PHAS2423: Mathematical Methods for Theoretical Physics

Alexander Nico-Katz

2017-03-11

Contents

1 Cartesian Tensors **4**
- 1.1 Introduction . . . 4
- 1.2 Preliminaries . . . 4
  - 1.2.1 Spaces, Subspaces, Surfaces, and Curves . . . 4
- 1.3 Einstein Summation Convention . . . 4
- 1.4 Properties of Tensors . . . 5
  - 1.4.1 Rank: . . . 5
  - 1.4.2 Sum and Difference: . . . 5
  - 1.4.3 Symmetric and Antisymmetric Tensors: . . . 5
  - 1.4.4 Tensor Decomposition: . . . 6
  - 1.4.5 Outer Product: . . . 6
  - 1.4.6 Contraction: . . . 6
  - 1.4.7 Inner Product: . . . 6
  - 1.4.8 Gradient of a Vector: . . . 6
  - 1.4.9 Transformation Rule: . . . 7
  - 1.4.10 Quotient Theorem: . . . 7
- 1.5 Changing Basis: . . . 7
- 1.6 Important Tensors: . . . 8
  - 1.6.1 Kronecker Delta . . . 8
  - 1.6.2 Levi-Civita Tensor: . . . 8

2 Linear Ordinary Differential Equations **9**
- 2.1 Definitions . . . 9
  - 2.1.1 Linearity: . . . 9
  - 2.1.2 Homogeneity: . . . 9
  - 2.1.3 Ordinary and Partial: . . . 9
- 2.2 Methods of Solving Linear ODEs . . . 9
  - 2.2.1 Integrating Factor: . . . 9
  - 2.2.2 Characteristic Equation: . . . 9
  - 2.2.3 Laplace Transform: . . . 10
  - 2.2.4 Variation of Parameters: . . . 10
  - 2.2.5 Green's Functions: . . . 11
- 2.3 What to use and when: . . . 14

3 Sturm-Liouville Theorem **15**
- 3.1 Self-Adjoint Hermitian Operators . . . 15
- 3.2 Properties of Eigenfunctions and Eigenvalues of $\mathcal{SL}(x)$ operator . . . 15
- 3.3 Gram-Schmidt Orthogonalisation . . . 15

4 Numerical Methods **16**
4.1 Tridiagonal Matrices 16
4.2 Kinds of Methods 16
4.3 Euler Method 17
4.3.1 Forwards Euler: 17
4.3.2 Backwards Euler: 17
4.3.3 Centered difference method: 17

5 Integral Transforms **17**
5.1 Fourier Transform 17
5.1.1 Formally 18
5.2 Laplace Transform 18
5.3 Convolution 19
5.3.1 Fourier Convolution 19
5.3.2 Laplace Convolution 19

6 Linear Partial Differential Equations **20**
6.1 1^{st} Order 20
6.2 2^{nd} Order 20
6.3 Common Linear PDEs 21

7 Fluid Mechanics **21**
7.1 Navier-Stokes' Equation 22

1 Cartesian Tensors

1.1 Introduction

Tensors are mathematical objects that describe linear relationships between other tensors. Because of this they must be independant of choice of basis/coordinate system. For cartesian tensors (the only kind we will consider), this is preserved trivially, as any new cartesian coordinate system is simply a translation and a rotation of any other, so transforming the tensor in the opposite way that we transform the coordinate system will render the entire object invariant. If we transform tensors appropriately, the scalar temperature of a point will still read 200K, and the vector position of a cat will still point at that same cat - regardless of what coordinate system we choose. Scalars, vectors, and linear maps (certain matrices) are all examples of tensors.

1.2 Preliminaries

1.2.1 Spaces, Subspaces, Surfaces, and Curves

An N-Dimensional space is simply defined as the set of all points corresponding to all possible values of N real, independant variables. A curve is defined as the set of points x_i where:

$$x_i = f_i(u) \tag{1}$$

So that all x_i are parameterised by a single variable u which is conceptually equivalent to the distance along the curve. A subspace is the set of points that exist within a space x_i where:

$$x_i = f_i(u_1, u_2, ...u_M) \tag{2}$$

Where M must be less than the dimension N of the space in which the subspace exists. Clearly this means that the subspace is parameterised by less independant variables u_j than the space is by x_i. If $M = N - 1$, then the subspace is called a hypersurface. A 2D plane in 3D space is an example of a subspace, and a hypersurface.

