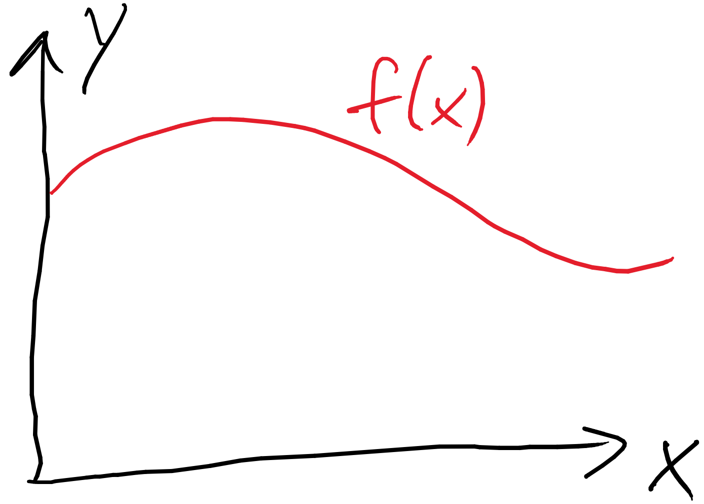
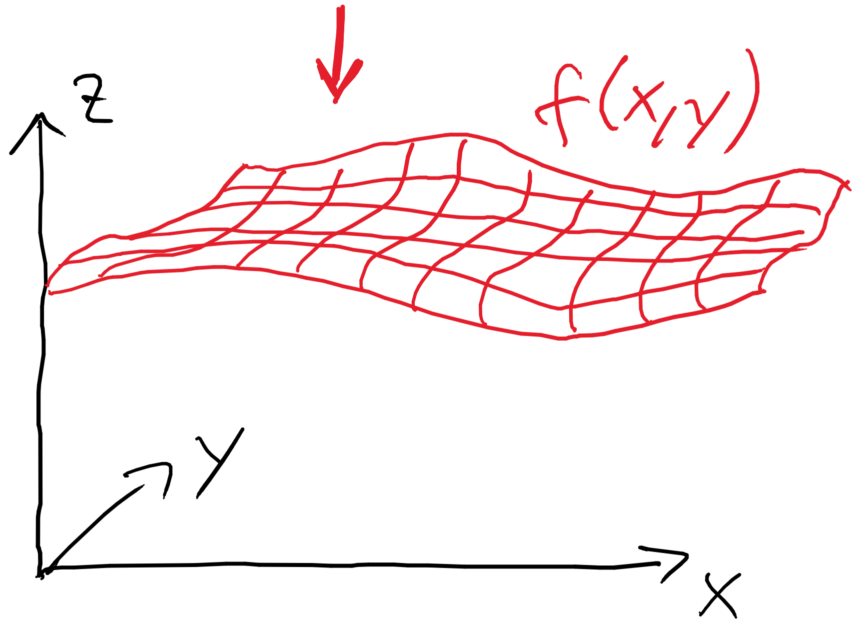
Multivariate Functions I

Dealing with functions of two or more variables, $z = f(x,y)$





Examples:

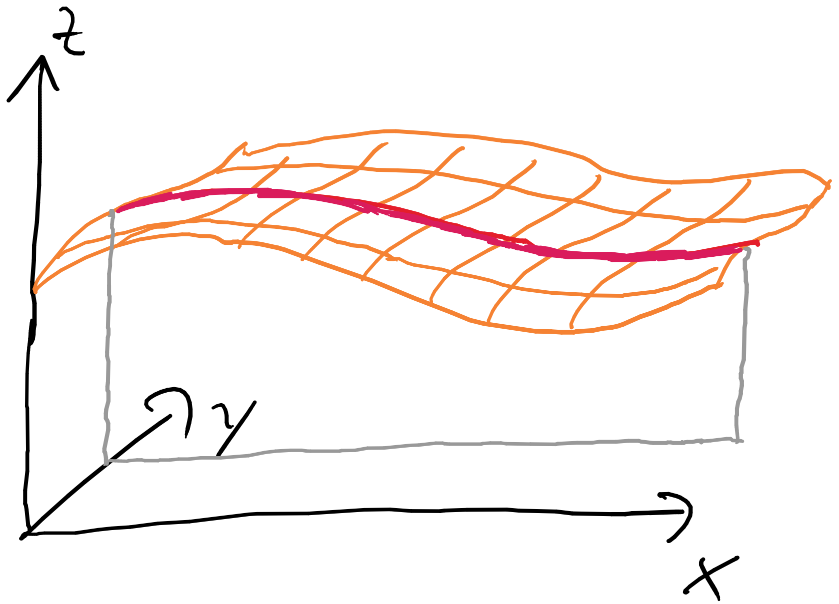
- Temperature, density as function of position x, y, z
- Altitude as function of position x, y

