Basic and transverse waves - short

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The wave equation for 1 dimension is:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$$

This describes the propagation of the wave, with c being the speed of proagation

We can solve the wave equation by letting $\psi(x,y) = f(u)$ where $u = x \pm ct$

We can then solve using the chain rule:

$$u = x \pm ct \rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial t} = \pm c$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u}\right) = \frac{\partial^2\psi}{\partial u^2}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\pm c\,\frac{\partial\psi}{\partial u}\right) = c^2\frac{\partial^2\psi}{\partial u^2}$$

$$\rightarrow \frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$$

As both solutions $u = x + ct$ and $u = x - ct$ are valid, we write:

$$\psi(x,t) = f(x+ct) + g(x-ct)$$

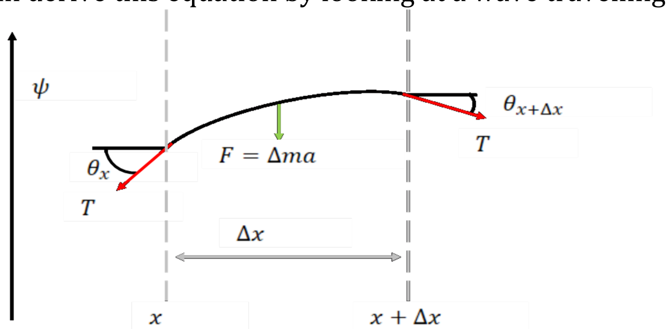
Where $g(u)$ represents a wave travelling in the opposite direction to $f(u)$

As we know the equation describes a sinusoidal function, we can incorporate 2π and λ, which we define as the distance between each repeated period.

$$k = \frac{2\pi}{\lambda}, \omega = \frac{2\pi c}{\lambda}$$

$$\psi(x,t) = f(kx+\omega t) + g(kx-\omega t)$$

We can derive this equation by looking at a wave travelling in a string.



The string has a mass per unit length of μ

We first look at the force acting to pull the string to equilibrium:

$$F = ma = (\mu\Delta x)\frac{\partial^2\psi}{\partial t^2}$$

We then resolve the tension forces with this force, and then equate:

$$F = Tsin(\theta_x + \Delta x) - Tsin(\theta_x) = T\left[\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial\psi}{\partial x}\right)_x\right]$$

$$\mu\Delta x\frac{\partial^2\psi}{\partial t^2} = T\left[\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial\psi}{\partial x}\right)_x\right]$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\mu}\left[\frac{\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial\psi}{\partial t}\right)_x}{\Delta x}\right] = \frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2}$$

$$\therefore c = \sqrt{\frac{T}{\mu}}$$

We can find the total energy wave by looking at the kinetic energy and the potential energy per unit length.

The potential energy comes from the string stretching, and is equal to the work done against tension. If ΔL is the amount the segment has stretched:

$$\Delta L = \sqrt{\Delta x^2 + \Delta\psi^2} - \Delta x \approx \frac{\Delta x}{2}\left(\frac{\partial\psi}{\partial x}\right)^2$$

$$PE = T * \frac{\Delta L}{\Delta x} = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 = \frac{1}{2}TA^2k^2\sin^2(kx-\omega t)$$

We can add this to the kinetic energy to find the total energy per unit length:

$$KE = \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{1}{2}\mu A^2\omega^2\sin^2(kx-\omega t)$$

$$E(x,t) = KE + PE = \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2$$

We can introduce impedance Z_0 for the waves, which is a measure of the resistance the wave encounters. For a stretched string $Z_0 = \sqrt{T\mu}$.

