Chapter 6 : Integral Transforms

This chapter will cover using the Laplace and Fourier integral transforms in order to solve partial differential equations (PDEs).

The Laplace transform

The Laplace transform is a very good way to approach a problem where a variable has a range from 0 to ∞.

Salt diffusion into water example.

Example. Consider a semi-infinite tube filled initially with pure water and brought in contact with a salt solution maintained at a fixed concentration. Find the total amount of salt diffused into the tube by the time t if the diffusion constant is k.

The Question

The semi-infinite tube is an one-dimensional system. Therefore, concentration of salt in the tube u can be described using two variables only: x – the distance from the salt container and t – duration of the contact. Hence, $u = u(x, t)$.

U = Concentration of salt in the tube

x = The distance from the salt container

t = duration of the contact

$\therefore \quad U = U(x,t)$

Using Fick's 2nd law we know that

$$k \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \qquad (1)$$

Thinking about how the system work we are then able to make the following conditions:

- $u(0, t) = u_0$, i.e., the concentration of the salt solution "at the source" is the same at all times.
- $u(x, 0) = 0$, i.e., the tube contained pure water prior to be brought in contact with the salt reservoir at $t = 0$.
- $u(\infty, t) = 0$, i.e., salt will never rich the far end of the semi-infinite tube.
- $u(x, t)$ is finite for all x and all t.

To approach this problem we will start by trying to convert this PDE into an ODE, by eliminating the dependence on one of the variables.

Take the Laplace transform of (1)

$$\int_0^\infty k\,\frac{\partial^2 u}{\partial x^2}\,e^{-st}\,dt = \int_0^\infty \frac{\partial u}{\partial t}\,e^{-st}\,dt.$$

Changing the order of the integral and partial derivative operators we now get

$$k\frac{\partial^2}{\partial x^2}\left(\int_0^\infty u(x,t)\,e^{-st}\,dt\right) = k\frac{\partial^2}{\partial x^2}U(x,s) = \int_0^\infty \frac{\partial u}{\partial t}\,e^{-st}\,dt, \quad (2)$$

(This come from the Leibniz Integral Rule)

Integrating the RHS by parts

$$\int_0^\infty \frac{\partial u}{\partial t} e^{-st}\,dt = \left[ue^{-st} - \int (-s)\,ue^{-st}\,dt\right]_0^\infty = \left[ue^{-st} + s\int ue^{-st}\,dt\right]_0^\infty$$

$$= ue^{-st}\Big|_{t=0}^{t=\infty} + s\int_0^\infty ue^{-st}\,dt = u(x,t)e^{-st}\Big|_{t=0}^{t=\infty} + s\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= u(x,\infty)e^{-s(\infty)} - u(x,0)e^{-s(0)} + s\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= 0 - u(x,0)e^{-s(0)} + s\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= -u(x,0) + s\,L[u(x,t)]$$

$$= sU(x,s) - u(x,0)$$

Now looking at boundary conditions we see that

$u(x,0) = 0 \quad (3)$

$$\therefore \int_0^\infty \frac{\partial u}{\partial t} e^{-st}\,dt = sU(x,s)$$

Looking at (2) we see that:

$$K \frac{\partial^2}{\partial x^2} \bar{U}(x,s) = s\,\bar{U}(x,s)$$

Which is the same as an ordinary differential equation with respect to $\bar{U}(x,s)$.

Solving this ODE the same way we would solve any other ODE leads to:

$$\bar{U}(x,s) = A(s)\exp\left(\sqrt{\tfrac{s}{K}}\,x\right) + B(s)\exp\left(-\sqrt{\tfrac{s}{K}}\,x\right)$$

Looking back at our conditions we see that...

$U(0,t) = U_0$ ⑤

$U(x,0) = 0$ ③

$U(\infty,t) = 0$ ④

- From ④ we can say that $\bar{U}(\infty,s) = 0$

$\therefore\ 0 = A(s)\exp\left(\sqrt{\tfrac{s}{K}}\,\infty\right) + B(s)\exp\left(-\sqrt{\tfrac{s}{K}}\,\infty\right)$

$x = \infty$, $\exp\left(-\sqrt{\tfrac{s}{K}}\,\infty\right) = \frac{1}{\infty} = 0$

$$\therefore\ 0 = A(s)\exp\left(\sqrt{\tfrac{s}{K}}\,\infty\right) \quad ⑥$$

with ⑥ in mind we see that $\bar{U}(x,s)$ simplifies too

$$\bar{U}(x,s) = B(s)\exp\left(-\sqrt{\tfrac{s}{K}}\,x\right) \quad ⑦$$

- From ⑤:

$U(0,t) = U_0$

to change any boundary condition from one domain to another, or in this example from the t domain to the s domain we simply apply the same transform which was applied to our original PDE

$$L[U(0,t)] = L[U_0] = \int_0^\infty U_0 e^{-st}\,dt = U_0 \int_0^\infty e^{-st}\,dt = U_0\left[\frac{e^{-st}}{(-s)}\right]_0^\infty$$

$$= -U_0\left[\frac{1}{e^{st}s}\right]_{t=0}^{t=\infty} = -U_0\left(\frac{1}{\infty} - \frac{1}{s}\right) = \frac{U_0}{s} = \bar{U}(0,s)$$

because $\bar{U}(0,s) = B(s)\exp(0) = B(s) = \frac{U_0}{s}$

we then also have then now that

$$\bar{U}(x,s) = \frac{U_0}{s}\exp\left(-\sqrt{\frac{s}{K}}\,x\right)$$

Using the inverse Laplace transform tables we arrive at

$$U(x,t) = U_0\left[1 - \text{erf}\left(\frac{x}{\sqrt{4Kt}}\right)\right]$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp$.

