PHAS1202 - Atoms, Stars and The Universe
Problem Solving Tutorial Sheet 1 - 2017

All questions (or variations of them) may appear in the In-Course-Assessment test. Questions are made available approximately one week before the PST. Please attempt the problem sheet in advance of the PST class. A solution sheet will be made available after all PSTs have taken place. **Please print this question sheet and bring it to the PST.**

Objectives:

1. Gain further experience in the energy scales of single photons in comparison with the energy scales of classical Physics.
2. Practise physicist's methods of order-of-magnitude calculation. Please attempt these questions **without** the use of a calculator, make suitable approximations and aim for your numerical answers to be correct to the first significant figure.
3. Gain insights into the wavelength scales in the Compton Effect.
4. Explore the mathematical properties of Rydberg's formula.
5. Explore how adding optic components affects the Mach-Zehnder interferometer.
6. See a modern application of Bohr's atomic model and gain intuition for the atomic size-scales the model predicts.

Useful constants

Planck's constant h is 6.6×10^{-34} Js (2 s.f.).
The speed of light is 3×10^{8} ms^{-1}.
The mass of a Hydrogen atom is 1.7×10^{-27} kg.
The mass of an electron is 9.1×10^{-31}kg.
1 electronVolt is 1.6×10^{-19} J.

1: Quantum scale

1.a) Without using a calculator, give an order-of-magnitude estimate of how many photons are emitted per second by a 5mW green laser pointer?

You may assume that 100% of the power supplied to the laser is converted to green light at a wavelength of 500 nm.

We are doing an order of magnitude calculation here, so we just need to get the answer to the nearest power of 10.

The green laser pointer has wavelength $\lambda = 500 \times 10^{-9}$m = 5×10^{-7}m .

The power (energy per second) is 5mW = 5×10^{-3}W (Joules per second). (We work in SI units).

The energy of a photon is given by the Planck-Einstein formula $E = hf = hc/\lambda$.

The number of photons per second is (energy per second) / (energy per photon) =

$$\frac{5 \times 10^{-3} \times 5 \times 10^{-7}}{6.6 \times 10^{-34} \times 3 \times 10^{8}} \approx 10^{16}\text{s}^{-1}$$

For reference the ”calculator solution” is $1.26 \times 10^{16}\text{s}^{-1}$.

Thus, to the closest order of magnitude there are 10^{16} photons per second in my laser pointer beam.

1.b)Many modern lighthouses utilise 1kW bulbs coupled with rotating mirror and lens assemblies. Assuming the wavelength of the light is 570nm calculate the energy of a single photon and the number of photons emitted per second.

The method is much the same as in part a). The calculator solution is 2.8×10^{21} per second.

2: Compton Effect

Compton’s derivation predicted that, after scattering from an electron in a material, if the incoming photon wavelength is λ, the outgoing wavelength λ' of a photon scattered through angle θ would satisfy

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta).$$

2.a) The quantity $\frac{h}{m_e c}$ is called the Compton wavelength and represents frequency shift of light scattered at 90°. Without using a calculator provide an estimate (in metres) for this wavelength.

The Compton wavelength is 2.4×10^{-12} m, so 10^{-12} m or 1 pm would be a good order of magnitude estimate.

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2 b.i) Consider xray light with a wavelength of 0.1nm. What is the ratio between the Compton wavelength and this wavelength?

0.1nm is 10^{-10} m so 10^{-12} is 1/100 of this wavelength.

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2 b.ii) Consider visible light with a wavelength of 500nm. What is the ratio between the Compton wavelength and this wavelength?

500nm is 5×10^{-7} m and $10^{-12}/5 \times 10^{-7} = 2 \times 10^{-6}$. A wavelength shift of 100th of a wavelength (the xray case) would be much easier to measure than a shift of 2×10^{-6} of a wavelength, so it is very fortunate that Compton was carrying out his experiment using xrays.

