VI

Series and Limits

30. Infinite Series (Riley 4.1)

30.1 Sequences, Partial Sums and Convergence

In the following we will be concerned with infinite sequences of numbers and their sums.

Definition 30.1 — Partial Sum. Given a sequence of numbers u_1, u_2, u_3, $\ldots$, u_k, $\ldots$, the sum of the first n numbers is called the n-th partial sum S_n,

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \equiv \sum_{k=1}^{n} u_k. \tag{30.1}$$

We are then especially interested in the behaviour of the partial sums as n approaches infinity.

Definition 30.2 — Infinite Sum. If the partial sums S_1, S_2, S_3,... converge to a finite limit S,

$$S = \lim_{n\to\infty} S_n, \tag{30.2}$$

then S is called the sum of the **infinite** series,

$$S = u_1 + u_2 + u_3 + \cdots = \sum_{k=1}^{\infty} u_k, \tag{30.3}$$

and the series is said to be **convergent**. Otherwise, the series may **diverge** (i.e. approach $\pm\infty$) or oscillate.

■ **Example 30.1** Consider the number sequence 1, $1/4$, $1/9$, $1/16$, $\ldots$, $1/k^2$, $\ldots$. The first few partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$ are listed in the table below. Clearly, the individual numbers in the sequence tend to zero as $k \to \infty$. On the other hand, it can be shown that the infinite series converges to $\pi^2/6$,

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

k	1	2	3	4	5	6	7	...
$\frac{1}{k^2}$	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$	$\frac{1}{36}$	$\frac{1}{49}$	...
$S_k=\sum_{i=1}^{k}\frac{1}{i^2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	...

■

30.2 The Arithmetic Series

In an arithmetic series there is a common difference d between successive terms in the sequence. The sum S_n of an arithmetic series of n terms can be written as

$$S_n=\sum_{k=0}^{n-1}(a+kd)=a+[a+d]+[a+2d]+\cdots+[a+(n-2)d]+[a+(n-1)d], \tag{30.4}$$

with a and d constants. If we add the first term to the last, the second to the next-to-last and so on, we find that each term in the addition is the same and equal to $(2a+(n-1)d)$. Altogether, there are $n/2$ of such pairs.

Formula 30.3 — Arithmetic Series. Thus

$$S_n=\sum_{k=0}^{n-1}(a+kd)=\frac{n}{2}(2a+(n-1)d). \tag{30.5}$$

For $n\to\infty$, the series will increase or decrease indefinitely, i.e. it diverges.

■ **Example 30.2** Find the sum of integers from 1 to N,

$$S_n=1+2+3+4+\cdots+n=\sum_{k=1}^{n}k.$$

By replacing $a=1$ and $d=1$ in the above general formula, we have

$$S_n=\frac{n}{2}(2+(n-1))=\frac{n(n+1)}{2}.$$

■

30.3 The Geometric Series

In a geometric series there is a common ratio r between successive terms in the sequence. The partial sum S_n of a geometric series is

$$S_n=\sum_{k=0}^{n-1}ar^k=a+ar+ar^2+\cdots+ar^{n-1}, \tag{30.6}$$

where a and r are constant numbers. The above series, when multiplied throughout by r, yields

$$rS_n=ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n. \tag{30.7}$$

Subtracting the second from the first equation above we find

$$(1-r)S_n=a-ar^n=a(1-r^n). \tag{30.8}$$

Formula 30.4 — Geometric Series. Thus, the partial sum of the geometric series is

$$S_n = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}. \tag{30.9}$$

For $|r| < 1$, the infinite series converges to the value

$$S = \sum_{k=0}^{\infty} ar^k = a\frac{1}{1-r}, \tag{30.10}$$

because $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$. For $|r| \geq 1$ it diverges (for $r < -1$ in an oscillatory fashion).

■ **Example 30.3** Consider a ball that drops from a height of 27 m and on each bounce retains only a third of its kinetic energy. Thus after one bounce it will return to a height of 9 m, after two bounces to 3 m, etc. Find the total distance travelled between the first bounce and the n-th bounce.

The distance is given by the sum of $n-1$ terms,

$$S_n = 2(9+3+1+\dots)\text{ m} = 2\text{ m}\sum_{k=0}^{n-2}\frac{9}{3^k},$$

where the factor of 2 takes into account the ball travelling up and down between bounces. This is a geometric series with a common ratio of $r = 1/3$. Applying the general formula above we find

$$S_n = 18\text{ m}\sum_{k=0}^{n-2}\frac{1}{3^k} = 18\text{ m}\frac{1-\left(\frac{1}{3}\right)^{n-1}}{1-\frac{1}{3}} = 27\text{ m}\left[1-\left(\frac{1}{3}\right)^{n-1}\right].$$

In the abstract and unrealistic limit of an infinite number of bounces, the ball would travel exactly $S = \lim_{n\to\infty} S_n = 27$ m after the first bounce. ■

30.4 The Arithmetic-Geometric Series

The arithmetic-geometric series is essentially a combination of the above with the form

$$S_n = \sum_{k=0}^{n-1}(a+kd)r^k = a+(a+d)r+(a+2d)r^2+\dots+[a+(n-1)d]r^{n-1}, \tag{30.11}$$

where a, d and r are constant numbers. To find the sum, we proceed as with a geometric series and multiply by r,

$$rS_n = ar+(a+d)r^2+(a+2d)r^3+\dots+[a+(n-1)d]r^n. \tag{30.12}$$

Subtracting the two equations yields

$$(1-r)S_n = a+dr+dr^2+dr^3+\dots+dr^{n-1}-[a+(n-1)d]r^n. \tag{30.13}$$

The terms do not cancel as nicely as for a geometric series, but we can identify the sum of the terms,

$$T_n = dr+dr^2+dr^3+\dots+dr^{n-1}, \tag{30.14}$$

with a standard geometric series, except that the first ($k=0$) term is missing. Its sum is thus

$$T_n = d\frac{1-r^n}{1-r}-d = rd\frac{1-r^{n-1}}{1-r}. \tag{30.15}$$

Thus we have

$$(1-r)S_n = a + T_n - [a+(n-1)d]r^n = a + rd\frac{1-r^{n-1}}{1-r} - [a+(n-1)d]r^n. \tag{30.16}$$

For the sum of the arithmetic-geometric series we therefore get

$$S_n = \sum_{k=0}^{n-1}(a+kd)r^k = rd\frac{1-r^{n-1}}{(1-r)^2} + \frac{a-[a+(n-1)d]r^n}{1-r}. \tag{30.17}$$

The infinite series will converge for $|r| < 1$, to the value

$$S = \frac{a}{1-r} + \frac{rd}{(1-r)^2} = \frac{a(1-r)+dr}{(1-r)^2}. \tag{30.18}$$

If $|r| \geq 1$, the series diverges or oscillates.