1.3 Einstein Summation Convention

The Einstein summation convention is the most appropriate method for describing and manipulating tensors. Indices are used to describe a tensor, repeated or 'dummy' indices are used to denote a sum over tensor(s). For example, the matrix:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{3}$$

Can be described using the following notation:

$$M = M_{i,j} \tag{4}$$

Where i and j take values between 1 and 3. Similarly, vectors can be described by a single index:

$$v = v_i = (v_1, v_2, v_3, ..., v_n) \tag{5}$$

And scalars by no index at all:

$$T = T = T_0 \tag{6}$$

Repeating an index indicates summation over that index. This is clear in the case of the matrix $M_{i,j}$ with the repeated index $M_{i,i}$:

$$M_{i,i} = \sum_i M_{i,i} = m_{11} + m_{22} + m_{33} \tag{7}$$

Which clearly gives the trace of a matrix. By analogy, the dot product of the vector a with the vector b is simply defined as:

$$a_i b_i = \sum_i a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + ...a_n b_n \tag{8}$$

1.4 Properties of Tensors

1.4.1 Rank:

The rank of a tensor is simply the number of indices required to describe it fully.

1.4.2 Sum and Difference:

The sum or difference of two tensors is simply the elementwise addition or subtraction of one to/from another. This can clearly only happen between tensors of equal rank.

1.4.3 Symmetric and Antisymmetric Tensors:

A tensor if symmetric with respect to the n^th and m^th indices if, when those indices are swapped, the resulting tensor is equal to the original. A tensor is antisymmetric with respect to the n^th and m^th indices if, when those indices are swapped, the resulting tensor is the negative of the original. A rank 2 tensor is symmetric/antisymmetric if it's transpose is equal to/the negative of the original tensor.

1.4.4 Tensor Decomposition:

All tensors can be written as an arithmetic sum of symmetric and antisymmetric ones by the following relation for rank 2 tensors, and it's logical extension to rank N tensors:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) = S_{ij} + A_{ij} \tag{9}$$

1.4.5 Outer Product:

The outer product takes two tensors of ranks N and M and generates a rank $N + M$ tensor from them, it is defined as:

$$T_{ij} = a_i b_j \tag{10}$$

1.4.6 Contraction:

By equating and summing over two subscripts, the contraction of a rank N tensor produces another tensor of rank $N - 2$. In the case of a rank 2 tensor, the contraction simply gives the trace as seen in equation 7. Vector contraction is ill-defined, but constructing a matrix $T_{ij} = a_i b_j$ and then contracting T will give the dot product of $\hat{a}$ and $\hat{b}$.

1.4.7 Inner Product:

The inner product of two tensors is defined by simply equating one or more of the indices in each tensor. Clearly if we take a rank 5 tensor A_{lmnop} and a rank 3 tensor B_ijk, the possible inner products are:

$$C_{lmnojk} = A_{lmnop} B_{pjk} \tag{11}$$

$$C_{lmnk} = A_{lmnop} B_{pok} \tag{12}$$

$$C_{lm} = A_{lmnop} B_{pon} \tag{13}$$

Tensors of rank 6, 4, and 2 respectively.

1.4.8 Gradient of a Vector:

The gradient of a vector $\hat{v}$ form the components of a rank 2 tensor by the following relation:

$$T_{ij} = \frac{\partial v_i}{\partial x_j} \tag{14}$$

1.4.9 Transformation Rule:

An object is only a tensor if it transforms according to the following transformation rule:

$$T'_{i_1 i_2 ... i_N} = L_{i_1 j_1} L_{i_2 j_2} ... L_{i_N j_N} T_{j_1 j_2 ... j_N} \tag{15}$$

Where the transformed tensor T' of the rank N tensor T is constructed by applying N linear maps to the tensor T. Only if this condition is met is T a tensor. From this, the fact that the inner product, outer product, and contraction of tensors produces another tensor.

1.4.10 Quotient Theorem:

The quotient theorem simply states that contraction of two objects only produces a tensor if both original objects are tensors. Thus, if A is an arbitrary tensor of rank N, C is shown to be a tensor of rank $N+M-2$, and:

$$C_{i_1 ... i_N j_1 ... j_M} = A_{i_1 ... \alpha ... i_N} B_{j_1 ... \alpha j_M} \tag{16}$$

Then B must be a tensor of rank M. This theorem just lets us assume that certain things are tensors, which is a useful tool to have in proofs, derivations etc..

