbf{v}_2 = \mathbf{v}_2' + \mathbf{V}. \tag{383}$$

The centre of the circle is therefore displaced through a vector $\mathbf{V}$.

- If $m_1 = m_2$ and $e = 1$ and mass m_2 is initially stationary, the initial velocity of the COM is $\mathbf{V} = \frac{1}{2}\mathbf{u}$ and the circle in the lab frame passes through the origin (particle 1 stationary after a head-on collision, as before);
- If $m_1 = m_2$ but $e < 1$ the radius of the circle is reduced and it no longer passes through the origin.

7.5.3 Glancing collision of two balls—elastic case

We now see how to relate the angles in the lab frame and the CM frame, in the case where one of the balls is initially at rest. We shall assume balls are smooth, so that the impulse on each ball can only be along line of their centres, and that the collision is elastic.

Suppose ball 2 is initially at rest as shown in fig 55. The impulse on it, $\mathbf{I}_2 = m_2\mathbf{v}_2$, is along the line of centres, so the direction of $\mathbf{v}_2$ is along line of ball centres at impact. Impulse on ball 1 is

$$\mathbf{I}_1 = -\mathbf{I}_2 = m\mathbf{v}_1 - m_1\mathbf{u}_1. \tag{384}$$

The direction of $\mathbf{v}_2$ is determined by angle ϕ, the angle between line of centres and the initial direction of $\mathbf{u}_1$.

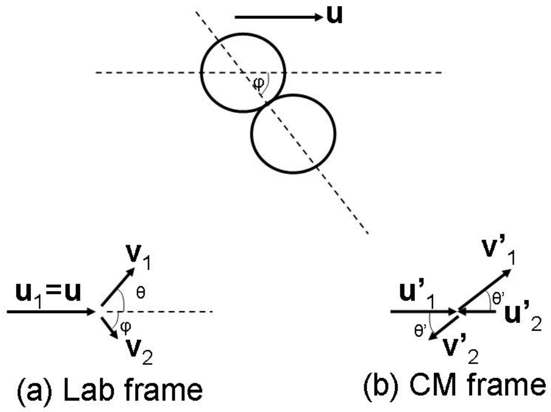

Figure 55: Glancing collision between two balls.

We suppose ball 1 emerges from the collision at an angle θ, as shown.

Viewed in the centre of mass frame, however, the velocities have to remain in opposite directions so both particles are deflected through the same angle θ'.

The relative velocity is initially $\mathbf{u} = \mathbf{u}_1$, and the CM velocity is $\mathbf{V} = \frac{m_1}{M}\mathbf{u}$. If the final relative velocity is $\mathbf{v}$, in an elastic collision

Therefore $m_1\mathbf{u}_1'$ and $m_1\mathbf{v}_1'$ have equal magnitudes; it follows that

$$\theta' = \pi - 2\phi \quad \text{or} \quad \phi = \frac{1}{2}(\pi - \theta'). \tag{385}$$

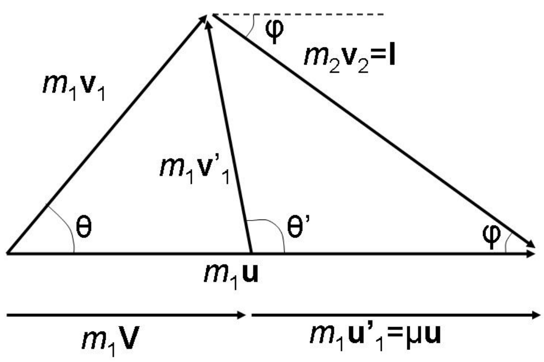

Figure 56: Momentum addition for glancing collision between two balls.

