2018 first attempt

02 April 2019 16:37

1. a) Give the definition of a conservative force. According to this definition, is friction a conservative force? **[2]**

b) Consider a hypothetical 3-dimensional gravitational force, still proportional to the particle mass, but with a 3-dimensional acceleration $\underline{a} = (3g, 2g, -g)$. Calculate the difference in gravitational potential energy between the origin and point $(1, 1, 1)$ for a body with mass m. **[3]**

c) Determine the expression for the three components of the velocity of a ball starting at rest, after being subject to this 3-dimensional gravity for 1 second. **[3]**

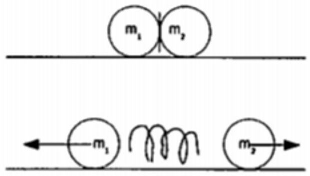
a) Work done is independent of path.
friction is dependent on distance ∴ no.

b)
$F = ma$
$V = mah$
$V(1,1,1) = 3g + 2g - g$
$= 4g$?

c)
$v = u + at$
$\frac{da}{dt} = 0$
$\Downarrow$
$v_x = 3g$
$v_y = 2g \rightarrow \underline{v} = g\begin{pmatrix}3\\2\\-1\end{pmatrix}$?
$v_z = -g$

2. a) In a two-body system, if the momentum of the first body in the centre of mass system is $-p$, what will the momentum of the second body be? [1]

b) Two balls with masses m_1 and m_2, with $m_1 > m_2$, are placed on a frictionless horizontal table, with a spring betwen them (see figure).



The spring has elastic constant k, and natural length ℓ. The spring is fully compressed until the balls touch, at which point the balls are released, starting their motion in two opposite directions; after the spring has expanded back to its natural length, it exerts no further force on the balls. Determine the expression for the speed of the two balls (assuming that after the spring has expanded to its natural length, it detaches from the balls). [4]

c) Determine an expression for the minimal mass ratio m_1/m_2 such that the absolute value of the speed of ball 2 is larger than that of ball 1. [1]

a) $\Sigma p = 0 \therefore p_2 = +p$

b) mometum in CM from $\Sigma = 0$:

$$v_1 m_1 = v_2 m_2 \qquad (v_1 < 0,\ v_2 > 0)$$

$$\frac{1}{2} k \ell^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$$

$$k \ell^2 = m_1 v_1^2 + m_2 v_2^2$$

$$v_1 = \frac{v_2 m_2}{m_1}$$

$$k \ell^2 = \frac{v_2^2 m_2^2}{m_1} + m_2 v_2^2$$

$$k \ell^2 = v_2^2 \left(\frac{m_2^2 + m_1 m_2}{m_1} \right)$$

$$v_2^2 = \frac{m_1 k \ell^2}{m_2^2 + m_1 m_2}$$

$$v_1^2 = v_2^2 \frac{m_2^2}{m_1^2}$$

$$v_1^2 = v_2 \frac{m_2}{m_1^2}$$

$$= \frac{m_2 k l^2}{m_1^2 + m_1 m_2}$$

c) $v_2 > v_1$

$$v_2 = \frac{m_1}{m_2} v_1$$

$$\therefore \frac{m_1}{m_2} > 1 \ ?$$

3. a) Give the definition of a central force. [2]

b) Consider a rod of length 2ℓ placed in a coordinate reference frame, with its centre at the origin, and a fixed angle θ (with $0 < \theta < \pi/2$) with respect to the x axis. A ball of mass m can freely move on the rod, pulled towards the centre by an elastic force $\underline{F} = -kr\hat{r}$. Assume that the ball is at rest and released at time $t = 0$ at one extreme of the rod, at the end of the rod in the positive x, positive y quadrant; calculate the ball's position $r(t)$ as a function of time (hint: solve the problem s that of a one-dimensional harmonic oscillator along the axis r). [2]

Find the (x, y) positions of the ball in the original reference frame. [1]

a) Force only has radial component, no transverse component.

b) $ma = -kr$

$$a = -\frac{k}{m} r$$

$$r = \cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$r = 2l \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$(x, y) = r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$? too easy?

$(x,y) = r \begin{pmatrix} - \\ \sin\theta \end{pmatrix}$! too easy?

