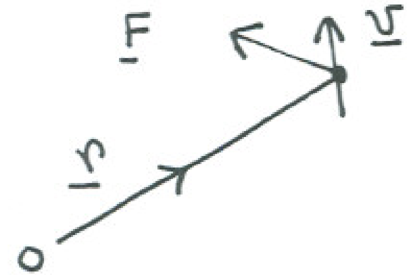
Solutions to 1B47 Exam, 2005

1.



Angular momentum $\underline{L} = \underline{r} \times \underline{p} = m\underline{r} \times \underline{v}$

Torque of force $\underline{\tau} = \underline{r} \times \underline{F}$

But $\underline{F} = \frac{d\underline{p}}{dt} = m\frac{d\underline{v}}{dt}$

$\therefore \underline{\tau} = m\underline{r} \times \frac{d\underline{v}}{dt}$

Consider $\frac{d}{dt}(\underline{r} \times \underline{v}) = \frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \frac{d\underline{v}}{dt}$

$= \cancel{\underline{v} \times \underline{v}} + \underline{r} \times \frac{d\underline{v}}{dt}$

$\therefore \underline{\tau} = m\frac{d}{dt}(\underline{r} \times \underline{v}) = \frac{d\underline{L}}{dt}$

A central force is such that

$\underline{F}(\underline{r}) = F(r)\hat{r}$

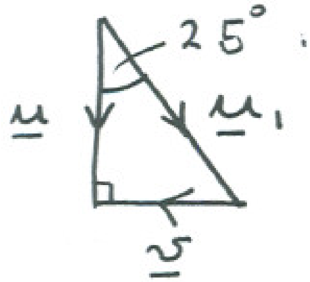
where $\hat{r}$ is radial outgoing unit vector, i.e. $\underline{F}$ is radially inwards or radially outwards.

Torque of central force is

$\underline{\tau} = \underline{r} \times (F(r)\hat{r}) = 0 \quad \therefore \frac{d\underline{L}}{dt} = 0$

$\therefore \underline{L}$ is constant

2.



Velocity of rain relative to athlete is $\underline{u}_1 = \underline{u} - \underline{v}$

$v = 20$ km/hr

From above velocity diagram

$$\frac{v}{u} = \tan 25^\circ$$

$$\therefore u = \frac{v}{\tan 25^\circ} = \frac{20}{0.466} = \underline{42.9 \text{ km/hr}}$$

Velocity of rain relative to ground when wind is blowing with velocity $\underline{w}$ is

$\underline{u}' = \underline{u} + \underline{w}$

$\therefore$ velocity of rain relative to runner is $\underline{u}_2 = \underline{u}' - \underline{v}$

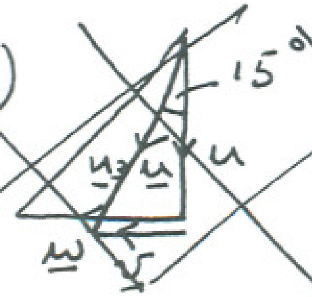
If rain hits ~~front~~ of athlete, velocity diagram is as above

$$\therefore \frac{(v - w)}{u} = \tan 15^\circ$$

$\therefore 20 - w = 42.9 \times 0.27 = 11.5$

$\therefore$ wind speed $\underline{w = 20 - 11.5 = 8.5 \text{ km/hr}}$

~~b)~~

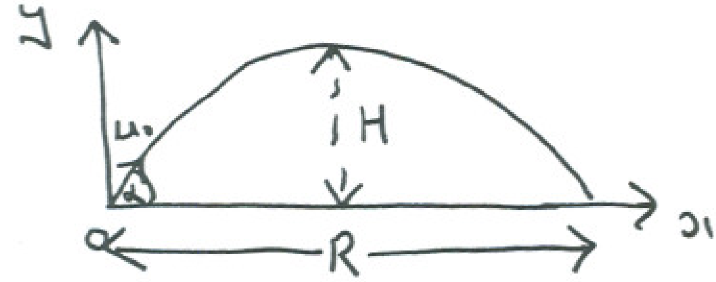


~~If rain hits back of athlete~~

$$\cancel{\frac{w - v}{u} = \tan 15^\circ = 0.27}$$

$\cancel{\therefore w - 20 = 42.9 \times 0.27}$ $\cancel{\therefore w = 31.5 \text{ km/hr}}$