$\Longrightarrow$ Need to generalize differentiation

Partial derivative

- Derivative with respect to one variable, keeping all other variables constant,

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_y \equiv f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}}$$



- For well-behaved functions, order of differentiation is interchangeable

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

Gradient

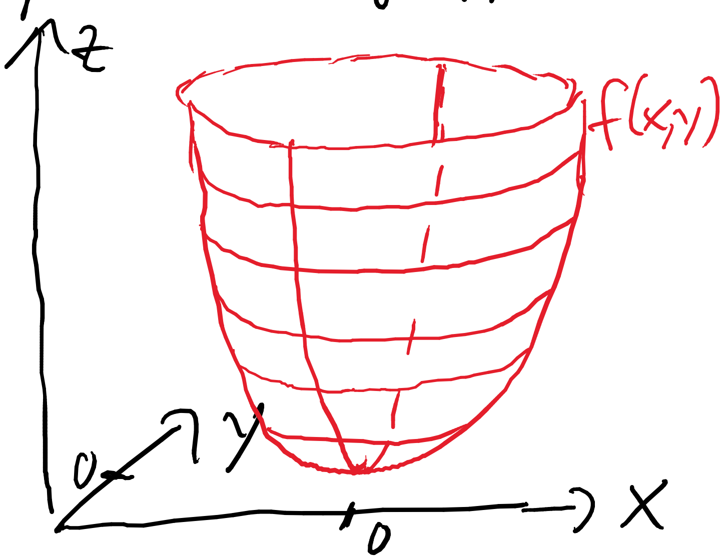
- Partial derivatives form component of a vector field $\equiv$ the gradient

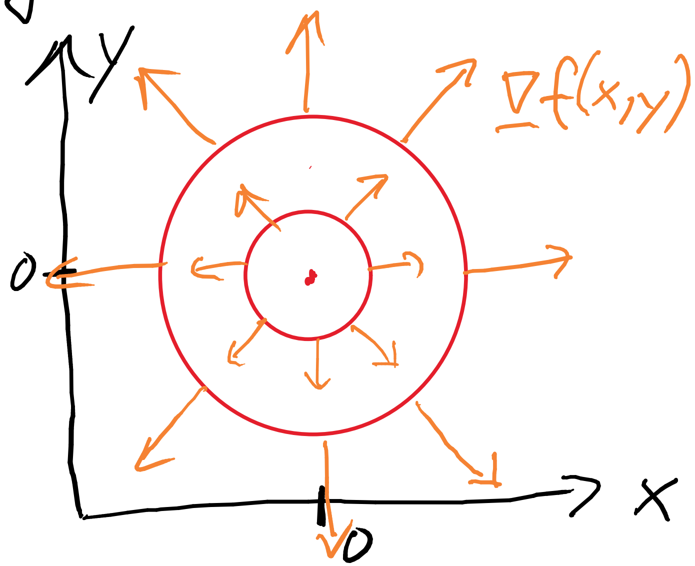
$$\boxed{\underline{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}\right)\underline{i} + \left(\frac{\partial f}{\partial y}\right)\underline{j} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}}$$