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2 c) Consider light scattering from a proton. Assuming a similar physical process to Compton scattering takes place calculate the equivalent "Compton wavelength" for scattering from a proton. If you wanted to build an experiment to measure the proton Compton wavelength what region of the electromagnetic spectrum would you use?

The "Compton wavelength" for scattering from a proton will be $h/(m_p c)$ where m_p is the proton mass. This mass is 1.7×10^{-27} kg. Putting the numbers in we find that this Compton wavelength is 1.3×10^{-15} m . This is 3 orders of magnitude larger than for the electron, so to see an effect analogous to xray electron Compton scattering one would use light with wavelength 3 orders of magnitude smaller than x-ray light, e.g. a gamma ray with wavelength 0.1pm.

3: Macroscopic photons?

In the lectures, we have always assumed that photons have an extremely small energy, and this is why we do not observe the "granularity" of photons in classical (i.e. human-scale) light. But why is this the case? Planck's photon energy formula does not have an upper bound. In principle at least, photons could have any energy. In this question, you will explore the reasons why we do not see photons with the energies of macroscopic objects in nature.

3a) Estimate (to the nearest order of magnitude) energy of a cricket ball, bowled by James Anderson (England's leading wicket taker).

> What order of magnitude is the mass of a cricket ball? 1 g? 10 g? A kilogram? How fast is it likely to go? Try to use your own intuition to estimate these (i.e. not wikipedia!). Without using a calculator, give an estimate for the kinetic energy in Joules.

Wikipedia tells us that the mass of a cricket ball is 160 g, so 100 g ($= 10^{-1}$ kg) is the best order of magnitude estimate. From just thinking about the size and the way cricketers throw the ball, it should be possible to determine that 10 g is too light and 1 kg is too heavy.

A cricket pitch is 20m long (it is actually 1 chain which is 22 yards) from where the bowler releases the ball to the wicket. A good estimate for how long the ball takes from the bowler to the wicket is maybe half a second. So this would give a speed of 40 ms^{-1}, which is around 90 miles-per-hour (this is the characteristic speed of a good fast bowler, James Anderson is typically a little slower than this). For an order of magnitude estimate either 10 ms^{-1} or 100 ms^{-1} would be acceptable. Since the velocity enters the kinetic energy as a squared term it is more important to have a good estimate of the velocity than the mass of the cricket ball.

With these numbers the kinetic energy $mv^2/2$ is $(1/2) \times 10^{-1} \times (40)^2 = 0.5 \times 10^{-1}$ =80 Joules. 10 J or 100 J are both acceptable order of magnitude solutions. We need to choose one, so below I will use 100 J, as the kinetic energy of a James Anderson cricket ball.

3b) What frequency would a photon need to have to possess the same energy?

Planck's photon energy law tells us $E = hf$ and hence $f = E/h$. Putting numbers in: $f = 100/(6.6 \times 10^{-34}) \approx (1/7) \times 10^2 \times 10^{34} \approx 1 \times 10^{35}$ Hz.

This is a very high frequency! For comparison, gamma rays are at approx. 10^{20} Hz.

3c) Wien's law gives the peak wavelength of the emission spectrum of a black-body at temperature T.

$$\lambda_{\text{max}} = \frac{2.9 \times 10^{-3}}{T} \text{m} \cdot \text{K}$$

where m $\cdot$ K (meter Kelvins) are the SI unit for this constant. How hot must a black body be such that its peak emission will be at the frequency of the photon you just calculated?

We invert this expression to give

$$T = \frac{2.9 \times 10^{-3}}{\lambda_{\text{max}}}.$$

Let's rewrite this in terms of the frequency $f_{\text{max}} = c/\lambda_{\text{max}}$.

$$T = \frac{2.9 \times 10^{-3} \times f_{\text{max}}}{c}$$

Putting numbers in $T = 2.9 \times 10^{-3} \times 10^{35}/3 \times 10^{8} \approx 10^{-3} \times 10^{35} \times 10^{-8} = 10^{24}$ K.

