PHAS2423 - Self Study - Fluid Dynamics - Problems and Solutions

(1) State the conditions under which the following theoretical tools are valid descriptions of fluid flow:

- Bernouilli's equation along a streamline
  - steady, inviscid, incompressible flow
- Bernoulli's equation throughout the region
  -steady, inviscid, incompressible, irrotational flow
- the continuity equation
  - a general flow
- Euler's equation
  - inviscid flow (or Re $\to \infty$)
- Laplace's equation for a scalar field
  - steady, inviscid, incompressible, irrotational flow
- the Navier-Stokes equation
  - a general flow of an isotropic Newtonian fluid
- Laplace's equation for a vector field
  - steady, incompressible, creeping flow (Re $\to 0$). It would actually be the inhomogeneous Laplace equation, also known as Poisson's equation.

---

(2) Using the properties of ϵ_{ijk}, prove that $\nabla \times (\nabla V) = 0$, where V is a scalar field.

Solution:

Since $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$ and $(\nabla V)_i = \partial_i V$, we write $(\nabla \times (\nabla V))_i = \epsilon_{ijk} \partial_j \partial_k V$. We use the anti-symmetry of the Levi-Civita symbol, namely $\epsilon_{ijk} = -\epsilon_{ikj}$, to write

$$\epsilon_{ijk} \partial_j \partial_k V = -\epsilon_{ikj} \partial_j \partial_k V.$$

The next step is to remember that j and k are dummy variables and are being summed over. Therefore, it doesn't matter how we label them, i.e. $\epsilon_{ikj} \partial_j \partial_k V = \epsilon_{inm} \partial_m \partial_n V$. In particular, we could relabel j as k and vice versa. Then $\epsilon_{ijk} \partial_j \partial_k V = \epsilon_{ikj} \partial_k \partial_j V$. Thus $\epsilon_{ijk} \partial_j \partial_k V = -\epsilon_{ikj} \partial_k \partial_j V = -\epsilon_{ijk} \partial_j \partial_k V$. A quantity that is its own negative must be zero, hence $(\nabla \times (\nabla V))_i = 0$

(3) The advective term in the Navier-Stokes equation is

$$\begin{aligned}(\mathbf{v}.\nabla\mathbf{v})_r &= v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\\ (\mathbf{v}.\nabla\mathbf{v})_\theta &= v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z}\\ (\mathbf{v}.\nabla\mathbf{v})_z &= v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\end{aligned}$$

in cylindrical polar coordinates. Show that for a flow field of the form $\mathbf{v} = (0, 0, v_z(r))$, each component of the advective term is equal to zero. Thus calculate the Poiseuille volumetric flow rate along a cylindrical pipe of radius a, subject to an axial pressure gradient $-\alpha$.

Solution:

- Since $v_r = v_\theta = 0$, it is clear that the r and θ components of $\mathbf{v}.\nabla\mathbf{v}$ are zero. As for the z component, the only possible non-zero contribution must come from $v_z\partial_z v_z$. However, we are told that v_z is a function of r only and hence this component of the inertial term is zero too.
- Actually, we might have concluded this without bothering to identify the components of $\mathbf{v}.\nabla\mathbf{v}$ in cylindrical polars since we have already seen that it is zero in channel flow, where we described it using Cartesian coordinates. Poiseuille flow down a pipe differs from channel flow only by the geometry of the boundary conditions. The equation of motion (the Navier-Stokes equation) is the same and the terms in this equation that are zero in one set of coordinates will remain zero in another.
- The Navier-Stokes equation for incompressible flow is

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{F}$$

and for a steady flow of the kind envisaged, and in the absence of body forces, this reduces to

$$\nabla p = \mu\nabla^2\mathbf{v}.$$

The only non-zero component of $\mathbf{v}$ is v_z, which is only a function of r. Therefore, the only non-zero component of $\nabla^2\mathbf{v}$ is

$$\left(\nabla^2\mathbf{v}\right)_z = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right).$$

