# 2014 first attempt

01 April 2019 17:39

1. Define the *momentum* of a particle of mass $m$. **[2]**

   For a system of two particles having masses $m_1$, $m_2$ and velocity vectors $\mathbf{v}_1$, $\mathbf{v}_2$, write down an expression for the velocity $\mathbf{V}$ of the centre of mass. Show how the total momentum $\mathbf{P}$ of the system is related to $\mathbf{V}$. **[4]**

$P = m\underline{v}$

Product of mass & velocity.

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$P = \sum m_i v_i = m_1 v_1 + m_2 v_2$$

$$P = M V_{cm}$$

$$M = m_1 + m_2$$

$$M V_{cm} = m_1 v_1 + m_2 v_2 =$$

2. What is the work $\delta W$ done by a force $\mathbf{F}$ when the body on which it is acting moves through a small displacement $\delta\mathbf{r}$? Generalise your answer to write down the total work done when the body moves from an initial position $\mathbf{r}_1$ to a final position $\mathbf{r}_2$. **[3]**

What is meant by the term *conservative force*? Explain how the work done is related to the potential energy, for a conservative force. **[3]**

$$\delta W = \underline{F} \cdot \underline{\delta r} = (F\cos\theta)|\delta \underline{r}|$$

$$\Rightarrow W = \int_{r_1}^{r_2} F \cdot dr$$

Conservative force - Work done independent of path, only start & end point matter.

$$W = V_2 - V_1$$

... ?

3. What is the definition of the *reduced mass* $\mu$ in a system of two particles with masses $m_1$ and $m_2$? [3]

Explain why the velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$ of the two particles in the centre-of-mass frame of the system obey the relation

$$m_1\mathbf{v}_1' + m_2\mathbf{v}_2' = 0.$$

[2]

Hence show that the momentum of mass 1 in the centre-of-mass frame can be written as

$$\mathbf{p}_1' = \mu\mathbf{v},$$

where $\mathbf{v} = \mathbf{v}_1' - \mathbf{v}_2'$ is the relative velocity of the two particles. [3]

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$V_1'$ & $V_2'$ are velocity relative to $V_{cm}$

$\rightarrow V_1' = V_1 - V_{cm}$

$V_2' = V_2 - V_{cm}$

$$m_1 v_1' = m_1 (v_1 - V_{cm})$$

$$m_2 v_2' = m_2 (v_2 - V_{cm})$$

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2 - (m_1 + m_2) V_{cm}$$

$$= m_1 v_1 + m_2 v_2 - m_1 v_1 - m_2 v_2$$

$$= 0$$

$$P_1' = m_1 v_1'$$

$$= m_1 (v_1 - V_{cm})$$

$$= m_1 \left( v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)$$

$$= m_1 \left[ \frac{m_1 v_1 + m_2 v_1 - m_1 v_1 - m_2 v_2}{m_1 + m_2} \right]$$

$$= \frac{m_1}{M} \times m_2 (v_1 - v_2)$$

$$= \frac{m_1 m_2}{m_1 + m_2} [v_1 - v_2]$$

$$v_1' = v_1 - v_{cm}$$

$$v_2' = v_2 - v_{cm}$$

$$\Rightarrow v_1' - v_2' = v_1 - v_2$$

$$= \mu \underline{v}$$

4. A particle of mass $m$ is located at a position $\mathbf{r}$ and has a velocity $\mathbf{v}$. It is acted on by a force $\mathbf{F}$. Write down expressions for:

(a) the torque, $\boldsymbol{\tau}$, about the origin associated with the applied force and [1]

(b) the angular momentum, $\mathbf{L}$, of the particle about the origin. [1]

Hence show that

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

[3]

Show that angular momentum is conserved if $\mathbf{F}$ is a *central force*. [2]

a) $\tau = r \times F$

b) $L = m r^2 \omega$

$v = r\omega$

$L = m \underline{r} \times \underline{v}$

$$\frac{dL}{dt} = m \underline{r} \times \left[\frac{d}{dt}(\underline{v})\right]$$

$= m \underline{r} \times \underline{a}$

$= \underline{r} \times \underline{F}$ $\quad (m\underline{a} = \underline{F})$

$= \tau$

$$= T$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \quad (\text{central fun})$$

$$L = mr^2\dot{\theta}$$

$$\frac{dL}{dt} = m\left[\frac{d}{dt}(r^2)\dot{\theta} + \frac{d}{dt}(\dot{\theta})r^2\right]$$

$$= m\left[2\dot{r}r\dot{\theta} + \ddot{\theta}r^2\right]$$

$$= r\,m\underbrace{\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]}_{= F_{\hat{\theta}} = 0}$$

$$\rightarrow \frac{dL}{dt} = 0$$

5. Draw a diagram showing the direction of the centripetal force acting on a particle of mass $m$ moving in a circular orbit of radius $r$. If the time to make one revolution, $T$, is constant, write down the equation for the centripetal force in terms of $T, m$ and $r$. **[3]**

Determine the radius of the Earth's orbit in km around the Sun assuming that the only gravitational force on the Earth is from the Sun and that the Earth moves in a circular orbit around the Sun. **[4]**

