PHYS1B24 (PHAS1224) Waves, Optics and Acoustics
Solutions to Final Exam 2006

Answer ALL SIX questions from section A and THREE questions from section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of part marks per sub-section of a question.

A "BW" beside the mark indicates that the answer is mainly "bookwork". A "US" indicates that the answer is to an "unseen" question.

SECTION A **[Part marks]**

1. (a) Rayleigh's criterion depends on the maximum of the diffraction pattern of one slit lying over or beyond the first zero of the other: this leads to the minimum resolvable detail. The angular separation θ at this point is given by: **[1 BW]**

$$\sin\theta \approx \theta \geq \frac{\lambda}{a}$$ **[2 BW]**

where a is the slit separation.

(b) For circular apertures:

$$\sin\theta \approx \theta \geq 1.22\frac{\lambda}{D}$$ **[2 BW]**

where D is the aperture diameter.

(c)

$$\begin{aligned}\theta &\geq 1.22\frac{\lambda}{D}\\ &= 1.22\frac{600\times 10^{-9}}{0.366}\\ &= 2.00\times 10^{-6}\text{ rads}\\ &= 2.00\times 10^{-6}\frac{180}{\pi}\\ &= (1.15\times 10^{-4})^{\circ}\end{aligned}$$ **[2 US]**

2. (a) **[2 US]**

$$\lambda_{\text{diamond}} = \frac{\lambda}{n_{\text{diamond}}} = \frac{550}{2.42} = 227.3 \text{ nm.}$$

(b) The appropriate geometry is shown below in Figure 1:

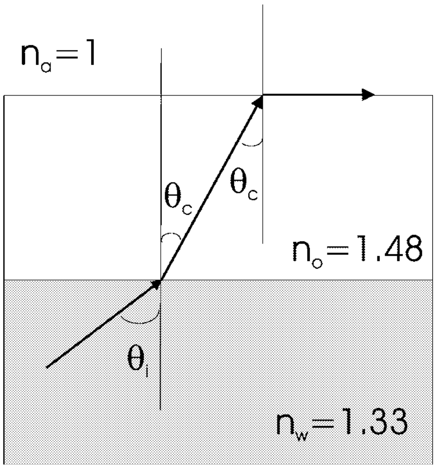

Figure 1 Geometry for Question 2b.

Total internal reflection can occur only at the oil-air interface since $n_w < n_o$. There **[1 US]**

$$\theta_c = \sin^{-1}\frac{1}{1.48} = 42.5^\circ$$ **[1 US]**

Snell's law applied to the water-oil interface is

$$1.33 \sin\theta_i = 1.48 \sin\theta_c = 1$$ **[2 US]**

and $\theta_i = 48.8^\circ$. I.e., for incident angles equal to or greater than 48.8° the beam will be reflected back into the water.

3. (a) A source is moving with velocity v_s towards an observer in a medium in which the wave speed is c. The frequency of the source is f, so in a time t it will have emitted ft waves, but in the region in front of the source these will have been emitted into a distance $(c - v_s)t$. Thus the wavelength is

$$\lambda' = \frac{(c - v_s)t}{ft}$$

or **[2 BW]**

$$\begin{aligned}\lambda' &= \frac{(c - v_s)}{f} \\ &= (c - v_s)\frac{\lambda}{c} \\ &= \lambda\left(1 - \frac{v_s}{c}\right),\end{aligned}$$

as required. **[2 BW]**

[In a time t a stationary observer will receive the waves in a distance ct, which will be ct/λ' wavelengths, so the observed frequency will be $f' = fc/(c - v_s)$.]