30.5 Difference Method of Summation

If the terms u_k in a series

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + u_3 + \cdots + u_n, \tag{30.19}$$

can be expressed as the difference of the form

$$u_k = f(k) - f(k-1), \tag{30.20}$$

then it is possible to cancel almost all pairs by rearranging,

$$\begin{aligned} S_n &= [f(1)-f(0)] + [f(2)-f(1)] + \cdots + [f(n-1)-f(n-2)] + [f(n)-f(n-1)] \\ &= -f(0) + [f(1)-f(1)] + [f(2)-f(2)] + \cdots + [f(n-1)-f(n-1)] + f(n) \\ &= f(n) - f(0). \end{aligned} \tag{30.21}$$

We illustrate this technique with a few variations in a couple of examples.

■ **Example 30.4** To calculate the series

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+2)},$$

we use partial fractions to rewrite the series elements

$$u_k = \frac{1}{k(k+2)} = \frac{1}{2k} - \frac{1}{2(k+2)} = f(k) - f(k+2).$$

where we define f as

$$f(k) = \frac{1}{2k}.$$

Thus

$$\begin{aligned} S_n &= u_1 + u_2 + u_3 + \cdots + u_{n-2} + u_{n-1} + u_n \\ &= [f(1)-f(3)] + [f(2)-f(4)] + [f(3)-f(5)] + \ldots \\ &\quad + [f(n-2)-f(n)] + [f(n-1)-f(n+1)] + [f(n)-f(n+2)] \\ &= f(1) + f(2) - f(n+1) - f(n+2), \end{aligned}$$

as all other terms cancel. Substituting f we obtain

$$S_n = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right).$$

Therefore the series converges, with the sum

$$S = \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}.$$

■

■ **Example 30.5** Consider the series

$$\sum_{k=1}^{\infty} \frac{4}{(k+1)(k+3)}.$$

Using partial fractions we can write

$$\frac{4}{(k+1)(k+3)} = \frac{2}{k+1} - \frac{2}{k+3} = f(k) - f(k+2),$$

with

$$f(k) = \frac{2}{k+1}.$$

This allows us to evaluate the partial sum using the same technique as before,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{4}{(k+1)(k+3)} \\ &= [f(1) - f(3)] + [f(2) - f(4)] + [f(3) - f(5)] + \dots \\ &+ [f(n-2) - f(n)] + [f(n-1) - f(n+1)] + [f(n) - f(n+2)] \\ &= f(1) + f(2) - f(n+1) - f(n+2) \\ &= 1 + \frac{2}{3} - \frac{2}{n+2} - \frac{2}{n+3}. \end{aligned}$$

This shows that the infinite series converges, with a sum

$$S = \sum_{k=1}^{\infty} \frac{4}{(k+1)(k+3)} = \frac{5}{3}.$$

■

■ **Example 30.6** Determine for which values of x the series

$$\sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)}$$

converges. For these values, determine the sum of the series.

We proceed as before and use partial fractions to express the series element as

$$u_k = \frac{1}{(x+k)(x+k-1)} = \frac{1}{x+k-1} - \frac{1}{x+k} = f(k-1) - f(k),$$

with $f(k) = 1/(x+k)$. Looking at the denominator of each element, we must require $x \neq 0, -1, -2, -3, \ldots, -k, \ldots$, or otherwise at least one of the denominators would be zero. With this constraint, we can calculate the partial sum as before,

$$\begin{aligned} S_n &= u_1 + u_2 + \cdots + u_{n-1} + u_n \\ &= [f(0) - f(1)] + [f(1) - f(2)] + \cdots + [f(n-2) - f(n-1)] + [f(n-1) - f(n)] \\ &= f(0) - f(n) = \frac{1}{x} - \frac{1}{x+n}. \end{aligned}$$

The infinite series thus converges to $1/x$ as long as $x \neq 0, -1, -2, -3, \ldots$ and the series can be thought of as an 'expansion' of $1/x$,

$$\frac{1}{x} = \sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)},$$

but only for $x \neq -1, -2, -3, \ldots$. The infinite series may be visualized as the function curve $y = 1/x$ with holes at $x = -1, -2, -3, \ldots$. ∎

31. Convergence Tests (Riley 4.3)

We have touched on the convergence of infinite series and although we can sometimes calculate the value of the infinite sum and thus determine directly what the series converges to, this is not always easy. Instead, it is very often useful to first check if a given series converges or not, without calculating the sum in a closed form. We will here discuss three such **convergence tests** that allow to assess whether certain series converge or not.

31.1 Preliminary Test

If the individual terms u_k in a series do not tend to zero as $k \to \infty$, a series cannot converge.

Formula 31.1 — Preliminary Test. The so-called preliminary test thus states that if

$$\lim_{k\to\infty} u_k \neq 0, \tag{31.1}$$

the infinite series $\lim_{n\to\infty} S_n$, $S_n = \sum_k^n u_k$ does not converge.

R We stress that the condition $\lim_{k\to\infty} u_k = 0$ is **necessary but not sufficient** for convergence. Thus if $\lim_{k\to\infty} u_k = 0$, one may not conclude that the series converges. In such a case, the preliminary test is **inconclusive**.

