2018

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1. (a) Determine the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, in terms of their components a_x, a_y, a_z and b_x, b_y, b_z. [2]

   (b) A plane is defined by a point A (position vector $\mathbf{a}$) on it and a unit vector $\hat{\mathbf{n}}$ perpendicular to it. Write down the equation of the plane, satisfied by any point R (position vector $\mathbf{r}$) on that plane. [1]

   (c) Determine x and y such that the vector $\mathbf{a} = x\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ has unit magnitude and is perpendicular to the vector $\mathbf{b} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$. [3]

a) $a \times b = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$

b) $\hat{n} \cdot (\underline{a} - \underline{r}) = 0$

c) perp $\therefore$ $a \cdot b = 0$

$\rightarrow \frac{x}{\sqrt{3}} + \frac{x}{\sqrt{3}} - \frac{y}{\sqrt{3}} = 0$

$2x = y$

$\left|\begin{pmatrix} x \\ x \\ 2x \end{pmatrix}\right| = 1$

$\rightarrow x \frac{1}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}$

$\rightarrow a = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$x = \frac{1}{\sqrt{6}}, \quad y = \frac{2}{\sqrt{6}}$

2. (a) Given two complex numbers, $z_1 = 3 + 7i$ and $z_2 = 6e^{-i\pi/2}$, determine **[3]**

i. $z_1 - z_2$,

ii. $z_1 z_2$,

iii. z_1/z_2.

Express each result in the form $x + iy$, where x, y are real numbers.

(b) Find all roots of **[3]**

$$z^3 = -4\sqrt{2}(1+i),$$

and express them in exponential form using the convention that $-\pi < \arg z \leq \pi$.

(c) Evaluate $\mathrm{Re}(e^{3iz})$, where $z = x + iy$ (x, y are real numbers). **[2]**

a) $Z_2 = 6(0 - i) = -6i$

i) $Z_1 - Z_2 = 3 + 13i$

ii) $Z_1 Z_2 = -18i + 42$

$\rightarrow 42 - 18i$

$= 6(7 - 3i)$

iii) $\dfrac{Z_1}{Z_2} = \dfrac{3+7i}{-6i} \times \dfrac{6i}{6i}$

$= \dfrac{18i - 42}{36}$

$= -\dfrac{7}{6} + \dfrac{1}{2}i$

b)

$Z^3 = -4\sqrt{2}(1+i)$

$= -8e^{i\frac{\pi}{4}}$

$Z_k = -2e^{i\frac{\frac{\pi}{4} + 2\pi k}{3}}$

$Z_1 = \sqrt{2} - \sqrt{2}i$

$Z_2 = \left(\dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{2}}{2}\right) + \left(\dfrac{\sqrt{6}+\sqrt{2}}{2}\right)i$

$Z_3 = Z_0 = -\left(\dfrac{\sqrt{6}+\sqrt{2}}{}\right) + \left(\dfrac{\sqrt{2}-\sqrt{6}}{}\right)i$

$$z_3 = z_0 = -\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}-\sqrt{6}}{2}\right)i$$

$$z_0 = \frac{\pi}{12}, \quad z_1 = \frac{3\pi}{4}$$

$$z_2 = \frac{-7}{12}\pi$$

c) $$e^{3iz} = e^{3i(x+iy)}$$
$$= e^{3xi - 3y}$$
$$= e^{-3y} e^{i(3x)}$$
$$Re\left(e^{3iz}\right) = e^{-3y} \cos(3x)$$

3. (a) State the formal definition of the derivative of a function $f(x)$. [1]

(b) Using the formal definition, calculate the derivative of [3]

$$f(x) = \frac{1}{x^2}.$$

(c) Find all stationary points of [3]

$$f(x) = x^4 + 6x^3 - 6,$$

and determine their nature.

