2016 First attempt

28 March 2019 09:27

1. Write the expression for the impulse, $\underline{\mathbf{I}}$, received by an object when acted on by a force, $\underline{\mathbf{F}}$. [2]
   Using Newton's second law, find an expression relating $\underline{\mathbf{I}}$ to the momentum of the object. [2]
   Express Newton's third law for two particles in terms of the impulses acting on the particles, and show, in the absence of an external force, that the combined momentum of the two particles is constant. [3]

$I = Ft$

$I = \frac{dp}{dt}$

$I = \frac{dp}{dt}$

$I_1 = -I_2 \rightarrow$

$p_1 + p_2 = I_1 + I_2$

$= 0$ 3.

2. Give one characteristic property of a non-conservative force. [1]
   Give an expression for the torque, $\underline{\boldsymbol{\tau}}$, on a particle at position, $\underline{\mathbf{r}}$, from a force, $\underline{\mathbf{F}}$. Show that $\underline{\boldsymbol{\tau}} = \frac{d\underline{\mathbf{L}}}{dt}$ where $\underline{\mathbf{L}}$ is the angular momentum and that $\underline{\boldsymbol{\tau}}$ is zero for a central force. [4]
   For a rigid, rotating body with a moment of inertia of I, write down an expression for the torque acting on the body. [1]

work done while passing through depends on path taken.

$$\tau = \underline{r} \times \underline{F}$$

$$L = mr^2\dot{\theta}$$

$$\frac{dL}{dt} = m\alpha r^2$$

$$\alpha r = \underline{a}$$

$$\underline{a}m = \underline{F}$$

$$\frac{dL}{dt} = \underline{r} \times \underline{F}$$

$$r = 0 \to mr^2\dot{\theta} = 0$$

$$\therefore \tau = 0$$

$$\tau = I\alpha$$

3. Give an expression for the reduced mass, μ, for two particles with masses m_1 and m_2. If $\underline{\mathbf{r}}_1$ and $\underline{\mathbf{r}}_2$ are the position vectors of the two particles, show that the position of the centre of mass, $\underline{\mathbf{R}}_{\text{CM}}$, is given by: [3]

$$\underline{\mathbf{R}}_{\text{CM}} = \mu\left(\frac{\underline{\mathbf{r}}_1}{m_2} + \frac{\underline{\mathbf{r}}_2}{m_1}\right).$$

Write down an expression for the velocity of the centre of mass and show that, in the absence of external forces, it is constant. [3]

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{cm} = \frac{r_1 m_1 + r_2 m_2}{m_1 + m_2}$$

$$= \frac{1}{m_1 + m_2}\left(r_1 m_1 + r_2 m_2\right)$$

$$= \frac{m_1 m_2}{m_1 + m_2}\left[\frac{r_1 m_1}{m_1 m_2} + \frac{r_2 m_2}{m_1 m_2}\right]$$

$$= \mu\left[\frac{r_1}{m_2} + \frac{r_2}{m_1}\right]$$

$$V_{cm} = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2}$$

4. Assuming θ increases in the counter-clockwise direction, draw a diagram illustrating the direction of the unit vectors: $\underline{\hat{\mathbf{r}}}$ and $\underline{\hat{\boldsymbol{\theta}}}$, with reference to the $\underline{\hat{\mathbf{i}}}$ and $\underline{\hat{\mathbf{j}}}$ vectors. [2]

   Define $\underline{\hat{\mathbf{i}}}$ and $\underline{\hat{\mathbf{j}}}$ in terms of $\underline{\hat{\mathbf{r}}}$ and $\underline{\hat{\boldsymbol{\theta}}}$. [2]

