Answer ALL SIX questions in Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional maximum credit per section of a question.

Section A

[Part marks]

1. (a) State the formal definition of the derivative of a function $f(x)$. **[2]**

Answer:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(b) Using the formal definition of the derivative, calculate from first principles the derivative of $f(x) = x^{1/4}$ at $x = 2$. **[4]**

Answer:

$$\sqrt[4]{2}/8$$

2. (a) State and derive de Moivre's theorem. **[3]**

Answer: See lecture notes

(b) Determine all the solutions of the equation $z^5 = 4 - 4i$, in polar form. **[5]**

Answer:

$z_0 = \sqrt{2}e^{-\pi i/20}$, $z_1 = \sqrt{2}e^{7\pi i/20}$, $z_2 = \sqrt{2}e^{3\pi i/4}$, $z_3 = \sqrt{2}e^{-17\pi i/20}$, $z_4 = \sqrt{2}e^{-9\pi i/20}$

3. (a) Given a function $y = f(x)$, state the condition for a point x_0 to be stationary. **[2]**

Answer: x_0 is solution to the equation

$$\frac{df}{dx} = 0$$

(b) Given a function $y = f(x)$, state the criteria to determine the nature of a stationary point. **[2]**

Answer: See lecture notes.

(c) Find the stationary point(s) of $f(x) = 3x^6$ and discuss its (their) nature. **[4]**

Answer: Stationary point at $x_0 = 0$. Its nature is strictly speaking undetermined using the criteria discussed in the lectures (sufficient for full marks) but by inspection of the function, it must be a minimum ($f(x)$ increases on either side of the stationary point).

[Part marks]

4. (a) Write the general expression for the Maclaurin expansion of a function $y = f(x)$. **[2]**

   Answer:

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=0} x^n$$

   (b) Determine the Maclaurin expansion up to the second order of the following functions: **[4]**

$$(i) \quad f(x) = \sin x + \cos x \ ;$$

$$(ii) \quad g(x) = \frac{e^x}{x-1} \ .$$

   Answer:

$$(i) \quad f(x) = 1 + x - \frac{1}{2}x^2 + \dots$$

$$(ii) \quad g(x) = -1 - 2x - \frac{5}{2}x^2 + \dots$$

5. Calculate the following limits: **[6]**

$$(i) \quad \lim_{x \to +\infty} \frac{2x^2+1}{4x^2+3x+1} \ ;$$

$$(ii) \quad \lim_{x \to \pi/2} \frac{\cos x}{x^2 - \pi^2/4} \ .$$

   Answer:

$$(i) \quad \frac{1}{2}$$

$$(ii) \quad -\frac{1}{\pi}$$

6. (a) For the geometric series, $S_N = a + ar + ar^2 + ... + ar^{N-1}$, sum the series to derive an expression for S_N. **[3]**

   Answer:

$$S_N = a\frac{1-r^N}{1-r}$$

   (b) Determine whether the following series converges or not: **[3]**

$$\sum_{n=0}^{\infty} \frac{2n}{5^n} \ .$$

   Answer: Series converges

Section B

7. (a) Express the vector $\underline{v} = 3\underline{i} - 3\underline{j} + 2\underline{k}$ in the form $\underline{v} = a\underline{r}_1 + b\underline{r}_2 + c\underline{r}_3$, where the vectors $\underline{r}_1, \underline{r}_2, \underline{r}_3$ are given as **[4]**

$$\underline{r}_1 = 3\underline{i} - 2\underline{j} - \underline{k} \ ,$$
$$\underline{r}_2 = \underline{i} + 2\underline{j} - 3\underline{k} \ ,$$
$$\underline{r}_3 = 2\underline{i} - \underline{j} + 4\underline{k} \ .$$

Answer: Vector $\underline{v}$ can be expressed as above with

$$a = \frac{7}{8}, \quad b = -\frac{17}{40}, \quad c = \frac{2}{5}$$

(b) Two vectors $\underline{a}$ and $\underline{b}$ have direction cosines $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $(0, \sqrt{3}/2, -1/2)$, respectively. Find the direction cosines of a third vector $\underline{c}$ that is perpendicular to both $\underline{a}$ and $\underline{b}$. **[4]**

Answer:

$$c_x = \frac{-1}{\sqrt{5}}, \quad c_y = \frac{1}{\sqrt{5}}, \quad c_z = \frac{1}{\sqrt{15}}$$

(c) The angle between two planes is defined as the angle formed by the respective vectors normal to the planes.

Given the two planes defined by

$$a_1x + b_1y + c_1z + d_1 = 0$$

and

$$a_2x + b_2y + c_2z + d_2 = 0 \ ,$$

with a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 real parameters, determine:

i. The angle between the two planes as a function of the parameters defining the planes. **[6]**

ii. The condition for the two planes to be parallel, expressed as a condition for the parameters defining the two planes. **[6]**

Answer:

$$(i) \quad \theta = \arccos\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

$$(ii) \quad \text{e.g. } (a_1b_2 - b_1a_2)^2 + (a_1c_2 - c_1a_2)^2 + (b_1c_2 - c_1b_2)^2 = 0$$

8. (a) Consider a function $y = f(x)$. Give a geometrical interpretation of the derivative of the function at a point x_0, and provide a relevant sketch. **[2]**

Answer: The derivative gives the slope of the tangent to the function at the point $x = x_0$. For a sketch, see lecture notes.

(b) Find the acute angles between the curves

$$y = 8x^2$$

and

$$y = 8x^3$$

at their points of intersection. **[5]**

Answer: There are two points of intersections, $x_0 = 0$ and $x_1 = 1$, and the acute angle θ_i between the functions at these points is given by $\theta_0 = 0$ and $\tan\theta_1 = \frac{8}{385}$, respectively.