4. a) Give the definition of the centre of mass of a system, and show that its velocity is constant if the system does not feel any external force, and its mass does not change. **[2]**

b) Consider a system of three pointlike balls, all with mass m_0, placed on a plane at (x, y) positions $(0,0)$, $(1,1)$, $(2,0)$. Give the expression for the (x, y) position of the centre of mass of the system. **[2]**

c) The mass of the ball at the origin is increased by pouring sand on top of it, according to the relation $m(t) = m_0(1 + bt)$, where t is the time since the start of pouring sand. The masses of the other balls are unchanged. Give the expression for the position of the centre of mass as a function of time, remembering that the total mass of the system changes. **[4]**

a) $R_{cm} = \dfrac{\sum R_i M_i}{\sum M_i}$

$V = \dfrac{\sum V_i M_i}{\sum m_i}$

$V = \dfrac{P}{M} \rightarrow F = 0 \rightarrow \dfrac{dp}{dt} = 0$

$\rightarrow Vm = P$

$1. \quad m \dfrac{dv}{dt} = 0$

$\rightarrow$

$\dfrac{d V_{cm}}{dt} = 0$

b) $R = \dfrac{m_0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + m_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + m_0 \begin{pmatrix} 2 \\ 0 \end{pmatrix}}{3m_0}$

$$= \frac{1}{3}\begin{pmatrix}3\\1\end{pmatrix}$$

c) $$R = \frac{m(t)\begin{pmatrix}0\\0\end{pmatrix} + m_0\begin{pmatrix}1\\1\end{pmatrix} + m_0\begin{pmatrix}2\\0\end{pmatrix}}{2m_0 + m(t)}$$

$$= \frac{m_0\begin{pmatrix}3\\1\end{pmatrix}}{2m_0 + m_0(1+bt)}$$

$$= \frac{\begin{pmatrix}3\\1\end{pmatrix}}{3+bt}$$

5. A stationary rod of length L has a non-uniform density, and is located on the x axis of a coordinate system, with one extreme at the origin. The mass density (mass per per unit length) $\lambda(x)$ varies as $\lambda(x) = \lambda_0 x/L$.

a) Determine the expression for the total mass of the rod. [2]

b) Determine the expression for the x position of its centre of mass. [2]

c) Determine the expression for its moment of inertia about an axis through the origin and perpendicular to the rod. [3]

a) $$M = \int_0^L x \, dm$$

$$\frac{M}{L} = \frac{\lambda_0 x}{L}$$

$$m = l_0 x$$

$$dm = l_0 \, dx$$

$$m = \int_0^L x \lambda_0 \, dx \quad ?$$

b) $CM = \frac{\sum m_i x_i}{\sum m_i}$

$=$?

6. Consider a particle with mass m, subject to a central force, described by a gravitational potential $V(r) = -GMm/r$.

a) Write down the total effective potential in the rotating system, as a function of the angular momentum L, the particle mass and the distance r, as well as the gravitational potential $V(r)$. **[2]**

b) Draw the graph of the effective potential as a function of the radius r, and indicate on it three horizontal lines representing possible values of the total energy of the system corresponding to the three different possible types of orbit. **[2]**

a) $m(\ddot{r} - r\dot{\theta}^2) = \frac{GMm}{r^2}$

$m\ddot{r} = \frac{GMm}{r^2} + mr\dot{\theta}^2$

$F_{eff} = ?$

$L = mr^2\dot{\theta}$

$\dot{\theta}^2 = \frac{L^2}{m^2 r^4}$

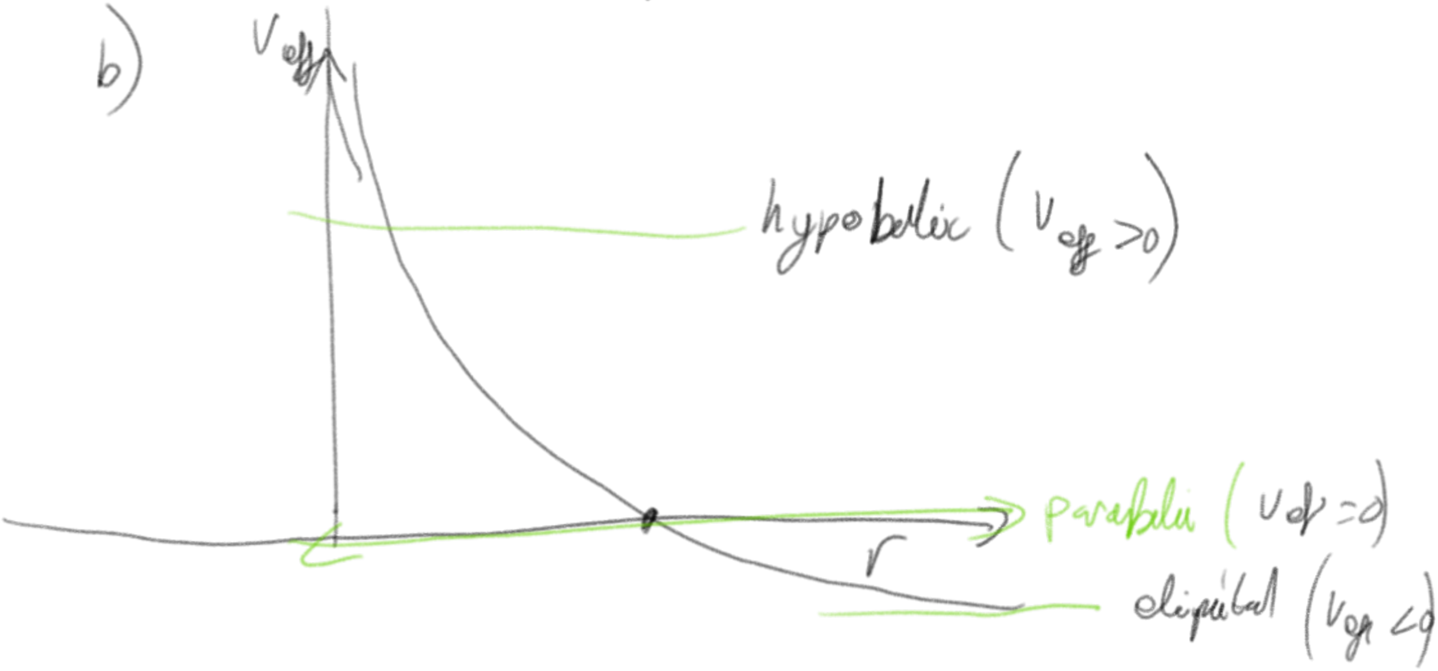
$m\ddot{r} = \frac{GMm}{r^2} + \frac{L^2}{mr^3}$