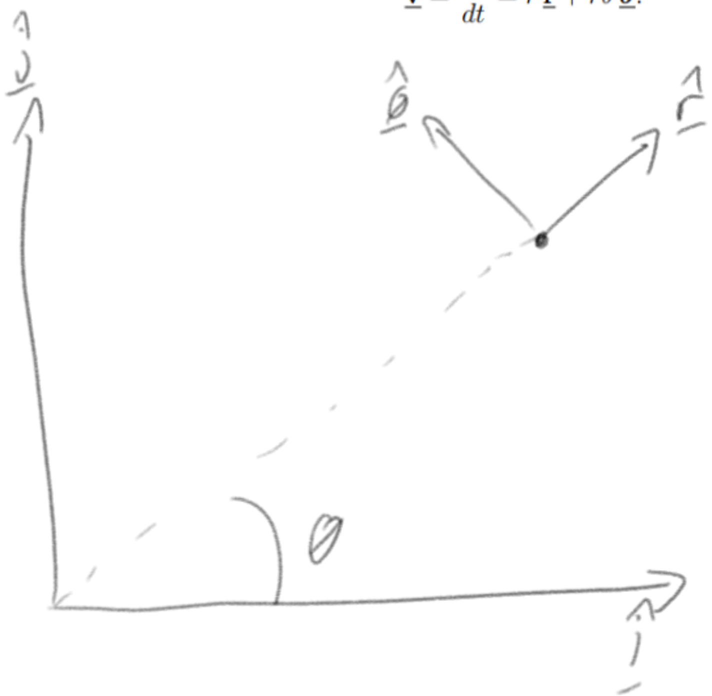
   Then by defining $\underline{\hat{\mathbf{r}}}$ and $\underline{\hat{\boldsymbol{\theta}}}$ in terms of $\underline{\hat{\mathbf{i}}}$ and $\underline{\hat{\mathbf{j}}}$, show that:

$$\frac{d\underline{\hat{\mathbf{r}}}}{dt} = \dot{\theta}\,\underline{\hat{\boldsymbol{\theta}}}$$

   and hence that:

$$\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\dot{\theta}\,\underline{\hat{\boldsymbol{\theta}}}.$$

[3]



$$\underline{\hat{i}} = \cos(\theta)\,\underline{\hat{r}} - \sin(\theta)\,\underline{\hat{\theta}}$$
$$\underline{\hat{j}} = \sin(\theta)\,\underline{\hat{r}} + \cos(\theta)\,\underline{\hat{\theta}}$$

$$\underline{\hat{r}} = \cos(\theta)\,\underline{\hat{i}} + \sin(\theta)\,\underline{\hat{j}}$$
$$\underline{\hat{\theta}} = -\sin(\theta)\,\underline{\hat{i}} + \cos(\theta)\,\underline{\hat{j}}$$

$$\frac{d}{dt}\left(\underline{\hat{r}}\right) = -\sin(\theta)\dot{\theta}\,\underline{\hat{i}} + \cos(\theta)\dot{\theta}\,\underline{\hat{j}}$$
$$= \dot{\theta}\,\underline{\hat{\theta}}$$

$$\underline{r} = r\,\underline{\hat{r}}$$

$$\frac{d}{dt}\left[\underline{r}\right] = \dot{r}\,\underline{\hat{r}} + r\,\dot{\underline{\hat{r}}}$$
$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

5. A particle of mass m when displaced from its equilibrium position by a distance $z\underline{\hat{\mathbf{k}}}$ is subject to a force: $-kz\,\underline{\hat{\mathbf{k}}}$.

Write down the equation of motion of the particle and show that: $z = A\cos(\omega t + \phi)$ is a solution to this equation where ω is the angular frequency of oscillation and ϕ is a constant phase. Hence determine the period of the particle's motion about the equilibrium position. [3]

Write down the equation of motion if an additional damping force: $-2m\alpha\dot{z}\,\underline{\hat{\mathbf{k}}}$ is applied.

By considering a solution to this equation of the form: $z(t) = Be^{qt}$, and by finding q, show that the maximum displacement from the equilibrium position reduces with time as: $e^{-\alpha t}$ provided $m\alpha^2 < k$. [4]

$$\frac{d^2z}{dt^2} = -kz$$

$$\rightarrow \ddot{z} + kz = 0$$

$$z = A\cos(\omega t + \phi)$$
$$\dot{z} = -A\omega\sin(\omega t + \phi)$$
$$\ddot{z} = -A\omega^2\cos(\omega t + \theta)$$

$$\ddot{z} + kz = 0$$

$$-A\omega^2\cos(\omega t + \theta) + kA\cos(\omega t + \theta) = 0$$

$$k = \omega^2$$

$$0 = 0$$

$$T = \frac{2\pi}{} - \frac{2\pi}{}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k}}$$

$$\ddot{z} + 2m\alpha\dot{z} + \omega^2 z = 0$$

$$q^2 + 2m\alpha q + \omega^2 = 0$$

$$q = -m\alpha \pm \sqrt{m^2\alpha^2 - \omega^2}$$

$$k = \omega^2 \to m\alpha^2 < \omega^2$$

$$q = -m\alpha \pm i\sqrt{\omega^2 - m^2\alpha^2}$$

$$z(t) = e^{-m\alpha t}\left(A\cos(\omega t) + B\sin(\omega t)\right)$$

6. A rigid body is composed of N particles with the i[th] particle having a mass m_i and being a distance r_i from the axis of rotation (A).

