PHAS1247 Classical Mechanics 2017–2018
Full Notes with Additional Worked Examples

1 Orbits

The radial equation of motion for a central force is

$$m\left(\ddot{r}-r\dot{\theta}^{2}\right)=F\left(r\right) \tag{1}$$

and

$$L=mr^{2}\dot{\theta}=\text{cons}\tan\text{t}. \tag{2}$$

Thus

$$\dot{\theta}=\frac{L}{mr^{2}} \tag{3}$$

and

$$m\left(\ddot{r}-r\left(\frac{L}{mr^{2}}\right)^{2}\right) = F\left(r\right) \tag{4}$$

$$m\ddot{r}-\frac{L^{2}}{mr^{3}} = F\left(r\right) \tag{5}$$

$$m\ddot{r}=F\left(r\right)+\frac{L^{2}}{mr^{3}}. \tag{6}$$

This is the same equation of motion for a particle moving in one dimension under an effective force

$$F_{eff}=F\left(r\right)+\frac{L^{2}}{mr^{3}}. \tag{7}$$

The second term $\frac{L^2}{mr^3}$ is called the **centrifugal force**

$$F_{C}=\frac{L^{2}}{mr^{3}}. \tag{8}$$

It is positive and therefore directed radially **outwards** (away from the centre of force). This is our first example of a **fictitious force**: it appears because the body is rotating (i.e. not in true one-dimensional motion) and the central force has to provide the centripetal force to maintain this rotation as well as producing any radial acceleration.

We can also introduce the **centrifugal potential**

$$V_{C}=\frac{1}{2}\frac{L^{2}}{mr^{2}} \tag{9}$$

so that

$$F_{C}\left(r\right)=-\frac{\mathrm{d}V_{C}}{\mathrm{d}r}. \tag{10}$$

It is often useful to think of the particle moving in the combined effective potential

$$V_{\text{eff}}(r)=V(r)+V_{C}(r)=V(r)+\frac{1}{2}\frac{L^{2}}{mr^{2}}. \tag{11}$$

Note that the centrifugal potential is positive and rises as we approach the centre of force. A stable circular orbit then corresponds to a minimum in the effective potential.

1.1 Nearly circular motion

Suppose the radius is not constant. If the force is central the angular momentum $L = mr^2\dot{\theta}$ is still conserved, so we have

$$F(r) = m(\ddot{r} - r\dot{\theta}^2) = m\ddot{r} - \frac{L^2}{mr^3}.$$

Suppose the force is an *attractive* power law:

$$F(r) = -Kr^n$$

and suppose there is a circular orbit at radius r_0. Its angular velocity ω and period τ are given by equating the centripetal force to the central force:

$$mr_0\dot{\theta}^2 = \frac{L^2}{mr_0^3} = Kr_0^n \quad \Rightarrow \quad \omega^2 = \dot{\theta}^2 = \frac{Kr_0^{n-1}}{m} \quad \Rightarrow \quad \text{Period} \quad \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{Kr_0^{n-1}}}.$$

For $r =$ close to r_0 we write $r = r_0 + x$ with x small; then we have

$$F(r) \approx F(r_0) + x\left.\frac{\mathrm{d}F}{\mathrm{d}r}\right|_{r_0} = F(r_0) + xnKr_0^{n-1}.$$

Similarly

$$\frac{L^2}{mr^3} \approx \frac{L^2}{mr_0^3} + x\frac{\mathrm{d}}{\mathrm{d}r}\left.\left[\frac{L^2}{mr^3}\right]\right|_{r_0} = \frac{L^2}{mr_0^3} - 3x\frac{L^2}{mr_0^4}.$$

Hence the equation of motion becomes

$$m\ddot{x} \approx -F(r_0) - xnKr_0^{n-1} + \frac{L^2}{mr_0^3} - 3x\frac{L^2}{mr_0^4}.$$

The first and third terms cancel by definition, so

$$m\ddot{x} = -x\left[nKr_0^{n-1} + 3\frac{L^2}{mr_0^4}\right] = -xKr_0^{n-1}[n+3] \quad \text{since} \quad \frac{L^2}{mr_0^4} = Kr_0^{n-1}.$$

Provided $n > -3$ this equation describes simple harmonic motion, with an angular frequency Ω and period T given by

$$\Omega = \sqrt{\frac{Kr_0^{n-1}[n+3]}{m}} = \sqrt{(n+3)}\omega; \qquad T = \frac{2\Pi}{\Omega} = \frac{\tau}{\sqrt{n+3}}.$$

respectively. The radius therefore oscillates about r_0 with period T. For $n < -3$ r rapidly diverges away from r_0 and the circular orbit is unstable.

Note that in general T is an irrational multiple of τ, so there is no rational relationship between the two periods and the orbit does not close on itself. Two important exceptions are:

- $n = +1$ (Hooke's law spring force), for which $T = \tau/2$, so the orbit closes on itself with two oscillation per revolution. (It corresponds to an ellipse with the centre of force at the centre.)
- $n = -2$ (inverse square law), for which $T = \tau$ and the orbit closes on itself with one oscillation per revolution. (We will see that this case corresponds to an ellipse with the centre of force at one focus.)

This simple harmonic motion is an example of the general property that small oscillations about a stable equilibrium are simple harmonic except in exceptional circumstances(see later).

1.2 Motion under inverse square law of force

Motion under an inverse square law of force has been extensively analysed (starting from Newton and Hooke) because gravitation is such a force. (So is electrostatic attraction, so this theory would also apply to the classical theory of an electron orbiting an atomic nucleus, although in fact we now know that classical mechanics cannot describe this case.) This is a central force with

$$\mathbf{F} = \frac{K}{r^2}\widehat{\mathbf{r}}. \tag{12}$$

Thus the potential energy function is

$$V = -\int \frac{\mathrm{d}r}{r^2}\,\mathrm{d}r = \frac{K}{r} + C. \tag{13}$$

Conventionally we take the potential energy to be zero when the two objects are well separated (i.e. as $r \to \infty$) so $C = 0$ and

$$V = \frac{K}{r}.$$

Now we have

$$\mathbf{F} = -\nabla V = -\widehat{\mathbf{r}}\frac{\partial V}{\partial r} = \frac{K}{r^2}\widehat{\mathbf{r}}, \tag{14}$$

as required.

If K is positive the force is repulsive; if K is negative the force is attractive. For the gravitational force $K = -GMm$ and is always attractive. The gravitational constant $G = 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$. For the electrostatic force $K = q_1 q_2 / (4\pi\varepsilon_0)$ where q_1 and q_2 are the charges which can be of either sign.

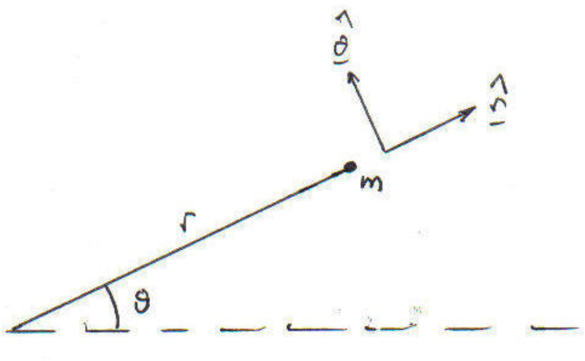

Figure 1: Rotational motion in a plane.

The equation of motion of the particle is (see eq(**??**) and fig 1)

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\theta}\right] = \frac{K}{r^2}\widehat{\mathbf{r}}. \tag{15}$$

Thus for the radial component

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \frac{K}{r^2} \tag{16}$$

and for the transverse component

$$\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{17}$$

It was noted earlier that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{18}$$

Thus as $r \neq 0$, then $r^2\dot{\theta}$ is constant, and so the angular momentum $L = mr^2\dot{\theta}$ is also constant. Hence

$$\dot{\theta} = \frac{L}{mr^2} \tag{19}$$

and the radial equation can therefore be written in the form

$$m\left(\ddot{r}-\frac{L^2}{m^2r^3}\right)=\frac{K}{r^2}. \tag{20}$$

Instead of solving for r and θ as functions of time we will consider the shape of the orbit, i.e. determine r as a function of θ. To do this we define the variable $u=1/r$. We then have

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}, \tag{21}$$

$$\dot{r} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\left(\frac{L}{mr^2}\right)=-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}. \tag{22}$$

Differentiating again we get

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{L}{m}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\left(\frac{L}{mr^2}\right) \tag{23}$$

$$\ddot{r} = -\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. \tag{24}$$

In terms of u the radial equation (20)is

$$m\left(-\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}-u^3\frac{L^2}{m^2}\right)=Ku^2. \tag{25}$$

Multiplying through by $-m/\left(L^2u^2\right)$ gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=-\frac{mK}{L^2}. \tag{26}$$

In terms of the variable $y=u+mK/L^2$ this is the equation of simple harmonic motion,

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+y=0 \tag{27}$$

with solution $y=A\cos\left(\theta-\theta_0\right)$. Thus the general solution of eq(26) is

$$u=A\cos\left(\theta-\theta_0\right)-\frac{mK}{L^2}=\frac{1}{r}. \tag{28}$$

We will choose $A>0$ so that $\theta=0$ corresponds to the distance of closest approach.

If K is negative, i.e. attractive force, the trajectory curves towards the centre of force and is of the form shown in fig 2.

If K is positive, i.e. repulsive force, the trajectory curves away from the centre of force and is of this form in fig 3.

These are of the same form as the general equation of a **conic section**, (see eq(32))

$$u=\frac{1}{r}=\frac{1}{h}\left(1+e\cos\theta\right). \tag{29}$$

We can choose $\theta_0=0$ as this defines the orientation of the trajectory.

1.2.1 Conic sections

A conic section is a locus of a point which moves in a plane such that its distance from a fixed point, the focus, is a constant ratio e, the eccentricity, to the distance from a fixed straight line, the directrix, see fig 4.

Consider an attractive force so that the relevant conic section is as in the diagram, Figure 2. The quantity h is the value of r when $\theta=\pi/2$. From the geometry and definitions of e and h, we have

$$r\cos\theta+\frac{r}{e}=\frac{h}{e}, \tag{30}$$

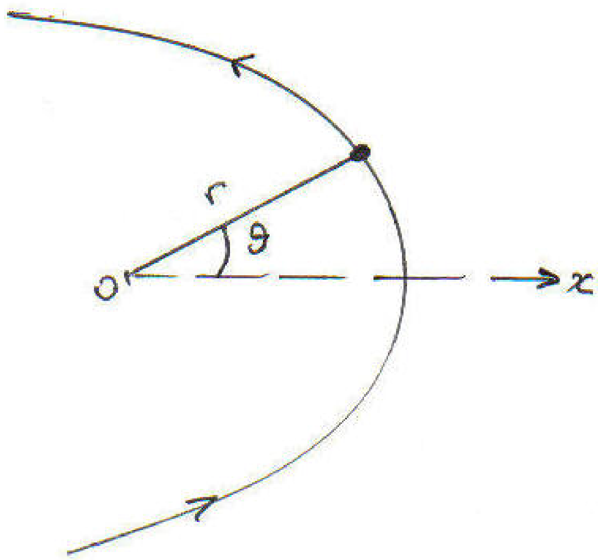

Figure 2: Trajectory for attractive inverse square force law, focus at O.

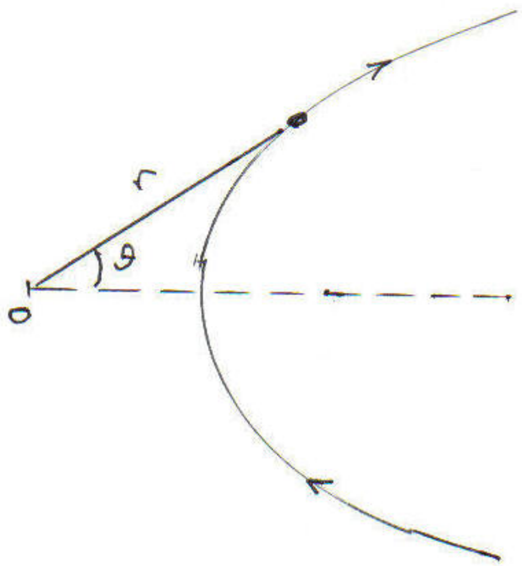

Figure 3: Trajectory for repulsive inverse square force law, focus at O.

