PHAS2423 - Problem Based Learning I - Problems and Solutions

(1) Determinant of a 3×3 matrix ($|A|$) can be expressed as

$$|A|\epsilon_{lmn} = A_{li}A_{mj}A_{nk}\epsilon_{ijk}.$$

(a) Demonstrate that determinant of the transpose matrix A^T is equal to determinant of the matrix A:

$$|A^T| = |A|.$$

(b) Show that if C is a product of two square matrices, $C = AB$, then

$$|C| = |AB| = |A|\,|B|.$$

Solution.

(a) For determinants of matrices A and A^T we have:

$$|A|\epsilon_{pqr} = A_{pi}A_{qj}A_{rk}\epsilon_{ijk},$$

$$|A^T|\epsilon_{pqr} = (A^T)_{pi}(A^T)_{qj}(A^T)_{rk}\epsilon_{ijk}.$$

Since $A_{ij} = (A^T)_{ji}$,

$$|A^T|\epsilon_{pqr} = A^T_{pi}A^T_{qj}A^T_{rk}\epsilon_{ijk} = A_{ip}A_{jq}A_{kr}\epsilon_{ijk}.$$

Contract both sides of this equation with ϵ_{pqr} and rearrange the terms:

$$\left(|A^T|\epsilon_{pqr}\right)\epsilon_{pqr} = (A_{ip}A_{jq}A_{kr}\epsilon_{ijk})\,\epsilon_{pqr} = (A_{ip}A_{jq}A_{kr}\epsilon_{pqr})\,\epsilon_{ijk}.$$

Note that all indices in this expression are dummy indices. Hence, we can rename any of them. Let us rename the indices as follows: $i \leftrightarrow p$, $j \leftrightarrow q$, $k \leftrightarrow r$. Then,

$$(A_{ip}A_{jq}A_{kr}\epsilon_{pqr})\,\epsilon_{ijk} = (A_{pi}A_{qj}A_{rk}\epsilon_{ijk})\,\epsilon_{pqr} = (|A|\epsilon_{pqr})\,\epsilon_{pqr}.$$

Thus,

$$\left(|A^T|\epsilon_{pqr}\right)\epsilon_{pqr} = (|A|\epsilon_{pqr})\,\epsilon_{pqr}.$$

Therefore,

$$|A^T| = |A|.$$

(b) For the determinant of C we have

$$|C|\epsilon_{pqr} = C_{pi}C_{qj}C_{rk}\epsilon_{ijk} = (A_{pa}B_{ai})(A_{qb}B_{bj})(A_{rc}B_{ck})\epsilon_{ijk}.$$

Contract both sides with ϵ_{pqr}, rearrange the terms, and use $A_{kn} = (A^T)_{nk}$:
$|C|\epsilon_{pqr}\epsilon_{pqr} = (A_{pa}A_{qb}A_{rc}\epsilon_{pqr})(B_{ai}B_{bj}B_{ck}\epsilon_{ijk}) = (|A^T|\epsilon_{abc})(|B|\epsilon_{abc}) = |A^T|\,|B|\epsilon_{abc}\epsilon_{abc}$.
Thus, using $|A^T| = |A|$, obtain

$$|C| = |A|\,|B|.$$

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(2) In a certain system of units the electromagnetic stress tensor is given by

$$M_{ij} = E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(E_kE_k + B_kB_k),$$

where E_i and B_i are components of the 1st-order tensors representing the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively.

(a) Demonstrate that components M_{ij} transform as components of a tensor.

(b–d) For $|\boldsymbol{E}|$=$|\boldsymbol{B}|$ (but $\boldsymbol{E} \neq \boldsymbol{B}$):
(b) show that $\boldsymbol{E} \pm \boldsymbol{B}$ are principal axes of the tensor $\boldsymbol{M}$;
(c) determine the third principal axis and
(d) find all principal values.

Solution. Let us open the parenthesis:

$$M_{ij} = E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}E_kE_k - \frac{1}{2}\delta_{ij}B_kB_k.$$

(a) We will show that $\boldsymbol{M}$ is a tensor using two methods.
(a.1) Analyse the nature of each of its terms and apply the knowledge of the tensor algebra discussed in the lecture notes. The first two terms E_iE_j and B_iB_j are components of the outer products $\boldsymbol{E} \otimes \boldsymbol{E}$ and $\boldsymbol{B} \otimes \boldsymbol{B}$, respectively. We have proven (see the lecture notes) that the outer product of two 1st-order tensors is a 2nd-order tensor. Hence, E_iE_j and B_iB_j are components of a tensor.