3.



$$x = (u_0 \cos\alpha)\, t$$

$$y = (u_0 \sin\alpha)\, t - \frac{1}{2} g t^2$$

a) When $y = 0$, $0 = u_0 \sin\alpha\, t - \frac{1}{2} g t^2$

$\therefore t = 0$ (at launch) or $\underline{t_1 = \frac{2 u_0 \sin\alpha}{g}}$

b) Range $R = u_0 \cos\alpha\, t_1$

$\therefore R = \frac{2 u_0^2 \sin\alpha \cos\alpha}{g} = \underline{\frac{u_0^2 \sin 2\alpha}{g}}$

c) Maximum height $y = H$ when

$$0 = u_0^2 \sin^2\alpha - 2 g H$$

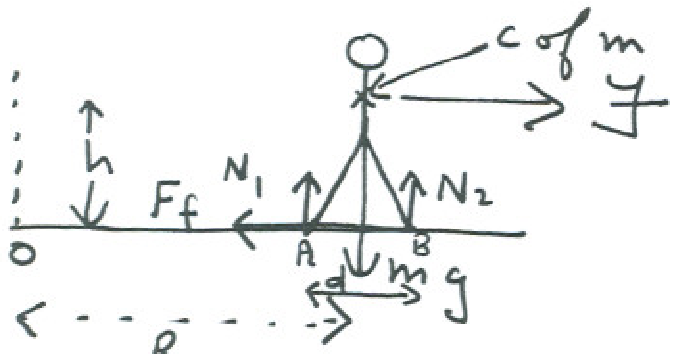
$$\therefore \underline{H = \frac{u_0^2 \sin^2\alpha}{2g}}$$

<u>or</u> $H = u_0 \sin\alpha \left(\frac{1}{2} t_1\right) - \frac{1}{2} g \left(\frac{1}{2} t_1\right)^2$

$$= \frac{u_0^2 \sin^2\alpha}{g} - \frac{1}{2} g \frac{u_0^2 \sin^2\alpha}{g^2}$$

$$\therefore H = \underline{u_0^2 \sin^2\alpha / (2g)}$$

4.



F_f is centripetal force of friction

Real forces are N_1, N_2, mg and F_f

$$F_f = mv^2/R \quad , \quad N_1 + N_2 = mg$$

$\mathcal{F}$ = fictitious centrifugal force

$= mv^2/R$

Take moments about A, for equilibrium.

$$mg\frac{d}{2} + \mathcal{F}h - N_2 d = 0$$

$$\therefore N_2 = \frac{1}{d}\left(mg\frac{d}{2} + \frac{mv^2}{R}h \right)$$

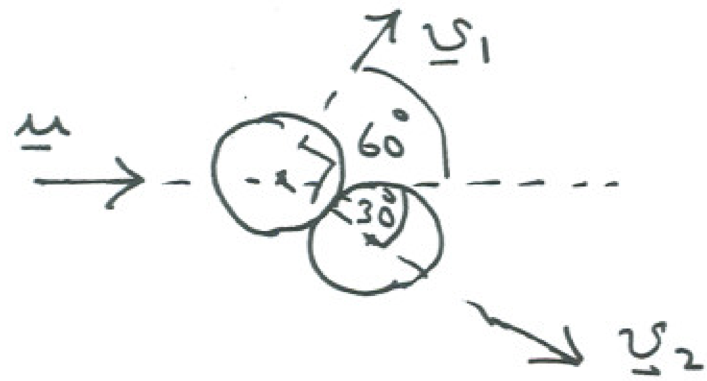
$$\therefore N_2 = \frac{mg}{2} + \frac{mv^2 h}{Rd}$$

$$\therefore N_1 = mg - N_2 = \frac{mg}{2} - \frac{mv^2 h}{Rd}$$

Speed at which $N_1 = 0$ is v_1

$$\therefore \frac{\cancel{m}g}{2} = \frac{\cancel{m}v_1^2 h}{Rd}$$

$$\therefore v_1 = \sqrt{\frac{gRd}{2h}}$$

5. 

