PHAS1247 Classical Mechanics
Problem Sheet 3—Model Answers
Submission deadline: Thursday 26 November 2009

1. The forces experienced by the mass are (i) its weight, which is of magnitude mg and vertically downwards; (ii) the normal reaction force from the surface, which (because the surface is smooth) must act normal to the surface. [**2 marks**]

   Let the magnitude of the reaction force be N. Hence the net (vector) force is

   $$\mathbf{F} = N\hat{\mathbf{r}} - mg\hat{\mathbf{k}} = (N - mg\sin\theta)\hat{\mathbf{r}} - mg\cos\theta\hat{\theta}.$$

   [**1 mark**] Since the radius is constant as long as the mass remains on the surface of the hemisphere, the radial equation of motion becomes

   $$N - mg\sin\theta = -ma\dot{\theta}^2,$$

   [**1 mark**] while the angular equation is

   $$-mg\cos\theta = ma\ddot{\theta}.$$

   [**1 mark**]

   If we take the zero of potential energy at ground level, the potential is

   $$V(\theta) = mag\sin\theta,$$

   while so long as the particle remains on the surface of the sphere its speed is simply $a\dot{\theta}$ and therefore its kinetic energy is

   $$K = \frac{1}{2}ma^2\dot{\theta}^2.$$

   Hence

   $$\dot{\theta}^2 = \frac{2g(1-\sin\theta)}{a}.$$

   (This result can also be obtained by multiplying the angular equation of motion by $\dot{\theta}$ and integrating.) [**2 marks**]

   Substituting in the radial equation of motion we find

   $$N - mg\sin\theta = -2mg(1-\sin\theta) \quad \Rightarrow \quad N = mg(3\sin\theta - 2).$$

   [**2 marks**] The mass remains on the surface of the sphere as long as N is positive, i.e. as long as $\sin\theta > \frac{2}{3}$, or its height $a\sin\theta > \frac{2a}{3}$. But once the height lost from the starting point reaches $a/3$, N drops to zero and the particle must fly off. [**2 marks**]

2. The angular momentum is

   $$L = mr_0^2\omega_0.$$

   [**2 marks**]

   The angular momentum of the particle about the hole is conserved, since the force is central. [**1 mark**]

Hence (a) the angular velocity ω is detemined by the requirement that

$$mr^2\omega = L = mr_0^2\omega_0 \quad \Rightarrow \quad \omega = \omega_0\frac{r_0^2}{r^2} = \omega_0\frac{r_0^2}{(r_0 - Vt)^2}.$$

[1 mark]

(b) The radial part of the velocity is $-V$, while the angular part is

$$\omega r = \omega_0\frac{r_0^2}{r} = \omega_0\frac{r_0^2}{(r_0 - Vt)}.$$

Thus, using plane polar coordinates, the velocity is

$$\mathbf{v} = -V\hat{\mathbf{r}} + \omega_0\frac{r_0^2}{(r_0 - Vt)}\hat{\theta}.$$

[2 marks]

The tension T can be determined from the radial acceleration of the particle:

$$-T = m(\ddot{r} - r\dot{\theta}^2) = -\omega^2 r = -m\omega_0^2\frac{r_0^4}{r^3} = -m\omega_0^2\frac{r_0^4}{(r_0 - Vt)^3}.$$

[2 marks]

The kinetic energy is found from the magnitude of the velocity vector:

$$K = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\left[V^2 + \omega_0^2\frac{r_0^4}{(r_0 - Vt)^2}\right].$$

[1 mark]

Thus

$$\frac{\mathrm{d}K}{\mathrm{d}t} = -m \times (-V)\omega_0^2\frac{r_0^4}{(r_0 - Vt)^3} = mV\omega_0^2\frac{r_0^4}{(r_0 - Vt)^3},$$

whereas the power developed when pulling the string is

$$TV = Vm\omega_0^2\frac{r_0^4}{(r_0 - Vt)^3}.$$

Hence the two are equal, as required. **[1 mark]**

3. (i) Let the mass of the satellite be m. From concservation of energy (kinetic plus potential) we have

$$\frac{1}{2}mv_A^2 - \frac{GM_\odot m}{R_1} = \frac{1}{2}mv_B^2 - \frac{GM_\odot m}{R_2},$$

or

$$v_A^2 - v_B^2 = 2GM_\odot\left[\frac{1}{R_1} - \frac{1}{R_2}\right].$$

[1 mark]

