PHAS1245—Problem Sheet 4, Solutions

1. (a)

$$\begin{aligned}\frac{d}{dx}\left(\sin(\cos\,x)\right) &= \cos(\cos\,x)\cdot -\sin\,x \\ &= -\sin\,x\,\cos(\cos\,x)\end{aligned}$$

(b)

$$\begin{aligned}y = x^{\cos\,x} \Rightarrow \ln\,y &= \ln\,x^{\cos\,x} \\ &= \cos\,x\,\ln\,x \\ \Rightarrow \frac{1}{y}\frac{dy}{dx} &= \frac{\cos\,x}{x} - \sin\,x\,\ln\,x \\ \frac{dy}{dx} &= x^{\cos\,x}\left(\frac{\cos\,x}{x} - \sin\,x\,\ln\,x\right)\end{aligned}$$

2.

$$I = \int_0^\infty x^n\,e^{-x^2}\,dx = \int_0^\infty x^{n-1}\cdot x\,e^{-x^2}\,dx$$

$$\begin{aligned}u &= x^{n-1} &\Rightarrow\quad & \frac{du}{dx} = (n-1)\,x^{n-2} \\ \frac{dv}{dx} &= x\,e^{-x^2} &\Rightarrow\quad & v = -\frac{1}{2}e^{-x^2}\end{aligned}$$

$$\Rightarrow I = \left[-\frac{1}{2}\,x^{n-1}\,e^{-x^2}\right]_0^\infty + \int_0^\infty \frac{(n-1)}{2}\,x^{n-2}\,e^{-x^2}\,dx$$

which given that the first term is zero, due to the exponential for $x = \infty$ and x^{n-1} for $x = 0$, implies

$$I = \frac{1}{2}(n-1)\int_0^\infty x^{n-2}\,e^{-x^2}\,dx$$

Therefore

$$\begin{aligned}\int_0^\infty x^5\,e^{-x^2}\,dx &= 2\int_0^\infty x^3\,e^{-x^2}\,dx \\ &= 2\int_0^\infty x\,e^{-x^2}\,dx \\ &= 2\left[-\frac{1}{2}e^{-x^2}\right]_0^\infty \\ &= -\left[e^{-\infty^2} - e^0\right] \\ &= 1\,.\end{aligned}$$

3. Given

$$f = \frac{x}{y} \quad \Rightarrow \quad \frac{\partial f}{\partial x} = \frac{1}{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}\,.$$

Therefore

$$\begin{aligned}
\Delta_f^2 &= \left(\frac{\partial f}{\partial x}\right)^2 \Delta_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta_y^2 \\
&= \left(\frac{1}{y^2}\right)\cdot x + \left(\frac{x^2}{y^4}\right)\cdot y \\
&= \frac{x}{y^2} + \frac{x^2}{y^3} \\
\Rightarrow \Delta_f &= \sqrt{\frac{x}{y^3}\,(x+y)} = \frac{1}{y}\sqrt{\frac{x}{y}\,(x+y)}
\end{aligned}$$

4. (a)

$$P(E=0) = \frac{1}{\pi}\,\frac{\Gamma/2}{0+\Gamma^2/4} = \frac{2}{\pi\Gamma}\,.$$

(b) Half-peak height is at half of the above, i.e. $P = 1/(\pi\Gamma)$. Therefore

$$\begin{aligned}
\frac{1}{\pi\Gamma} &= \frac{1}{\pi}\,\frac{\Gamma/2}{E^2+\Gamma^2/4} \\
\Rightarrow 2\,E^2 + \frac{\Gamma^2}{2} &= \Gamma^2 \\
E^2 &= \frac{\Gamma^2}{4} \\
E &= \pm\frac{\Gamma}{2}\,.
\end{aligned}$$

Hence the width at half-peak height is from $-\Gamma/2$ to $\Gamma/2$, i.e. Γ.

(c) We will make the substitution

$$E = \frac{\Gamma}{2}\tan\theta \quad \Rightarrow \quad dE = \frac{\Gamma}{2}\sec^2\theta\,d\theta$$

and for the limits, when $E = \pm\infty$, then $\theta = \pm\frac{\pi}{2}$. So

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{\Gamma/2}{E^2+(\Gamma/2)^2}\,dE &= \int_{-\pi/2}^{\pi/2} \frac{1}{\pi}\,\frac{\Gamma/2\cdot\Gamma/2\,\sec^2\theta\,d\theta}{\Gamma^2/4\,\tan^2\theta+\Gamma^2/4} \\
&= \int_{-\pi/2}^{\pi/2}\frac{1}{\pi}\,d\theta \\
&= 1\,.
\end{aligned}$$

where we have used $\sec^2\theta = 1+\tan^2\theta$.