PHAS1247 Classical Mechanics: Problem-Solving Tutorial 3, 2010 Model Answers

Week 13 (22–26 November): Rotational motion and orbits

Section A: energy and equations of motion in polar coordinates

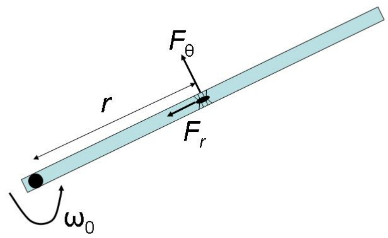
1. (Demonstrators may find they need to explain the rules of conkers to some students!)

   (i) The height of the conker above the centre of the circle is is $l\sin\theta$, so its gravitational potential energy (taking the zero at the centre—other choices also work) is
$$V = mgl\sin\theta.$$
   Its total energy is initially
$$E = \frac{1}{2}mv_0^2 + mgl,$$
   so the speed v at angle θ is given by the requirement
$$\frac{1}{2}mv^2 + mgl\sin\theta = E \quad \Rightarrow \quad v^2 = v_0^2 + 2gl(1-\sin\theta).$$

   (ii) The centripetal force required for the circular motion (radius l) is then
$$-\frac{mv^2}{l}\hat{\mathbf{r}} = -\left[\frac{mv_0^2}{l} + 2mg(1-\sin\theta)\right]\hat{\mathbf{r}}.$$

   (iii) This force has to be provided by the combination of the tension T in the string and the radial component of the weight. Hence
$$-(T + mg\sin\theta)\hat{\mathbf{r}} = -\frac{mv^2}{l}\hat{\mathbf{r}} = -\left[\frac{mv_0^2}{l} + 2mg(1-\sin\theta)\right]\hat{\mathbf{r}},$$
   so
$$T = \frac{mv_0^2}{l} + mg(2 - 3\sin\theta).$$
   The maximum tension is at the lowest point on the orbit when $\sin\theta = -1$; equating this to the breaking tension T_{break} we find
$$\frac{mv_{0,\,\mathrm{max}}^2}{l} + 5mg = T_{\mathrm{break}},$$
   so
$$v_{0,\,\mathrm{max}} = \sqrt{\frac{l}{m}(T_{\mathrm{break}} - 5mg)} = \sqrt{\frac{0.3\times(10 - 5\times 9.81\times 0.02)}{0.02}}\,\mathrm{ms}^{-1} = 11.6\,\mathrm{ms}^{-1}.$$
   The minimum tension is at the launch point when $\sin\theta = +1$, so the string will be always taut provided
$$T_{\mathrm{min}} = \frac{mv_0^2}{l} - mg > 0 \quad \Rightarrow \quad v_0 > v_{0\,\mathrm{min}} = \sqrt{gl} = \sqrt{9.81\times 0.3}\,\mathrm{ms}^{-1} = 1.72\,\mathrm{ms}^{-1}.$$

2. Diagram showing forces (which arise from the friction between the rod and the spider's feet):



The spider's velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} = v_0\hat{\mathbf{r}} + r\omega_0\hat{\theta}.$$

Hence the speed is

$$v = |\mathbf{v}| = \sqrt{v_0^2 + r^2\omega_0^2},$$

and the kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_0^2 + r^2\omega_0^2).$$

The equation of motion is

$$\begin{aligned}\mathbf{F} &= m\mathbf{a} = m[(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}] \\ &= -mr\omega_0^2\hat{\mathbf{r}} + 2mv_0\omega_0\hat{\theta}.\end{aligned}$$

Thus the power is

$$P = \mathbf{F}\cdot\mathbf{v} = m[-r\omega_0^2 v_0 + 2r\omega_0^2 v_0] = mr\omega_0^2 v_0.$$

Comparing the earlier result for the kinetic energy, we see that

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m(v_0^2 + r^2\omega_0^2)\right] = mr\dot{r}\omega_0^2 = mrv_0\omega_0^2,$$

which is equal to the power as expected.

Problem for general discussion For this motion,

$$\mathbf{F} = -\frac{K}{r^3}\hat{\mathbf{r}} = m\left(-r\dot{\theta}^2\right)\hat{\mathbf{r}}, \tag{1}$$

so

$$\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}. \tag{2}$$

The initial value of the angular momentum at $r = r_0$ is

$$L = mr^2\dot{\theta} = \sqrt{mK}, \tag{3}$$

which is independent of r_0 and so is the only possible value for the angular momentum for motion in a circle under an inverse cube force law.

The impulse does not change the angular momentum so it remains at a value $\sqrt{mK}$ for the subsequent motion. The radial equation of motion after the impulse is

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{K}{r^3}.$$

But $\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}$ so

$$\begin{aligned} m\left[\ddot{r} - r\left(\frac{1}{r^2}\sqrt{\frac{K}{m}}\right)^2\right] &= -\frac{K}{r^3}, \\ m\ddot{r} &= 0. \end{aligned}$$

Hence $\dot{r}$ is constant. Immediately after the outward impulse we have

$$\dot{r} = \frac{I}{m}.$$

Therefore at time t we have

$$r(t) = r_0 + \frac{It}{m}.$$

So the particle moves in an outward spiral with constant radial component of velocity. [This illustrates that an inverse cube law marks the boundary where circular orbits become unstable to small perturbations.]

Section B: orbits in different central forces

1. Since this is a central force depending only on the distance from the centre of force, it is automatically conservative. The potential energy associated with it can be found from the indefinite integral of the force with respect to r:

$$V(r) = -\int F(r)\,\mathrm{d}r = -\int (-Kr)\,\mathrm{d}r = \frac{Kr^2}{2} + C$$

where C is an integration constant. Taking $V = 0$ at $r = 0$ we find $V(r) = \frac{Kr^2}{2}$.

