ODE

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1) a) $$y'' + 4y' + 4y = e^{-2x}$$

$$L[y'' + 4y' + 4y] = L[e^{-2x}]$$

$$L[y''] + L[4y'] + L[4y] = L[e^{-2x}]$$

$$p^2L(y) - py(0) - y'(0) + 4pL(y) - 4y(0) + 4L(y) = L(e^{-2x})$$

$$p^2Y - py(0) - y'(0) + 4pY - 4y(0) + 4Y = L[e^{-2x}]$$

$$p^2Y - 4p - 4 + 4pY - 16 + 4Y = L[e^{-2x}]$$

$$p^2Y + 4pY + 4Y - 4p - 20 = L[e^{-2x}]$$

$$Y(p^2 + 4p + 4) - 4p - 20 = \frac{1}{p+2}$$

$$(p+2)^2Y = \frac{1}{p+2} + 4p + 20$$

$$Y = \frac{1}{(p+2)^3} + \frac{4p}{(p+2)^2} + \frac{20}{(p+2)^2}$$

$$\frac{p}{(p+2)^2} = \frac{p+2-2}{(p+2)^2} = \frac{1}{p+2} - \frac{2}{(p+2)^2}$$

$$Y = \frac{1}{(p+2)^3} + \frac{4}{p+2} - \frac{8}{(p+2)^2} + \frac{20}{(p+2)^2}$$

$$= \frac{1}{(p+2)^3} + \frac{12}{(p+2)^2} + \frac{4}{p+2}$$

$$= \frac{1}{(p+2)^3} + \frac{12}{(p+2)^2} + \frac{4}{p+2}$$

$$\mathcal{L}(y_1) = \frac{1}{(p+2)^3}$$

$$\mathcal{L}(y_2) = \frac{12}{(p+2)^2}$$

$$\mathcal{L}(y_3) = \frac{4}{p+2}$$

$$\mathcal{L}(x^n e^{ax}) = \frac{n!}{(s-a)^{n+1}}$$

y_1 : $n+1 = 3 \to n = 2$

$\to y_1 = \frac{1}{2} x^2 e^{-2x}$

y_2: $n+1 = 2 \to n = 1$

$\to y_2 = 12 x e^{-2x}$

y_3: $n+1 = 1 \to n = 0$

$\to y_3 = 4 e^{-2x}$

$\to y = y_1 + y_2 + y_3$

$= \left(\frac{1}{2}x^2 + 12x + 4\right) e^{-2x}$

b) ① $f'(x) + g'(x) - 3g(x) = 0$

② $f''(x) + g'(x) = 0$

① : $-f(0) + pF - g(0) + pG - 3G = 0$

① : $-f(0) + pF - g(0) + pG - 3G = 0$

② $-f'(0) - pf(0) + p^2 F - g(0) + pG = 0$

$f(0) = f'(0) = 0$
$g(0) = \frac{4}{3}$

$\rightarrow$ ① $0 + pF - \frac{4}{3} + pG - 3G = 0$

② $0 - 0 + p^2 F - \frac{4}{3} + pG = 0$

①: $pF + pG - 3G = \frac{4}{3}$

② $p^2 F + pG = \frac{4}{3}$

$\rightarrow$ ① = ②

$pF + pG - 3G = p^2 F + pG$

$\rightarrow$ $-3G = p^2 F - pF$

$\rightarrow$ $G = \frac{1}{3}(p - p^2) F$

③ $G = \frac{1}{3} p(1-p) F$

③ $\Rightarrow$ ②

$p^2 F + p\left(\frac{1}{3} p(1-p) F\right) = \frac{4}{3}$

$p^2 F + \frac{1}{3} p^2 F - \frac{1}{3} p^3 F = \frac{4}{3}$

$$F\left(\frac{4}{3}p^2 - \frac{1}{3}p^3\right) = \frac{4}{3}$$

$$F = \frac{4}{4p^2 - p^3} = \frac{4}{p^2(4-p)}$$

$$G = \frac{1}{3}p(1-p)F = \frac{4}{3}\frac{1-p}{p(4-p)}$$

F: $$\frac{4}{p^2(4-p)} = \frac{A}{4-p} + \frac{Bp+C}{p^2}$$

$$4 = p^2 A + (4-p)(Bp+C)$$

$p=4$:

$$4 = 16A \to A = \frac{1}{4}$$

$p=0$

$$4 = 4C \to C=1$$

$$4 = \frac{1}{4}p^2 + (4-p)(Bp+1)$$

$p=1$

$$4 = \frac{1}{4} + 3(B+1)$$

$$16 = 1 + 12B + 12$$

$$12B = 3$$

$$B = \frac{1}{4}$$

$$\to F = \frac{1}{4(4-p)} + \frac{\frac{1}{4}p+1}{p^2}$$

$$= \frac{1}{4(4-p)} + \frac{1}{4p} + \frac{1}{p^2}$$

$$G = \frac{4}{3} \frac{1-p}{p(4-p)}$$

$$\rightarrow \frac{4(1-p)}{3.p(4-p)} = \frac{A}{4-p} + \frac{B}{3p}$$

$$4(1-p) = 3pA + (4-p)B$$

$p = 0$

$4 = 4B \rightarrow B = 1$

$p = 4$

$-12 = 12A$

$\rightarrow A = -1$

$$G = \frac{1}{p-4} + \frac{1}{3p}$$

$$F = \frac{1}{4(4-p)} + \frac{1}{4p} + \frac{1}{p^2}$$

$$F: \mathcal{L}[x^n e^{ax}] = \frac{n!}{(p-a)^{n+1}}$$

$\mathcal{L}^{-1}\left[\frac{1}{4(4-p)}\right]$: $n+1 = 1 \rightarrow n = 0$

$f_1 = -\frac{1}{4} e^{4x}$

$\mathcal{L}^{-1}\left[\frac{1}{4p}\right]$: $n+1 = 1 \rightarrow n = 0$

$a = 0$

$\rightarrow f_2 = \frac{1}{4}$

$\mathcal{L}^{-1}\left[\frac{1}{p^2}\right]$ $a = 0$, $n+1 = 2$

$\rightarrow n = 1$

$f_3 = x$

$$\rightarrow f = -\frac{1}{4}e^{4x} + \frac{1}{4} + x$$

$$G: \; L^{-1}\left[\frac{1}{p-4}\right] \qquad a = -4 \qquad \begin{array}{l} n+1=1 \\ \rightarrow n=0 \end{array}$$

$$\rightarrow g_1 = e^{-4x}$$

$$L^{-1}\left[\frac{1}{3p}\right] = \frac{1}{3}$$

$$\rightarrow g = e^{4x} + \frac{1}{3}$$

2) a)

$$y'' + \omega^2 y = \sin(\omega x)$$

CF: $y'' + \omega^2 y = 0$

$$\rightarrow y(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$y'' = -\omega^2 y$$

$$\rightarrow -\omega^2 y + \omega^2 y = 0 \quad \checkmark$$

PI: $y_p(x) = k_1(x)\cos(\omega x) + k_2(x)\sin(\omega x)$

Impose: ① $k_1'(x)\cos(\omega x) + k_2'(x)\sin(\omega x) = 0$

② $\omega\left[-k_1'(x)\sin(\omega x) + k_2'(x)\cos(\omega x)\right] = \sin(\omega x)$