a) $$\frac{d}{dx}\left(f(x)\right) = \lim \Delta x \to 0 \quad \frac{f(x) - f(x+\Delta x)}{\Delta x}$$

b) $$f(x) = \frac{1}{x^2}$$
$$f(x+\Delta x) = \frac{1}{x^2 + \Delta x}$$

$$\frac{d}{dx}\left(f(x)\right) = \frac{\frac{1}{x^2} - \frac{1}{(x+\Delta x)^2}}{}$$

$$\Delta x$$

$$= \frac{\dfrac{(x+\Delta x)^2 - x^2}{x^2(x+\Delta x^2)}}{\Delta x}$$

$$= \frac{\dfrac{\cancel{x^2} + 2x\Delta x + \cancel{\Delta x^2} - \cancel{x^2}}{x^2\left(x^2 + 2x\Delta x + \cancel{\Delta x^2}\right)}}{\Delta x}$$

$$= \frac{\dfrac{2x\cancel{\Delta x}}{x^4 + 2x^3\Delta x}}{\cancel{\Delta x}}$$

$$= \frac{2x}{x^4 + \cancel{2x^3\Delta x}}$$

$$= \frac{2}{x^3}$$

c) $f(x) = x^4 + 6x^3 - 6$

$$\frac{df}{dx}(x) = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 = 0$$

$$x = 0$$

$$4x = -18$$

$$x = -\frac{9}{2}$$

$$f''(x) = 12x^2 + 36x$$

$f''(0) = 0 \rightarrow$ unknown

$f''(1) = 48$

$f''(-1) = -24 \rightarrow$

point of inflection

$f''\left(-\frac{9}{2}\right) = 81 \rightarrow$ min

4. (a) Determine the following indefinite integrals: [4]

i. $\int x^{5/2} dx$,

ii. $\int x^n \ln x \, dx$, ($n > 0$ is a positive integer).

(b) Determine the definite integral [2]

$$\int_{-1}^{1} \frac{\sin x}{1+x^2} dx,$$

and justify your answer.

a) i)

$$\int x^{5/2} \, dx = \frac{2}{7} x^{\frac{7}{2}} + C$$

ii)

$u = \ln(x)$ $\qquad \frac{dv}{dx} = x^n$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = \frac{x^n}{n+1}$

$$I = \frac{x^n}{n+1} \ln(x) - \int \frac{x^{n-1}}{n+1} dx$$

$$= \frac{x^n}{n+1} \ln(x) - \frac{1}{n(n-1)} x^n + C$$

b)

$$\int_{-1}^{1} \frac{\sin x}{1+x^2} dx$$

0 - graph is symmetrical with half +ve & half -ve

5. (a) Determine the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function [3]

$$f(x, y) = \ln\left(1 + xy^2\right),$$

and calculate the total derivative $\frac{df}{dt}$ for the parametrized path defined by $x(t) = t$, $y(t) = \sqrt{t}$.

(b) Show that the above function $f(x, y)$ in (a) satisfies the equation [3]

$$2\frac{\partial^2 f}{\partial x^2} + y^3 \frac{\partial^2 f}{\partial x \partial y} = 0.$$

a)

$$u = 1 + xy^2$$

$$\frac{\partial u}{\partial x} = y^2$$

$$\frac{\partial u}{\partial y} = 2yx$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{1+xy^2} \qquad \frac{\partial f}{\partial y} = \frac{2xy}{1+xy^2}$$

$$x(t) = t \quad y(t) = \sqrt{t}$$

$$f(t) = \ln(1+t^2)$$

$$\frac{\partial f}{\partial t} = \frac{2t}{1+t^2}$$

b)

$$\frac{\partial f}{\partial x} = \frac{y^2}{1+xy^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2 \times \frac{y^2}{(1+xy^2)^2}$$

$$\frac{\partial x^2}{\ } = 0 \quad \frac{\ }{(1+xy^2)^2}$$

$$= \frac{y^4}{(1+xy^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2xy}{1+xy^2}\right)$$

$u = 2xy \qquad v = 1+xy^2$

$\frac{du}{dx} = 2y \qquad \frac{dv}{dx} = y^2$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2y(1+xy^2) - 2xy(y^2)}{(1+xy^2)^2}$$

$$= \frac{2y}{(1+xy^2)^2}$$

$$2x \frac{-y^4}{(1+xy^2)^2} + y^3 \frac{2y}{(1+xy^2)^2}$$

$$\rightarrow \frac{-2y^4}{(1+xy^2)^2} + 2y^4 \ldots$$

$$= 0$$

6. (a) Write down the general form of the Maclaurin series of a function $f(x)$. **[2]**