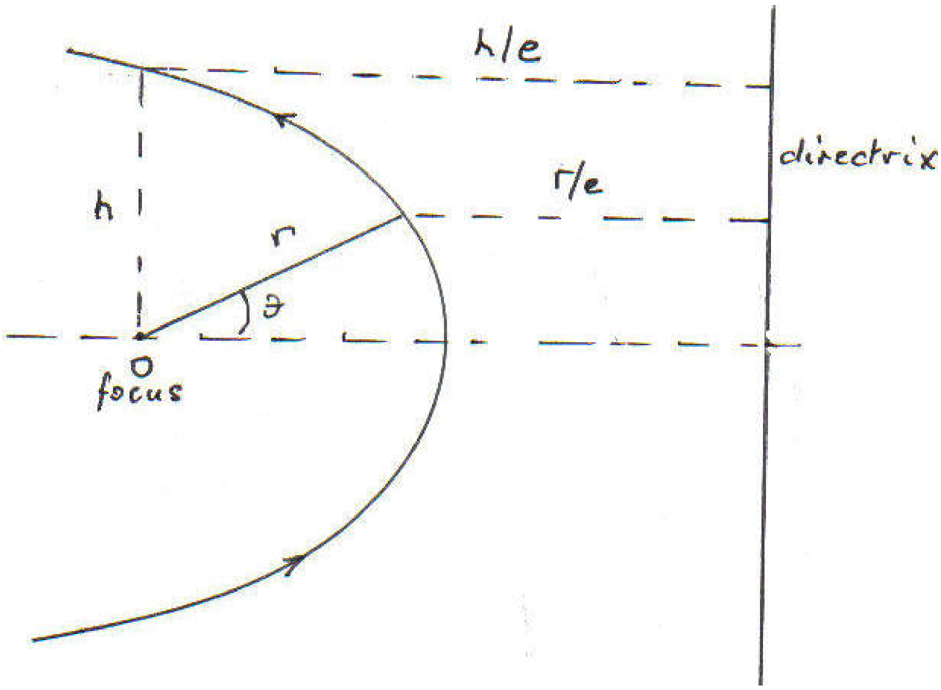

Figure 4: Definition of terms for a conic section.

and therefore

$$r\,(1 + e\cos\theta) = h, \tag{31}$$

or in terms of $u = 1/r$,

$$u = \frac{1}{r} = \frac{1}{h}\,(1 + e\cos\theta)\,. \tag{32}$$

We shall see that possible forms of the trajectory depend on whether the force is repulsive or attractive. For the attractive force it matters whether the total energy is positive, zero or negative.

1.2.2 Trajectories - Attractive force (K is negative)

For an attractive central force such as gravity there are three types of orbit. We shall show below that they arise as follows:

1. If total energy $E > 0$, eccentricity $e > 1$ the trajectory is a hyperbola, as in fig 5.
2. If $E = 0$, $e = 1$, the trajectory is a parabola, as in fig 6.
3. If $E < 0$, $e < 1$, the trajectory is an ellipse, as in fig 7.

A circle is a special case of an ellipse with zero eccentricity.

1.2.3 Trajectory—repulsive force

For a repulsive force, the trajectory curves away from the centre of force and the corresponding conic section is

$$u = \frac{1}{r} = \frac{1}{h}\,(e\cos\theta - 1) \quad \text{or} \quad h = r(e\cos\theta - 1). \tag{33}$$

Note this requires $e > 1$ for there to be a solution. This corresponds to the other branch of a hyperbolic orbit.

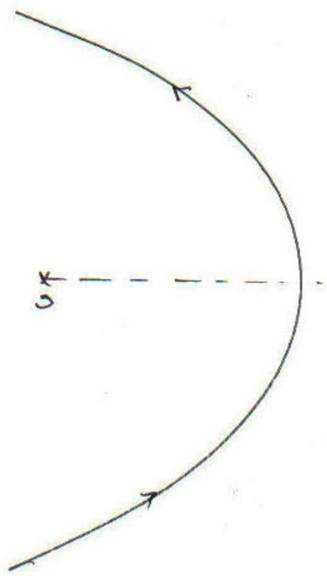

Figure 5: Hyperbola trajectory, $E > 0$, $e > 1$.

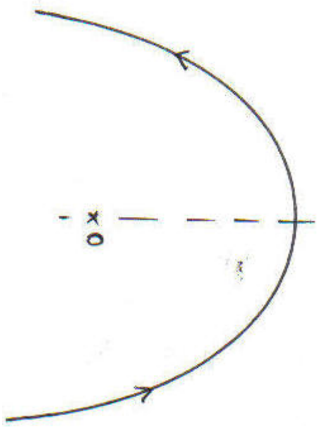

Figure 6: Parabola trajectory, $E = 0$, $e = 1$.

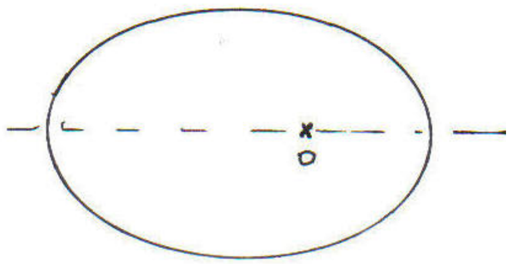

Figure 7: Elliptical trajectory, $E < 0$, $0 \leq e < 1$.

1.2.4 Determination of eccentricity

We shall determine the properties of the orbit for given total energy E and angular momentum L in the case of an attractive force (so $K < 0$). In summary, we will find that:

- The semi-latus rectum is determined solely by the angular momentum of the orbit:
$$h = -\frac{L^2}{mK}; \tag{34}$$
- In the case of a bound orbit ($E < 0$) the semi-major axis and the period of the ellipse are determined solely by the energy:
$$a = \frac{K}{2E}; \quad T = 2\pi\sqrt{\frac{ma^3}{|K|}} \tag{35}$$
- The eccentricity of the orbit is
$$e = \sqrt{\left(1 + \frac{2EL^2}{mK^2}\right)}. \tag{36}$$

The total energy of the particle is
$$E = \frac{1}{2}mv^2 + V \tag{37}$$
and constant throughout the motion (gravitation is a conservative force). So
$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r}. \tag{38}$$
But $\dot{r} = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$, and $\dot{\theta} = \frac{L}{mr^2} = \frac{L}{m}u^2$ so
$$\begin{aligned} E &= \frac{1}{2}m\left[\left(\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{1}{u^2}\left(\frac{L}{m}u^2\right)^2\right] + Ku && (39)\\ &= \frac{1}{2}m\left[\frac{L^2}{m^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{L^2}{m^2}u^2\right] + Ku && (40)\\ &= \text{[positive kinetic energy]} + \text{potential energy}. && (41)\end{aligned}$$

Hence we have that

(i) If K is positive (repulsive force) then E must also be positive;

(ii) if K is negative (attractive force) then E may be positive, zero or negative.

The equation of the orbit is (see eq(28))
$$u = A\cos\theta - \frac{mK}{L^2}, \tag{42}$$
so
$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -A\sin\theta \tag{43}$$
and the expression for the total energy becomes
$$E = \frac{1}{2}m\left[\frac{L^2}{m^2}(-A\sin\theta)^2 + \frac{L^2}{m^2}\left(A\cos\theta - \frac{mK}{L^2}\right)^2\right] + K\left(A\cos\theta - \frac{mK}{L^2}\right). \tag{44}$$

Since the energy is constant throughout the motion we can evaluate the expression for E at any convenient point. Choose $\theta = \pi/2$. The energy is
$$\begin{aligned} E &= \frac{1}{2}m\left[\frac{L^2A^2}{m^2} + \frac{L^2}{m^2}\frac{m^2K^2}{L^4}\right] - \frac{mK^2}{L^2} && (45)\\ &= \frac{A^2L^2}{2m} - \frac{mK^2}{2L^2}. && (46)\end{aligned}$$

We can use this to determine the arbitrary constant A, since re-arranging

$$A^2 = \frac{2m}{L^2}\left(E + \frac{mK^2}{2L^2}\right) = \frac{m^2K^2}{L^4}\left(1 + \frac{2EL^2}{mK^2}\right), \tag{47}$$

and

$$A = \frac{m\,|K|}{L^2}\sqrt{\left(1 + \frac{2EL^2}{mK^2}\right)}. \tag{48}$$

Thus the orbit is determined completely if the values of E and L are known for a given K and m.

The solution for the motion is

$$u = A\cos\theta - \frac{mK}{L^2} \tag{49}$$

and the general form of the conic section (attractive case) is

$$\begin{aligned} u &= \frac{1}{h}\left(1 + e\cos\theta\right) && (50)\\ &= \frac{e}{h}\cos\theta + \frac{1}{h}. && (51) \end{aligned}$$

Comparing these two, we have in the attractive case

$$\begin{aligned} \frac{e}{h} &= A, && (52)\\ \frac{1}{h} &= -\frac{mK}{L^2}, && (53)\\ e &= -\frac{L^2A}{mK} = \sqrt{\left(1 + \frac{2EL^2}{mK^2}\right)}. && (54) \end{aligned}$$

Note h is determined entirely by the angular momentum (for a given force law).

Clearly

(a) if $E > 0$then $e > 1$ and the motion is a hyperbola,

(b) if $E = 0$ then $e = 1$ and motion is a parabola,

(c) if $E < 0$ then $e < 1$ and motion is an ellipse. In this case the particle cannot escape to infinity because at infinity $V = 0$ and the kinetic energy ≥ 0. Therefore to escape to infinity requires $E \geq 0$.

Consider an elliptic orbit as in Figure 8.

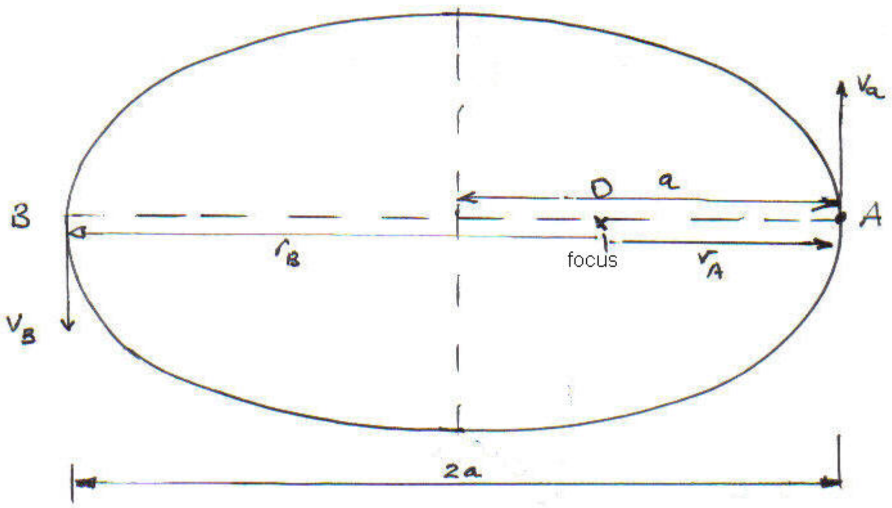

Figure 8: Parameters for an elliptical orbit.

If we know speed of particle at the position of closest approach, we know everything about the orbit. The angular momentum is

$$L = mr_Av_A \tag{55}$$

and energy is

$$E = \frac{1}{2}mv_A^2 - \frac{|K|}{r_A}. \tag{56}$$

We can determine the eccentricity e and the length of the semi-major axis, a. From the general equation

$$r\,(1 + e\cos\theta) = h \tag{57}$$

we have for r_A $(\theta = 0)$ and r_B $(\theta = \pi)$,

$$\begin{aligned} r_A\,(1+e) &= h; \qquad r_B\,(1-e) = h, & (58)\\ 2a &= r_A + r_B = \frac{h}{(1+e)} + \frac{h}{(1-e)} = \frac{2h}{(1-e^2)}, & (59)\\ a &= \frac{h}{(1-e^2)}. & (60) \end{aligned}$$

From the expressions for h and e above in terms of E and L,

$$a = \frac{-\frac{L^2}{mK}}{-\frac{2EL^2}{mK^2}} = \frac{K}{2E} = \left|\frac{K}{2E}\right|. \tag{61}$$

i.e. it depends only on the energy (for a given force law).

For the gravitational force, $K = -GMm$ with G the gravitational constant, $6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$. The point A is the **perihelion** if the Sun is at the focus, and **perigee** if the Earth is at the focus. Similarly point B is the **aphelion** (Sun at the focus) or **apogee** (Earth at the focus).

