PHAS2423 - Problem Based Learning II - Problems

(1) Green's function. Find the Green's function $G(t,\tau)$ that satisfies

$$\frac{d^2G(t,\tau)}{dt^2} + \alpha\frac{dG(t,\tau)}{dt} = \delta(t-\tau)$$

under the boundary conditions

$$G(0,\tau) = 0 \qquad \text{and} \qquad \frac{dG(t,\tau)}{dt} = 0 \quad \text{for } t = 0.$$

Then, solve

$$\frac{d^2x(t)}{dt^2} + \alpha\frac{dx(t)}{dt} = f(t)$$

for

$$f(t) = \begin{cases} 0 & \text{if } \ t < 0 \\ Ae^{-\beta t} & \text{if } \ t \geq 0 \end{cases}.$$

Solution.
To find the Green's function $G(t,\tau)$ we first need to find solutions of the corresponding homogeneous equation

$$\frac{d^2y(t)}{dt^2} + \alpha\frac{dy(t)}{dt} = 0.$$

One can go about it in two ways.
(i) Form and solve the auxiliary equation:

$$\lambda^2 + \alpha\lambda = \lambda(\lambda+\alpha) = 0.$$

The roots of this equation are $\lambda_1 = -\alpha$ and $\lambda_2 = 0$. Thus (see notes)

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} = Ae^{-\alpha t} + B,$$

where A and B are arbitrary constants.

(ii) Alternatively, we can notice that the original homogeneous equation does not contain $y(x)$. Hence, introduce a new function $k(t) = y'(t)$. Then the original homogeneous ODE is transformed into a first oder ODE

$$\frac{dk(t)}{dt} + \alpha k(t) = 0.$$

Solution of this ODE is $k(t) = Ae^{-\alpha t}$. Now obtain $y(t)$ by integrating $k(t)$:

$$y(t) = \int k(t)\ dt = -\frac{A}{\alpha}e^{-\alpha t} + B,$$

which is equivalent to the solution found in **(i)**.

As before, the Green's function is represented using the solutions of the homogeneous ODE and unknown functions:

$$\begin{aligned} G(t,\tau) &= A(\tau)e^{-\alpha t} + B(\tau) && (t > \tau) \\ G(t,\tau) &= C(\tau)e^{-\alpha t} + F(\tau) && (t < \tau) \end{aligned}$$

Apply the boundary conditions:

$$\begin{array}{lcl} G(0,\tau) = 0 & \rightarrow & C + D = 0 \\ G'(t,\tau)_{(t=0)} = 0 & \rightarrow & -\alpha C = 0 \end{array}.$$

Thus, $C = D = 0$. Apply the continuity and discontinuity conditions

$$\begin{array}{lcl} Ae^{-\alpha\tau} + B = 0 & \rightarrow & B = -Ae^{-\alpha\tau} \\ -\alpha Ae^{-\alpha\tau} = 1 & \rightarrow & A = -e^{\alpha\tau}/\alpha \end{array}$$

Hence, $B = 1/\alpha$ and the Green's function for $t > \tau$ is

$$G(t,\tau) = -\frac{1}{\alpha}e^{\alpha\tau}e^{-\alpha t} + \frac{1}{\alpha} = \frac{1}{\alpha}\left[1 - e^{\alpha(\tau - t)}\right]$$

and for $t < \tau$

$$G(t,\tau) = 0.$$

To solve the ODE with respect to $x(t)$ for the given $f(t)$, calculate

$$x(t) = \int G(t,\tau)f(\tau)\,d\tau.$$

Since $f(t) = 0$ for $t < 0$ and $G(t,\tau) = 0$ for $t < \tau$, it is sufficient to calculate the integral for $0 \leq \tau \leq t$:

$$x(t) = \int_0^t \frac{1}{\alpha}\left[1 - e^{\alpha(\tau - t)}\right] Ae^{-\beta\tau}\,d\tau = \frac{A}{\alpha}\int_0^t e^{-\beta\tau}\,d\tau - \frac{Ae^{-\alpha t}}{\alpha}\int_0^t e^{\tau(\alpha-\beta)}\,d\tau.$$

$$x(t) = -\left[\frac{A}{\alpha\beta}e^{-\beta\tau}\right]_0^t - \left[\frac{Ae^{-\alpha t}}{\alpha(\alpha-\beta)}e^{\tau(\alpha-\beta)}\right]_0^t = A\left[\frac{1}{\alpha\beta} - \frac{1}{\alpha\beta}e^{-\beta t} - \frac{e^{-\beta t}}{\alpha(\alpha-\beta)} + \frac{e^{-\alpha t}}{\alpha(\alpha-\beta)}\right].$$

Simply it:

$$x(t) = A\left[\frac{1 - e^{-\beta t}}{\alpha\beta} + \frac{e^{-\alpha t} - e^{-\beta t}}{\alpha(\alpha-\beta)}\right] = A\left[\frac{(\alpha-\beta)(1 - e^{-\beta t}) + \beta e^{-\alpha t} - \beta e^{-\beta t}}{\alpha\beta(\alpha-\beta)}\right].$$

$$x(t) = A\left[\frac{\alpha(1-e^{-\beta t}) - \beta(1-e^{-\alpha t})}{\alpha\beta(\alpha-\beta)}\right].$$

---

(2) Eigenfunctions method. Eigenfunctions of the operator

$$\mathcal{L} = x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{1}{4} \qquad\qquad (1 \le x \le e),$$

which satisfy the boundary conditions $y_n(1) = y_n(e) = 0$ are

$$y_n(x) = A_n\frac{1}{\sqrt{x}}\sin(n\pi\ln x),$$

where n are positive integers.
(a) Calculate coefficients A_n, such that functions $y_n(x)$ are normalised to 1.
(b) Show that these functions are orthogonal for $n \ne m$.
(c) Calculate eigenvalues λ_n in $\mathcal{L}y_n = \lambda_n y_n$.
(d) Find a solution of

$$\mathcal{L}y(x) = \frac{1}{\sqrt{x}}$$

as a series of $y(x) = \sum c_n y_n(x)$.

Solution.
(a) To normalise y_n, calculate

$$I_{nn} = \int_1^e y_n y_n\,dx = \int_1^e \frac{A_n^2}{x}\sin^2(n\pi\ln x)\,dx$$

and require that $I_{nn} = 1$. Substitute variables and change the integration limits:

$$\ln x = t, \qquad x = e^t, \qquad \frac{dx}{x} = dt, \qquad \frac{dt}{dx} = \frac{1}{x} = e^{-t}, \qquad \ln 1 = 0, \qquad \ln e = 1.$$

Then

$$I_{nn} = A_n^2\int_0^1 \sin^2(n\pi t)dt = \frac{A_n^2}{2}\int_0^1 (1-\cos(2n\pi t)dt = \frac{A_n^2}{2}\left[t - \frac{\sin(2n\pi t)}{2n\pi}\right]_0^1 = \frac{A_n^2}{2}.$$

Thus, $A_n = \sqrt{2}$ and the normalised eigenfunctions are

$$y_n(x) = \sqrt{\frac{2}{x}}\sin(n\pi\ln x).$$

(b) Show that these functions are orthogonal for $m \ne n$:

$$I_{nm} = 2\int_0^1 \sin(n\pi t)\sin(m\pi t)\,dt = \int_0^1 [\cos(n\pi t - m\pi t) - \cos(n\pi t + m\pi t)]dt = 0.$$

(c) To find eigenvalues λ_n, evaluate

$$\mathcal{L}y_n(x) = \left(x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{1}{4}\right)\sqrt{\frac{2}{x}}\sin(n\pi\ln x).$$