(ii) From conservation of angular momentum we have

$$mv_A R_1 = mv_B R_2 \quad \text{or} \quad v_B = v_A \frac{R_1}{R_2}.$$

[1 mark]

Eliminating v_B from these two equations gives us

$$v_A^2 \left[1 - \left(\frac{R_1}{R_2}\right)^2\right] = v_A^2 \left[\frac{R_2^2 - R_1^2}{R_2^2}\right] = 2GM_\odot \left[\frac{1}{R_1} - \frac{1}{R_2}\right] = 2GM_\odot \left[\frac{R_2 - R_1}{R_1 R_2}\right],$$

so

$$v_A^2 = 2GM_\odot \left[\frac{R_2}{R_1(R_1 + R_2)}\right].$$

The speed V_1 of the original circular orbit is given by

$$\frac{mV_1^2}{R_1} = \frac{GM_\odot m}{R_1^2} \quad \Rightarrow \quad V_1^2 = \frac{GM_\odot}{R_1}.$$

[1 mark] Therefore the velocity boost required is

$$\Delta v = v_A - V = \sqrt{2GM_\odot \left[\frac{R_2}{R_1(R_1 + R_2)}\right]} - \sqrt{\frac{GM_\odot}{R}} = \sqrt{\frac{GM_\odot}{R_1}} \left[\frac{2R_2}{(R_1 + R_2)} - 1\right],$$

as required. **[2 marks]**

In the lectures we saw that the time for a complete period of rotation of an elliptical orbit for a mass m in an central force $-|K|/r^2$ is

$$T = 2\pi \sqrt{\frac{ma^3}{|K|}},$$

where a is the semi-major axis of the ellipse. In this case we have

$$2a = R_1 + R_2 \quad \text{and} \quad K = -GM_\odot m.$$

The transfer time is the time required for one half of the elliptical orbit, so

$$T_{\text{transfer}} = \frac{T}{2} = \pi \sqrt{\frac{(R_1 + R_2)^3}{8GM_\odot}},$$

as required. **[2 marks]**

Using the data given we have

$$T_{\text{transfer}} = \pi \sqrt{\frac{(R_1 + R_2)^3}{8GM_\odot}} = 2.24 \times 10^7\,\text{s} \quad \text{or} \quad 259\,\text{days},$$

[1 mark] and

$$\Delta v = \sqrt{\frac{GM_\odot}{R_1}} \left[\frac{2R_2}{(R_1 + R_2)} - 1\right] = 2.93 \times 10^3\,\text{ms}^{-1}.$$

[1 mark]

The eccentricity of the orbit can be found by noting that R_1 is the minimum radius and R_2 the maximum distance from the focus, so applying the defining equation of an ellipse as used in the lectures

$$r(1 + e\cos\theta) = h,$$

we have

$$R_1(1+e) = R_2(1-e) \quad \Rightarrow \quad e(R_1+R_2) = R_2 - R_1 \quad \Rightarrow \quad e = \frac{R_2 - R_1}{R_2 + R_1}.$$

In this case we have

$$e = \frac{2.28 - 1.50}{2.28 + 1.50} = 0.206.$$

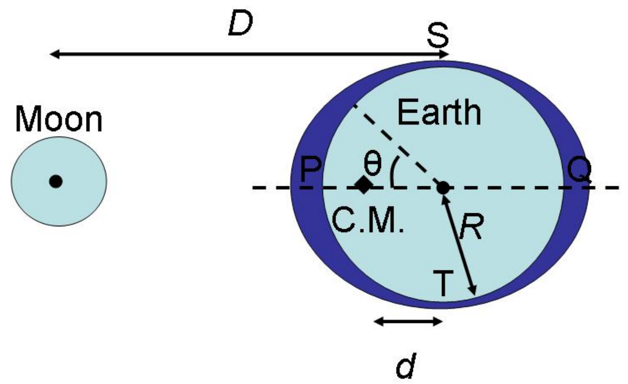
[1 mark]