(c) Consider a circle around the origin defined by the equation

$$x^2 + y^2 = 25 \ .$$

i. Determine the slope of the tangent to the above circle at a given point (x_0, y_0) on the circle. **[3]**

ii. Find the value x_i where such a tangent intersects with the x-axis of the coordinate system. Express x_i as a function of x_0. **[4]**

Answer:

$$(i) \quad \left.\frac{dy}{dx}\right|_{x_0,y_0} = -\frac{x_0}{y_0}$$

$$(ii) \quad x_i = \frac{25}{x_0}$$

(d) Determine the derivative of $\sec^{-1} x$. **[6]**

Answer:

$$\frac{df}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

[Part marks]

9. (a) Using $e^{i\theta} = \cos\theta + i\,\sin\theta$, express $\cos\theta$ and $\sin\theta$ in terms of exponentials. **[3]**

Answer:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

(b) Write down the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials. Using these and the relationships derived in (a), express $\sinh(ix)$ in terms of $\sin x$ and $\cosh(ix)$ in terms of $\cos x$. **[4]**

Answer:

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}), \quad \sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\cosh(ix) = \cos x, \quad \sinh(ix) = i\sin x$$

(c) Express $\cosh(x + i\,y)$ in the form $u + i\,v$, where x, y, u and v are all real, and show that **[7]**

$$|\cosh(x + i\,y)|^2 = \sinh^2 x + \cos^2 y\,.$$

Answer:

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$$

Derivations not shown as answers.

(d) Show that $y = (\cosh^{-1} x)^2$ satisfies the equation **[6]**

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2\,.$$

Answer: Derivations not shown as answers.

[Part marks]

10. (a) A linear wave propagating in the x-direction can be expressed using the complex function **[5]**

$$f(x,t) = Ae^{i(kx-\omega t)} \, .$$

Here, the amplitude A, the wave number k and the angular frequency ω are real constants whereas t denotes time. Show that this function satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 f}{\partial t^2} \, .$$

Answer: Derivations not shown as answers.

(b) If $V = xf(u)$ and $u = \frac{y}{x}$ show that **[8]**

$$x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} = 0 \, .$$

Answer: Derivations not shown as answers.

(c) A vector field is given by $\underline{F} = -y\underline{i} + x\underline{j} + x^2\underline{k}$. In addition, a path C is defined by a counter-clockwise circular helix around the positive z axis starting at $\underline{r}_A = 2\underline{i}$ and ending at $\underline{r}_B = 2\underline{i} + 4\underline{k}$ (see figure below). As it goes from $\underline{r}_A$ to $\underline{r}_B$, the helix makes two full cycles around the z axis. Write down a parametrisation of the path and determine the line integral **[7]**

$$I = \int_C \underline{F} \cdot d\underline{r} \, .$$

Answer: Parametrisation

$$\underline{r}(t) = (2\cos t)\underline{i} + (2\sin t)\underline{j} + (t/\pi)\underline{k} \quad \text{with} \quad t_A = 0,\, t_B = 4\pi$$

Integral

$$I = 8(2\pi + 1)$$

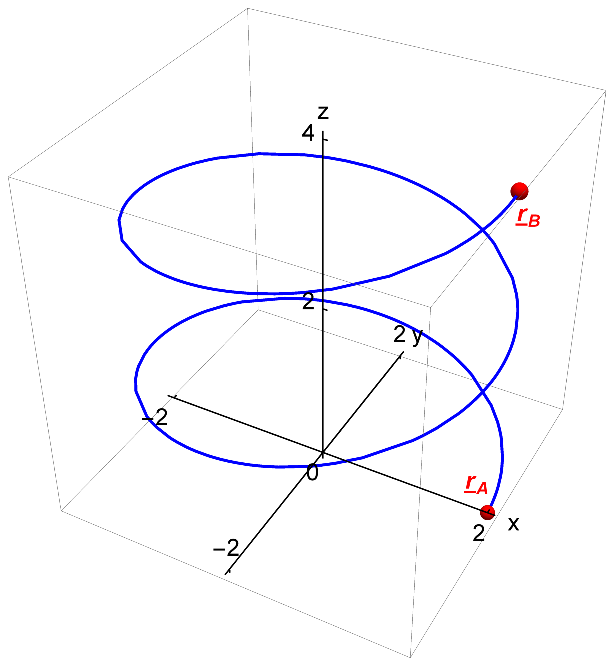

Figure 1: Helical path described in question 10 (c).

[Part marks]

11. (a) The distribution of speeds, v, for the molecules of an ideal gas is given by $(v \geq 0)$ **[6]**

$$f(v) = 4\pi \left[\frac{M}{2\pi RT}\right]^{\frac{3}{2}} v^2 \exp\left[-\frac{Mv^2}{2RT}\right] ,$$

where M, R and T are constants. Determine the average speed v_a.
Note: The distribution $f(v)$ is properly normalised, i.e. $\int_0^\infty f(v)dv = 1$.
Answer:

$$v_a = 4\sqrt{\frac{RT}{2\pi M}}$$

(b) A cylinder of radius r and height h is constructed such that its volume is minimised while satisfying the condition **[6]**

$$\left(\frac{\pi}{2r}\right)^2 + \left(\frac{\pi}{h}\right)^2 = \text{constant}\,.$$

Find the relationship between r and h for such a condition.
Answer:

$$\frac{h}{r} = 2\sqrt{2}$$

(c) Determine by any means the definite integral **[8]**

$$I(\alpha) = \int_0^\infty \frac{\ln(1+\alpha^2 x^2)}{1+x^2} dx \,,$$

where $\alpha \geq 0$ is a parameter.

Answer:

$$I(\alpha) = \pi \ln(\alpha + 1)$$