

## PHAS1245—Problem Sheet 1, Solutions

1.

$$\text{We have } 2x^2 - 3x - 5 = x^2 - \frac{3}{2}x - \frac{5}{2} = \left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{10}{4} = 0 .$$

$$\text{Thus } x = \frac{3 \pm 7}{4} .$$

2. In  $2^{2x} + 3(2^x) - 4 = 0$  substitute  $y = 2^x$  then

$$y^2 + 3y - 4 = (y + 4)(y - 1) = 0 ,$$

$$\text{i.e. } y = -4 \text{ or } y = 1 \text{ and } 2^x = -4 \text{ or } 2^x = 1 .$$

There are no real values of  $x$  for which  $2^x = -4$ . But if  $2^x = 1$  then  $x = 0$  is the only solution to this equation.

3. Since

$$E = E_1 + E_2 \text{ and } \sin \frac{\phi}{2} = \frac{m}{2\sqrt{E_1 E_2}} , 4E_1 E_2 = \frac{m^2}{\sin^2 \frac{\phi}{2}} , \text{ or } 4E_1(E - E_1) = \frac{m^2}{\sin^2 \frac{\phi}{2}}$$

$$\text{from which we obtain } E_1^2 - EE_1 + \frac{m^2}{4 \sin^2 \frac{\phi}{2}} = 0 .$$

Completing the square we find

$$\left(E_1 - \frac{E}{2}\right)^2 - \frac{E^2}{4} + \frac{m^2}{4 \sin^2 \frac{\phi}{2}} = 0 \text{ whence } E_1 = \frac{E}{2} \pm \frac{1}{2} \sqrt{E^2 - \frac{m^2}{\sin^2 \frac{\phi}{2}}} .$$

From the square root term we find  $\sin \frac{\phi}{2} = \frac{m}{E}$  .

4. Using the factor theorem three times

$$x^3 + 2x^2 - x - 2 = (x - 1)(x + 2)(x + 1) .$$

(Some students will do this in part by long division—full credit if they get the correct result.)

For  $x^3 - x^2 - x - 2$  the factor theorem gives  $(x - 2)$  as a factor .

Long division or other methods, which the students should do, leads to

$$x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1) .$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x + 1)(x - 1)(x^2 + 1) .$$

$$x^{\frac{3}{2}} - a^{\frac{3}{2}} = (x^{\frac{1}{2}})^3 - (a^{\frac{1}{2}})^3 = (\sqrt{x} - \sqrt{a})(x + \sqrt{x}\sqrt{a} + a) .$$

(The expansion for  $x^3 \pm a^3$  has been done in the lectures.)

5. We have

$$u = v_1 \cos \theta + \sqrt{2}v_2 \text{ and } v_1 \sin \theta = \sqrt{2}v_2 .$$

From the above

$$v_1 = \frac{u}{\sin \theta + \cos \theta} \text{ and } v_2 = \frac{u \sin \theta}{\sqrt{2}(\sin \theta + \cos \theta)} .$$

Putting these into the third equation yields

$$u^2 = \frac{u^2}{(\sin \theta + \cos \theta)^2} + \frac{2u^2 \sin^2 \theta}{2(\sin \theta + \cos \theta)^2}$$

$$\begin{aligned} \Rightarrow (\sin \theta + \cos \theta)^2 &= 1 + \sin^2 \theta \\ \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta &= 1 + \sin^2 \theta \\ 2 \sin \theta \cos \theta &= \sin^2 \theta \\ \sin^2 \theta (1 - 2 \cot \theta) &= 0 \end{aligned}$$

$$\text{i.e. } \sin \theta = 0 \text{ or } \tan \theta = 2 .$$

The value  $\tan \theta = 2$  corresponds to  $\sin \theta = \frac{2}{\sqrt{5}}$  and  $\theta = 63.4^\circ$ .