PHAS2423 - Self-Study - Cartesian Tensors - Problems and Solutions

(1) The summation convention.
(a) Express the following using the summation convention.

$$\text{(a.1) } x'_i = \sum_{j=1}^{3} a_{ij}x_j; \qquad \text{(a.2) } T'_{kl} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ki}a_{lj}T_{ij}; \qquad \text{(a.3) } B'_{pqr} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} L_{pi}L_{qj}L_{rk}B_{ijk}.$$

(b) Write the following using explicit summation.

$$\text{(b.1) } C_{jknm}A_nB_{jk}; \qquad \text{(b.2) } (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})A_{ik}.$$

(c) Show that

$$\text{(c.1) } \delta_{ij}\delta_{jk} = \delta_{ik}; \qquad \text{(c.2) } \delta_{ii} = N.$$

(d) Evaluate

$$\text{(d.1) } \delta_{ij}\delta_{jk}\delta_{km}\delta_{im}; \qquad \text{(d.2) } \epsilon_{jk2}\epsilon_{k2j}; \qquad \text{(d.3) } \epsilon_{23i}\epsilon_{2i3}.$$

Solutions

(a.1) $x'_i = a_{ij}x_j$ **(a.2)** $T'_{kl} = a_{ki}a_{lj}T_{ij}$ **(a.3)** $B'_{pqr} = L_{pi}L_{qj}L_{rk}B_{ijk}$

(b.1)

$$C_{jknm}A_nB_{jk} = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{n=1}^{3} C_{jknm}A_nB_{jk}$$

(b.2)

$$(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj})A_{ik} = \sum_{i=1}^{3}\sum_{k=1}^{3} (\delta_{ij}\delta_{kl}A_{ik} + \delta_{il}\delta_{kj}A_{ik}) = \sum_{i=1}^{3} (\delta_{ij}A_{il} + \delta_{il}A_{ij}) = A_{jl}+A_{lj}$$

(c.1)

$$\delta_{ij}\delta_{jk} = \sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = \left\{ \begin{array}{ll} 1 & (\text{if} \quad i = k) \\ 0 & (\text{if} \quad i \neq k) \end{array} \right\} = \delta_{ik}$$

(c.2)

$$\delta_{ii} = \delta_{11} + \delta_{22} + ...\delta_{NN} = N$$

(d.1) Use the results of (c.1) and (c.2):

$\delta_{ij}\delta_{jk}\delta_{km}\delta_{im} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$ (or N in the case of N-dimensional space).

(d.2) It is sufficient to consider all combinations in which $j \neq k$ (all other contributions are equal to zero), and recall that $\epsilon_{ijk} = -\epsilon_{jik}$. Then,

$$\epsilon_{jk2}\epsilon_{k2j} = -\epsilon_{jk2}\epsilon_{kj2} = \epsilon_{jk2}\epsilon_{jk2} = 2.$$

(d.3) It is sufficient to consider all combinations in which i=1:

$$\epsilon_{23i}\epsilon_{2i3} = -\epsilon_{23i}\epsilon_{23i} = -(\epsilon_{23i})^2 = -(\epsilon_{231})^2 = -1.$$

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(2) Rotation.
(a) Orthogonality. An orthogonal matrix L has components L_{ij}. Evaluate the following:

(a.1) $L_{ij}L_{jk}$; (a.2) $L_{ji}L_{kj}$; (a.3) $L_{ij}L_{ik}$; (a.4)$L_{ij}L_{kj}$.

(b) Rotation. Show that the transformation matrix L for a rotation of the coordinate system by an angle θ about $\boldsymbol{e}_3$ axis is

$$L = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Consecutive rotations. Show that two consecutive rotations of the coordinate system by an angle θ about $\boldsymbol{e}_3$ axis is also a rotation about the same axis with the value of the rotation angle of 2θ.