Take the motion in the xy plane with the original velocity along the x-axis. The final velocities are now easy to write down: in the CM frame,

$$\mathbf{v}_1' = \frac{um_2}{M}(\cos\theta'\hat{\mathbf{i}} + \sin\theta'\hat{\mathbf{j}}); \qquad \mathbf{v}_2' = \frac{um_1}{M}(-\cos\theta'\hat{\mathbf{i}} - \sin\theta'\hat{\mathbf{j}}). \tag{386}$$

In the lab frame, we have

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{v}_1' + \mathbf{V} = \frac{um_2}{M}(\cos\theta'\hat{\mathbf{i}} + \sin\theta'\hat{\mathbf{j}}) + \frac{m_1}{M}u\hat{\mathbf{i}} \\
&= \frac{um_2}{M}\left[\left(\cos\theta' + \frac{m_1}{m_2}\right)\hat{\mathbf{i}} + \sin\theta'\hat{\mathbf{j}}\right] \qquad (387) \\
\mathbf{v}_2 &= \mathbf{v}_2' + \mathbf{V} = \frac{um_1}{M}(-\cos\theta'\hat{\mathbf{i}} - \sin\theta'\hat{\mathbf{j}}) + \frac{m_1}{M}u\hat{\mathbf{i}} \\
&= \frac{um_1}{M}\left[(1-\cos\theta')\,\hat{\mathbf{i}} - \sin\theta'\hat{\mathbf{j}}\right]. \qquad (388)
\end{aligned}$$

Hence the angle of deflection of particle 1 is given by

$$\tan\theta = \frac{\sin\theta'}{\frac{m_1}{m_2} + \cos\theta'} = \frac{\sin(2\phi)}{\frac{m_1}{m_2} - \cos(2\phi)}. \qquad (389)$$

Now recall that

$$\mathbf{V} = \frac{m_1}{M}\mathbf{u} = \frac{m_1}{m_2}\mathbf{u}_1' \quad \Rightarrow \quad V = \frac{m_1}{m_2}u_1' = \frac{m_1}{m_2}v_1'. \qquad (390)$$

If $m_2 > m_1$ then $V < v_1'$ and the left-hand vertex of the triangle in Figure 56 lies inside the circle swept out by the top vertex as θ' varies from 0 to π. Hence the maximum value of θ is also π. However if $m_1 > m_2$ the far-left vertex lies outside the circle; the maximum deflection θ of mass 1 is then when the vectors $\mathbf{v}_1$ and $\mathbf{v}_1'$ are perpendicular, and is given by

$$\sin\theta_{\max} = \frac{v_1'}{V} = \frac{m_2}{m_1}. \qquad (391)$$

The final speed of particle 2 in the lab frame can be found from the length of the above vector, or more simply from the geometry of the isosceles (right-hand) triangle in Figure 56:

$$v_2 = \frac{2\mu u}{m_2}\sin\left(\frac{\theta'}{2}\right) = \frac{2m_1}{M}u\sin\left(\frac{\theta'}{2}\right) = \frac{2m_1}{M}u\cos(\phi). \qquad (392)$$

Hence the fraction of the original kinetic energy transferred is

$$\frac{m_2 v_2^2}{m_1 u^2} = \frac{4m_1 m_2}{M^2}\sin^2\left(\frac{\theta'}{2}\right) = \frac{4m_1 m_2}{M^2}\cos^2(\phi). \qquad (393)$$

This is a maximum for the case of a head-on collision ($\phi = 0$, $\theta' = \pi$), in which case we have

$$\mathbf{v}_1 = \frac{um_2}{M}\left(\frac{m_1}{m_2} - 1\right)\hat{\mathbf{i}} = \frac{m_1 - m_2}{M}u\hat{\mathbf{i}}; \qquad \mathbf{v}_2 = \frac{2m_1}{M}u\hat{\mathbf{i}} \qquad (394)$$

as before.

Finally in the case where $m_1 = m_2$, we have

$$\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2 \qquad (395)$$

and

$$\frac{1}{2}m_1 u^2 = \frac{1}{2}m_1(\mathbf{v}_1 + \mathbf{v}_2)^2 = \frac{1}{2}m_1(v_1^2 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + v_2^2) = \frac{1}{2}m_1(v_1^2 + v_2^2) \quad \Rightarrow \quad \mathbf{v}_1\cdot\mathbf{v}_2 = 0, \qquad (396)$$

i.e. the exterior triangle in Figure 56 is right-angled, so

$$\theta + \phi = \pi/2 \qquad (397)$$

and the two objects emerge from the collision at right angles (in the lab frame). Hence also

$$\theta' = \pi - 2\phi = \pi - 2(\pi/2 - \theta) = 2\theta. \qquad (398)$$

Example 50 *Snooker balls.*