PHAS1202 - Atoms, Stars and The Universe
Extra Problem Sheet
Model Answer

These problems are provided for extra practise and exam preparation. Solutions will be made available in Moodle.

Useful constants

Planck's constant h is 6.6×10^{-34} Js (2 s.f.).

The mass of an electron is 9.1×10^{-31}kg.

1 Electron Volt (eV) is 1.6×10^{-19}Joules.

1 Angstrom is 10^{-10}m.

1: Rayleigh-Jeans law vs Planck's law

Prior to Planck's black-body radiation law, the best theoretical model for Black Body radiation led to a formula known as the Rayleigh-Jeans law. While Rayleigh-Jeans law had unphysical consequences for high-frequency light (the ultraviolet catastrophe), for low-frequency light it did match predictions very well. 3. By expanding the exponential as a series, show that in the limit that the wavelength λ is very

long, Planck's law converges to the Rayleigh-Jeans law.

The Rayleigh-Jeans law is:

$$I(\lambda, T) = \frac{2\pi\, c\, k\, T}{\lambda^4}$$

Planck's law is:

$$I(\lambda, T) = \frac{2\pi\, h\, c^2}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

The series expansion for e^x is:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

If λ is very long, the exponent in Planck's law is very small. Therefore we can truncate, to good approximation, the power series for the exponential to its second term.

$$e^{hc/\lambda kT} \approx 1 + \frac{hc}{\lambda kT}$$

.

Substituting this into

$$\frac{2\pi\, h\, c^2}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

reduces, after cancellations, to

$$\frac{2\pi\, c\, k\, T}{\lambda^4}$$

as required.

2: de Broglie wavelength

Felix Baumgartner was in the news recently for performing the highest ever sky-dive.

2.a) Felix ascended in his balloon at a rate of around 2 ms^{-1}. What was the de Broglie wave-length of Felix and his balloon. (Treat Felix, his suit and balloon as a single object).

2.b) Felix then dropped from his balloon and fell towards Earth reaching a top speed of 372 ms^{-1}. What was the de Broglie wave-length of Felix at this point. (Treat Felix and his suit as a single object).

2.c) What would be the de Broglie wave-length of an electron travelling at Felix's top speed?

Assume Felix Baumgarnter weighs 90 kg. His space-suit weighed 118 kg and his balloon had mass 1315 kg.

Key equations:

$$\lambda = \frac{h}{p}$$
$$p = mv$$

2.a) Total mass of Felix, his suit and balloon: $90 + 118 + 1315 = 1523$ kg, which we can round to 2 significant figures to 1500 kg. $p = mv = 1500 \times 2 = 3000$ kg ms^{-1} (2 s.f.).

$$\lambda = \frac{6.6 \times 10^{-34}}{3 \times 10^3} = 2.2 \times 10^{-37}\text{m}.$$

2.b) Total mass of Felix and his suit $90 + 118 = 208$ kg, which we can round to 2 significant figures to 210 kg. $p = mv = 210 \times 372 = 78000$ kg ms^{-1} (2 s.f.).

$$\lambda = \frac{6.6 \times 10^{-34}}{7.8 \times 10^4} = 8.5 \times 10^{-39}\text{m}.$$

2.c) Electron mass: 9.1×10^{-31} kg (2 s.f.) $p = mv = 9.1 \times 10^{-31} \times 372 = 3.4 \times 10^{-28}$ kg ms^{-1} (2 s.f.).

$$\lambda = \frac{6.6 \times 10^{-34}}{3.4 \times 10^{-28}} = 1.94 \times 10^{-6}\text{m}$$

or approximately 2 μm microns.

3: The Scanning Tunnelling Microscope (STM)

In lectures, you saw that the probability for a quantum particle with mass m energy E, to tunnel through a square barrier of width L and height U, was given by:

$$P = \exp[-2CL]$$

where

$$C = \frac{\sqrt{2m(U-E)}}{\hbar}.$$

In an STM, electrons tunnel across a potential barrier to a surface, completing a circuit which then has a current I proportional to the tunnelling probability.

$I \propto \exp[-2CL]$

The barrier height is approximately the same as the work-function of the electron, typically on the order of a few electron Volts, while the barrier width is the distance between electrode and surface.

3.a) For an STM, with electron energy 1eV and barrier height 4eV calculate C.

Our starting point is the definition of C

$$C = \frac{\sqrt{2m(U-E)}}{\hbar}$$

The mass $m = 0.1 \times 10^{-31}$kg, U is 4eV and E is 1eV, so $U - E = 3\text{eV} = 3 \times 1.6 \times 10^{-19}$J.

Putting the numbers in we find:

$$C = 8.9 \times 10^{9}\text{m}^{-1}$$

.

Perhaps the hardest part of this calculation is getting the units right, we can see from the definition of P that CL must be dimensionless (all numbers inside a mathematical function $f(x) = e^x$ or $f(x) = \sin(x) etc.$ must always be dimensionless) hence C must have the inverse units of L. If we work solely in SI units, these units must be the SI unit of inverse length, e.g. inverse metres.