Notice that the third term does not depend on subscript k (k is a dummy index). The E_kE_k in the third term is the contraction of the tensor formed by the outer product $\boldsymbol{E} \otimes \boldsymbol{E}$. We have proven (see lecture notes) that contraction of a tensor: (i) produces a tensor and (ii) reduces the order of a tensor by 2. Hence E_kE_k is a 0th-order tensor, i.e., a scalar, and, therefore, it remains the same in all rotated coordinate systems. It follows that the third term can be represented as

$$-\frac{1}{2}\delta_{ij} \times Constant,$$

which is a 2nd-order tensor (the prove that δ_{ij} is a tensor is in the lecture notes). Exactly the same considerations apply to the fourth term.

Finally, since a sum of two tensors of **the same order** and calculated in **the same coordinate system** is a tensor, components M_{ij} are components of the 2nd-order tensor.

(a.2) Let us use the definition of how components of Cartesian tensors transform upon rotation of a coordinate system. For scalars:

$$a' = a.$$

For the 1st order tensors:

$$v'_i = L_{ij} v_j.$$

For the 2nd order tensors:

$$T'_{ij} = L_{ik} L_{jm} T_{km}.$$

Apply these rules to investigate transformation of M_{ij}. In a new coordinate system components of $\boldsymbol{M}$ should be

$$M'_{ij} = E'_i E'_j + B'_i B'_j - \frac{1}{2}\delta'_{ij} E'_k E'_k - \frac{1}{2}\delta'_{ij} B'_k B'_k,$$

where E'_i, B'_i, and δ'_{ij} are components of $\boldsymbol{E}$, $\boldsymbol{B}$, and δ_{ij} in the new coordinate system. Since $\boldsymbol{E}$, $\boldsymbol{B}$ are vectors, they transform as

$$E'_i = L_{ij} E_j \qquad \text{and} \qquad B'_i = L_{ij} B_j.$$

Hence, the first two terms transform as

$$E'_i E'_j = (L_{im} E_m)(L_{jn} E_n) = L_{im} L_{jn} (E_m E_n)$$

and

$$B'_i B'_j = (L_{im} B_m)(L_{jn} B_n) = L_{im} L_{jn} (B_m B_n)$$

For the third term we write

$$\begin{aligned}
\delta'_{ij} E'_k E'_k &= (L_{ip} L_{jq} \delta_{pq})(L_{kr} E_r)(L_{ks} E_s) \quad \text{(rearrange the terms)} \\
&= L_{ip} L_{jq} (L_{kr} L_{ks})(\delta_{pq} E_r E_s) \quad \text{(use orthogonality of } L\text{: } L_{kr} L_{ks} = \delta_{rs}\text{)} \\
&= L_{ip} L_{jq} (\delta_{rs})(\delta_{pq} E_r E_s) \quad \text{(} \delta_{rs} \text{ eliminates summation over } s\text{)} \\
&= L_{ip} L_{jq} (\delta_{pq} E_r E_r) \quad \text{(rename dummy indices } p \to m,\ q \to n\text{)} \\
&= L_{im} L_{jn} (\delta_{mn} E_r E_r).
\end{aligned}$$

Similarly,

$$\delta'_{ij} B'_k B'_k = L_{im} L_{jn} (\delta_{mn} B_r B_r).$$

Combine the four terms together and move $L_{im}L_{jn}$ outside the parenthesis:

$$\begin{aligned} M'_{ij} &= L_{im}L_{jn}\left(E_mE_n + B_mB_n - \tfrac{1}{2}\delta_{mn}E_rE_r - \tfrac{1}{2}\delta_{mn}B_rB_r\right) \\ &= L_{im}L_{jn}M_{mn}. \end{aligned}$$

Thus, quantities M_{ij} transform as components of a tensor of the 2nd order, hence $\boldsymbol{M}$ is a tensor.

(b) Check that $\boldsymbol{E} \pm \boldsymbol{B}$ are principal axes. If vector $\boldsymbol{v}$ is a principal axis of tensor $\boldsymbol{T}$, $T_{ij}v_j = \lambda v_i$, where λ is a constant. Check that this is correct for M_{ij} (use $E^2 = B^2$):

$$\begin{aligned} M_{ij}(E_j \pm B_j) &= E_iE_j(E_j \pm B_j) + B_iB_j(E_j \pm B_j) - \tfrac{1}{2}\delta_{ij}(E^2 + B^2)(E_j \pm B_j) \\ &= E_iE^2 \pm E_i(\boldsymbol{E}\cdot\boldsymbol{B}) + B_i(\boldsymbol{B}\cdot\boldsymbol{E}) \pm B_iB^2 - \tfrac{1}{2}(E^2 + B^2)(E_i \pm B_i) \\ &= E_iE^2 \pm B_iE^2 + (\boldsymbol{E}\cdot\boldsymbol{B})(\pm E_i + B_i) - \tfrac{1}{2}(2E^2)(E_i \pm B_i) \\ &= E^2(E_i \pm B_i) + (\boldsymbol{E}\cdot\boldsymbol{B})(\pm E_i + B_i) - E^2(E_i \pm B_i) \\ &= (\boldsymbol{E}\cdot\boldsymbol{B})(\pm E_i + B_i) \\ &= \pm(\boldsymbol{E}\cdot\boldsymbol{B})(E_i \pm B_i) = \pm\lambda(E_i \pm B_i) \end{aligned}$$