Write down an expression for the moment of inertia about A in terms of m_i and r_i and an expression for the rotational kinetic energy if the angular speed of rotation is ω. **[2]**

What is the moment of inertia of a ring of mass M and radius R about an axis through the centre of the ring and perpendicular to the plane of the ring?
If the rotation axis is again perpendicular to the plane of the ring but is instead through the ring itself i.e. displaced by R from the centre, what is the new moment of inertia? **[2]**

An impulse, $\underline{\mathbf{J}}$, is applied for a duration Δt at a distance R from the centre of a uniform, rigid rod of length L $(R < L/2)$ which is initially at rest. The direction of the impulse is perpendicular to the plane of the rod and the magnitude of the torque provided by the impulse is denoted by τ.
Show that the magnitude of the change in the rod's angular momentum about an axis through the rod's centre of mass is RJ and that $RJ = \tau\Delta t$. **[3]**

$$I = \sum_{1}^{N} m_i r_i^2 = \int_{vol} r_\perp^2 \rho \, dV$$

$$KE = \frac{1}{2} I \omega^2$$

$$\sigma = \frac{M}{\pi R^2}$$

$$dm = \sigma \, dA$$
$$= \sigma \, 2\pi r \, dr$$

$$I = \int_0^R r^2 \, dm$$

$$= \sigma 2\pi \int_0^R r^3 \, dr$$

$$= \frac{2\pi}{\pi} M \int_0^R r \, dr$$

$$= \frac{2}{2} M \left[r^2 \right]_0^R$$

$$= MR^2$$

$$I_2 = MR^2 + MR^2$$

$$= 2MR^2$$

$$T = R \times F$$

$$T = \frac{dL}{dt}$$

$$L = T\,dt$$

$$= R \times F\,dt$$

$$= RJ$$

$$RJ = T\,dt$$

$$= T\,\Delta t$$

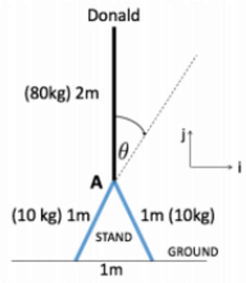
7. (a) Write the definition of the moment of inertia, I, of a rigid body, in the form of an integral. [1]

(b) Show that the moment of inertia of a uniform rod of length L and mass M about an axis through the end of the rod and perpendicular to the plane containing the rod is: [4]

$$I = \frac{1}{3}ML^2.$$

The rod has a negligible height and width.

(c) A clown, Donald, has his feet nailed to the top of a supporting stand. Donald can be represented as a uniform rod of mass 80 kg and length 2 m. The stand is composed of two rods each of mass 10 kg and length 1 m. The two rods of the stand are 1 m apart on the ground. Donald and the stand meet at point A and Donald stands vertically upright on the stand and he faces in the $\hat{\mathbf{i}}$ direction (see diagram).



At what height above the ground is the centre of mass (CM) of the combined Donald+stand system? [3]

(d) Determine the moment of inertia of the combined Donald+stand system about the axis A (into the page) and hence the moment of inertia about the CM of the combined Donald+stand system. **[3]**

(e) Donald leans forward and so rotates due to gravity in the $\hat{\underline{\mathbf{i}}}$-$\hat{\underline{\mathbf{j}}}$ plane through an angle θ to the vertical (see diagram). His feet remain at A and the stand remains stationary. Draw a sketch showing the new position of the combined Donald+stand CM relative to Donald's CM and the CM of the stand. Show that the distance moved in metres by the combined CM in the $\underline{\mathbf{i}}$ direction, X, is given by:

$$X = \frac{4}{5}\sin\theta.$$

[4]

(f) This movement of the CM results in a net torque about A. At the angle θ, what is the value of this torque? Hence show, at the moment Donald is horizontal, that the translational acceleration at the point on the rod furthest from A (i.e. Donald's head) is larger than g. **[5]**

7) a) $I = \int_{vol} r^2 \, dV$

b)

