Taylor-Series

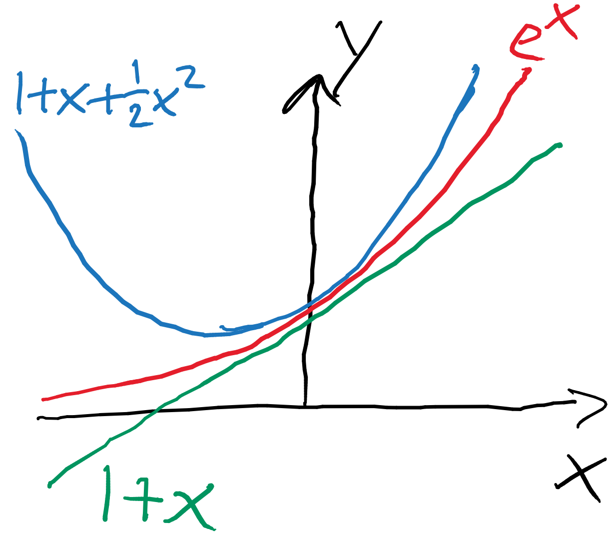
Express given function $f(x)$ in terms of power series

$$\boxed{\begin{aligned} f(x) &= f(x_0) + \overset{\text{1st order}}{f'(x_0)(x-x_0)} + \overset{\text{2nd order}}{\tfrac{1}{2} f''(x_0)(x-x_0)^2} + \ldots \\ &= f(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x_0} (x-x_0)^k \end{aligned}}$$

"Taylor-expand $f(x)$ around $x = x_0$"

- If $x_0 = 0 \longrightarrow$ Maclaurin-Series
- Example: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$



Note: Convergence of Taylor series of $\sin x$ and $\cos x$

Problem: Every other term in series is zero, e.g.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$\Rightarrow$ d'Alembert Ratio test not directly applicable!

Solution: Express series as

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\Rightarrow \rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{|x^{2k+3}|}{(2k+3)!} \cdot \frac{(2k+1)!}{|x^{2k+1}|}$$

$$= \lim_{k\to\infty} \frac{x^2}{(2k+2)(2k+3)} = x^2 \cdot 0 \Rightarrow \text{converges for all } x$$

Taylor Expansion of multi-variate functions $z=f(x,y)$

- Taylor-expand first in x then in y to required order

$$f(x,y) = f(x_0,y) + \frac{\partial f}{\partial x}(x_0,y)\cdot(x-x_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y)\cdot(x-x_0)^2 + \ldots$$

(1) (2) (3)

now only functions in y

(1) to 2nd order

$$= \left[f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdot(y-y_0)^2 + \ldots\right]$$

(2) to 1st order

$$+ \left[\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial y \partial x}(x_0,y_0)\cdot(y-y_0) + \ldots\right](x-x_0)$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x_0,y_0) + \ldots\right](x-x_0)^2 + \ldots$$

(3) to 0th order

$$= f(x_0,y_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)$$

(all derivatives at $x=x_0$, $y=y_0$)

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + \frac{\partial^2 f}{\partial y^2}(y-y_0)^2 + 2\frac{\partial^2 f}{\partial y \partial x}(x-x_0)(y-y_0)\right]$$