(c) Find a vector orthogonal to both $\boldsymbol{E} + \boldsymbol{B}$ and $\boldsymbol{E} - \boldsymbol{B}$:

$$(\boldsymbol{E} + \boldsymbol{B}) \times (\boldsymbol{E} - \boldsymbol{B}) = \boldsymbol{E}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{B} + \boldsymbol{B}\times\boldsymbol{E} - \boldsymbol{B}\times\boldsymbol{B} = 2\boldsymbol{B}\times\boldsymbol{E}$$

(d) Principal values

$$\begin{array}{lcl} \text{for } \boldsymbol{E}+\boldsymbol{B} & \rightarrow & +(\boldsymbol{E}\cdot\boldsymbol{B}) \\ \text{for } \boldsymbol{E}-\boldsymbol{B} & \rightarrow & -(\boldsymbol{E}\cdot\boldsymbol{B}) \\ \text{for } \boldsymbol{B}\times\boldsymbol{E} & \rightarrow & ? \end{array}$$

First recall that outer product of vector $\boldsymbol{a}$ with itself ($\boldsymbol{a}\otimes\boldsymbol{a}$) is a symmetric tensor of the 2nd order with components

$$T_{ij} = a_ia_j$$

and notice that contraction of a symmetric tensor with an antisymmetric tensor produces zero for any combination of subscripts:

$$c_j = \epsilon_{ijk}T_{ik} = \epsilon_{ijk}a_ja_k = 0.$$

With this in mind, consider

$$\begin{aligned} M_{ij}(\boldsymbol{B}\times\boldsymbol{E})_j &= M_{ij}(\epsilon_{jlm}B_lE_m) \\ &= E_iE_j\epsilon_{jlm}B_lE_m + B_iB_j\epsilon_{jlm}B_lE_m - \tfrac{1}{2}\delta_{ij}2E^2\epsilon_{jlm}B_lE_m \\ &= E_iB_l(\epsilon_{jlm}E_jE_m) + B_iE_m(\epsilon_{jlm}B_jB_l) - E^2(\epsilon_{ilm}B_lE_m) \\ &= 0 + 0 - E^2(\boldsymbol{B}\times\boldsymbol{E})_i \end{aligned}$$

Hence, principal values

$$\begin{array}{lcl} \text{for } \boldsymbol{E}+\boldsymbol{B} & \rightarrow & +(\boldsymbol{E}\cdot\boldsymbol{B}) \\ \text{for } \boldsymbol{E}-\boldsymbol{B} & \rightarrow & -(\boldsymbol{E}\cdot\boldsymbol{B}) \\ \text{for } \boldsymbol{B}\times\boldsymbol{E} & \rightarrow & -E^2 = -B^2 \end{array}$$

Alternatively, to calculate λ_3, recall that contraction of an Nth order tensor produces an $N-2$ order tensor. In the case of the 2nd order tensor, the contraction gives a scalar equal to the trace of this tensor. Hence, the 3rd eigenvalue can be also calculated from the value of the trace of the tensor $\boldsymbol{M}$. For the trace of $\boldsymbol{M}$ we have:

$$Tr(M) = M_{ii} = E_iE_i + B_iB_i - \frac{\delta_{ii}}{2}\left(E_kE_k + B_kB_k\right).$$

Since

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3,$$

we have

$$M_{ii} = E^2 + B^2 - \frac{3}{2}\left(E^2 + B^2\right) = -\frac{E^2+B^2}{2} = -E^2 = -B^2.$$

On the other hand,

$$M_{ii} = \lambda_1 + \lambda_2 + \lambda_3 = (\boldsymbol{E}\cdot\boldsymbol{B}) - (\boldsymbol{E}\cdot\boldsymbol{B}) + \lambda_3 = \lambda_3.$$

Hence, $\lambda_3 = -E^2 = -B^2$.

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(3) A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate system the particles are at positions

$$\pm a(3,1,-1) \qquad \pm\, a(1,-1,3) \qquad \pm\, a(1,3,-1) \qquad \pm\, a(-1,1,3).$$

The body rotates about an axis passing through the origin. Show that, if the angular velocity and angular momentum vectors are parallel, then their ratio must be $40ma^2$, $64ma^2$, or $72ma^2$.