■ **Example 31.1** We consider the series

$$\sum_{k=0}^{\infty} \frac{k^2}{4(k+1)(k+2)}.$$

As

$$\lim_{k\to\infty} \frac{k^2}{4(k+1)(k+2)} = \frac{1}{4} \neq 0,$$

the series does not converge. ■

■ **Example 31.2** For the series

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

we have

$$\lim_{k\to\infty} \frac{1}{k} = 0.$$

Thus the preliminary test is inconclusive and we do not know whether the series converges or not. We will actually show, using the next method, that it diverges. ■

31.2 Comparison Test

The comparison test determines the convergence behaviour by comparing it with another series which is known to be either convergent or divergent.

Formula 31.2 — Comparison Test. Thus the test works in one of two directions, where we consider only positive elements u_k and v_k for definiteness:

- Test for convergence

$$\text{If } \sum_k^{\infty} v_k \ \textbf{converges}\text{, and } u_k \le v_k \ \text{ for } k > N \quad \Rightarrow \quad \sum_k^{\infty} u_k \ \textbf{converges}.$$

- Test for divergence

$$\text{If } \sum_k^{\infty} v_k \ \textbf{diverges}\text{, and } u_k \ge v_k \ \text{ for } k > N \quad \Rightarrow \quad \sum_k^{\infty} u_k \ \textbf{diverges}.$$

Here, N is some fixed number dependent on the series. It expresses the fact that the condition that u_k is less/greater than v_k is not required for every k, but only for k above a certain but finite value N.

■ **Example 31.3** Consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}.$$

We can group elements in the series as follows,

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots,$$

where every bracket is clearly larger than $1/2$. Thus by comparison with the obviously divergent series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots,$$

we can conclude that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. ■

■ **Example 31.4** Does

$$S = \sum_{k=0}^{\infty} \frac{1}{k!+1} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{25} + \ldots$$

converge?

We use the fact (which will be demonstrated later) that

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

converges. Comparing:

$$\frac{1}{k!+1} < \frac{1}{k!}$$

we can conclude that S converges. ■

31.3 D'Alembert Ratio Test

The third test we discuss is based on the comparison of a given series with a geometric series. We do not prove its validity but merely state its applicability.

Formula 31.3 — d'Alembert Ratio Test. For a given infinite series $S = \sum_k^\infty u_k$, we define the quantity

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right|, \tag{31.2}$$

i.e. the absolute value of the ratio of subsequent terms, in the limit $k \to \infty$. Depending on the value of ρ, the **d'Alembert ratio test** states that

- If $\rho < 1$, the series S converges.
- If $\rho > 1$, the series S diverges.
- If $\rho = 1$, the test is inconclusive, i.e. the convergence behaviour of S remains undetermined.

■ **Example 31.5** Determine if

$$S = \sum_{k=0}^{\infty} \frac{1}{k!}$$

converges.

We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{k(k-1)(k-2)\ldots 3\cdot 2\cdot 1}{(k+1)k(k-1)(k-2)\ldots 3\cdot 2\cdot 1} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

Therefore $\rho < 1$ and S converges. ■

■ **Example 31.6** Determine if

$$S = \sum_{k=1}^{\infty} \frac{k!}{10^k}$$

converges.

We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{(k+1)!}{10^{k+1}} \frac{10^k}{k!} = \lim_{k\to\infty} \frac{k+1}{10} \to \infty.$$

So $\rho \to \infty$, certainly larger than one, and therefore S diverges. ■

■ **Example 31.7** Determine if

$$S = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

converges.

We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{k+1}{2^{k+1}} \frac{2^k}{k} = \frac{1}{2} \lim_{k\to\infty} \frac{k+1}{k} = \frac{1}{2} \lim_{k\to\infty} \left(1 + \frac{1}{k}\right) = \frac{1}{2}.$$

So $\rho = 1/2 < 1$ and therefore S converges. ■

■ **Example 31.8** Determine if

$$S = \sum_{k=1}^{\infty} \frac{1}{k^r}$$

converges, where $r > 0$ is a constant positive real number.

We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{k^r}{(k+1)^r} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^r = \lim_{k\to\infty} \left(\frac{1}{1+1/k}\right)^r = 1.$$

So $\rho = 1$ and the d'Alembert ratio test is inconclusive, i.e. we cannot determine whether series of the above form converge or not. For $r = 1$ we have in fact the harmonic series which was shown to diverge using the comparison test. ■

32. Power and Taylor Series

32.1 Power Series and their Convergence (Riley 4.5)

Definition 32.1 — Power Series. An infinite power series is a polynomial to infinite order, defining a function $P(x)$ of the form

$$P(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots, \tag{32.1}$$

where a_k are constant, real coefficients.

To test the convergence of a given power series we use the d'Alembert ratio test, i.e. calculate the quantity

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1} x^{k+1}}{a_k x^k} \right| = |x| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|. \tag{32.2}$$

Formula 32.2 — Interval of Convergence. We thus see that the convergence of $P(x)$ depends on the value of x. Specifically, $P(x)$ converges if $\rho < 1$ and thus if

$$|x| < \left(\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right)^{-1} \equiv R. \tag{32.3}$$

This defines an **interval of convergence** $-R < x < R$, within which the power series converges to a finite value. Outside this interval, the series diverges.

■ **Example 32.1** For what values of x does the power series

$$P(x) = \sum_{k=0}^{\infty} (2x)^k = 1 + 2x + 4x^2 + 8x^3 + \dots$$

converge?

We first calculate ρ,

$$\rho = \lim_{k\to\infty} \left| \frac{(2x)^{k+1}}{(2x)^k} \right| = |2x|.$$

Thus $P(x)$ converges for

$$|x| < \frac{1}{2},$$

and the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$. ■

■ **Example 32.2** Study the convergence of the series

$$P(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots.$$

We first calculate ρ,

$$\rho = \lim_{k\to\infty} \left| \frac{x^{k+1}}{k+1} \frac{k}{x^k} \right| = \lim_{k\to\infty} \left| x \frac{k}{k+1} \right| = |x|.$$

Thus the power series converges for $|x| < 1$. ■

■ **Example 32.3** Study the convergence of the series

$$P(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

where $k!$ denotes the **factorial** of k,

$$k! \equiv 1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k,$$

with $0! = 1$ by definition.