The semi-minor axis can be found by noticing that the ellipse crosses the y-axis when

$$\cos\theta = -\frac{ae}{r}, \tag{62}$$

so

$$r(1+\cos\theta) = r - ae^2 = h \quad \Rightarrow \quad r = h + ae^2 = h\left[1 + \frac{e^2}{1-e^2}\right] = a. \tag{63}$$

Hence

$$b^2 = r^2 - a^2e^2 = a^2(1-e^2) = \frac{h}{1-e^2} \quad \Rightarrow \quad b = \frac{h}{\sqrt{1-e^2}}. \tag{64}$$

If the origin of coordinates is taken at the mid-point of the major axis, the focus is at $(ae, 0)$ and the Cartesian equation of the orbit is

$$\begin{aligned} \frac{x^2}{\left[\frac{h^2}{(1-e^2)^2}\right]} + \frac{y^2}{\left[\frac{h^2}{1-e^2}\right]} &= 1, & (65)\\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 & (66) \end{aligned}$$

with semi-major and semi-minor axes

$$a = \frac{h}{1-e^2}, \qquad b = \frac{h}{\sqrt{1-e^2}}. \tag{67}$$

1.2.5 Understanding the orbit from the effective potential

For an attractive inverse square law, $F\,(r) = -\,|K|\,/r^2$, with potential function $V = -\,|K|\,/r$. The total effective potential function (see fig 9) is

$$V_{eff} = -\frac{|K|}{r} + \frac{L^2}{2mr^2}. \tag{68}$$

If total energy $E < 0$ then the energy line crosses V_{eff} at $r = r_{\min}$ and $r = r_{\max}$, with $r_{\min} + r_{\max} = 2a$, as in fig 9. If total energy $E > 0$ then $r_{\max}$ is infinite and orbit is a hyperbola. If $E = 0$, $r_{\max}$ is infinite but particle has zero kinetic energy at infinity.

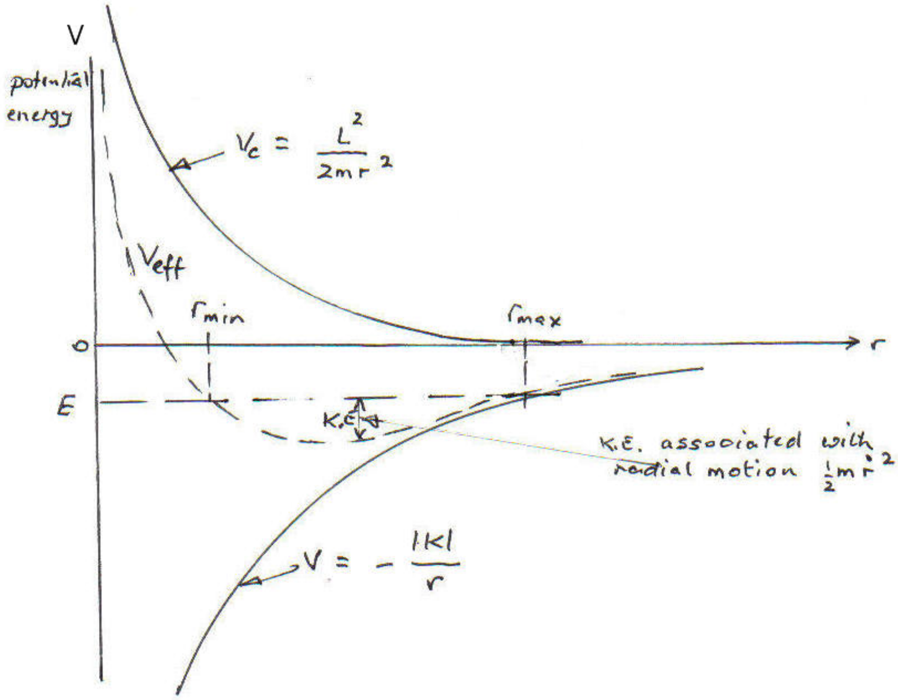

Figure 9: Effective one-dimensional potential for an inverse square law force.

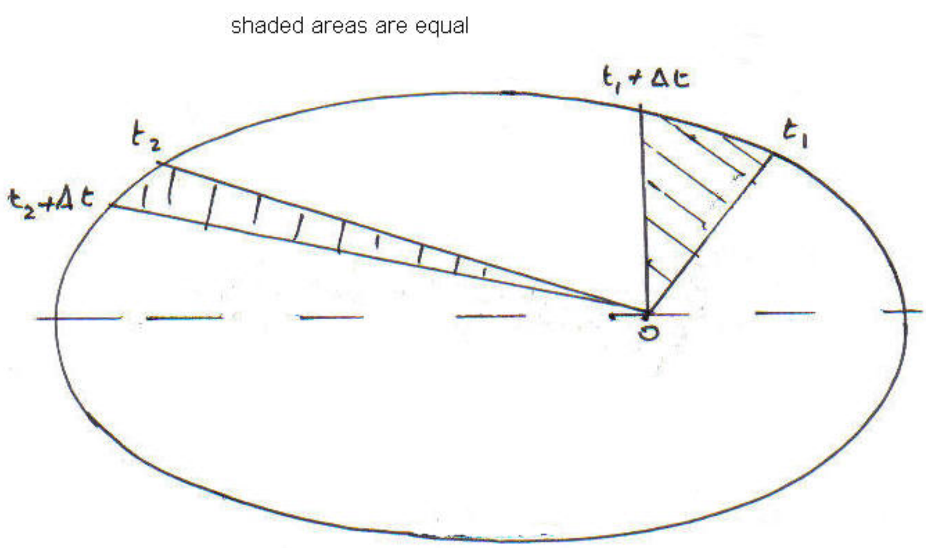

Figure 10: Kepler's first law.

1.2.6 Kepler's laws of planetary motion

1. Planets move in elliptic orbits with Sun at a focus,
2. The radius vector sweeps out equal areas in equal times as illustrated in Figure 10.
3. (Period of rotation)$^2 \propto$ (semi-major axis)^3.

From fig 11, the area of triangle is

$$\begin{aligned} \mathrm{d}A &= \frac{1}{2} r v_\theta \mathrm{d}t = \frac{1}{2} r r \dot{\theta} \mathrm{d}t \qquad (69) \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2} r^2 \dot{\theta}. \qquad (70) \end{aligned}$$

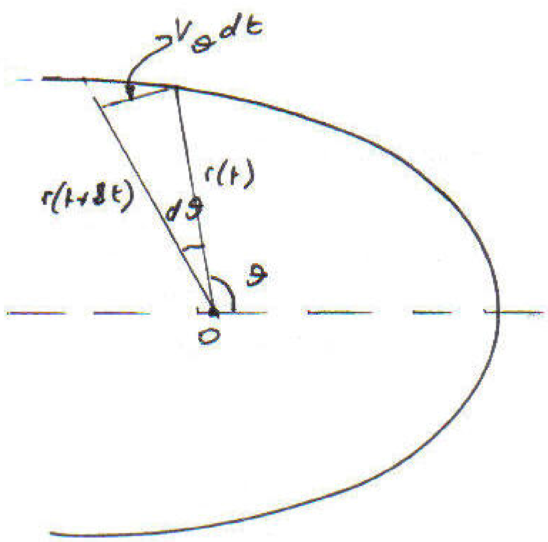

Figure 11: Area swept out as a planet revolves.

But the angular momentum $L = mr^2\dot{\theta}$ is a constant for any central force so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\frac{L}{m} = \mathtt{const}. \qquad (71)$$

Hence we have Kepler's second law.

The third law can be found by considering the total area of the ellipse

$$A = \pi ab = \frac{\pi h^2}{(1-e^2)^{3/2}}. \qquad (72)$$

The period can therefore be found as the time after which the area swept out is equal to the total area:

$$T = \frac{A}{\frac{\mathrm{d}A}{\mathrm{d}t}} = \frac{2\pi abm}{L} = \frac{2\pi mh^2}{L(1-e^2)^{3/2}} \qquad (73)$$

But $h = L^2/m|K|$, so $L = \sqrt{m|K|h}$ and

$$T = 2\pi\sqrt{\frac{mh^3}{|K|(1-e^2)^3}} = 2\pi\sqrt{\frac{ma^3}{|K|}}. \qquad (74)$$

Thus $T^2 \propto a^3$, as expected.

1.2.7 Unbound orbits—Rutherford scattering

Now we consider the case of positive-energy orbits ($E > 0$ and $e > 1$). The distance from the centre of force can now diverge to infinity; this happens when $\theta = \theta_\infty$, where (for attractive forces)

$$1 + e\cos\theta_\infty = 0 \quad \Rightarrow \quad \cos\theta_\infty = -1/e. \tag{75}$$

and for repulsive forces

$$1 - e\cos\theta_\infty = 0 \quad \Rightarrow \quad \cos\theta_\infty = 1/e. \tag{76}$$

We express the collision in terms of the speed at infinity v_∞ and the **impact parameter** b (the distance of closest approach to the centre of force if the object did not deviate from its initial trajectory). Then

$$L = mv_\infty b \tag{77}$$

and

$$E = \frac{1}{2}mv_\infty^2. \tag{78}$$

Therefore the eccentricity is

$$e^2 = 1 + \frac{2EL^2}{mK^2} = 1 + \frac{m^2v_\infty^4b^2}{K^2} \tag{79}$$

Note that b replaces the semi-minor axis for positive-energy orbits: the equation of the hyperbolic orbit can be written in Cartesian coordinates as

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{80}$$

The x-component of the force on the body is

$$\frac{K}{r^2}\cos\theta = \frac{mK\dot{\theta}}{L}\cos\theta, \quad \text{since} \quad L = mr^2\dot{\theta}. \tag{81}$$

Therefore

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{K\dot{\theta}}{L}\cos\theta = \frac{K}{L}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta) \tag{82}$$

so

$$v_x = \frac{K}{L}\sin\theta + C. \tag{83}$$

But $v_x = 0$ at $\theta = 0$, so $C = 0$.

However, at infinity we have $v_x = v_\infty\cos\theta$, so it follows that

$$\tan\theta_\infty = \frac{v_\infty L}{K} = \frac{mv_\infty^2 b}{K} = \frac{2Eb}{K}. \tag{84}$$

This is valid for either attractive or repulsive forces: for attractive forces ($K < 0$) we have $\tan\theta_\infty < 0$ and thus $|\theta_\infty| > \pi/2$, whereas for repulsive forces $|\theta_\infty| < \pi/2$

Angle of deflection is $\phi = \pi - 2\theta_\infty$, so $\phi/2 = \pi/2 - \theta_\infty$ and

$$\tan\left(\frac{\phi}{2}\right) = \frac{K}{2Eb} \quad \text{or} \quad \cot\left(\frac{\phi}{2}\right) = \frac{2Eb}{K}. \tag{85}$$

As $b \to 0$ we have $\phi \to \pm\pi$ (Rutherford back-scattering).

1.3 Reduced mass and the two-body problem

When considering a planet orbiting the Sun we have assumed that the Sun is fixed at the focus. However the mass of the Sun, M, is finite and therefore the Sun and planet **both** move with respect to their overall centre of mass.

We showed previously that the centre of mass of two bodies is at point O in fig 12 such that the position of object 1 relative to it is

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{(m_1 + m + 2)}\mathbf{r}. \tag{86}$$

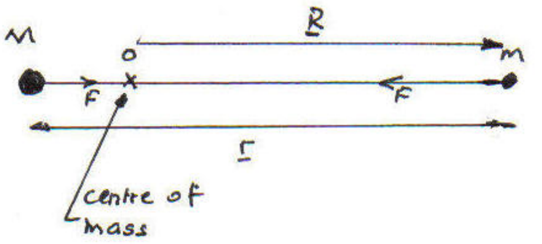

Figure 12: Rotation about centre of mass for two bodies.

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative position of the two bodies.

We also showed that the momentum of particle 1 as measured relative to the centre of mass is

$$\mathbf{p}'_1 = m_1\mathbf{r}'_1 = \mu\dot{\mathbf{r}} = \mu\mathbf{v}, \tag{87}$$

where μ is the reduced mass

$$\mu = \frac{m_1 m_2}{(m_1 + m_2)} \tag{88}$$

where $\mathbf{v}$ is the relative velocity, and that the kinetic energy of relative motion, the force law (if only internal forces act) and the angular momentum about the centre of mass could also be expressed in terms of the relative motion and μ:

$$K_{\text{rel}} = \frac{1}{2}\mu v^2; \tag{89}$$

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v}; \tag{90}$$

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mu\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mu\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{91}$$

where $\mathbf{F}$ is the force on particle 1 from particle 2.

Hence the real system in which both bodies orbit about a common centre of mass is equivalent to a body of the reduced mass μ orbiting at a distance r from the (centre of mass) **fixed** point—see fig 13.

Figure 13: Equivalence of the motion of two-body system and one-body system with reduced mass

Hence all dynamical properties of the two-body system (in the absence of external forces) are equivalent to the dynamics of a single body whose position vector is the relative coordinate $\mathbf{r}$ and whose mass is the reduced mass μ.

