PHAS1247 Classical Mechanics
Formula Summary
2016–17

Glossary of symbols for PHAS1247

Bold symbols refer to vector quantities (underlined in handwritten lecture notes). If a symbol referring to a vector quantity is used in an ordinary font, it signifies the magnitude of the vector. Primes are used to denote position $\mathbf{r}'$, velocity $\mathbf{v}'$, momentum $\mathbf{p}'$ etc relative to the centre of mass.

$\mathbf{a}$	Acceleration
a	Semi-major axis of elliptical orbit
b	Semi-minor axis of elliptical orbit, impact parameter of hyperbolic orbit
e	Coefficient of restitution (in collisions) or eccentricity (of elliptical orbit)
E	Total energy
$\mathbf{F}$	Force
g	Acceleration due to gravity
G	Gravitational constant
h	Semi-latus rectum of conic section, height
k	Spring constant (in Hooke's law), radius of gyration (in rotational motion)
K	Kinetic energy, constant in inverse square law ($F(r) = K/r^2$)
$\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$	Cartesian basis vectors
$\mathbf{I}$	Impulse
I	Moment of inertia matrix, or moment of inertia for a fixed axis
$\mathbf{L}$	Angular momentum
m	Mass of single object
M	Total mass of a number of objects
$\mathbf{N}$	Normal force between two surfaces
$\mathbf{p}$	Momentum of an object
p	Pressure
$\mathbf{P}$	Total momentum of a number of objects
q	Electric charge
$\mathbf{r}$	Position vector (for a single particle) or relative position vector (for two objects)
r	Distance from origin in polar coordinates; magnitude of vector $\mathbf{r}$
$\mathbf{R}$	Position vector of centre of mass
R	Amplitude (of simple harmonic motion)
s	Distance moved (in one dimension)
t	Time
T	Period of oscillation
$\mathbf{u}$	Initial velocity
$\mathbf{v}$	Velocity (for a single particle) or relative velocity (for two objects)
V	Potential energy
$\mathbf{V}$	Velocity of centre of mass
W	Work done
x, y, z	Cartesian coordinates
δ	Denotes infinitesimal change in a quantitiy
Δ	Denotes finite change in a quantity
ϵ_0	Permittivity of the vacuum (appears in Coulomb's electrostatic force law)
ϕ	Phase angle (in simple harmonic motion)
γ	Damping rate (for simple harmonic oscillator)
λ	Constant of proportionality for velocity-dependent damping force: $\mathbf{F} = -\lambda\mathbf{v}$.
μ	Coefficient of friction or reduced mass
ν	Frequency
ρ	Density (mass per unit volume)
σ	Mass per unit length or mass per unit area
$\boldsymbol{\tau}$	Torque
θ	Angle in polar coordinates
ω	Angular frequency or angular velocity

1 The laws of motion and simple examples

1.1 Newton's second law

Momentum

$$\mathbf{p} = m\mathbf{v} \tag{1}$$

Newton's second law:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{2}$$

1.2 Constant acceleration

Motion under a constant force (general case):

$$\begin{aligned}
\mathbf{v} &= \mathbf{u} + \mathbf{a}t, && (3)\\
\Delta\mathbf{r} &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2, && (4)\\
v^2 &= u^2 + 2\mathbf{a}\cdot\Delta\mathbf{r}, && (5)\\
\Delta\mathbf{r} &= \frac{1}{2}\left(\mathbf{u}+\mathbf{v}\right)t. && (6)
\end{aligned}$$

Motion under a constant force (one dimension):

$$\begin{aligned}
v &= u + at, && (7)\\
s &= ut + \frac{1}{2}at^2, && (8)\\
v^2 &= u^2 + 2as, && (9)\\
s &= \frac{1}{2}\left(u+v\right)t. && (10)
\end{aligned}$$

Impulse

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t \tag{11}$$

Impulse is change of momentum during the interval:

$$\mathbf{I} = \left(\mathbf{p}_2 - \mathbf{p}_1\right). \tag{12}$$

1.3 Coefficient of friction

Sliding friction: frictional force $\mathbf{F}_f$ acts opposing the sliding motion of the two surfaces and has magnitude $|\mathbf{F}_f| = \mu|\mathbf{N}|$, where $\mathbf{N}$ is the normal force pressing the surfaces together.
Static friction: acts opposing any net force that would have the effect of moving one surface relative to the other. Has maximum magnitude $|\mathbf{F}_f| = \mu|\mathbf{N}|$; when applied force exceeds this, the surfaces start to slide.