① $$k_1' = -k_2' \frac{\sin(\omega x)}{\cos(\omega x)}$$

② $$\omega\left[k_2' \frac{\sin^2(\omega x)}{\cos(\omega x)} + k_2' \cos(\omega x)\right] = \sin(\omega x)$$

$$② \quad \omega\left[k_2' \frac{\sin^2(\omega x)}{\cos(\omega x)} + k_2' \cos(\omega x)\right] = \sin(\omega x)$$

$$\frac{\omega k_2'}{\cos(\omega x)} = \sin(\omega x)$$

$$\rightarrow k_2' = \frac{\cos(\omega x)\sin(\omega x)}{\omega} = \frac{\sin(2\omega x)}{\omega}$$

$$k_1' = \frac{-\sin^2(\omega x)}{\omega} = \frac{\cos(2\omega x) - 1}{2\omega}$$

$$k_1 = \int k_1' \, dx = \frac{1}{2\omega}\int \cos(2\omega x) - 1 \, dx$$

$$= \frac{\sin(2\omega x)}{4\omega^2} - \frac{x}{2\omega} + C_1$$

$$k_2 = \int k_2' \, dx = \frac{1}{\omega}\int \sin(2\omega x)\, dx$$

$$= \frac{-1}{2\omega^2}\cos(2\omega x) + C_2$$

$$y_{PI}(x) = \left(\frac{\sin(2\omega x)}{4\omega^2} - \frac{x}{2\omega} + C_1\right)\cos(\omega x) + \left(-\frac{\cos(2\omega x)}{2\omega^2} + C_2\right)\sin(\omega x)$$

$\downarrow$ C_1 & C_2 get absorbed into A & B for general solution.

$$y_{PI} = \frac{\sin(\omega x)\cos^2(\omega x)}{4\omega^2} - \frac{x\cos(\omega x)}{2\omega} - \frac{\cos^2(\omega x)\sin(\omega x)}{2\omega^2} + \frac{\sin^3(\omega x)}{2\omega^2}$$

$$= \frac{1}{2\omega^2}\left[\sin(\omega x)\cos^2(\omega x) - \omega x\cos(\omega x) + \sin^3(\omega x)\right]$$

$= 1 - \sin^2(\omega x)$

$$= \frac{1}{2\omega^2}\left[\sin(\omega x) - \omega x\cos(\omega x)\right]$$

$$y = A\cos(\omega x) + B\sin(\omega x) + \frac{\sin(\omega x)}{2\omega^2} - \frac{x\cos(\omega x)}{2\omega}$$

$$y' = -A\omega\sin(\omega x) + B\omega\cos(\omega x) + \cancel{\frac{1}{2\omega}\cos(\omega x)} - \cancel{\frac{\cos(\omega x)}{2\omega}} + \frac{x\sin(\omega x)}{2}$$