Example 1 *We find the reduced mass of the Earth and Sun, and the consequent change in the period of the Earth's orbit from what we would have calculated assuming the Sun was fixed.*

The reduced mass is

$$\mu = \frac{M_\odot M_{\text{Earth}}}{M_\odot + M_{\text{Earth}}} = M_{\text{Earth}}\frac{1}{1 + M_{\text{Earth}}/M_\odot}.$$

We have $M_\odot = 1.99 \times 10^{30}\,\text{kg}$ *and* $M_{\text{Earth}} = 5.97 \times 10^{24}\,\text{kg}$, *so*

$$M_{\text{Earth}}/M_\odot = 3 \times 10^{-6}$$

and

$$\mu = 0.999997 M_{\text{Earth}}.$$

The period was

$$T = 2\pi\sqrt{\frac{ma^3}{|K|}}.$$

The constant $K = GM_\odot M_{\text{Earth}}$ is determined by the law of gravitation and is unchanged. If a is kept the same, we have $T \propto \sqrt{m}$ so the period is reduced by a factor $\sqrt{\mu/M_{\text{Earth}}}$. This corresponds to a fractional change

$$\frac{\Delta T}{T} = \frac{1}{2}(1 - \mu/M_{\text{Earth}}) = 1.5 \times 10^{-6}.$$

In a year this corresponds to $\Delta T = 34\,\text{s}$, which is small but measurable.

2 Rigid bodies

Reading:

- Jewett and Serway Chapter 10
- Kibble and Berkshire Chapter 9 (not §9.9 on Euler angles)
- Kleppner and Kolenkow Chapter 7.

2.1 Centre of mass

Rigid bodies can be considered as a system of particles rigidly joined together. In the limit of a very large number of particles we describe them by their density as a function of position, rather than giving the coordinates of each particle separately.

An example is a uniform thin rod of length L and mass M as in fig 14.

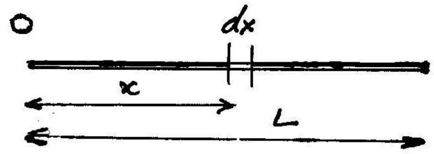

Figure 14: Uniform thin rod

We can determine the position of the centre of mass as follows:

Mass per unit length is

$$\sigma = \frac{M}{L} \tag{92}$$

so the mass of an element of length $\mathrm{d}x$ is $\mathrm{d}m = \sigma \mathrm{d}x$. Therefore from the definition, the position of the centre of mass is given by

$$X = \frac{\int_0^L x\,\mathrm{d}m}{\int_0^L \mathrm{d}m} = \frac{\int_0^L x\,\sigma\mathrm{d}x}{\int_0^L \sigma\mathrm{d}x} = \frac{\frac{1}{2}\sigma x^2|_0^L}{M} = \frac{\frac{1}{2}\sigma L^2}{M} = \frac{1}{2}\frac{\sigma L^2}{M} = \frac{1}{2}\frac{M}{L}\frac{L^2}{M} = \frac{1}{2}L \tag{93}$$

as might have been expected. Note this procedure works even when σ is not constant.

Example 2 *Position of centre of mass of a thin wedge shaped rod (as shown in fig 15) where the mass per unit length is proportional to distance from one end, i.e. $\sigma = kx$ with k a constant.*

Then mass of element of length δx is $\delta m = \sigma\delta x = kx\delta x$ and total mass is

$$M = \int_0^L \mathrm{d}m = \int_0^L \sigma\mathrm{d}x = \int_0^L kx\mathrm{d}x = \frac{1}{2}kL^2. \tag{94}$$

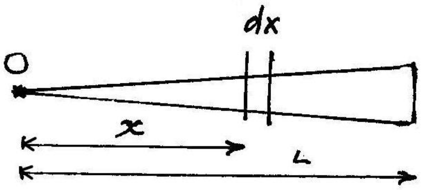

Figure 15: Centre of mass of wedge rod

The position of the centre of mass is

$$X = \frac{\int_0^L x\,\mathrm{d}m}{\int_0^L \mathrm{d}m} = \frac{\int_0^L x\,\sigma \mathrm{d}x}{M} = \frac{\int_0^L kx^2\,\mathrm{d}x}{M} = \frac{\frac{1}{3}kL^3}{\frac{1}{2}kL^2} = \frac{2}{3}L. \tag{95}$$

In the general case of a three-dimensional body with density $\rho(\mathbf{r})$ the total mass is

$$M = \int_{\text{body}} \rho(\mathbf{r})\,\mathrm{d}V \tag{96}$$

and position vector of the centre of mass is

$$\mathbf{R} = \frac{1}{M}\int_{\text{body}} \rho(\mathbf{r})\mathrm{r}\,\mathrm{d}V \tag{97}$$

2.2 Rotation of a rigid body about a fixed axis

Consider rotation of a rigid body as depicted in fig 16, with the axis of rotation perpendicular to the page. Let ω be the angular velocity. A mass element m_i is at a perpendicular distance r_i from the pivot. For a rigid body, all elements must have the **same** angular velocity. The only velocity is transverse to this direction of magnitude $v_i = \omega r_i$. More generally, if $\mathbf{r}_i$ is the position vector relative to a point on the rotation axis, we have

$$\mathbf{v}_i = \omega \times \mathbf{r}_i. \tag{98}$$

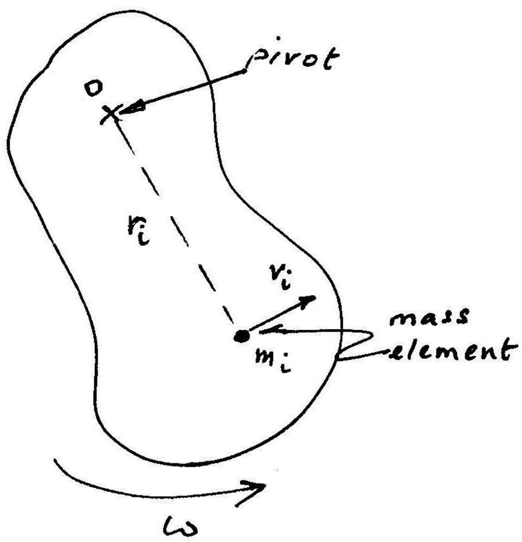

Figure 16: Rotating rigid body

The most general motion of a rigid body is a uniform velocity $\mathbf{V}$ plus a rigid rotation about some axis; we can therefore generalise the last equation to

$$\mathbf{v}_i = \mathbf{V} + \omega \times \mathbf{r}_i, \tag{99}$$

with again the understanding that $\mathbf{r}_i$ is measured with respect to an origin on the rotation axis.

2.3 Angular momentum of a rotating rigid body.

The magnitude of the angular momentum of a point mass m moving in a circle of radius $r_\perp$ (the distance perpendicular to the rotation axis) with angular velocity ω is

$$|L| = mr_\perp^2\omega = mvr_\perp. \tag{100}$$

But more strictly $\mathbf{L}$ is a vector, so we should write (for purely rotational motion)

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times (\omega \times \mathbf{r}). \tag{101}$$

For a system of many particles we would therefore have

$$\begin{aligned} \mathbf{L} &= \sum_i m_i \mathbf{r_i} \times (\omega \times \mathbf{r_i}) &(102)\\ &= \sum_i m_i [r_i^2 \omega - (\mathbf{r}_i \cdot \omega)\mathbf{r}_i]. &(103) \end{aligned}$$

Consider one component, say L_z and write in terms of components of $\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$ and $\omega = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$:

$$\begin{aligned} L_z &= \sum_i m_i [r_i^2\omega_z - (x_i\omega_x + y_i\omega_y + z_i\omega_z)z_i] &(104)\\ &= \sum_i m_i [(r_i^2 - z_i^2)\omega_z - x_i z_i \omega_x - y_i z_i \omega_y]. &(105) \end{aligned}$$

The whole thing can therefore be summarised as

$$\mathbf{L} = \sum_i m_i \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & r_i^2 - y_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & r_i^2 - z_i^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \underline{\underline{\mathbf{I}}}.\omega \tag{106}$$

The matrix $\underline{\underline{\mathbf{I}}}$ is called the **moment of inertia matrix** (or moment of inertia tensor) for the object.

For a continuous mass distribution with density ρ which may vary with position, an element of volume $\mathrm{d}V$ has mass $\mathrm{d}m = \rho\,\mathrm{d}V$. Then we can replace the sum by an integral and we get

$$\underline{\underline{\mathbf{I}}} = \int_{vol} \rho(\mathbf{r}) \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & r_i^2 - y_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & r_i^2 - z_i^2 \end{pmatrix} \mathrm{d}V. \tag{107}$$

It depends only on the choice of origin and the distribution of mass within the body—we will see how to calculate it shortly.

Note that if the rotation is about one axis ($\omega = \omega_z\hat{\mathbf{k}}$, say) and we just want the z component of the angular momentum then

$$L_z = I_{zz}\omega_z = \sum_i m_i (r_i^2 - z_i^2)\omega_z = \sum_i m_i (x_i^2 + y_i^2)\omega_z = \sum_i m_i r_{i,\perp}^2 \omega_z \tag{108}$$

in agreement with the simple result.

In general, however, because of the 'off-diagonal' elements in the matrix $\underline{\underline{\mathbf{I}}}$ the angular momentum and the angular velocity will **not necessarily be parallel**.

For any given object there are particular directions (usually high-symmetry directions) called the 'principal axes of inertia', where $\mathbf{L}$ and ω will be parallel. We will generally assume in this course that the rotations we consider are along these principal axes.

2.4 Kinetic energy of rotation and moment of inertia

The kinetic energy of rotation is

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\omega \times \mathbf{r}_i\right)^2 . \tag{109}$$

Now

$$\left(\omega \times \mathbf{r}_i\right)^2 = (\omega \times \mathbf{r}_i) \cdot (\omega \times \mathbf{r}_i) = \omega \cdot [\mathbf{r}_i \times (\omega \times \mathbf{r}_i)], \tag{110}$$

so

$$K = \frac{1}{2}\omega \cdot \sum_i m_i [\mathbf{r}_i \times \omega \times \mathbf{r}_i] = \frac{1}{2}\omega \cdot \mathbf{L} = \frac{1}{2}\omega \cdot \underline{\underline{\mathbf{I}}} \cdot \omega. \tag{111}$$

Hence the kinetic energy can also be expressed in terms of the moment of inertia.

For rotation around a principal axis (say the z-axis) we would have

$$K = \frac{1}{2} I_{zz} \omega_z^2. \tag{112}$$

2.5 Correspondence between rotational and translational motion

The previous sections show that we can establish a table of correspondences between quantities related to linear motion and to rotational motion.

Linear			Rotation
mass	m	I	moment of inertia
velocity	$\mathbf{v}$	ω	angular velocity
linear momentum	$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \underline{\underline{\mathbf{I}}} \cdot \omega$	angular momentum
linear kinetic energy	$\frac{1}{2} m v^2$	$\frac{1}{2}\omega \cdot \underline{\underline{\mathbf{I}}} \cdot \omega$	rotational kinetic energy
force	$F = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$	$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$	torque
	$F = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$	$F = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$	

2.6 Compound pendulum

The mass of the body is M. The centre of mass is a distance ℓ from the pivot as shown in fig17. It will be assumed that the moment of inertia, I, of this body about an axis through the pivot point A is known.

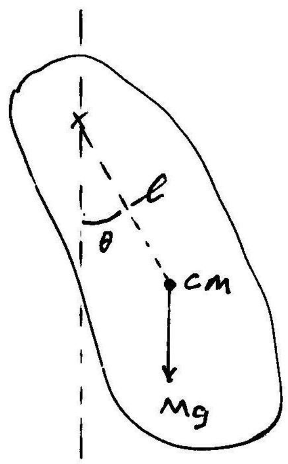

Figure 17: Compound pendulum

Torque of the weight about A is

$$\tau = -Mg\ell \sin\theta. \tag{113}$$

Note the negative sign as the torque is in the sense of decreasing θ.