1.4 Conservation of momentum

Total momentum $\mathbf{P}$ changes only as a result of application of external forces:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{ext,tot}}$$

In absence of external forces, $\mathbf{P}$ is constant.

2 Work, power and energy

Work done by force $\mathbf{F}$ when object moves through infinitesimal displacement $\delta\mathbf{r}$:

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} = (F\cos\theta)\ |\delta\mathbf{r}|. \tag{13}$$

Work done by force $\mathbf{F}$ along a path from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} [F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z]. \tag{14}$$

If the force is constant,

$$W = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1). \tag{15}$$

Power developed by force $\mathbf{F}$

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v} = F_x v_x + F_y v_y + F_z v_z. \tag{16}$$

Kinetic energy

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}. \tag{17}$$

Total work done on object (by all forces acting on it) is the change in kinetic energy:

$$W = K(t_2) - K(t_1), \tag{18}$$

If the force depends only on the position (and not on e.g. velocity), the potential energy change between two points is the work done by an external force counterbalancing the physical force, i.e. $F_{\text{ext}} = -F$, in moving the system from the first point to the second. In one dimension,

$$W_{\text{ext}} = \int_{x_1}^{x_2} F_{\text{ext}}\, \mathrm{d}x = -\int_{x_1}^{x_2} F\, \mathrm{d}x = V(x_2) - V(x_1), \tag{19}$$

Work done by the physical force is

$$W = -W_{\text{ext}} = V(x_1) - V(x_2). \tag{20}$$

and force is related to potential energy by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{21}$$

In three dimensions, can only define potential energy if the force is conservative, i.e. if work done to move betweeen two endpoints is independent of path taken. If the force is conservative, then

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) = \text{change of P.E.} \tag{22}$$

and the force is related to the potential energy by

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial z}\hat{\boldsymbol{k}}. \tag{23}$$

The condition for a force to be conservative can be expressed as

$$\int_{\text{loop}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0. \tag{24}$$

or

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

(and similarly for the other pairs of coordinates). For a motion under a conservative force the total energy $E = T + V$ is conserved.

3 Collisions

3.1 Centre of mass

Centre of mass defined as

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \tag{25}$$

where $M = \sum_i m_i$ is the total mass. Then

$$M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = M\mathbf{V} = \mathbf{P}. \tag{26}$$

Positions and velocities relative to centre of mass:

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R} \tag{27}$$

and

$$\mathbf{v}'_i = \frac{\mathrm{d}\mathbf{r}'_i}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{v}_i - \mathbf{V}. \tag{28}$$

Total momentum relative to centre of mass is zero:

$$\sum_i m_i \mathbf{v}'_i = 0. \tag{29}$$

Kinetic energy is

$$K = \sum_i \frac{m_i v_i'^2}{2} + \frac{MV^2}{2} = K_{\mathrm{rel}} + K_{\mathrm{cm}}. \tag{30}$$

3.2 Relative position and velocity; reduced mass

For two objects define relative position

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{31}$$

and velocity

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2. \tag{32}$$

Displacements and velocities in centre of mass frame

$$\mathbf{r}'_1 = \frac{m_2}{M}\mathbf{r}; \qquad \mathbf{r}'_2 = \frac{-m_1}{M}\mathbf{r}, \tag{33}$$

and

$$\mathbf{v}'_1 = \frac{m_2}{M}\mathbf{v}; \qquad \mathbf{v}'_2 = \frac{-m_1}{M}\mathbf{v}. \tag{34}$$

Reduced mass

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{35}$$

Momentum of each particle in centre of mass frame

$$\mathbf{p}'_1 = -\mathbf{p}'_2 = \frac{m_1 m_2}{M}\mathbf{v} = \mu\mathbf{v}. \tag{36}$$

The relative part of the kinetic energy is

$$K_{\mathrm{rel}} = \frac{1}{2}\mu v^2, \tag{37}$$

and the total angular momentum about the centre of mass is

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v}. \tag{38}$$

In the absence of external forces, equation of motion for the relative separation $\mathbf{r}$ of two bodies is

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_1, \tag{39}$$

where $\mathbf{F}_1$ is the force exerted on particle 1 by particle 2. This is just the equation for a single particle of mass μ moving under the force $\mathbf{F}_1$.