(b) Determine the first three non-zero terms in the Maclaurin series of the following functions: **[5]**

i. $f(x) = \sqrt{1+2x}$,

ii. $f(x) = \sin(2x^2)$.

a) $f(x) \simeq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 \ldots$

$= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

b) i) $f(x) = \sqrt{1+2x}$

$f(0) = 1$

$f'(0) = \frac{1}{2} \times 2 (1+2x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+2x}}$

$f''(0) = -\frac{1}{(2x+1)^{\frac{3}{2}}}$

$f(x) = 1 + x - \frac{1}{2}x^2$

ii) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\rightarrow f(x) = 2x^2 - \frac{8x^6}{6} + \frac{32x^{10}}{5!}$

7. (a) Find the minimal distance d between the point $P = (1, 1, 1)$, and the line passing through the points $A = (2, 1, 5)$ and $B = (3, 4, 3)$. **[5]**

(b) Find the equation of the line formed by the intersection of the two planes **[5]**

$$3x + y - z = 3,$$
$$2y + 4z = -4.$$

Express the equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of a point on the line, $\mathbf{b}$ is a vector in the direction of the line and λ is a real parameter.

(c) Calculate the scalar and vector product between the vectors **[5]**

$$\mathbf{a} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j},$$
$$\mathbf{b} = \cos\phi\ \mathbf{i} + \sin\phi\ \mathbf{j},$$

and hence prove that

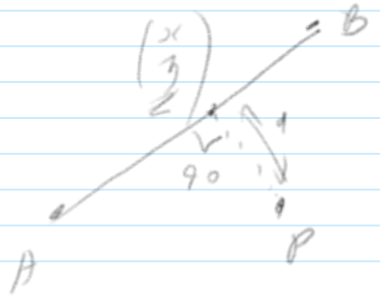
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi,$$
$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

(d) Sketch (in separate Argand diagrams) and describe the regions on the complex z plane, defined by the following inequalities: **[5]**

i. $|z + 2 - 3i| \leq 2,$

ii. $\text{Re}(z^2) > 0.$

a)



$$(x - p) \cdot (B - A) = 0$$

$$(x-1)(1) + (y-1)3 + (z-1)(-2) = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A + \lambda (B - A)$$

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \cdot (B - A) = 0$$

$$\begin{matrix} x \\ y \end{matrix} = \begin{matrix} 2 + \lambda \\ 1 + 2\lambda \end{matrix}$$

$$\begin{aligned} x &= 2 + \lambda \\ y &= 1 + 3\lambda \\ z &= 5 - 2\lambda \end{aligned}$$

$$(1+\lambda)\times 1 + (3\lambda)\times 3 + (4-2\lambda)\times -2 = 0$$

$$6\lambda - 7 = 0$$

$$\lambda = \frac{7}{6}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{19}{6} \\ \frac{9}{2} \\ \frac{8}{3} \end{pmatrix}$$

$$\left|\begin{matrix} x-1 \\ y-1 \\ z-1 \end{matrix}\right| = \sqrt{\left(\frac{13}{6}\right)^2 + \left(\frac{7}{2}\right)^2 + \left(\frac{5}{3}\right)^2}$$

$$= \frac{\sqrt{710}}{6}$$

b)

$$3x + y - z = 3$$

$$2y + 4z = -4 \rightarrow$$

$$y = -2(1+z)$$

$$3x - 2(1+z) - z = 3$$

$$3x - 2 - 3z = 3$$

$$3(x-z) = 5$$

$$x - z = \frac{5}{3}$$

$$z = x - \frac{5}{3}$$

$$y = -2(1+z)$$
$$= -2\left(x - \frac{2}{3}\right)$$
$$= -2x + \frac{4}{3}$$

Let $x = 0$

$$y = \frac{4}{3}, \quad z = \frac{-5}{3}$$

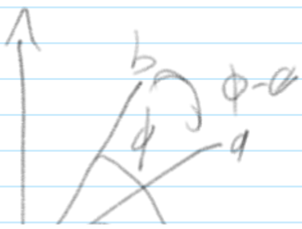
$$\underline{r} = \frac{1}{3}\begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