The equation of motion for rotation about the axis through A is

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{114}$$

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -Mg\ell\sin\theta. \tag{115}$$

If the angle θ is always small, then $\sin\theta \simeq \theta$ (in radians) and

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + Mg\ell\theta = 0, \tag{116}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{Mg\ell}{I}\theta = 0, \tag{117}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0, \tag{118}$$

where

$$\omega = \sqrt{\frac{Mg\ell}{I}}. \tag{119}$$

This is the equation for simple harmonic motion (in θ) with a period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mg\ell}}. \tag{120}$$

If we express the moment of inertia by

$$I = Mk^2 \tag{121}$$

with k called the **radius of gyration** $\left(k = \sqrt{I/M}\right)$ then

$$T = 2\pi\sqrt{\frac{Mk^2}{Mg\ell}} = 2\pi\sqrt{\frac{k^2}{g\ell}}. \tag{122}$$

A **simple pendulum** is a point mass M on the end of a massless inextensible string (!). The moment of inertia of a point mass M at a distance ℓ from a pivot is $I = M\ell^2$, so $k = \ell$. The period becomes

$$T = 2\pi\sqrt{\frac{\ell^2}{g\ell}} = 2\pi\sqrt{\frac{\ell}{g}}. \tag{123}$$

2.7 Determination of moment of inertia

The basic definition is

$$I = \sum_{i=1}^{n} m_i r_i^2 \longrightarrow \int_{vol} r_\perp^2 \rho\,\mathrm{d}v. \tag{124}$$

1. Point mass M at distance ℓ from axis of rotation, $I = M\ell^2$.
2. Ring of mass M and of radius R about an axis through O, the centre of the ring as in fig 18, and perpendicular to the plane of the ring.

   As all the mass is the same distance from O, then eq(124) immediately gives $I = MR^2$. It is important to note that for all other axes the moment of inertia is different.
3. Uniform thin rod, mass M, length L about an axis through one end of the rod and perpendicular to the rod, as in fig 14.

   Mass per unit length is $\rho = \frac{M}{L}$ so mass of length $\mathrm{d}x$ is $\mathrm{d}m = \rho\,\mathrm{d}x$. Thus from eq(124)

$$I_A = \int x^2\,\mathrm{d}m = \int_0^L x^2\rho\,\mathrm{d}x = \frac{1}{3}\rho L^3 = \frac{1}{3}\frac{M}{L}L^3 = \frac{1}{3}ML^2. \tag{125}$$

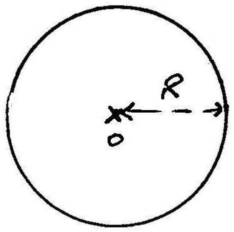

Figure 18: Moment of inertia of a ring

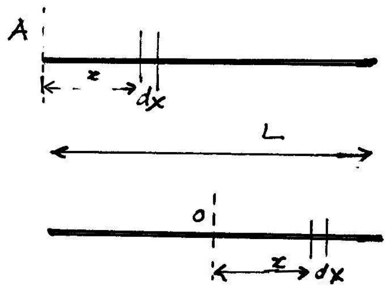

Figure 19: Moment of inertia of a uniform rod about one end.

If the axis were through the centre of the rod, point O, and perpendicular to it then

$$I_O = \int_{-L/2}^{L/2} x^2 \rho \, \mathrm{d}x = \frac{1}{3}\rho x^3|_{-L/2}^{L/2} = \frac{1}{24}\rho L^3 + \frac{1}{24}\rho L^3 = \frac{1}{12}\rho L^3 = \frac{1}{12}\frac{M}{L}L^3 = \frac{1}{12}ML^2. \tag{126}$$

This clearly illustrates that the moment of inertia is not a fixed quantity for a rigid body but depends on the choice of axis about which rotation is to occur. Two important theorems help us determine moments of inertia. These theorems are called (a) the theorem of perpendicular axes and (b) the theorem of parallel axes.

2.8 Theorem of perpendicular axes

It is important to note that this theorem **only applies to plane lamina**. Consider a lamina (sheet) lying in the $x - y$ plane, with mass per unit area ρ (not necessarily constant) as shown in fig 20. Mass of an element of area $\mathrm{d}x\mathrm{d}y$ is $\mathrm{d}m = \rho \mathrm{d}x\mathrm{d}y$.

Thus moment of inertia of lamina about the z-axis is

$$\begin{aligned}
I_z &= \int_{area} r^2 \, \mathrm{d}m = \int \left[\rho \left(x^2 + y^2\right)\right] \, \mathrm{d}x\mathrm{d}y, && (127)\\
&= \int \left[\rho \left(x^2 + y^2\right)\right] \, \mathrm{d}x\mathrm{d}y = && (128)\\
&= \int \rho x^2 \, \mathrm{d}x\mathrm{d}y + \int \rho y^2 \, \mathrm{d}x\mathrm{d}y && (129)\\
&= \int_x x^2 \left(\int_y \rho \, \mathrm{d}x\mathrm{d}y\right) + \int_y y^2 \left(\int_x \rho \mathrm{d}x\mathrm{d}y\right). && (130)
\end{aligned}$$

Since

$$\left(\int_y \rho \, \mathrm{d}x\mathrm{d}y\right) = \mathrm{d}M \tag{131}$$

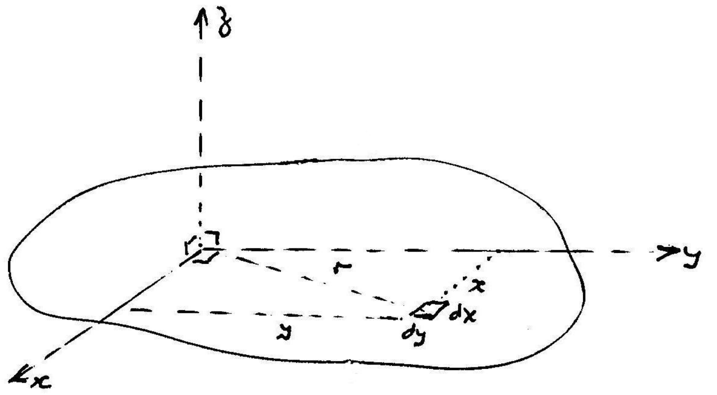

Figure 20: Diagram for perpendicular axes theorem of moments of inertia.

is the mass of a strip of width dx parallel to the y-axis, the first integral on the right-hand side

$$\int_x x^2 \,\mathrm{d}M = I_y, \tag{132}$$

is the moment of inertia of the lamina about the y-axis as shown in fig 21.

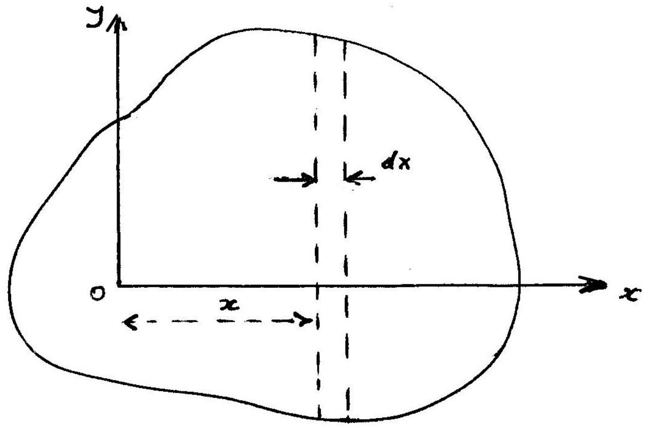

Figure 21: Diagram for identification of integrals

Similarly the second integral gives I_x the moment of inertia of the lamina about the x-axis. Therefore

$$I_z = I_x + I_y. \tag{133}$$

Note the x and y axes can be chosen anywhere in the lamina, but the z-axis **must** be taken through the point of intersection of the x and y axes.

2.9 Theorem of parallel axes

This theorem applies to any solid body. Consider any axis through the centre of mass. Let I_0 be the moment of inertia of the body about this axis. Now consider a parallel axis at a distance a from the axis through the centre of mass (see fig 22).

The moment of inertia about a given axis depends only on the perpendicular distance from the axis of each mass element. Therefore we may squash the three-dimensional body down onto a plane perpendicular to the axis. Suppose this is the $x - y$ plane. Then the distance of an element from the axis through O is r and

$$r^2 = x^2 + y^2, \tag{134}$$

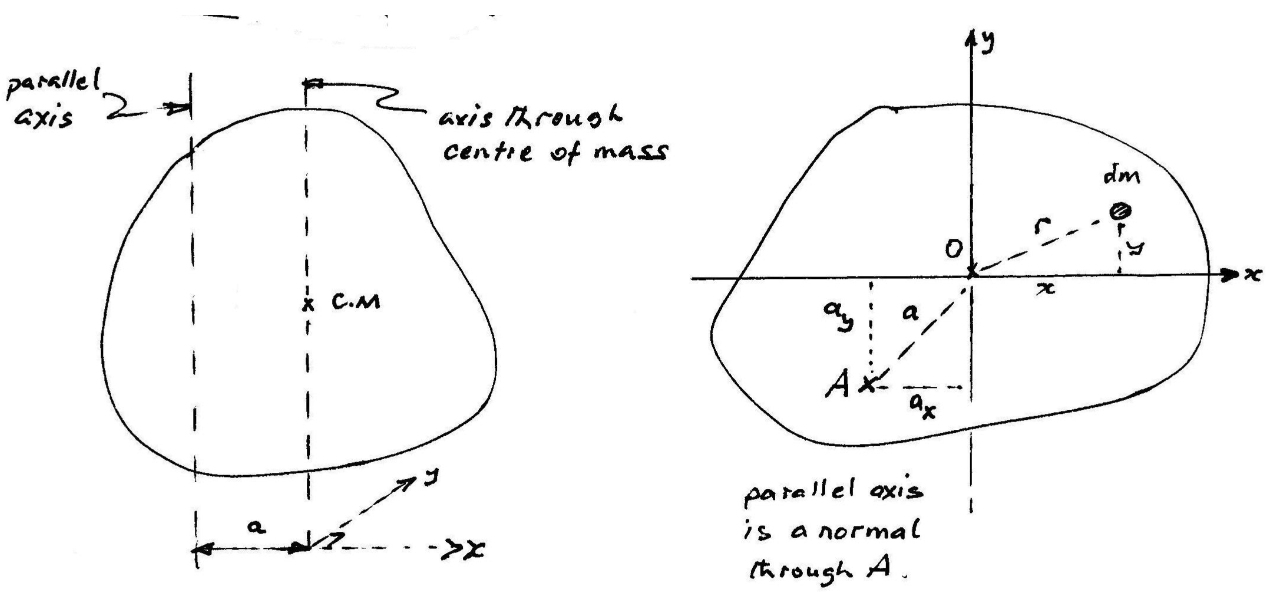

Figure 22: Diagram for parallel axes theorem of moments of inertia

and

$$a^2 = a_x^2 + a_y^2. \tag{135}$$

Distance from A to mass element is R such that

$$R^2 = (x + a_x)^2 + (y + a_y)^2 . \tag{136}$$

Moment of inertia of body about axis through O perpendicular to the $x - y$ plane is

$$I_0 = \int_{vol} r^2 \, \mathrm{d}m = \int_{vol} \left(x^2 + y^2\right) \, \mathrm{d}m \tag{137}$$

and the total mass

$$M = \int \mathrm{d}m. \tag{138}$$

Moment of inertia about parallel axis through A is

$$\begin{aligned}
I_A &= \int_{vol} R^2 \, \mathrm{d}m & (139)\\
&= \int_{vol} \left[(x + a_x)^2 + (y + a_y)^2\right] \, \mathrm{d}m & (140)\\
&= \int_{vol} \left[\left(x^2 + y^2\right) + \left(a_x^2 + a_y^2\right) + 2xa_x + 2ya_y\right] \, \mathrm{d}m & (141)\\
I_A &= \int_{vol} \left(x^2 + y^2\right) \, \mathrm{d}m + a^2 \int \mathrm{d}m + 2a_x \int x \, \mathrm{d}m + 2a_y \int y \, \mathrm{d}m. & (142)
\end{aligned}$$

But by definition of the centre of mass

$$\int x \, \mathrm{d}m = \int y \, \mathrm{d}m = 0 \tag{143}$$

and so

$$I_A = I_0 + Ma^2. \tag{144}$$

This theorem is true for any body.

Example 3 *Consider previous example of a uniform rod of length L. We showed explicitly that $I_0 = \frac{1}{12}ML^2$. Thus by the parallel axes theorem the moment of inertia about a parallel axis through one end of the rod is $I_A = I_0 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$ as explicitly shown previously.*

2.10 Kinetic energy of rigid body with rotation and translation

We shall find an expression for the kinetic energy of a body which is rotating through its centre of mass and also is in rectilinear motion.