Momentum is conserved

$\therefore m\underline{u} = m\underline{v}_1 + m\underline{v}_2$

Kinetic energy is conserved

$\therefore \frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$

$\therefore \underline{u} = \underline{v}_1 + \underline{v}_2 \quad - - - (1)$

and $u^2 = v_1^2 + v_2^2 \quad - - - (2)$

~~From~~ (1), $u^2 = \underline{u}.\underline{u} = (\underline{v}_1 + \underline{v}_2).(\underline{v}_1 + \underline{v}_2)$

$\therefore u^2 = v_1^2 + v_2^2 + 2\underline{v}_1.\underline{v}_2$

$\therefore$ comparing with (2), $\underline{\underline{v}_1.\underline{v}_2 = 0}$

$\therefore \underline{v}_1$ is orthogonal to $\underline{v}_2$.

Angles as shown in diagram

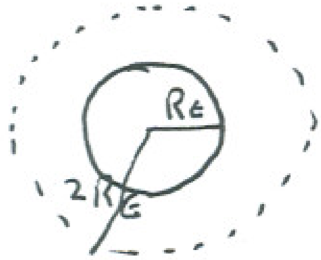
$\therefore u = v_2\sqrt{3}/2 + v_1/2$

$v_1\sqrt{3}/2 = v_2/2$

$\therefore v_2 = \sqrt{3}\,v_1$, $\therefore u = v_1 3/2 + v_1/2 = 2v_1$

$\underline{\therefore v_1 = u/2}$ and $\underline{v_2 = u\sqrt{3}/2}$

6.



For particle on Earth's surface

$$\frac{GmM_E}{R_E^2} = mg$$

$$\therefore g = \frac{GM_E}{R_E^2}$$

For satellite in circular orbit of radius $2R_E$

$$\frac{G\cancel{m}M_E}{(2R_E)^{\cancel{2}}} = \frac{\cancel{m}v^2}{\cancel{2R_E}}$$

$$\therefore v = \sqrt{\frac{GM_E}{2R_E}} = \sqrt{\frac{gR_E^2}{2R_E}} = \underline{\sqrt{\frac{gR_E}{2}}}$$

$$\therefore K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{gR_E}{2} = \frac{1}{4}mgR_E$$

Potential energy of satellite is

$$V = -\frac{GM_Em}{2R_E} = -\frac{gR_E^2m}{2R_E} = \underline{-\frac{1}{2}gmR_E}$$

$$\therefore \text{total energy is } E = V + K.E. = \underline{-\frac{1}{4}gmR_E}$$

If $v \to \sqrt{2}\,v$

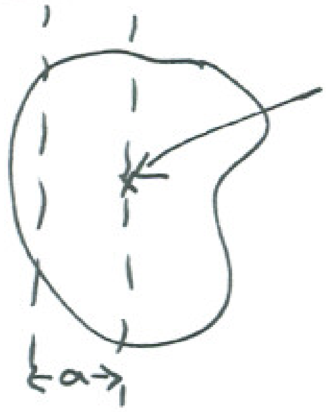
$$K.E. \to 2\times\left(\frac{1}{4}mgR_E\right) = \frac{1}{2}mgR_E$$

Change in potential energy during increase in speed is negligible

$$\therefore E = \frac{1}{2}mgR_E - \frac{1}{2}mgR_E = \underline{0}$$

$\therefore$ trajectory is a <u>parabola</u>.