3.3 Head-on collisions

In absence of external forces, total momentum is always conserved. This is automatically ensured by working in the centre-of-mass frame and ensuring total momentum there is zero. For elastic collisions, kinetic energy is also conserved; otherwise, for inelastic collisions, kinetic energy reduces as energy is dissipated.
For an elastic collision between a particle of mass m_1 and initial velocity u_1 and a stationary particle of mass m_2, final velocities are

$$v_1 = \frac{m_1 - m_2}{M} u_1; \qquad v_2 = \frac{2m_1}{M} u_1.$$

Limiting cases:

- If the masses are equal, $m_1 = m_2$,
$$v_1 = 0, \quad v_2 = u_1.$$
- If $m_2 \gg m_1$ then
$$v_1 \approx -u_1; \qquad v_2 \approx 0.$$
- If $m_1 \gg m_2$, then
$$v_1 \approx u_1; \qquad v_2 \approx 2u_1.$$

Inelastic collision: coefficient of restitution is factor by which relative velocity is reduced following the collision. For the same collision as above but with a coefficient of restitution e, the final velocities are

$$v_1 = \frac{m_1 - em_2}{M} u_1; \qquad v_2 = \frac{(1+e)m_1}{M} u_1.$$

The case $e = 0$ gives a 'sticking collision'.

3.4 Glancing collision of two balls

Impulse is along the line joining the centres. If ϕ is angle made by final velocity $\mathbf{v}_2$ of ball 2 (initially at rest) to initial velocity $\mathbf{u}_1$ of ball 1, and θ' is the angle of deflection of ball 1 in the centre-of-mass frame, then

$$\begin{aligned} \theta' &= \pi - 2\phi; \\ v_2 &= \frac{2m_1}{M} u \sin\left(\frac{\theta'}{2}\right) = \frac{2m_1}{M} u \cos(\phi). \end{aligned}$$

Proportion of kinetic energy transferred to particle 2 is

$$\frac{m_2 v_2^2}{m_1 u^2} = \frac{4m_1 m_2}{M^2} \sin^2\left(\frac{\theta'}{2}\right) = \frac{4m_1 m_2}{M^2} \cos^2(\phi). \tag{40}$$

Maximum for the case of a head-on collision ($\phi = 0$, $\theta' = \pi$).
If the masses are equal, the final velocities are perpendicular to one another: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

4 Orbits, angular momentum and central forces

4.1 Motion in plane polar coordinates

Unit basis vectors are

$$\widehat{\mathbf{r}} = \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}} \tag{41}$$
$$\widehat{\theta} = -\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}. \tag{42}$$

Position vector is

$$\mathbf{r} = r\widehat{\mathbf{r}}.$$

Velocity is

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\widehat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{43}$$

Acceleration is

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}. \tag{44}$$

For circular motion with constant angular velocity $\omega = \dot{\theta}$ and constant radius r, the speed is $v = \omega r$ and the centripetal acceleration is

$$\mathbf{a} = -r\dot{\theta}^2\widehat{\mathbf{r}} = -\frac{v^2}{r}\widehat{\mathbf{r}}. \tag{45}$$

4.2 Angular momentum and torque vectors

The angular momentum (about the origin) is defined as the vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}. \tag{46}$$

Equation of motion (Newton's second law) becomes is

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}, \tag{47}$$

where τ is the applied torque (also about the origin) defined by

$$\tau = \mathbf{r} \times \mathbf{F}. \tag{48}$$

The angular momentum and the torque can also be though of as the moments of the momentum and the force respectively about the origin.