We first calculate ρ,

$$\rho = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \frac{k!}{x^k} \right| = |x| \lim_{k\to\infty} \left| \frac{1}{k+1} \right| = 0,$$

irrespective of the value of x. Thus $\rho = 0 < 1$ always and the series converges for all values of x, i.e. the interval of convergence is $-\infty < x < +\infty$. ■

32.2 Power Series of Complex Numbers

The above discussion may be generalized to power series of complex numbers,

$$P(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \dots, \tag{32.4}$$

where z is a complex number variable and the coefficients a_k can in general be complex, constant numbers as well.

The converge is determined as before, namely $P(z)$ converges if

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| = |z| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1, \tag{32.5}$$

with the only difference being that we now interpret $|\cdot|$ as the **modulus** of the corresponding complex number inside the vertical lines, instead of the absolute value. This now defines a **radius of convergence** $|z| < R$ with

$$R = \left(\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right)^{-1}, \tag{32.6}$$

meaning that $P(z)$ converges for z values that are situated within a circle (but not on the circle) of radius R around the origin in an Argand diagram.

■ **Example 32.4** Determine the radius of convergence of the series

$$P(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^k = 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \dots.$$

We first calculate ρ,

$$\rho = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} z^{k+1}}{2^{k+1}} \frac{2^k}{(-1)^k z^k} \right| = \frac{1}{2}|z|.$$

Thus $P(z)$ converges if z lies inside a circle with radius equal to 2 around the origin on an Argand diagram, $|z| < 2$. ■

32.3 Taylor and Maclaurin Series (Riley 4.6)

The **Maclaurin series**, and the more general **Taylor series**, allow to represent a given function $f(x)$ in terms of a power series, $f(x) = P(x) = \sum_k^\infty a_k (x-x_0)^k$. It can be applied to functions that are continuous and sufficiently differentiable within the x range of interest. The Taylor and Maclaurin series are extremely useful as we can for example substitute a complicated function $f(x)$ with its more easily manageable power series form $P(x)$. The power series is often cut off at a certain power of x^k, acting as an approximation $f(x) \approx P(x)$ of the function close to the point x_0 of expansion.

To determine the Taylor series, we assume that a given function $f(x)$ can be represented as a power series of the following form,

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots, \tag{32.7}$$

where $x = x_0$ is the point around which $f(x)$ is being expanded. We now need to identify the coefficients a_k. For a_0, we evaluate both sides at $x = x_0$, giving

$$f(x_0) = a_0, \tag{32.8}$$

as all other terms on the right-hand side become zero. To determine a_1, we differentiate both sides,

$$\frac{df}{dx} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots, \tag{32.9}$$

and evaluate again at $x = x_0$,

$$\left.\frac{df}{dx}\right|_{x=x_0} = a_1. \tag{32.10}$$

For the other coefficients we repeat the procedure of differentiation and evaluation at $x = x_0$, e.g.

$$\frac{d^2 f}{dx^2} = 2a_2 + 6a_3(x-x_0) + \dots \quad \Rightarrow \quad \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} = 2a_2. \tag{32.11}$$

For the general coefficient a_k, this repeated process yields

$$\left.\frac{d^k f}{dx^k}\right|_{x=x_0} = 2 \cdot 3 \dots (k-1) \cdot k\, a_k = k!\, a_k. \tag{32.12}$$

Formula 32.3 — Taylor and Maclaurin Series. We can thus represent a function $f(x)$ in terms of a **Taylor expansion** around a point $x = x_0$,

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots \\ &= f(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k f}{dx^k}\bigg|_{x=x_0} (x-x_0)^k, \end{aligned} \tag{32.13}$$

where the derivatives are all to be evaluated at the point $x = x_0$ as indicated.

For the important special case $x_0 = 0$, this expansion is called the **Maclaurin series**,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= f(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k f}{dx^k}\bigg|_{x=0} x^k. \end{aligned} \tag{32.14}$$

R The function $f(x)$ must be differentiable at the point $x = x_0$ of expansion. For example, $f(x) = \sqrt{x}$ has no Maclaurin series at $x = 0$. It can still be Taylor-expanded for any other $x > 0$.

The Taylor expansion is a special case of a power series and it is thus only valid within the corresponding interval of convergence. The calculation proceeds in the same way by calculating the parameter ρ of the d'Alembert ratio test,

$$\rho = |x - x_0| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \equiv \frac{|x-x_0|}{R}, \tag{32.15}$$

and the Taylor series is converging to $f(x)$ within the interval $-R < x - x_0 < R$ around x_0.

Determination of Taylor Series from Definition

To calculate the Taylor expansion of a given function we can apply the general formula above.

■ **Example 32.5** Determine the Maclaurin series for $f(x) = \sqrt{1+x}$ up to the cubic term x^3.

To apply the general formula for the Maclaurin series, we repeatedly calculate the derivatives and evaluate them at $x = 0$,

$$\begin{aligned} f(x) &= (1+x)^{1/2} &\Rightarrow\quad f(0) &= 1, \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2} &\Rightarrow\quad f'(0) &= \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2} &\Rightarrow\quad f''(0) &= -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2} &\Rightarrow\quad f'''(0) &= \frac{3}{8}. \end{aligned}$$

The Maclaurin series up to the cubic term thus reads

$$\begin{aligned} \sqrt{1+x} &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 - \mathcal{O}(x^4) \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \mathcal{O}(x^4). \end{aligned}$$

The above result uses the so-called 'Big-O' notation $\mathcal{O}(x^4)$ to denote that the term proportional to x^4 and higher powers have been neglected. ■

■ **Example 32.6** Determine the complete Maclaurin series for $f(x)=\ln(1+x)$ and its interval of convergence.

