Vectors

1. Basic Vector Properties

R Throughout these notes, we denote vectors by boldface symbols, e.g. $\mathbf{a}$. Examples of other notations found in the literature are an arrow above the symbol ($\vec{a}$) or an underline ($\underline{a}$).

R Vectors can be defined in any number of abstract dimensions. Unless noted otherwise, we will focus on the **three-dimensional (3D)** case, most relevant to our physical reality.

Exercise 1.1 These notes are missing crucial diagrams and figures. Draw suitable graphics as you go through the course to illustrate the equations and formulas. ■

Exercise 1.2 Try to find as many typos and errors in these notes and report to the lecturer. ■

1.1 Scalars vs Vectors (Riley 7.1)

Definition 1.1 — Scalars. These are the simplest kind of physical quantity that can be completely specified by its magnitude, a single number together with the units in which they are measured. Examples include temperature, time, density, etc.

Definition 1.2 — Vectors. A quantity that requires both a magnitude and a direction in space to specify it completely. Examples include force, velocity, electric field, etc.

1.2 Addition and Subtraction of Vectors (Riley 7.2)

The vector sum,

$$\mathbf{c} = \mathbf{a} + \mathbf{b}, \tag{1.1}$$

of two displacement vectors is the displacement vector that results from performing first one, then the other displacement. Vector addition is commutative

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}. \tag{1.2}$$

When adding three vectors, this leads to the associativity property of addition, i.e.

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \tag{1.3}$$

In fact, in general, it is immaterial in what order any number of vectors are added.

The subtraction of two vectors is very similar to their addition:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \tag{1.4}$$

where $-\mathbf{b}$ is a vector of equal magnitude but exactly the opposite direction to $\mathbf{b}$.

The subtraction of two equal vectors yields the zero vector, $\mathbf{0}$, which has zero magnitude and no associated direction.

1.3 Multiplication of a Vector by a Constant (Riley 7.3)

Multiplication of a vector by a scalar changes the magnitude but not the direction, although if the scalar is negative, we obtain a vector pointing in the opposite direction. Multiplication by a scalar is commutative and distributive over addition. Therefore for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ and arbitrary scalars λ and μ:

$$(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a}) = \mu(\lambda\mathbf{a}), \qquad \text{[commutative]}, \tag{1.5}$$

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}, \qquad \text{[distributive over sum of vectors]}, \tag{1.6}$$

$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}, \qquad \text{[distributive over sum of factors]}. \tag{1.7}$$

2. Basis, Position and Unit Vectors (Riley 7.4)

Given any three different vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, which do not all lie in a plane, we can, in 3D space, write any other vector in terms of scalar multiples of them,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \ . \tag{2.1}$$

The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are said to form a **basis** (for the 3D space). The scalars which may be positive, negative or zero are called the components of the vector $\mathbf{a}$ with respect to this basis.

Formula 2.1 — Decomposition of a Vector in the Cartesian Basis. In the Cartesian coordinate system (x,y,z) we introduce the **unit vectors i**, **j** and **k** which point along the positive $x-$, $y-$ and $z-$axis, respectively. A vector **a** may then be written as the sum of three vectors,

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_k\mathbf{k}, \tag{2.2}$$

or in short

$$\mathbf{a} = (a_x, a_y, a_z). \tag{2.3}$$

The basis vectors themselves may thus be represented by

$$\mathbf{i} = (1,0,0), \tag{2.4}$$
$$\mathbf{j} = (0,1,0), \tag{2.5}$$
$$\mathbf{k} = (0,0,1). \tag{2.6}$$

They are therefore called the **Cartesian unit vectors**.

Formula 2.2 — Position Vector. A special case of the general vector is a **position vector r** which starts at the origin and goes to the point $P(x,y,z)$,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x,y,z).$$

To add and subtract vectors, we just add/subtract the components:

$$\mathbf{a}+\mathbf{b} = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})+(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}) = (a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k} \tag{2.7}$$

$$\mathbf{a}-\mathbf{b} = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})-(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}) = (a_x-b_x)\mathbf{i}+(a_y-b_y)\mathbf{j}+(a_z-b_z)\mathbf{k}, \tag{2.8}$$

or in short,

$$\mathbf{a}+\mathbf{b} = (a_x+b_x, a_y+b_y, a_z+b_z), \tag{2.9}$$

$$\mathbf{a}-\mathbf{b} = (a_x-b_x, a_y-b_y, a_z-b_z). \tag{2.10}$$

Multiplication by a scalar leads to multiplication of each component,

$$\lambda\mathbf{a} = \lambda a_x\mathbf{i}+\lambda a_y\mathbf{j}+\lambda a_z\mathbf{k} = (\lambda a_x, \lambda a_y, \lambda a_z). \tag{2.11}$$

3. Magnitude of a Vector (Riley 7.5)

The magnitude of a vector **a** is denoted by $|\mathbf{a}|$ and it gives the "length" of the vector (in the units of the physical quantity that **a** represents).