7 <u>Theorem of parallel axes</u>



Body of mass M

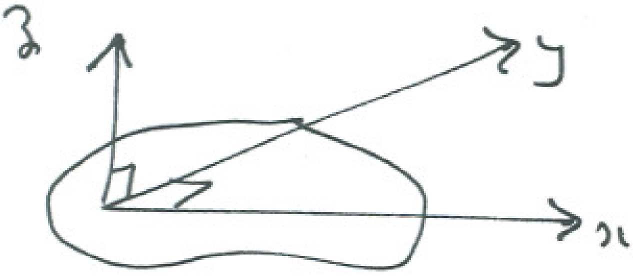
I_0 = moment of inertia about axis through c. of m.

I_A = moment of inertia about parallel axis, distance a apart

$$\underline{I_A = I_0 + Ma^2}$$

applies to any body

<u>Theorem of perpendicular axes</u>



Consider lamina in x - y plane

Moments of inertia of lamina about any x- and y- axes in plane of lamina are I_x and I_y respectively

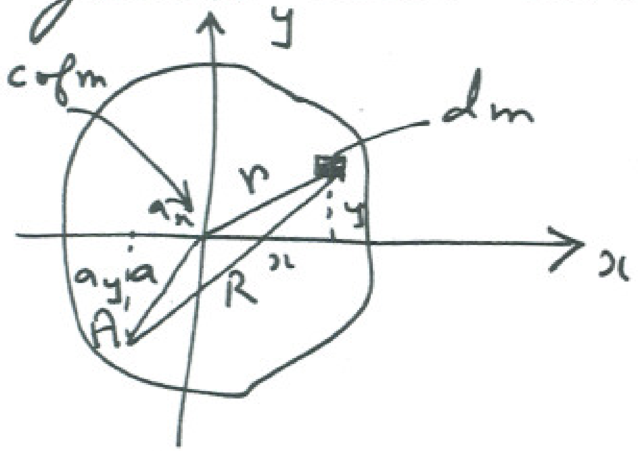
$$\therefore \underline{I_z = I_x + I_y}$$

where I_z = moment of inertia about axis $\perp$ plane of lamina through point of intersection of x- and y- axes.

7 (continued)

Proof of parallel axes theorem.

Moment of inertia depends only on perpendicular distance of each mass element from the axis under consideration, ∴ project all mass elements onto x-y plane for which given axes are normals.



$$I_0 = \int_{\text{area}} r^2\,dm$$

$$= \int_{\text{area}} (x^2+y^2)\,dm$$

$$I_A = \int_{\text{area}} R^2\,dm = \int\left[(x+a_x)^2 + (y+a_y)^2\right]dm$$

$$= \int (x^2+y^2)\,dm + \int (a_x^2+a_y^2)\,dm$$

$$+ \int_{\text{area}} 2a_x x\,dm + \int_{\text{area}} 2a_y y\,dm$$

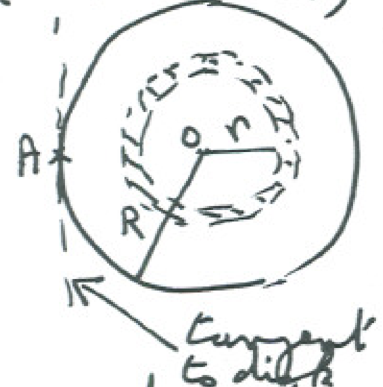
But $\int a_x x\,dm = 0$ and $\int a_y y\,dm = 0$

as body balances about any axis through centre of mass

also $\int dm = M$

$$\therefore \underline{I_A = I_0 + Ma^2}$$

7 (continued)



Mass per unit area is $\rho = \frac{M}{\pi R^2}$

$\therefore$ mass of shaded ring is

$$dm = 2\pi r\, dr\, \rho$$

and its moment of inertia about axis through O $\perp$ plane of disk is

$$dI_o = r^2 dm = 2\pi r^3 dr\, \rho$$

$$\therefore I_o = 2\pi\rho\int_0^R r^3 dr = 2\pi \frac{M}{\pi R^2}\frac{1}{4}R^4$$

$$\therefore \underline{I_o = \frac{1}{2}MR^2}$$

From perpendicular axes theorem

$$I_o = I_{ox} + I_{oy} = 2I_{ox} \quad \text{by symmetry}$$

$$\therefore \underline{I_{ox} = \frac{1}{4}MR^2} = \text{moment of inertia about a diameter}$$