1.5 Changing Basis:

If we change the basis of a vector according to:

$$\hat{v}_i = x_i \hat{e}_i = x'_i \hat{e}'_i \tag{17}$$

$$\hat{e}'_i = S_{ji} \hat{e}_j \tag{18}$$

Then:

$$\hat{v}_i = x'_i S_{ji} \hat{e}_j \tag{19}$$

$$x_i = S_{ij} x'_i \tag{20}$$

$$x'_i = S^{-1}_{ij} x_j \tag{21}$$

Assuming the linear map S is a rotation (orthonormal), it's inverse is just it's transpose, so:

$$x_i = S_{ij} x'_j \tag{22}$$

$$x'_i = S_{ji} x_j \tag{23}$$

Which gives us a method of changing basis without changing where the object points, thus the object is a vector. S is defined sensibly as:

$$S_{ij} = \frac{\partial x_i}{\partial x'_j} \tag{24}$$

Which remains sensible when considering the inverse and transpose of the rotation S. Note that - for an orthogonal transformation L_{ij} we get:

$$L_{ij}^{-1} = L_{ij}^{T} = L_{ji}$$

And - very importantly:

$$L_{ij}L_{jk} = L_{ik}^{2}$$

$$L_{ij}L_{ik} =$$

1.6 Important Tensors:

1.6.1 Kronecker Delta

The Kronecker delta is the discretised Dirac delta function, it is defined as below:

$$\delta_{ij} = \begin{cases} i = j \ \rightarrow \ \delta_{ij} = 1 \\ i \neq j \ \rightarrow \ \delta_{ij} = 0 \end{cases} \tag{25}$$

The following property of δ_{ij} should be clear:

$$\delta_{ij}L_{ik} = \delta_{1j}L_{1k} + \delta_{2j}L_{2k} + \delta_{3j}L_{3k} + ... \tag{26}$$

Thus, for an example value $j = 1$, the value of L_1k is extracted, all other terms fall to zero. Thus:

$$\delta_{ij}L_{ik} = L_{jk} \tag{27}$$

Moreover, it should be clear that:

$$\delta_{ii} = N \tag{28}$$

Where N is the maximum value of the indices i and j.

1.6.2 Levi-Civita Tensor:

Also called the rank 3 permutation tensor, it is defined as follows:

$$\epsilon_{ijk} = \begin{cases} \text{Even permutation of indices} \rightarrow \epsilon_{ijk} = 1 \\ \text{Odd permutation of indices} \rightarrow \epsilon_{ijk} = -1 \\ \text{Neither permutation of indices} \rightarrow \epsilon_{ijk} = 0 \end{cases} \tag{29}$$

It can also be generalised up to any rank. Clearly the rank of the tensor is the same as the maximum value of each of it's indices. Otherwise the tensor wouldn't make sense. The following properties are all properties of ϵ_{ijk}:

$$\delta_{ij}\epsilon_{ijk} = 0 \tag{30}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{31}$$

$$\epsilon_{ipq}\epsilon_{jpq} = 2\delta_{ij} \tag{32}$$

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{33}$$

The cross product is also defined as:

$$\hat{c} = \hat{a} \times \hat{b} = \hat{e}_i a_j b_k \epsilon_{ijk} \tag{34}$$

2 Linear Ordinary Differential Equations

2.1 Definitions

2.1.1 Linearity:

A differential equation is linear if each term contains no instances or a single instance of the function or it's derivatives. Summarised as the following series:

$$\sum_n a_n(x)y^{(n)}(x) = b(x) \tag{35}$$

2.1.2 Homogeneity:

A differential equation is homogeneous if each term contains an instance of the function or it's derivatives. In short, if $b(x) = 0$ in equation 35.

2.1.3 Ordinary and Partial:

A differential equation is ordinary if the function is a function of a single variable, and all derivatives are total derivatives. A differential equation is partial if the function is a function of multiple variables, and the derivatives are partial derivatives.

2.2 Methods of Solving Linear ODEs

If it's separable, just solve it. It's not that hard.