Formula 3.1 — Vector Magnitude. For a general vector **a** with given Cartesian components, the magnitude is determined using Pythagoras

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{3.1}$$

■ **Example 3.1** Two particles have velocities $\mathbf{v}_1 = \mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v}_2 = \mathbf{i} - 2\mathbf{k}$. Find the velocity **u** of the second particle relative to the first.

The relative velocity is given by the difference

$$\mathbf{u} = \mathbf{v}_2 - \mathbf{v}_1 = (1-1)\mathbf{i} + (0-3)\mathbf{j} + (-2-6)\mathbf{k} = -3\mathbf{j} - 8\mathbf{k}$$

and thus magnitude of the relative velocity is $|\mathbf{u}| = \sqrt{(-3)^2 + (-8)^2} = \sqrt{73}$. ■

In general, a vector whose magnitude equals unity is called a **unit vector**. The Cartesian basis vectors in the previous section are examples of unit vectors. In general, the unit vector in the direction **a** is

$$\hat{\mathbf{a}} = \hat{\mathbf{e}}_a = \frac{\mathbf{a}}{|\mathbf{a}|} = (1/|\mathbf{a}|)\mathbf{a}. \tag{3.2}$$

By construction, $\hat{\mathbf{a}}$ has a length of one. Note if we have a vector of the form $\lambda \hat{\mathbf{e}}_a$, then we have the magnitude $(= \lambda)$ and direction $(\hat{\mathbf{e}}_a)$ explicitly separated.

■ **Example 3.2** A point P divides a line segment AB in the ratio $\lambda : \mu$. If the position vectors of the points A and B are **a** and **b**, respectively, find the position vector of point P.

The vector connecting **a** and **b** is

$$\mathbf{AB} = \mathbf{b} - \mathbf{a}.$$

Now, note the distances:

$$\frac{AP}{PB} = \frac{\lambda}{\mu} \Rightarrow \frac{BP}{AB} = \frac{\mu}{\mu+\lambda}, \; \frac{AP}{AB} = \frac{\lambda}{\mu+\lambda}.$$

Consider going from the origin O to A and then from A to P:

$$\begin{aligned}
\mathbf{OP} &= \mathbf{a} + \mathbf{AP} \\
&= \mathbf{a} + \frac{\lambda}{\mu+\lambda}\mathbf{AB} \\
&= \mathbf{a} + \frac{\lambda}{\mu+\lambda}(\mathbf{b}-\mathbf{a}) \\
&= \left(1 - \frac{\lambda}{\mu+\lambda}\right)\mathbf{a} + \frac{\lambda}{\mu+\lambda}\mathbf{b} \\
&= \frac{\mu}{\mu+\lambda}\mathbf{a} + \frac{\lambda}{\mu+\lambda}\mathbf{b} \\
&= \frac{1}{\mu+\lambda}(\mu\mathbf{a} + \lambda\mathbf{b}).
\end{aligned}$$

If λ and μ are normalized such that $\mu+\lambda = 1$ (only the ratio $\lambda : \mu$ is relevant here), one simply has $\mathbf{OP} = \mu\mathbf{a} + \lambda\mathbf{b}$. ■

4. The Scalar (or Dot) Product (Riley 7.6.1)

As the name suggests, the dot product yields a scalar quantity, i.e. a number. It is defined as follows:

Definition 4.1 — Scalar Product.