$\sigma = \frac{M}{L}$

$I = \int r^2 \, dm$

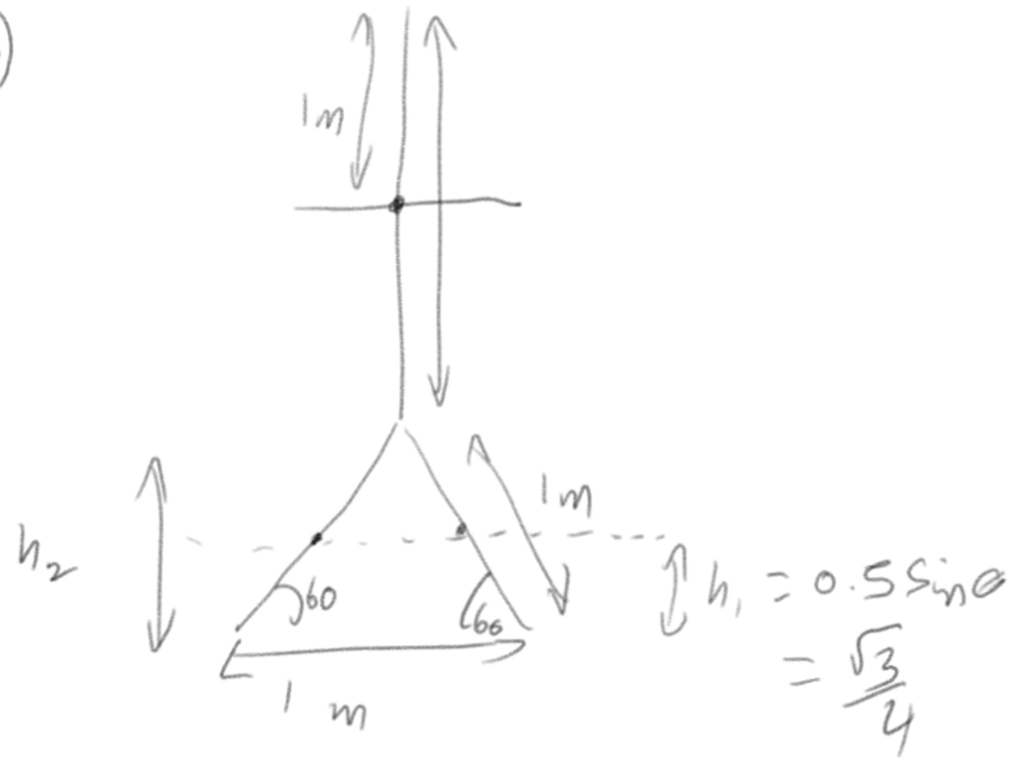
$dm = L \, d\sigma$

$$I = \frac{M}{L}\int_0^L r^2 \, dr$$

$$= \frac{1}{3}\frac{M}{L}\left[r^3\right]_0^L$$

$$= \frac{1}{3} M L^2$$

c)



$h_2 = \frac{\sqrt{3}}{2}$

$$CM = \frac{\left(80\,kg \times \left(\frac{\sqrt{3}}{2} + 1\right) + 20 \times \frac{\sqrt{3}}{4}\right)}{100}$$

$= 1.579\,m$

d)

$\frac{1}{3} \times 80 \times 2^2 = 106.\cancel{7}$

$\frac{1}{3} \times 10 \times 1^2 =$?

$\times 2 = 6.666$

$\times \cos(30) = 5.774$

$$\ldots = 113.33\ kg\,m^2$$

CM →

$$x = 1.579 - \sin(60)$$
$$= 0.713$$

$$I_{cm} = 113.33 - (0.713)^2 \times 100$$
$$= 62.5$$

a)

$$CM_x = \frac{20 \times 0 + x \times 80}{100}$$

$$CM_x = \frac{20\times 0 + }{100}$$

$$= \frac{4}{5}x$$

$$\underline{\underline{x = \frac{4}{5}\sin\theta \times 1}}$$

f)

$$\tau = 1 \times mg\sin\theta$$
$$= mg\sin\theta$$

$$\alpha = ra$$

$\alpha = ra$

$\tau = I\alpha$

$= I ra$

$r = 2$

$I = \frac{1}{3} \times 80 \times 2^2$

$= 106.66$

$\tau = mg\sin\theta \quad \theta = 90$

$\tau = mg$

$2mg = 106.66 \quad > a$

$a = \frac{2 \times 80}{106.66} g$

$= 1.5g$

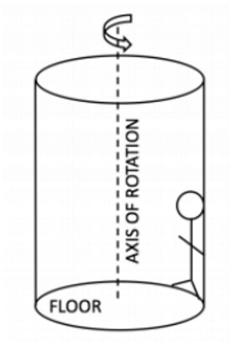
8. (a) The water in a river moves with velocity $4\hat{\underline{\mathbf{i}}}\,\mathrm{ms}^{-1}$. A boat starts at the bank and this bank, bank-A, is along the $\hat{\underline{\mathbf{i}}}$ direction. The boat's captain wants to cross the river to the opposite bank, bank-B, which runs parallel to bank-A. The captain sets the engine such that the boat would have a speed of $5\,\mathrm{ms}^{-1}$ in still water. What is the velocity vector of the boat and the angle of the boat's travel relative to bank-A for the boat to cross the river in the shortest distance? **[3]**