2.2.1 Integrating Factor:

$$y(x) = e^{-S(x)}\left[\int Q(x)e^{S(x)}dx + C\right] \tag{36}$$

$$S(x) = \int P(x)dx \tag{37}$$

2.2.2 Characteristic Equation:

The characteristic equation of a differential equation is basically just replacing the derivatives with λs like so:

$$y''(x) + ay'(x) + by(x) = g(x) \;\rightarrow\; \lambda^2 + a\lambda + b = 0 \tag{38}$$

Solving will give the homogeneous (complimentary) solution as follows:

$$y(x) = \begin{cases} \lambda_1 \neq \lambda_2 & \rightarrow & Ae^{\lambda_1 x} + Be^{\lambda_2 x} \\ \lambda_1 = \lambda_2 & \rightarrow & Ae^{\lambda x} + Bxe^{\lambda x} \\ \lambda_{1,2} = a \pm ib & \rightarrow & e^{ax}A\sin(bx + B) \end{cases} \tag{39}$$

2.2.3 Laplace Transform:

The Laplace transform changes the variable that the parameter takes, it can be used to turn differential equations into algebraic ones. The transform is defined as:

$$L[f(x)] = \int_0^\infty f(x)e^{-px}dx = F(p) \tag{40}$$

This transform is linear and relatively easy to compute, for inverse laplace transforms, consult a table - you can't do it by yourself. Clearly, by integration by parts, the following is true:

$$L[f'(x)] = \int_0^\infty f'(x)e^{-px}dx = pL[f(x)] - f(0) \tag{41}$$

Now assuming $f(x)$ and all of it's derivatives are 0 at $x = 0$, we get:

$$L[af''(x) + bf'(x) + cf(x)] = L[g(x)] \tag{42}$$

$$F(p) = \frac{G(p)}{ap^2 + bp + c} \tag{43}$$

Then consult a table of inverse Laplace transforms to find out what the hell $F(p)$ really is.

2.2.4 Variation of Parameters:

2.2.4.1 Wronskian:

To find out if a set of functions $\{y_1, y_2, ...y_n\}$ are linearly independant, we must find only a single solution to the equation:

$$c_1y_1 + c_2y_2 + ... + c_ny_n = 0 \tag{44}$$

In which all constants c_i are 0. To this end we construct the Wronskian as the determinant of the following matrix:

$$W(y_1, y_2, ...y_n)(x) = \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{vmatrix} \tag{45}$$

From what we know of matrices and simultaneous equations, if $W \neq 0$, then they're linearly independant, otherwise, they may not be. We use the Wronskian to determine whether or not a set of functions (like those that comprise complimentary solutions to differential equations) are linearly independant.

2.2.4.2 Solution:

If there exists an inhomogenous ordinary differential equation of the form:

$$y^{(n)}(x) + \sum_{i=0}^{n-1} a_i y^{(i)}(x) = f(x) \tag{46}$$

If the homogeneous solution (complimentary solution) is of the form:

$$y_c = \sum_{i=0}^{n} b_i y_i(x) \tag{47}$$

Where all y_i are linearly independant ($W \neq 0$), we can derive the particular solution as:

$$y_p = \sum_{i=0}^{n} k_i(x) y_i(x) \tag{48}$$

Where the k_i can be determined by imposing a further $n-1$ conditions (n including the original ODE) on them. We can construct the following:

$$\mathbf{W}\hat{k'} = \hat{R} \tag{49}$$

Where $\mathbf{W}$ is the matrix used to form the wronskian, $\hat{k'}$ is the vector containing all derivatives k_i', and $\hat{R}$ is the residue vector in which all elements are zero except the last one which is $b(x)$. By Cramer's rule, the following is then true:

$$k_i' = \frac{W_i}{W} \tag{50}$$

Where W_i is the Wronskian with the i^{th} column replaced by the residue column vector $\hat{R}$, and W is the unchanged Wroksian. Integrating these solutions gives the coefficients k_i of the particular solution terms. This allows us to derive the particular solution from the complimentary one.