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, \qquad (0 \leq \theta \leq \pi), \tag{4.1}$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

The scalar product can be positive ($0 \leq \theta < \pi/2$), negative ($\pi/2 \leq \theta \leq \pi$) or zero ($\theta = \pi/2$ or $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$). It follows that two non-zero vectors are perpendicular if

$$\mathbf{a}\cdot\mathbf{b} = 0. \tag{4.2}$$

Note that as we are dealing with just a scalar number, the following properties hold:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \qquad [\text{commutative}], \tag{4.3}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}, \qquad [\text{distributive}], \tag{4.4}$$

$$(\lambda\mathbf{a})\cdot(\mu\mathbf{b}) = \lambda\mu(\mathbf{a}\cdot\mathbf{b}), \qquad [\lambda,\mu \text{ are scalars}]. \tag{4.5}$$

For the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we have:

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \qquad \text{and} \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0. \tag{4.6}$$

Formula 4.2 — Scalar Product in Cartesian Coordinates. From these relations, we can then write the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$, in terms of the components:

$$\mathbf{a}\cdot\mathbf{b} = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\cdot(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}) = a_xb_x+a_yb_y+a_zb_z. \tag{4.7}$$

From the above it follows that:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_xb_x+a_yb_y+a_zb_z}{|\mathbf{a}||\mathbf{b}|}. \tag{4.8}$$

Also, the magnitude of a vector can be found from the scalar product of the vector with itself,

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos\theta = |\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2. \tag{4.9}$$

The scalar product of a vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ with one of Cartesian unit vectors gives the simple result:

$$\begin{aligned} \mathbf{a}\cdot\mathbf{i} &= a_x, \\ \mathbf{a}\cdot\mathbf{j} &= a_y, \\ \mathbf{a}\cdot\mathbf{k} &= a_z, \end{aligned} \tag{4.10}$$

thus the scalar product with a unit vector 'projects' out the corresponding component of **a**. On the other hand, the original definition of the scalar product applied in this case gives ($|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$)

$$\begin{aligned} \mathbf{a}\cdot\mathbf{i} &= |\mathbf{a}|\cos\theta_x, \\ \mathbf{a}\cdot\mathbf{j} &= |\mathbf{a}|\cos\theta_y, \\ \mathbf{a}\cdot\mathbf{k} &= |\mathbf{a}|\cos\theta_z, \end{aligned} \tag{4.11}$$

where θ_x, θ_y and θ_z are the angles of **a** against the three Cartesian axes. The vector components and angles are thus related by

$$\begin{aligned} a_x &= |\mathbf{a}|\cos\theta_x, \\ a_y &= |\mathbf{a}|\cos\theta_y, \\ a_z &= |\mathbf{a}|\cos\theta_z. \end{aligned} \tag{4.12}$$

The cosines of these three angles are called the **direction cosines**. The unit vector in the direction of **a** can then be expressed as

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \mathbf{i}\cos\theta_x + \mathbf{j}\cos\theta_y + \mathbf{k}\cos\theta_z. \tag{4.13}$$

■ **Example 4.1** Find the angle between the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.
We need to calculate:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

with

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 1\times 2 + 2\times 3 + 3\times 4 = 20, \\ |\mathbf{a}| &= \sqrt{1^2+2^2+3^2} = \sqrt{14}, \\ |\mathbf{b}| &= \sqrt{2^2+3^2+4^2} = \sqrt{29}, \end{aligned}$$

from which follows

$$\cos\theta = \frac{20}{\sqrt{406}} \approx 0.9926.$$

The angle between the vectors is thus approximately $\theta \approx 0.12$ rad. ■

■ **Example 4.2** We can also derive the cosine rule in trigonometry from the scalar product. Let

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

Then

$$\begin{aligned}
\mathbf{c}\cdot\mathbf{c} &= |\mathbf{a}|^2+|\mathbf{b}|^2+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}\\
&= |\mathbf{a}|^2+|\mathbf{b}|^2+2|\mathbf{a}||\mathbf{b}|\cos\theta\\
&= |\mathbf{a}|^2+|\mathbf{b}|^2-2|\mathbf{a}||\mathbf{b}|\cos\gamma\\
\Rightarrow c^2 &= a^2+b^2-2ab\cos\gamma.
\end{aligned}$$

where γ is the angle inside the triangle opposite c, $\gamma=\pi-\theta\Rightarrow\cos\gamma=-\cos\theta$. ■

5. The Vector (or Cross) Product (Riley 7.6.2)

The vector product is defined as follows:

Definition 5.1 — Vector Product.

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\hat{\mathbf{n}} \tag{5.1}$$

where the magnitude is $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ ($0 \le \theta \le \pi$).