(b) If the average speed of an atom in a low-pressure cadmium vapour discharge tube is $8.6\sqrt{T}$ m s^{-1} where T is the absolute temperature, then at 300 K the average speed, which we take as the source speed, v_s, is **[1 US]**

$$v_s = 8.6\sqrt{300} = 149 \text{ m s}^{-1}.$$

The wavelength is related to the frequency by $\lambda = c/f$, so

$$\lambda'_{\text{approaching}} = \frac{c - v_s}{c}\lambda; \qquad \lambda'_{\text{retreating}} = \frac{c + v_s}{c}\lambda,$$ **[1 BW]**

and the difference in wavelength between atoms approaching with the average speed and those retreating is

$$\Delta\lambda' = 2\frac{v_s}{c}\lambda = 2\frac{149}{3 \times 10^8} \times 480 \text{ nm} = 4.8 \times 10^{-13} \text{ m}.$$ **[1 US]**

(Award full marks for other methods, e.g., going via frequency.)

4. (a) Two of

- selective absorption (Polaroid),
- double refraction (birefringence),
- scattering, or, **[2 BW]**
- reflection.

(b) Malus' Law: $I = I_0 \cos^2 \theta$. Here we have **[1 US]**

$$\begin{aligned} 1 &= 3\cos^2\theta \\ \cos^2\theta &= \frac{1}{3} \\ \cos\theta &= \frac{1}{\sqrt{3}} \end{aligned}$$ **[1 US]**

so

$$\theta = 54.7^\circ$$ **[1 US]**

(c) No – sound waves are *longitudinal* waves. **[2 US]**

5. (a) From graph: $f = -60$ cm. **[1 US]**

(b) Radius of curvature:

$$R = 2f = 2 \times (-60) = -120 \text{ cm.}$$ **[1 US]**

(c) Convex mirror **[1 BW]**

(d) Virtual image **[1 BW]**

(e) Magnification is given by

$$m = -\frac{\text{image distance}}{\text{object distance}} = -\frac{q}{p}$$ **[1 BW]**

From graph: $p = 60$ cm and $q = -30$ cm, so **[1 US]**

$$m = -\frac{-30}{60} = \frac{1}{2} = 0.5.$$

6. (a)

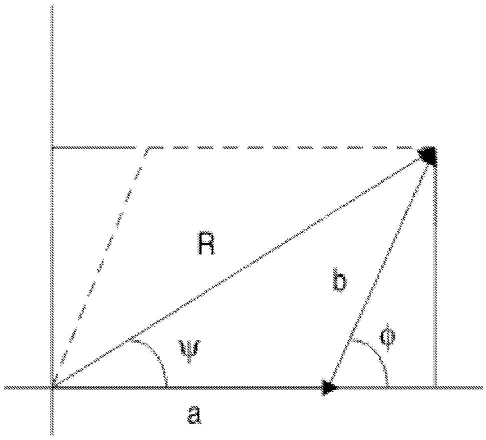

Figure 2 Two phasors differing in phase by ϕ, for Question 6a.

Let the continuous simple harmonic signals with the same fixed frequency ω be $a\cos(\omega t)$ and $b\cos(\omega t+\phi)$. Then the phasor diagram at $t=0$ is as in Figure 2. Using the cosine rule for the triangle with sides a, b and R we have **[1 US]**

$$\begin{aligned} R^2 &= a^2+b^2-2ab\cos(\pi-\phi) \\ &= a^2+b^2+2ab\cos(\phi), \end{aligned}$$ **[2 US]**

so the amplitude is

$$R=\sqrt{a^2+b^2+2ab\cos(\phi)}.$$

The phase of the resultant relative to the first phasor, ψ, is found from the projections of the phasor onto the y and x axes:

$$\tan(\psi)=\frac{b\sin(\phi)}{a+b\cos(\phi)}.$$ **[2 BW]**

(b) If $a=1$, $b=2$ and $\phi=30°$ we have

$$R=\sqrt{1^2+2^2+2\times 1\times 2\times\cos(30)}=2.91$$ **[1 US]**

and

$$\psi=\tan^{-1}\left[\frac{2\sin(30)}{1+2\cos(30)}\right]=20.1°.$$ **[1 US]**

SECTION B

7. (a) i. At $x = 0$ the two waves are given by **[1 US]**

$$
\begin{aligned}
y_1 &= A\cos\omega_1 t \\
y_2 &= A\cos\omega_2 t
\end{aligned}
$$