Solutions
(a.1)

$$L_{ij}L_{jk} = (L^2)_{ik} = M_{ik} \qquad \text{(matrix } M = L^2 \text{ is also orthogonal)}$$

(a.2)

$$L_{ji}L_{kj} = L_{kj}L_{ji} = (L^2)_{ki} = M_{ki} = (M^T)_{ik} = (M^{-1})_{ik}$$

Alternatively,

$$L_{ji}L_{kj} = (L^T)_{ij}(L^T)_{jk} = (L^{-1})_{ij}(L^{-1})_{jk} = \left(\left(L^{-1}\right)^2\right)_{ik} = \left(\left(L^2\right)^{-1}\right)_{ik} = (M^{-1})_{ik}$$

(a.3)

$$L_{ij}L_{ik} = \delta_{jk} \qquad \text{(scalar product of columns } j \text{ and } k\text{)}$$

(a.4)

$$\text{(a.4) } L_{ij}L_{kj} = \delta_{ik} \qquad \text{(scalar product of rows } i \text{ and } k\text{)}$$

(b) See Fig. 2.2 of the lecture notes:

$$\begin{array}{rcr} x_1' &=& x_1\cos\theta + x_2\sin\theta \\ x_2' &=& -x_1\sin\theta + x_2\cos\theta \\ x_3' &=& x_3 \end{array}, \qquad \text{i.e.,} \qquad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c)

$$L^2 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & 0 \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$L^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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(3) Transformation of tensors.

(a) Contraction. Given that T_{ijk} and V_n are components of the 3rd order and 1st order tensors, respectively,
(a.1) Show that T_{iij} is a 1st-order tensor.
(a.2) Show that $T_{ijk}V_k$ is a 2nd-order tensor.

(b) Outer product. If quantities A_{ij} and B_{kl} are components of 2nd order tensors, show that quantities T_{ijkl} formed by $T_{ijkl} = A_{ij}B_{kl}$ is a 4th-order tensor.

(c) Vectors. For the case of a two-dimensional space
(c.1) Show that $\boldsymbol{v} = (x_2, -x_1)$ transforms as a vector under rotation of the coordinate system.
(c.2) Show that $\boldsymbol{v} = (x_2, x_1)$ is not a vector.

(d) Scalars.
(d.1) Show that the scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is, indeed, a scalar.
(d.2) Show that $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ is a scalar (assume that $\boldsymbol{v}$ is a vector).

(e) Higher order tensors. Demonstrate that matrix T represents a 2^{nd} order tensor:

$$\boldsymbol{T} = \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}.$$

Solutions

(a.1) Let $P_j = T_{iij}$. Then,

$$P_j' = T_{iij}' = L_{ip}L_{iq}L_{jr}T_{pqr} = \delta_{pq}L_{jr}T_{pqr} = L_{jr}T_{ppr} = L_{jr}P_r.$$

(a.2) Let $P_{ij} = T_{ijk}V_k$. Then,

$$P_{ij}' = T_{ijk}'V_k' = (L_{ip}L_{jq}L_{kr}T_{pqr})\,(L_{ks}V_s) = L_{ip}L_{jq}\delta_{rs}T_{pqr}V_s = L_{ip}L_{jq}(T_{pqs}V_s) = L_{ip}L_{jq}P_{pq}.$$

(b)

$$T_{ijkl}' = A_{ij}'B_{kl}' = (L_{ip}L_{jq}A_{pq})\,(L_{kr}L_{ls}B_{rs}) = L_{ip}L_{jq}L_{kr}L_{ls}(A_{pq}B_{rs}) = L_{ip}L_{jq}L_{kr}L_{ls}T_{pqrs}.$$

(c.1)

$$v_1' = x_2' = -x_1\sin\theta + x_2\cos\theta = v_2\sin\theta + v_1\cos\theta$$
$$v_2' = -x_1' = -(x_1\cos\theta + x_2\sin\theta) = v_2\cos\theta - v_1\sin\theta.$$

Thus,

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

i.e., upon rotation of the coordinate system components of $\boldsymbol{v}$ transform in the same way as coordinates. Hence, $\boldsymbol{v}$ is a vector.

(c.2)

$$v_1' = x_2' = -x_1\sin\theta + x_2\cos\theta = -v_2\sin\theta + v_1\cos\theta$$
$$v_2' = x_1' = x_1\cos\theta + x_2\sin\theta = v_2\cos\theta + v_1\sin\theta.$$

Thus,

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

i.e., components of $\boldsymbol{v}$ do not transform as coordinates. Hence, $\boldsymbol{v}$ is not a vector.