The unit vector $\hat{\mathbf{n}}$ is in a direction perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. The direction of $\hat{\mathbf{n}}$ is given by the right-hand rule: if your index finger points in the direction of $\mathbf{a}$ and your middle finger in the direction of $\mathbf{b}$, then your thumb gives the direction of $\hat{\mathbf{n}}$.

The vector product is distributive over addition, but **anti-commutative** and **non-associative**:

$$(\mathbf{a}+\mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}), \qquad \text{[distributive]}, \tag{5.2}$$

$$(\mathbf{b} \times \mathbf{a}) = -(\mathbf{a} \times \mathbf{b}) \qquad \text{[anti-commutative]}, \tag{5.3}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \qquad \text{[non-associative]}. \tag{5.4}$$

Also, if two vectors are non-zero, then if $\mathbf{a} \times \mathbf{b} = 0$, then $\mathbf{a}$ is parallel (or anti-parallel) to $\mathbf{b}$. It thus simply follows that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. For the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we have,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}, \tag{5.5}$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \tag{5.6}$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \tag{5.7}$$

$$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}. \tag{5.8}$$

Therefore, for general vectors $\mathbf{a}$, $\mathbf{b}$ given in terms of their components with respect to the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector product is calculated as

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\
&= a_xb_y(\mathbf{i} \times \mathbf{j}) + a_xb_z(\mathbf{i} \times \mathbf{k}) + a_yb_x(\mathbf{j} \times \mathbf{i}) + a_yb_z(\mathbf{j} \times \mathbf{k}) + a_zb_x(\mathbf{k} \times \mathbf{i}) + a_zb_y(\mathbf{k} \times \mathbf{j}) \\
&= (a_yb_z - a_zb_y)\mathbf{i} + (a_zb_x - a_xb_z)\mathbf{j} + (a_xb_y - a_yb_x)\mathbf{k}.
\end{aligned} \tag{5.9}$$

Formula 5.2 — Vector Product in Cartesian Coordinates. Using short-hand notation, this can be written compactly,

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}. \tag{5.10}$$

R You might be familiar with the following way of calculating a vector product using a matrix determinant:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

While matrices and determinants are not covered in this lecture course (they will be introduced in PHAS1246: Mathematical Methods II), you can of course use the above equation as a mnemonic.

■ **Example 5.1** Construct a vector $\mathbf{n}$ that is perpendicular to both $\mathbf{a} = (1,-1,1)$ and $\mathbf{b} = (0,-1,1)$.
Given two vectors, we can construct a perpendicular vector by taking the vector product,

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{pmatrix} (-1)\times 1 - 1\times(-1) \\ 1\times 0 - 1\times 1 \\ 1\times(-1) - (-1)\times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

Clearly, the choice is not unique and multiplying $\mathbf{n}$ with any number will give a vector that is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ as well. It is often convenient to construct a unit vector $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$, which in this case yields $\hat{\mathbf{n}} = (-\mathbf{j}-\mathbf{k})/\sqrt{(-1)^2+(-1)^2} = -\frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$. ■

The cross product has an important geometric property: Its magnitude gives the **area of a parallelogram** spanned by the given two vectors. In fact, the area of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is

$$A_{\text{parallelogram}} = |\mathbf{a}|h = |\mathbf{a}||\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|, \tag{5.11}$$

where $h = |\mathbf{b}|\sin\theta$ is the height of parallelogram perpendicular to $\mathbf{a}$.

Similarly, it is then easy to see that the **area of a triangle** spanned by $\mathbf{a}$ and $\mathbf{b}$ is

$$A_\Delta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|. \tag{5.12}$$

■ **Example 5.2** Calculate the are of a triangle formed by the points $A(6,4,-2)$, $B(2,4,1)$ and $C(-2,2,4)$.
In order to apply the above equation, we first need to determine two vectors that span the triangle from one of the points. Arbitrarily choosing point A as 'anchor', the spanning vectors are

$$\mathbf{AB} = \mathbf{b}-\mathbf{a} = (-4,0,3), \quad \text{and}$$
$$\mathbf{AC} = \mathbf{c}-\mathbf{a} = (-8,-2,6).$$

The area of the triangle is then

$$A_\Delta = \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left|\begin{pmatrix} 0\times 6 - 3\times(-2) \\ 3\times(-8) - (-4)\times 6 \\ (-4)\times(-2) - (-8)\times 0 \end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}\right| = \frac{1}{2}\sqrt{6^2+8^2} = 5.$$

Choosing either B or C as 'anchor' points and constructing the corresponding vectors spanning the triangle will yield the same result. ■

6. Triple Products (Riley 7.6.3, 7.6.4)

Dot and cross products can be combined in the following manner.