Adding gives us

$$y_1 + y_2 = A\cos\omega_1 t + A\cos\omega_2 t$$

Using the sum of cosines **[1 BW]**

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

we get

$$y = y_1 + y_2 = 2A\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$ **[1 BW]**

as required.

ii. For the carrier wave: **[2 US]**

$$
\begin{aligned}
\tau_{\text{carrier}} = \frac{2\pi}{\omega} &= \frac{2\pi}{\frac{\omega_1+\omega_2}{2}} \\
&= \frac{4\pi}{\omega_1+\omega_2} \\
&= \frac{4\pi}{\frac{2\pi}{\tau_1}+\frac{2\pi}{\tau_2}} \\
&= \frac{2\tau_1\tau_2}{\tau_2+\tau_1} \\
&= \frac{2(19)(20)}{19+20} \\
\tau_{\text{carrier}} &= 19.49 \text{ s}
\end{aligned}
$$

Similarly: **[1 US]**

$$
\begin{aligned}
\tau_{\text{envelope}} &= \frac{2\tau_1\tau_2}{\tau_2-\tau_1} \\
&= \frac{2(19)(20)}{20-19} \\
\tau_{\text{envelope}} &= 760 \text{ s}
\end{aligned}
$$

iii. $y = 0$ when either **[1 US]**

$$\frac{\omega_1 + \omega_2}{2} t = \frac{\pi}{2}$$
$$t = \frac{\pi}{\omega_1 + \omega_2} \quad \text{or} \quad t = \frac{\tau_{\text{carrier}}}{4}$$

or **[1 US]**

$$\frac{\omega_1 - \omega_2}{2} t = \frac{\pi}{2}$$
$$t = \frac{\pi}{\omega_1 - \omega_2} \quad \text{or} \quad t = \frac{\tau_{\text{envelope}}}{4}$$

Obviously, the former value of t is smaller, so, with $\omega = 2\pi/\tau$,

$$\begin{aligned} t &= \frac{\pi}{\omega_1 + \omega_2} = \frac{\pi}{\frac{2\pi}{\tau_1} + \frac{2\pi}{\tau_2}} \\ &= \frac{1}{2}\left(\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}\right) = \frac{1}{2}\frac{19 \times 20}{20 + 19} = 4.9 \text{ sec} \\ &\text{or} \\ t &= \frac{\tau_{\text{carrier}}}{4} = \frac{19.95}{4} = 4.9 \text{ s.} \end{aligned}$$ **[1 US]**

(b) i. dispersion relation **[2 BW]**

ii. *Phase velocity* is the speed at which points of constant phase (e.g. maxima) of the wave travel: $v_{\text{p}} = \omega/k$ where ω is the angular frequency and k is the wavevector. *Group velocity* is the speed at which an envelope function, or a variation in amplitude, travels (and hence the speed at which a signal, or energy, travels): $v_{\text{g}} = \partial\omega/\partial k$. **[1 BW]** **[1 BW]**

iii.

$$v_p = \frac{\omega}{k} = \sqrt{\frac{gk}{2}}/k = \sqrt{\frac{g}{2k}}$$ **[2 US]**

while

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\sqrt{\frac{gk}{2}} = \sqrt{\frac{g}{2}}\frac{\partial}{\partial k}k^{\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{g}{2}}k^{-\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{g}{2k}} = \frac{1}{2}v_p$$ **[2 US]**

as required.