As $c = \sqrt{\frac{T}{\mu}}$ we can re-write our energy as:

$$E(x,t) = \frac{1}{2}\frac{Z_0}{c}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 + c^2\left(\frac{\partial\psi}{\partial t}\right)^2\right]$$

We can find the rate at which energy moves down the string, the power. The power is equal to the force multiplied with the velocity:

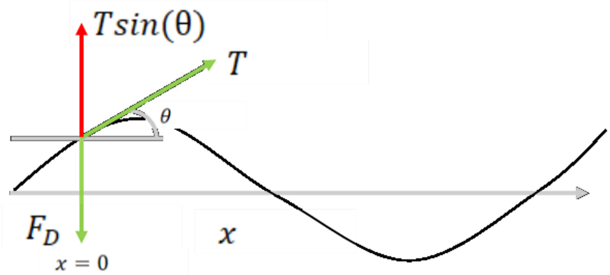
$$F \approx -T\frac{\partial\psi}{\partial x}, v = \frac{\partial\psi}{\partial t}$$

$$P(x,t) = -Z_0 c\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}$$

So far we have assumed the waves to travel for infinite distance and time with no defined start point. In the real world, a wave propagating must have a driving force.

If we imagine a stretched string of length L which starts at $x = 0$ and ends at $x = L$

For a transverse wave, the driving force must act perpendicular to the propagation of the wave, and must balance the component tension at this point.



$$F_D = T\left(\frac{\partial\psi}{\partial x}\right)_{x=0} \qquad (sin(\theta) = \left(\frac{\partial\psi}{\partial x}\right)_{x=0})$$

We can also find the driving force in terms of the velocity of the wave:

$$\frac{\partial\psi}{\partial x} = \frac{1}{c}\frac{\partial\psi}{\partial t} \therefore F_D = \frac{T}{c}\left(\frac{\partial\psi}{\partial t}\right)_{x=0}$$

$$\frac{T}{c} = Z_0 \therefore F_D = Z_0\left(\frac{\partial\psi}{\partial t}\right)_{x=0}$$

The impedance here is due to the energy that is being transported away from the driving force by the propagation of the wave.

To fully terminate a wave, a force at the end point must balance the component of tension. If this is the case, the wave will be completely terminated.

$$F_L = Z_0\left(\frac{\partial\psi}{\partial t}\right)_{x=L}$$

If the force at the point of termination perfectly balances the component of tension no reflection will occur. This allows us to treat the rest of the string as though it were infinite. This is called impedance matching.

If the force at the end isn't perfectly balances, we need to consider the boundaries between the two media.

If there is no other medium (or one with infinite impedance), the wave will reflect

$$\psi = \psi_i + \psi_r = f(kx - \omega t) + g(kx + \omega t)$$

If the end of the string is fixed at a wall, then at this point the total wave will be 0.

$$\psi = \psi_i + \psi_r = 0 \rightarrow \psi_r = -\psi_i$$

A wave reflected from the wall will have a phase difference of π compared to the incident wave.

If instead the end of the string is free to move, the transverse force at this point is 0. Therefore:

$$T\left(\frac{\partial \psi}{\partial x}\right) = 0 \rightarrow \frac{\partial \psi}{\partial x} = 0$$

As the force at force at this point is 0, we can write:

$$\psi = \psi_i + \psi_r = 2\psi_i \rightarrow \psi_r = \psi_i$$

The end point reaches an amplitude of $2A$ and the reflection is in phase with the incident wave

If the boundary connects to another string, we can find the reflected wand transmitted waves.

Imagine 2 strings. The first of length L_1 stretched between $x = -L_1 \rightarrow x = 0$, with a driving force at $x = -L_1$. It has a density of μ_1 and impedence $Z_1 = \sqrt{T\mu_1}$. The second string has length L_2 stretched between $x = 0 \rightarrow x = L_2$. It has a density μ_2 and impedence $Z_2 = \sqrt{T\mu_2}$

When an incidence wave from the first string reaches the boundary, some will be reflected and some transmitted.

$$\psi_i(0,t) + \psi_r(0,t) = \psi_t(0,t)$$

At the boundary, the first string will receive a drag force from the second string equal to the impedance multiplied by the transverse velocity:

$$F_{drag} = Z_2\left(\frac{\partial \psi_t}{\partial t}\right) = Z_2\left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_r}{\partial t}\right)$$

