Problem Set 3.b - Answers.

1. $r_\parallel = 0$ if $n_t \cos\theta_i = n_i \cos\theta_t$ But from Snell's law $\sin\theta_t = (n_i/n_t)\sin\theta_i$. Therefore, $r_\parallel = 0$ if

$$n_t^2 \cos^2\theta_i = n_i^2(1 - \sin^2\theta_t),$$

and using Snell's law this becomes

$$n_t^2 \cos^2\theta_i = n_i^2\left(1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i\right).$$

And so dividing through by n_i^2,

$$\frac{n_t^2}{n_i^2}\cos^2\theta_i = 1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i. \qquad [3]$$

Using the identity $1 = \sin^2\theta + \cos^2\theta$ in the above

$$\frac{n_t^2}{n_i^2}\cos^2\theta_i = \cos^2\theta_i + \sin^2\theta_i - \frac{n_i^2}{n_t^2}\sin^2\theta_i.$$

Putting all common terms on the same side

$$\cos^2\theta_i\left(\frac{n_t^2}{n_i^2} - 1\right) = \sin^2\theta_i\left(1 - \frac{n_i^2}{n_t^2}\right).$$

Therefore

$$\cos^2\theta_i \frac{n_t^2 - n_i^2}{n_i^2} = \sin^2\theta_i \frac{n_t^2 - n_i^2}{n_t^2},$$

which quickly leads to

$$\frac{\cos^2\theta_i}{n_i^2} = \frac{\sin^2\theta_i}{n_t^2} \quad \rightarrow \quad \frac{\sin^2\theta_i}{\cos^2\theta_i} \equiv \tan^2\theta_i = \frac{n_t^2}{n_i^2}.$$

taking the square root of both sides then $r_\parallel = 0$ for the angle $\theta_p = \tan^{-1}(n_t/n_i)$ (Brewster's angle). [4]

2. From the expression in lectures we get constructive interference when

$$x = \frac{\lambda(p + \frac{1}{2})}{2\alpha},$$

where α is the wedge angle and p is an integer. So the distance between two fringes is when p increases by 1, i.e.

$$\Delta x = \frac{\lambda}{2\alpha} \quad \rightarrow \quad \alpha = \frac{\lambda}{2\Delta x}.$$

Here $\lambda = 500$nm and $\Delta x = 1$mm. Therefore $\alpha = 2.5 \times 10^{-4}$ radians. But if d is the diameter of the hair,

$$\alpha = d/100\text{mm},$$

and so $d = 0.025$mm. [3]

If the gap is now filled with oil of refractive index $n = 1.5$ the optical path difference of the light travelling through the gap is increased by a factor of 1.5, i.e. the separation between fringes is given by

$$1.5\Delta x = \frac{\lambda}{2\alpha}.$$

So the separation of the fringes reduces to $1\text{mm}/1.5 = 0.666$mm. [2]

3. The radius of the p_{th} bright band is given from the lecture notes as $r^2 = (p + \frac{1}{2})R\lambda/n$, where R is the lens curvature. It is $p + \frac{1}{2}$ because the phase shift at one boundary means constructive interference needs a half-integer number of wavelengths path difference. Dark bands require destructive interference and an integer number of wavelengths path difference so $r^2 = pR\lambda/n$. Hence,

$$r_p = \sqrt{p\lambda R/n}.$$

Denoting the radius of the band in air by r_p^a and that in the liquid by r_p^l and similarly for the refractive indices then we find that

$$\frac{r_p^a}{r_p^l} = \sqrt{\frac{(p\lambda R/n^a)}{(p\lambda R/n^l)}}.$$

Using $n^a = 1$ then the cancellation of other terms leads to

$$\frac{r_p^a}{r_p^l} = \sqrt{n^l},$$

independent of the order of the fringe. In this case this is

$$\frac{r_p^a}{r_p^l} = \sqrt{1.461} = 1.209,$$

The radius of the 13th dark band in the second case is

$$r_{13}^l = \sqrt{13 \times 589.29 \times 10^{-9} \times 0.2/1.461} = 1.02\text{mm}.$$

Extra Questions not for Assessment.

4. The right-circularly polarised light has equal components of the field vector along the x and y axes but the component along the y axis is retarded by $\pi/2$. If it travels through the quarter wave plate again the phase of the light with field along the y axis is delayed by a further $\pi/2$ i.e. by a total of π. But a phase change of π simply reverses the direction of the field. Thus the part of the field polarised along the y axis is now polarised along the negative y axis. The component polarised along the x axis is unchanged, so after passing through

the wave plate the second time the light is linearly polarised along the line $x = -y$. But this is the axis of polarisation for the linear polariser in the receiver so all the light from the right-circular generator is transmitted by the right-circular receiver. [3]