$y(0) = 0$

$\Rightarrow 0 = A \quad \Rightarrow A = 0$

$y'(0) = 0$

$0 = B\omega \Rightarrow B = 0$

$$y(x) = \frac{\sin(\omega x)}{2\omega^2} - \frac{x\cos(\omega x)}{2\omega}$$

b) $x^2 y''(x) - 2xy'(x) + 2y = x\ln(x)$

CF: $x^2 y''(x) - 2xy'(x) + 2y = 0$

$y_c(x) = y_1(x) + y_2(x)$

$y_1 = x \ , \ y_2 = x^2$

$y_1: \ x^2 \times 0 - 2x \times 1 + 2x = -2x + 2x = 0$ ✓

$y_2: \ 2x^2 - 4x^2 + 2x^2 = 0$ ✓

$y_p = h_1(x)x + h_2(x)x^2$

$h_1' x + h_2' x^2 = 0 \quad a_2(h_1' + 2h_2' x) = x\ln(x)$

$h_1' x + h_2' x^2 = 0 \quad x^2(h_1' + 2h_2' x) = x\ln(x)$

$\Rightarrow$ ① $h_1' = -x h_2'$ ② $h_1' + 2x h_2' = \ln(x)$

$h_1'x + h_2'x^2 = 0 \qquad x^2(h_1' + 2h_2'x) = x\ln(x)$

$\rightarrow ① \; h_1' = -xh_2' \qquad ② \; h_1' + 2xh_2' = \dfrac{\ln(x)}{x}$

$\underline{① \Rightarrow ②}$

$$-xh_2' + 2xh_2' = \frac{\ln(x)}{x}$$

$$h_2' = \frac{\ln(x)}{x^2} \qquad h_1' = -\frac{\ln(x)}{x}$$

$$h_1 = \int h_1' \, dx = -\int \ln(x) \frac{1}{x} \, dx$$

Let $u = \ln(x) \quad \dfrac{du}{dx} = \dfrac{1}{x} \rightarrow dx = x\,du$

$$h_1 = -\int u \, du = -\frac{1}{2}u^2$$
$$= -\frac{1}{2}\ln^2(x) + c_1$$

$$h_2 = \int h_2' = \int \frac{\ln(x)}{x^2} \, dx$$

$u(x) = \ln(x) \qquad dv = \dfrac{1}{x^2}$

$du = \dfrac{1}{x} \qquad v = -\dfrac{1}{x}$

$$\rightarrow h_2 = -\frac{\ln(x)}{x} + \int \frac{1}{x^2} \, dx$$

$$= -\frac{\ln(x)}{x} - \frac{1}{x} \, dx$$

$$y(x) = h_1x + h_2x^2 + Ax + Bx^2$$
$$= -\frac{1}{2}\ln^2(x)x - \ln(x)x - x + Ax + Bx^2$$
$$= -\frac{1}{2}\ln^2(x)x - \ln(x)x + Ax + Bx^2$$

$A-1$

3) a) $\displaystyle\int_0^3 (5x-2)\,\delta(2-x)\,dx = f(2) = 10-2$
$= \underline{8}$

b)

$t-\quad \displaystyle\int^{\infty} f(x)\,\phi'(x)\,dx$.

b)

$$I = \int_{-\infty}^{\infty} f(x)\, \Theta'(x)\, dx .$$

$u = f(x) \qquad dv = \Theta'$

$du = f'(x) \qquad v = \Theta$

$$I = \Big[f(x)\Theta(x) \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\, \Theta(x)\, dx$$

$$I = f(\infty)\underbrace{\Theta(\infty)}_{=1} - f(-\infty)\underbrace{\Theta(-\infty)}_{=0} - \int_{a}^{\infty} f'(x)\, dx$$

$$I = f(\infty) - \Big[f(x) \Big]_{a}^{\infty}$$

$$= \cancel{f(\infty)} - \cancel{f(\infty)} + f(a)$$

$$= f(a)$$

$$\rightarrow \int f(x)\, \Theta'(x)\, dx = f(a)$$

$$\therefore \Theta'(x) = \delta(x-a)$$

c)

$$x^m \delta^n (x)$$

4) $y''(x) + \omega^2 y(x) = e^{-x}$

$$G''(x,t) + \omega^2 G''(x,t) = \delta(x-t)$$

$$G_{x<t}(x,t) = A(t)\cos(\omega x) + B(t)\sin(\omega x)$$

$$G_{x>t}(x,t) = C(t)\cos(\omega x) + D(t)\sin(\omega x)$$

$x<t$: $x=0$ $G(0,t)=0$

$$\rightarrow 0 = A$$

$$G'(x,t) = -\omega A\sin(\omega x) + \omega B\cos(\omega x)$$

$$G'(0,t)=0$$

$$\rightarrow B=0$$

$$\rightarrow A=B=0$$

if $x=t$:

$$G''(x,t) + \omega^2 G(x,t) = \delta(x-t)$$

$$= 0$$

$$C\cos(\omega t) + D\sin(\omega t) = 0$$

$$\rightarrow C = -D\frac{\sin(\omega t)}{\cos(\omega t)}$$

$$-\omega C\sin(\omega t) + \omega D\cos(\omega t) = 0$$

$$-\omega\left[-\frac{\sin(\omega t)}{\cos(\omega t)}D\right]\sin(\omega t) + \omega D\cos(\omega t) = 0$$