iv. With $\omega = kv_{\text{p}}$ we can write

$$\begin{aligned} v_{\text{g}} &= \frac{\mathrm{d}\omega}{\mathrm{d}k} \\ &= \frac{\mathrm{d}(kv_{\text{p}})}{\mathrm{d}k} \\ v_{\text{g}} &= v_{\text{p}} + k\frac{\mathrm{d}v_{\text{p}}}{\mathrm{d}k} \end{aligned}$$ **[1 BW]**

as required. **[2 BW]**

8. (a) i. π phase change

ii. 0 phase change **[2 BW]**

(b) i. The condition for bright fringes is

$$\left(p+\frac{1}{2}\right)\lambda_n = 2\alpha x \qquad (1)$$ **[1 BW]**

where $\lambda_n = \lambda_0/n$ and α is the angle between the plates. **[1 BW]**
We can write for a fringe:

$$\left(p+\frac{1}{2}\right)\frac{\lambda_0}{n} = 2\alpha x \quad \text{and the next one :} \quad \left(p+\frac{3}{2}\right)\frac{\lambda_0}{n} = 2\alpha(x+\Delta x).$$ **[1 US]**

Subtracting,

$$\frac{\lambda_0}{n} = 2\alpha\Delta x \qquad (2)$$ **[1 US]**

or

$$\alpha = \frac{\lambda_0}{2n\Delta x} = \frac{589\times10^{-9}}{2\times1.329\times0.2\times10^{-3}} = 0.0011 \text{ rad} = 0.0635°.$$ **[2 US]**

ii. From Equation 1, as α decreased, so would p, so the fringes would move down. **[2 US]**
Also, from Equation 2, as α decreased, Δx would increase, i.e., the fringe separation would increase. **[2 US]**

(c) i. Figure 3 shows the formation of interference rings in Newton's experiment *(diagram not necessary)*.

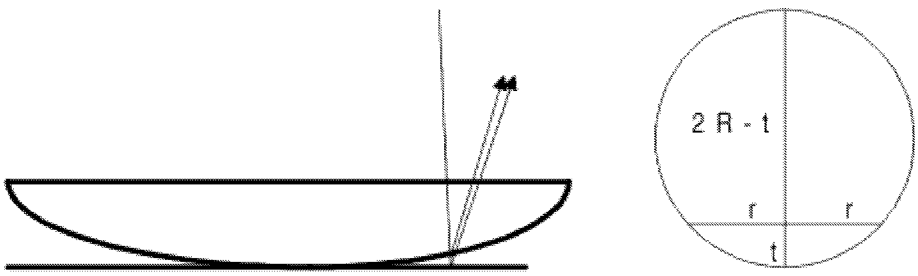

Figure 3 Newton's rings for Question 8(c)i.

At a radius r, the thickness of the air layer t is given by

$$r^2 = t(2R - t) \qquad \text{(the chord formula)}$$

or

$$r^2 = R^2 - (R - t)^2 \qquad \text{(Pythagoras)}$$

so, for small t,

$$r^2 = 2Rt \qquad \text{(or straight from the Note).}$$

[2 BW]

The different phase changes on reflection from the bottom of the lens and the top of the plate are equivalent to a half wavelength optical path, so there will be destructive interference if the path length difference $2t$ is equal to an even number of half wavelengths *(it is essential that this is explained).* Thus bright rings occur when

$$r^2 = R(2m)(\lambda/2)$$

or **[2 BW]**

$$r_m = \sqrt{mR\lambda},$$

as required.

ii. The radius of the p^{th} bright fringe is given by

$$r^2 = \left(p + \frac{1}{2}\right)\lambda R, \quad \text{where } R \text{ is the radius of curvature of the lens.}$$

[2 BW]

In air, we have, with $p = 4$:

$$2.52^2 = 4.5\lambda R$$

and with the liquid inserted: **[1 US]**

$$2.21^2 = 4.5\frac{\lambda}{n_{\text{liquid}}}R.$$

Dividing: **[1 US]**

$$\left(\frac{2.52}{2.21}\right)^2 = n_{\text{liquid}} \quad \text{so that} \quad n_{\text{liquid}} = 1.30.$$