These derivatives are straightforward to calculate. However, it is a little easier to do so using the variable t, which we have introduce above. Then, the eigenfunctions become (here we omit the normalisation factor of $\sqrt{2}$ – this does not affect the eigenvalues):

$$u_n(t) = e^{-t/2}\sin(n\pi t).$$

Then contributions to $\mathcal{L}u_n(t)$ are:

$$x^2\frac{d^2u_n}{dx^2} = e^{2t}\frac{dt}{dx}\frac{d}{dt}\left[\frac{dt}{dx}\frac{du_n}{dt}\right] = e^t\frac{d}{dt}\left[e^{-t}\frac{du_n}{dt}\right] = e^t\left[-e^{-t}\frac{du_n}{dt} + e^{-t}\frac{d^2u_n}{dt^2}\right] =$$

$$= -\frac{du_n}{dt} + \frac{d^2u_n}{dt^2}.$$

$$2x\frac{du_n}{dx} = 2x\frac{dt}{dx}\frac{du_n}{dt} = 2x\frac{1}{x}\frac{du_n}{dt} = 2\frac{du_n}{dt}.$$

$$\frac{1}{4}u_n = \frac{1}{4}e^{-t/2}\sin(n\pi t)$$

$$\frac{du_n}{dt} = \left[-\frac{e^{-t/2}}{2}\sin(n\pi t) + n\pi e^{-t/2}\cos(n\pi t)\right] = e^{-t/2}\left[-\frac{1}{2}\sin(n\pi t) + n\pi\cos(n\pi t)\right].$$

$$\frac{d^2u_n(t)}{dt^2} =$$

$$= -\frac{1}{2}e^{-t/2}\left[-\frac{1}{2}\sin(n\pi t) + n\pi\cos(n\pi t)\right] + e^{-t/2}\left[-\frac{n\pi}{2}\cos(n\pi t) - (n\pi)^2\sin(n\pi t)\right]$$

Combining these contributions gives

$$\mathcal{L}u_n = e^{-t/2}\left[\sin(n\pi t)\left(\frac{1}{4} - n^2\pi^2 - \frac{1}{2} + \frac{1}{4}\right) + \cos(n\pi t)\left(n\pi - \frac{n\pi}{2} - \frac{n\pi}{2}\right)\right] = (-n^2\pi^2)u_n,$$

i.e., $\lambda_n = -n^2\pi^2$.

(d) Substitute $y(x) = \sum c_n y_n(x)$ into the ODE:

$$\mathcal{L}y(x) = \sum_{n=1}^{\infty} c_n(-n^2\pi^2)y_n = \sum_{n=1}^{\infty} c_n(-n^2\pi^2)\sqrt{\frac{2}{x}}\sin(n\pi\ln x) = \frac{1}{\sqrt{x}}.$$

Multiply by y_m and integrate:

$$\sum_{n=1}^{\infty} c_n(-n^2\pi^2)\int_1^e y_m y_n\, dx = \sum_{n=1}^{\infty} c_n(-n^2\pi^2)\delta_{mn} = -c_m\, m^2\pi^2 = \int_1^e \frac{1}{\sqrt{x}} y_m\, dx.$$

Hence,

$$c_m = -\frac{1}{m^2\pi^2}\int_1^e \frac{1}{\sqrt{x}} y_m\, dx = -\frac{\sqrt{2}}{m^2\pi^2}\int_1^e \frac{\sin(m\pi\ln x)}{x}dx = -\frac{\sqrt{2}}{m^2\pi^2}\int_0^1 \sin(m\pi t)dt =$$

$$= -\frac{\sqrt{2}}{m^2\pi^2}\left[-\frac{\cos(m\pi t)}{m\pi}\right]_0^1 = -\frac{\sqrt{2}}{m^3\pi^3}\left[1-(-1)^m\right].$$