6.1 Scalar Triple Product

Definition 6.1 — Scalar Triple Product. This is the dot product of a vector **a** with the cross product formed from two other vectors **b** and **c**, i.e.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}), \tag{6.1}$$

with the result being a number.

Expressed in terms of the components of each vector with respect the Cartesian basis set, the scalar triple product is

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_x(b_yc_z - b_zc_y) - a_y(b_xc_z - b_zc_x) + a_z(b_xc_y - b_yc_x). \tag{6.2}$$

We also use the equivalent notation

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}), \tag{6.3}$$

which emphasizes the **cyclic property**

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}], \tag{6.4}$$

i.e. the scalar triple product is unchanged under cyclic permutation of the vectors **a**, **b**, **c**. On the hand, the scalar triple product **switches sign** for any other permutations (switching position of two vectors)

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{c},\mathbf{b},\mathbf{a}]. \tag{6.5}$$

The scalar triple product has an important geometric interpretation: Its magnitude gives the **volume of a parallelepiped** spanned by the three vectors **a**, **b** and **c**. This can be seen as follows:

The vector $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to the base formed by $\mathbf{a}$ and $\mathbf{b}$, and has the magnitude $A_{\text{parallelogram}} = ab \sin\theta$, i.e. the area of the base parallelogram. Also, $\mathbf{v} \cdot \mathbf{c} = vc\cos\phi$, where ϕ is the angle between $\mathbf{c}$ and $\mathbf{v}$. As $h = c\cos\phi$ is the height above the base of the parallelepiped, then

$$V_{\text{parallelepiped}} = A_{\text{parallelogram}} h = (ab\sin\theta)(c\cos\phi) = |\mathbf{a} \times \mathbf{b}||\hat{\mathbf{n}} \cdot \mathbf{c}| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \tag{6.6}$$

is the area of the base multiplied by the perpendicular height, i.e. the volume of parallelepiped.

R The scalar triple product can result in a negative number. Geometrically, this corresponds to the case where the defining vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ do not form a right-handed set of vectors, e.g. $\mathbf{c}$ would be on the 'opposite side' to the perpendicular vector of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$, and thus $\hat{\mathbf{n}} \cdot \mathbf{c} < 0$. This is why the volume in Equation (6.6) is defined as the absolute value of the triple scalar product.

The interpretation as the volume of a parallelepiped illustrates an important property of the triple scalar product: If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lie on a common plane, the scalar triple product is zero, as the parallelepiped collapses,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0 \Leftrightarrow \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ are situated on a common plane.} \tag{6.7}$$

This includes the special cases that any two of the vectors are parallel, or any of the vectors is the null vector $\mathbf{0}$.

■ **Example 6.1** Find the volume V of a parallelepiped with sides $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ and $\mathbf{c} = 7\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$.

We first calculate

$$\mathbf{a} \times \mathbf{b} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k},$$

from which

$$\begin{aligned} V &= |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \\ &= |(-3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) \cdot (7\mathbf{i} + 8\mathbf{j} + 10\mathbf{k})| \\ &= |(-3 7 + 6 8 + (-3) 10| = 3 \end{aligned}$$

follows. As $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -3 < 0$, $\mathbf{c}$ is on the 'left-handed' side of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$. ■

6.2 Vector Triple Product

Definition 6.2 — Vector Triple Product. This is the cross product of a vector $\mathbf{a}$ with the cross product formed from two other vectors $\mathbf{b}$ and $\mathbf{c}$, i.e.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \tag{6.8}$$

with the result being a vector.

Formula 6.3 — Vector Triple Product Identities. The triple vector product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is necessarily perpendicular to $\mathbf{a}$ and lies in the plane containing $\mathbf{b}$ and $\mathbf{c}$. It thus can be expressed in terms of them, i.e.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{6.9}$$

Likewise, a different vector triple product is

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}. \tag{6.10}$$

These identities can be most easily (but tediously) shown by expressing the vectors **a**, **b** and **c** using their Cartesian components.

■ **Example 6.2** Show the

Formula 6.4 — Lagrange Identity.