Writing $\mathbf{r}$ in terms of the Cartesian basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and remembering that they do not vary with time, we find

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = m(\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}) = -K(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$

Therefore

$$m\ddot{x} = -Kx; \qquad m\ddot{y} = -Ky.$$

These are the equations for independent simple harmonic motion in x and y, with an angular frequency $\omega^2 = K/m$. A possible pair of solutions are

$$x = a\cos(\omega t); \qquad y = b\sin(\omega t)$$

with a and b constant; this gives

$$\mathbf{r}(t) = a\cos(\omega t)\hat{\mathbf{i}} + b\sin(\omega t)\hat{\mathbf{j}}$$

as required.

Since the force is a conservative central force, both E and L will be conserved. We can therefore evaluate them at any time we like; let us choose $t = 0$, at which time the position and velocity are

$$\mathbf{r}(0) = a\hat{\mathbf{i}}; \qquad \mathbf{v} = \dot{\mathbf{r}}(0) = b\omega\hat{\mathbf{j}}.$$

Thus the energy E is the sum of kinetic and potential parts:

$$E = K + V = \frac{m\omega^2}{2}b^2 + \frac{K}{2}b^2 = \frac{m\omega^2}{2}(a^2 + b^2), \quad \text{since} \quad K = m\omega^2,$$

and the z-component of the angular momentum is equal to the product of the distance from the origin and the linear momentum, since the velocity and the displacement are perpendicular:

$$L_z = m\omega ab.$$

Hence the angular momentum is in the positive z-direction if both a and b are positive, as expected for an anto-clockwise rotation in the xy-plane; this can also be seen from writing

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = m\omega ab\hat{\mathbf{i}} \times \hat{\mathbf{j}} = m\omega ab\hat{\mathbf{k}}.$$

(Of course the students can also do this calculation at a general time if they wish; since $\mathbf{v}(t) = \omega[-a\sin(\omega t)\hat{\mathbf{i}} + b\cos(\omega t)\hat{\mathbf{j}}]$, we have

$$E(t) = \frac{m\omega^2}{2}[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)] + \frac{K}{2}[a^2\cos^2(\omega t) + b^2\sin^2(\omega t)] = \frac{m\omega^2}{2}(a^2 + b^2)$$

and

$$\begin{aligned}\mathbf{L}(t) &= m\omega[a\cos(\omega t)\hat{\mathbf{i}} + b\sin(\omega t)\hat{\mathbf{j}}] \times [-a\sin(\omega t)\hat{\mathbf{i}} + b\cos(\omega t)\hat{\mathbf{j}}] \\ &= m\omega ab[\cos^2(\omega t) + \sin^2(\omega t)]\hat{\mathbf{k}} = m\omega ab\hat{\mathbf{k}},\end{aligned}$$

so one obtains the same results as expected.)

For this orbit we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\omega t) + \sin^2(\omega t) = 1,$$

as required. This is the equation of an ellipse with semi-major axis equal to the larger of a and b and semi-minor axis equal to the smaller of the two. Hence—as stated but not proved in the lectures—the orbit is elliptical.

(The axes come out along the x and y directions in this example because we arbitrarily chose the x and y oscillations to be $\pi/2$ out of phase with one another. Students with some time on their hands might like to convince themselves that different choices of the relative phase would still have led to an ellipse, but with the axes rotated.)

2. The cross-sectional area of the beam corresponding to a small range of impact parameters from b ro $b + \delta b$ is approximately $2\pi b \delta b$. Hence the number of particles per unit time with this range of b is

$$\delta N = 2\pi F b \delta b.$$

In order to find the number of particles scattered into a range of angles we need to know how $\delta\phi$ and δb are related. We know that

$$b = \frac{K}{2E} \cot\left(\frac{\phi}{2}\right),$$

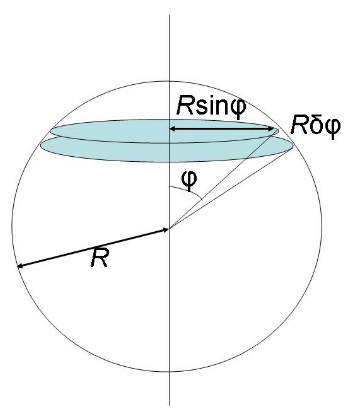
so

$$\delta b = \frac{\mathrm{d}b}{\mathrm{d}\phi}\delta\phi = -\frac{K}{2E}\frac{1}{2\sin^2(\phi/2)}\delta\phi.$$

Hence the number scattered through angles ϕ to $\phi + \delta\phi$ is

$$\delta N = \pi F \frac{|K|}{2E}\frac{\cot(\phi/2)}{\sin^2(\phi/2)}|\delta\phi| = \pi F \frac{|K|}{2E}\frac{\cos(\phi/2)}{\sin^3(\phi/2)}|\delta\phi|.$$

On the sphere of radius R, this range of angles ϕ correponds to a circular strip of circumference $2\pi R \sin\phi$ and width $R\delta\phi$—see diagram.



So its area is

$$\delta A = 2\pi R^2 \sin\phi\delta\phi = 4\pi R^2 \sin(\phi/2)\cos(\phi/2)\delta\phi,$$

as required. Hence

$$\delta\phi = \frac{1}{4\pi \sin(\phi/2)\cos(\phi/2)}\delta\Omega,$$

and therefore the number of particles scattered into this rangle of angles can be writen

$$\delta N = F\left(\frac{K}{2E}\right)^2 \frac{1}{4\sin^2(\phi/2)}\delta\Omega.$$

The result is proportional to K^2 and therefore is sensitive to the magnitude, but not the sign, of K; it therefore does not distinguish between attractive and repulsive forces of the same strength.