$F = \frac{mv^2}{r}$ $\quad v = \omega r$


$\Rightarrow F = m\omega^2 r$

$\omega = \frac{2\pi}{T}$

$F = \frac{4\pi^2 m r}{T^2}$

$F_{grav} = \frac{G M m}{r^2}$

$= \frac{4\pi^2 m r}{\phantom{T^2}}$

$$\frac{\quad}{T^2}$$

$$\frac{GMm}{r^2} = \frac{4\pi^2 m r}{T^2}$$

$$\frac{GMT^2}{4\pi^2} = r^3$$

$$r = \frac{\sqrt[3]{\frac{GMT^2}{4\pi^2}}}{1000}$$

$G = 6.67 \times 10^{-11}$

$M = 1.99 \times 10^{30}$

$T = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^{7}$

$r = 149 \times 10^{6}$ km

6. Define what is meant by the phenomenon of *resonance* and what condition is satisfied when an undamped system undergoes resonance. **[2]**

A horizontal spring with spring constant $k$ is attached to a fixed point at one end and to an object of mass $m$ at the other. The object lies on a frictionless horizontal surface. The mass is then moved (such that the spring is extended a horizontal distance $x$ from equilibrium) and then released. Write down the equation of motion of the spring and show that is has a solution of the form:

$$x(t) = A\cos(\omega t)$$

where $\omega^2 = \frac{k}{m}$. **[2]**

Show that the total mechanical energy of the system, $U$, is given by:

$$U = \frac{1}{2}kA^2.$$

**[2]**

When an oscillating driving force has a frequency close to / equal to the resonant frequency of an object the total amplitude of oscillation will reach a maximum.

what's 0 damping can't have resonance?


$$m\ddot{x} = -kx \rightarrow \ddot{x} = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$x = A\cos(\omega t)$$

$$\dot{x} = -A\omega\sin(\omega t)$$

$$\ddot{x} = -A\omega^2\cos(\omega t)$$

$$= -\omega^2 x$$

$$\Downarrow$$

$$\ddot{x} = -\omega^2 x$$

$$U = KE + PE$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$PE = \frac{1}{2}kx^2$$

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(A^2\omega^2\sin^2(\omega t)\right)$$

$$\omega^2 = \frac{k}{m} \rightarrow m\omega^2 = k$$

$$\omega^2 = \frac{k}{m} \to m\omega^2 = k$$

$$\to KE = \frac{1}{2} k A^2 \sin^2(\omega t)$$

$$PE = \frac{1}{2} k x^2$$

$$= \frac{1}{2} k A^2 \cos^2(\omega t)$$

$$U = KE + PE = \frac{1}{2} k A^2 \left(\sin^2(\omega t) + \cos^2(\omega t)\right)$$

$$= \frac{1}{2} k A^2$$

7. Given the acceleration vector $\mathbf{a}(t)$ of an object at time $t$, state how would you determine (i) the velocity vector $\mathbf{v}(t)$ and (ii) the displacement vector $\Delta\mathbf{r}(t)$ of the object from its starting point. **[2]**

An object has initial velocity $\mathbf{u}$ at time $t = 0$ and has a constant acceleration vector $\mathbf{a}$; derive an expression for its velocity $\mathbf{v}$ at a later time $t$. Derive also an expression for its displacement $\Delta\mathbf{r}$ at time $t$, and show that

$$\Delta\mathbf{r} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t.$$

**[4]**

Astronauts George and Sandra have masses $m_G = 80\,\mathrm{kg}$ and $m_S = 60\,\mathrm{kg}$ respectively. During a spacewalk they are moving with a common initial velocity $\mathbf{u}_0 = 2 \times 10^3 \hat{\boldsymbol{\imath}}\,\mathrm{m\,s^{-1}}$ (relative to Earth) when the repair-kit they are handling suddenly explodes. As a result of the impulse received from the explosion, Sandra's velocity becomes $\mathbf{u}_S = (2.012 \times 10^3 \hat{\boldsymbol{\imath}} + 24\hat{\boldsymbol{k}})\,\mathrm{m\,s^{-1}}$ .

[The coordinate system is defined with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ horizontal, and $\hat{\boldsymbol{k}}$ vertically upwards. You may neglect the curvature and rotation of the Earth and take the acceleration $-g\hat{\boldsymbol{k}}$ due to gravity as locally constant, with $g$ equal to its value at the Earth's surface.]

(a) Assuming the explosion can be treated as an internal force, i.e. no object apart from the two astronauts is involved, find George's velocity $\mathbf{u}_G$ immediately after the explosion. **[4]**

(b) Both astronauts are temporarily unconscious as a result of the explosion. Sandra recovers consciousness after 60 seconds; what is her velocity vector at this point, and how far has she moved from the point of the explosion (in each case as viewed by an observer on Earth)? **[4]**

(c) What is George's velocity relative to Sandra at this time, and how far away from her is he? [2]

(d) Sandra has a thruster pack capable of delivering a constant force of 120 N in an arbitrary direction, but only has 30 s supply of fuel remaining. To help the still unconscious George she must reach him and ensure her velocity is equal to his as she gets to him. She is prepared to use up all her fuel in order to do this as quickly as possible. In what directions and for how long should she apply the force from her thruster pack? If she starts at once, how long will it take her to reach George adopting this strategy? [4]