As a result we must solve the ordinary differential equation

$$\frac{dp}{dz} = -\alpha = \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

subject to the no-slip boundary condition $v_z(a) = 0$. The first integral of this ordinary differential equation is

$$r\frac{dv_z}{dr} = -\frac{1}{2}\frac{\alpha}{\mu}r^2 + A,$$

where A is a constant, so $dv_z/dr = -\alpha r/(2\mu) + A/r$ and the second integral is

$$v_z = -\frac{1}{4}\frac{\alpha}{\mu}r^2 + A\ln r + B,$$

where B is another constant. We are obliged to set $A = 0$ since the flow velocity is finite at the origin and B is then identified from the no-slip condition, giving

$$v_z = \frac{1}{4}\frac{\alpha}{\mu}(a^2 - r^2).$$

The volumetric flow rate is $\int v_z dS$ integrated over the cross-sectional area, namely $\int_0^a v_z 2\pi r dr$, and this is

$$Q = 2\pi\frac{\alpha}{\mu}\frac{1}{4}\left[\frac{1}{2}a^2r^2 - \frac{1}{4}r^4\right]_0^a = \frac{\pi\alpha a^4}{8\mu}.$$

---

(4) The figure illustrates the flow of an inviscid fluid around a sphere. The drag force on the sphere may be written as an integral involving the local pressure on the surface. What is the magnitude of the drag force? [Hint: you might think that you need to substitute the derived potential flow solution into Euler's equation to obtain the pressure at the surface and then perform the integral, but the answer (which might not be what you expect) can be obtained by a simpler route based upon imagining a reversal of all velocities.]

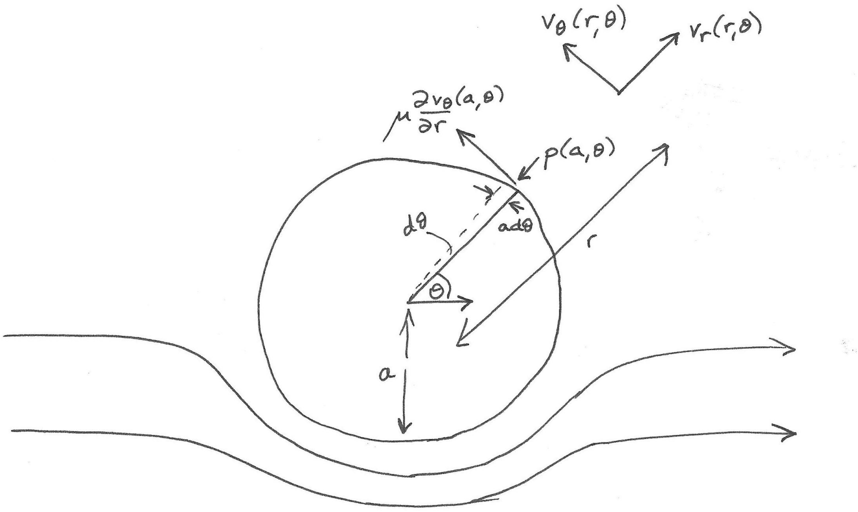
Solution:

There is a long and a short way to solve this problem. Let's start with the long way

- We recognise that the flow is irrotational, since well upstream of the sphere, the flow is uniform, vorticity is zero and the fluid is inviscid. Therefore, Bernouilli's equation holds throughout the space. The pressure at the surface of the sphere is therefore given by

  $$p(a,\theta) + \frac{1}{2}\rho v_\theta^2(a,\theta) = C,$$

  where C is a constant, having recognised that the velocity at the surface of the sphere has a non-zero component only in the $\hat{\theta}$ direction. Given the



potential flow solution in the notes (also see one of the problems in the Tutorial), we get

$$p(a,\theta) = C - \frac{1}{2}\rho U^2 \frac{9}{4} \sin^2\theta$$

and then the drag force on the sphere is

$$F_d = -\int_0^\pi 2\pi a \sin\theta a d\theta p(a,\theta)\cos\theta = 0.$$

The integral vanishes since the integrand is antisymmetric about $\theta = \pi/2$. Maybe we could have guessed this: drag is fundamentally due to viscous stresses and therefore an inviscid flow solution cannot generate drag.