By repeatedly differentiating we establish the following pattern,

$$\begin{aligned}
f(x) &= \ln(1+x) & &\Rightarrow & f(0) &= 0,\\
f'(x) &= (1+x)^{-1} & &\Rightarrow & f'(0) &= 1,\\
f''(x) &= -(1+x)^{-2} & &\Rightarrow & f''(0) &= -1,\\
f'''(x) &= 2(1+x)^{-3} & &\Rightarrow & f'''(0) &= 2,\\
f^{(4)}(x) &= -6(1+x)^{-4} & &\Rightarrow & f^{(4)}(0) &= -6,\\
&\vdots & &\vdots & &\vdots\\
f^{(k)}(x) &= (-1)^{k-1}(k-1)!(1+x)^{-k} & &\Rightarrow & f^{(k)}(0) &= (-1)^{k-1}(k-1)!.
\end{aligned}$$

We can thus write

$$\ln(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!}{k!}x^k=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k,$$

which can also be expressed by redefining $k \to k+1$,

$$\ln(1+x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots.$$

To determine the interval of convergence, we calculate

$$\rho=|x|\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=|x|\lim_{k\to\infty}\left|\frac{k+1}{k+2}\right|=|x|.$$

Thus the interval of convergence is $-1<x<1$. ■

■ **Example 32.7** Determine the complete Maclaurin series and its interval of convergence for

$$f(x)=\frac{1}{1-x}.$$

We have

$$\begin{aligned}
f(x) &= (1-x)^{-1} & &\Rightarrow & f(0) &= 1,\\
f'(x) &= (1-x)^{-2} & &\Rightarrow & f'(0) &= 1,\\
f''(x) &= 2(1-x)^{-3} & &\Rightarrow & f''(0) &= 2,\\
f'''(x) &= 6(1-x)^{-4} & &\Rightarrow & f'''(0) &= 6,\\
&\vdots & &\vdots & &\vdots\\
f^{(k)}(x) &= k!(1-k)^{-k-1} & &\Rightarrow & f^{(k)}(0) &= k!,
\end{aligned}$$

from which follows

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}\frac{k!}{k!}x^k=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+x^4+\dots.$$

This result is rather unsurprising as $\frac{1}{1-x}$ is the infinite sum of a geometric series with common ratio x. We can calculate the interval of convergence as before but as we have a geometric series we can immediately conclude that it converges if $|x|<1$. ■

■ **Example 32.8** Determine the Maclaurin series for the natural exponential e^x and its interval of convergence.

Because the natural exponential is its own derivative, $\frac{d^k(e^x)}{dx^k} = e^x$, we have

$$\left.\frac{d^k e^x}{dx^k}\right|_{x=0} = 1.$$

Formula 32.4 — Maclaurin Series of e^x. Thus the Maclaurin series is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \dots. \tag{32.16}$$

We saw in Example 32.3 that the series converges for all values of x, i.e. the interval of convergence is $-\infty < x < +\infty$. ■

■ **Example 32.9** Determine the Maclaurin series of $f(x) = \sin x$.

$$\begin{aligned}
f(x) &= \sin x &\Rightarrow\quad f(0) &= 0,\\
f'(x) &= \cos x &\Rightarrow\quad f'(0) &= 1,\\
f''(x) &= -\sin x &\Rightarrow\quad f''(0) &= 0,\\
f'''(x) &= -\cos x &\Rightarrow\quad f'''(0) &= -1,\\
f^{(4)}(x) &= \sin x &\Rightarrow\quad f^{(4)}(0) &= 0,
\end{aligned}$$

from which point on the pattern repeats.

Formula 32.5 — Maclaurin Series of sin x. Thus the Maclaurin series is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{32.17}$$

■

■ **Example 32.10** For $\cos x$, we can proceed in the same way,

$$\begin{aligned}
f(x) &= \cos x &\Rightarrow\quad f(0) &= 1,\\
f'(x) &= -\sin x &\Rightarrow\quad f'(0) &= 0,\\
f''(x) &= -\cos x &\Rightarrow\quad f''(0) &= -1,\\
f'''(x) &= \sin x &\Rightarrow\quad f'''(0) &= 0,\\
f^{(4)}(x) &= \cos x &\Rightarrow\quad f^{(4)}(0) &= 1,
\end{aligned}$$

from which point on the pattern repeats.

Formula 32.6 — Maclaurin Series of cos x. Thus the Maclaurin series is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \tag{32.18}$$

Notice that the same result can been found by differentiating the series for $\sin x$ term by term,

$$\cos x = (\sin x)' = \frac{d}{dx}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \dots\right) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \dots.$$

■

Proof of Euler's Equation

We are now in a position to prove Euler's equation,

$$e^{ix} = \cos x + i \sin x, \tag{32.19}$$

by using the Maclaurin series developed above,

$$\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\
&= \cos x + i \sin x.
\end{aligned} \tag{32.20}$$

Taylor Series around Stationary Point

We can Taylor-expand a function $f(x)$ around one of its stationary points $x = x_0$. As the first derivative vanishes by definition at x_0, the Taylor series has the general form

$$f(x) = f(x_0) + a_m(x - x_0)^m + a_{m+1}(x - x_0)^{m+1} + \dots, \tag{32.21}$$

where $m \geq 2$ is the power of the *lowest non-vanishing term* with $a_m \neq 0$ (apart from the constant term $f(x_0)$ giving the function value at the stationary point). We can recast and generalize the conditions to determine the nature of the stationary point based on m and its associated coefficient a_m: if m is even, x_0 is either a local minimum ($a_m > 0$) or a local maximum ($a_m < 0$); if m is odd, x_0 is a point of inflection.

Determination of Taylor Series using Basic Functions

Instead of starting from the definition, we can build the Taylor expansion of more complicated functions from the known expansions of its component, simple functions such as e^x, $\sin x$ and $\cos x$.

■ **Example 32.11** Determine the Maclaurin series of

$$f(x) = \frac{4+5x}{(2+x)(1-x)},$$

up to and including the third power.