The right-circularly polarised light has equal components of the field vector along the x and y axes but the component along the y axis is retarded by $\pi/2$. Passing through the wave plate for the left circular polariser this retardation is reversed and the field components along the x and y axes are back in phase. Thus we have light linearly polarised along the line $x = y$. But the axis of polarisation for the linear polariser in the receiver is the line $x = -y$ which is perpendicular, so no light is transmitted when light from the right-circular generator is incident on the left-circular receiver. [3]

a. When face A of one set is adjacent to face B of the other (i.e. one set is simply on top of the other) unpolarised light enters the top pair of glasses, passes through the wave plate, and hence remains unpolarised. It then passes through the polariser so becomes linearly polarised along the x axis. Passing through either wave plate when entering the second pair of glasses produces either right or left circularly polarised light. This then hits a second polariser with axis along the x axis. Since circularly polarised light is rotating, on average half of it is transmitted by a linear polariser along any axis in the plane of its rotation, so half the light gets through regardless of the orientation of the linear polariser. [2]

b. When face B of one pair is adjacent to face B of the other the linear polariser of the two pairs are adjacent with no wave plate between them. The axis of this is the same in each case, it is the wave plates which have opposite effects in each lens. Hence, this is just like having two equal linear polarisers next to each other, whichever lens is next to the other. If they have the same orientation all light is transmitted. If the are at right angles none is. [2]

c. When face A of one pair is adjacent to face A of the other (i.e. glasses face each other), the light entering one is circularly polarised, i.e. the light enters from the side where the lens acts as a generator of circularly polarised light. However, the second lens is still acting as a receiver. Hence, if light enters the right-hand lens it will be right-circularly polarised. It will then be transmitted fully by the second right-hand lens acting as a receiver. This does not depend on orientation since there is no preferred axis for circularly polarised light. If light enters the right-hand lens, becoming right-circularly polarised and then this enters the left-hand lens, the latter is the receiver for left-circularly polarised light and nothing is transmitted. Clearly the argument is the same if the incoming light enters the left-hand lens. It is fully transmitted by the second left-hand lens, but not transmitted if the second lens is the right-hand lens. [2]

5. Here we have to take into account that the generation of circularly polarised light relies on introducing a $\pi/2$ phase difference. This means there is a quarter wavelength difference in light transmitted along the two perpendicular directions. This can only be exactly true for one wavelength, which will be somewhere near the middle of the visible range i.e $\lambda \approx 450$nm.

In part a the light transmitted by the first pair of glasses is linearly polarised along the line x axis. Passing through the quarter wave plate the light at 450nm will be circularly polarised. That at the blue end, i.e. shorter wavelength will have one component with a phase shift $> \pi/2$, and so this component will be nearer to being π out of phase, i.e. the light will be nearer to linear polarisation at right angles to the initial polarisation. That near the red end, i.e. longer wavelength will have a component with phase shift $< \pi/2$ so will have a degree of linear polarisation of the same orientation as that transmitted. Hence, if the two lenses have the same orientation, so the two linear polarisers are along the same axis, there will be preferential transmission at the redder end compared to the blue. If the second lens is at right angles to the first the axis of transmission of the second polariser is now at right angles to the first and there will be preferential transmission at the blue end of the spectrum.

In case c only the light at 450nm will be generated by the first lens with exact circular polarisation. That at lower wavelength will have a component with phase shift $> \pi/2$ and that at higher wavelength will have a component with phase shift $< \pi/2$. Consider the first lens as the right-hand lens. The wave plate in this lens delays the field component along the line $x = y$. If the second lens is the left-hand lens and is parallel to the first the wave plate in this delays the component along the line $x = -y$ axis, but rotation by 90^0 about the y axis to get the lenses facing each other swaps x for $-x$ axes so the delay is actually again along the line $x = y$. This means that the light with $\lambda = 450$nm has had the phase of the field along the line $x = y$ changed by π, so its direction is reversed and on reaching the second polariser there is a component along the line $x = -y$, i.e. $\propto \mathbf{i} - \mathbf{j}$ and along the negative part of the line $x = y$, i.e. $\propto -\mathbf{i} - \mathbf{j}$. Hence, the resultant is along the (negative) y axis, and is perpendicular to the linear polariser so no light is transmitted. However, for λ not exactly equal to 450nm the phase shift is not exactly π. So the component along $x = y$ is not completely reversed and the resultant is not exactly along the y axis. Hence, at the red and blue end of the spectrum some light gets through giving a purple transmission.

If the second lens is perpendicular to the first, the axis where the phase shift is introduced is further rotated by another 90^0. Hence, one lens delays along the line $x = y$ and the other along the line $x = -y$. Hence, in total both field components are delayed by the same amount and the light arriving at the second linear polariser has both components back in phase and the polarisation is the same as transmitted by the first, i.e. along the x axis. However, if the second lens is oriented perpendicular to the first the linear polarisation of the second lens is along the y axis, and no light is transmitted independent of wavelength.