Now to find

The Inverse Laplace Transform

After applying the Laplace transform one may want to go backwards with the information they found in the s-domain.
To do this we introduce the idea of convolution.

Convolution

Consider the product $H(p)G(p)$

$H(p) = L(h(s))$

$G(p) = L(g(t))$

$$H(p)G(p) = \int_0^\infty h(s)e^{-ps}\,ds \int_0^\infty g(t)e^{-pt}\,dt = \int_0^\infty \int_0^\infty h(s)g(t)e^{-p(s+t)}\,ds\,dt$$

let $r = s+t$

$\Longrightarrow s = r - t$

$\Longrightarrow ds = dr$

$$\Rightarrow H(p)G(p) = \int_0^\infty \left[\int_t^\infty h(r-t)g(t)\exp(-pr)\,dr\right] dt$$

When $s=0$, $0 = r - t \Rightarrow t = r$

We Now rearrange this integral so that only objects which are effected by the integral stay inside the integral:

$$\therefore H(p)G(p) = \int_0^\infty \left[\int_0^r h(r-t)g(t)\,dt\right] e^{-pr}\,dr$$

This inner integral is then defined as the convolution of functions $g(x)$ and $h(x)$ or as is commonly denoted $h*g$

This equation can then be written as

$$H(P)G(P) = \int_0^\infty [h*g]\,e^{-pr}\,dr$$

Fourier Transform

Convolution Theorem

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \, dx \, , \quad G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-ikx} \, dx$$

respectively. **Note** that other definitions of the Fourier transform may use a different factor in front of the integral.

Chapter 7

Linear Partial Differential Equations

Linear first order PDE

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y) \quad (1)$$

Case 1:

$C(x,y) = R(x,y) = 0$

(1) becomes $$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0 \quad (2)$$

if we let the solution $u(x,y) = f(p)$. where $p = p(x,y)$

$\therefore u(x,y) = f(p(x,y))$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial y}$$

with this information (2) becomes

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{\partial f}{\partial p} = 0 \quad (3)$$

$$\Longrightarrow \frac{\partial f(p)}{\partial p} = 0$$ which implies that $f(p)$ is a constant

alternatively it could also imply that

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0 \quad (5)$$

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = 0 \quad (4)$$

For p to remain a constant combination of x and y (4) must be true

if we set $dx = \frac{A(x,y)}{B(x,y)} dy$ then (4) becomes (5)

Hence instead of solving Eqns (4) and (5) we can integrate equation

$$\int \frac{dx}{A(x,y)} = \int \frac{dy}{B(x,y)}$$

__Case 2:__

__$R(x,y) = 0$__

(1) becomes $\quad A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = 0 \quad (2)$

Considering a solution where $u(x,y) = h(x,y)\,f(p)$
which means:

$$\frac{\partial u}{\partial x} = \frac{\partial h(x,y)}{\partial x} f(p) + h(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial h(x,y)}{\partial y} f(p) + h(x,y)\frac{df(p)}{dp}\frac{\partial p}{\partial y}$$

substituting this back into (2) gives

$$\left[A(x,y)\left(\frac{\partial h}{\partial x}f(p) + h\frac{df(p)}{dp}\frac{\partial p}{\partial x}\right) + B(x,y)\left(\frac{\partial h}{\partial y}f(p) + h\frac{df(p)}{dp}\frac{\partial p}{\partial y}\right) + C(x,y)hf(p)\right] = 0.$$

this then rearranges to

$$\left[A(x,y)\frac{\partial h}{\partial x} + B(x,y)\frac{\partial h}{\partial y} + C(x,y)h\right] f(p) + \left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right] h\frac{df(p)}{dp} = 0. \quad (3)$$

The first term turns to zero if function $h(x,y)$ is a solution of the original PDE. No matter how simple.