[Neglect the change in Sandra's mass arising from the discharge of the thruster fuel.]

i) $\underline{v}(t) = \int a(t)\,dt + v_0$

ii) $\delta \underline{r}(t) = \int \underline{v}(t)\,dt$

$= \iint a(t)\,dt$

$\underline{v} > \underline{u} + \underline{a}t$

$\underline{r} = \int \underline{v}(t)\,dt$

$= ut + \frac{1}{2}at^2$

$= \frac{1}{2}(2u + at)t$

$$= \frac{1}{2}(2u + at)t$$

$$= \frac{1}{2}(u + v)t$$

a) $u_4 = 2\times10^3\hat{i} - \dfrac{60\times(2.012\times10^3\hat{i} + 24\hat{k})}{80}$

$= 1491\hat{i} - 18\hat{k}$

b) $v_5 = u + at$

$= 2012\hat{i} + (24 - gt)\hat{k}$

$= 2012\hat{i} - 564.6\hat{k}$

c) $v_4 = u + at$

$= 1491\hat{i} - 606.6\hat{k}$

$V_{rel} = V_4 - V_5 = -21\hat{i} - 42\hat{k}$

$$V_{rel} = V_G - V_S = -21\hat{i} - 42\hat{k}$$

$$\Delta \underline{r} = \frac{1}{2}(u+v)t$$

$$u = v$$

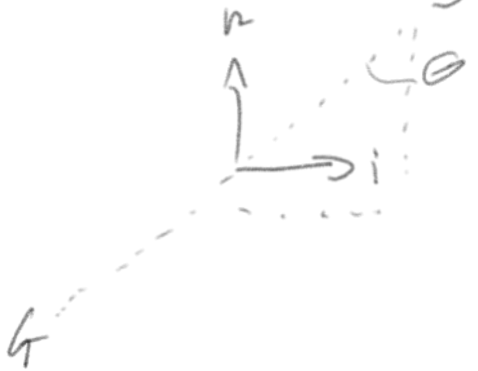
$$\Delta \underline{r} = vt$$

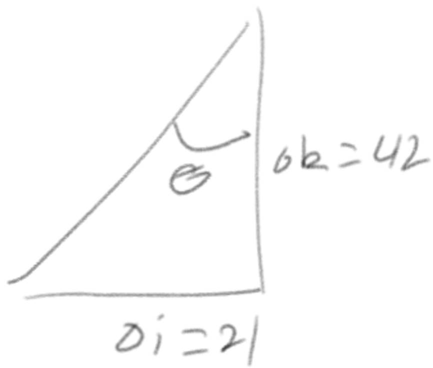
$$= (-21\hat{i} - 42\hat{k})60$$

$$|\Delta r| = \sqrt{21^2 + 42^2} \times 60$$

$$= 2817 m$$

d)




$$Ft = \Delta P$$

$$\tan = 21$$

$Ft = \Delta v$

$$\tan\theta = \frac{21}{42}$$

$$\theta = 26.6°$$

$$\frac{Ft}{m} = \Delta v$$

$$\frac{120t}{60} = \sqrt{21^2 + 42^2}$$

$$t = 23.47s$$

$\rightarrow$

this bring rel velocity to 0.

she has 6.52s of fuel left,

so can burn 3.26 more ∴

total burn of 26.73s

the acceleration felt in this time is

$$\frac{120}{\ } = 2ms^{-2}$$

$$\frac{120}{6} = 2ms^{-2}$$

$$u = -\sqrt{21^2 + 42^2}$$
$$= -46.1$$

$v = 0$

$a = 2$

$$v^2 = u^2 + 2as$$

$$\frac{21^2 + 42^2}{2a} = s$$

$$s = 551.25$$

$$v = u + at$$
$$= 2 \times 3.26$$
$$= 6.52$$

$$s = (551.25 + 2817)$$
$$= 3368$$

$a = 0$ , $u = v$

$s = ut$

$$t = \frac{3368}{6.52} = 516.6\,s$$

8. A particle moves in two dimensions in the $(x, y)$-plane under the action of a force described by the potential energy function

$$V(x, y) = (x^2 + 2y^2 + \alpha y^4)\,\mathrm{J},$$

where $\alpha$ is a non-negative constant , $x$ and $y$ are measured in metres and $V(x, y)$ in Joules. What is the work done on the object by the force as it moves from the point $(2, 0)$ to the point $(0, 0)$? **[4]**

Find a vector expression for the force on the object when it located at a general position $(x, y)$. **[3]**

Explain which of the following quantities you expect to be conserved during the object's motion, and why: (i) the total mechanical energy and (ii) the angular momentum about the origin. **[2]**

Suppose the object has mass $m = 2\,\mathrm{kg}$. Write down Newton's second law for the object and show that it implies the coordinates $x$ and $y$ obey the following differential equations:

$$\ddot{x} = -x; \qquad \ddot{y} = -2y - 2\alpha y^3.$$

**[3]**

At time $t = 0$ the mass has position vector $\mathbf{r} = 2\hat{\boldsymbol{\imath}}\,\mathrm{m}$ and velocity $\mathbf{v} = 4\hat{\boldsymbol{\jmath}}\,\mathrm{m\,s^{-1}}$. Compute its total mechanical energy at $t = 0$. **[2]**

