Integration Techniques

- Find anti-derivative

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left[\int_a^x f(u)\,du\right] = f(x)$$

- Substitution ($\leftarrow$ chain rule)

- Partial fractions

- Integration by parts ($\leftarrow$ product rule)

$$\int uv'\,dx = uv - \int u'v\,dx$$

- Differentiation with respect to parameter

  e.g. $\int x \cdot e^{-ax}\,dx = -\frac{d}{da}\left[\int e^{-ax}\,dx\right]$

- Reduction formulae

- Extension to complex numbers

  e.g. $e^{ax} \cdot \cos bx = \mathrm{Re}\left[e^{(a+ib)x}\right]$

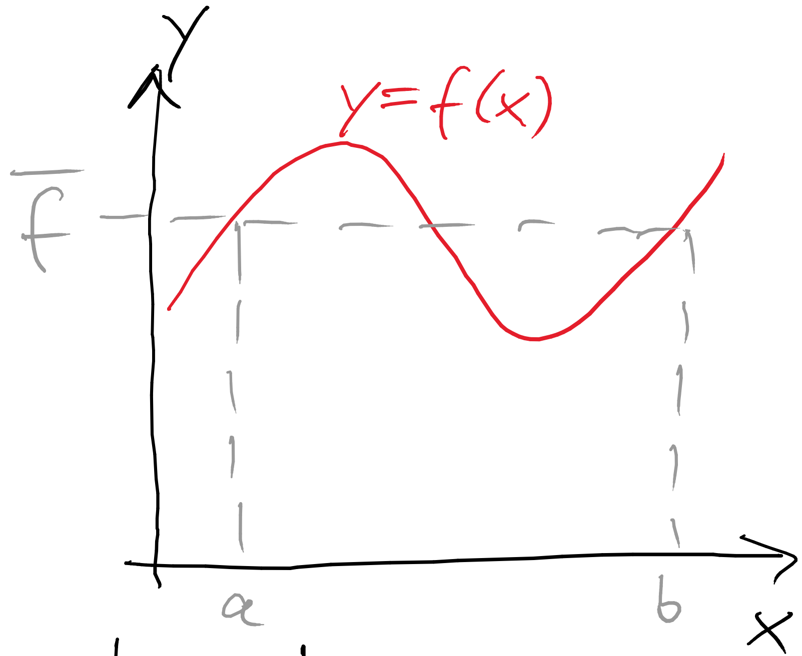
- Symmetries in definite integrals

  e.g. $\int_{-a}^{a} f_{odd}(x)\,dx = 0 \qquad f_{odd}(-x) = -f_{odd}(x)$

Average of a function

- Average of a series of numbers

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$



- Average of a function

$$\boxed{\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

→ Rectangle with same area as integral

$$\bar{f}(b-a) = \int_a^b f(x)\,dx$$

Distributions

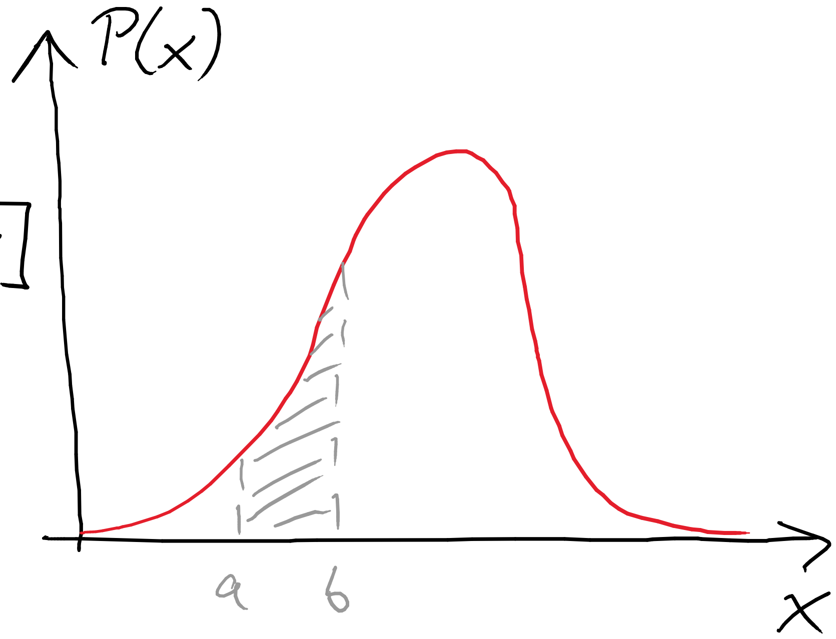
- Discrete random variable with values $x_1, x_2, \ldots, x_n$ and probabilities $P(x_1), P(x_2), \ldots, P(x_n)$