We can relate this series to the one for $\frac{1}{1-x}$ previously examined. To do so we decompose into partial fractions,

$$\frac{4+5x}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x},$$

which implies $4 + 5x = A(1-x) + B(2+x)$ and thus $A = -2$ and $B = 3$, leading to

$$\frac{4+5x}{(2+x)(1-x)} = \frac{-2}{(2+x)} + \frac{3}{(1-x)}.$$

Using the previous result

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

with an interval of convergence $|x| < 1$. Likewise, we can write

$$\frac{2}{2+x} = \frac{1}{1-\left(-\frac{x}{2}\right)} = 1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \left(-\frac{x}{2}\right)^3 + \dots = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots,$$

with an interval of convergence $|x/2| < 1$, i.e. $|x| < 2$. Combining both results in the partial decomposition,

$$\begin{aligned}\frac{4+5x}{(2+x)(1-x)} &= -\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\dots\right)+3(1+x+x^2+x^3+\dots)\\ &= 2+\frac{7}{2}x+\frac{11}{4}x^2+\frac{25}{8}x^3+\dots,\end{aligned}$$

which converges for $|x| < 1$, i.e. in the interval within which both series converge. ■

■ **Example 32.12** Determine the Maclaurin series of

$$f(x) = e^{-3x}\cos(2x)$$

up to the third order.

Using the previous result,

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots,$$

with convergence for all values of x. Thus

$$e^{-3x} = 1+(-3x)+\frac{(-3x)^2}{2}+\frac{(-3x)^3}{3!}+\dots = 1-3x+\frac{9x^2}{2}-\frac{9x^3}{2}+\dots.$$

Also, for $\cos x$ we have (the next term will already be of fourth order)

$$\cos x = 1-\frac{x^2}{2}+\dots,$$

with convergence for all values of x. Thus

$$\cos(2x) = 1-\frac{(2x)^2}{2}+\dots = 1-2x^2+\dots.$$

Combining both we have

$$\begin{aligned}e^{-3x}\cos(2x) &= \left(1-3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\dots\right)(1-2x^2+\dots)\\ &= \left(1-3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\dots\right)-2x^2(1-3x+\dots)+\dots\\ &= 1-3x+\frac{5}{2}x^2+\frac{3}{2}x^3+\dots,\end{aligned}$$

by carefully calculating the coefficients of the individual terms. The series converges for all x. ■

■ **Example 32.13** Determine the Maclaurin series of

$$f(x) = \sin x\,\ln(1-2x),$$

up to the fourth order.

Using the previous result (the next term is already fifth order),

$$\sin x = x-\frac{x^3}{3!}+\dots = x-\frac{x^3}{6}+\dots,$$

which converges for all x. We also saw that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

which converges for $|x| < 1$. Therefore

$$\ln(1-2x) = (-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots = -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \dots,$$

which converges for $|-2x| < 1$, i.e. $|x| < 1/2$. Combining both we have

$$\begin{aligned}\sin x\ \ln(1-2x) &= \left(x - \frac{x^3}{6} + \dots\right)\left(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \dots\right)\\ &= -2x^2 - 2x^3 - \frac{7}{3}x^4 + \dots,\end{aligned}$$

which converges for $|x| < 1/2$. ■

32.4 Taylor Series of Multivariate Functions

The construction of Taylor series can be extended to multivariate functions. By successively expanding $f(x,y)$ around a point (x_0,y_0), first in x and then in y, one can show that up to second order (i.e. including terms up to x^2, y^2 and xy), f can be expressed as

$$\begin{aligned}f(x,y) &= f(x_0,y_0) + [f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)]\\ &\quad + \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + f_{yy}(x_0,y_0)(y-y_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)\right] + \dots,\end{aligned} \tag{32.22}$$

with the usual notation $f_{xy}(x_0,y_0)$ etc. indicating partial derivatives of f, evaluated at the point of expansion (x_0,y_0).

■ **Example 32.14** Find the quadratic approximation to the function $f(x,y) = \sin x \sin y$ near the origin.

This corresponds to finding the Maclaurin series at $(0,0)$,

$$\begin{aligned}f(0,0) &= \sin 0\ \sin 0 = 0,\\ f_x(0,0) &= \cos 0\ \sin 0 = 0,\\ f_y(0,0) &= \sin 0\ \cos 0 = 0,\\ f_{xx}(0,0) &= -\sin 0\ \sin 0 = 0,\\ f_{yy}(0,0) &= -\sin 0\ \sin 0 = 0,\\ f_{xy}(0,0) &= \cos 0\ \cos 0 = 1.\end{aligned}$$

Thus up to the second order, we have

$$\sin x \sin y = xy + \dots.$$

■

■ **Example 32.15** Find the second order Maclaurin expansion of $f(x,y) = e^{2x}\sin(3y)$.

This can be done in two different ways. The first is to compute all partial derivatives up to the second order as above,

$$\begin{aligned}
f(0,0) &= e^{2x}\sin(3y)|_{(0,0)} = 0,\\
f_x(0,0) &= 2e^{2x}\sin(3y)|_{(0,0)} = 0,\\
f_y(0,0) &= 3e^{2x}\cos(3y)|_{(0,0)} = 3,\\
f_{xx}(0,0) &= 4e^{2x}\sin(3y)|_{(0,0)} = 0,\\
f_{yy}(0,0) &= -9e^{2x}\sin(3y)|_{(0,0)} = 0,\\
f_{xy}(0,0) &= 6e^{2x}\cos(3y)|_{(0,0)} = 6.
\end{aligned}$$