$V_{eff} = V(r) + -\frac{1}{dr}\left(\frac{L^2}{mr^3}\right)$

$= V(r) + \frac{1}{2}\frac{L^2}{mr^2}$

$$= V(r) + \frac{1}{2}\frac{L^2}{mr^2}$$

b)



c) Determine the expression for the value of the radius r corresponding to a circular orbit (minimum of the effective potential) as a function of L, $G\,M$ and m, and explain your result. **[2]**

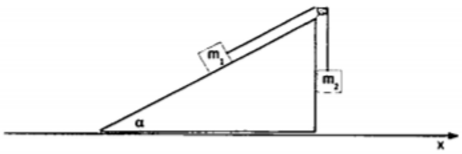
Lich →

$$\frac{GMm}{r} = \frac{1}{2}\frac{L^2}{mr^2}$$

$$r = \frac{L^2}{2GM}$$

trivial

7. A body with mass m_1 is laying on an inclined plane, that has an angle α with respect to the horizontal (see figure).



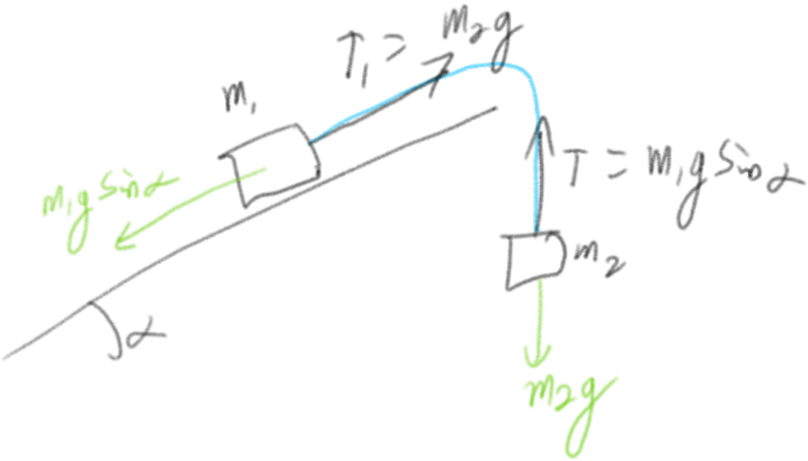
Assume initially no friction between the body and the plane. The body is attached to a massless rope, connected to a massless and frictionless pulley, placed at the top of the plane. On the other side of the pulley, the rope is attached to another body of mass m_2, hanging verically below it. Both masses are subject to the earth's gravitational field, and the plane is initially not moving.

a) Give the expressions for the forces acting on both masses, as well as the tension of the rope and the mass ratio needed to have the system in equilibrium. [5]

b) Consider now the case where there is a friction coefficient μ between mass m_1 and the inclined plane. Write the forces acting on the system, clearly separating the case when the rope is moving towards the left and when it is moving towards the right. [5]

c) Show that while in the frictionless case the equilibrium between the forces is only possible for a specific value of m_1/m_2, in the presence of friction, due to the opposite sign of the friction force in the two directions, the system is static over a range of mass ratios m_1/m_2, and give the expression for this range. [5]

d) Neglect friction again, and assume that the mass ratios are such that the system moves in the direction of mass m_1 (to the left in the figure). Now consider that the inclined plane and the masses feel a constant acceleration in the negative x-direction. Find the acceleration of the plane needed to keep both masses in the same position with respect to the plane (neglect any change in the angle of the rope holding mass m_2). Hint: think about the horizontal component of the acceleration of m_1 when the plane is not moving. [5]



in equilibrium → $m_2 = m_1 \sin\alpha$

$$\frac{m_1}{m_2} = \frac{1}{\sin\alpha}$$