Solution. If a body rotates about a fixed axis, the vector of the angular velocity $\boldsymbol{\omega}$, vector of the orbital momentum $\boldsymbol{L}$, and the moment of inertia are related as

$$\boldsymbol{L} = I\boldsymbol{\omega},$$

where scalar I is defined for that rotation axis. If the rotation axis is not fixed, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ may not be parallel and the relation between them becomes

$$L_i = I_{ij}\omega_j,$$

where I_{ij} is a tensor of inertia. Hence, vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ become parallel if the rotation axis coincides with a principal axis of the tensor of inertia and, therefore, the ratio between $\boldsymbol{L}$ and $\boldsymbol{\omega}$ is equal to the principal moment of the tensor of inertia.

$$I_{ij} = \sum m(r^2\delta_{ij} - x_i x_j)$$

$$\begin{aligned}
I_{11} &= 2ma^2(4\times 11 - 9 - 1 - 1 - 1) = 64ma^a \\
I_{22} &= 2ma^2(4\times 11 - 1 - 1 - 9 - 1) = 64ma^2 \\
I_{33} &= 2ma^2(4\times 11 - 1 - 9 - 1 - 9) = 48ma^2 \\
I_{12} &= -2ma^2(3 - 1 + 3 - 1) = -8ma^2 \\
I_{13} &= -2ma^2(-3 + 3 - 1 - 3) = 8ma^2 \\
I_{23} &= -2ma^2(-1 - 3 - 3 + 3) = 8ma^2
\end{aligned}$$

Hence,

$$I = 8ma^2 \begin{pmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

Solve characteristic equation $\det(\boldsymbol{I} - \lambda\boldsymbol{E}) = 0$:

$$\begin{vmatrix} 8-\lambda & -1 & 1 \\ -1 & 8-\lambda & 1 \\ 1 & 1 & 6-\lambda \end{vmatrix} = (8-\lambda)(\lambda^2 - 14\lambda + 47 - 2) = (8-\lambda)(\lambda - 9)(\lambda - 5) = 0.$$

Thus, principal moments are $40ma^2$, $64ma^2$, and $72ma^2$.

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(4) Quantities $x(t)$ and $y(t)$ satisfy a system of equations

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0 \qquad\qquad \frac{d^2y}{dt^2} + 2n\frac{dy}{dt} + n^2y = \mu\frac{dx}{dt}$$

with the following boundary conditions at $t = 0$:

$$x(0) = y(0) = \frac{dy(t)}{dt} = 0 \qquad \text{and} \qquad \frac{dx}{dt} = \lambda.$$

Use the Laplace transform method to show that

$$y(t) = \frac{1}{2}\mu\lambda t^2 \left(1 - \frac{1}{3}nt\right) e^{-nt}.$$

Solution.

Let X and Y be the Laplace transforms of $x(t)$ and $y(x)$, respectively. Apply the Laplace transform to both equations. Taking into account the boundary conditions, obtain:

$$p^2X - \lambda + 2npX + n^2X = 0$$
$$p^2Y + 2npY + n^2Y = \mu pX.$$

From the first equation obtain:

$$X = \frac{\lambda}{p^2 + 2np + n^2} = \frac{\lambda}{(p+n)^2}.$$

Substituting X into the second equation gives

$$(p^2 + 2np + n^2)Y = \mu p \frac{\lambda}{(p+n)^2}.$$

Thus,

$$Y = \mu\lambda \frac{p}{(p+n)^4}.$$

Represent $p/(p+n)^4$ as partial fractions (see PHAS1245 notes):

$$\frac{p}{(p+n)^4} = \frac{A}{(p+n)^3} + \frac{B}{(p+n)^4} = \frac{Ap + An + B}{(p+n)^4}.$$

Thus,

$$A = 1 \qquad\qquad B = -n$$

and

$$Y = \frac{\mu\lambda}{(p+n)^3} - \frac{\mu\lambda n}{(p+n)^4} = \frac{\mu\lambda}{2}\frac{2!}{(p-(-n))^{2+1}} - \frac{\mu\lambda}{6}\frac{3!}{(p-(-n))^{3+1}}.$$

Refer to the table of Laplace transforms and find that the Laplace transform of

$$x^n e^{ax} \qquad \text{is} \qquad \frac{n!}{(p-a)^{n+1}}.$$

Hence,

$$y(t) = \frac{\mu\lambda t^2}{2} e^{-nt} - \frac{\mu\lambda t^3}{6} e^{-nt} = \frac{\mu\lambda t^2}{2}\left(1 - \frac{nt}{3}\right) e^{-nt}.$$