(b) A model car has four wheels and a total mass of 100 g. It is initially at rest at one side of a horizontal table. It is given an impulse in the $+\hat{\underline{\mathbf{i}}}$ direction and reaches the other side of the table in 4 s. The coefficient of kinetic (sliding) friction is 0.1 and g is $9.81\,\mathrm{ms}^{-2}$.

If the wheels are locked and the car slides across the table, what is the minimum impulse required for the car to cross the table?

The car is placed back at its starting position and the wheels are unlocked and the wheels, and hence car, roll immediately after an impulse is applied in the $+\hat{\underline{\mathbf{i}}}$ direction. If the radius of the wheels is 5 cm, what is the minimum impulse for the car to roll 1 m in 4 s and what is the angular speed of rotation of the wheels? **[7]**

(c) A fairground ride comprises a cylindrical drum of radius 5 m that is rotated in a horizontal plane about a vertical axis through its centre so that the people standing inside the drum are held against its wall by friction when the horizontal floor initially supporting them is removed: see diagram. The coefficient of static friction between the person and the drum's wall is 0.4.



Calculate the minimum rotational speed for a person of mass 70 kg to remain fixed against the wall. [5]

(d) A small cockroach, Nigel, of mass m, clings to the rim of a rotating disc of mass M and radius R. The disc rotates (in the horizontal plane) around a vertical (frictionless) axis through its centre which is perpendicular to the horizontal floor. When Nigel is at the rim the disc rotates with angular speed ω_0. Nigel then walks with uniform velocity in a straight line to the centre of the disc. The moment of inertia of a disc is $\frac{1}{2}MR^2$.

Determine how the angular speed of the disc, ω, changes as a function of Nigel's radial distance, r, from the centre. How does the kinetic energy of the system change and why? [5]

a)

$$|v| = \sqrt{5^2+4^2}$$
$$= \sqrt{41}$$

90^0 to Bank

$$\cos(\theta) = \frac{4}{5}$$
$$\theta = 36.9^0$$

b)

$$0.01g \times 4s = I$$
$$= 0.392$$

$$I = \frac{dP}{\Delta t}$$

$I = \frac{dp}{dt}$

$v = \frac{1}{4} = 0.25 m/s$

$p = 0.25 \times 0.1 kg$

$\rightarrow$ $I = 0.025$

$\longrightarrow$

$0.25 m/s$

$circ = 2\pi r$

$= 0.314 m$

$\frac{0.25}{0.314} = 0.796 \ rad/s$

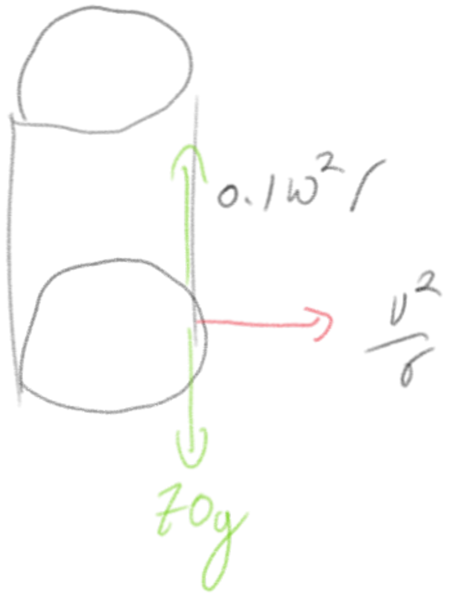
$\times 2\pi$

$\omega = 5$

$\times 2\pi$

$\omega = 5$

c)



$v = \omega r$

$\omega^2 r$

$$\cancel{70} \times 0.4\omega^2 r > \cancel{70} g$$

$$\omega^2 > \frac{\cancel{70\times}9.8}{0.4\times 5}$$

$$\omega^2 > 4.9$$

$$\omega > 2.21 \text{ rad s}^{-1}$$

ω_0

d)

$$L = \frac{1}{2} m R^2 \omega + r m \omega$$

$$= \omega \left[\frac{1}{2} M R^2 + r m \right]$$

L is constant

$r = R$

$\omega_0 = \omega$

$$[M + m] R^2 \omega_0 = \omega \left[\frac{1}{2} M R^2 + r m \right]$$

?

ke is constant

9. (a) Using the definition of angular momentum, $\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$, show that $|\underline{\mathbf{L}}| = mr^2\dot{\theta}$ and state the direction of $\underline{\mathbf{L}}$. [4]

(b) Hence, if a particle is subject to a central force show that this requires: [3]

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

(c) The Earth's radius is $R = 6371\,\text{km}$ and the Earth can be treated as a sphere of uniform density $\rho = 5513\,\text{kgm}^{-3}$. The gravitational constant, $G = 6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$.