(d.1) Let $s = \boldsymbol{a}\cdot\boldsymbol{b}$. Then,

$$s' = \boldsymbol{a}'\cdot\boldsymbol{b}' = a_i'b_i' = (L_{ik}a_k)(L_{in}b_n) = (L_{ik}L_{in})a_kb_n = \delta_{kn}a_kb_n = a_nb_n = s.$$

(d.2)

$$\boldsymbol{\nabla}' \cdot \boldsymbol{v}' = \frac{\partial}{\partial x'_i} v'_i = \left(\frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} \right) (L_{ik} v_k) = L_{ij} L_{ik} \frac{\partial}{\partial x_j} v_k = \delta_{jk} \frac{\partial}{\partial x_j} v_k = \frac{\partial}{\partial x_k} v_k = \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

(e) In order to check whether T_{ij} are components of a tensor, one has to show that

$$T'_{ij} = L_{ip} L_{jq} T_{pq}, \quad \text{where} \quad T' = \begin{pmatrix} x'_2 x'_2 & -x'_1 x'_2 \\ -x'_1 x'_2 & x'_1 x'_1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

First, express T'_{ij} in terms of x_1 and x_2:

$$\begin{aligned}
T'_{11} &= (x'_2)^2 \\
&= (-x_1 \sin\theta + x_2 \cos\theta)^2 \\
&= x_1^2 \sin^2\theta + x_2^2 \cos^2\theta - 2 x_1 x_2 \sin\theta \cos\theta \\
\\
T'_{22} &= (x'_1)^2 \\
&= (x_1 \cos\theta + x_2 \sin\theta)^2 \\
&= x_1^2 \cos^2\theta + x_2^2 \sin^2\theta + 2 x_1 x_2 \sin\theta \cos\theta \\
\\
T'_{12} &= -x'_1 x'_2 \\
&= -(x_1 \cos\theta + x_2 \sin\theta)(-x_1 \sin\theta + x_2 \cos\theta) \\
&= x_1^2 \cos\theta \sin\theta - x_2^2 \sin\theta \cos\theta - x_1 x_2 (\cos^2\theta - \sin^2\theta) \\
\\
T'_{21} &= T'_{12}
\end{aligned}$$

Now calculate components of the transformed tensor T:

$$T'_{11} = \sum_{i=1}^{2} \sum_{j=1}^{2} L_{1i} L_{1j} T_{ij} \qquad T'_{22} = \sum_{i=1}^{2} \sum_{j=1}^{2} L_{2i} L_{2j} T_{ij}$$

$$T'_{12} = \sum_{i=1}^{2} \sum_{j=1}^{2} L_{1i} L_{2j} T_{ij} \qquad T'_{21} = \sum_{i=1}^{2} \sum_{j=1}^{2} L_{2i} L_{1j} T_{ij}$$

Then,

$$\begin{aligned}
T'_{11} &= (\cos\theta)^2(x_2^2) + \cos\theta\sin\theta(-x_1x_2) + \sin\theta\cos\theta(-x_1x_2) + (\sin\theta)^2(x_1^2) \\
&= x_1^2\sin^2\theta + x_2^2\cos\theta - 2x_1x_2\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
T'_{22} &= (-\sin\theta)^2(x_2^2) + (-\sin\theta)\cos\theta(-x_1x_2) + \cos\theta(-\sin\theta)(-x_1x_2) + (\cos\theta)^2(x_1^2) \\
&= x_1^2\cos^2\theta + x_2^2\sin\theta + 2x_1x_2\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
T'_{12} &= \cos\theta(-\sin\theta)(x_2^2) + \cos\theta\cos\theta(-x_1x_2) + \sin\theta(-\sin\theta)(-x_1x_2) + \sin\theta\cos\theta(x_1^2) \\
&= x_1^2\sin\theta\cos\theta - x_2^2\sin\theta\cos\theta - x_1x_2(\cos^2\theta - \sin^2\theta)
\end{aligned}$$

$$T'_{21} = T'_{12}$$

Clearly, both methods give the same expression for the values of T'_{kn}. Thus, $\boldsymbol{T}$ is a tensor.

