PHAS2443 PRACTICAL MATHEMATICS II

ORBITING BODIES: THE DANCE OF THE STARS

Introduction

We are used to thinking of orbits in terms of one or more light bodies orbiting a heavy one (planets round a star or satellites round a planet), but interesting orbital patterns arise when we look at the interactions of several bodies with equal masses. An article in New Scientist (August 4^{th} 2001) looked at new developments in this field. Figure 1 shows some stable orbits for three or more bodies of equal mass.

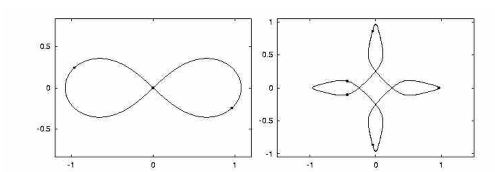

Figure 1: Three bodies in a figure-of-eight orbit and five bodies in a flower-shaped orbit.

The three-body case was solved by Euler (right) and Lagrange (left). Euler found collinear solutions for point masses, and Lagrange found a non-collinear three-body configuration in the form of an equilateral triangle. Lagrange thought his solution did not apply to the solar system but we now know that the both the Earth and Jupiter have asteroids sharing their orbits in the equilateral triangle solution configuration. For Jupiter these bodies are called Trojan planets, the first to be discovered being Achilles in 1908. Even today the system of three orbiting bodies has the potential to surprise[1].

[1] Milovan Šuvakov and V. Dmitrašinović. Three classes of Newtonian three-body planar periodic orbits. *Phys. Rev. Lett.*, 110(11):114301, March 2013

Figure 2: Euler (left) and Lagrange(right).

Project

Classical Problems

You should explore the accuracy of the numerical procedure, by starting with a simple case. Two equal masses, for example, will orbit in a circle about their overall centre of mass with the masses on either end of a diameter. Find the radius and velocity for this case (the algebra is simple, but use Mathematica as an exercise), and hence get suitable starting values for a numerical solution of the differential equations. For how long can you follow the motion with the numerical procedure? What control does Mathematica offer over the procedure.

Then examine the Sun-Moon-Earth system. It is convenient to start the problem off with the Sun, Moon and Earth in a straight line (as in an eclipse). Suitable values of the initial velocities can be deduced from the known periods and distances (readily available in reference books or on-line), and

$$
\begin{aligned}
GM_{\odot} &= AU^3 (2\pi/year)^2 \\
GM_{\oplus} &= d_{EM}^3 (2\pi/month)^2 \\
v_{CM} &= \sqrt{\frac{GM_{\odot}(1+\epsilon_{\oplus})}{AU(1-\epsilon_{\oplus})}} \\
v_R &= \sqrt{\frac{(GM_{\oplus}+GM_{\leftmoon})(1+\epsilon_{\leftmoon})}{d_{EM}(1-\epsilon_{\leftmoon})}},
\end{aligned}
$$

where AU is the Earth-Sun mean distance, d_{EM} the Earth-Moon mean distance, M represents the mass of the Sun ($\odot$), Earth ($\oplus$) or Moon($\leftmoon$), and ϵ the orbital eccentricity. In the velocities, CM refers to the velocity of the centre of gravity of the Earth-Moon system, R to the relative Earth-Moon velocity. You may assume the Sun to be stationary. Initially, ignore the inclination of the Moon's orbit. Then include the inclination (5.1°). If you run this simulation for long enough and analyse the eccentricity of the Moon's orbit, you should be able to see the key periods (including the Saros cycle (18 years, 11.3 days). The eccentricity of the Moon's orbit is proportional to the length of the Laplace-Runge-Lenz-Pauli vector,

$$
L = \frac{\mathbf{r}_R}{r_R} - \frac{v_R^2 \mathbf{r}_R - \mathbf{r}_R.\mathbf{v}_R \; \mathbf{v}_R}{G(M_{\oplus}+M_{\leftmoon})}.
$$

A corresponding formula with $\mathbf{r}_{CM}$, $\mathbf{v}_{CM}$, and $M_{\odot}$ instead of $M_{\leftmoon}$ would give the eccentricity of the Earth's orbit.

Other simple cases to explore are the Lagrangian one (three particles equally spaced round a circle; three particles with one at rest in the centre of a circle round which the other two orbit). Find suitable starting conditions and follow these.

See what happens if you start particles off under slightly the wrong conditions. See if you can find other stable orbits by trial and error (in many cases you will not need to run the calculation for long before the masses go rushing off to as close as the computer can take them towards infinity).

Exotic Orbits

You should set up and solve the equations of motion for several bodies, each moving in the gravitational field of the others. For simplicity, only consider cases in which all the bodies are in the same plane, and all have the same mass. It is convenient (why?) to work in units in which the mass of each body is unity and the gravitational constant G is also unity, but you should explain how to translate your results into other units.

You should write down the equations of motion in Cartesian coordinates, which will be second order differential equations, but then convert these to twice as many first order differential equations by writing the acceleration as dv/dt, the velocity as dx/dt. You will find that Mathematica cannot solve the equations analytically, so you will have to use the numerical differential equation solver. This means that you will have to specify the boundary conditions, that is, the initial positions and velocities of the particles.

When you first tackle this project you will probably find it easiest to think up names for all the variables and write out all the equations yourself, and for two or three bodies this is certainly a practical approach. In general, though, you can use Mathematica's ability to convert from expressions to strings and back again in order to generate variable names 'on the fly'. This would mean that you could write a function that took as input a set of coordinates and velocities, and automatically sorted out how many bodies there were, generated the equations, and solved them numerically.

There are at least two ways of plotting your output: you can give the orbits in the plane, with time as a parameter, or you can give 'phase plots', which plot one component of velocity against the corresponding component of position, again with time as a parameter. It is useful to distinguish the orbits for the different bodies using colour or line types.

As well as looking for stable orbiting systems, you could explore collisions. For example, try setting up two masses in a stable orbit and letting another mass collide with them: this might represent a third star interacting with a binary system. Can you find conditions for which the binary remains after the collision?

The n-body system can also exhibit chaotic behaviour, where minute changes in initial conditions lead to totally different final states. Often there are regions in the space of starting conditions for which the behaviour is chaotic and other regions which are stable.

Summary

Even the world of three-body dynamics is extremely rich, and exhibits many different kinds of behaviour. In this mini-project you cannot expect to undertake an exhaustive exploration of n-body systems. It is important to be methodical, to test your Mathematica code carefully before using it extensively, and to record and comment on your results.

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References

[1] Milovan Šuvakov and V. Dmitrašinović. Three classes of Newtonian three-body planar periodic orbits. *Phys. Rev. Lett.*, 110(11):114301, March 2013.