By parallel axes theorem

$$I_A = I_{oy} + MR^2$$

$$= \frac{1}{4}MR^2 + MR^2 = \underline{\frac{5}{4}MR^2}$$

For the 'doughnut' we must <u>remove</u> a circular disk of radius r_o and mass

$$m = \pi r_o^2 \rho = \pi r_o^2 \frac{M}{\pi R^2} = \frac{r_o^2}{R^2}M$$

7 (continued)

$\therefore$ moment of inertia of this disk about a diameter is

$$I_{0x} = \frac{1}{4} m r_0^2 = \frac{1}{4} M \frac{r_0^4}{R^2}$$

$\therefore$ moment of inertia of 'doughnut' about diameter is

$$I_1 = \frac{1}{4} M R^2 - \frac{1}{4} M \frac{r_0^4}{R^2} = \underline{\frac{1}{4} M \frac{(R^4 - r_0^4)}{R^2}}$$

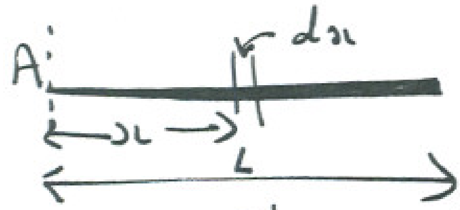
~~and moment of inertia about a tangent is~~

$$\cancel{I_2 = I_1 + (M-m) R^2}$$

$$\cancel{= \frac{1}{4} M \frac{(R^4 - r_0^4)}{R^2} + M \left(\frac{R^2 - r_0^2}{R^2}\right) R^2}$$

$$\cancel{= \underline{\frac{1}{4} M \left(\frac{5R^4 - 4R^2 r_0^2 - r_0^4}{R^2}\right)}}$$

8.



Mass of element dx is $dm = \rho\, dx = Kx^2 dx$

$$\therefore M = \int_0^L Kx^2\,dx = \frac{1}{3}KL^3$$

$$\therefore K = \frac{3M}{L^3}$$

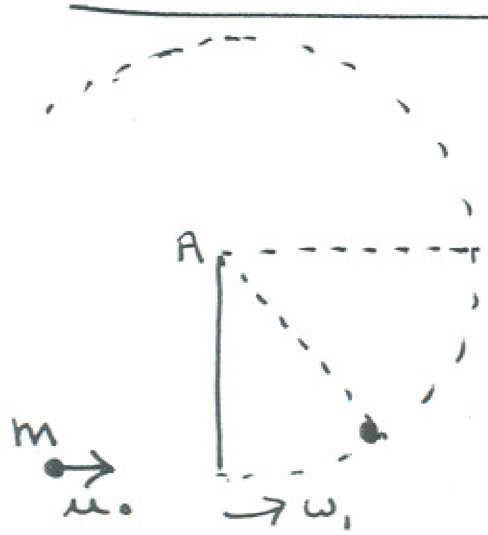
Distance of centre of mass from A is $X = \dfrac{\int x\,dm}{M} = \dfrac{\int_0^L Kx^3\,dm}{M} = \dfrac{\frac{1}{4}KL^4}{M}$

$$\therefore X = \frac{\frac{1}{4}L^4\frac{3M}{L^3}}{M} = \frac{3}{4}L$$

Moment of inertia about axis through A $\perp$ rod is

$$I_A = \int x^2\,dm = \int_0^L Kx^4\,dx = \frac{1}{5}KL^5$$

$$\therefore I_A = \frac{3}{5}ML^2$$



Angular momentum is conserved

8 (continued)