If you knew only the initial total energy you have just calculated, what could you say about the greatest displacement from the origin the object could reach along: (a) the $x$-axis and (b) the $y$-axis during its subsequent motion? **[3]**

$$W = \int F \cdot dr$$

$$= V(2,0) - V(0,0)$$

$$= V(2,0)$$

$$= 2^2 = 4$$

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$$F = \frac{dV}{dr}$$

$$= \nabla V$$

$$= -\frac{d}{dx}\hat{i} - \frac{d}{dy}\hat{j}\left(x^2 + 2y^2 + \alpha y^4\right)$$

$$F = -2x\,\hat{i} - 4\left[y + \alpha y^3\right]\hat{j}$$

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i) total mechanical E is conserved - conservative force (as $F = \nabla V$)

force (as $F = \nabla V$)

Angular momentum is not - force is not always central

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$F = ma$

$ma = 2\ddot{x} = F\begin{pmatrix} \hat{i} \end{pmatrix}$

$= -2x$

$\therefore \ddot{x} = -x$

$F = ma \qquad F = -4[y + \alpha y^3]\hat{j}$

$= 2\ddot{y} \quad \rightarrow 2\ddot{y} = -4[y + \alpha y^3]$

$\ddot{y} = -2y - 2\alpha y^3$

---

$KE = \frac{1}{2}mv^2$

$= \frac{1}{2} \times 2 \times 4^2$

$= 16J$

=

$i \to x$ ? (not obvious...)

$PE = 2^2$

$= 4$

$U = 20$

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if $y = 0$ $\to$ $x_{max} \to$

$x^2 = 20$

$x_{max} = \sqrt{20}$

if $x = 0$

$2y^2 + \alpha y^4 = 20$

$p = y^2$

$\alpha p^2 + 2p - 20 = 0$

$$p = \frac{-2 \pm \sqrt{4 + 80\alpha}}{2\alpha}$$

$= -1 \pm \sqrt{1 + 20\alpha}$

$$= \frac{-1 \pm \sqrt{1+20\alpha}}{\alpha}$$

$$y_{max} = \sqrt{\frac{-1 \pm \sqrt{1+20\alpha}}{\alpha}}$$

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?

9. What is the definition of an elastic collision? [2]

For a system of two particles with masses $m_1$ and $m_2$, show how the kinetic energy as viewed in the centre-of-mass frame (i.e. the kinetic energy of relative motion) can be expressed in terms of the reduced mass of the two particles and their relative velocity $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$. (You may use the relationships from question A3.) [4]

Hence describe how the relative velocity of the two particles can change during an elastic collision between them. [2]

Two particles of masses $m_1 = 3\,\mathrm{kg}$ and $m_2 = 2\,\mathrm{kg}$ are moving with initial velocities $\mathbf{u}_1 = 5\hat{\boldsymbol{\imath}}\,\mathrm{m\,s^{-1}}$ and $\mathbf{u}_2 = 2\hat{\boldsymbol{\jmath}}\,\mathrm{m\,s^{-1}}$ in the laboratory frame. Show that the velocity of the system's centre of mass is $(3\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}})\,\mathrm{m\,s^{-1}}$; find the total kinetic energy of the system, and divide it into parts associated with the motion of the system's centre of mass and the relative motion of the particles. [4]

The two particles undergo an elastic collision, following which they move apart in such a way that mass 1 moves along the positive $z$-axis when viewed from mass 2. Find the final velocities of both particles in the lab frame. [5]

Suppose the collision were inelastic, with a coefficient of restitution $e = 0.5$ and all other details remaining the same. How would the final velocities in the lab frame change? [3]

KE is conserved - sep = app

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$v_1' = v_1 - v_{cm}$$

$$KE_1' = \frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2$$

$$= \frac{1}{2} m_1 \left( v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)^2$$

$$= \frac{1}{2} m_1 \left( v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right) \cdots$$

$$= \frac{1}{2} m_1 \left( \frac{m_1 v_1 + m_2 v_1 - m_1 v_1 - m_2 v_2}{m_1 + m_2} \right)^2 \cdots$$

$$= \frac{1}{2} m_1 \left( \frac{m_2 (v_1 - v_2)}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left( \frac{m_1 (v_1 - v_2)}{m_1 + m_2} \right)^2$$

$$=$$

$$KE' = \frac{1}{2} \sum m_i v_i'^2$$

$$= \frac{1}{2} \sum m (v_i - v_{cm})^2$$

$$= \frac{1}{2} \left[ m_1 \left( v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)^2 + m_2 \left( v_2 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)^2 \right]$$

$$= \frac{1}{2} \left[ m_1 \left( \frac{m_2 (v_1 - v_2)}{m_1 + m_2} \right)^2 + m_2 \left( \frac{m_1 (v_2 - v_1)}{m_1 + m_2} \right)^2 \right]$$

$$= \frac{1}{2}\left[\frac{\mu m_2}{m_1+m_2} v^2 + \frac{\mu m_1}{m_1+m_2} v^2\right]$$