This is a very hot temperature! In comparison, the centre of the sun is believed to have a temperature of approx. 10^7K.

3d) Consider Rydberg's formula for the Hydrogen atom emission spectrum.

$$\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$

where $R_H = 1.1 \times 10^7 \text{m}^{-1}$, where n and m are positive integers and where $n > m$. What is the highest frequency of of all Hydrogen emission lines?

To maximise the frequency, we need to maximise the positive part of this expression (by minimising m) and minimise the negative part of the expression (by maximising n). Therefore we set $m = 1$ and $n \to \infty$. Then Rydberg's formula reduces to:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{1} - 0 \right) = R_H$$

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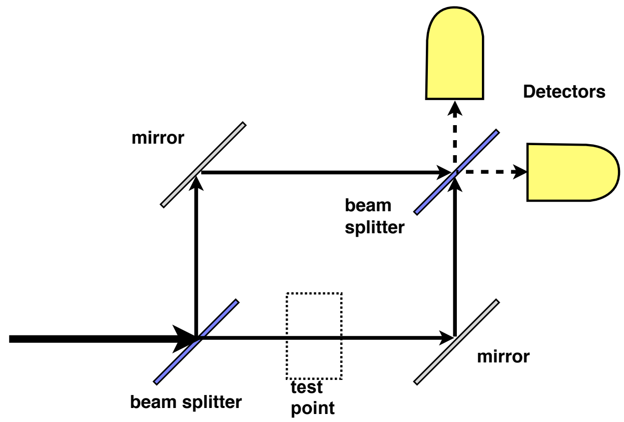
Thus $f = c/\lambda = cR_H = 3 \times 10^8 \times 1.1 \times 10^7 \approx 3.3 \times 10^{15}$Hz.

4: Mach-Zehnder Interferometer

In the lectures we discussed the Mach-Zehnder interferometer, as pictured in the image below. As a reminder the beam splitters transmit 50% of the incident light with no phase shift, whilst the remaining 50% is reflected with a phase shift of $\lambda/4$. The mirrors reflect 100% of the incident light with no phase shift.



4a) What is the probability that a photon will reach the right-most detector if there is nothing placed in the test point?

This was exactly the case that we described in the lectures, the easiest way to think about all these questions is to write down the phase shift along each of the four paths.

- Path A to Top Detector – Transmit through 1st beamsplitter ($\mathbf{+0}\lambda$), reflect off mirror ($\mathbf{+0}\lambda$), transmit through 2nd beamsplitter ($\mathbf{+0}\lambda$) == Phase shift = $\mathbf{0}\lambda$
- Path B to Top Detector – Reflect off 1st beamsplitter (λ**/4**), reflect off mirror ($\mathbf{+0}\lambda$), reflect off 2nd beamsplitter ($\mathbf{+}\lambda$**/4**) == Phase shift = λ**/2**
- Path C to Right Detector – Transmit through 1st beamsplitter ($\mathbf{+0}\lambda$), reflect off mirror ($\mathbf{+0}\lambda$), reflect off 2nd beamsplitter ($+\lambda/4$) == Phase shift = λ**/4**
- Path D to Right Detector – Reflect off 1st beamsplitter (λ**/4**), reflect off mirror ($\mathbf{+0}\lambda$), transmit through 2nd beamsplitter ($\mathbf{+0}\lambda$) == Phase shift = λ**/4**

Clearly the two paths to the top detector (A & B) are in anti-phase and completely destructively interfere. The two paths to the right detector (C & D) are in phase and constructively interfere. So 100% of the intensity will arrive at the right detetctor, meaning that the probability of a photon arriving at the right detector is 100%.

4b) What is the probability that a photon will reach the right-most detector if an absorbing block is placed in the test point (the absorbing block absorbs 100% of the light incident upon it).