After putty has stuck, moment of inertia about axis through A $\perp$ rod is

$$I_1 = I_A + mL^2 = L^2\left(\frac{3}{5}M + m\right)$$

$\therefore$ from angular momentum conservation

$$m u_0 L = I_1 \omega_1$$

$$\therefore \omega_1 = \frac{m u_0 \cancel{L}}{L^{\cancel{2}}\left(\frac{3}{5}M+m\right)} = \underline{\frac{m u_0}{\left(\frac{3}{5}M+m\right)L}}$$

Kinetic energy immediately after collision is

$$\frac{1}{2} I_1 \omega_1^2 = \frac{1}{2}\cancel{L^2}\cancel{\left(\frac{3}{5}M+m\right)}\frac{m^2 u_0^2}{\left(\frac{3}{5}M+m\right)^{\cancel{2}}\cancel{L^2}}$$

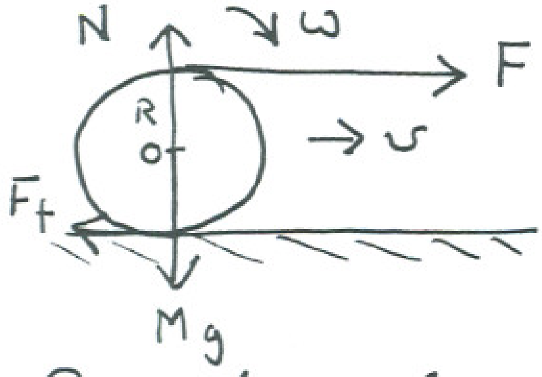
$$\therefore \underline{K.E. = \frac{1}{2}\frac{m^2 u_0^2}{\left(\frac{3}{5}M+m\right)}}$$

If rod, with mass m attached, is just to reach horizontal, then from conservation of energy

$$\frac{1}{2}\frac{m^2 u_0^2}{\left(\frac{3}{5}M+m\right)} = mgL + MgX = gL\left(m + \frac{3}{4}M\right)$$

$$\therefore \underline{u_0 = \frac{1}{m}\sqrt{\left(\frac{3}{5}M+m\right)\left(m+\frac{3}{4}M\right)2gL}}$$

9.



F_f = force of friction

$N = Mg$

Equation of motion of centre of mass is

$$F - F_f = M\frac{dv}{dt} = Ma \quad \text{---} \; ①$$

Equation of motion for rotation about O

$$FR + F_f R = I_0 \frac{d\omega}{dt} \quad \text{--} \; ②$$

If no sliding

$$v = R\omega$$

$$\therefore \frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$\therefore \frac{F - F_f}{M} = \frac{R^2(F + F_f)}{I_0}$$

$$\therefore F_f (MR^2 + I_0) = F(I_0 - MR^2)$$

$$\therefore F_f = F\frac{(I_0 - MR^2)}{(I_0 + MR^2)}$$

$\therefore$ from eqn 1, acceleration $a = (F - F_f)/M$

$$= \frac{F}{M}\left[1 - \frac{(I_0 - MR^2)}{(I_0 + MR^2)}\right]$$

$$\therefore a = \frac{F}{M}\frac{2MR^2}{(I_0 + MR^2)} = \frac{2F}{M(I_0/MR^2 + 1)}$$

9 (continued)

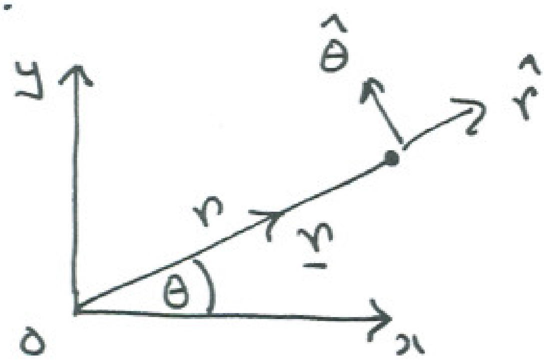
For a hollow cylinder

$I_o = MR^2$

$\therefore \underline{F_f = 0}$ and $\underline{a = \dfrac{F}{M}}$

[Note: because for any cylinder other than a hollow cylinder, $I_o < MR^2$, the expression for the frictional force yields a negative value, i.e. the frictional force is in the opposite direction to that indicated in the diagram.]