Thus up to the second order,

$$e^{2x}\sin(3y) = 3y + 6xy + \dots.$$

The second way exploits the single variable Maclaurin expansions,

$$e^x = 1 + x + \frac{x^2}{2} + \dots,$$

and

$$\sin y = y + \dots.$$

Replacing x by $2x$ in the first and y by $3y$ in the second we obtain after multiplication,

$$e^{2x}\sin(3y) = \left(1 + (2x) + \frac{(2x)^2}{2} + \dots\right)(3y + \dots) = 3y + 6xy + \dots,$$

where we retain terms up to the second order. ■

We can use the general Taylor expansion to determine the nature of a stationary point in multivariate functions, which we just stated previously. At a stationary point, the first partial derivatives vanish, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, thus using the Taylor expansion yields

$$f(x,y) - f(x_0,y_0) = \frac{1}{2}\left[f_{xx}(x-x_0)^2 + 2f_{xy}(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right] + \dots, \tag{32.23}$$

where we neglected higher-order terms. This can be rewritten in two ways,

$$\begin{aligned}
f(x,y) - f(x_0,y_0) &= \frac{1}{4f_{xx}}\left[\left(f_{xx}(x-x_0) + f_{xy}(y-y_0)\right)^2 - (y-y_0)^2\left(f_{xy}^2 - f_{xx}f_{yy}\right)\right],\\
&= \frac{1}{4f_{yy}}\left[\left(f_{yy}(y-y_0) + f_{xy}(x-x_0)\right)^2 - (x-x_0)^2\left(f_{xy}^2 - f_{xx}f_{yy}\right)\right]. \qquad (32.24)
\end{aligned}$$

Either way, if $f_{xy}^2 - f_{xx}f_{yy} < 0$, then the terms in the square brackets are always positive and so the sign of $f(x,y) - f(x_0,y_0)$ is completely determined by the sign of f_{xx} and f_{yy}, i.e. there is a maximum or minimum. Otherwise, there exists a direction in which the square bracket can become negative and the function increases in one, but decreases in another direction, corresponding to a saddle point. The nature is determined as previously stated: We have a maximum if

$$f_{xx} < 0, \quad f_{yy} < 0, \quad f_{xx}f_{yy} > f_{xy}^2, \tag{32.25}$$

and a minimum if

$$f_{xx} > 0, \quad f_{yy} > 0, \quad f_{xx}f_{yy} > f_{xy}^2. \tag{32.26}$$

Finally, we have a saddle point if

$$f_{xy}^2 > f_{xx}f_{yy}. \tag{32.27}$$

33. Limits (Riley 4.7)

We have already encountered expressions of the form

$$\lim_{x \to a} f(x), \tag{33.1}$$

i.e. the limit of an expression $f(x)$ as x approaches a certain value a, where $f(x)$ is not necessarily defined at $x = a$. We have used basic techniques and arguments to calculate such limits but we now develop more precise means to do so for more complicated expressions.

33.1 Definition and Basic Properties

Definition 33.1 — Limit of a Function. Consider a function $f(x)$. If we can make $f(x)$ arbitrarily close to a given number L, i.e. $|f(x) - L|$ becomes arbitrarily small, by choosing x sufficiently near to a number a, then L is said to be the limit of $f(x)$ as $x \to a$ and this is expressed as

$$\lim_{x \to a} f(x) = L. \tag{33.2}$$

If this is not possible, we say that the limit does not exist, i.e. in most cases, the function value may approach $\pm\infty$ as $x \to a$.

If $f(x)$ is defined at $x = a$, one simply has $\lim_{x \to a} f(x) = f(a)$. The limit of a function $f(x)$ as x tends to a being $f(a)$ seems reasonable. However $f(x)$ may be undefined for $x = a$. For example,

$$\lim_{x \to 0} \frac{\sin x}{x}, \tag{33.3}$$

which can be shown to be equal to 1 (see below), although $\sin x / x$ is undefined for $x = 0$. In the following, we will develop techniques to calculate the limit in such cases, and to start, we state a few basic properties of limits.

Formula 33.2 — Basic Properties of Limits. If $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then the following relations hold:

- Limit of a Sum/Difference and Factors

$$\lim_{x\to a}[c_1 f(x) \pm c_2 g(x)] = c_1 \lim_{x\to a} f(x) \pm c_2 \lim_{x\to a} g(x) \quad [c_1, c_2 \text{ are constants}]. \tag{33.4}$$

- Limit of a Product

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x). \tag{33.5}$$

- Limit of a Quotient

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \tag{33.6}$$

provided that $\lim_{x\to a} g(x) \neq 0$.

33.2 Evaluating Limits: Taylor Expansion

The main step of calculating a given limit, is to simplify the expression sufficiently such that the value of the limit becomes obvious by inspection. As the limit only concerns the expression $f(x)$ near a point a, we can use Taylor expansion for this task.

■ **Example 33.1** Consider the limit

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

We know that the Maclaurin expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

where terms of order x^7 and higher are neglected. Thus for small values of x we have the combined result

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots.$$

From this we easily derive that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

as all other terms become arbitrarily small when $x \to 0$. ■

■ **Example 33.2** Consider the limit

$$\lim_{x\to 0} \frac{e^x - 1}{x}.$$

We use the Maclaurin expansion of e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

and thus for small values of x,

$$\frac{e^x - 1}{x} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\right) - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots.$$

From this we again derive that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

■

33.3 Evaluating Limits: L'Hôpital's Rule

Limits of quotients as the above frequently occur and we can formalize their treatment within a useful rule. To evaluate a limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} \qquad \text{when} \qquad f(a) = g(a) = 0, \tag{33.7}$$

i.e. when the expression is undefined and of the undetermined form $\frac{0}{0}$, we perform a Taylor expansion of both the numerator and denominator around $x = a$,

$$\frac{f(x)}{g(x)} = \frac{f(a) + (x-a)f'(a) + [(x-a)^2/2!]f''(a) + \ldots}{g(a) + (x-a)g'(a) + [(x-a)^2/2!]g''(a) + \ldots}.$$

As $f(a) = g(a) = 0$, and dividing by $(x-a)$ this gives

$$\frac{f(x)}{g(x)} = \frac{f'(a) + [(x-a)/2!]f''(a) + \ldots}{g'(a) + [(x-a)/2!]g''(a) + \ldots}.$$

Formula 33.3 — L'Hôpital's Rule: $\frac{0}{0}$. We then arrive at L'Hôpital's rule for calculating a limit of the undetermined form $\frac{0}{0}$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \qquad \text{when} \qquad f(a) = g(a) = 0, \tag{33.8}$$

provided that $g'(a) \neq 0$.

