PHAS2423 - Problem Based Learning - Numerical Methods - Partial Differential Equations - Problems and Solutions

(1) An differential equation is given in the form

$$\frac{\partial y(x)}{\partial x} = f\left[(x, y(x)\right],$$

where function $f\left[(x, y(x)\right]$ is known. To solve it numerically, one can use a two-step finite-difference scheme:

$$y_{i+1} = \alpha y_i + \beta y_{i-1} + (\mu f_i + \nu f_{i-1})\,\Delta x,$$

where $\Delta x = x_i - x_{i-1}$, $y_i = y(x_i)$, $f_i = f(x_i, y_i)$, and α, β, μ and ν are constants.

(a) Show that for α=1, β=0, μ=3/2 and ν=–1/2, this scheme gives errors of the order of $(\Delta x)^3$.
(b) Find the values of α, β, μ and ν that will give the greatest accuracy.

Solution.
(a) According to this scheme,

$$y_{i+1} = y_i + \left(\frac{3}{2}f_i - \frac{1}{2}f_{i-1}\right)\Delta x,$$

Expand f_{i-1} near x_i in the Taylor series:

$$f_{i-1} = f_i - f_i'\Delta x + \frac{1}{2!}f_i''(\Delta x)^2 - \frac{1}{3!}f_i'''(\Delta x)^3 + ...$$

and use $f = y'$:

$$f_{i-1} = y_i' - y_i''\Delta x + \frac{1}{2!}y_i^{(3)}(\Delta x)^2 - \frac{1}{3!}y_i^{(4)}(\Delta x)^3 + ...$$

Thus, according to this scheme,

$$y_{i+1} = y_i + \left[\frac{3}{2}y_i' - \frac{1}{2}\left(y_i' - y_i''\Delta x + \frac{1}{2!}y_i^{(3)}(\Delta x)^2 - \frac{1}{3!}y_i^{(4)}(\Delta x)^3 + ...\right)\right]\Delta x,$$

which is

$$y_{i+1} = y_i + y_i'\Delta x + \frac{1}{2}y_i''(\Delta x)^2 - \frac{1}{4}y_i^{(3)}(\Delta x)^3 + \frac{1}{12}y_i^{(4)}(\Delta x)^4 + ...$$

Compare this with the Taylor series for $y(x)$ near x_i:

$$y_{i+1} = y_i + y_i'\Delta x + \frac{1}{2}y_i''(\Delta x)^2 + \frac{1}{3!}y_i^{(3)}(\Delta x)^3 + \frac{1}{4!}y_i^{(4)}(\Delta x)^4 + ...$$

The difference between these two expression for y_{i+1} is

$$\left(\frac{1}{3!}+\frac{1}{4}\right)y_i^{(3)}(\Delta x)^3 + \left(\frac{1}{4!}-\frac{1}{12}\right)y_i^{(4)}(\Delta x)^4 + ... = \frac{5}{12}y_i^{(3)}(\Delta x)^3 - \frac{1}{24}y_i^{(4)}(\Delta x)^4 + ...$$

Thus, the highest order contribution to the error is of the order of $(\Delta x)^3$.

(b) For the best accuracy choose the coefficients so as

$$y_{i+1} = \alpha y_i + \beta y_{i-1} + (\mu f_i + \nu f_{i-1})\,\Delta x$$

is as close to the Taylor expansion for y_{i+1} as possible. Using a shorter notation for $x_{i+1} - x_i = \Delta$ and taking into account

$$y_{i-1} = y_i - y_i'\Delta + \frac{1}{2}y_i''\Delta^2 - \frac{1}{3!}y_i^{(3)}\Delta^3 + ...$$

and

$$f_{i-1} = y_i' - y_i''\Delta + \frac{1}{2}y_i^{(3)}\Delta^2 - \frac{1}{3!}y_i^{(4)}\Delta^3 + ...,$$

we have $y_{i+1} =$

$$\alpha y_i + \beta\left(y_i - y_i'\Delta + \frac{1}{2}y_i''\Delta^2 - \frac{1}{3!}y_i'''\Delta^3 + ...\right) + \mu y_i'\Delta + \nu\Delta\left(y_i' - y_i''\Delta + \frac{1}{2}y_i^{(3)}\Delta^2 - \frac{1}{3!}y_i^{(4)}\Delta^3 + ...\right).$$