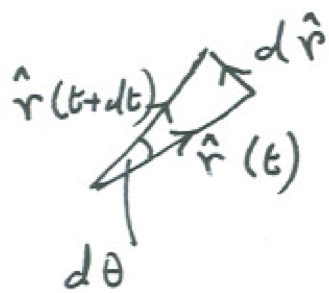
10.



$$\underline{r} = r\,\hat{r}$$

$$\therefore \underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

Consider change in $\hat{r}$ in time dt



From diagram

$$d\hat{r} = 1\,d\theta\,\hat{\theta}$$

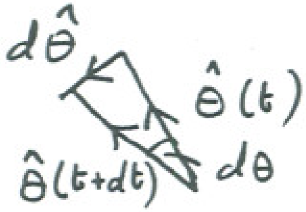
$$\therefore \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}$$

$$\therefore \underline{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = \underline{\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right)$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt}$$

Consider change in $\hat{\theta}$ in time dt



From diagram

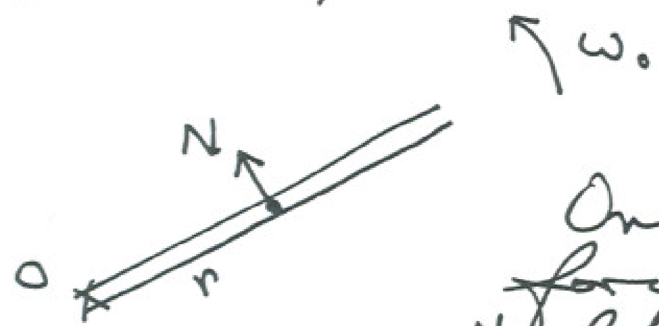
$$d\hat{\theta} = 1\,d\theta(-\hat{r})$$

$$\therefore \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}$$

$$\therefore \underline{a} = \frac{d^2r}{dt^2}\hat{r} + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\right)\hat{r}$$

$$\therefore \underline{\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$$

10 (continued)



Only horizontal ~~force~~ on particle is N after particle released.

$$\therefore m\left[(\ddot r - r\dot\theta^2)\hat r + (2\dot r\dot\theta + r\ddot\theta)\hat\theta\right] = N\hat\theta$$

But $\dot\theta = \omega_0 = \text{const}$, $\therefore \ddot\theta = 0$

$\therefore$ radial eqn of motion is

$$\underline{m\left[\ddot r - r\omega_0^2\right] = 0}$$

Transverse eqn of motion is

$$\underline{m\left[2\dot r\omega_0\right] = N}$$

If $r = Ae^{\omega_0 t} + Be^{-\omega_0 t}$

$\ddot r = +\omega_0^2\left(Ae^{\omega_0 t} + Be^{-\omega_0 t}\right)$

$\therefore$ this function trivially is a solution to the radial eqn.

$\dot r = \omega_0 Ae^{\omega_0 t} - \omega_0 Be^{-\omega_0 t}$

$\therefore$ inserting initial conditions

$r_0 = A + B$

$0 = \omega_0(A - B)$

$\therefore \underline{A = B = \frac{1}{2} r_0}$

$\therefore r = \frac{1}{2} r_0\left(e^{\omega_0 t} + e^{-\omega_0 t}\right)$

$\dot r = \frac{1}{2} r_0\omega_0\left(e^{\omega_0 t} - e^{-\omega_0 t}\right)$

$\therefore v = \sqrt{\dot r^2 + r^2\omega_0^2}$

10 (continued)

$$\therefore v^2 = \frac{1}{4} r_0^2 \omega_0^2 \left(e^{2\omega_0 t} + e^{-2\omega_0 t} - 2\right)$$
$$+ \frac{1}{4} r_0^2 \omega_0^2 \left(e^{2\omega_0 t} + e^{-2\omega_0 t} + 2\right)$$
$$= \frac{1}{2} r_0^2 \omega_0^2 \left(e^{2\omega_0 t} + e^{-2\omega_0 t}\right)$$

$$\therefore \text{kinetic energy} = \frac{1}{4} m r_0^2 \omega_0^2 \left(e^{2\omega_0 t} + e^{-2\omega_0 t}\right)$$