  → Total probability: $\sum_{i=1}^{n} P(x_i) = 1$

  → Average / expectation value:

$$\boxed{\overline{x} = \sum_{i=1}^{n} x_i P(x_i)}$$

- Continuous probability distribution



$P(x)dx \rightarrow$ Probability of finding x in the interval $[x, x+dx]$

Normalization:

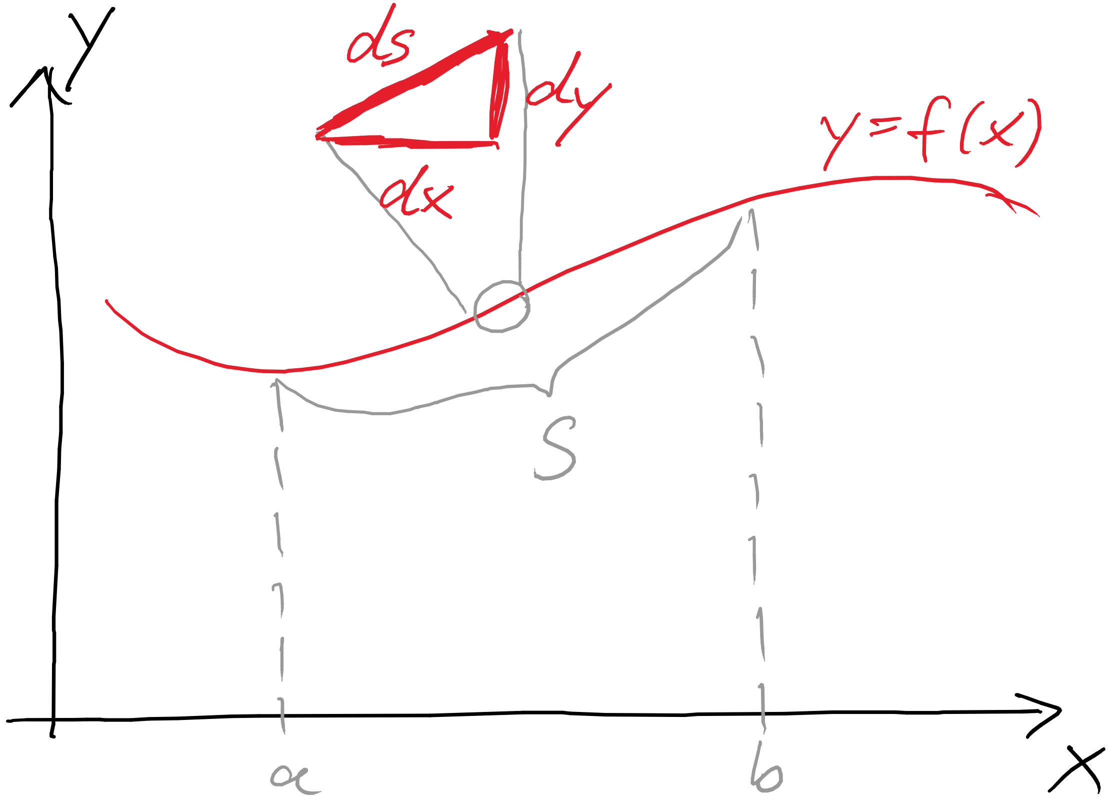
$$\int_{x_{min}}^{x_{max}} P(x) = 1$$

Probability of finding x in $[a,b] = \int_a^b P(x)dx$

Average / expectation value $\boxed{\bar{x} = \int_{x_{min}}^{x_{max}} x \cdot P(x)dx}$

Length of curve

defined by function $f(x)$ between a and b.



Curve element:

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$\Rightarrow$ Length:

$$\boxed{S = \text{"}\int ds\text{"} = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx}$$