Again this example was discussed in the lectures. With an absorbing block in place 50% of the photons will transmit through the first beamsplitter and be absorbed by the block. The remaining 50% will reflect off the first beamsplitter, and then off the mirror and arrive at the second beamsplitter. Here 50% of the 50% (so 25% of the incoming photons) will reflect up to the top detector, whilst the other 50% of the 50% (so 25% of the incoming photons) will transmit through tot the right detector. So the chance of an incoming photon reaching the right detector is 25%.

A quarter-wave plate is a device that transmits 100% of the light but does so whilst adding a phase shift of $\lambda/4$.
4c) What is the probability that a photon will reach the right-most detector if a single quarter-wave plate is placed in the test point?

We have not introduced quarter-wave plates in the lectures, but as described above they just add an extra phase shift of $\lambda/4$, hence the name. Again if we describe the four paths we can determine what happens when one of these quarter-wave plates is introduced to the system.

- Path A to Top Detector – Transmit through 1st beamsplitter (**+0**λ), transmit through quarter-wave plates ($\boldsymbol{\lambda/4}$), reflect off mirror (**+0**λ), transmit through 2nd beamsplitter (**+0**λ) == Phase shift = $\boldsymbol{\lambda/4}$

- Path B to Top Detector – Reflect off 1st beamsplitter ($\boldsymbol{\lambda/4}$), reflect off mirror (**+0**λ), reflect off 2nd beamsplitter ($\boldsymbol{+\lambda/4}$) == Phase shift = $\boldsymbol{\lambda/2}$

- Path C to Right Detector – Transmit through 1st beamsplitter (**+0**λ), transmit through quarter-wave plates ($\boldsymbol{\lambda/4}$), reflect off mirror (**+0**λ), reflect off 2nd beamsplitter (+$\lambda/4$) == Phase shift = $\boldsymbol{\lambda/2}$

- Path D to Right Detector – Reflect off 1st beamsplitter ($\boldsymbol{\lambda/4}$), reflect off mirror (**+0**λ), transmit through 2nd beamsplitter (**+0**λ) == Phase shift = $\boldsymbol{\lambda/4}$

Now in this case the two paths to the right detector have a phase difference of $\lambda/4$, so are neither in-phase nor in anti-phase. So at first it might seem tricky to determine teh probability. However, when we look at the two paths to the top detector we see the phase difference is also $\lambda/4$. Sicne both detectors have the same phase differences, they must have the same probabilities – 50%.

4d) What is the probability that a photon will reach the right-most detector if two quarter-wave plates are placed in the test point?

- Path A to Top Detector – Transmit through 1st beamsplitter (**+0**λ), transmit through both quarter-wave plates (λ**/2**), reflect off mirror (**+0**λ), transmit through 2nd beamsplitter (**+0**λ) == Phase shift = λ**/2**
- Path B to Top Detector – Reflect off 1st beamsplitter (λ**/4**), reflect off mirror (**+0**λ), reflect off 2nd beamsplitter (**+**λ**/4**) == Phase shift = λ**/2**
- Path C to Right Detector – Transmit through 1st beamsplitter (**+0**λ), transmit through both quarter-wave plates (λ**/2**), reflect off mirror (**+0**λ), reflect off 2nd beamsplitter (**+**λ/4) == Phase shift = **3**λ**/4**
- Path D to Right Detector – Reflect off 1st beamsplitter (λ**/4**), reflect off mirror (**+0**λ), transmit through 2nd beamsplitter (**+0**λ) == Phase shift = λ**/4**

Now in this case the two paths to the right detector have a phase difference of $\lambda/2$, so they are in anti-phase resulting in total destructive interference. Whilst the two paths to the top detector are in phase. In this setup the probability of photons reaching the right hand detector is 0%.