9. (a) i. The diagram (Figure 4) shows a compound microscope with an objective with focal length 4 mm and an eyepiece with focal length 20 mm, forming an image, 250 mm from the eye, of an object placed 4.1 mm from the objective. *(The diagram must show the intermediate and final images, which must be located using two or more principal rays for each image.)* **[6 US]**

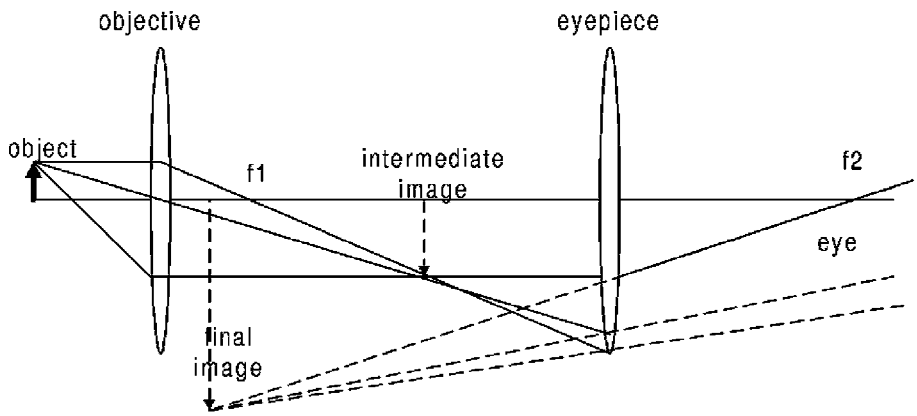

Figure 4 Ray diagram for the compound microscope of Question 9(a)i.

ii. The image at 250 mm gives us, from the lens formula for the eyepiece

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

the position of the intermediate image as **[2 US]**

$$\frac{1}{p}=\frac{1}{230}+\frac{1}{20} \quad \text{or} \quad p=4600/250 \text{ mm}=18.4 \text{ mm}.$$

This point, 18.4 mm to the left of the eyepiece, is $(L-18.4)$ mm to the right of the objective. But for the objective **[1 US]**

$$\frac{1}{q}=\frac{1}{f}-\frac{1}{p}=\frac{1}{4}-\frac{1}{4.1}=\frac{0.1}{16.4},$$

or $q=164$ mm. Thus the separation of the lenses in this setting is $18.4+164=182.4$ mm. **[1 US]**

iii. The magnification of the microscope when adjusted in this way is the product of the objective and eyepiece magnifications

$$M = M_o m_e = -\frac{L}{f_o}\frac{250}{f_e} = -\frac{182.4}{4}\frac{250}{20} = -570.$$

(The sign denotes that the image is inverted.) **[2 US]**

iv. It is more usual to adjust the microscope so that the image is at infinity because the eye is then relaxed, and so use of the microscope is less tiring. **[1 US]**

(b) i. Final image is virtual. **[1 US]**

ii. Final image is at infinity since $p_2 = f_\mathrm{e}$, or the object for the eyepiece is at the eyepiece's focal point. **[2 US]**

iii. Since $m = -f_\mathrm{o}/f_\mathrm{e}$, we have $f_\mathrm{o} = 3|f_\mathrm{e}|$.
Also, $L = f_\mathrm{o} - |f_\mathrm{e}|$, so $f_\mathrm{o} = 10 + |f_\mathrm{e}|$. **[1 US]**
Combining with the above equation gives $3|f_\mathrm{e}| = 10+|f_\mathrm{e}|$, or $|f_\mathrm{e}| = 5$ cm, so $f_\mathrm{e} = -5$ cm . **[2 US]**
Then $f_\mathrm{o} = 3|f_\mathrm{e}|$ gives $f_\mathrm{o} = 15$ cm. **[1 US]**