Explicitly the units are $\sqrt{\text{kg} \times \text{J}}/(\text{kg m}^2\text{s}^{-1})$. From $E = mc^2$ a Joule is can be expressed in kg, metres and seconds as kg m^{2}s^{-2}, and hence the units of C are inverse m as required. Note that C is almost exactly an inverse Angstrom. This is what gives the TEM sensitivity on the Angstrom scale.

3.b) If initially the current is 1 Amp what will be the current if the surface height increases (and thus the barrier width decreases) by 1 Angstrom?

First, let's define some notation. Let I be current, and let I_0 be the original current, when the distance from probe to surface (= barrier width) is L_0, and let I_1 and L_1 be the final current and barrier width respectively.

We are given that

$$I \propto e^{-2CL}$$

we can therefore introduce a constant of proportionality A:

$$I = Ae^{-2CL}$$

Therefore

$$I_0 = Ae^{-2CL_0}$$

and

$$I_1 = Ae^{-2CL_1}$$

Dividing these equations:

$$\frac{I_1}{I_0} = \frac{Ae^{-2CL_1}}{Ae^{-2CL_0}} = \frac{e^{-2CL_1}}{e^{-2CL_0}} = e^{2C(L_0-L_1)}$$

Note that the unknown constant A has cancelled from this ratio. We see that the ratio depends has an exponential dependance on the difference in barrier width $L_0 - L_1$. The question tells us that $L_0 - L_1 = 1\text{Å} = 10^{-10}$m.

Putting in the numbers we see that:

$$I_1 = I_0 e^{1.78} = 5.93 I_0 = 5.93\text{Amps.}$$

Thus a change in height of an Angstrom on the surface causes a huge change in current. This is what gives the STM its Angstrom scale resolution.

4: Quantum Hydrogen Atom

In the final lecture of this course, we saw that wavefunction solutions to the TISE were indexed by three integer *quantum numbers* n, l and m, where

- n is any non-negative non-zero integer, e.g. $1, 2, 3, \ldots$.
- l is any non-negative integer less than n, e.g. $0, 1, 2, \ldots, n-1$.
- m is any integer such that $|m| \leq l$, e.g. $-l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$.

The energy of the Hydrogen atom states is a function of n (the principle quantum number) only, and satisfies a formula given in lectures.

4.a) What is the energy of the $n = 3$, $l = 2$, $m = -2$ state?

The key equation is

$$E_n = \frac{-13.6}{n^2}\text{eV}$$

.

This is the energy of the energy states of Hydrogen. Importantly it does not depend on l and m. Thus to answer this question we consider $n = 3$ and find:

$$E_3 = \frac{-13.6}{9}\text{eV} = 1.51\text{eV}$$

.

4.b) Which combinations of quantum numbers correspond states of energy $E = -13.6$eV? (Without taking spin into account) how many different wavefunctions are there with this energy?

From the above equation, $E = -13.6$eV when $n = 1$. For $n = 1$, there is only a single possible value of l, $l = 0$ and also only a single value of m, $m = 0$. There is thus just 1 wavefunction. (If we included spin as well, there would be two possible states at this energy, but we are not considering spin in this question).

4.c) (Without considering spin) how many different wavefunctions have quantum number $n = 2$?

For $n = 2$, there are two possible values of l, $l = 0$ and $l = 1$. $l = 0$ corresponds to just 1 value of m, $m = 0$, and $l = 1$ corresponds to the 3 values $m = -1, 0, +1$. There are thus 1+3=4 wavefunctions.

4.d) Find a formula which describes the the number of different wavefunctions for a general n.
Hint: Write out the possible allowed combinations of quantum numbers for n =1, 2, 3, 4 etc. and look for a pattern - in particular look for an arithmetic series. You may use the following identity for an arithmetic series without proving it:

$$\sum_{j=1}^{n} a_j = \frac{n}{2}(a_1 + a_n)$$

where $a_j = a_{j-1} + d$, and d is a constant.

For $n = 3$, there are three possible values of l, $l = 0, 1, 2$. There are thus 1+3=4 wavefunctions for $l = 0, 1$ so we need only consider $l = 2$, where there are $m = -2, -1, 0, 1, 2$ 5 wave functions. Thus there are in total 1+3+5=9 wave functions.

There is a pattern here (1, 4, 9) which looks suggestively like squares of the integers. Let us prove this. As we increase n by 1 we add a further set of wavefunctions for $l = n - 1$. There are $2l + 1$ of these. Thus for general n, the number of wave functions is

$$\sum_{l=0}^{n-1}(2l + 1).$$

Let us introduce the new index $p = l + 1$ to bring this expression into the form needed for the identity:

$$\sum_{l=0}^{n-1}(2l + 1) = \sum_{p=1}^{n}(2(p - 1) + 1) = \sum_{p=1}^{n}(2p - 1) = \frac{n}{2}(1 + 2n - 1) = n^2.$$

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