Write an expression for the mass of the fraction of the Earth contained within a sphere of radius r where $r \leq R$.

A hollow evacuated tunnel of radius 500 km is dug between the North and South Poles through the centre of the Earth. At time $t = 0$ a ball is dropped from rest vertically down the tunnel at the North Pole. Neglecting air resistance, temperature effects and the mass of Earth excavated to make the tunnel, show that the position of the ball at time t, $r(t)$, satisfies: [6]

$$r(t) = R\cos(\omega t)$$

and determine the period of oscillation to the nearest minute.

(d) Kepler's laws are valid for any central force. Derive the second law i.e.

$$\frac{dA}{dt} = \frac{L}{2m}$$

for a particle of mass m and angular momentum L subject to a central force, where dA is the area traced out in a time dt by the trajectory of m and the line joining m to the origin of the force. **[2]**

(e) A second tunnel, also of radius 500 km, is now dug from a point on the Equator through the centre of the Earth and a particle of mass m is dropped from rest down the tunnel. The equation of motion now includes a term due to the Earth's rotation, show that the equation of motion is still described by:

$$\ddot{r} = f(r).$$

In accordance with Kepler's first law for a central force, the particle follows an elliptical trajectory through the Earth. Using this and the form of Kepler's second law given above, determine, neglecting the Coriolis force, the closest distance the ball dropped down the Equatorial tunnel gets to the centre of the Earth. You may assume that the period of oscillation is the same (to the nearest minute) as calculated in part (c). **[5]**

d)

$$\underline{L} = \underline{r} \times \underline{p}$$

$$p = \underline{v} m$$

$$\underline{v} = \dot{\theta} r$$

$$|L| = |\underline{r}| \times |p|$$

$$|p| = \dot{\theta} r m$$

$$|L| = |r||r||\dot{\theta}| m$$

$$= m r^2 \dot{\theta}$$

L is positive ~~out~~ out from paper (Right hand rule)

Perp to $\underline{r}$ & $\underline{p}$

b) central force $\rightarrow$ $\underline{\hat{\theta}} = 0$

$$\underline{F} = m(\ddot{r} - r\dot{\theta}^2)\underline{\hat{r}} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{\hat{\theta}}$$

L is constant:

↳ L is constant:

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$= m\left[\frac{d}{dt}\left(r^2\right)\dot{\theta} + \frac{d}{dt}\left(\dot{\theta}\right)r^2\right] = 0$$

$$= m\left[2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right] = 0$$

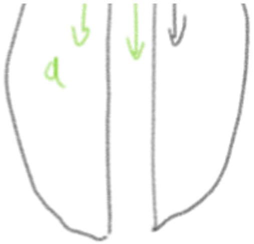
$$\to 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

c) $m = \rho V$

$V = \frac{4}{3}\pi r^2$

$$\frac{m}{M} = \frac{\frac{4}{3}\pi r^2}{\frac{4}{3}\pi R^2} = \frac{r^2}{R^2}$$

$$a = \frac{GM}{R^2}$$



$$a = \frac{4\pi}{R^2}$$

$$m = \frac{r^3}{R^2}$$

$$a = \rho \frac{4 r^2}{R^4}$$

d)

$$\frac{dL}{dt} = 0$$

$$L = m r^2 \dot{\theta}$$

$$\frac{dL}{dt} = 0$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

$$dA = \frac{r^2}{2}\dot{\theta}\delta t$$

r

$$dA = \frac{r^2}{2}\dot{\theta}\,\delta t$$

$$\frac{dA}{dt} = \frac{r^2\dot{\theta}}{2}$$

$$= \frac{mr^2\dot{\theta}}{2m}$$

$$= \frac{L}{2m}$$?

?