If we then assume that we have found the function $h(x,y)$ then only the second term remains, and we have:

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right] h \frac{\partial f(p)}{\partial p} = 0 \quad (4)$$

Non-trivial solutions of (4) can be found by solving

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0$$

In other words, the problem reduced to the previously considered case of $C(x,y) = R(x,y) = 0$

Case 3:

Solutions of Non-Homogeneous equations

Given (1)

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u(x,y) = R(x,y)$$

and a boundary condition

$$u(0,y) = g(y)$$

we can write that the general solution of this equation is

$$u(x,y) = v(x,y) + w(x,y)$$

where

$v(x,y)$ is any solution of the non-homogenous PDE and

$w(x,y)$ is a general solution of the homogenous equation

Homogeneity — C is a constant

1) Homogenous PDEs:

if $u(x,y)$ is a solution of a PDE then

$V(x,y) = C \cdot u(x,y)$ is too

2) Homogeneous Boundary Conditions

if $u(x,y)$ satisfies the boundary conditions, then $V(x,y) = C \cdot u(x,y)$ also satisfies the same boundary conditions.

Linear 2nd order PDEs

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = R(x,y)$$

is the general form of a 2nd order PDE.

- In this section we will be working with equation of the type:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

, A, B, C are constants with respect to x and y.

- $B^2 > 4AC$ – *hyperbolic* equations; they describe propagating oscillations (waves);
- $B^2 = 4AC$ – *parabolic* equations; they describe transport processes, such as heat conduction and diffusion;
- $B^2 < 4AC$ – *elliptic* equations describe stationary systems, such as steady electric fields and temperature distributions.

Similarly to the case of the first order PDE. Consider solutions in the form $u(x,y) = f(p)$.

$u(x,y) = f(p)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{df(p)}{dp}\frac{\partial p}{\partial x}\right) = \frac{d^2 f(p)}{dp^2}\frac{\partial p}{\partial x}\frac{\partial p}{\partial x} + \frac{df(p)}{dp}\frac{\partial^2 p}{\partial x^2} \quad (2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{df(p)}{dp}\frac{\partial p}{\partial y}\right) = \frac{d^2 f(p)}{dp^2}\frac{\partial p}{\partial y}\frac{\partial p}{\partial y} + \frac{df(p)}{dp}\frac{\partial^2 p}{\partial y^2} \quad (3)$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{df(p)}{dp}\frac{\partial p}{\partial x}\right) = \frac{d^2 f(p)}{dp^2}\frac{\partial p}{\partial y}\frac{\partial p}{\partial x} + \frac{df(p)}{dp}\frac{\partial^2 p}{\partial y\partial x} \quad (4)$$

- If we then let

$\frac{\partial p}{\partial x} = \text{constant}$, $\frac{\partial p}{\partial y} = \text{constant}$ $(\Rightarrow)$

it follows that p is a linear function of x and y $\therefore$ $p = ax + by$ (5)

$\therefore$ $u(x,y) = f(ax + by)$ (6)

- All second derivatives will have the same common factor of $\frac{d^2 f(p)}{dp^2}$
- After substituting in ⑤ to ②, ③, and ④, we then arrive at the fact that

$$(A^2 a + Bab + Cb^2)\frac{\partial^2 f(p)}{\partial p^2} = 0$$

$$\Rightarrow \underbrace{\frac{d^2 f(p)}{dp^2} = 0}_{(\Rightarrow)\ \downarrow \text{(implies)}} \quad \text{or/and} \quad A a^2 + Bab + Cb^2 = 0$$

$f(p) = kp + m$, where k and m are constants with respect to p

∴ Refering back to ⑤ we can say that

$u(x,y) = f(p) = kp + m = k(ax + by) + m = kax + kby + m$

or

$$u(x,y) = \alpha x + \beta y + \gamma$$

This solution solves our PDE but only for the condition that $\frac{df^2(p)}{dp^2} = 0$.

Making it a trivial solution.

To find non-trivial solutions we must then solve

$$Aa^2 + Bab + Cb^2 = 0$$

$$A\frac{a^2}{a^2} + B\frac{ab}{a^2} + C\frac{b^2}{a^2} = 0$$

$$A + B\frac{b}{a} + C\left(\frac{b}{a}\right)^2 = 0$$

let

$$\lambda = \frac{b}{a}$$

$$C\lambda^2 + B\lambda + A = 0$$

$$\therefore \lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4CA}}{2C} = \left(\frac{b}{a}\right)_{1,2}$$

hence p may have two values, depending on the the discriminant $D = B^2 - 4AC$

$$p_1 = a(x + \lambda_1 y) \quad , \quad p_2 = a(x + \lambda_2 y)$$

Which consequently means we get two solutions for $U(x,y)$, which are then superposed to give us:

$$U(x,y) = f(a(x+\lambda_1 y)) + g(a(x + \lambda_2 y))$$

Note that we used the discriminant

$$D = B^2 - 4AC$$

of the quadratic equation

$$C\lambda^2 + B\lambda + A = 0.$$

The classification of the 2nd order PDE we have introduced above, corresponds to the different values of this discriminant:

1. If $B^2 > 4AC$, the discriminant is positive and λ_1 and λ_2 are real; such equations are called *hyperbolic*.
2. If $B^2 < 4AC$, the discriminant is negative, i.e. λ_1 and λ_2 are complex; such equations are called *elliptic*.
3. If $B^2 = 4AC$, the discriminant is zero and $\lambda_1 = \lambda_2$; such equations are called *parabolic*.