$$= \frac{1}{2}\left[\frac{\mu (m_2+m_1)}{m_1+m_2} v^2\right]$$

$$= \frac{1}{2}\mu v^2$$

Rel velocity after = rel velocity before × e

$e = 1$

∴ rel v is constant

$\xrightarrow{5\hat{i}}$


$$V_{cm} = \frac{3\times 5\hat{i} + 2\times 2\hat{j}}{5}$$

$$= 3\hat{i} + \frac{4}{5}\hat{j}$$

$$KE = KE_{rel} + KE_{cm}$$

$$KE_{rel} = \frac{1}{2}\mu v^2$$

$$= \frac{1}{2}\left(\frac{3\times 2}{3+2}\right)\begin{pmatrix}5\\2\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}$$

$$= \frac{1}{}\ 6$$

$$= \frac{1}{2} \times \frac{6}{5} \left(29\right)$$

$$= 17.4 J$$

$$KE_{cm} = \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2}(3+2)\left(3\hat{i} + \frac{4}{5}\hat{j}\right)^2$$

$$= \frac{5}{2}\left(9 + \frac{16}{25}\right)$$

$$= 24.1 J$$

$$\therefore KE = 24.1 + 17.4$$

$$= 41.5$$

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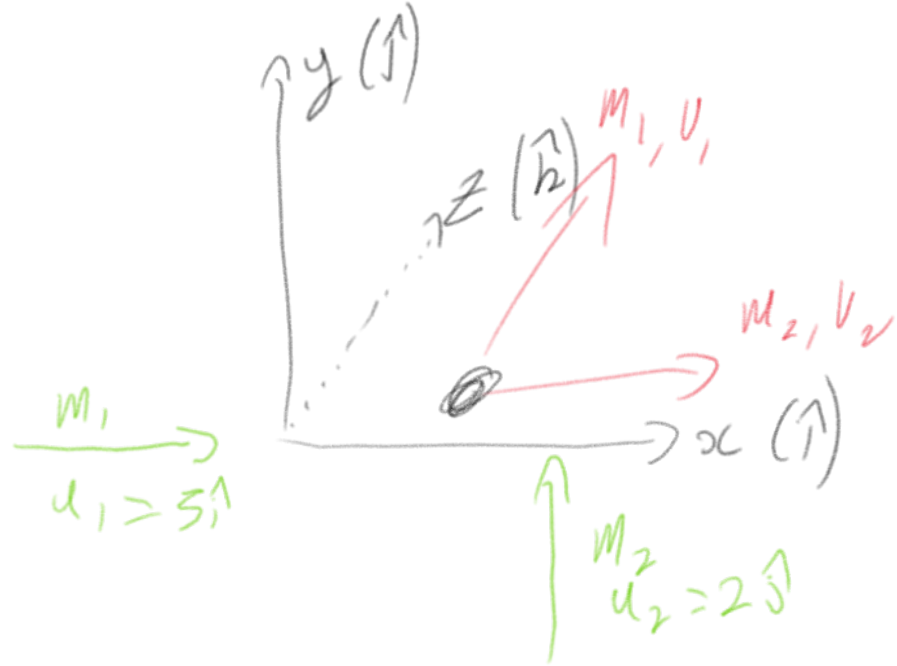
Elastic $\therefore$ KE is constant

Elastic ∴ KE is constant




$$v_2 - v_1 = \alpha \hat{k}$$

∴ $x$ & $y$ components equal.

momentum conserved:

$$m_1 u_1 = m_1 v_{1x} + m_2 v_{2x}$$

$$m_2 u_2 = m_1 v_{1y} + m_2 v_{2y}$$

$$0 = m_1 v_{1z} + m_2 v_{2z}$$

$$v_{1x} = v_{2x}$$

$$v_{1y} = v_{2y}$$

$$m_1 u_1 = (m_1 + m_2) v_{1x}$$

$$m_2 u_2 = (m_1 + m_2) v_{1y}$$

$V_{cm}$ is constant. $= 3\underline{\hat{i}} + \frac{4}{5}\underline{\hat{j}}$

KE is constant.

$$m_1 v_{1z} = -m_2 v_{2z}$$

$$\frac{1}{2}\left(3kg\times\left(9+\frac{16}{25}+v_{1z}^2\right)+2kg\left(9+\frac{16}{25}+v_{2z}^2\right)\right)=41.5$$

$$41.5=\frac{3}{2}v_{1z}^2+v_{2z}^2+24.1$$

$$\frac{3}{2}v_{1z}^2+v_{2z}^2=17.4$$

$$m_1v_{1z}=-m_2v_{2z}$$

$$v_{1z}=-\frac{2}{3}v_{2z}$$

$$\frac{3}{2}\left(\frac{4}{9}\right)v_{2z}^2+v_{2z}^2=17.4$$

$$\frac{5}{3}v_{2z}^2=17.4$$

$$v\ \ 2=\ \ 261$$

5

$$v_{2z}^2 = \frac{261}{25}$$

$$v_{2z} = \frac{3\sqrt{29}}{5}$$

$$v_{1z} = -\frac{-2\sqrt{29}}{5}$$

$$v_1 = 3\hat{\underline{i}} + \frac{4}{5}\hat{\underline{j}} - \frac{2\sqrt{29}}{5}\hat{\underline{k}}$$

$$v_2 = 3\hat{\underline{i}} + \frac{4}{5}\hat{\underline{j}} + \frac{3\sqrt{29}}{5}\hat{\underline{k}}$$