Thus, $c_m = 0$ for even m and $c_m = -2\sqrt{2}/m^3\pi^3$ for odd m. We can represent odd positive m as $m = 2k+1$, where $k = 0, 1, 2, \ldots$ Then,

$$y(x) = \sum_{k=0}^{\infty} -\frac{2\sqrt{2}}{(2k+1)^3\pi^3}\sqrt{\frac{2}{x}}\sin\left[(2k+1)\pi\ln x\right] = -\frac{4}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin\left[(2k+1)\pi\ln x\right]}{(2k+1)^3\sqrt{x}}.$$

---

(3) ODE: non-homogeneous boundary conditions. Consider the non-homogeneous equation

$$y''(x) + y(x) = x,$$

where $0 \le x \le \pi/2$ and function $y(x)$ satisfies non-homogeneous boundary conditions:

$$y(0) = 8 \qquad \text{and} \qquad y'(0) = -1.$$

Convert the non-homogeneous boundary conditions to homogeneous, then find the function $y(x)$. (*Hint: substitute $y(x)$ with another function $u(x)$ which satisfy homogeneous boundary conditions.*)

Solution.

The general strategy is to convert the non-homogeneous boundary conditions to the homogeneous form. To this end we define a function $u(x) = y(x) - h(x)$, where $h(x)$ is such that $u(x)$ satisfies the homogeneous boundary conditions. To choose the function $h(x)$ we notice that the two boundary conditions can be satisfied if $h(x)$ is a polynomial function with two independent parameters. Hence, consider

$$h(x) = C_1 x + C_0.$$

In order to find parameters C_0 and C_1, request that $u(x)$ satisfies the homogeneous conditions and use the given boundary conditions for $y(x)$.

$$u(0) = y(0) - h(0) = 8 - C_0 = 0 \qquad \text{and} \qquad u'(0) = y'(0) - h'(0) = -1 - C_1 = 0.$$

Hence, $C_0 = 8$ and $C_1 = -1$, i.e., $h(x) = -x + 8$ and $y(x) = u(x) - x + 8$.

Now write the ODE with respect to $u(x)$. First, notice that $y'' = u'' + h'' = u''$. Thus,

$$u''(x) + u(x) - x + 8 = x.$$

Rewrite it so as the left-hand side depends on $u(x)$ only:

$$u''(x) + u(x) = 2x - 8.$$

Solutions of the homogenous ODE are $u_1 = \cos x$ and $u_2 = \sin x$. In order to find a particular solution we can use, for example, the method of variation of parameter (see the lecture notes):

$$u_p = k_1 \cos x + k_2 \sin x.$$

The system of equations with respect to k_1' and k_2' is:

$$k_1' \cos x + k_2' \sin x = 0 \qquad \text{and} \qquad -k_1' \sin x + k_2' \cos x = 2x - 8.$$

Solving this system equations gives

$$k_1' = -(2x - 8) \sin x \qquad \text{and} \qquad k_2' = (2x - 8) \cos x.$$

Integrate k_1' and k_2' and obtain:

$$k_1(x) = 2x \cos x - 2 \sin x - 8 \cos x + A_1, \qquad k_2(x) = 2x \sin x + 2 \cos x - 8 \sin x + A_2,$$

where A_1 and A_2 are arbitrary constants.

Finally, find the general form of $u(x)$:

$$u = k_1 u_1 + k_2 u_2 + B_1 \cos x + B_2 \sin x = 2x - 8 + D_1 \cos x + D_2 \sin x,$$

where constants D_1 and D_2 have to be determined from the boundary conditions for $u(x)$:

$$u(0) = -8 + D_1 = 0 \qquad \text{and} \qquad u'(0) = 2 + D_2 = 0.$$

Hence $D_1 = 8$, $D_2 = -2$.

Therefore,

$$u(x) = 2x - 8 + 8 \cos x - 2 \sin x$$

and, finally,

$$y(x) = u(x) + h(x) = x + 8 \cos x - 2 \sin x.$$