The force will be balanced by the transverse force from the first string:

$$F = -T\left(\frac{\partial \psi}{\partial x}\right) = Z_1\left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_r}{\partial t}\right)$$

If we equate these two forces we come to a final solution of:

$$\psi_r(0,t) = \frac{Z_1 - Z_2}{Z_1 + Z_2}\psi_i(0,t)$$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$\psi_t = \psi_i + \psi_r = (1 + R)\psi_i = T\psi_i$$

$$T = 1 + R = \frac{2Z_1}{Z_1 + Z_2}$$

The value R is the reflection coefficient, and T is the transmission coefficient:

$$-1 \leq R \leq 1, 0 \leq T \leq 2$$

When a wave is transmitted at a boundary, the frequency of the transmitted and reflected wave are equal. With this, we can find the speed, wavelength, and wave number:

$$c_1 = \sqrt{\frac{T}{\mu_1}}, k_1 = \frac{\omega}{c_1}, \lambda_1 = \frac{c_1}{f}$$

$$c_2 = \sqrt{\frac{T}{\mu_2}}, k_2 = \frac{\omega}{c_2}, \lambda_2 = \frac{c_2}{f}$$

When a wave is reflected, it can form a standing wave by interacting with the incident wave.

For standing waves to form on a stretched string it must be fixed at both ends.

$$\psi(0,t) = \psi(L,t) = 0$$

$$\therefore \left(\frac{\partial\psi}{\partial t}\right)_{x=0} = \left(\frac{\partial\psi}{\partial t}\right)_{x=L} = 0$$

We can assume the form of the wave equation, and then use the boundary conditions to find the final solution:

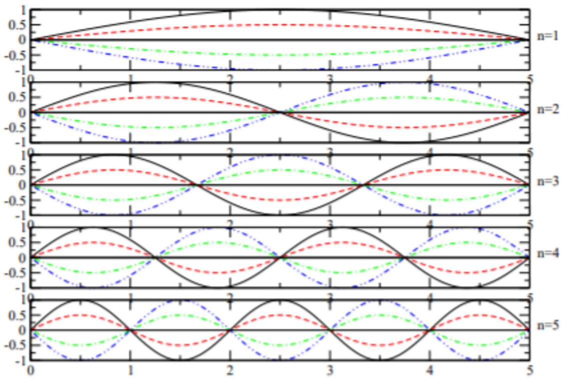
$$\psi(x,t) = Ae^{i(\omega t+kx)} + Be^{i(\omega t-kx)}$$
$$\psi(0,t) = Ae^{i\omega t} + Be^{i\omega t} = 0 \rightarrow A = -B$$
$$\therefore \psi(x,t) = Ae^{i\omega t}\left(e^{ikx} - e^{ikx}\right) = 2iAe^{i\omega t}\sin(kx)$$
$$\therefore \psi(x,t) = -2Asin(\omega t)\sin(kx)$$

$$When\ x = L, \psi(L,t) = 0$$
$$-2Asin(\omega t)\sin(kL) = 0$$
$$\therefore kL = n\pi$$

As the product between the wave number and the total length must be a multiple of π, we can find the allowed wave numbers and the associated wave lengths and frequencies:

$$k_n = \frac{n\pi}{L} \qquad \lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n},$$
$$\omega_n = \frac{2\pi c}{\lambda_n} = \frac{n\pi c}{L}, f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L}$$

These allowed values give rise to a set of waves for varying values of n:



The points with no movement are nodes, the points which fluctuate between maxima are antinodes.

For the nth wave, there are n antinodes, and $n-1$ nodes

f_1 is known as the fundamental frequency, and any larger values of n are the harmonics.

If we look at the wave equation for a standing wave, we can find the positions of the nodes and antinodes:

$$\psi(x,t) = -2Asin(\omega t)\sin(kx)$$
$$x_{node} \rightarrow kx = n\pi \rightarrow x_{node} = \frac{n\pi}{k} = n\frac{\lambda}{2}$$
$$x_{antinode} \rightarrow kx = \left(n + \frac{1}{2}\right)\pi \rightarrow x_{antinode} = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$$