5: Nobel-prize winning Bohr model atoms (unseen)

In your lectures, you learnt about Niels Bohr's model of the Hydrogen atom, where electrons lie circular orbits, which satisfy an angular momentum quantisation rule – $l = n\hbar = nh/(2\pi)$ where n is an integer from 1 to infinity. Although the model has been superceded by a fully quantum mechanical treatment (which you'll study in detail in PHAS2222), it remains a good approximation for certain cases.

Prof Serge Haroche was one of the winners of the 2012 Nobel Prize in Physics. Prof Haroche performs experiments with *circular Rydberg states* of Rubidium atoms. We say an atom is in a Rydberg state when one of the electrons has been promoted into a very high energy state. Circular Rydberg states are states whose outer electron's properties are well approximated by the Bohr model for Hydrogen.

Why does the Hydrogen Bohr model work so well for these states? Rubidium has atomic number 37 - i.e. it has 37 electrons orbiting a nucleus containing 37 protons. In a circular Rydberg state, the outer electron is very far away from all other electrons and the nucleus. The outer electron, therefore, experiences the other electrons and protons together as a single object of charge $+e$ (this effect is known as *electron shielding* of the nucleus). The energy of these states and average radius of the outer electron are very well approximated by Bohr's Hydrogen model, and we shall use Bohr's model to study them in this question.

5.a) In Prof Haroche's experiments, photons were generated via the transition from the $n = 51$ circular Rydberg state to the $n = 50$ state (n here can be taken to mean the same n that appears in Bohr's angular momentum rule). What is the radius of these orbits in the Bohr model?

We start again with an easy-to-remember formula

$$r_n = 0.5n^2\text{Angstroms}$$

$50^2 = 2500$ and $51^2 = 2601$ so we obtain $r_{50} = 1250\mathring{A}$ or approx 0.12 micron. This is very large for an atom!

5.b) What is the frequency of the photons emitted in this experiment? In what region of the electromagnetic spectrum do they lie (e.g. are they infrared? ultraviolet? etc.)?

A starting point is the easy-to-remember equation for the energies of Bohr atom orbits:

$$E_n = -\frac{13.6}{n^2} eV$$

We convert this to Joules to get

$$E_n = -\frac{13.6 \times 1.6 \times 10^{-19}}{n^2} J = -\frac{2.2 \times 10^{-18}}{n^2} J$$

Hence the energy difference between $n = 51$ and $n = 50$ is

$$E_{51} - E_{50} = (2.2 \times 10^{-18}) \left(\frac{-1}{51^2} - \frac{-1}{50^2} \right)$$

Putting the numbers in we get $E_{51} - E_{50} = 3.4 \times 10^{-23}$ Joules.

We obtain a photon frequency from Planck-Einstein's formula $E = hf$, hence $f = E/h$. Putting the numbers in again we obtain $f = 50$ GHz (to 1 s.f.). This is in the microwave region.

5.c) A human hair has a thickness of approx 10^{-5}m. Approximately which orbit (i.e. which value of n) would you need to excite the electron to make an atom with a diameter similar to that of a human hair? What frequency of light would drive the atom into this orbit from its ground state (i.e. from its lowest energy orbit)?

First we use

$$r_n = 0.5 n^2 \text{Angstroms}.$$

This question asks for the diameter of the atom, which is simply n^2 Angstroms. 10^{-5}m = $10^5 \mathring{A}$. Thus n=$\sqrt{10^5}$. This is approx 320.

Finally we use the frequency version of Rydberg's formula:

$$f = cR_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$

A full calculation will give $\lambda = 3.3 \times 10^{15}(1 - 10^-5) = 3.3 \times 10^{15}$Hz. However, we get a good enough approximation by saying that $1/320^2$ is close enough to 0. This means we get a good approximation from the frequency we already calculated in question (3d) - namely 3.3×10^{15}Hz. This indicates that these high lying energy orbits have energies extremely close to each other.

Ryan Nichol, October 2017 (adapted from Dan Browne)