5 Motion under a central force

Force directed along line joining particle to centre of force:

$$\mathbf{F}(\mathbf{r}) = F(r)\widehat{\mathbf{r}}. \tag{49}$$

Hence the torque $\tau = \mathbf{r} \times \mathbf{F} = 0$ and all components of the angular momentum $\mathbf{L}$ are conserved for a central force.
Important examples:

(a) electrostatic force

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}. \tag{50}$$

Like charges repel and unlike charges attract.

(b) gravitational force

$$\mathbf{F}(\mathbf{r}) = -G\frac{M_1 M_2}{r^2}\hat{\boldsymbol{r}} \tag{51}$$

between point or spherically symmetric masses; this is always attractive.

Then

$$m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0, \tag{52}$$

independent of the form of $F(r)$. The component of angular momentum perpendicular to the plane is

$$L = mr^2\dot{\theta}. \tag{53}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}t} = mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{54}$$

5.1 Circular motion in a gravitational field

Attractive centripetal acceleration provided by gravitational force, so

$$\omega^2 = \frac{GM}{r^3}, \tag{55}$$

where M is the mass of the central body (assumed fixed).

5.2 Potential energy for central force

A central force depending only on r is conservative: potential is

$$V(r) = -\int F(r)\,\mathrm{d}r \quad \text{so} \quad F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}.$$

The radial equation of motion is then the same as that for a particle moving in the effective potential

$$V_{\text{eff}}(r) = V(r) + V_C(r) = V(r) + \frac{1}{2}\frac{L^2}{mr^2}. \tag{56}$$

V_C is known as the *centrifugal potential.*

5.3 Nearly-circular motion in a central force

For an attractive central force varying as a power law

$$F(r) = -Kr^n$$

(where K is a positive constant), a slight perturbation to a circular orbit of period τ produces an oscillation of the radius whose period is

$$T = \frac{\tau}{\sqrt{n+3}}. \tag{57}$$

In general this produces an orbit that does not close on itself; exceptions are a Hooke's-law force ($n = 1$, $T = \tau/2$) and an inverse square law ($n = -2$, $T = \tau$). Circular orbits are unstable if $n < -3$.

5.4 General motion in an inverse-square-law force

Write

$$\mathbf{F} = \frac{K}{r^2}\hat{\mathbf{r}}. \tag{58}$$

Repulsive forces have $K > 0$; attractive forces have $K < 0$. Potential energy is

$$V = -\int \frac{\mathrm{d}r}{r^2}\,\mathrm{d}r = \frac{K}{r}; \tag{59}$$

with the zero chosen so the potential goes to zero as $r \to \infty$ (i.e. as objects become separated).
Radial equation of motion

$$m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) = \frac{K}{r^2}. \tag{60}$$

or

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{mK}{L^2}. \tag{61}$$

where $u = 1/r$. This describes simple harmonic motion in the variable u, so (choosing closest approach to the centre of force to be at $\theta = 0$) the orbit becomes

$$u = \frac{1}{r} = A\cos(\theta) - \frac{mK}{L^2}. \tag{62}$$

The orbit is a **conic section**

$$u = \frac{1}{r} = \frac{1}{h}(e\cos\theta \pm 1) \tag{63}$$

(plus sign applies to attractive force, minus sign applies to repulsive force). The **semi-latus rectum** is

$$h = \frac{L^2}{m|K|}, \tag{64}$$

and the **eccentricity** is

$$e = \sqrt{\left(1 + \frac{2EL^2}{mK^2}\right)} \tag{65}$$

For an attractive force ($K < 0$) there are three possibilities:

- $E > 0$, $e > 1$—hyperbola;
- $E = 0$, $e = 1$—parabola;
- $E < 0$, $e < 1$—ellipse (with $e = 0$ for the special case of a circular orbit).

For a repulsive force ($K > 0$) always have $E \geq 0$ and $e \geq 1$ (hyperbolic or parabolic orbits).

5.4.1 Elliptical orbits

Can also characterize an ellipse by the **semi-major axis**

$$a = \frac{h}{1 - e^2} = \left|\frac{K}{2E}\right| \tag{66}$$

and the **semi-minor axis**

$$b = \frac{h}{\sqrt{1 - e^2}} \tag{67}$$

The minimum radius (at $\theta = 0$) is $h/(1 - e)$; the maximum radius (at $\theta = \pi$) is $h/(1 + e)$.
Kepler's laws of planetary motion:

1. Planets move in an ellipse with the Sun at one focus;
2. Orbits sweep out equal areas in equal times, since

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\frac{L}{m} = \texttt{const.}; \tag{68}$$

3. Period of orbit

$$T = \frac{2\pi abm}{L} = 2\pi\sqrt{\frac{ma^3}{|K|}}, \tag{69}$$

so $T^2 \propto a^3$.