2.2.5 Green's Functions:

2.2.5.1 Dirac Delta Function:

The Dirac delta function is defined as:

$$\delta(x) = \begin{cases} x = 0 \ \rightarrow \ \delta_(0) = \infty \\ x \neq 0 \ \rightarrow \ \delta(x) = 0 \end{cases} \tag{51}$$

And:

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \tag{52}$$

By substituting $t - t_0$ for x, it is clear that this produces an infinitesimally thin, infinitely tall distribution centered about t_0 and 0 everywhere else. This function can only be investigated by using probe

functions inside integrals, using which, the following properties can be discovered:

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \tag{53}$$

Which is equivalent to multiplying the function $f(x)$ at x_0 by the entire area underneath $\delta(x-x0)$. This clearly gives $1f(x_0) = f(x_0)$. Furthermore:

$$\delta(x) = \delta(-x) \tag{54}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{55}$$

$$L[\delta(x-x_0)] = e^{-px_0} \tag{56}$$

The properties of the derivatives of the dirac delta function are tested using probe functions and integration by parts to achieve the following results:

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-x_0)dx = (-1)^n f^{(n)}(x_0) \tag{57}$$

$$x\delta(x) = 0 \tag{58}$$

$$x\delta'(x) = -\delta(x) \tag{59}$$

$$x^2\delta''(x) = 2\delta(x) \tag{60}$$

Note that the Gaussian taken to infinitesimal width approaches the dirac delta function, but it is a true function as opposed to this weird stuff we need to probe.

2.2.5.2 Solutions:

An ODE can be represented by the application of an operator to a function, equated to a separate, independant function in the same variable:

$$\mathcal{L}(x)y(x) = f(x) \tag{61}$$

Assuming that $y(x)$ can be determined as follows by a general function $G(x,t)$ as follows:

$$y(x) = \int_a^b G(x,t)f(t)dt \tag{62}$$

Using this, and the Dirac delta function we can clearly rewrite equation 61 as:

$$\int_a^b \mathcal{L}(x)G(x,t)f(t)dt = f(x) = \int_a^b \delta(t-x)f(t)dt \tag{63}$$

Thus:

$$\mathcal{L}(x)G(x,t) = \delta(t-x) \tag{64}$$

And we can solve for the unknown $G(x,t)$, that will then let us solve for $y(x)$. The following conditions must first be imposed on $G(x,t)$ however:

- If $y(a) = y(b) = 0$ we can impose the same conditions on $G(a,t) = G(b,t) = 0$. For non-zero boundary conditions, we must be cleverer. Also kinda swept under the rug of this course.

- As $x = t$ gives a singularity (infinity), Green's function is defined differently for $x > t$ and $x < t$.

- These two definitions of Green's function are continuous, but their derivatives aren't, with the difference between $G^{(n)}_{x>t}(x,t) \text{and} G^{(n)}_{x<t}(x,t)$ at $x = t$ being $\frac{1}{a_{n+1}(x)}$. Where $a_{n+1}(x)$ is the coefficient of the $(n+1)^{th}$ derivative in the operator $\mathcal{L}$.

2.3 What to use and when:

- **Separation of variables** if trivial:
  - Separate variables.
  - Solve.
  - If you can't do this, just give up now.
- **Integrating factor** for simple first order ODE's:
  - Separate into appropriate form.
  - Find $S(x)$.
  - Use to find $y(x)$.
- **Characteristic equation** for simple second order ODE's:
  - Isolate second derivative term (coefficient = 1).
  - Solve characteristic equation and select appropriate complimentary solution.
  - Guess particular solution.
- **Laplace transform** if you're told to use the Laplace transform:
  - Check that Laplace transform conditions are met.
  - Apply Laplace transform.
  - Rearrange for transformed function.
  - Look up solution in inverse Laplace transform tables.
- **Variation of Parameters** if you have the time, and there's no other way, this WILL get you the answer eventually:
  - Isolate highest derivative term (coefficient = 1).
  - Find complimentary solution to homogeneous equation.
  - Construct Wronskian matrix, make sure wronskian doesn't equal zero. If it does, do Green's or something idfk smh.
  - Solve Cramer's rule and integrate for coefficients of corresponding terms in particular solution.
- **Green's Function** is surprisingly easy and useful, use as much as possible, especially if boundary conditions are nice and derivatives are hard:
  - Extract operator.
  - $\mathcal{L}(x)G(x,t) = \delta(t-x)$ and solve like standard ODE for G(x, t).
  - Impose contraints and boundary conditions, remember the different functions for $x > t$, $x < t$.
  - $y(x) = \int_a^b G(x,t)f(t)dt$ and solve for y(x).