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(4) Quotient theorem. Given that $\boldsymbol{A}$ is an arbitrary tensor and $\boldsymbol{B}$ is a non-zero tensor, prove the quotient theorem for the following cases:

(a) $X_iA_{ij} = B_j$ **(b)** $X_{ij}A_k = B_{ijk}$

Solutions

(a) $X_iA_{ij} = B_j$.

$$\begin{aligned}
X'_iA'_{ij} &= B'_j = L_{jp}B_p = L_{jp}(X_qA_{qp}) \\
&= L_{jp}X_q(L_{mq}L_{np}A'_{mn}) = (L_{jp}L_{np})L_{mq}X_qA'_{mn} = (\delta_{jn})L_{mq}X_qA'_{mn} \\
&= L_{mq}X_qA'_{mj}.
\end{aligned}$$

Replace the dummy index m by i and subtract the right hand side from the left hand side:

$$(X'_i - L_{iq}X_q)A'_{ij} = 0.$$

Since $\boldsymbol{A}$ is an arbitrary tensor,

$$X'_i = L_{iq}X_q,$$

i.e., $\boldsymbol{X}$ is a 1st order tensor.

(b) $X_{ij}A_k = B_{ijk}$.

$$
\begin{aligned}
X'_{ij}A'_k &= B'_{ijk} = L_{ip}L_{jq}L_{kr}B_{pqr} = L_{ip}L_{jq}L_{kr}X_{pq}A_r \\
&= L_{ip}L_{jq}L_{kr}X_{pq}(L_{mr}A'_m) = L_{ip}L_{jq}(L_{kr}L_{mr})X_{pq}A'_m \\
&= L_{ip}L_{jq}(\delta_{km})X_{pq}A'_m = L_{ip}L_{jq}X_{pq}A'_k.
\end{aligned}
$$

Subtract the right hand side from the left hand side:

$$(X'_{ij} - L_{ip}L_{jq}X_{pq})A'_k = 0.$$

Since $\boldsymbol{A}$ is an arbitrary tensor,

$$X'_{ij} = L_{ip}L_{jq}X_{pq},$$

i.e., $\boldsymbol{X}$ is a 2nd order tensor.

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(5) Application of tensors ϵ_{ijk} and δ_{ij} Use properties of the Levi-Civita and Kronecker tensors to prove the following identities for vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$:

(a)

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

(b)

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d}]\, \boldsymbol{c} - [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}]\, \boldsymbol{d}$$

(c) Find an explicit expression for the i^{th} component of vector $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$.

Solutions

(a) Using the expression for the k^{th} component of the vector product $\boldsymbol{u} = \boldsymbol{a} \times \boldsymbol{b}$:

$$u_k = \epsilon_{kmn}a_m b_n,$$

we obtain

$$\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) &= (\boldsymbol{a}\times\boldsymbol{b})_i(\boldsymbol{c}\times\boldsymbol{d})_i \\
&= (\epsilon_{ijk}a_jb_k)(\epsilon_{ilm}c_ld_m) = (\epsilon_{ijk}\epsilon_{ilm})a_jb_kc_ld_m \\
&= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})a_jb_kc_ld_m = \delta_{jl}\delta_{km}a_jb_kc_ld_m - \delta_{jm}\delta_{kl}a_jb_kc_ld_m \ . \\
&= a_jb_kc_jd_k - a_jb_kc_kd_j = (a_jc_j)(b_kd_k)-(a_jd_j)(b_kc_k) \\
&= (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})
\end{aligned}$$

(b) The i^{th} component of the vector product $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})$ is

$$\begin{aligned}
[(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})]_i &= \epsilon_{ijk}(\boldsymbol{a}\times\boldsymbol{b})_j(\boldsymbol{c}\times\boldsymbol{d})_k \\
&= \epsilon_{ijk}(\epsilon_{jpq}a_pb_q)(\epsilon_{krs}c_rd_s) \\
&= \epsilon_{jpq}(\epsilon_{kij}\epsilon_{krs})a_pb_qc_rd_s = \epsilon_{jpq}(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})a_pb_qc_rd_s \\
&= \epsilon_{jpq}\delta_{ir}\delta_{js}a_pb_qc_rd_s - \epsilon_{jpq}\delta_{is}\delta_{jr}a_pb_qc_rd_s \\
&= \epsilon_{jpq}a_pb_qc_id_j - \epsilon_{jpq}a_pb_qc_jd_i = [(\epsilon_{jpq}a_pb_q)d_j]\,c_i - [(\epsilon_{jpq}a_pb_q)c_j]\,d_i \\
&= [(\boldsymbol{a}\times\boldsymbol{b})_jd_j]\,c_i - [(\boldsymbol{a}\times\boldsymbol{b})_jc_j]\,d_i \\
&= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}]\,c_i - [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}]\,d_i.
\end{aligned}$$