- At a given point x, y, the gradient vector points in the direction of the steepest rise of f; it's magnitude is the slope

$$f(x,y) = x^2 + y^2$$

$$\underline{\nabla} f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

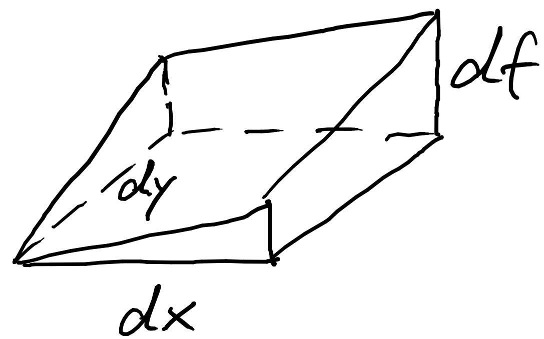




Total differential

- Infinitesimal change df for small steps dx, dy

$$\boxed{df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = (\underline{\nabla} f) \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}}$$



- Linear approximation of f near (x_0, y_0)

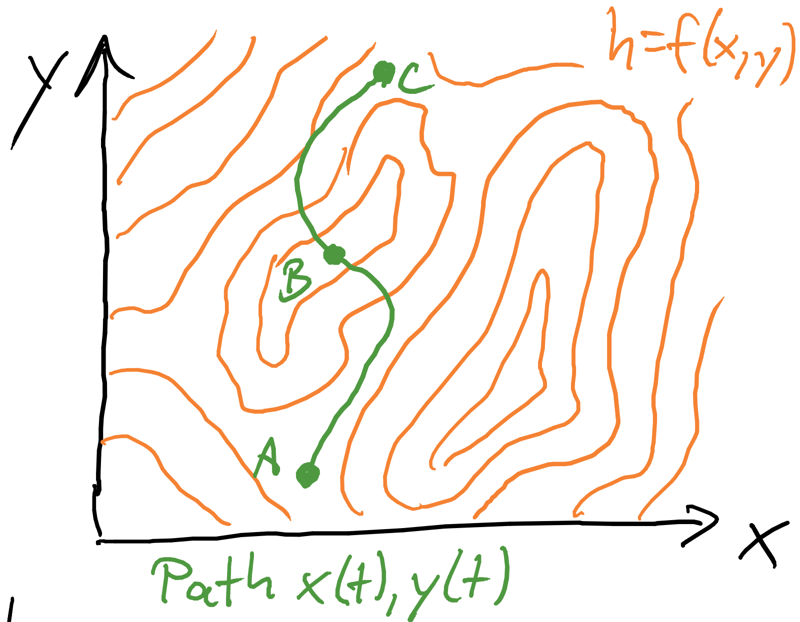
$$f(x,y) \approx f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} \cdot (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} \cdot (y - y_0)$$

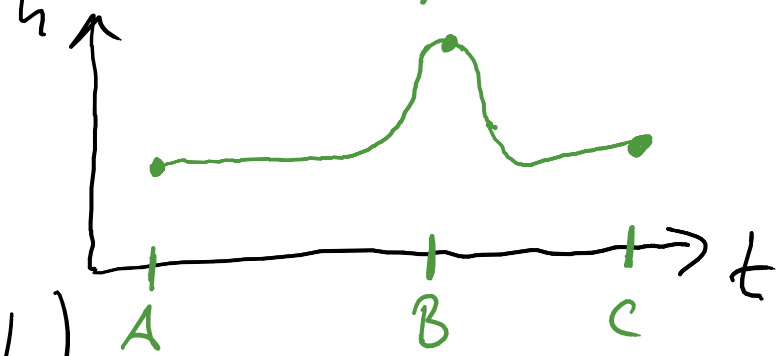
$\longrightarrow$ Equation of tangential plane at (x_0, y_0)

Total derivative

→ Derivative of $f(x,y)$ along a prescribed parametrized path

$$\underline{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$





⇒ $f(x(t), y(t))$ describes function of t

⇒ Total derivative (of f along path)

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}}$$

→ present if f depends <u>explicitly</u> on t, $f(x,y,t)$