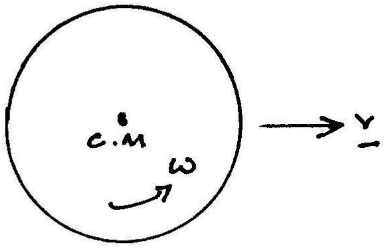

Figure 23: Rotational and translational motion

Body has mass M and moment of inertia about axis through centre of mass of $\underline{\underline{\mathbf{I}}}$. The kinetic energy of rotation about axis through centre of mass is $\frac{1}{2}\omega \cdot \underline{\underline{\mathbf{I}}}_0 \omega$. Kinetic energy of rectilinear motion is $\frac{1}{2}Mv^2$. So total kinetic energy is

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}\omega \cdot \underline{\underline{\mathbf{I}}}_0 \omega. \tag{145}$$

If the rotation axis is a principal axis, the kinetic energy is simply

$$K = \frac{1}{2}I_0\omega^2 + \frac{1}{2}Mv^2. \tag{146}$$

Example 4 *Consider a wheel* ***rolling*** *on a surface* ***without skidding****. As the point in contact with the surface is not slipping, it must be momentarily stationary. Thus the speed of the centre of mass $v = \omega R$.*

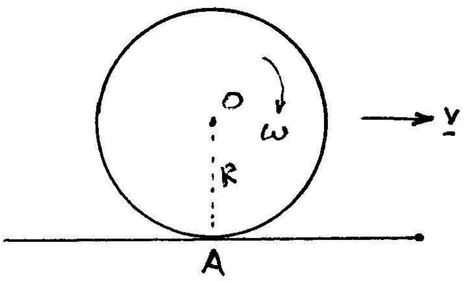

Figure 24: Rolling wheel on a rough surface.

The total kinetic energy is

$$\begin{aligned} K_E &= \frac{1}{2}I_0\omega^2 + \frac{1}{2}Mv^2 &(147)\\ &= \frac{1}{2}I_0\omega^2 + \frac{1}{2}M\omega^2R^2 &(148)\\ &= \frac{1}{2}\left(I_0 + MR^2\right)\omega^2 &(149)\\ &= \frac{1}{2}I_A\omega^2, &(150)\end{aligned}$$

where

$$I_A = I_0 + MR^2 \tag{151}$$

is the moment of inertia of the wheel about an axis through A (the point of contact with the surface) perpendicular to the plane of the wheel. Therefore the wheel can be considered as momentarily rotating *about point A (the point of contact with the surface) with angular velocity ω. The moment of inertia about axis through centre of mass of the wheel (ring) perpendicular to the wheel is $I_0 = MR^2$. In this case*

$$K_E = \frac{1}{2}\left(I_0 + MR^2\right)\omega^2 = \frac{1}{2}\left(MR^2 + MR^2\right)\omega^2 = M\omega^2 r^2 = Mv^2. \tag{152}$$

2.11 Effect of external force

We wish to see what is the effect of an external force applied to a **free** rigid body if the force does not act through the centre of mass of the body, see fig 25.

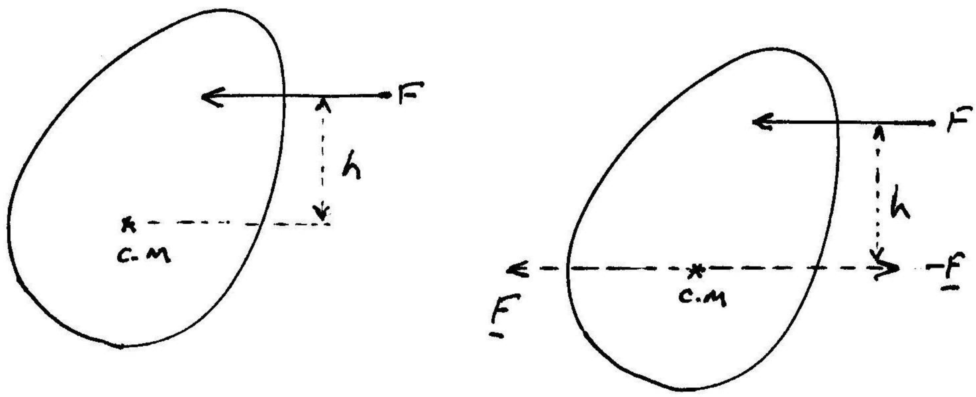

Figure 25: Left: External force not through C.o.M; Right: equivalent force system

If we now add forces $\mathbf{F}$ and $-\mathbf{F}$ acting through the centre of mass we now have a system with a force $\mathbf{F}$ through the centre of mass and a couple made up from the force $-\mathbf{F}$ through the centre of mass and the original force $\mathbf{F}$. Force acting through the centre of mass accelerates the mass according to

$$\mathbf{F} = M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2}. \tag{153}$$

The couple, with a torque $\tau = Fh$, produces an angular acceleration

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I_0\frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{154}$$

about an axis through the centre of mass and perpendicular to the plane defined by the direction of the force and the centre of mass.

2.11.1 Centre of percussion

Suppose force $\mathbf{F}$ is applied for a very short time $\mathrm{d}t$ to a system initially at rest. Then

$$\mathbf{F}\,\mathrm{d}t = M\mathbf{V} \tag{155}$$

where $\mathbf{V}$ is velocity of centre of mass, and

$$\tau\mathrm{d}t = I_0\omega, \tag{156}$$

where ω is the angular velocity about axis through the centre of mass. Immediately after the impulse is applied the speeds of the points A and B (see fig 26) are

$$v_A = V + \ell_A\omega \tag{157}$$
$$v_B = V - \ell_B\omega. \tag{158}$$

It may be that $v_B = 0$. The point at which the impulse is applied for which $v_B = 0$ is called the **centre of percussion**. If a body, e.g. a door is hinged at B and is struck at the centre of percussion there is no impulsive reaction at the hinge. Similarly if a cricket bat or tennis racket is held at one end (i.e. B) and the ball strikes the centre of percussion there is no painful jarring sensation at the player's hand.

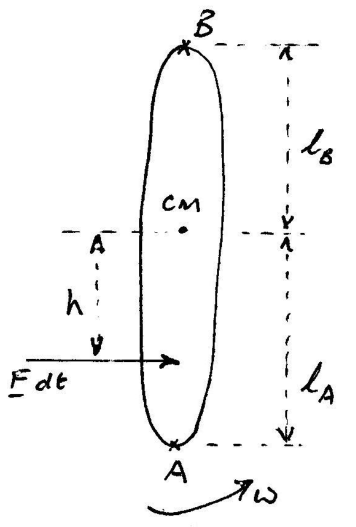

Figure 26: Impulse given to body free to rotate.

Determination of position of centre of percussion If $v_B = 0$ then $V = \ell_B\omega$. But

$$MV = F\mathrm{d}t \tag{159}$$

and

$$I_0\omega = \tau\mathrm{d}t = Fh\mathrm{d}t = MVh \tag{160}$$

so

$$I_0\omega = M\ell_B\omega h \tag{161}$$

$$h = \frac{I_0}{M\ell_B}. \tag{162}$$

For a uniform rod of length L and mass M, we have $I_0 = \frac{1}{12}ML^2$ and $\ell_B = \frac{1}{2}L$, so $h = \frac{1}{6}L$, i.e. 2/3 of the length of the rod from B.

2.12 Simple theory of the gyroscope

Consider spinning a disc with moment of inertia I_0 about an axis perpendicular to the disc, as in fig27.

The torque of the weight about O is $\tau = Mgx$. The direction of the torque vector τ is into the paper. Since

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{163}$$

then in time dt the angular momentum of the spinning disc changes by an amount

$$\mathrm{d}\mathbf{L} = \tau\,\mathrm{d}t \tag{164}$$

in the direction of τ. Thus viewed from above we have the vector diagram in Figure 28.

$\mathbf{L}(t)$ is the angular momentum at time t. As can be seen from the diagram, the angular momentum vector $\mathbf{L}$ is rotated through an angle $\mathrm{d}\alpha$ in time $\mathrm{d}t$ but it is still horizontal and of the same magnitude. Hence

$$\mathrm{d}\alpha = \frac{\mathrm{d}L}{L} = \frac{\tau\mathrm{d}t}{L} = \frac{Mgx\mathrm{d}t}{I_0\omega}. \tag{165}$$

The rate of rotation of $\mathbf{L}$, or **rate of precession**, is

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\tau}{L} = \frac{Mgx}{I_0\omega}. \tag{166}$$

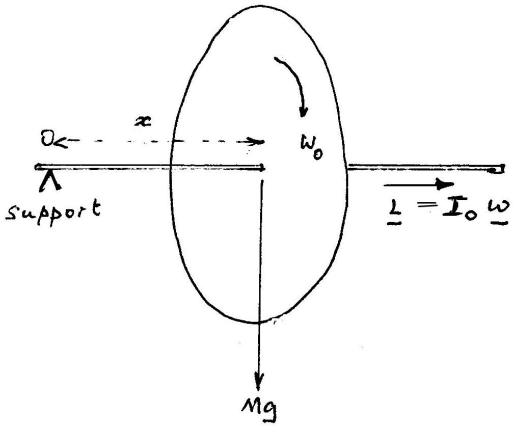

Figure 27: A spinning disc

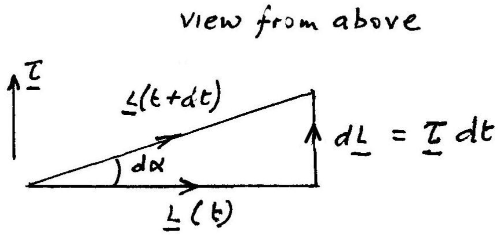

Figure 28: Vector diagram for the gyroscope.

This simple theory is valid if $\frac{\mathrm{d}\alpha}{\mathrm{d}t} \ll \omega$.

There are many applications of gyroscopes in maintaining stability of rotating bodies and in various control systems.

3 Accelerating and rotating frames of reference

Reading

- Jewett and Serway: §6.3;
- Kleppner and Kolenkow: Chapter 8;
- Kibble and Berkshire: Chapter 5.

3.1 Transformation of velocity and acceleration

Consider again a car travelling with velocity **v** and a bird flying with velocity **u** relative to the ground as in fig 29.

Velocity of bird relative to car is

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}. \tag{167}$$

Suppose **v** is constant but bird is accelerating relative to the ground, then

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \tag{168}$$

and the **acceleration** of the bird is the **same** in both frames of reference. Frames for which this is true are known as **inertial frames**.

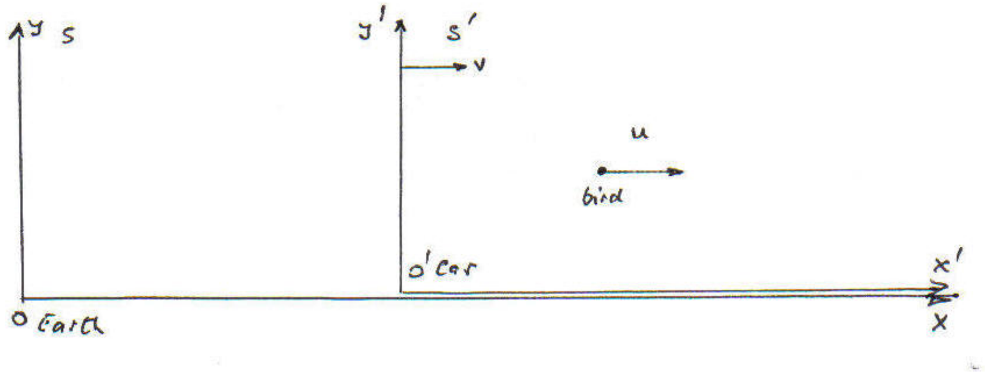

Figure 29: Diagram for acceleration as seen in two frames

However if the car is also accelerating relative to the Earth, then

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{169}$$

with $\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t}$ being the acceleration of the bird relative to the car, $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ being the acceleration of the bird relative to the Earth, and $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ being the acceleration of the car relative to the Earth. The force on the bird necessary to accelerate it in frame S (Earth) is

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}. \tag{170}$$

The equation of motion of the bird in the reference frame of the car, S', is **NOT**

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t}. \tag{171}$$

It is correctly

$$\mathbf{F}' = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{172}$$

i.e. the effective force on the bird in the car's frame of reference, S', is

$$\mathbf{F}' = \mathbf{F} - m\mathbf{a}, \tag{173}$$

where $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ is the acceleration of the car (in the Earth frame S). Thus if we want to apply Newton's laws of motion in an **accelerating frame** we must **add** to the **real** force **F** the **fictitious** force ($-m\mathbf{a}$). An accelerating frame of reference is also called a **non-inertial** frame. An **inertial** (or non-accelerating) frame of reference is one in which Newton's laws of motion apply **without** needing to introduce fictitious forces.

Example 5 *To an observer in a lift accelerating upwards with acceleration $\mathbf{a} = a_z\hat{\mathbf{k}}$, objects behave as if they were subject to an apparent weight (fictitious force)*

$$\mathbf{F}' = -m(g + a_z)\hat{\mathbf{k}}. \tag{174}$$

Einstein's Equivalence Principle (the cornerstone of General Relativity) states that this additional 'apparent weight' is indistinguishable from the force of gravity by any experiment we can perform purely within the lift.