5.4.2 Hyperbolic orbits

The angular momentum is determined by the **impact parameter** b and the velocity at infinity v_∞:

$$L = mv_\infty b. \tag{70}$$

The angle of deflection ϕ from the scattering is given by

$$\tan\left(\frac{\phi}{2}\right) = \frac{K}{2Eb} \quad \text{or} \quad \cot\left(\frac{\phi}{2}\right) = \frac{2Eb}{K}. \tag{71}$$

6 Frames of reference and fictitious forces

Newton's laws are the same in all **inertial frames** (frames moving at a constant velocity with respect to one another). In a frame moving with a non-constant velocity $\mathbf{v}$ (i.e. one that is accelarating) the apparent acceleration of an object of velocity $\mathbf{u}$ is

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{72}$$

so Newton's second law $\mathbf{F}' = m\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t}$ applies only if the force is modified from the real physical force $\mathbf{F}$ as

$$\mathbf{F}' = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} - m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{73}$$

The second term is the **ficititious force** from the accelerating frame.
The rate of change of a vector $\mathbf{r}$ resulting from rotation of angular velocity ω about an axis through the origin (represented as a vector of length ω along the rotation axis) is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \omega \times \mathbf{r}, \tag{74}$$

Hence, in a frame rotating at uniform angular velocity ω, the apparent acceleration is

$$\mathbf{a}_{\text{apparent}} = \mathbf{a}_{\text{true}} - 2\omega \times \mathbf{v}_{\text{apparent}} - \omega \times (\omega \times \mathbf{r}). \tag{75}$$

There are two fictitious forces:

- The **centrifugal force**
$$\mathbf{F}_{\text{centrifugal}} = -m\omega \times (\omega \times \mathbf{r}) = m\omega^2 \mathbf{r}_\perp \tag{76}$$
(where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ perpendicular to ω), which acts radially outward in the rotating frame;
- The **Coriolis force**
$$\mathbf{F}_{\text{Coriolis}} = -2m\omega \times \mathbf{v}_{\text{apparent}}, \tag{77}$$
which acts at right angles to ω and to $\mathbf{v}_{\text{apparent}}$ (acts to the right on the Earth in the northern hemisphere).

7 Simple harmonic motion

7.1 Undamped

Mass m, attached to spring of force constant k. Equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \quad \text{where} \quad \omega^2 = \frac{k}{m}. \tag{78}$$

General solution

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{79}$$

or

$$x(t) = R\cos(\omega t + \phi). \tag{80}$$

The angular frequency is ω and the period of the oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{81}$$

Potential energy of spring

$$V(x) = \frac{1}{2}kx^2 + V_0. \tag{82}$$

7.2 Damped

Frictional force proportional to velocity: $F_f = -\lambda \frac{\mathrm{d}x}{\mathrm{d}t}$. Equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0, \tag{83}$$

where the natural (undamped) angular frequency is $\omega_0 = \sqrt{\frac{k}{m}}$, and

$$\gamma = \frac{\lambda}{2m}. \tag{84}$$

Solution is

$$x(t) = C\mathrm{e}^{q_1 t} + D\mathrm{e}^{q_2 t}, \tag{85}$$

where q_1 and q_2 are the roots of

$$q^2 + 2\gamma q + \omega_0^2 = 0. \tag{86}$$

Write

$$q = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} = -\gamma \pm i\omega, \tag{87}$$

with

$$\omega = \sqrt{\omega_0^2 - \gamma^2}. \tag{88}$$

- Underdamped case $\gamma < \omega_0$: ω is real and

  $$x(t) = \mathrm{e}^{-\gamma t}\left[A\cos\omega t + B\sin\omega t\right], \tag{89}$$

  giving damped oscillations.
  Quality factor

  $$Q = \frac{\omega}{2\gamma} \tag{90}$$

  so fraction $1/Q$ of energy is lost in time interval $1/\omega$.