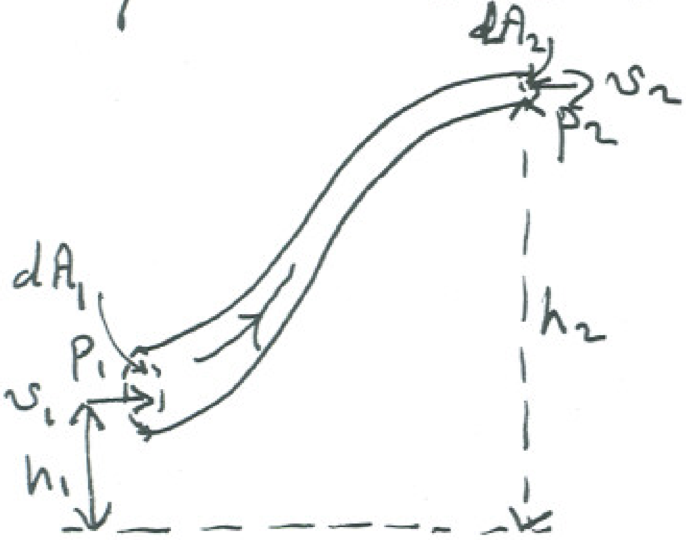
From transverse eqn of motion

$$N = 2m\dot{r}\omega_0$$
$$= 2m\omega_0 \frac{1}{2} r_0 \omega_0 \left(e^{\omega_0 t} - e^{-\omega_0 t}\right)$$
$$\therefore N = m r_0 \omega_0^2 \left(e^{\omega_0 t} - e^{-\omega_0 t}\right)$$

11.

A <u>streamline</u> is a line in the fluid along which the velocity vector of the fluid is everywhere a tangent to the streamline. In a steady state situation (no time variation) a streamline is the path taken by an element of the fluid.

A stream-tube is a small bundle of streamlines, and no fluid flows out of the sides of the stream-tube.



The equation of continuity is the equation of mass flow conservation along a stream tube. In time dt, mass entering end 1 of stream-tube is

$$\rho\, dA_1 v_1 dt$$

and mass leaving end 2 is

$$\rho\, dA_2 v_2 dt$$

$$\therefore \rho\, dA_1 v_1 dt = \rho\, dA_2 v_2 dt$$

for incompressible liquid

$$\therefore \underline{dA_1 v_1 = dA_2 v_2} \quad \text{continuity eqn.}$$

11 (continued)

With reference to figure, net work done by pressure in time dt is

$$p_1\, dA_1\, v_1\, dt - p_2\, dA_2\, v_2\, dt$$

Change in kinetic energy of mass $\rho\, dA_1 v_1\, dt = \rho\, dA_2 v_2\, dt$ is

$$\frac{1}{2}(\rho\, dA_1 v_1\, dt)(v_2^2 - v_1^2)$$

Change in potential energy is

$$(\rho\, dA_1 v_1\, dt)\, g\, (h_2 - h_1)$$

$\therefore$ from energy conservation, and using continuity eqn.

$$\therefore (p_1 - p_2)\cancel{dA_1 v_1 dt} = \frac{1}{2}\rho \cancel{dA_1 v_1 dt}(v_2^2 - v_1^2) + \rho \cancel{dA_1 v_1 dt}\, g(h_2 - h_1)$$

$$\therefore p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

i.e. $p + \frac{1}{2}\rho v_1^2 + \rho g h =$ constant along a streamline.