$$\omega D\left[\frac{\sin^2(\omega t)}{\cos(\omega t)} + \cos(\omega t)\right]$$

$$\sin^2 = 1-\cos^2$$

$$= \omega D\left[\frac{1-\cancel{\cos^2(\omega t)}}{\cos(\omega t)} + \frac{\cancel{\cos^2(\omega t)}}{\cos(\omega t)}\right]$$

$$= \frac{\omega D}{\cos(\omega t)} = 1$$

$\rightarrow D = \frac{\cos(\omega t)}{\omega} \qquad C = -\frac{\sin(\omega t)}{\omega}$

$$\rightarrow G_{x>t}(x,t) = C(t)\cos(\omega x) + D(t)\sin(\omega x)$$
$$= \frac{1}{\omega}\left[\sin(\omega t)\cos(\omega x) - \cos(\omega t)\sin(\omega x)\right]$$

$$\rightarrow y(x) = \int_0^x G_{x>t}(x,t)\, e^{-t}\, dt$$

$$= \frac{1}{\omega}\cos(\omega x)\int_0^x \sin(\omega t)e^{-t}\, dt - \frac{1}{\omega}\sin(\omega x)\int_0^x \cos(\omega t)e^{-t}\, dt$$

a) (Again):

$$y''(x) + \omega^2 y(x) = e^{-x}$$

① Replace $y^{(n)}$ with $G(x,t)^{(n)}$ & solve homogenious eq

$$\rightarrow G(x,t)'' + \omega^2 G(x,t) = 0$$

$$\rightarrow G(x,t) = A(t)\cos(\omega x) + B(t)\sin(\omega x).$$

② Note, function is discontinuous at $x = t$.
write solution for $x<t$ & $x>t$

$$G_{x<t}(x,t) = A(t)\cos(\omega x) + B(t)\sin(\omega x)$$
$$G_{x>t}(x,t) = C(t)\cos(\omega x) + D(t)\sin(\omega x)$$

③ Notice, bounds are $0 \le x \le \infty$. Boundry Conditions are
$$y(0) = y'(0) = 0$$
$$x=0 \therefore G_{x<t}(0,t) = G'_{x<t}(0,t) = 0$$

G: $\quad 0 = A(t)\cdot 1 + B(t)\cdot 0 \rightarrow A = 0$

G': $\quad 0 = \omega B(t)\cos(\omega x) = 0 \rightarrow B = 0$

④ Discontinuity equation: for n^{th} order equation, we must satisfy: $\frac{d^{n-1}G(x,t)}{dx^{n-1}} = \frac{1}{a_n}$

This Question: order = 2 ∴

$$\frac{dG}{dx} = \frac{1}{a_1}$$

$$a_1 = 1$$

$$\frac{dG}{dx} = \frac{d}{dx}\Big[C(t)\cos(\omega x) + D(t)\sin(\omega x)\Big]_{x=t} = \frac{1}{a_1}$$

① $$= -\omega C\sin(\omega t) + \omega D\cos(\omega t) = 1$$

N = order

⑤ Satisfy the continuity conditions: For $0 \leq n \leq N-2$

$$\left.\frac{d^nG}{dx^n}\right|_{x=t} = 0$$

Our order is 2 → Only one condition:

$$G(x,t)\big|_{x=t} = 0$$

→ ② $$C(t)\cos(\omega t) + D(t)\sin(\omega t) = 0$$

⑥ Rearrange to find $C(t)$ & $D(t)$

② : $$c(t) = -D(t)\frac{\sin(\omega t)}{\cos(\omega t)}$$

① $$-\omega C(t)\sin(\omega t) + \omega D(t)\cos(\omega t) = 1$$

② ⇒ ①

$$-\omega\left[-D(t)\frac{\sin(\omega t)}{\cos(\omega t)}\right]\sin(\omega t) + \omega D(t)\cos(\omega t) = 1$$