If the right-hand side is again of the form $\frac{0}{0}$, i.e. $f'(a) = g'(a) = 0$, the process can be repeated,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}, \tag{33.9}$$

provided that $g''(a) \neq 0$. If the limit exists, it is usually (but not always) possible to stop the process at a finite number of derivatives,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}, \tag{33.10}$$

with $g^{(n)}(a) \neq 0$.

Formula 33.4 — L'Hôpital's Rule: $\frac{\infty}{\infty}$, $\mathbf{x \to \infty}$. Note that l'Hôpital's also applies for limits when $x \to \infty$ and also for the case $f(x \to a), g(x \to a) \to \infty$ (and the combination), i.e.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{when} \qquad f(x \to a), g(x \to a) \to \pm\infty, \tag{33.11}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)}, \tag{33.12}$$

where $\frac{f(x)}{g(x)}$ can be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to \infty$ in the latter case.

This is for example because we can relate it to the standard form of l'Hôpital's rule by introducing a new variable y such that $x = \frac{1}{y}$,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{y \to 0} \frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)} = \lim_{y \to 0} \frac{-\frac{1}{y^2} f'\left(\frac{1}{y}\right)}{-\frac{1}{y^2} g'\left(\frac{1}{y}\right)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}. \tag{33.13}$$

■ **Example 33.3** Determine the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}.$$

This is an indeterminate form of the type $\frac{0}{0}$. L'Hôpital's rule gives

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2.$$

■

■ **Example 33.4** Determine the limit

$$\lim_{x \to 0} \frac{\sin(3x)}{\sinh x}.$$

This is an indeterminate form of the type $\frac{0}{0}$. L'Hôpital's rule gives

$$\lim_{x \to 0} \frac{\sin(3x)}{\sinh x} = \lim_{x \to 0} \frac{3\cos(3x)}{\cosh x} = 3.$$

■

■ **Example 33.5** Limits are not always given in term of quotients: Determine the limit

$$\lim_{x \to 0} (x \ln x).$$

This is an indeterminate form of the type $0 \cdot \infty$. In order to apply l'Hôpital's rule, we re-arrange as follows,

$$\lim_{x \to 0} (x \ln x) = \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{(\ln x)'}{(1/x)'} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} (-x) = 0.$$

■

■ **Example 33.6** Determine the limit

$$\lim_{x\to 1} \frac{1+\cos(\pi x)}{x^2-2x+1}.$$

This is an indeterminate form of the type $\frac{0}{0}$. Here, we need to apply l'Hôpital's rule twice,

$$\lim_{x\to 1} \frac{1+\cos(\pi x)}{x^2-2x+1} = \lim_{x\to 1} \frac{-\pi\sin(\pi x)}{2x-2} = \lim_{x\to 1} \frac{-\pi^2\cos(\pi x)}{2} = \frac{\pi^2}{2}.$$

■

■ **Example 33.7** Determine the limit

$$\lim_{x\to\infty} \left(x^2 e^{-x}\right).$$

This is an indeterminate form of the type $\infty \cdot 0$. In order to apply l'Hôpital's rule, we re-arrange,

$$\lim_{x\to\infty} \left(x^2 e^{-x}\right) = \lim_{x\to\infty} \frac{x^2}{e^x},$$

which is an indeterminate form of the type $\frac{\infty}{\infty}$. We apply l'Hôpital's rule twice,

$$\lim_{x\to\infty} \left(x^2 e^{-x}\right) = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

■

■ **Example 33.8** Determine the limit

$$\lim_{x\to\infty} x^{1/x}.$$

This is an indeterminate form of the type ∞^0. To apply l'Hôpital's rule we take the exponential of the logarithm of the expression,

$$\lim_{x\to\infty} x^{1/x} = \exp\left[\ln\left(\lim_{x\to\infty} x^{1/x}\right)\right] = \exp\left[\lim_{x\to\infty}\left(\ln x^{1/x}\right)\right] = \exp\left[\lim_{x\to\infty} \frac{\ln x}{x}\right].$$

The limit inside the exponential is an indeterminate form of the type $\frac{\infty}{\infty}$. We can therefore apply l'Hôpital's rule,

$$\lim_{x\to\infty} x^{1/x} = \exp\left[\lim_{x\to\infty} \frac{\ln x}{x}\right] = \exp\left[\lim_{x\to\infty} \frac{1}{x}\right] = e^0 = 1.$$

■

■ **Example 33.9** Determine the limit

$$\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right).$$

This is an indeterminate form of the type $\infty - \infty$. We re-arrange to a form appropriate for the application of l'Hôpital's rule,

$$\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x},$$

which is of the form $\frac{0}{0}$. We now need to apply l'Hôpitals rule repeatedly,

$$
\begin{aligned}
\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} &= \lim_{x\to 0}\frac{2x-\sin(2x)}{2x\sin^2 x+x^2\sin(2x)}\\
&= \lim_{x\to 0}\frac{2-2\cos(2x)}{2\sin^2 x+4x\sin(2x)+2x^2\cos(2x)}\\
&= \lim_{x\to 0}\frac{4\sin(2x)}{6\sin(2x)+12x\cos(2x)-4x^2\sin(2x)}\\
&= \lim_{x\to 0}\frac{8\cos(2x)}{24\cos(2x)-24x\sin(2x)-8x\sin(2x)-8x^2\cos(2x)}\\
&= \frac{1}{3}.
\end{aligned}
$$

Alternatively, the limit can be calculated by Taylor expansion,

$$
\begin{aligned}
\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} &= \lim_{x\to 0}\frac{x^2-\left(x-\frac{x^3}{6}+\dots\right)^2}{x^2\left(x-\frac{x^3}{6}+\dots\right)^2} = \lim_{x\to 0}\frac{x^2-\left(x^2-\frac{x^4}{3}+\dots\right)}{x^2\left(x^2-\frac{x^4}{3}+\dots\right)}\\
&= \lim_{x\to 0}\frac{\frac{x^4}{3}+\dots}{x^4-\frac{x^6}{3}+\dots} = \lim_{x\to 0}\frac{\frac{1}{3}+\dots}{1-\frac{x^2}{3}+\dots}\\
&= \frac{1}{3}.
\end{aligned}
$$

■