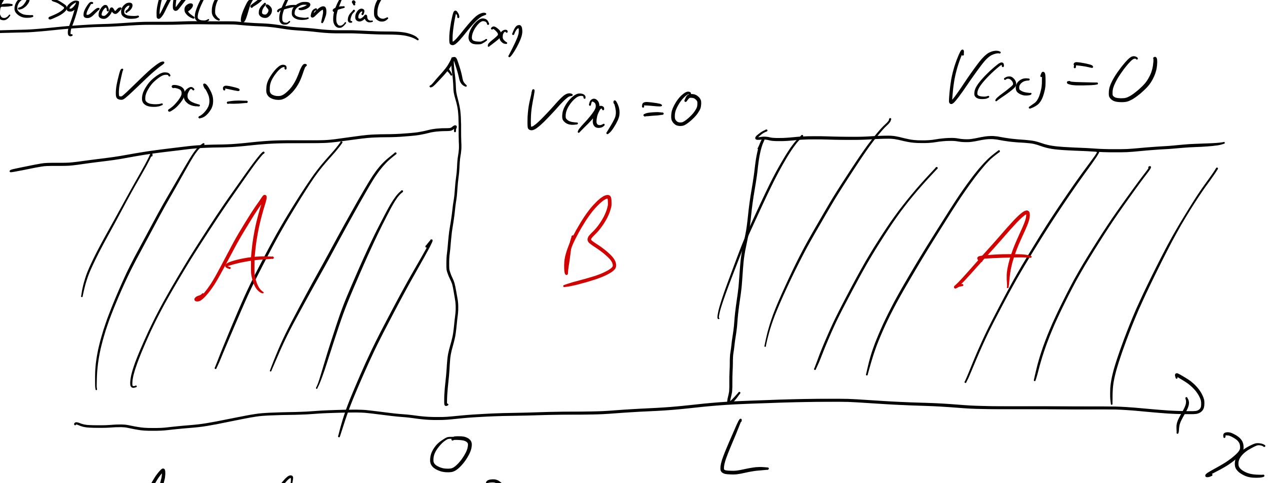
# Finite Square Well Potential




$$\psi_B(x) = A\sin\left(\frac{px}{\hbar} + C\right)$$

## Region A

TISE $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} = (E-U)\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-U)\psi(x)$$

Two cases:

1. $U > E \Rightarrow \frac{d^2\psi}{dx} = +k\psi(x)$
2. $U < E \Rightarrow \frac{d^2\psi}{dx^2} = -k\psi(x)$

Consider $U > E$:

Try solution of the form:

$$\psi(x) = A e^{Bx + C}$$

$$\frac{d\psi}{dx} = A B e^{Bx + C}$$

$$\frac{d^2\psi}{dx^2} = A B^2 e^{Bx + C} = B^2 \psi(x)$$

TISE

$$\frac{d^2\psi}{dx^2} = \frac{2m(U-E)}{\hbar^2}\psi(x)$$

$$B^2\psi(x) = \frac{2m(U-E)}{\hbar^2}\psi$$

$$\therefore B = \pm\sqrt{\frac{2m(U-E)}{\hbar^2}} = \pm K$$

where $K = \frac{\sqrt{2m(U-E)}}{\hbar}$

$$\therefore \psi_1(x) = A e^{\pm Kx + C}$$

Boundary Conditions

Aside: $Ae^{+kx+C} = De^{kx}$

@ $x = 0$

$\psi_A(0) = \psi_B(0)$

$\psi_A(x) = De^{+kx}$ , $\psi_B(x) = A\sin\left(\frac{px}{\hbar} + C\right)$

$\psi_A(0) = D$

$\psi_B(0) = A\sin C$

$\therefore D = A\sin C$ 1

$\frac{d\psi}{dx}$ - continuous

$\frac{d\psi_A}{dx} = Dke^{+kx}$ $\frac{d\psi_B}{dx} = A\frac{p}{\hbar}\cos\left(\frac{px}{\hbar} + C\right)$

