PHAS0010 Classical Mechanics
Problem-Solving Tutorial 3, 19 Nov – 23 Nov 2018
CoM, Collisions, central forces

1. Two objects of masses m_1 and m_2 are separated by a distance d. Starting from the definition of centre of mass given in the lectures, show that its position vector can be written in the form

$$\mathbf{R} = (1-\lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

and find the appropriate value of λ. Hence show that their centre of mass lies on a line between the two objects and that its distances from the two masses are $dm_2/(m_1+m_2)$ from mass 1 and $dm_1/(m_1+m_2)$ from mass 2.

2. A mass $m_1 = 4\,\mathrm{kg}$ and a mass $m_2 = 1\,\mathrm{kg}$ approach each other along a line: mass m_1 has a velocity $u_1 = +10\,\mathrm{ms}^{-1}$ and mass m_2 has a velocity $u_2 = -5\,\mathrm{ms}^{-1}$. What is the velocity of the centre of mass? What is the momentum of each particle in the centre-of-mass frame?

   The two masses undergo a head-on elastic collision. Describe the collision in the centre-of-mass frame and hence find the final velocities v_1 and v_2 of the two masses, assuming they continue to move along the same line.

3. Two elastic balls are placed vertically above each other with a small gap between and released to fall from a height h under gravity. The top ball has mass m_1 and the bottom m_2. Neglecting air resistance and the size of the balls relative to h, determine the velocity, u, of the balls when the bottom one reaches the ground.

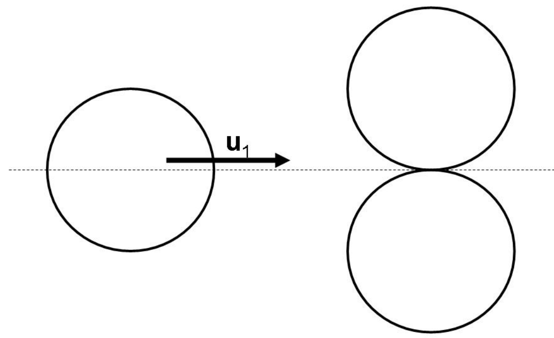
   Assuming the collision with the ground is elastic and the subsequent collision between the balls is also elastic, determine the value of m_1/m_2 which results in the upper ball (m_1) having the largest possible share of the final kinetic energy after the collision of the two balls and for this value of m_1/m_2 determine the maximum height the upper ball reaches after the collision.

   Determine the velocity of m_2 in the centre of mass frame after it has bounced from the ground and just before it collides with m_1.

   The maximum possible height that can be attained by the upper ball occurs when $m_2 \gg m_1$. Show in this case that the upper ball reaches a height of $9h$.

   How do the above considerations change if, while the balls remain perfectly elastic, the collision with the ground of the lower mass has a restitution coefficient e? (no need to repeat the whole derivation, just indicate what changes in the equations after the collision with the ground).

4. A smooth spherical ball of mass m moving with an initial velocity $\mathbf{u}_1$ along the x-axis makes simultaneous elastic collisions with two other balls, identical to the first, that are initially at rest and positioned as in the diagram. Sketch the three balls at the moment of collision and indicate the directions of the impulse on each ball. Hence determine the velocities (magnitudes and directions) of all three balls after the collision.



[HINT—this three-body problem is more easily solved if you do **not** transform to the centre-of-mass frame.]

5. Consider the motion of a particle of mass m in an attractive central force of the form

$$\mathbf{F}(\mathbf{r}) = -K\mathbf{r} = -Kr\hat{\mathbf{r}},$$

where K is a positive constant, r is the distance from the centre of force, and as usual $\hat{\mathbf{r}}$ is a unit vector in the radially outwards direction.

What is the potential energy associated with the force $\mathbf{F}$? (Choose the zero of potential energy to be at the centre of force, $r = 0$.)

Writing the position vector as $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, find the differential equations satisfied by the coordinates x and y. Hence show that one possible class of solutions to the equation of motion is of the form

$$\mathbf{r}(t) = a\cos(\omega t)\hat{\mathbf{i}} + b\sin(\omega t)\hat{\mathbf{j}},$$

where a, b and ω are constants. Find the value of ω.

Find the total energy E and of the particle in terms of m, a, b and ω, when it moves in this orbit.

For this orbit, show that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

What is the shape of the orbit?

Problem for general discussion.

Find how the kinetic energy of a system of particles changes when we transform from one frame of reference to another inertial frame moving with respect to it at a constant velocity $\mathbf{v}_0$. Show that the change depends only on $\mathbf{v}_0$, the total mass M of the system, and the centre-of-mass velocity $\mathbf{V}_{\mathbf{CM}}$.

Hence show also that of all possible such inertial frames, the kinetic energy is a minimum when viewed in the centre-of-mass frame.