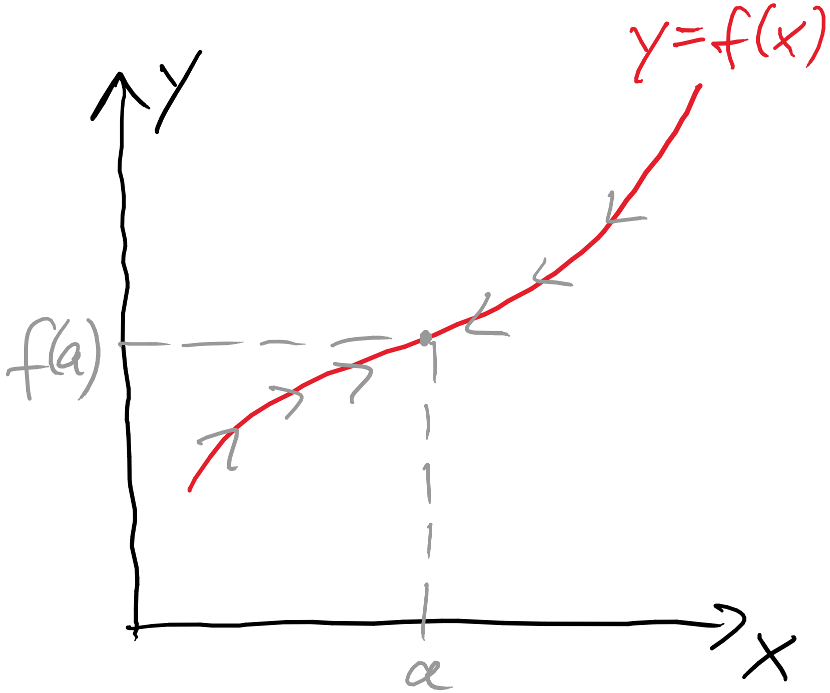
Limits

Consider "limit" of function value $f(x)$ as x "approaches" a:

$$\lim_{x \to a} f(x)$$

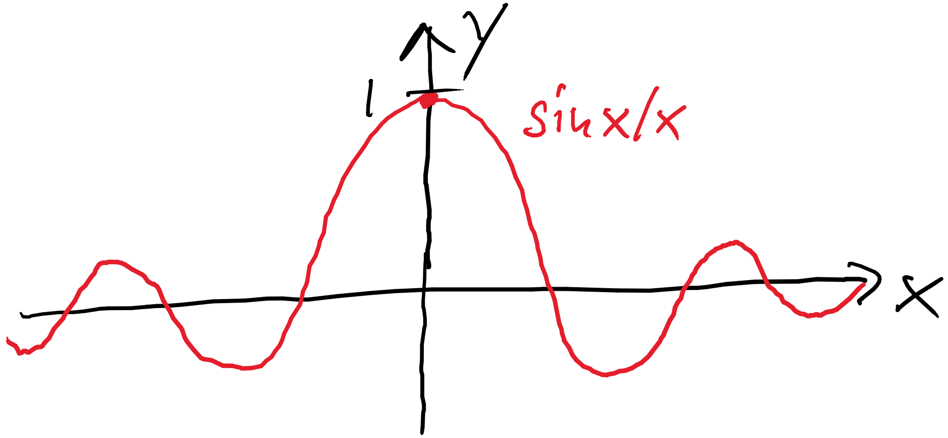
- If $f(x)$ is defined at $x = a$ and well-behaved (continuous):

$$\lim_{x \to a} f(x) = f(a)$$



- Can also be defined if $f(x)$ is undefined at $x = a$, e.g.

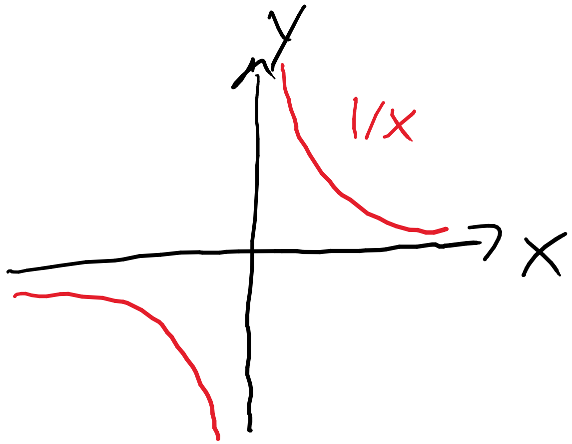
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



- Possible issues

  - Limit may not exist, e.g.

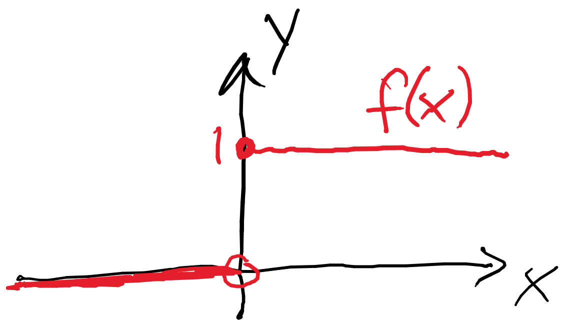
$$\lim_{x \to 0} \frac{1}{x} \to \pm \infty$$



  - Limit maybe different on each "side"

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \Rightarrow \lim_{x \to 0+} f(x) = 1$$

$$\lim_{x \to 0-} f(x) = 0$$



"Exact" definition:

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \left(0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right)$$

Basic properties $\left(\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x) \text{ exist} \right)$

- $$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
- $$\lim_{x \to a} \left[f(x) g(x) \right] = \left[\lim_{x \to a} f(x) \right] \cdot \left[\lim_{x \to a} g(x) \right]$$
- $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$