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Sep > 0.5 app

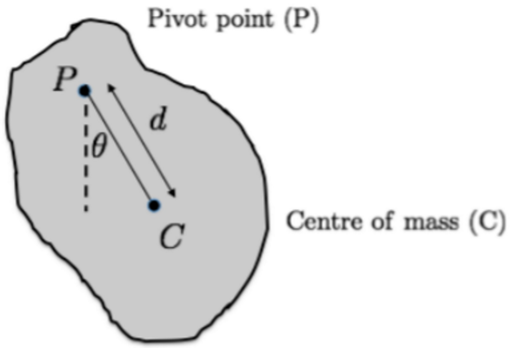
$V_{cm}$ is constant

$$Sep = \frac{1}{2} app$$

$$\vec{v}_1 = 3\hat{i} + \frac{4}{5}\hat{j} - \frac{2\sqrt{29}}{10}\hat{k}$$

$$v_2 = 3\hat{i} + \frac{4}{5}\hat{j} + \frac{3\sqrt{29}}{10}\hat{k}$$

10. Consider an extended body of mass $M$ (see figure below) with a hole drilled through it at position P. The body is suspended from a fixed peg through the hole at $P$ such that it is free to swing from side to side. In equilibrium its centre of mass is at a position C located vertically below P by a distance $d$.



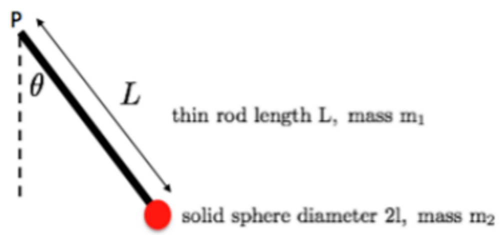

If the body is swung through a small displacement show that the angular equation of motion is given by:

$$I\ddot{\theta} = -(Mgd)\theta$$

where $I$ is the moment of inertia of the body about the pivot point (P) and $\theta$ is the angle the line CP makes with the vertical. [4]

Show that this extended body oscillates with the same angular frequency as a simple pendulum (string of negligible mass supporting a mass $m$) of length $\frac{I}{Md}$. [3]

An extended body of total mass $M$ is composed of a thin rod of mass $m_1$ and length $L$ and a solid spherical ball of mass $m_2$ and radius $l$ attached to the bottom end of the rod such that the entire length of the system is $L + 2l$.



The pivot position P is at the top of the rod. You can assume the rod's mass is distributed uniformly.

State the parallel axis theorem for the moments of inertia of a body. **[2]**

Derive an expression for the moment of inertia of the rigid rod about its centre of mass and hence show that the moment of inertia about the pivot position P is:

$$I_{\mathrm{P}} = \frac{m_1 L^2}{3}.$$

**[3]**

In the limit that $l \ll L$ show that the kinetic energy of the system, $T$, when oscillating is:

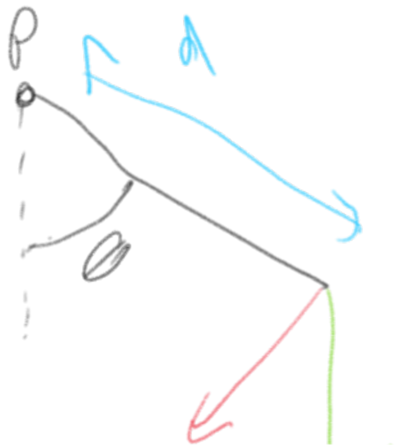
$$T = \frac{1}{2}\left(m_2 + \frac{m_1}{3}\right) L^2 \dot{\theta}^2.$$

**[3]**

By considering the conservation of energy, show for small displacements that:

$$\left(m_2 + \frac{m_1}{3}\right) L^2 \ddot{\theta} + \left(m_2 + \frac{m_1}{2}\right) g L \theta = 0$$

and determine the frequency of oscillation. Show this frequency becomes that of a simple pendulum when $m_1 \to 0$. **[5]**

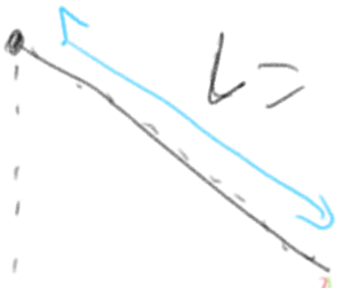


$-mg\sin\theta$ $\quad mg$

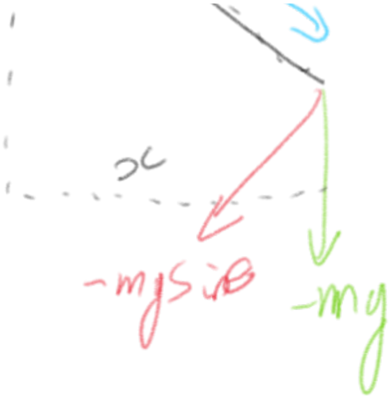
$$\tau = I\ddot{\theta} = r \times F$$
$$= -d(mg\sin\theta)$$