4. **Problem for tutorial discussion—no marks.** The quantitative argument is quite complex but the qualitative point is simple—there are two tides each day because of the centrifugal force resulting from rotation of the the Earth and Moon about their common centre of mass (which is actually within the Earth, as shown in the figure and as calculated in the second problem-solving tutorial). The distance from the centre of the Earth the common centre of mass is

$$d = \frac{M_{\mathrm{Moon}}}{M_{\mathrm{Moon}} + M_{\mathrm{Earth}}} D,$$

where the Earth-Moon distance is D.

The Moon's gravity attracts a mass of water on the side nearest to the Moon; the cetrifugal force produces a second accumulation of water on the far side. Equivelantly, if you don't like centrifugal force, the gravity of the Moon pulls the water closest to it away from the Earth, and then pulls the Earth away from the water on the far side. Each 'bulge' of water passes a given point once each day as the Earth rotates on its own axis, making a total of two high tides per day. (There are places such as the Solent with a larger number of tides but these are the result of complicated local flow patterns in the ocean.)



The rotation occurs about an axis through the centre of mass (marked 'C.M.' in the diagram). Suppose this axis is perpendicular to the plane of the page. The angular frequency Ω of rotation is given by the requirement that the gravitational attraction of the Earth and Moon exactly balances the centrifugal force (or equivalently, supplies the necessary centripetal force):

$$\mu\Omega^2 D = M_{\text{Earth}}\Omega^2 d = \frac{GM_{\text{Earth}}M_{\text{Moon}}}{D^2} \quad \Rightarrow \quad \Omega^2 = \frac{GM_{\text{Moon}}d}{D^2}.$$

I give two versions of the argument from here on—one based on forces, one based on potentials.

Argument from forces. The Moon's gravitational field in the neighbourhood of the Earth is not constant. Its magnitude at a point on the surface of the Earth making an angle θ can be approximated as

$$|\mathbf{g}_{\text{Moon}}| = \frac{GM_{\text{Moon}}}{D^2 + R^2 - 2DR\cos\theta}.$$

Its direction varies in a complicated way and we will evaluate it just at four points in the diagram (P, Q, S, T).

At P ($\theta = 0$) the magnitude of the Moon's gravitational field is

$$g_P = \frac{GM_{\text{Moon}}}{(D-R)^2} \approx \frac{GM_{\text{Moon}}}{D^2}(1 + 2R/D)$$

where R is the Earth's radius and the centrifugal acceleration is

$$\Omega^2(R-d)$$

(acting to the left). Hence the net force per unit mass (also to the left) is

$$\frac{GM_{\text{Moon}}}{D^2}(1 + 2R/D) + \Omega^2(R-d) = \frac{2GM_{\text{Moon}}R}{D^3}.$$

Similarly at Q ($\theta = \pi$)the magnitude of the Moon's gravitational field is

$$g_Q = \frac{GM_{\rm Moon}}{(D+R)^2} \approx \frac{GM_{\rm Moon}}{D^2}(1 - 2R/D)$$

and the centrifugal acceleration is

$$\Omega^2(R+d)$$

(acting to the right). Hence the net force per unit mass (also to the right) is

$$\frac{GM_{\rm Moon}}{D^2}(1+2R/D) + \Omega^2(R+d) - \frac{GM_{\rm Moon}}{D^2}(1-2R/D) = \frac{2GM_{\rm Moon}R}{D^3}.$$