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

First, we treat the left-hand side as a scalar triple product of $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then we use the cyclic property of the scalar triple product,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{d} \cdot ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}).$$

Using the result

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a},$$

we can then derive

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{d} \cdot ((\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

which is the Lagrange identity. ■

■ **Example 6.3** Express the magnitude of a vector product, $|\mathbf{a} \times \mathbf{b}|$, in terms of scalar products.

The magnitude is a special case of the above Lagrange identity,

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= \sqrt{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})} \\ &= \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a})} \\ &= \sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \end{aligned}$$

Trigonometrically, this is easy to understand because the above reads

$$\begin{aligned} ab \sin\theta &= \sqrt{a^2 b^2 - (ab\cos\theta)^2} \\ &= ab\sqrt{1 - \cos^2\theta}, \end{aligned}$$

where $a = |\mathbf{a}|$, $b = |\mathbf{b}|$ and θ is the angle between **a** and **b**. ■

7. Lines, Planes and Spheres (Riley 7.7)

7.1 Equation of a Line

Formula 7.1 — Parametric Equation of a Line. One possibility to specify a line is by defining a 'start' point A (with position vector $\mathbf{a}$) and then moving continuously along the direction of a vector $\mathbf{b}$. Any point R (with position vector $\mathbf{r}$) on the line can then be described by

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}. \tag{7.1}$$

Here, λ is an arbitrary parameter (hence the above is called a **parametrization** of a line) specifying a point on the line, e.g. $\mathbf{r}(\lambda = 0) = \mathbf{a}$ or $\mathbf{r}(\lambda = -1) = \mathbf{a} - \mathbf{b}$. Writing the above equation in component form gives

$$\begin{aligned} x &= a_x + \lambda b_x, \\ y &= a_y + \lambda b_y, \\ z &= a_z + \lambda b_z. \end{aligned} \tag{7.2}$$

The equation of a line going through two fixed points A and C with position vectors $\mathbf{a}$ and $\mathbf{c}$, respectively, can be easily derived from equation (7.1). Since the direction $\mathbf{AC}$ is given by $\mathbf{c} - \mathbf{a}$, the position vector of a general point on the line is:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{c} - \mathbf{a}). \tag{7.3}$$

Formula 7.2 — Implicit Equation(s) of a Line. By solving the three equations (7.2) for λ, we can describe a line with the following set of equations:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z} = \lambda. \tag{7.4}$$

Finally, we can take the vector product of Equation (7.1) with $\mathbf{b}$. As $\mathbf{b} \times \mathbf{b} = \mathbf{0}$ we obtain the equivalent line equation

$$(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}. \tag{7.5}$$

■ **Example 7.1** The line through the point $A(2,1,5)$ and direction $\mathbf{b}=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ is represented by the standard line equation

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}.$$

If $x=4$ for a point C on the line, what are the values of y and z for this point? Is the origin $(0,0,0)$ on the line?

In components the line equation corresponds to

$$x=2+\frac{\lambda}{\sqrt{3}},$$
$$y=1+\frac{\lambda}{\sqrt{3}},$$
$$z=5+\frac{\lambda}{\sqrt{3}}.$$

The equation(s) of the line can also be written as

$$\frac{x-2}{\frac{1}{\sqrt{3}}}=\frac{y-1}{\frac{1}{\sqrt{3}}}=\frac{z-5}{\frac{1}{\sqrt{3}}}=\lambda.$$

Dividing by $\sqrt{3}$ throughout gives the equivalent form:

$$x-2=y-1=z-5=\lambda'=\frac{\lambda}{\sqrt{3}}.$$

These equations relate the coordinates, i.e. if $x=4$ then $y=3$ and $z=7$, and $C(4,3,7)$ is a point on the line. Similarly, it is easy to check that the origin is not on the line, i.e. $x=y=z=0$ does not satisfy the equations. ■

7.2 Equation of a Plane

Formula 7.3 — Parametric Equation of a Plane. Similar to a line, we can find a parametric equation of a plane by specifying a point on the line $\mathbf{a}$ and *two non-parallel* direction vectors $\mathbf{u}$ and $\mathbf{w}$ (as a plane is a two-dimensional object compared to a 1-D line),

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v}. \tag{7.6}$$

Here, λ and μ are two independent and free parameters.

We can also define a plane by three points which are contained within it. For three points given by position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we have

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})+\mu(\mathbf{c}-\mathbf{a}). \tag{7.7}$$

Again, $\mathbf{a}$ is the starting point and all other points on the plane may be reached by defining two non-parallel directions, i.e. $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$.