Write down the terms with the same power of Δ^n and equate them to the corresponding terms in the Taylor series:

$$\begin{array}{ll} n=0: & (\alpha+\beta)\,y_i = y_i \\ n=1: & (-\beta+\mu+\nu)\,y_i'\Delta = y_i'\Delta \\ n=2: & \left(\frac{1}{2}\beta-\nu\right)y_i''\Delta^2 = \frac{1}{2}y_i''\Delta^2 \\ n=3: & \left(-\frac{1}{3!}\beta+\frac{1}{2}\nu\right)y_i^{(3)}\Delta^3 = \frac{1}{3!}y_i^{(3)}\Delta^3 \end{array}.$$

Thus, four equations with respect to the coefficients are:

$$\alpha+\beta = 1 \qquad -\beta+\mu+\nu = 1 \qquad \frac{1}{2}\beta - \nu = \frac{1}{2} \qquad -\frac{1}{6}\beta + \frac{1}{2}\nu = \frac{1}{6}.$$

Solving them yields:

$$\alpha = -4 \qquad \beta = 5 \qquad \mu = 4 \qquad \nu = 2.$$

Thus,

$$y_{i+1} = -4y_i + 5y_{i-1} + (4f_i + 2f_{i-1})\,\Delta x.$$

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(2) Show that if

$$u(x, y) = f(x + \lambda y) + x \cdot g(x + \lambda y),$$

where f and g are some functions of $x + \lambda y$ and a is a constant, then $u(x, y)$ can be also represented as

$$u(x, y) = v(x + \lambda y) + y \cdot w(x + \lambda y),$$

where v and w are some other functions of $x + \lambda y$.

Solution

Introduce a new variable $p = x + \lambda y$. Then, $x = p - \lambda y$ and

$$u(x, y) = f(p) + (p - \lambda y) \cdot g(p) = f(p) + p \cdot g(p) - (\lambda y) \cdot g(p).$$

One can put

$$v(p) = f(p) + p \cdot g(p)$$

and

$$w(p) = (-\lambda) \cdot g(p).$$

Thus,

$$u(x, y) = v(p) + yw(p) = v(x + \lambda y) + yw(x + \lambda y).$$

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(3) For the partial differential equation

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy(2y^2 - x^2)$$

(1) find the general solution of the corresponding homogeneous equation;

(2) find a particular solution of the inhomogeneous equation (*Hint: use a probe solution of the form* $u(x, y) = Ax^n y^m$);

(3) find the general solution $u(x, y)$ of the inhomogeneous equation for $x \geq 0$, subject to the boundary condition $u(x, 0) = x$.

Solution

(1) To solve the homogeneous equation, integrate

$$\frac{dx}{2y} = -\frac{dy}{x},$$

i.e.,

$$x\,dx = -2y\,dy.$$

The integration gives

$$\frac{1}{2}x^2 = -y^2 + C/2,$$

where C is an arbitrary constant. Identify the parameter p with C, i.e., $p = x^2 + 2y^2$. Hence, any function f that depends on p, $f(p) = f(x^2 + 2y^2)$, is a solution of the homogeneous equation.

(2) Substitute the suggested form of the probe solution into the equation:

$$2yAnx^{n-1}y^m - xmAx^n y^{m-1} = 2xy^3 - x^3y.$$

Rearrange:

$$A(2nx^{n-1}y^{m+1} - mx^{n+1}y^{m-1}) = 2xy^3 - x^3y.$$

Compare the powers of x and y and the coefficients in the left hand side (LHS) and right hand side (RHS) of the equation. Observe that the LHS = RHS if the following equations are satisfied simultaneously:

$$2An = 2 \qquad Am = 1$$

for the coefficients and

$$n - 1 = 1 \qquad m + 1 = 3 \qquad n + 1 = 3 \qquad m - 1 = 1$$

for the powers of x and y.