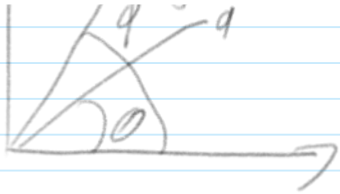
c)

$$\underline{a} = \cos(\theta)\,\hat{i} + \sin\theta\,\hat{j}$$
$$\underline{b} = \cos(\phi)\,\hat{i} + \sin\phi\,\hat{j}$$

$$a \cdot b = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$= |a||b|\cos(\theta - \phi)$$
$$\Rightarrow \cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$a \times b = \begin{pmatrix} 0 \\ 0 \\ \cos\theta\sin\phi - \cos\phi\sin\theta \end{pmatrix}$$





$a \times b = |a||b| \hat{n} \sin(\phi - \theta)$

$\Rightarrow \sin(\theta - \phi) = -a \times b$

$= \sin\theta \cos\phi - \cos\theta \sin\phi$

d) ?

8. (a) Calculate the derivative $\frac{df}{dx}$ of the following functions: **[6]**

   i. $f(x) = \arctan x$,

   ii. $f(x) = x^{(x^2)}$,

   iii. $f(x) = e^{-x^2} + \int_0^x e^{-t^2} dt$.

   (b) Calculate the volume of revolution formed by rotating the curve **[3]**

$$f(x) = \frac{1}{\sqrt{1+x^2}},$$

   around the x-axis in a full circle. The volume extends over the range $-\infty < x < \infty$.

   (c) i. Given a function $f(x, y)$, state the condition for a point (x_0, y_0) to be stationary, and the criteria to determine its nature. **[3]**

   ii. Find all stationary points of the function **[4]**

$$f(x, y) = x^3 - yx^2 + y^2,$$

   and determine their nature.

   (d) A tilted ellipse in the x-y plane is described by the implicit relation

$$x^2 + xy + y^2 = 12.$$

   Find the location (x, y) of the **[4]**

   i. top-most (largest y value),

   ii. bottom-most (smallest y value),

   iii. right-most (largest x value), and

   iv. left-most (smallest x value)

   point on the ellipse.

a) i)

$y = \tan^{-1}(x)$

$\tan(y) = x$

$y = \ldots$

$\tan(y) = x$

$$\frac{dx}{dy} = \sec^2(y)$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2(y)}$$

$$= \frac{1}{1+x^2}$$

ii) $y = x^{x^2}$

$\ln(y) = x^2 \ln(x)$

$u = x^2 \qquad v = \ln(x)$

$du = 2x \qquad dv = \frac{1}{x}$

$$\frac{1}{y}\frac{dy}{dx} = x + 2x\ln(x)$$

$$\frac{dy}{dx} = x^{x^2}\left(2x\ln(x) + x\right)$$

iii) $-2xe^{-x^2} + \frac{d}{dx}\left[\int_0^x e^{-t^2}\,dt\right]$

$-2xe^{-x^2} + e^{-x^2} \to e^{-x^2}(1-2x)$

b)

$$y = \frac{1}{\sqrt{1+x^2}}$$

$$I = \pi \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$$

$$= \pi \left[\arctan(x) \right]_{-\infty}^{\infty}$$

$$= \pi \pi$$

$$= \pi^2$$

c) i) $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 0$ &

$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0$

ii)

$$f(x,y) = x^3 - yx^2 + y^2$$

$$\frac{\partial f}{\partial x} = 3x^2 - 2xy = 0$$

$$\frac{\partial f}{\partial y} = 2y - x^2 = 0$$

$$\rightarrow 2y = x^2$$

$\downarrow$

$$3x^2 - x^3 = 0$$

$$x = 0$$

$$x = 3$$

$$\rightarrow y = \frac{x^2}{2}$$

$\rightarrow y = \frac{x^2}{2}$

$(0,0)$

$\left(3, \frac{9}{2}\right)$

d)