3 Sturm-Liouville Theorem

First of all the operator looks like this:

$$\mathcal{SL}(x) = -\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] \tag{65}$$

And:

$$\mathcal{SL}(x)y(x) = \lambda\rho(x)f(x) \tag{66}$$

Where y(x) is an eigenfunction of the $\mathcal{SL}(x)$ operator, and λ is an eigenvalue. It's linear, and all second-order linear ODEs can be recast as equation 66 using a Sturm-Liouville operator.

3.1 Self-Adjoint Hermitian Operators

What the fuck is this I don't understand.

3.2 Properties of Eigenfunctions and Eigenvalues of $\mathcal{SL}(x)$ operator

3.3 Gram-Schmidt Orthogonalisation

For a set of vectors $v_1, v_2, v_3, ...$, they can be orthogonalised by taking each vector in succession, and subtracting from it it's own projection onto each vector before:

$$h_1 = v_1$$

$$h_2 = v_2 - \text{proj}_{v_2}(v_1)v_1$$

$$h_3 = v_3 - \text{proj}_{v_3}(v_1)v_1 - \text{proj}_{v_3}(v_2)v_2$$

Where the 'proj' function is the projection of a vector onto another vector - it is simply the inner product (dot product for vectors, integration for functions) of the two vectors/functions.

And so on so forth. Clearly this ensures that no part of h_2 projects onto h_1, and no part of h_3 projects onto h_1 or h_2. These vectors clearly aren't normalised, so we must normalise as follows:

$$g_n = \frac{1}{C}h_n$$

$$C^2 = v_1 \cdot v_2$$

For vectors, or:

$$C^2 = \int_b^a f_1 f_1 dx$$

This generates an orthonormal basis from a given set of vectors/functions.

4 Numerical Methods

4.1 Tridiagonal Matrices

Tridiagonal matrices have elements on the leading diagonal and immediate off-diagonals. By applying this matrix to a vector $\hat{x}$, we can consider the effect of the neighbours of each vector element. This is useful for considering physical systems which evolve based on their nearest neighbours. Clearly the following equation is obtained:

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = y_i$$

Now assuming that each x_{i-1} is related to x_i by a set of numbers δ_k and λ_k, we get the following:

$$x_{i-1} = \delta_{i-1} x_i + \lambda_{i-1}$$

We can substitute this into the original equation to obtain a recurrence relation between x_i and x_{i+1}. If we then rearrange this to isolate x_i, we can define the next values of our number sets: δ_i and λ_i in terms of $a_i, b_i, c_i, y_i, \delta_{i-1}$, and λ_{i-1}:

$$\delta_i = -\frac{c_i}{a_i \delta_{i-1} + b_i}, \qquad \lambda_i = \frac{y_i - a_i \lambda_{i-1}}{a_i \delta_{i-1} + b_i}$$

This lets us generate our original vector $\hat{x}$ numerically from the resultant vector $\hat{y}$, all without inverting the tridiagonal matrix - a handy tool!

Please note that this procedure is **correctly defined** if the denominator isn't zero for all i, and is **stable** if $|\delta_i| < 1$. Both of these conditions must be met for the procedure to work, and both conditions are satisfied if the matrix is **diagonal-dominant**, i.e: $|b_i| > |a_i| + |c_i|$ for all i.

4.2 Kinds of Methods

- **Explicit methods** are methods in which $y(x + \Delta x)$ is generated from $y(x)$ directly, and they are typically sensitive to the size of Δx.
- **Implicit methods** are methods in which $y(x + \Delta x)$ is generated from $y(x)$ as a solution of another function containing both of these terms. They are typically less sensitive to the size of Δx than explicit methods, but are trickier to compute and implement.
- **Stiff equations** are just differential equations for which numerical solutions are unstable unless Δx is very small indeed.

4.3 Euler Method

4.3.1 Forwards Euler:

This method is explicit and very simple. By taking Δx to be very small on a grid of x-coordinates, we get:

$$y_{i+1} = y_i + y_i' \Delta x \tag{67}$$

4.3.2 Backwards Euler:

This method is implicit but very simple still. By taking Δx to be very small on a grid of x-coordinates, we get:

$$y_i = y_{i-1} + f[x_i, y_i] \Delta x \tag{68}$$

Note how the equation must be solved for y_i rather than y_i being entirely dependant on known terms. Thus it is implicit.