$\theta \ll 1 \rightarrow \theta \simeq \sin\theta$

$$I\ddot{\theta} = -dmg\theta$$
$$= -(Mgd)\theta$$



$$L = \frac{I}{Md}$$




$$m\ddot{x} = -\frac{mg}{L}x$$

$$\ddot{x} = \frac{Mgd}{I}x$$

$$\omega_x^2 = \frac{Mgd}{I}$$

= ✓

Parallel axis theorem:

$$I_A = I_0 + MA^2$$

$$I = \int_{vol} r^2 \, dm$$

$$\sigma = \frac{M}{L}$$

$$dm = \sigma \, dr$$

$$I_{cm} = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, dr$$

$$= \frac{M}{3L} \left[ r^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{M}{3L} \left( \frac{L^3}{8} - - \frac{L^3}{8} \right)$$

$$= \frac{[illegible]}{3L}\left(\frac{[illegible]}{8} - \frac{[illegible]}{8}\right)$$

$$= \frac{mL^2}{12}$$

$$I_p = I_{cm} + m\left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{4}$$

$$= \frac{mL^2}{3}$$

$$KE = \frac{1}{2}mv^2 \qquad v = r\omega$$

$$KE = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$KE = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{2}\frac{m_1 L^2}{3}\omega^2 + \frac{1}{2} m_2 (L+l)^2 \omega^2$$

$$l \to 0$$

$$= \frac{1}{2}\frac{m_1 L^2}{3}\omega^2 + \frac{1}{2} m_2 L^2 \omega^2$$

$$= \frac{1}{2}\left[m_2 + \frac{m_1}{3}\right] L^2 \dot{\theta}^2$$

---

$$KE_0 + PE_0 = KE_1 + PE_1$$

$$PE = mgh$$


$h = L\sin\theta$

$$PE = m_1 g \frac{L}{2}\sin\theta + m_2 g L\sin\theta$$

$$\theta \doteq \sin\theta$$

$$PE = \left(m_1 g \frac{L}{2} + m_2 g L\right)\theta$$

$$= \left(m_2 + \frac{m_1}{2}\right) g L\theta$$

$$\frac{dE}{dt} = 0$$

$$E = \frac{1}{2}\left(m_2 + \frac{m_1}{2}\right) L^2\dot{\theta}^2 + \left(m_2 + \frac{m_1}{2}\right) g L\theta$$

$$E = \frac{1}{2}\left(m_2 + \frac{m_1}{3}\right) l \theta + \left(m_2 + \frac{m_1}{2}\right) g l \theta$$

$$\frac{dE}{dt} = 2 \times \theta \times \frac{1}{2}\left(m_2 + \frac{m_1}{3}\right) l^2 \ddot{\theta} + \left(m_2 + \frac{m_1}{2}\right) g l$$

$$\left(m_2 + \frac{m_1}{3}\right) \dot{\theta} l^2 \ddot{\theta} + \left(m_2 + \frac{m_1}{2}\right) g l \frac{d}{dt}(\theta)$$

?

$$\frac{d}{dt}(\theta) = \dot{\theta} \ldots ?$$

$$\ddot{\theta} = - \frac{\left(m_2 + \frac{m_1}{2}\right) g l}{\left(m_2 + \frac{m_1}{3}\right) l^2} \theta$$

$$= - \frac{\left(m_2 + \frac{m_1}{2}\right) g}{\left(m_2 + \frac{m_1}{3}\right) l} \theta$$

$$\omega = \sqrt{\frac{\left(m_2 + \frac{m_1}{2}\right) g}{\left(m_2 + \frac{m_1}{3}\right) l}}$$

$m_1 \to 0$

$$\omega = \sqrt{\frac{g}{l}}$$

11. State Kepler's second law for a planet orbiting the Sun. On which conservation law does its validity rely? **[2]**

Derive Kepler's second law assuming that the only gravitational interaction is between the Sun and the planet. **[3]**

The magnitude of the velocity of a planet in a circular orbit about the Sun is $v_C$ and the velocity at which the planet is no longer constrained in a orbit about the Sun (the escape velocity) is $v_E$. Show that:

$$v_E = \sqrt{2}v_C.$$

**[2]**

Draw the orbits for these cases: $v_C < v < v_E$, $v = v_E$ and $v > v_E$. **[3]**

An asteroid is initially in a circular orbit around the Earth at a radius $R_0$ where $R_0$ is $1.5R_E$ and $R_E$ is the radius of the Earth. A small craft is injected into the same orbit as the asteroid. Once in the orbit no boosters are fired. Unfortunately the craft is put into the orbit in the wrong direction and collides with the asteroid head on. The craft coalesces with the asteroid and the combination is put into a new orbit.

The mass of the asteroid is $m_A$ and the mass of the craft is $m_{KY}$. The only gravitational interactions you need consider are between the Earth and the asteroid and the Earth and the craft.