3.2 Rotating frames of reference

Rotating frames of reference are an important example of **non-inertial frames**. The laws of physics appear different in them, leading to the introduction of so-called **fictitious forces**.

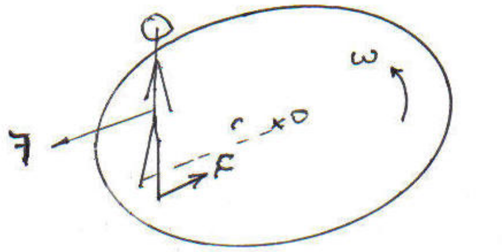

Figure 30: Observer in a rotating frame

Example 6 *Consider a person standing on a rotating roundabout as in fig 30.*

Real force of friction at the feet provides the centripetal acceleration $r\omega^2$, therefore $F = mr\omega^2$. ***Relative to the roundabout*** *the person is* ***at rest*** *under the influence of a real centripetal force $F = mr\omega^2$ and a* ***fictitious centrifugal force*** *$\mathfrak{F} = mr\omega^2$ acting through the centre of mass. If ω is large enough, the person may fall over outwards under the influence of these two forces, which constitute a couple acting on the person (see section on rotational motion and couples later).*

3.3 Rotating frames of reference—general case

We saw that the change $\delta\mathbf{r}$ in a position vector resulting from a small (right-handed) rotation $\delta\phi$ about an axis through the origin with unit vector $\hat{\mathbf{n}}$ could be written

$$\delta\mathbf{r} = \delta\phi\hat{\mathbf{n}} \times \mathbf{r} = \delta\theta \times \mathbf{r}. \tag{175}$$

We can also define an **angular velocity vector** $\omega = \omega\hat{\mathbf{n}}$ such that

$$\delta\theta = \omega\delta t. \tag{176}$$

Then the change in vector $\mathbf{r}$ arising from the rotation is

$$\delta\mathbf{r} = \omega \times \mathbf{r}\delta t, \tag{177}$$

so

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \omega \times \mathbf{r}, \tag{178}$$

In a rotating frame of reference the Cartesian unit vectors obey this relationship because they are 'fixed' relative to that frame (i.e. they change only because of the rotation):

$$\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t} = \omega \times \hat{\mathbf{i}}'; \quad \frac{\mathrm{d}\hat{\mathbf{j}}'}{\mathrm{d}t} = \omega \times \hat{\mathbf{j}}'; \quad \frac{\mathrm{d}\hat{\mathbf{k}}'}{\mathrm{d}t} = \omega \times \hat{\mathbf{k}}'. \tag{179}$$

As usual we can express any vector $\mathbf{v}$ either in terms of the original basis vectors $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ or in terms of the rotating set $\{\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'\}$:

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} = v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}'. \tag{180}$$

The components (v'_x, v'_y, v'_z) are the components of the apparent vector $\mathbf{v}'$ in the rotating frame.

Now consider the (true) time derivative of a general vector $\mathbf{v}$. From the product rule we have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}v'_x}{\mathrm{d}t}\hat{\mathbf{i}}' + \frac{\mathrm{d}v'_y}{\mathrm{d}t}\hat{\mathbf{j}}' + \frac{\mathrm{d}v'_z}{\mathrm{d}}t\hat{\mathbf{k}}' + v'_x\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t} + v'_y\frac{\mathrm{d}\hat{\mathbf{j}}'}{\mathrm{d}t} + v'_z\frac{\mathrm{d}\hat{\mathbf{k}}'}{\mathrm{d}t} &(181)\\
&= \frac{\mathrm{d}v'_x}{\mathrm{d}t}\hat{\mathbf{i}}' + \frac{\mathrm{d}v'_y}{\mathrm{d}t}\hat{\mathbf{j}}' + \frac{\mathrm{d}v'_z}{\mathrm{d}t}\hat{\mathbf{k}}' + \omega \times \left[v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}'\right] &(182)\\
&= \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{\text{apparent}} + \omega \times \mathbf{v}. &(183)
\end{aligned}$$

This applies to **any** vector $\mathbf{v}$ in the rotating frame (we have not assumed, for example, that $\mathbf{v}$ is the velocity of the particle).

Now let's apply it to the position vector:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\text{apparent}} + \omega \times \mathbf{r}, \tag{184}$$

so

$$\mathbf{v}_{\text{true}} = \mathbf{v}_{\text{apparent}} + \omega \times \mathbf{r}. \tag{185}$$

Now apply our rule to the velocity vector:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}_{\text{true}}}{\mathrm{d}t} &= \left.\frac{\mathrm{d}\mathbf{v}_{\text{true}}}{\mathrm{d}t}\right|_{\text{apparent}} + \omega \times \mathbf{v}_{\text{true}} &(186)\\
&= \left.\frac{\mathrm{d}\mathbf{v}_{\text{apparent}}}{\mathrm{d}t}\right|_{\text{apparent}} + \omega \times \mathbf{v}_{\text{apparent}} + \omega \times [\mathbf{v}_{\text{apparent}} + \omega \times \mathbf{r}] &(187)\\
&= \left.\frac{\mathrm{d}\mathbf{v}_{\text{apparent}}}{\mathrm{d}t}\right|_{\text{apparent}} + 2\omega \times \mathbf{v}_{\text{apparent}} + \omega \times (\omega \times \mathbf{r}). &(188)
\end{aligned}$$

So the true acceleration is

$$\mathbf{a}_{\text{true}} = \mathbf{a}_{\text{apparent}} + 2\omega \times \mathbf{v}_{\text{apparent}} + \omega \times (\omega \times \mathbf{r}). \tag{189}$$

The equation of motion will be

$$\mathbf{F} = m\mathbf{a}, \tag{190}$$

where $\mathbf{F}$ is the true applied force, so

$$m\mathbf{a}_{\text{apparent}} = \mathbf{F} - 2m\omega \times \mathbf{v}_{\text{apparent}} - m\omega \times (\omega \times \mathbf{r}). \tag{191}$$

The second and third terms are the apparent forces that appear because of the acceleration of the rotating frame: they are known as the **Coriolis force** and **centrifugal force** respectively.

Let's try and develop some intuition for them. First the centrifugal force: we can expand the triple product as

$$\begin{aligned}
\omega \times (\omega \times \mathbf{r}) &= (\omega \cdot \mathbf{r})\omega - \omega^2\mathbf{r} &(192)\\
&= -\omega^2\left[\mathbf{r} - (\hat{\omega} \cdot \mathbf{r})\hat{\omega}\right] &(193)\\
&= -\omega^2\mathbf{r}_\perp, &(194)
\end{aligned}$$

where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ perpendicular to ω. Hence

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2\mathbf{r}_\perp. \tag{195}$$

It is always directed radially outwards, and depends on how far we are from the rotation axis.

Example 7 *Consider a geostationary satellite.*

Satellite appears stationary relative to an observer on a rotating Earth as shown in fig 31.

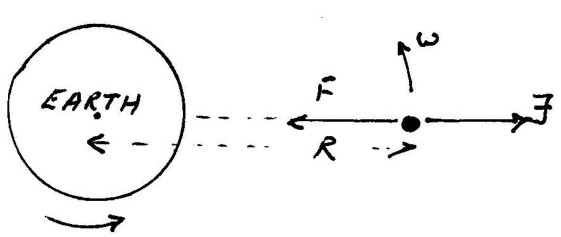

Figure 31: Geostationary satellite

Real force on the satellite is the gravitational force (centripetal force)

$$F = -G\frac{Mm}{R^2} = -mR\omega^2. \tag{196}$$

The observer on Earth thinks there is also a fictitious centrifugal force

$$F = mR\omega^2 = G\frac{Mm}{R^2} \tag{197}$$

so there is no net force, and so the satellite is at rest and in equilibrium in this frame.

*This also explains why an astronaut in an orbiting spacecraft feels weightless: the Earth's gravity has **not** fallen to zero, but the apparent centrifugal force cancels his weight.*

3.4 Observing a moving body in a rotating frame

Consider a person B at rest on the roundabout observing a moving body, e.g. a flying bird, as in fig 32.

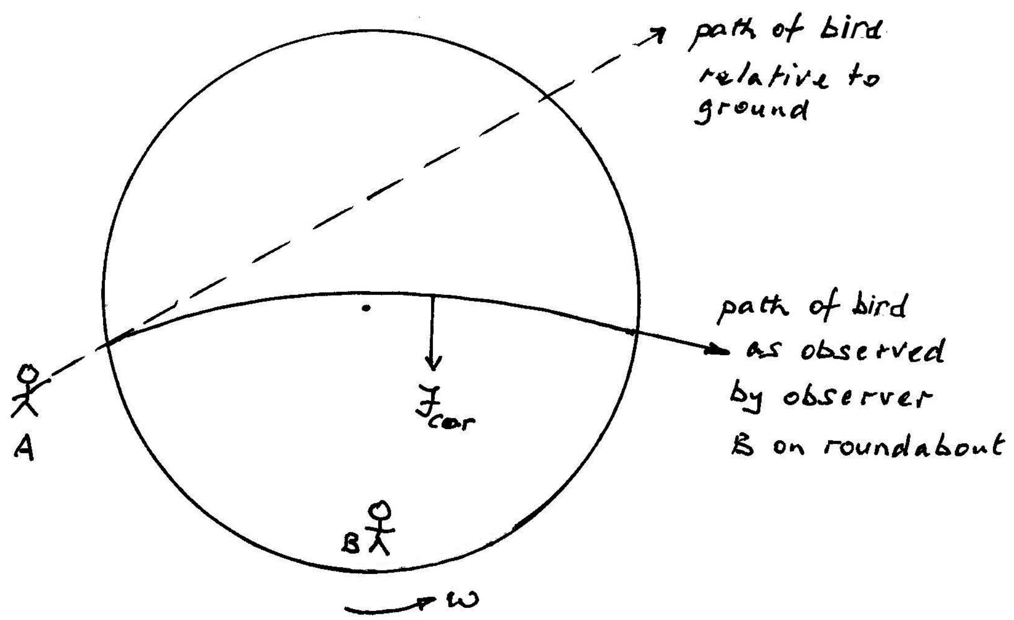

Figure 32: Observers in rotating and inertial frames

Consider the same bird being observed by a second person A at rest on the ground and not on the roundabout. To observer A the path of the bird is a straight line. Relative to observer B on the roundabout the path of the bird is curved. Hence observer B assumes there is a transverse horizontal force $\mathfrak{F}_{cor}$ acting on the bird deflecting the bird to the right for the given sense of rotation. This is the **Coriolis force**.

The general expression for the Coriolis force is

$$\mathfrak{F}_{cor} = -2m\omega \times \mathbf{v}, \tag{198}$$

where ω is the angular velocity vector of rotating frame, $\mathbf{v}$ is velocity of particle of mass m relative to the rotating frame of reference. Note $\mathfrak{F}_{cor}$ is perpendicular to both ω and $\mathbf{v}$ and so always transverse to the direction of motion. There are several interesting consequences of the Coriolis force:

Consider the Coriolis force of the rotating Earth (Figure 33):

Throw a ball horizontally due East at three different places on the Earth's surface, A, B, and C as in fig 33 - dotted line is the local horizontal at each place on the Earth. In the northern hemisphere (point A) the Coriolis force deflects the ball to the right as it has a component in this direction. On the equator (point B) the Coriolis force has no horizontal component so the ball is not deflected. (Coriolis force is

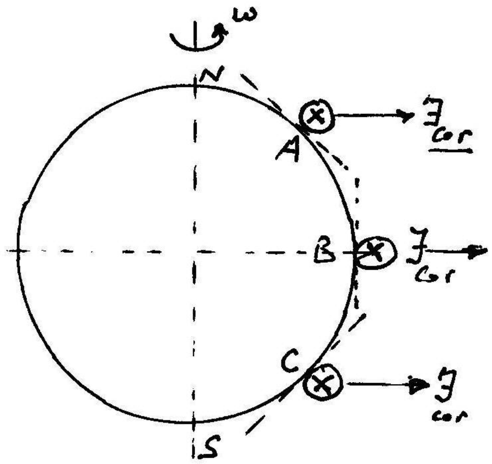

Figure 33: Effect of Coriolis force at Earth's surface'

vertical so affects the local effective force due to gravity.) In the southern hemisphere the ball is deflected to the left.

Weather systems are also determined by the Coriolis force. Wind blows anticlockwise (clockwise) around region of low pressure in the northern (southern) hemispheres. Winds blow approximately along isobars instead of down the pressure gradient.