We examine now the equation of a plane, passing through point A, of position vector $\mathbf{a}$, and with unit vector $\hat{\mathbf{n}}$ normal to the plane. With $\mathbf{w}$ representing a vector that is situated in the plane, one can write the general position vector of a point on the plane as

$$\mathbf{r} = \mathbf{a} + \mathbf{w} \tag{7.8}$$

or

$$\mathbf{r} - \mathbf{a} = \mathbf{w}. \tag{7.9}$$

Taking the scalar product with $\hat{\mathbf{n}}$ gives

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = \mathbf{w} \cdot \hat{\mathbf{n}}. \tag{7.10}$$

By construction, $\mathbf{w}$ and $\hat{\mathbf{n}}$ are perpendicular and the equation of a plane can be expressed as

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0. \tag{7.11}$$

This can also be re-written as

Formula 7.4 — Equation of Plane using Perpendicular Unit Vector.

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}}, \tag{7.12}$$

where $d = \mathbf{a} \cdot \hat{\mathbf{n}}$ is the perpendicular distance of the plane from the origin. This is the definition of a plane when a point in the plane and a vector perpendicular to the plane is given.

In component form, the above equation of a plane can be expressed as follows. We first write the unit vector as

$$\hat{\mathbf{n}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}. \tag{7.13}$$

Formula 7.5 — Equation of Plane in Cartesian Coordinates. This yields the plane equation in component form,

$$lx + my + nz = d, \tag{7.14}$$

where l, m, n are the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the plane. Thus, $|\hat{\mathbf{n}}|^2 = l^2 + m^2 + n^2 = 1$.

■ **Example 7.2** Find the direction of the line of intersection of the two planes $x + 3y - z = 5$ and $2x - 2y + 4z = 3$.

As above, for a plane given by $lx + my + nz = d$, the vector $\mathbf{v} = (l, m, n)$ is perpendicular to the plane (although not **normalized**, i.e. not a unit vector, as $l^2 + m^2 + n^2 \neq 1$). Therefore the normal vector to the first and second plane is

$$\mathbf{v}_1 = \mathbf{i} + 3\mathbf{j} - \mathbf{k} \text{ and } \mathbf{v}_2 = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k},$$

respectively. The direction $\mathbf{w}$ of the intersecting line must lie in both planes and is thus perpendicular to both normals. Such a direction can be constructed by taking the vector product of the normals,

$$\mathbf{w} = \mathbf{v}_1 \times \mathbf{v}_2 = 10\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}.$$

Alternatively, the plane equations can be used to eliminate one of the coordinates, e.g. x,

$$2(x + 3y - z) - (2x - 2y + 4z) = 8y - 6z = 2 \times 5 - 3 = 7 \Rightarrow y = \frac{6}{8}z + \frac{7}{8},$$

or y,

$$2(x+3y-z)+3(2x-2y+4z)=8x+10z=2\times 5+3\times 3=19 \Rightarrow x=-\frac{10}{8}z+\frac{19}{8}.$$

In this way, z can be viewed as the parameter of a line equation with components

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19/8 \\ 7/8 \\ 0 \end{pmatrix} + \begin{pmatrix} -10/8 \\ 6/8 \\ 1 \end{pmatrix} z.$$

The direction vector $(-10/8, 6/8, 1)$ derived in this way is anti-parallel to $\mathbf{w}$ (multiply by (-8)) and can thus equally serve as the direction vector of the intersecting line. ■

7.3 Equation of a Sphere

All points $\mathbf{r}$ on a sphere have the property of being equidistant from a fixed point $\mathbf{c}$ (the center of the sphere), with the distance equal to the radius a.

Formula 7.6 — Equation of a Sphere. Thus all points on the sphere satisfy

$$|\mathbf{r}-\mathbf{c}|^2 = (\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c}) = a^2. \tag{7.15}$$

i.e. the equation of a sphere (by using the squared distance, we can avoid having a square root in the equation). In component form, this is written as ($\mathbf{c}=c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}$)

$$(x-c_x)^2+(y-c_y)^2+(z-c_z)^2 = a^2. \tag{7.16}$$

■ **Example 7.3** The unit sphere (i.e. with radius $a=1$) around the origin is described by the equation

$$|\mathbf{r}|^2 = x^2+y^2+z^2 = 1.$$

■

■ **Example 7.4** What is the shape of the intersection of the unit sphere around the origin with the plane described by $z=3/5$?