10. (a) From

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}\frac{\mu}{T}$$

we obtain, using $y = A\cos(kx - \omega t)$, **[2 BW]**

$$\begin{aligned} -A^2k^2\cos(kx-\omega t) &= -A^2\omega^2\cos(kx-\omega t)\frac{\mu}{T} \\ k^2 &= \omega^2\frac{\mu}{T} \\ \frac{\omega^2}{k^2} &= \frac{T}{\mu} \\ \frac{\omega}{k} &= \sqrt{\frac{T}{\mu}} \end{aligned}$$

So, since $v = \omega/k$, $v = \sqrt{T/\mu}$, as required. **[1 BW]**

(b) For a string stretched between two points L apart, sliding freely vertically at $x = 0$ and fixed at $x = L$ the boundary conditions are

$$T\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0, \quad \text{[transverse force is zero since it moves freely];}$$ **[2 BW]**

$$y(L,t) = 0, \quad \text{[no transverse displacement since wave is fixed].}$$ **[1 BW]**

(c) The superposition of waves with a single frequency moving to the left and to the right may be written as

$$y(x,t) = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)},$$ **[2 BW]**

and imposing the boundary condition at $x = 0$ we find

$$-ikae^{i(\omega t - k0)} + ikbe^{i(\omega t + k0)} = 0,$$ **[1 BW]**

whence $a = b$, so that **[1 BW]**

$$\begin{aligned} y(x,t) &= ae^{i(\omega t - kx)} + ae^{i(\omega t + kx)} \\ &= ae^{i\omega t}\left[e^{-ikx} + e^{ikx}\right] \\ &= 2ae^{i\omega t}\cos(kx). \end{aligned}$$ **[2 BW]**

Then the other boundary condition gives

$$\begin{aligned} \cos(kL) &= 0 \\ kL &= (2n+1)\frac{\pi}{2} \\ \frac{2\pi}{\lambda}L &= (2n+1)\frac{\pi}{2} \\ \lambda &= \frac{4L}{2n+1} \end{aligned}$$ **[1 BW]** **[1 BW]**

where n is an integer .

(d) i. If the tension in the string is 80 N and the mass per unit length is 10 gm m^{-1} then the wave speed is $c = \sqrt{80/10 \times 10^{-3}} = 89.4$ m s^{-1}. The frequency of vibration, f, is related to the wavelength by $f = c/\lambda$, so here the frequencies are **[1 US]**

$$f_n = \frac{(2n+1)c}{4L}$$ **[2 US]**

or 22.4 Hz, 67.1 Hz, and 112 Hz. **[1 US]**

ii. The corresponding patterns of displacement of the string are shown in Figure 5. **[2 US]**

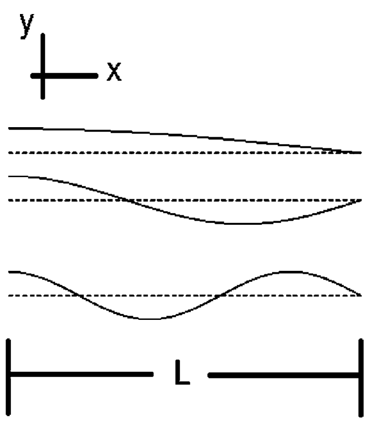

Figure 5 Patterns of displacement of the first three modes for the string of Question 10(d)ii.

11. (a) The *coherence* of a plane-wave pattern describes the extent to which it may be represented as an perfectly plane wave of infinite extent in time. *Longitudinal* coherence is a measure of the time (or distance along the direction of travel) over which the wave is sinusoidal with a constant amplitude. *Transverse* coherence is a measure of the distance perpendicular to the direction of propagation over which the phase and amplitude remain the same. **[1 BW]** **[1 BW]** **[1 BW]**

(b) In a Young's slits experiment if light of wavelength λ is passed through slits a distance h apart and the fringe pattern is observed on a screen a distance x in front of the slits, bright lines will be observed on the screen at distances y from the axis given by

$$h \sin\theta = m\lambda \quad \Rightarrow \quad \sin\theta = \frac{m\lambda}{h} \quad \Rightarrow \quad \tan\theta = \frac{y}{x}$$

For small θ $\tan\theta \approx \sin\theta$ so

$$\frac{y}{x} = \frac{m\lambda}{h} \quad \text{or} \quad y = \frac{m\lambda x}{h},$$

[2 BW]

where m is an integer.