One should set the size of the Coriolis force on the rotating Earth into context. The maximum value of the Coriolis acceleration is $a = -2\omega v$. Since $\omega = (2\pi)/(24 \times 60 \times 60)\,\text{rads}^{-1} = 7.272\,2 \times 10^{-5}\text{rads}^{-1}$ then for a speed of 15ms^{-1} ($\simeq$ 33mph), $a = 2.2 \times 10^{-3}\text{ms}^{-2}$, (quite small compared to the vertical component of acceleration due to gravity on the Earth's surface).

3.5 A simple viewpoint on the Coriolis force

Consider a massless smooth rod on which there is a small ring of mass m. The rod is rotating with constant angular velocity ω in a horizontal plane. As the rod is smooth there can be no radial force (from friction) on the ring. The only force on the ring is a transverse normal reaction force N as shown in fig 34.

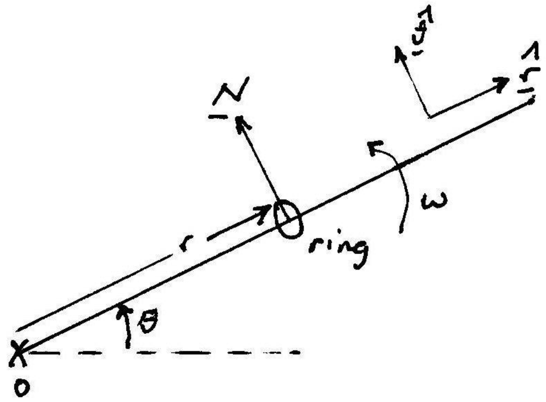

Figure 34: Diagram for simple derivation of expression for Coriolis force

The equation of motion of the ring is, relative to the inertial reference frame,

$$m\mathbf{a} = \mathbf{N} \tag{199}$$

$$m\left[\left(\ddot{r}-r\dot{\theta}^2\right)\widehat{\mathbf{r}}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\widehat{\theta}\right]=N\widehat{\theta}. \tag{200}$$

Therefore the radial equation of motion is

$$m\left(\ddot{r}-r\dot{\theta}^2\right)=0 \tag{201}$$

and the transverse one is

$$m\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=N. \tag{202}$$

But $\dot{\theta}=\omega=$ constant, so

$$m\left(\ddot{r}-r\omega^2\right) = 0, \tag{203}$$

$$2m\dot{r}\omega = N. \tag{204}$$

Now consider the equation of motion of the ring relative to the rotating rod. In this frame the ring can only move in one dimension along the rod. If we work in this frame we must add to the real forces the fictitious centrifugal force $\mathfrak{F}_c$ and, because the ring is moving relative to the rotating frame, the Coriolis force $\mathfrak{F}_{cor}$ as in fig 35

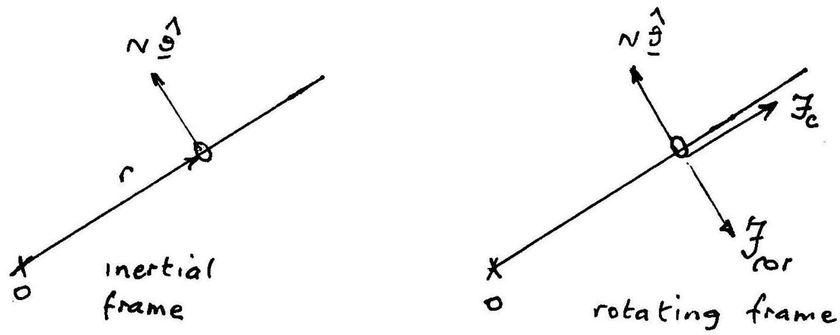

Figure 35: Forces on ring in inertial and rotating frames

In the rotating frame the equations of motion is radially

$$m\ddot{r}=\mathfrak{F}_c \tag{205}$$

and transversely,

$$N-\mathfrak{F}_{cor}=0 \tag{206}$$

as there is no transverse motion in the rotating frame. Comparing eq(203) with eq(205) gives a centrifugal force

$$\mathfrak{F}_c=mr\omega^2 \tag{207}$$

and comparing eq(204) and eq(206) gives a Coriolis force

$$\mathfrak{F}_{cor}=2m\dot{r}\omega. \tag{208}$$

In the rotating frame the ring has only radial velocity $v=\dot{r}$. Thus the magnitude of the Coriolis force

$$\mathfrak{F}_{cor}=2mv\omega. \tag{209}$$

The direction of $\mathfrak{F}_{cor}$ and its magnitude is consistent with the general expression quoted previously, i.e.

$$\mathfrak{F}_{cor}=-2m\omega\times\mathbf{v} \tag{210}$$

as ω is out of the page and $\mathbf{v}$ is along the rod.

3.6 Rotational equilibrium

For a system in equilibrium, $\mathbf{L} = 0$ and constant so there is no net torque about **any** point. This is the basis of taking moments to determine the magnitude of forces.

Example 8 *A massless beam is supported at points A and B and masses m_1 and m_2 are attached as in the fig 36.*

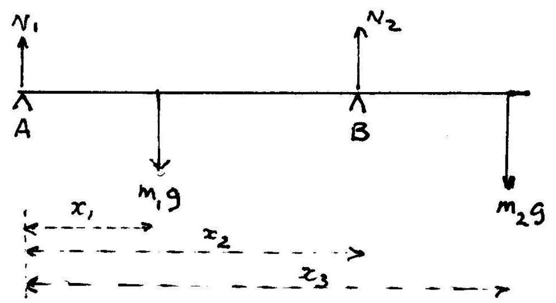

Figure 36: Masses on supported beam

Resolving vertically,

$$N_1 + N_2 = m_1 g + m_2 g = (m_1 + m_2)\, g. \tag{211}$$

Take moments about A, (assume clockwise moment is positive, anti-clockwise moment is negative)

$$m_1 g x_1 - N_2 x_2 + m_2 g x_3 = 0 \tag{212}$$

$$N_2 = \frac{(m_1 x_1 + m_2 x_3)\, g}{x_2}, \tag{213}$$

$$N_1 = (m_1 + m_2)\, g - N_2. \tag{214}$$

Example 9 *Motorcyclist with uniform acceleration such that front wheels lift off the ground as in fig 37*

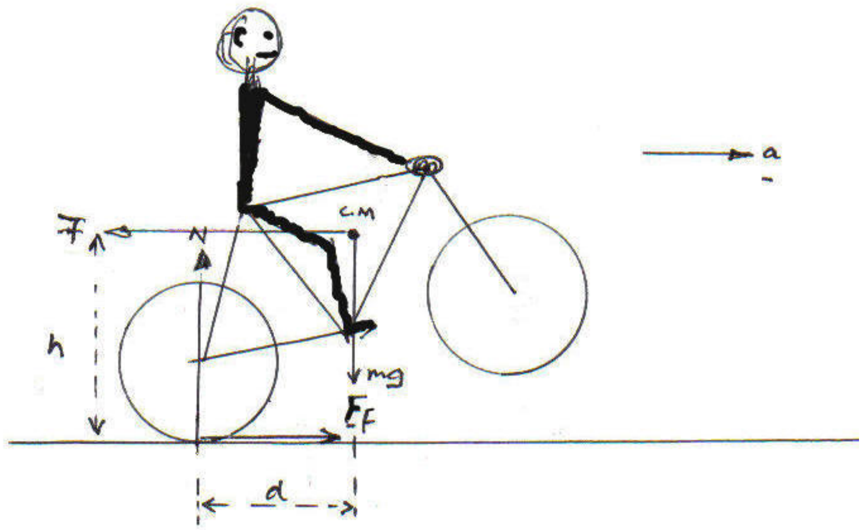

Figure 37: Accelerating cyclist

The real forces are, weight mg, friction $F = ma$ and normal reaction $N = mg$. In motorcyclists' own frame of reference (which is accelerating relative to the ground) he is in equilibrium under influences of *real **and** fictitious force $\mathfrak{F} = ma$ acting through the centre of mass as shown. Taking moments about point of contact of rear wheel with the ground,*

$$mgd - \mathfrak{F}h = 0. \tag{215}$$

Example 10 *An accelerating car as shown in fig 38.*

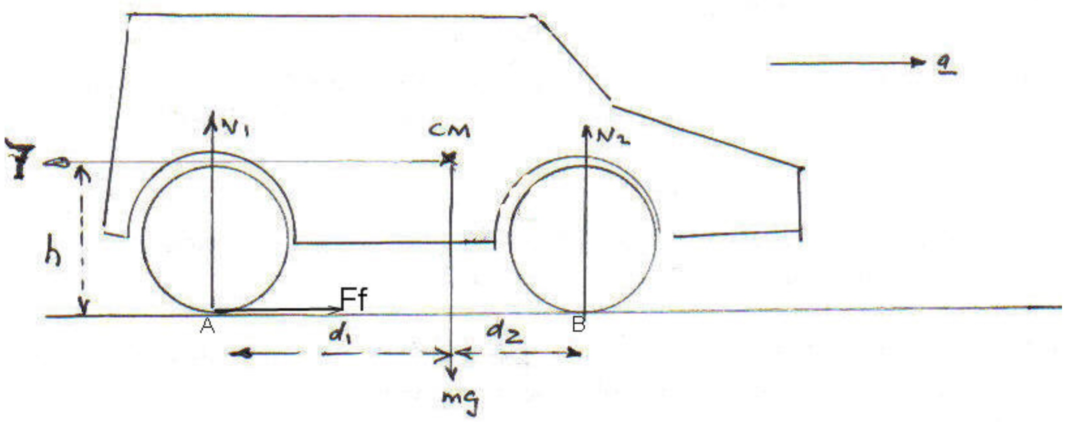

Figure 38: Accelerating vehicle

We will assume rear-wheel drive; that centre of mass is at height h above the ground; distances d_1 and d_2 from wheels as shown in the diagram. Real forces are: weight mg; friction $F = ma$ at the driven rear wheels contact with the ground (There is also friction between front wheels and the ground acting in the opposite sense causing the front wheels to turn rather than skid.); normal reactions at wheels, $N_1$and N_2. We have

$$N_1 + N_2 = mg. \tag{216}$$

In the car's accelerating frame we add a fictitious force $\mathfrak{F} = ma$ as shown. In this frame the car is now in equilibrium. Taking moments about point A,

$$mgd_1 - N_2\left(d_1 + d_2\right) - \mathfrak{F}h = 0, \tag{217}$$

$$N_2 = \frac{1}{(d_1 + d_2)}\left(mgd_1 - mah\right) = \frac{m}{(d_1 + d_2)}\left(gd_1 - ah\right), \tag{218}$$

and

$$N_1 = mg - N_2 = mg - \frac{m}{(d_1 + d_2)}\left(gd_1 - ah\right) \tag{219}$$

$$N_1 = \frac{m}{(d_1 + d_2)}\left(gd_2 + ah\right). \tag{220}$$

For a rear-wheel drive car the maximum acceleration of the car is either when (a) $F = ma = \mu N_1$ where μ is coefficient of friction, whence

$$a = \frac{\mu}{(d_1 + d_2)}\left(gd_2 + ah\right), \tag{221}$$

or (b) if μ is large enough when $N_2 = 0$ whence

$$a = g\frac{d_1}{h}. \tag{222}$$

(Hence the need for a low centre of mass in racing cars!) Any higher acceleration than this will cause the car to somersault backwards. For a front-wheel drive car, maximum acceleration occurs for $F = ma = \mu N_2$. The front wheels can never leave the ground because they will skid first.

3.7 Centre of gravity and centre of mass

We defined the position of the centre of mass by the (weighted) average of the position vectors of the component masses. We now show this is equivalent to requiring that the weight of the object should have no moment about the centre of mass.

The total torque exerted by the weights of the masses i about a point $\mathbf{R}$ in a uniform gravitational field $\mathbf{g}$ is

$$\tau = \sum_i m_i(\mathbf{r}_i - \mathbf{R}) \times \mathbf{g}. \tag{223}$$

This will be equal to zero only if

$$\sum_i m_i \mathbf{r}_i \times \mathbf{g} = \sum_i m_i \mathbf{R} \times \mathbf{g} = M\mathbf{R} \times \mathbf{g}. \tag{224}$$

This will be true for arbitrary directions of the gravitational acceleration $\mathbf{g}$ only if

$$\mathbf{R} = \sum_i \frac{m_i}{M}\mathbf{r}_i, \tag{225}$$

which was our original definition.

It follows that the total weight of a system of masses can be taken to act through the centre of mass; the centre of mass is the point about which the masses will balance in a uniform gravitational field if they form a rigid body (i.e. their relative positions are held fixed by the internal forces). It also follows that the fictitious forces on a set of masses can also be taken to act through the centre of mass, since the fictitious forces are also proportional to the mass of the particles.