- The shorter way to get the answer is to invoke symmetry. If a velocity and pressure field satisfy the Euler equation

$$\rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p$$

then the time-reversed velocity field $-\mathbf{v}$ and the same pressure field also satisfy it (with a reversal of the asymptotic velocity boundary condition). Had we neglected to put arrows on the streamlines in the figure, we would have been unable to work out which direction the fluid was moving along the streamlines. Now, since the pressure distribution on the sphere would be the same in both cases, the drag forces would be the same. However, the drag forces are expected to change sign when the direction of the flow is changed - this is only consistent if the drag force is zero. This argument holds for inviscid flow about a body of any shape - a result known as D'Alembert's paradox (in fact, it isn't a paradox and simply means that in order to calculate drag, we must include viscous terms in the Navier-Stokes equation).

---

(5) Show that the flow field $\mathbf{v} = (0, K(2\pi)^{-1}(1 - \exp[-r^2\rho/(4\mu t)]), 0)$ in cylindrical polar coordinates satisfies the Navier-Stokes equation. Assume the fluid is incompressible. You probably need to know that ∇^2 in cylindrical polars takes the form:

$$\begin{aligned}
(\nabla^2\mathbf{v})_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \\
(\nabla^2\mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \\
(\nabla^2\mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}
\end{aligned}$$

Calculate the radial pressure gradient. Calculate the vorticity. Describe what is going on physically in this flow field. **Solution:**

- The Navier-Stokes equation is

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v}.$$

Since $v_r = v_z = 0$ and v_θ is a function of r and t only, the components of the viscous term reduce to

$$\begin{aligned}
(\nabla^2\mathbf{v})_r &= 0 \\
(\nabla^2\mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} \\
(\nabla^2\mathbf{v})_z &= 0.
\end{aligned}$$

and the components of the advective term reduce to

$$\begin{aligned}
(\mathbf{v}.\nabla\mathbf{v})_r &= -\frac{v_\theta^2}{r} \\
(\mathbf{v}.\nabla\mathbf{v})_\theta &= 0 \\
(\mathbf{v}.\nabla\mathbf{v})_z &= 0.
\end{aligned}$$

The *radial component* of the Navier-Stokes equation is then

$$-\rho\frac{v_\theta^2}{r} = -\frac{dp}{dr}$$

and this identifies the radial pressure gradient as

$$\frac{dp}{dr} = \frac{\rho}{r}\left(\frac{K}{2\pi r}(1 - \exp[-r^2\rho/(4\mu t)]\right)^2$$

The θ *component* of the Navier-Stokes equation is

$$\rho \frac{\partial v_\theta}{\partial t} = \mu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}.\right)$$

Direct substitution of the flow field and a bit of perseverance verifies that form of $v_\theta(r,t)$ solves this equation.

- Next, we need to obtain the vorticity. The components of the curl of $\mathbf{v}$ in cylindrical polars are:

$$\begin{aligned}
(\nabla \times \mathbf{v})_r &= \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \\
(\nabla \times \mathbf{v})_\theta &= \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\
(\nabla \times \mathbf{v})_z &= \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)
\end{aligned}$$

For a flow of the form considered here, the only the z component is non-zero;

$$\omega_z = (\nabla \times \mathbf{v})_z = \frac{v_\theta}{r} + \partial_r v_\theta = \frac{K\rho}{4\pi\mu t} \exp\left[-\rho r^2/(4\mu t)\right].$$

- Finally, lets consider what type of flow this solution describes. As $t \to 0$, the exponential term in v_θ vanishes at all $r \neq 0$ and so $v_\theta \to K/(2\pi r)$: the initial flow is a free vortex. Now consider ω_z: it is a half-Gaussian in r (since $r \geq 0$) with a width proportional to $(\mu t)^{1/2}$. As t increases, ω_z spreads out: vorticity leaks away from the vortex axis into the initially irrotational flow around it as a result of viscous effects. Now look at v_θ as $t \to \infty$: the exponential term goes to unity and the circulating flow eventually stops. What we have here is a decaying free vortex. The lifetime of the vortex is controlled by the fluid viscosity.