If the separation of the slits is 1.9 mm and the fringe separation on a screen 1 m from the slits is 0.31 mm, then the wavelength of the light is

$$\lambda = \frac{h\Delta y}{x} = \frac{1.9 \times 10^{-3} \times 0.31 \times 10^{-3}}{1} = 5.89 \times 10^{-7}\ \text{m} = 589\ \text{nm}.$$

[2 US]

(c) In Figure 6 there are
$N-2$, or 3 subsidiary peaks; **[1 US]**
the second principal peak is missing because the slit separation is twice the slit width; **[1 US]**
general shape of envelope with pattern underneath. **[2 BW]**

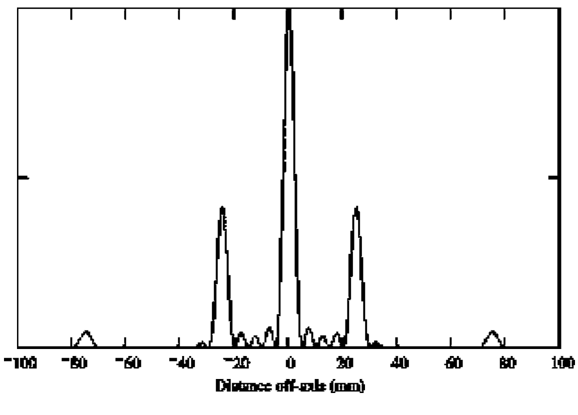

Figure 6 Intensity pattern for 5-slit grating of Question 11c.

(d) The Michelson interferometer. *Subtract mark for each missing or mislabelled component.* **[4 BW]**

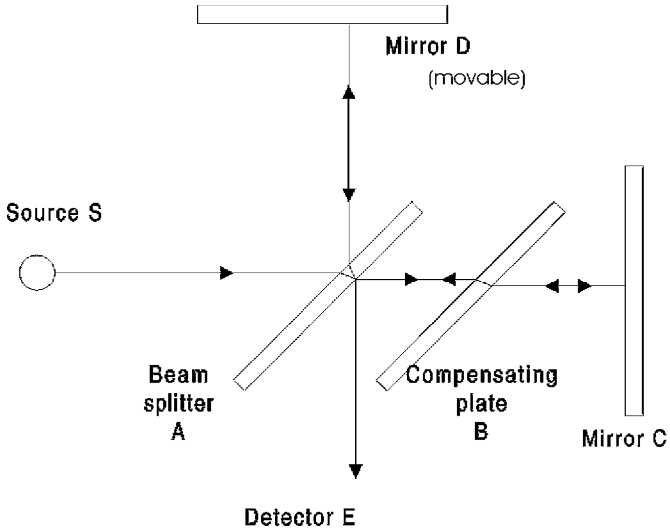

Figure 7 Diagram of Michelson interferometer for Question 11d.

(e) The change in optical path length is caused by the change in refractive index over twice the length of the container (beam goes to and fro once each): if p fringes pass then **[1 US]**

$$p\lambda = 2(n-1)L$$ **[2 US]**

and so with $L = 10$ cm, $\lambda = 600$ nm and $p = 100$ we have

$$n - 1 = \frac{100 \times 600\ 10^{-9}}{2 \times .1} = 0.0003,$$ **[1 US]**

so the refractive index of the air is 1.0003. **[1 US]**

 END OF PAPER