At S and T, on the other hand ($\theta = \pm\pi/2$), the magnitude of the Moon's field is almost exactly

$$g_{S,T} = \frac{GM_{\rm Moon}}{D^2 + R^2} \approx \frac{GM_{\rm Moon}}{D^2}$$

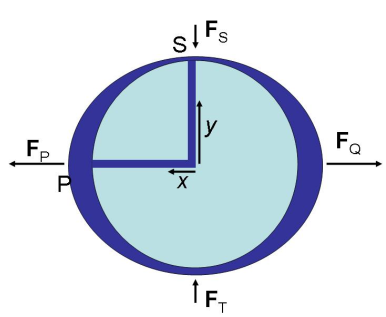
while the centrifugal acceleration is

$$\Omega^2 d$$

to the right. This cancels the horizontal (in the diagram) component of the gravitational field leaving a vertical component (towards the centre of the earth in each case) of magnitude

$$\frac{R}{D}\frac{GM_{\rm Moon}}{D^2}.$$

This pattern of forces (inward at $\theta = \pm\pi/2$, outward at $\theta = 0, \pi$) explains the shape of the oceans shown in the diagram. In order to find the corresponding difference in height we need to find a way of connecting the columns of water in different directions (the pressures of the water at the sea bed are **not** the same in all directions because the Moon's field produces pressure gradients with a component parallel to the sea bed). A nice thought experiment to do this was devised by Newton, who imagined connecting the points at S and P by thin tunnels drilled at right angles to the centre of the Earth. The total force (including the Earth's own gravity) exerted on each column of water from its surface along each tunnel to the centre of the Earth must be equal if the pressures are to balance.



The net fields from the centrifugal force and the Moon's gravity at distances x towards P and y towards S (see diagram) can be calculated by substituting x and y respectively in place of R in the formulae we have already derived. Hence if the heights of the water at S and P are h_S and h_P respectively, we have

$$\int_0^{h_P} [g_{\text{Earth}}(x) - \frac{2GM_{\text{Moon}}x}{D^3}]\,\mathrm{d}x = \int_0^{h_S} [g_{\text{Earth}}(y) + \frac{GM_{\text{Moon}}y}{D^3}]\,\mathrm{d}y.$$

The parts of the integral involving g_{Earth} differ only around the upper limit, where g_{Earth} is approximately constant, while the other parts of the integral are dominated by the contribution from the range $0 \le x, y \le R$, so we can write

$$g_{\text{Earth}}(R)[h_P - h_S] \approx \int_0^R \frac{3GM_{\text{Moon}}x}{D^3}\,\mathrm{d}x = \frac{3GM_{\text{Moon}}R^2}{2D^3}.$$

Recalling that

$$g_{\text{Earth}}(R) = \frac{GM_{\text{Earth}}}{R^2}$$

we finally find that the height difference is

$$h_P - h_S = \frac{3}{2}\frac{M_{\text{Moon}}}{M_{\text{Earth}}}\frac{R^3}{D^3}R.$$

Argument from potentials. It is convenient to work with the gravitational potential $\Phi(\mathbf{r})$, defined as the potential energy that a unit test mass would have if placed at the point $\mathbf{r}$ due to the presence of the other bodies. The gravitational potential from the Moon at a point on the surface of the Earth making an angle θ to the Moon (see diagram) is

$$\begin{aligned}
\Phi_{\text{Moon}} &= -\frac{GM_{\text{Moon}}}{\sqrt{D^2 + R^2 - 2DR\cos\theta}} \\
&= -\frac{GM_{\text{Moon}}}{D}\left[1 - 2\frac{R\cos\theta}{D} + \frac{R^2}{D^2}\right]^{-1/2} \\
&= -\frac{GM_{\text{Moon}}}{D}\left[1 - \frac{1}{2}\left(-2\frac{R\cos\theta}{D} + \frac{R^2}{D^2}\right) + \frac{3}{8}\left(-2\frac{R\cos\theta}{D} + \frac{R^2}{D^2}\right)^2 + \ldots\right] \\
&= -GM_{\text{Moon}}\left[\frac{1}{D} + \frac{R}{D^2}\cos\theta + \frac{R^2}{2D^3}\left(3\cos^2\theta - 1\right) + \mathrm{O}\left(\frac{R^3}{D^4}\right)\right].
\end{aligned}$$

Similarly the centrifugal potential can be expanded as

$$\Phi_{\text{cent}} = \frac{1}{2}\Omega^2(R^2 + d^2 - 2Rd\cos\theta).$$

(Note that this is not the same as the form of centrifugal potential introduced in the lectures when we were discussing planetary motion—the difference is that here the angular velocity of rotation is being held constant, there the angular momentum was held constant.)