Surface of revolution I

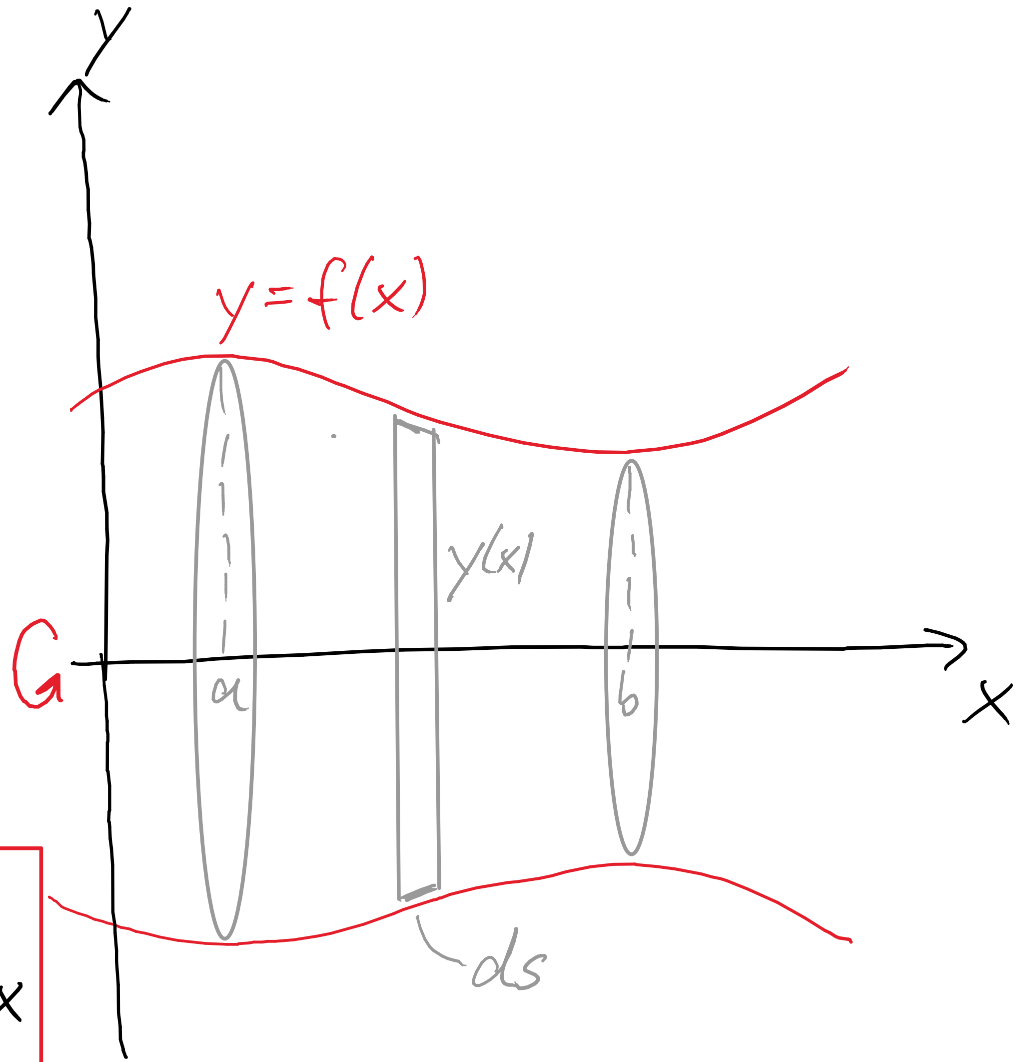
formed by rotating f(x) around the x-axis

Surface of ring of width ds:

$$dA = 2\pi y(x)\, ds$$

$\Rightarrow$ Total surface:

$$\boxed{A = \text{“}\int dA\text{''} = 2\pi \int_a^b f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx}$$