$\frac{d\psi_A(0)}{dx} = Dk$ $\frac{d\psi_B(0)}{dx} = \frac{Ap}{\hbar}\cos C$

$\therefore Dk = \frac{Ap}{\hbar}\cos C$ 2

$@ x = L$

$$\psi_A(x) = F e^{-kx}$$

$$\psi_B(L) = \psi_A(L)$$

$$\frac{d\psi_B(L)}{dx} = \frac{d\psi_A(L)}{dx}$$

$$A \sin\left(\frac{pL}{\hbar} + C\right) = F e^{-kL} \quad (3)$$

$$\frac{Ap}{\hbar} \cos\left(\frac{pL}{\hbar} + C\right) = -kF e^{-kL} \quad (4)$$

Attempt to solve

$$\frac{(1)}{(2)} \Rightarrow \frac{1}{k} = \frac{\hbar}{p} \tan C \Rightarrow \tan C = \frac{p}{k\hbar}$$

$$\frac{(3)}{(4)} \Rightarrow \tan\left(\frac{pL}{\hbar} + C\right) \frac{\hbar}{p} = \frac{-1}{k}$$

To simplify we will ignore $C$ for now:

$$\tan\left(\frac{pL}{\hbar}\right) = -\frac{p}{k\hbar}$$

$$p\cot\left(\frac{pL}{\hbar}\right) = -k\hbar$$

$$pL\cot\left(\frac{pL}{\hbar}\right) = -kL\hbar$$

Search for inspiration in the form of change of variables:

Let $z = \frac{pL}{\hbar}$, $z_1 = kL$

$$z\cot z = -z_1$$

$$z^2 = \frac{p^2L^2}{\hbar^2} \qquad z_1^2 = k^2L^2$$

Let $z_0^2 = \frac{2mUL^2}{\hbar^2}$

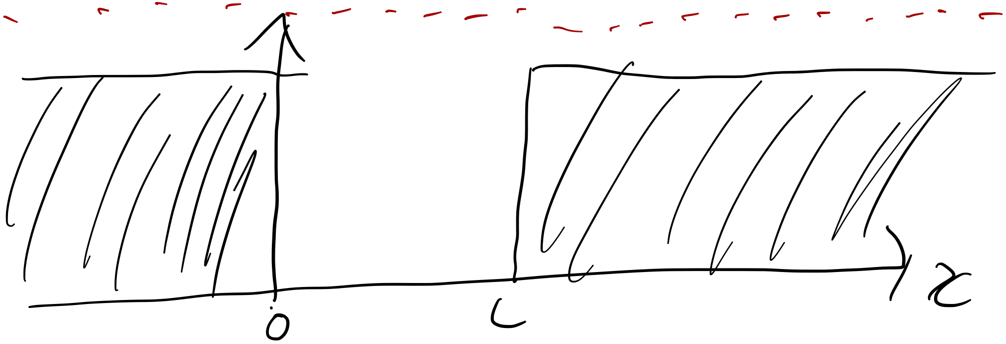
$$k^2 = \frac{2m(U-E)}{\hbar^2} \qquad p^2 = 2mE$$

$$\therefore z_1^2 = z_0^2 - z^2$$

$$k^2L^2 = \frac{2mUL^2}{\hbar^2} - \frac{p^2L^2}{\hbar^2}$$

Can not solve analytically but can graphically.

## Case 2 $E > U$




TISE

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-U)}{\hbar^2}\psi(x)$$

Solution is:

$$\psi(x) = A\sin\left(\frac{\tilde{p}x}{\hbar} + c\right)$$

$$\frac{d^2\psi}{dx^2} = -\frac{\tilde{p}^2}{\hbar^2}\psi(x)$$

$$\tilde{p} = \sqrt{2m(E-U}$$

This is just the classical energy equation.

$$E = \frac{\tilde{p}^2}{2m} + U$$