4.3.3 Centered difference method:

5 Integral Transforms

The purpose of an integral transform is to shift an equation in a certain variable to a corresponding equation in another one that is easier to solve. For example,

5.1 Fourier Transform

First let's start with a weird question: how much of a certain frequency is there in a periodic function $f(t)$? On a really crappy intuitive level we get the dreadful equation:

$$\text{Amount of } e^{i2\pi\nu t} \text{ in } f(t) = \frac{f(t)}{e^{i2\pi\nu t}} = f(t)e^{-i2\pi\nu t}$$

Where we've divided what we have by what we want. This is conceptually identical to "how many 8's are there in 48". This gives us a meaningless value for a given frequency at a specific time. But what if we integrate over the whole time range?

$$\int_{-\infty}^{\infty} f(t)e^{-i2\pi\nu t} dt$$

Gives us the total 'amount' of $e^{i2\pi\nu t}$ in $f(t)$ across all values of t. This will hold true no matter what ν we plug in, and so converts $f(t)$ into $g(h)$ giving us a function in frequency rather than time. So rather than seeing $f(t)$ as a function of time, we see it as a function of frequency - or rather, for a given frequency, how much of that frequency is in the function $f(t)$.

Clearly, if certain frequencies have very high values of $g(\nu)$, they dominate the wave, and less common frequencies can be dropped to smooth out the waveform.

5.1.1 Formally

Formally we define the Fourier transform as:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

And the inverse Fourier transform as:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{it\omega} d\omega$$

Which is just a shift back in the opposite direction. Note that the normalisation constant $\frac{1}{\sqrt{2\pi}}$ is due to the fact that us physicists use angular frequency instead of frequency:

$$\omega = 2\pi\nu$$

Instead of breaking the symmetry of the Fourier and inverse Fourier transforms by assigning them different normalising coefficients (namely giving the inverse Fourier a coefficient of $\frac{1}{2\pi}$ instead of 1), we normalise each by dividing by $\sqrt{2\pi}$, so that they 'meet in the middle' and basically just become easier to remember.

Note that the Fourier transform *must* transform a function across the entire real line, and so the function cannot grow exponentially - it must be periodic or normalisable. Upon Fourier transform: functions in time become functions in angular frequency, and functions in position become functions in momentum.

5.2 Laplace Transform

Laplace transform is a special case of the Fourier transform where we define: $s = i\omega$[1]:

$$L(p) = \int_{0}^{\infty} f(x) e^{-sx} dx$$

Where we only take the real line from zero to infinity - this is because, clearly, the exponential term will grow *pause for suspense* exponentially as $x \to -\infty$. Because of this, the inverse Laplace is very difficult to do, so we can set the normalisation constant to 1 rather than $\frac{1}{\sqrt{2\pi}}$. Note that - along the positive part of the real line - we are effectively multiplying $f(x)$ by an exponentially decaying function. This allows us to Laplace transform exponentially growing functions that we normally couldn't with Fourier transforms.

[1] Not always, but for our purposes, why not.

Another hugely important trick is that it converts differential equations into algebraic ones. As to conceptually how this happens, I have no earthly clue.

Conveniently, the Laplace transforms are rather unique for a given $f(x)$, so there is a table of extant inverse Laplace transforms that we can use to 'guess' the original function given it's transform.

5.3 Convolution

The convolution of a two functions $g(t)$ and $f(t)$ is:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Note by transformation of variables that:

$$(f * g)(t) = (g * f)(t)$$

Conceptually, this simply says that we move a kernel function $g(x-\tau)$ over the function $f(\tau)$ for all τ. We then find the area bounded by this intersection each τ, and call that $(f * g)(t = \tau)$. Clearly, the function $(f * g)(t)$ just gives the area bounded by both functions for a given t where t is 'how far into the convolution we are'.

5.3.1 Fourier Convolution

Consider the multiplication of two Fourier transforms:

$$F_f(k)F_g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t)e^{-ik(s+t)}dsdt$$

Now by the substitution of $p = s+t$ and changing ds to dp we get what looks like a convolution inside a Fourier transform:

$$\sqrt{2\pi}F_f(k)F_g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(p-t)g(t)dt\right]e^{-ikp}dp$$

$$\sqrt{2\pi}F_f(k)F_g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[(g * f)(p)\right]e^{-ikp}dp$$

$$\sqrt{2\pi}F_f(k)F_g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[(f * g)(p)\right]e^{-ikp}dp$$

So the product of the Fourier transforms of two functions is equal to $\frac{1}{2\pi}$ times the Fourier transform of their convolution.