- Critical damping $\gamma = \omega_0$:

  $$x(t) = (A + Bt)\, e^{-\gamma t}. \tag{91}$$

- Overdamped case $\gamma > \omega_0$:

  $$x(t) = A\mathrm{e}^{-|q_1|t} + B\mathrm{e}^{-|q_2|t} \tag{92}$$

  with

  $$q_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{93}$$

7.3 Forced damped oscillator

Write response to driving force $F = F_0 \cos\omega_f t$ in form (**??**). Then amplitude of response is

$$R = \frac{F_0}{m\left[\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2\right]^{1/2}} \tag{94}$$

and phase angle is

$$\tan\phi = \frac{2\gamma\omega_f}{\left(\omega_f^2 - \omega_0^2\right)}. \tag{95}$$

Work done per oscillation period by driving force

$$W = -\pi F_0 R \sin\phi,$$

and average power developed

$$\overline{P} = -\frac{1}{2} F_0 v_{\max} \sin\phi,$$

where $v_{\max} = r\omega_f$ is the maximum speed reached.

8 Rigid bodies

8.1 Centre of mass and moment of inertia

The centre of mass of a rigid body with mass distribution $\rho(\mathbf{r})$ is at

$$\mathbf{R} = \frac{1}{M}\int_{\text{body}} \rho(\mathbf{r})\mathrm{r}\,\mathrm{d}V. \tag{96}$$

The angular momentum and rotational kinetic energy of such a body are related to its angular velocity by

$$\mathbf{L} = I.\omega \tag{97}$$

and

$$K = \frac{1}{2}\omega \cdot I \cdot \omega \tag{98}$$

where I is the moment of inertia matrix

$$I = \sum_i m_i \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & r_i^2 - y_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & r_i^2 - z_i^2 \end{pmatrix} = \int_{vol} \rho(\mathbf{r}) \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & r_i^2 - y_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & r_i^2 - z_i^2 \end{pmatrix} \mathrm{d}V. \tag{99}$$

(The origin is assumed to be a point on the rotation axis.) In general $\mathbf{L}$ and ω are not parallel, but for rotations about a **principal axis** (or about a fixed axis) we can write

$$\mathbf{L} = I\omega \quad \text{and} \quad K = \frac{1}{2}I\omega^2, \tag{100}$$

where I is the appropriate diagonal element of the moment of inertia matrix:

$$I = \sum_i m_i r_{i,\perp}^2 = \int_{vol} \rho(\mathbf{r}) r_\perp^2\,\mathrm{d}V. \tag{101}$$

Hence I depends on mass distribution and on the rotation axis. The radius of gyration k is defined by the relationship $I = Mk^2$, where M is the total mass.

8.2 Analogy between linear and rotational motion

Linear			Rotation
mass	m	I	moment of inertia
velocity	$\mathbf{v}$	ω	angular velocity
linear momentum	$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p} = I \cdot \omega$	angular momentum
linear kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}\omega \cdot I \cdot \omega$	rotational kinetic energy
force	$F = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$	$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$	torque
	$F = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$	$\tau = I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}$	

8.3 Parallel axes theorem

If I_0 is the moment of inertia sbout an axis through the centre of mass of an arbitrary body, and I is the moment of inertia about a parallel axis a distance a away, then

$$I = I_0 + Ma^2. \tag{102}$$

8.4 Perpendicular axes theorem

For any lamina in the xy-plane,

$$I_z = I_x + I_y, \tag{103}$$

where x and y are two perpendicular axes in the plane and z is a third perpendicular axis passing through the intersection of x and y (which need not be the centre of mass of the object).

8.5 Combined rotational and translational motion

General motion of a rigid body is a combination of a translation of the centre of mass and a rotation about it, so the velocity of a component i of the body is

$$\mathbf{v}_i = \mathbf{V} + \omega \times (\mathbf{r}_i - \mathbf{R}). \tag{104}$$

The effect of any set of forces acting on the body can be decomposed into a total force acting through the centre of mass, plus a torque acting about the centre of mass.