What is the initial orbital speed of the asteroid ($v_0$) and the craft ($v_1$)? **[2]**

What is the speed of the coalesced asteroid and craft immediately after they collide? Express your answer in terms of $m_A$, $m_{KY}$ and $v_0$. **[2]**

After the collision, the coalesced system is in a new orbit. Use conservation of energy and angular momentum to show that the distance of closest approach, $R_{\mathrm{X}}$, of the system to the Earth satisfies the relation:

$$f^2 \frac{R_0^2}{R_{\mathrm{X}}^2} - \frac{2R_0}{R_{\mathrm{X}}} = (f^2 - 2) \text{ where } f = \frac{m_{\mathrm{A}} - m_{\mathrm{KY}}}{m_{\mathrm{A}} + m_{\mathrm{KY}}}.$$

[4]

Hence show that when $R_{\mathrm{X}} = R_{\mathrm{E}}$, resulting in a collision with the Earth that:

$$m_{\mathrm{KY}} = m_{\mathrm{A}} \left( \frac{\sqrt{5} - 2}{\sqrt{5} + 2} \right).$$

[2]

Conservation of energy & momentum ?

$$F = \frac{mv^2}{r} = m\omega^2 r$$

$$F = \frac{GMm}{r^2}$$

$$\frac{GMm}{r^2} = m\omega^2 r$$

$$\frac{GM}{r^3} = \omega^2$$

$$\frac{GM}{r^3} = \omega$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{GM}{r^3} = \frac{4\pi^2}{T^2}$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$T^2 \propto r^3$$

---

$$KE_{VE} + PE > 0$$

escape when $E > 0$

$$PE = \int \frac{GMm}{r^2}$$

$$\int \frac{}{r^2}$$

$$= -\frac{GMm}{r}$$

$$KE_{vc} + PE = -\frac{1}{2}PE$$

$$KE_{vc} = \frac{1}{2}PE$$

$$? \quad KE_{Escape} = KE_{vc} + \frac{1}{2}PE$$

$$= 2KE_{vc}$$

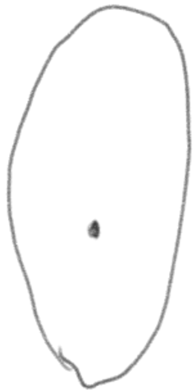
$$\frac{1}{2}mv_E^2 = 2 \times \frac{1}{2}mv_c^2$$

$$v_E^2 = 2v_c^2$$

$$v_E = \sqrt{2}\,v_c$$

$$v_c < v < v_E \qquad v > v_E$$

$v_c < v < v_E$



elliptical

$v > v_E$

papabolic - closes at $r = \infty$

$v > v_E$

hyperbolic

$$\frac{m v^2}{r} = \frac{G M m}{r^2}$$

$$v_0 = v_1 = \sqrt{\frac{GM}{r_0}}$$


$$m_A v_0 - m_{ky} v_1 = v_f \left(m_A + m_{ky}\right)$$

$$v_f = \frac{m_A - m_{ky}}{m_A + m_{ky}} v$$

Before Collision

$\Rightarrow$


$$mr^2\dot{\theta} = mr^2\dot{\theta}$$

$$v = r\dot{\theta} \rightarrow \dot{\theta} = \frac{v}{r} \rightarrow \boxed{v_f r_0 = V_x r_x}$$

---

$$KE + PE = KE + PE$$

$$\frac{1}{2}\cancel{m}v_f^2 - \frac{GM\cancel{m}}{r_0} = \frac{1}{2}\cancel{m}V_x^2 - \frac{GM\cancel{m}}{r_x}$$

$$v_0 = \sqrt{\frac{GM}{r_0}} \rightarrow GM = v_0^2 r_0$$

$$v_f = f v_0$$

$$\frac{1}{2} f^2 v_0^2 - v_0^2 = \frac{1}{2} v_x^2 - \frac{v_0^2 r_0}{r_x}$$

$$v_x^2 = \left(\frac{v_f r_0}{r_x}\right)^2 = \frac{f^2 v_0^2 r_0^2}{r_x^2}$$

$$\frac{1}{2} f^2 \cancel{v_0^2} - \cancel{v_0^2} = \frac{1}{2} \frac{f^2 \cancel{v_0^2} r_0^2}{r_x^2} + \cancel{v_0^2} \frac{r_0}{r_x}$$

$$f^2 - 2 = f^2 \frac{r_0^2}{r_x^2} - \frac{2 r_0}{r_x}$$

$R_x = R_E$

$R_o = 1.5 R_E$

$$f^2\left(\frac{(1.5R_E)^2}{R_E^2}\right) - \frac{2\times 1.5 R_E}{R_E} = f^2 - 2$$

$$\frac{9}{4}f^2 - 3 = f^2 - 2$$

$$\frac{5}{4}f^2 - 1 = 0$$

$$f^2 = \frac{4}{5}$$

$$f = \frac{2}{\sqrt{5}}$$

$$\frac{m_A - m_{py}}{m_A + m_{py}} = \frac{2}{\sqrt{5}}$$

$m_A - m_{py} = 2\,(m_A + m_{py})$

$$\overset{0}{m_A - m_{ry}} = \frac{2}{\sqrt{3}}\left(m_A + m_{ry}\right)$$

$$m_A - \frac{2}{\sqrt{3}} m_A = \left(\frac{2}{\sqrt{3}} + 1\right) m_r$$

$$m_r = \frac{\left(1 - \frac{2}{\sqrt{3}}\right) m_A}{\left(1 + \frac{2}{\sqrt{3}}\right)}$$

$$= \left(\frac{\sqrt{3} - 2}{\sqrt{3} + 2}\right) m_A$$