Inserting $z=3/5$ in the sphere equation gives

$$x^2+y^2 = 1-(3/5)^2 = (4/5)^2,$$

i.e. the equation of a *circle* around $x=0$, $y=0$ with radius $4/5$. ■

8. Calculating Distances (Riley 7.8)

8.1 Distance between Two Points

Formula 8.1 — Distance between two Points. The geometric distance between two points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ is uniquely defined and simply given by

$$d = |\mathbf{b}-\mathbf{a}| = \sqrt{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})} = \sqrt{(b_x-a_x)^2+(b_y-a_y)^2+(b_z-a_z)^2}. \tag{8.1}$$

8.2 Distance between a Point and a Line

A line consists of an infinite number of points and thus the distance to a given point P is not uniquely defined, but it is meaningful to ask for the *minimal* distance d between P and any of the points on the line. Defining the line with direction vector $\mathbf{b}$ and passing through a point A whose position vector is $\mathbf{a}$, we can find the minimal distance d as

$$d = |\mathbf{p}-\mathbf{a}|\sin\theta, \tag{8.2}$$

where θ is the angle between the vector difference $\mathbf{p}-\mathbf{a}$ and the line direction $\mathbf{b}$.

Formula 8.2 — Distance between Point and Line. This can be written in terms of the cross product

$$d = |(\mathbf{p}-\mathbf{a})\times\hat{\mathbf{b}}| \tag{8.3}$$

where $\mathbf{p}$ and $\mathbf{a}$ are the position vector of the point and a point on the line, respectively, and $\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}$ is the unit vector along the direction of the line.

■ **Example 8.1** Find the minimal distance from the point P with coordinates $(1,2,1)$ to the line $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}$ where $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

The line passes through $(1,1,1)$ and has direction $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$. The unit vector in this direction is

$$\hat{\mathbf{b}} = \frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}).$$

The position vector $\mathbf{p}$ of P is

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Thus

$$(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{b}} = \frac{1}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{k}).$$

and

$$d = \frac{1}{\sqrt{14}}|3\mathbf{i} - 2\mathbf{k}| = \sqrt{\frac{13}{14}}$$

for the distance between the given point and line. ■

8.3 Distance between a Point and a Plane

Formula 8.3 — Distance between Point and Plane. To find the minimal distance d from a point P (with position vector $\mathbf{p}$) to the plane defined by a point A (with position vector $\mathbf{a}$) and unit vector $\hat{\mathbf{n}}$ perpendicular to the plane, imagine viewing the situation from the 'edge' of the plane. One can then see that the distance is perpendicular on the plane to P and given by

$$d = |(\mathbf{a} - \mathbf{p}) \cdot \hat{\mathbf{n}}|. \tag{8.4}$$

■ **Example 8.2** Find the distance from the point $P(1,2,3)$ to the plane that contains the points $A(0,1,0)$, $B(2,3,1)$ and $C(5,7,2)$.

Two vectors in the plane are

$$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \text{ and } \mathbf{c} - \mathbf{a} = 5\mathbf{i} + 6\mathbf{j} + 2\mathbf{k},$$

and so a normal to the plane is

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

The unit vector $\hat{\mathbf{n}}$ perpendicular to the plane thus is

$$\hat{\mathbf{n}} = \frac{1}{3}(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}).$$

Hence,

$$d = |(\mathbf{a} - \mathbf{p}) \cdot \hat{\mathbf{n}}| = |(-\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot \frac{1}{3}(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k})| = |-\frac{5}{3}| = \frac{5}{3},$$

for the minimal distance between the given point and plane. ■

9. Polar Coordinates in Two Dimensions

So far we only used a Cartesian system of coordinates. We notice here that other systems of coordinates are commonly used. An example are **polar coordinates** in two dimensions. In two dimensions the position vector is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}. \tag{9.1}$$

Definition 9.1 — 2D Polar Coordinates. The transformation from the Cartesian coordinates x, y to the polar coordinates is defined by

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned} \tag{9.2}$$

where $r \geq 0$ is the distance from the origin and $0 \leq \theta < 2\pi$ is the angle between the vector and the positive x axis.

R Polar coordinates in two dimensions, as well as other coordinate systems in 3D, will be examined in more detail in the Chapter on "Partial Differentiation".