PHAS1202 - Atoms, Stars and The Universe
Extra Problem Sheet
Model Answer

These problems are provided for extra practise and exam preparation. Solutions will be made available in Moodle.

Useful constants

Planck's constant h is 6.6×10^{-34} Js (2 s.f.).

The mass of an electron is 9.1×10^{-31}kg.

1 Electron Volt (eV) is 1.6×10^{-19}Joules.

1 Angstrom is 10^{-10}m.

1: Rayleigh-Jeans law vs Planck's law

Prior to Planck's black-body radiation law, the best theoretical model for Black Body radiation led to a formula known as the Rayleigh-Jeans law. While Rayleigh-Jeans law had unphysical consequences for high-frequency light (the ultraviolet catastrophe), for low-frequency light it did match predictions very well. 3. By expanding the exponential as a series, show that in the limit that the wavelength λ is very

long, Planck's law converges to the Rayleigh-Jeans law.

The Rayleigh-Jeans law is:

$$I(\lambda, T) = \frac{2\pi\, c\, k\, T}{\lambda^4}$$

Planck's law is:

$$I(\lambda, T) = \frac{2\pi\, h\, c^2}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

The series expansion for e^x is:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

2: de Broglie wavelength

Felix Baumgartner was in the news recently for performing the highest ever sky-dive.

2.a) Felix ascended in his balloon at a rate of around 2 ms^{-1}. What was the de Broglie wave-length of Felix and his balloon. (Treat Felix, his suit and balloon as a single object).

2.b) Felix then dropped from his balloon and fell towards Earth reaching a top speed of 372 ms^{-1}. What was the de Broglie wave-length of Felix at this point. (Treat Felix and his suit as a single object).

2.c) What would be the de Broglie wave-length of an electron travelling at Felix's top speed?

Assume Felix Baumgarnter weighs 90 kg. His space-suit weighed 118 kg and his balloon had mass 1315 kg.

3: Solving the Time Independent Schrödinger Equation for a free particle

In lectures we saw that a general sinusoidal function:

$$\psi(x) = A \sin\left(\frac{2\pi(x-\phi)}{\lambda}\right)$$

was a solution of the TISE for a free particle, and we calculated the associated energy. In this question, you are going to attempt to solve the TISE for a free particle yourselves, using a variety of trial functions.

The TISE for a free particle is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

3.a) In lectures you heard that the only classes of functions which are proportional to their second derivative have sine-like and exponential-like forms. Let's look at some other functions and see how they fail. First let's consider

$$\psi(x) = x^2$$

Show that this function is *not* a solution to the free particle TISE.

3.b) Is there any (finite) value of n for which

$$\psi(x) = x^n$$

could be a solution to the TISE for a free particle?

3.c) Now consider the following function:

$$\psi(x) = A\sin(ax) + B\cos(bx)$$

Find an expression for the ratio

$$\frac{d^2\psi(x)}{dx^2} \div \psi(x).$$

Show that if $b = a$ this ratio is a constant, and that the wavefunction is solution to the TISE and calculate the energy of the particle as a function of parameters a, A and B.

4: The Scanning Tunnelling Microscope (STM)

In lectures, you saw that the probability for a quantum particle with mass m energy E, to tunnel through a square barrier of width L and height U, was given by:

$$P = \exp[-2CL]$$

where

$$C = \frac{\sqrt{2m(U-E)}}{\hbar}.$$

In an STM, electrons tunnel across a potential barrier to a surface, completing a circuit which then has a current I proportional to the tunnelling probability.

$I \propto \exp[-2CL]$

The barrier height is approximately the same as the work-function of the electron, typically on the order of a few electron Volts, while the barrier width is the distance between electrode and surface.

4.a) For an STM, with electron energy 1eV and barrier height 4eV calculate C.

4.b) If initially the current is 1 Amp what will be the current if the surface height increases (and thus the barrier width decreases) by 1 Angstrom?

5: Quantum Hydrogen Atom

In the final lecture of this course, we saw that wavefunction solutions to the TISE were indexed by three integer *quantum numbers* n, l and m, where

- n is any non-negative non-zero integer, e.g. $1, 2, 3, \ldots$.
- l is any non-negative integer less than n, e.g. $0, 1, 2, \ldots, n-1$.
- m is any integer such that $|m| \leq l$, e.g. $-l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$.

The energy of the Hydrogen atom states is a function of n (the principle quantum number) only, and satisfies a formula given in lectures.

5.a) What is the energy of the $n = 3$, $l = 2$, $m = -2$ state?

5.b) Which combinations of quantum numbers correspond states of energy $E = -13.6$eV? (Without taking spin into account) how many different wavefunctions are there with this energy?

5.c) (Without considering spin) how many different wavefunctions have quantum number $n = 2$?

5.d) Find a formula which describes the the number of different wavefunctions for a general n.
Hint: Write out the possible allowed combinations of quantum numbers for n =1, 2, 3, 4 etc. and look for a pattern - in particular look for an arithmetic series. You may use the following identity for an arithmetic series without proving it:

$$\sum_{j=1}^{n} a_j = \frac{n}{2}(a_1 + a_n)$$

where $a_j = a_{j-1} + d$, and d is a constant.

Dan Browne, November 2015