These equations are satisfied if $n = m = 2$ and $A = 1/2$. Thus, the general solution is

$$u(x, y) = f(x^2 + 2y^2) + \frac{1}{2}x^2y^2$$

(3) At the boundary

$$u(x, 0) = f(x^2 + 0) + 0 = x = \sqrt{x^2 + 0} = \sqrt{p},$$

i.e.,

$$f(p) = p^{1/2}.$$

Therefore, the general solution of the inhomogeneous equation, which satisfies the boundary condition is

$$u(x, y) = (x^2 + 2y^2)^{1/2} + \frac{1}{2}x^2y^2.$$

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(4) Convolution of functions $g(x)$ and $f(x)$ is denoted as $g * f$, where

$$g * f = \int_0^r g(r - t)f(t)\, dt.$$

Show that $g * f = f * g$.

Solutions

Introduce a new variable: $s = r - t$, $ds = -dt$. Then, work it out:

$$g * f = \int g(r - t)f(t)\, dt = \int_r^0 g(s)f(r - s)\,(-ds) = \int_0^r g(s)f(r - s)\, ds = f * g.$$

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(5) Given that the inverse Laplace transform of

$$\frac{1}{(p + a)(p + b)} \qquad \text{is} \qquad \frac{e^{-ax} - e^{-bx}}{b - a},$$

use the method of convolution to solve the differential equation

$$2y''(x) + 2y'(x) - 12y(x) = x^2,$$

where $y(0) = y'(0) = 0$.

Solution

Take the Laplace transform of both parts of the equation:

$$L[y''(x) + y'(x) - 6y(x)] = L[\frac{x^2}{2}].$$

Taking into account the boundary conditions for $y(x)$ and $y'(x)$, obtain:

$$L[y'] = -y(0) + pL[y(x)] = pL[y(x)] = pY$$

and

$$L[y''] = -y'(0) - py(0) + p^2L[y(x)] = p^2Y,$$

where Y is a short notation for $L[y(x)]$. Hence, the Laplace transform of the original ODE is

$$p^2Y + pY - 6Y = \frac{x^2}{2}$$

and, therefore,

$$Y = \frac{1}{p^2+p-6} L[\frac{x^2}{2}] = \frac{1}{(p-2)(p+3)} L[\frac{x^2}{2}].$$

Since the inverse Laplace transform of

$$\frac{1}{(p+a)(p+b)} \quad \text{is} \quad \frac{e^{-at}-e^{-bt}}{b-a}$$

and, in our case, $a=-2$ and $b=3$, the function Y can be represented as

$$Y = L\left[\frac{e^{2x}-e^{-3x}}{5}\right] \cdot L\left[\frac{x^2}{2}\right] = L[f(x)]L[g(x)].$$

Use the convolution theorem to find $y(x)$:

$$y(x) = \int_0^x g(t)f(x-t)dt = \frac{1}{10}\int_0^x t^2(e^{2(x-t)} - e^{-3(x-t)})dt = \frac{e^{2x}}{10}\int_0^x t^2e^{-2t}dt - \frac{e^{-3x}}{10}\int_0^x t^2e^{3t}dt.$$

Calculate the first integral:

$$I_1 = \int_0^x t^2e^{-2t}dt = -\frac{t^2e^{-2t}}{2}|_0^x + \int_0^x te^{-2t}dt = -\frac{x^2e^{-2x}}{2} + \int_0^x te^{-2t}dt.$$

$$\int_0^x te^{-2t}dt = -\frac{te^{-2t}}{2}|_0^x + \frac{1}{2}\int_0^x e^{-2t}dt = -\frac{xe^{-2x}}{2} + \frac{1}{2}\int_0^x e^{-2t}dt.$$

$$\frac{1}{2}\int_0^x e^{-2t}dt = -\frac{e^{-2t}}{4}|_0^x = -\frac{1}{4}e^{-2x} + \frac{1}{4}.$$

Thus,

$$I_1 = -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + \frac{1}{4} = -\frac{e^{-2x}}{2}\left(x^2 + x + \frac{1}{2}\right) + \frac{1}{4}.$$

