Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. (a) State the formal definition of a derivative. **[3]**

   (b) Using the formal definition of a derivative, derive from first principles the derivative of $y = x^2$. **[3]**

2. (a) State the definition of a definite integral. **[2]**

   (b) State the definition of an indefinite integral. **[2]**

   (c) Using the above definitions, demonstrate that integration and differentiation are inverse operations, i.e. that **[4]**

$$\frac{d}{dx}\left[\int_a^x f(u)du\right] = f(x)$$

3. (a) Given a function $y = f(x)$, state the condition for a point to be stationary. **[2]**

   (b) Given a function $y = f(x)$, state the criteria to determine the nature of the stationary point(s). **[2]**

   (c) Find the stationary point(s) of $f(x) = x^6$ and discuss its/their nature. **[4]**

4. (a) Given a general differential, df, of the form : **[2]**

$$df = A(x, y)\,dx + B(x, y)\,dy$$

   state the condition that means that df is exact.

   Hence determine whether the following are exact differentials :

   (b) $df = x\,dy + 3y\,dx$; **[2]**

   (c) $df = (3x + 2)y\,dx + x(x + 1)\,dy$. **[2]**

[Part marks]

5. (a) Given two complex numbers, $z_1 = 4 + 7\,i$ and $z_2 = -3\,i$, determine **[4]**

$$\text{(i)}\;\; z_1 + z_2 \qquad \text{(ii)}\;\; z_1 - z_2 \qquad \text{(iii)}\;\; z_1\, z_2 \qquad \text{(iv)}\;\; z_1/z_2\,.$$

(b) Determine the real and imaginary parts of the following : **[2]**

$$\text{(i)}\;\; \frac{1}{i^3} \qquad\qquad \text{(ii)}\;\; \left(-2 + \sqrt{5}\,i\right)^2 .$$

6. (a) Write down the general form of the Taylor series for a function $f(x)$. **[4]**

(b) Determine the Taylor series of $\cos x$ up to the cubic power, about $x = \pi/3$. **[2]**

Section B

7. (a) Write down the general form of the Maclaurin series for a function $f(x)$. **[4]**

   (b) Determine the first non-zero term in the Maclaurin series of the following two functions :

   i. $f(x) = \sin x^4$. **[3]**

   ii. $f(x) = \ln(1 + x)$. **[3]**

   (c) Hence or otherwise determine the limit of

   i. $\frac{\sin x^4}{x^4}$ as $x \to 0$. **[4]**

   ii. $\left(1 - \frac{a^2}{x^2}\right)^{x^2}$ as $x \to +\infty$. **[6]**

8. (a) State Euler's equation. **[2]**

   (b) Determine the Maclaurin series for e^x, $\sin\theta$, and $\cos\theta$. **[6]**

   (c) Hence derive Euler's equation. **[2]**

   (d) By considering the real and imaginary parts of the product $e^{i\theta}e^{i\phi}$ prove the standard formulae for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$. **[4]**

   (e) State and derive de Moivre's theorem. **[2]**

   (f) Prove that **[4]**

   $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$,

   and

   $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

[Part marks]

9. (a) Derive an expression for the arithmetic series $S_N = a + (a+d) + (a+2d) + ... + (a + (N-1)d)$. **[3]**

   (b) Derive an expression for the geometric series $S_N = a + ar + ar^2 + ... + ar^{N-1}$. **[3]**

   (c) Show, by any means, whether the following series converges or diverges : **[4]**

$$\sum_{n=1}^{\infty} n^4 e^{-n^2} .$$

   (d) Show, by any means, whether the following series converges or diverges : **[4]**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2} .$$

   (e) Determine for which values of x the series : **[6]**

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n-1)} .$$

   converges. For these values, determine the sum of the series.

[Part marks]

10. (a) Given two vectors, $\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$ and $\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$, write down the scalar and vector products of $\underline{a}$ and $\underline{b}$ in terms of the components. **[4]**

(b) State the formal definition of the derivative of a vector function $\underline{a}(t)$ with respect to t. **[2]**

(c) Using the formal definition of derivative of a vector function, prove that **[4]**

$$\frac{d}{dt}(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \left(\frac{d}{dt}\underline{b}\right) + \left(\frac{d}{dt}\underline{a}\right) \cdot \underline{b}$$

(d) Consider the time-dependent vector

$$\underline{r} = \underline{a} \cos \omega t + \underline{b} \sin \omega t$$

where $\underline{a}$, $\underline{b}$ are non-collinear constant vectors, ω is a constant, and t is time. Show that

i. **[5]**

$$\underline{r} \times \frac{d\underline{r}}{dt} = \omega(\underline{a} \times \underline{b})$$

ii. **[5]**

$$\frac{d^2\underline{r}}{dt^2} + \omega^2 \underline{r} = \underline{0} \; .$$

11. (a) Given a function $y = f(x, y)$, state the condition for a point to be stationary. **[3]**

(b) Given a function $y = f(x, y)$, state the criteria to determine the nature of the stationary point(s). **[5]**

(c) Find the stationary point(s) of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ and discuss its/their nature. **[6]**

(d) Determine by any means the integral **[6]**

$$I = \int_0^1 \frac{x^\alpha - 1}{\ln x} dx$$

where $\alpha > 0$ is a parameter.

END OF PAPER