Exact and inexact differentials

General differential expression

$$\boxed{A(x,y)\,dx + B(x,y)\,dy}$$

$$\boxed{\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \; ?}$$

Yes

No

Differential is exact, corresponds to function differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Differential is inexact; no function f can be found

Finding function f of exact differential:

- „Partially" integrate $\frac{\partial f}{\partial x} = A(x,y)$, $\frac{\partial f}{\partial y} = B(x,y)$

Example: $df = \left(-\frac{1}{x} - \frac{1}{y}\right) dx + \frac{x}{y^2}\, dy$

$$\frac{\partial f}{\partial x} = -\frac{1}{x} - \frac{1}{y} \Rightarrow f(x,y) = \int \left(-\frac{1}{x} - \frac{1}{y}\right) dx + C(y)$$

$$= \underline{-\ln x - \frac{x}{y} + C(y)}$$

$$\frac{\partial f}{\partial y} = \frac{x}{y^2} \Rightarrow f(x,y) = \int \frac{x}{y^2}\, dy + D(x) = \underline{-\frac{x}{y} + D(x)}$$

$$\Rightarrow -\ln x \cancel{- \frac{x}{y}} + C(y) \overset{\text{for all } x,y}{=} \cancel{-\frac{x}{y}} + D(x)$$

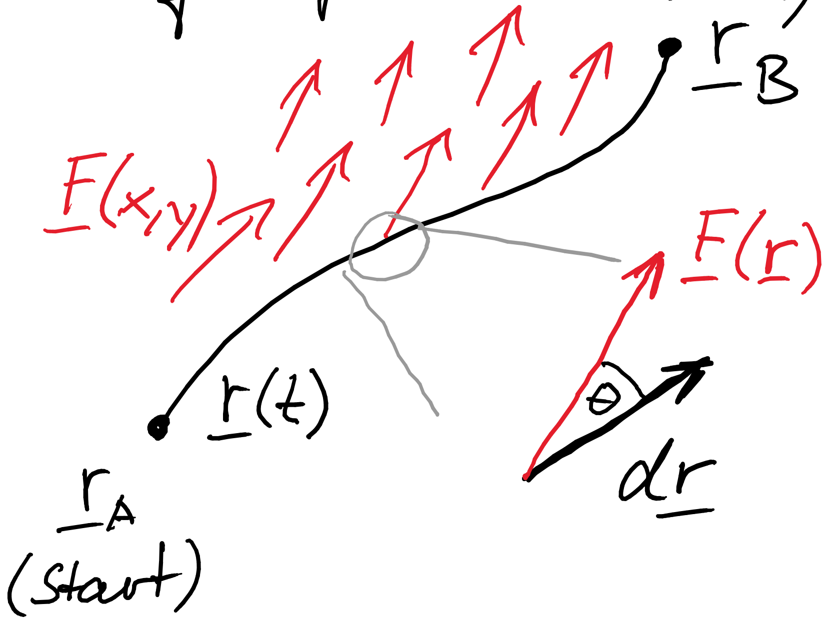
$$\Rightarrow D(x) = -\ln x + D$$
$$C(y) = C$$

$$\Rightarrow f(x,y) = -\ln x - \frac{x}{y} + C$$

Line Integrals

- Integration over a vector field along a path

$$\boxed{I = \int_c \underline{F} \cdot d\underline{r}} = \lim_{d\underline{r} \to 0} \sum_c \underline{F}(\underline{r}_i) \cdot d\underline{r}$$



- Transform to ordinary integral

$$\boxed{I = \int_{t_A}^{t_B} \left[\underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} \right] dt}$$

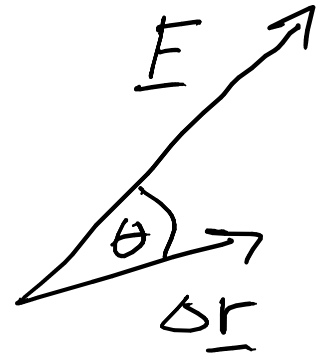
with $\underline{r}_A = \underline{r}(t_A)$

$\underline{r}_B = \underline{r}(t_B)$

Physics application: Work done in a force field

- Work = Force × Displacement

$$W = -F \cdot \Delta x$$



- Vector Form

$$W = -\underline{F} \bullet \Delta\underline{r} = -\underbrace{\left(|\underline{F}|\cos\theta\right)}_{\text{component of } \underline{F} \text{ along } \Delta\underline{r}} |\Delta\underline{r}|$$

- Position-dependent force $\rightarrow$ Vector field

$$W = -\int_C \underline{F} \cdot d\underline{r}$$

Example: $\underline{F} = \begin{pmatrix} xy \\ -y^2 \end{pmatrix}$, C: Straight line from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Parametrization: $\underline{r}(t) = \begin{pmatrix} 2t \\ t \end{pmatrix}$

$$\Rightarrow I = \int_C \underline{F} \cdot d\underline{r}$$

$$= \int_{t_A}^{t_B} \underline{F} \cdot \frac{d\underline{r}}{dt} \, dt$$

$$= \int_0^1 \begin{pmatrix} xy \\ -y^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} dt = \int_0^1 (2xy - y^2) \, dt$$

$$= \int_0^1 \left(2(2t)t - t^2\right) dt = 1$$