10. (a) A spring is compressed by a distance x from its equilibrium position. Write down an expression for the restoring force if the spring constant is k, and hence show that the potential energy of the spring is $\frac{1}{2}kx^2$. [2]

(b) A particle, A, with speed v_o collides with a stationary particle, B, of the same mass. After the collision, A is deflected through an angle θ with respect to its original direction and B moves at an angle ϕ to A's original direction with a speed v. Show that:

$$\tan\theta = \frac{v\sin\phi}{v_o - v\cos\phi}$$

and if the collision is elastic that: [9]

$$v = v_o\cos\phi.$$

For the elastic collision, show that the angle between the directions of A and B after the collision is 90°.

(c) Show that the kinetic energy (KE′) of a system of two particles calculated in the centre of mass (CM) frame is given by:

$$\mathrm{KE}' = \frac{1}{2}\mu v^2$$

where μ is the reduced mass of the two particles and v is the relative speed of the two particles. Hence show that the KE for the system calculated in the lab-frame ($\mathrm{KE}_{\mathrm{LAB}}$) which moves at a speed V_{CM} relative to the CM frame is:

$$\mathrm{KE}_{\mathrm{LAB}} = \mathrm{KE}' + \frac{1}{2}\mathrm{M}\mathrm{V}_{\mathrm{CM}}^2$$

where M is the total mass of the two particle system. [5]

(d) A mass $m_1 = 0.3\,\mathrm{kg}$ slides on a frictionless horizontal surface with speed $2\,\mathrm{ms}^{-1}$. It collides with a stationary mass, $m_2 = 0.7\,\mathrm{kg}$. A spring of negligible mass with a spring constant k $= 20\,\mathrm{Nm}^{-1}$ attached to m_1 compresses and then relaxes during the collision during which energy is conserved. See the figure below.

By considering the KE in the CM system, determine the maximum compression of the spring during the collision. [4]

a)

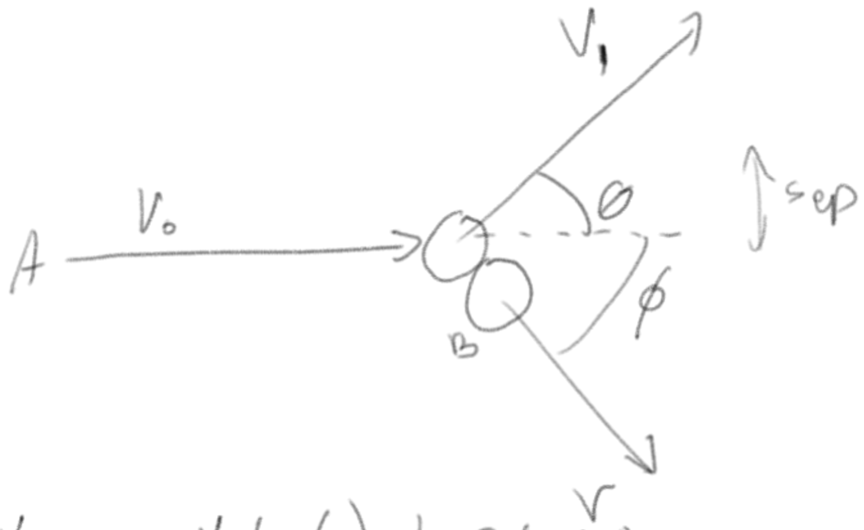
$$F = -kx$$

$$PE = -\int F \, dx$$

$$= \int kx^2 \, dx$$

$$= \frac{1}{2} kx^2$$

b)



$$V_0 = V_1 \cos(\theta) + v \cos(\phi)$$

$$V_1 \sin(\theta) = V \sin(\phi)$$

$$v_1 \sin(\theta) = V \sin(\phi)$$

$$v_1 \cos(\theta) = V_0 - V \cos(\phi)$$

$$\tan(\theta) = \frac{V \sin(\phi)}{V_0 - V \cos(\phi)}$$

$$e = \frac{sep}{app} \qquad app = V_0$$

$$\therefore \quad sep = V_0$$

$$sep = \sqrt{(V_1 \sin\theta - V \sin\phi)^2 + (V_1 \cos\theta - V \cos\phi)^2}$$

$$= \sqrt{\begin{array}{l} v_1^2 \sin^2\theta - 2v_1 v \sin\theta \sin\phi + v^2 \sin^2\phi \\ v_1^2 \cos^2\theta - 2v_1 v \cos\theta\cos\phi + v^2\cos^2\phi \end{array}}$$

