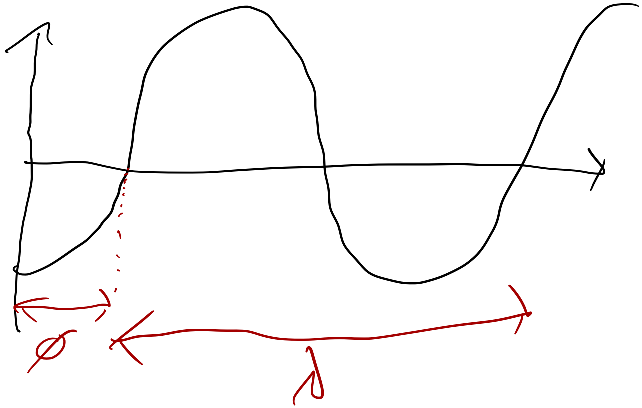
Time-independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

For a free particle $V(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$



Try, $\psi(x) = A\sin\left(\frac{2\pi(x-\phi)}{\lambda}\right)$

$$\frac{d\psi(x)}{dx} = \left(\frac{2\pi}{\lambda}\right)A\cos\left(\frac{2\pi(x-\phi)}{\lambda}\right)$$

$$\frac{d^2\psi(x)}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi(x-\phi)}{\lambda}\right) = -\left(\frac{2\pi}{\lambda}\right)^2\psi(x)$$

TISE $$\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -\left(\frac{2\pi}{\lambda}\right)^2\psi(x)$$

$$\therefore \quad \frac{2mE}{\hbar^2} = \left(\frac{2\pi}{\lambda}\right)^2$$

$$\frac{2\pi}{\lambda} = \frac{\sqrt{2mE}}{\hbar}$$

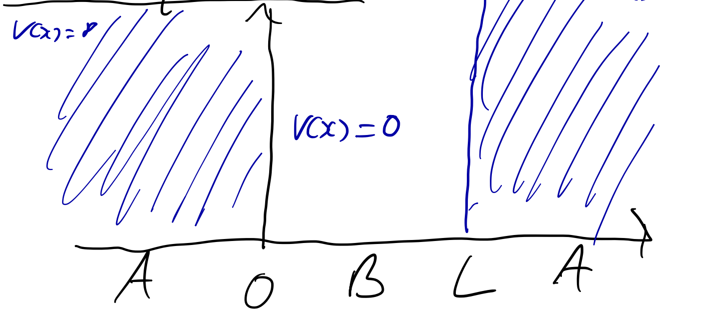
$$\frac{h}{\lambda} = \sqrt{2mE} = p \quad \left(\text{from de Broglie } p = \frac{h}{\lambda}\right)$$

$$\Rightarrow E = \frac{p^2}{2m}$$

But $$\frac{2\pi}{\lambda} = \frac{p}{\hbar} \Rightarrow \psi(x) = A\sin\left(\frac{px}{\hbar} + C\right) \qquad -\frac{2\pi\phi}{\lambda}$$

Since $$\psi(x) = A\sin\left(\frac{2\pi}{\lambda}(x-\phi)\right)$$

Infinite Square Well



$$\psi_A(x) = 0$$

$$\psi_B(x) = A \sin\left(\frac{px}{\hbar} + C\right)$$

Boundary Conditions

$$\psi_A(0) = \psi_B(0) \quad \text{and} \quad \psi_A(L) = \psi_B(L)$$

$\underline{x=0}$.

$$0 = A \sin\left(\frac{p0}{\hbar} + C\right) = A \sin C$$

$\therefore$ $A=0$ or $\sin C = 0$ $\Rightarrow$ $\sin C = 0$ $\Rightarrow$ $C = n\pi$, where $n = 0, 1, 2, 3, \ldots$

↳ Trivial no particle solution

$\underline{x=L}$

$$0 = A \sin\left(\frac{pL}{\hbar} + n\pi\right)$$

$A=0$ is the trivial no particle solution

$$\sin\left(\frac{pL}{\hbar} + n\pi\right) = 0 \Rightarrow \frac{pL}{\hbar} + n\pi = m\pi \Rightarrow \frac{pL}{\hbar} = n\pi$$

↑ Another integer

We have,

$$\psi_B(x) = A \sin\left(\frac{px}{\hbar} + n\pi\right) = \pm A \sin\left(\frac{px}{\hbar}\right)$$

Can ignore the $\pm$ since probability is $|\psi|^2$

$$\frac{pL}{\hbar} = n\pi \qquad \frac{p}{\hbar} = \frac{n\pi}{L}$$

$$\therefore \psi_B(x) = A \sin\left(\frac{n\pi x}{L}\right)$$ where n is an integer.

Energies in the infinite square well

TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{d\psi}{dx} = \left(\frac{n\pi}{L}\right)A\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{d^2\psi}{dx^2} = -\left(\frac{n\pi}{L}\right)^2 A\sin\left(\frac{n\pi x}{L}\right) = -\left(\frac{n\pi}{L}\right)^2\psi(x)$$

TISE

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -\left(\frac{n\pi}{L}\right)^2\psi(x)$$

$$\therefore \frac{2mE}{\hbar^2} = \left(\frac{n\pi}{L}\right)^2 \Rightarrow E = \frac{n^2\hbar^2\pi^2}{2mL^2}$$