Recalling that

$$M_{\text{Earth}}\Omega^2 d = \frac{GM_{\text{Earth}}M_{\text{Moon}}}{D^2},$$

we see that the terms in $\cos\theta$ in these two expressions cancel out (this reflects the fact that the total gravitational attraction of the centre of mass to the Moon is exactly balanced by the total centrifugal force on the Earth). Hence the total potential due the Moon and the centrifugal term is

$$\Phi_{\mathrm{Moon}} + \Phi_{\mathrm{cent}} = \mathrm{const.} - GM_{\mathrm{Moon}}\frac{3R^2}{2D^3}\cos^2\theta.$$

Now we include the Earth's gravitational field, which is approximately constant over the surface of the Earth:

$$|g_{\mathrm{Earth}}| = \frac{GM_{\mathrm{Earth}}}{R^2}.$$

If the water has flowed to a position of equilibrium in the rotating frame of the Earth, then the surface of the fluid will be an equipotential (i.e. a surface of constant potential); if this were not so then the water could lower its energy further by flowing from the high-potential regions to the low-potential regions. The height of the surface must therefore contain a θ-dependent component $h(\theta)$ such that

$$h(\theta)|g_{\mathrm{Earth}}| - GM_{\mathrm{Moon}}\frac{3R^2}{2D^3}\cos^2\theta = 0$$

or

$$h(\theta) = \frac{3}{2}\frac{M_{\mathrm{Moon}}}{M_{\mathrm{Earth}}}\left(\frac{R}{D}\right)^3 R\cos^2\theta,$$

giving a difference between high and low tide of

$$\Delta h = \frac{3}{2}\frac{M_{\mathrm{Moon}}}{M_{\mathrm{Earth}}}\left(\frac{R}{D}\right)^3 R$$

with two high tides per day (as a given point on the Earth passes through $\theta = 0$ and $\theta = \pi$). This is the same result we obtained from considering the forces.

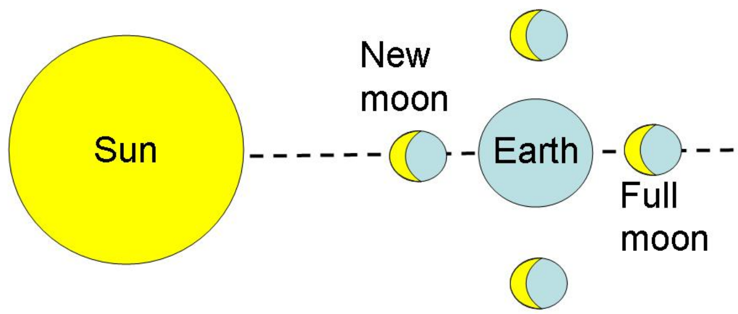
The estimate produces

$$\Delta h = 0.55\,\mathrm{m},$$

which sounds too small—however, remember that the tides we see near coasts may be strongly amplified by local geography. There is also some further amplification by effects we have neglected such as the gravitational self-interaction of the oceans. In reality also the Ocean takes a finite time to respond as the Moon rotates and so the bulge of water lags somewhat. However satellite measurments of tides in mid-ocean confirm that the value of Δh is approximtely correct.

Note the importance of the centrifugal force in this calculation—its cancellation of the term in $\cos\theta$ from the gravitational potential not only gave us two tides per day, but also prevented the tides from being larger in magnitude by a factor $\frac{2D}{3R} \approx 40$.

The largest tides are obtained at full moon and new moon because at those times the Sun is aligned on the same axis as the Moon and therefore its gravitational effect enhances that of the Moon.



By a remarkable coincidence of masses and lengthscales the tidal contribution of the Sun is of a similar order, giving

$$\Delta h_\odot = \frac{3}{2}\frac{M_\odot}{M_{\mathrm{Earth}}}\left(\frac{R}{D_\odot}\right)^3 R = 0.24\,\mathrm{m},$$

where $D_\odot$ is the Earth-Sun distance—hence the noticeable distance between spring tides and 'neap' tides (when the Sun and Moon are oriented at right angles relative to the Earth).