$$= \sqrt{\begin{array}{l} v_1^2\left(\sin^2\theta + \cos^2\theta\right) + v^2\left(\sin^2\phi + \cos^2\phi\right) + \\ -2v_1 v\left[\sin\theta\sin\phi + \cos\theta\cos\phi\right] \end{array}}$$

$$= \sqrt{2v_1 v \cos(\theta-\phi)}$$

3

$$V = V_0 \cos\phi$$

$$\tan\theta = \frac{V\sin\phi}{V_0 - V\cos\phi}$$

$$= \frac{V_0 \sin\phi\cos\phi}{V_0 - V_0\cos^2\phi}$$

$$= \frac{V_0}{V_0}\frac{\sin\phi\cos\phi}{1-\cos^2\phi}$$

$$c^2+s^2=1$$

$$\rightarrow 1-c^2 = s^2$$

$$\rightarrow \tan\theta = \frac{\sin\phi\cos\phi}{\sin^2\phi}$$

$$\tan\theta = \cot\phi$$

$$\tan\theta \tan\phi = 1$$

$$\therefore \quad \theta + \phi = 90$$

c)

$$K_{\text{rel}} = \frac{1}{2}\left(m_1 v_1'^2 + m_2 v_2'^2\right)$$

$$v_1' = \frac{m_2}{M} v$$

$$v_2' = -\frac{m_1}{M} v$$

$$KE_{rel} = \frac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right) v^2$$

$$= \frac{1}{2}\frac{m_1 m_2}{M^2}\left(m_2 + m_1\right) v^2$$

$$= \frac{1}{2}\frac{m_1 m_2}{M} v^2$$

$$= \frac{1}{2}\mu v^2.$$

$$KE_{lab} = KE_{rel} + KE_{cm}$$

$$KE_{cm} = \frac{1}{2} M V_{cm}^2$$

$KE_{cm} = \frac{1}{2}$

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$KE_{lab} = KE' + \frac{1}{2}\frac{(m_1+m_2)}{(m_1+m_2)^2}\left[m_1^2 v_1^2 + 2 m_1 m_2 v_1 v_2 + m_2 v_2^2\right]$$

$$KE_{lab} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

d)

$$v_{cm} = \frac{0.3 \times 2}{1}$$

$$= 0.6$$

$$\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (0.6)^2 = \frac{1}{2} k x^2$$

$$x = \sqrt{\frac{\mu (v_1 - v_2)}{k}}$$

$$\mu = \frac{m_1 + m_2}{m_1 m_2} = 4.76$$

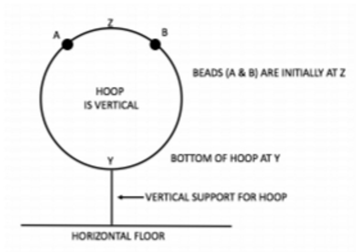
$$x = \sqrt{\frac{4.76 \times 2^2}{20}}$$

$$= 0.145$$

?

3

11. (a) Two beads each of mass m are at the top (Z) of a frictionless hoop of mass M and radius R which lies in the vertical plane. The hoop is supported by a frictionless vertical support. The beads are given a tiny impulse and due to gravity they slide down the hoop: one clockwise and one anti-clockwise.



Determine the minimum value of $X = \frac{m}{M}$, X_{MIN}, for which the hoop will rise up off the support before the beads reach the bottom of the hoop (Y). [7]

(b) If $X < X_{\text{MIN}}$ and the beads collide inelastically at the bottom of the hoop with a coefficient of restitution $e_R = 0.98$, determine the minimum angle (θ) to the nearest degree with respect to the initial position ($\theta = 0$) achieved by the clockwise moving bead. [3]

(c) How many collisions are required so that the maximum height achieved by the beads is less than 0.01R? [4]

(d) A satellite of mass m_2 is launched with the help of a rocket from a spherical and non-rotating planet of mass m_1 and radius r_1. The satellite is launched from point A on the planet's surface with a launch speed of V_A at an angle α with respect to the planet's radial direction. Unfortunately the rocket is not powerful enough for the satellite to achieve a stable orbit and it reaches a maximum height at a point B before crashing back down onto the planet. B is at a distance $r_2 = Xr_1$ from the centre of the planet and at B the satellite's speed is V_B. All forces on the satellite are radial with respect to the centre of the planet.

What quantities are conserved during the satellite's motion? Use this to find an expression for $\frac{V_B}{V_A}$ in terms of α and X. **[3]**

(e) Show that the satellite will always return to the planet's surface provided:

$$\frac{X(X-1)}{X^2 - \sin^2\alpha} < 1.$$

[3]

a) what?