$$\omega D\left[\frac{\sin^2(\omega t)}{\cos(\omega t)} + \cos(\omega t)\right] = 1$$

$$\sin^2\theta = 1-\cos^2\theta \qquad \cos\theta = \frac{\cos^2\theta}{\cos\theta}$$

$$\omega D\left[\frac{1-\cancel{\cos^2\omega t}}{\cos(\omega t)} + \frac{\cancel{\cos^2(\omega t)}}{\cancel{\cos(\omega t)}}\right] = 1$$

$$\frac{\omega D}{\cos(\omega t)} = 1$$

$$\rightarrow \quad D = \frac{\cos(\omega t)}{\omega} \qquad C = \frac{-\sin(\omega t)}{\omega}$$

7 we now know all coefficient functions (A, B, C, D)

$$\rightarrow \quad G_{x<t}(x,t) = 0$$

$$G_{x>t}(x,t) = C\cos(\omega x) + D\sin(\omega x)$$

$$= \frac{1}{\omega}\left[-\sin(\omega t)\cos(\omega x) + \cos(\omega t)\sin(\omega x)\right]$$

8 we now know $G(x,t) \Rightarrow$ solve:

$$y(x) = \int_a^b G(x,t)\, f(t)\, dt$$

where $f(t) = \int f(x)\,\delta(x-t)\,dx$

where $f(x)$ = RHS of non homogenious eq $(= e^{-x})$

For general:

$$y(x) = \int_a^b G(x,t)\, f(t)\, dt = \int_a^x G_{x>t}(x,t)\, f(t)\, dt + \int_x^b G_{x<t}(x,t)\, f(t)\, dt$$

NOTE THE $a \rightarrow x : G_{x>t}$

$x \rightarrow b : G_{x<t}$

For this question: $G_{x<t}(x,t) = 0$

So we must solve:

$$y(x) = \int^x \frac{1}{\omega}\left[-\sin(\omega t)\cos(\omega x) + \cos(\omega t)\sin(\omega x)\right] e^{-t}\, dt$$

$$y(x) = \int_0 \frac{1}{\omega}\left[-\sin(\omega t)\cos(\omega x) + \cos(\omega t)\sin(\omega x)\right] e^{-t} \, dt$$

$$= -\frac{\cos(\omega x)}{\omega}\int_0^x \sin(\omega t) e^{-t} \, dt + \frac{\sin(\omega x)}{\omega}\int_0^x \cos(\omega t) e^{-t} \, dt$$

We solve by integrating by parts, twice for each integral

$$y(x) = \frac{-\cos(\omega x)}{\omega} \times \frac{1}{\omega^2+1}\left(\omega e^{-x}\sin(\omega x) + e^{-x}\cos(\omega x) + 1\right) +$$

$$\frac{\sin(\omega x)}{\omega} \times \frac{1}{\omega^2+1}\left(-\omega e^{-x}\cos(\omega x) - e^{-x}\sin(\omega x) + 1\right)$$

$$= \frac{1}{\omega^2+1}\left(e^{-x} - \cos(\omega x) + \frac{1}{\omega}\sin(\omega x)\right)$$

4) b) $\quad (x^2+1)y''(x) - 2xy'(x) + 2y = (x^2+1)^2$

Check $y_1 = x$, $y_2 = 1-x^2$ are solutions of homogeneous equation:

$$(x^2+1)y''(x) - 2xy'(x) + 2y(x) = 0$$

$y(x) = x$: $\quad y'(x) = 1$, $y''(x) = 0$

$\Rightarrow \quad 0 - 2x + 2x = 0 \Rightarrow 0 = 0$ ✓

$y(x) = 1-x^2 \quad y'(x) = 1-2x \quad y''(x) = -2$

$\Rightarrow \quad -2(x^2+1) - 2x(1-2x) + 2(1-x^2) = 0$

$-2x^2 - 2 - 2x + 4x^2 + 2x - 2x^2 = 0$

$0 = 0$ ✓

We know solution for homogeneous $\to$ Form our greens function

$$G_{x<t}(x,t) = A(t)x + B(t)(1-x^2)$$

$$G_{x>t}(x,t) = C(t)x + D(t)(1-x^2)$$

$$G_{x<t}(x,t) = A(t)x + B(t)(1-x^2)$$
$$G_{x>t}(x,t) = C(t)x + D(t)(1-x^2)$$

$G_{x<t}$ Apply boundary condition

$$G_{x<t}(0,t) = B(t) = 0 \Rightarrow \underline{B=0}$$

$$G_{x>t}(1,t) = C(t)x = 0 \Rightarrow C=0$$

$$\Rightarrow G_{x<t}(x,t) = A(t)x$$
$$G_{x>t}(x,t) = D(t)(1-x^2)$$

Order $\Rightarrow 2$: Discontinuity :

$$\left[\frac{dG}{dx}\right]_{x<t}^{x>t} = \frac{1}{a_2} = \frac{1}{x^2+1}$$

$$\Downarrow$$

$$= \frac{d}{dx}(1-x^2)\,D(t) - \frac{d}{dx}(x)\,A(t) = \frac{1}{x^2+1}$$

$$\rightarrow -2xD - A = \frac{1}{x^2+1}$$

$x=t$

$$\rightarrow -2tD - A = \frac{1}{t^2+1}$$

Continuity condition:

$$G_{x>t}(x=t,t) - G_{x<t}(x=t,t) = 0$$
$$(1-t^2)D - tA = 0$$
$$\rightarrow D = A\frac{t}{1-t^2}$$

$$A\left(-2t\left[\frac{t}{1-t^2}\right] - 1\right) = \frac{1}{t^2+1}$$

$$A\left[\frac{-2t^2}{1-t^2} - \frac{1-t^2}{1-t^2}\right] = \frac{1}{t^2+1}$$

$$A\left[\frac{-2t^2}{1-t^2} - \frac{1-t^2}{1-t^2}\right] = \frac{1}{t^2+1}$$

$$A\left[\frac{-(t^2+1)}{1-t^2}\right] = \frac{1}{t^2+1}$$

$$\Rightarrow A = \frac{t^2-1}{(t^2+1)^2}$$

$$D = \frac{t^2-1}{(t^2+1)^2} \times \frac{t}{1-t^2}$$

$$= \frac{-t}{(t^2+1)^2}$$

$$G_{x<t}(x,t) = \frac{t^2-1}{(t^2+1)^2}\, x$$

$$G_{x>t}(x,t) = \frac{-t}{(t^2+1)^2}(1-x^2)$$

$$y(x) = \int_0^1 G(x,t)\, f(t)\, dt$$

$$= \int_0^x G_{x>t}(x,t)\, f(t)\, dt + \int_x^1 G_{x<t}(x,t)\, f(t)\, dt$$

$$= -(1-x^2)\int_0^x \frac{t}{\cancel{(t^2+1)^2}}\cancel{(t^2+1)^2}\, dt + x\int_x^1 \frac{t^2-1}{\cancel{(t^2+1)^2}}\cancel{(t^2+1)^2}\, dt$$

$$= (x^2-1)\left[\frac{t^2}{2}\right]_0^x + x\left[\frac{t^3}{3} - t\right]_x^1$$

$$= \frac{x^2}{2}(x^2+1) + x\left[\left(\frac{1}{3}-1\right) - \left(\frac{x^3}{3} - x\right)\right]$$

$$= \frac{x^4}{2} + \frac{x^2}{2} - \frac{2}{3}x - \frac{x^4}{3} + x^2$$

$$= \frac{x^4}{6} + \frac{x^2}{2} - \frac{2}{3}x$$