$\frac{\partial f}{\partial x} = 2x + y = 0$

$\frac{\partial f}{\partial y} = x + 2y = 0$

$\downarrow$

$x = -2y$

$4y^2 - 2y^2 + y^2 = 12$

$3y^2 = 12$

$y = \pm 2$

$x = \pm 4$

$(-4, 2)$ (left most)

$(4, -2)$ (right)

$y = -2x$

$x^2 - 2x^2 + 4x^2 = 12$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

$y = \mp 4$
$(2,-4)$ (~~bottom~~)
$(-2,4)$ (top)

9. (a) i. Given a general differential of the form

$$A(x,y)dx + B(x,y)dy,$$

state the condition that means that the differential is exact. **[1]**

ii. Hence determine whether the following differentials are exact or not: **[5]**

1) $\left(2x + y^2 + \frac{1}{x}\right)dx + \left(2xy - \frac{1}{y}\right)dy,$

2) $\frac{x}{x^2+y^2}dy - \frac{xy}{x^2+y^2}dx.$

In case a differential is exact, determine the corresponding function $f(x,y)$ such that $df = A(x,y)dx + B(x,y)dy$.

(b) A vector field in two-dimensional Cartesian coordinates is given by

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2+y^2)^{3/2}}.$$

Calculate the line integral $W = \int_C \mathbf{F}\cdot d\mathbf{r}$ for the path defined by a clockwise half-circle around the origin, from $\mathbf{r}_A = -2\mathbf{i}$ to $\mathbf{r}_B = 2\mathbf{i}$. **[3]**

(c) Show that the sum of squared integers, $\sum_{k=1}^{N} k^2$, is given by **[4]**

$$\sum_{k=1}^{N} k^2 = \frac{1}{6}N(N+1)(2N+1).$$

Hint: Use the identity $(k+1)^3 - k^3 = 3k^2 + (3k+1)$ to express the given series in terms of two other, explicitly summable, series.

(d) Evaluate the following limits: **[7]**

i.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}},$$

ii.

$$\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x\right)^x.$$

a) i) $\frac{\partial}{\partial y} A(x,y) = \frac{\partial}{\partial x} B(x,y)$

ii) 1) $\frac{\partial}{\partial y}\left(2x + y^2 + \frac{1}{x}\right) = 2y$

$\frac{\partial}{\partial x}\left(2xy - \frac{1}{y}\right) = 2y$

$\therefore$ exact

2) $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$

$u = x$ $\qquad v = x^2+y^2$

$\frac{du}{dx} = 1$ $\qquad \frac{dv}{dx} = 2x$

$$\frac{\partial}{\partial x}(\ldots) = \frac{2x^2 - x^2 - y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)$$

$u = xy \qquad v = x^2+y^2$

$\frac{du}{dy} = x \qquad \frac{dv}{dy} = 2y$

$$\frac{\partial}{\partial y}(\ldots) = \frac{2xy^2 - x^3 - xy^2}{(x^2+y^2)^2}$$

$$= \frac{xy^2 - x^3}{(x^2+y^2)^2} \quad X$$

Not exact

b)

$x = -\cos(t)$

$y = \sin(t)$

$$t \to 0 \to M$$

$$r \Rightarrow \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$$

$$w = \int_0^M \cancel{\frac{1}{(\cos^2+\sin^2)^{\frac{3}{2}}}} \left(-\sin(t)\hat{i} - \cos(t)\hat{j}\right) \cdot \left(\sin(t)\hat{i} + \cos(t)\hat{j}\right)$$

$$= -\sin^2(t) - \cos^2(t)$$

$$= -\int_0^M 1 \, dt$$

$$= -M$$

c) $3k^2 + (3k+1) = (k+1)^3 - k^3$

$\to 3k^2 = (k+1)^3 - k^3 - 3k - 1$

?

d) i) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}} \to \frac{x}{\sqrt{x^2}} \to \frac{x}{x} = 1$

ii) $\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x\right)^x$?

$\to \left(\sqrt{x^2} - x\right)^x \to (x-x)^x$

$= 0$

$= 0$