Hence, in the vector form:

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}]\,\boldsymbol{c} - [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}]\,\boldsymbol{d}.$$

(c) The i^{th} component of vector $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a})$ is

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a}))_i = \epsilon_{ijk}(\boldsymbol{\nabla})_j(\boldsymbol{\nabla}\times\boldsymbol{a})_k = \epsilon_{ijk}\epsilon_{kmn}(\boldsymbol{\nabla})_j(\boldsymbol{\nabla})_m a_n = \epsilon_{ijk}\epsilon_{kmn}\frac{\partial^2 a_n}{\partial x_j\partial x_m}.$$

Use

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}.$$

Then,

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a}))_i = (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\frac{\partial^2 a_n}{\partial x_j\partial x_m} = \frac{\partial^2 a_n}{\partial x_i\partial x_n} - \frac{\partial^2 a_i}{\partial x_m\partial x_m} = \frac{\partial}{\partial x_i}(\boldsymbol{\nabla}\cdot\boldsymbol{a}) - \frac{\partial^2}{\partial x_m^2}a_i.$$

Finally,

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}))_i = [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a})]_i - \nabla^2 a_i,$$

i.e.,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a}.$$

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(6) A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate system the particles are at positions

$$\pm a(3,1,-1) \qquad \pm a(1,-1,3) \qquad \pm a(1,3,-1) \qquad \pm a(-1,1,3).$$

The body rotates about an axis passing through the origin. Show that, if the angular velocity and angular momentum vectors are parallel, then their ratio must be $40ma^2$, $64ma^2$, or $72ma^2$.

Solution. If a body rotates about a fixed axis, the vector of the angular velocity $\boldsymbol{\omega}$, vector of the orbital momentum $\boldsymbol{L}$, and the moment of inertia are related as

$$\boldsymbol{L} = I\boldsymbol{\omega},$$

where scalar I is defined for that rotation axis. If the rotation axis is not fixed, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ may not be parallel and the relation between them becomes

$$L_i = I_{ij}\omega_j,$$

where I_{ij} is a tensor of inertia. Hence, vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ become parallel if the rotation axis coincides with a principal axis of the tensor of inertia and, therefore, the ratio between $\boldsymbol{L}$ and $\boldsymbol{\omega}$ is equal to the principal moment of the tensor of inertia.

$$I_{ij} = \sum m(r^2\delta_{ij} - x_i x_j)$$

$$\begin{aligned}
I_{11} &= 2ma^2(4 \times 11 - 9 - 1 - 1 - 1) = 64ma^a \\
I_{22} &= 2ma^2(4 \times 11 - 1 - 1 - 9 - 1) = 64ma^2 \\
I_{33} &= 2ma^2(4 \times 11 - 1 - 9 - 1 - 9) = 48ma^2 \\
I_{12} &= -2ma^2(3 - 1 + 3 - 1) = -8ma^2 \\
I_{13} &= -2ma^2(-3 + 3 - 1 - 3) = 8ma^2 \\
I_{23} &= -2ma^2(-1 - 3 - 3 + 3) = 8ma^2
\end{aligned}$$

Hence,

$$I = 8ma^2 \begin{pmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

Solve characteristic equation $\det(\boldsymbol{I} - \lambda \boldsymbol{E}) = 0$:

$$\begin{vmatrix} 8-\lambda & -1 & 1 \\ -1 & 8-\lambda & 1 \\ 1 & 1 & 6-\lambda \end{vmatrix} = (8-\lambda)(\lambda^2 - 14\lambda + 47 - 2) = (8-\lambda)(\lambda-9)(\lambda-5) = 0.$$

Thus, principal moments are $40ma^2$, $64ma^2$, and $72ma^2$.