5.3.2 Laplace Convolution

As the Laplace transform is a special case of the Fourier transform, and one with a normalisation constant of 1, by analogy: The product of the Laplace transforms of two functions is equal to the Laplace transform of their convolution.

6 Linear Partial Differential Equations

6.1 1^{st} Order

First order PDEs take the form:

$$A\partial_x u + B\partial_y u + Cu = R \tag{69}$$

Where $u \equiv u(x, y)$ is the function being investigated, and A, B, C and R can all be functions of x and y as well. We seek solutions of the form:

$$u(x, y) = \underbrace{h(x, y)f(p)}_{\text{Homogeneous}} + \underbrace{g(x, y)}_{\text{Particular}} \tag{70}$$

Where the following is the method used, go through the steps one by one to solve:

- Find $h(x, y)$ first: it is *any* solution (no matter how simple) to the original, homogeneous equation. Clearly for $C = 0$, $h(x, y)$ is just a constant.
- Find $f(p)$ by the following method:
  - $p \equiv p(x, y)$
  - Solve $\frac{dx}{A} = \frac{dy}{B}$, and set the constant of integration to be p. This requires the assumption that p is a constant combination of x and y.
  - For bonus points, derive this from the total differential of p, and the chain rule of $f(p)$ subbed into equation 69.
- Find $g(x, y)$: it is *any* solution (no matter how simple) to the original, inhomogeneous equation. Clearly for $R = 0$, $g(x, y) = 0$.

6.2 2^{nd} Order

We only consider equations of the form:

$$A\partial_{xx} u + B\partial_{xy} u + C\partial_{yy} u = 0 \tag{71}$$

Where we define the discriminant as $D = B^2 - 4AC$:

D > 0 -*hyperbolic* equation, describes propagating oscillations (waves).

D = 0 -*parabolic* equation, describes transport processes (diffusion etc.).

D < 0 -*elliptic* equation, describes stationary systems (temperature distributions etc.).

We solve them by seeking solutions to the characteristic equation:

$$A\lambda^2 + B\lambda + C = 0$$

Where the roots λ_1, λ_2 form two separate variables:

$$p_1 = x + \lambda_1 y$$

$$p_2 = x + \lambda_2 y$$

$$u = f(p_1) + g(p_2)$$

6.3 Common Linear PDEs

Learn how to solve all of these using as many methods as possible:

- The Wave Equation:

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

- The Diffusion Equation:

$$k\nabla^2 u = \frac{\partial u}{\partial t}$$

- Poisson's Equation[2]:

$$\nabla^2 u = \rho(\hat{r})$$

- Schrodinger's Equation:

$$\hat{H}u = i\hbar\frac{\partial u}{\partial t}$$

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\hat{r})u = i\hbar\frac{\partial u}{\partial t}$$

7 Fluid Mechanics

The newtonian fluids that we'll be dealing with will all be steady-state, and non-turbulent. They can have the following properties:

- Viscosity: a viscous fluid
- Rotationality:
- Compressibility:

[2]Laplace's equation is the special case: $\rho(\hat{r}) = 0$.

7.1 Navier-Stokes' Equation

The Navier-Stokes' equation describes the motion of a general newtonian fluid. It is as follows:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{F}$$

Where the terms represent quantities as follows:

- $\rho\frac{\partial \mathbf{v}}{\partial t}$

  Time-dependant term, clearly zero for steady-state solutions.

- $\rho\mathbf{v}\cdot\nabla\mathbf{v}$

  Advection term, describes how the fluid is carried along. Clearly zero for

- $-\nabla p$

  Pressure term, describes how the fluid is being forced to move.

- $\mu\nabla^2\mathbf{v}$

  Time-dependant term, clearly zero for steady-state solutions.

- $\rho\frac{\partial \mathbf{v}}{\partial t}$

  Time-dependant term, clearly zero for steady-state solutions.

- **Continuity Equation**: derived from conservation of mass when considering an infinitesimal region, we get:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\hat{v})$$

- **Euler Equation**: derived from conservation of mass when considering an infinitesimal region, we get:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\hat{v})$$

- **Continuity Equation**: derived from conservation of mass when considering an infinitesimal region, we get:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\hat{v})$$

- **Continuity Equation**: derived from conservation of mass when considering an infinitesimal region, we get:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\hat{v})$$