Calculate the second integral:

$$I_2 = \int_0^x t^2e^{3t}dt = \frac{t^2e^{3t}}{3}|_0^x - \frac{2}{3}\int_0^x te^{3t}dt = \frac{x^2e^{3x}}{3} - \frac{2}{3}\int_0^x te^{3t}dt.$$

$$\int_0^x te^{3t}dt = \frac{te^{3t}}{3}|_0^x - \frac{1}{3}\int_0^x e^{3t}dt = \frac{xe^{3x}}{3} - \frac{1}{3}\int_0^x e^{3t}dt.$$

$$\frac{1}{3}\int_0^x e^{3t}dt = \frac{e^{3t}}{3}|_0^x = \frac{1}{9}e^{3x} - \frac{1}{9}.$$

Thus,

$$I_2 = \frac{x^2e^{3x}}{3} - \frac{2}{3}\cdot\frac{xe^{3x}}{3} + \frac{2}{3}\cdot\frac{e^{3x}}{9} - \frac{2}{3}\cdot\frac{1}{9} = \frac{e^{3x}}{3}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) - \frac{2}{27}.$$

Finally,

$$y(x) = \frac{e^{2x}I_1 - e^{-3x}I_2}{10} = -\frac{1}{20}\left(x^2 + x + \frac{1}{2}\right) + \frac{e^{2x}}{40} - \frac{1}{30}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) + \frac{2e^{-3x}}{270}$$

Collect the coefficients in front of x^2, x^1, and x^0:

$$y(x) = \frac{1}{40} \cdot e^{2x} + \frac{2}{270} \cdot e^{-3x} - \frac{5}{60} \cdot x^2 - \frac{5}{180} \cdot x - \frac{1}{40} - \frac{2}{270}.$$

Not required (unless stated otherwise) but recommended.
To check that this is the correct solution, calculate $y'(x)$ and $y''(x)$:

$$y'(x) = \frac{1}{20} \cdot e^{2x} - \frac{2}{90} \cdot e^{-3x} - \frac{5}{30} \cdot x - \frac{5}{180},$$
$$y''(x) = \frac{1}{10} \cdot e^{2x} + \frac{2}{30} \cdot e^{-3x} - \frac{5}{30}.$$

Examine the boundary conditions for $y(0)$ and $y'(0)$:

$$y(0) = \frac{1}{40} + \frac{2}{270} - \frac{1}{40} - \frac{2}{270} = 0,$$
$$y'(x) = \frac{1}{20} - \frac{2}{90} - \frac{5}{180} = \frac{9}{180} - \frac{4}{180} - \frac{5}{180} = 0,$$

i.e., the boundary conditions are satisfied.

Check whether $y(x)$ is the solution of the equation. Calculate the coefficients in front of all linearly independent functions in $2y'' + 2y' - 12y$:

– for e^{2x}:

$$2 \cdot \frac{1}{10} + 2 \cdot \frac{1}{20} - 12 \cdot \frac{1}{40} = 2 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} - 3 \cdot \frac{1}{10} = 0;$$

– for e^{-3x}:

$$2 \cdot \frac{2}{30} - 2 \cdot \frac{2}{90} - 12 \cdot \frac{2}{270} = \frac{12}{90} - \frac{4}{90} - 4 \cdot \frac{2}{90} = 0;$$

– for x^2:

$$+12 \cdot \frac{5}{60} = 1;$$

– for x^1:

$$-2 \cdot \frac{5}{30} + 12 \cdot \frac{5}{180} = -\frac{10}{30} + 2 \cdot \frac{5}{30} = 0;$$

– for x^0:

$$-2 \cdot \frac{5}{30} - 2 \cdot \frac{5}{180} + 12 \cdot \frac{1}{40} + 12 \cdot \frac{2}{270} = -\frac{30}{90} - \frac{5}{90} + 3 \cdot \frac{1}{10} + 4 \cdot \frac{2}{90} = -\frac{27}{90} + \frac{3}{10} = 0.$$

Hence, $y(x)$ satisfies the ODE.