Volume of revolution

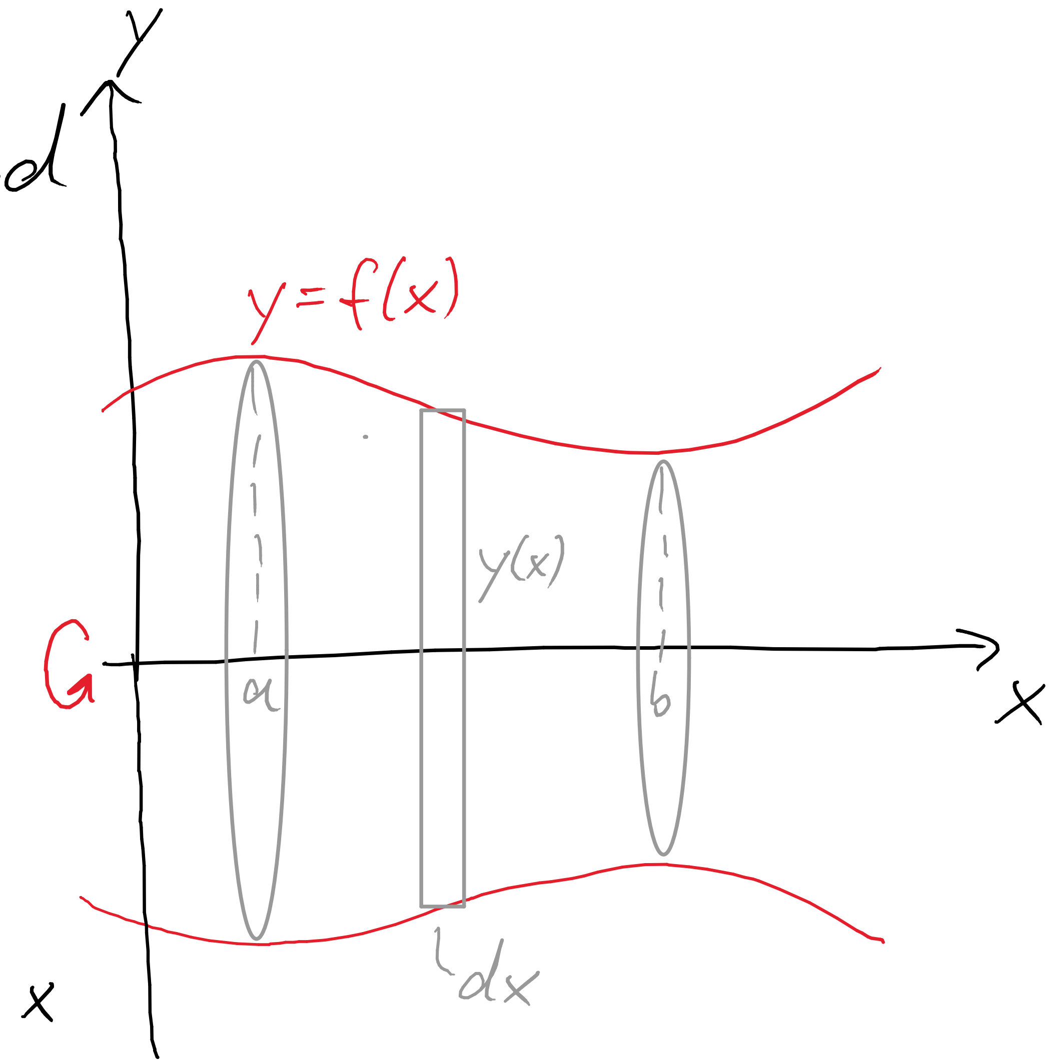
formed by rotating $f(x)$ around the x-axis

Volume of cylinder with width dx:

$$dV = \pi y^2(x)\, dx$$

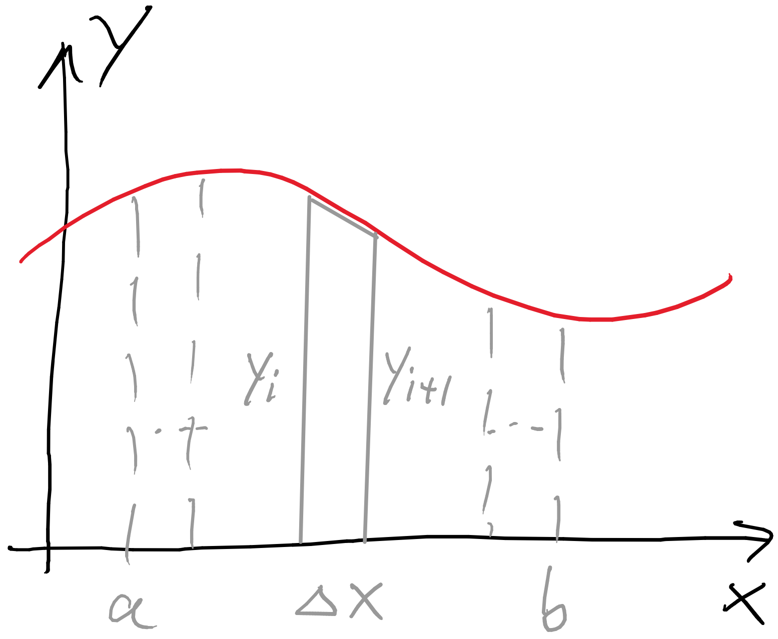
$\Rightarrow$ Total volume:

$$\boxed{V = \text{"}\int dV\text{"} = \pi \int_a^b f^2(x)\, dx}$$



Numerical integration

- Integrals often not solvable in practice
  $\Longrightarrow$ approximate using basic definition



- Subdivide area into N strips of equal width $\Delta x = (b-a)/N$

- Area of trapezium

$$\Delta A_i = \frac{1}{2}\left(f(x_i) + f(x_i + \Delta x)\right)\Delta x$$

$\Longrightarrow$ Sum of all trapezia $\approx$ integral

$$A = \int_a^b f(x)\,dx \approx \sum_{i=0}^{N-1} \Delta A_i = \frac{b-a}{N}\left[\frac{1}{2}y_0 + y_1 + \ldots + y_{N-1} + \frac{1}{2}y_N\right]$$

with $y_i = f(a + i\Delta x)$