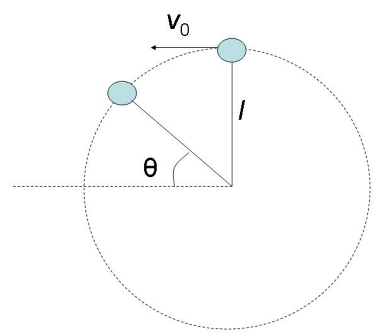
PHAS1247 Classical Mechanics: Problem-Solving Tutorial 3, 2010

Week 13 (22–26 November): Rotational motion and orbits

Section A: energy and equations of motion in polar coordinates

1. Health and safety concerns notwithstanding, I have decided to enter for a conker championship and I am practising my conker swing. My conker has mass m and my string has length l; I launch the conker horizontally with speed v_0 at the top of a circular arc, with the string vertical and held rigidly at the bottom with my other hand (see diagram). Assuming the string remains taut and the conker moves in a circle, find (i) the speed of the conker, (ii) the centripetal force required to keep it moving in a circle, and (iii) the tension in the string, all as a function of the angle θ.



If the breaking tension in my string is 10 N, $l = 0.3$ m and $m = 0.02$ kg, what is the maximum speed at which I can safely launch my conker? What is the minimum speed that will keep the string taut at all times?

[The acceleration due to gravity is $g = 9.81\,\mathrm{ms}^{-2}$.]

2. A spider of mass m is crawling at a constant rate v_0 along a thin sraight horizontal rod that is revolving in the horizontal plane with constant angular velocity ω_0 about a vertical axis through one end of the rod. Sketch the system and indicate the directions of the horizontal forces acting **on the spider**.

   Find the radial and transverse components of the velocity $\mathbf{v}$ of the spider when it is a distance r from the pivot, and hence determine its speed and kinetic energy.

   Also, write down the radial and transverse components of the spider's equation of motion and hence determine the horizontal force $\mathbf{F}$ acting upon it.

   Hence determine the power $\mathbf{F} \cdot \mathbf{v}$ required to keep the rod (with the spider on it) revolving at a constant angular velocity ω_0. Show that this expression for the power is equal to the time derivative of the spider's kinetic energy.

Problem for general discussion A particle of mass m moves on a smooth horizontal table subjected to an attractive central force

$$\mathbf{F} = -\frac{K}{r^3}\hat{\mathbf{r}},$$

where K is a positive constant. If particle is initially moving uniformly in a circle of radius r_0, determine the speed of the particle v and its angular momentum L. Comment on how L depends on the radius r_0.

At time $t = 0$ the particle is given a radially outward impulse $\mathbf{I} = I\hat{\mathbf{r}}$. Determine its distance from the origin at subsequent times t and hence describe its subsequent motion.

Section B: orbits in different central forces

1. Consider the motion of a particle of mass m in an attractive central force of the form
$$\mathbf{F}(\mathbf{r}) = -K\mathbf{r} = -Kr\hat{\mathbf{r}},$$
where K is a positive constant, r is the distance from the centre of force, and as usual $\hat{\mathbf{r}}$ is a unit vector in the radially outwards direction. (This is an example of the 'power law' forces considered in the lectures but with the power-law exponent n set equal to 1.)

   What is the potential energy associated with the force $\mathbf{F}$? (Choose the zero of potential energy to be at the centre of force, $r = 0$.)

   Writing the position vector as $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, find the differential equations satisfied by the coordinates x and y. Hence show that one possible class of solutions to the equation of motion is of the form
$$\mathbf{r}(t) = a\cos(\omega t)\hat{\mathbf{i}} + b\sin(\omega t)\hat{\mathbf{j}},$$
where a, b and ω are constants. Find the value of ω.

   Find the total energy E and the angular momentum L of the particle in terms of m, a, b and ω, when it moves in this orbit. What is the direction of the angular momentum vector $\mathbf{L}$, assuming a and b are both positive?

   For this orbit, show that
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
What is the shape of the orbit?

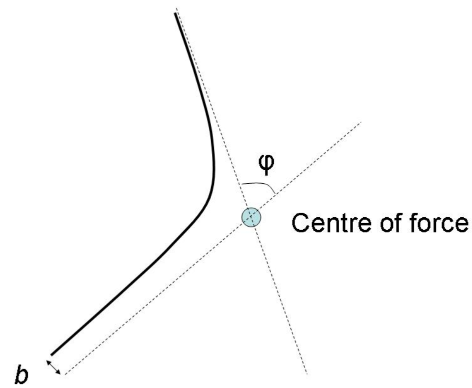
2. It is often conventient to describe collision problems by the so-called **impact parameter** b; this is the distance that the scattered particle would have passed from the centre of force had it been undeflected.

   We will show in the lectures that for a particle with positive initial kinetic energy scattered by an inverse-square-law force
$$\mathbf{F} = \frac{K}{r^2}\hat{\mathbf{r}},$$

the impact parameter b, the energy E and the ultimate angle of deflection ϕ are related by

$$b = \frac{K}{2E} \cot\left(\frac{\phi}{2}\right).$$

The diagram illustrates these quantities (for the case of a repulsive force, i.e. $K > 0$):



Consider the scattering of a beam of particles by the force field. Suppose the flux of incoming particles (i.e. the number per unit area per unit time) is F. Show that the number of particles per unit time having impact parameters in a small range between b and $b + \delta b$ is

$$\delta N = 2\pi F b \delta b.$$

Hence show that the number of particles scattered per unit time through angles between ϕ and $\phi + \delta\phi$ is

$$\delta N = \pi \left(\frac{K}{2E}\right)^2 F \frac{\cos(\phi/2)}{\sin^3(\phi/2)} \delta\phi.$$

Show that if these particles are detected on a sphere of radius R, the area of the sphere corresponding to this range of angles ϕ is

$$\delta A = 2\pi R^2 \sin(\phi)\delta\phi = 4\pi R^2 \sin(\phi/2)\cos(\phi/2)\delta\phi.$$

The so-called 'solid angle' corresponding to this area is defined as $\delta\Omega = \delta A/R^2$ (so that the solid angle corresponding to a whole sphere is 4π). Hence show that the number of particles scattered into this range of angles can be written

$$\delta N = F \left(\frac{K}{2E}\right)^2 \frac{1}{4\sin^4(\phi/2)} \delta\Omega.$$

This is the famous **Rutherford scattering formula**, derived by Ernest Rutherford and used to intepret the results of the Geiger-Marsden experiment that gave evidence for the existence of the atomic nucleus. Does it enable one to distinguish between attractive and repulsive forces?