PHAS2423 - Self Study - Fluid Dynamics - Problems and Solutions

(1) State the conditions under which the following theoretical tools are valid descriptions of fluid flow:

- Bernouilli's equation along a streamline
- Bernoulli's equation throughout the region
- the continuity equation
- Euler's equation
- Laplace's equation for a scalar field
- the Navier-Stokes equation
- Laplace's equation for a vector field

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(2) Using the properties of ϵ_{ijk}, prove that $\nabla \times (\nabla V) = 0$, where V is a scalar field.

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(3) The advective term in the Navier-Stokes equation is

$$\begin{aligned}
(\mathbf{v}.\nabla\mathbf{v})_r &= v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \\
(\mathbf{v}.\nabla\mathbf{v})_\theta &= v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \\
(\mathbf{v}.\nabla\mathbf{v})_z &= v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}
\end{aligned}$$

in cylindrical polar coordinates. Show that for a flow field of the form $\mathbf{v} = (0, 0, v_z(r))$, each component of the advective term is equal to zero. Thus calculate the Poiseuille volumetric flow rate along a cylindrical pipe of radius a, subject to an axial pressure gradient $-\alpha$.

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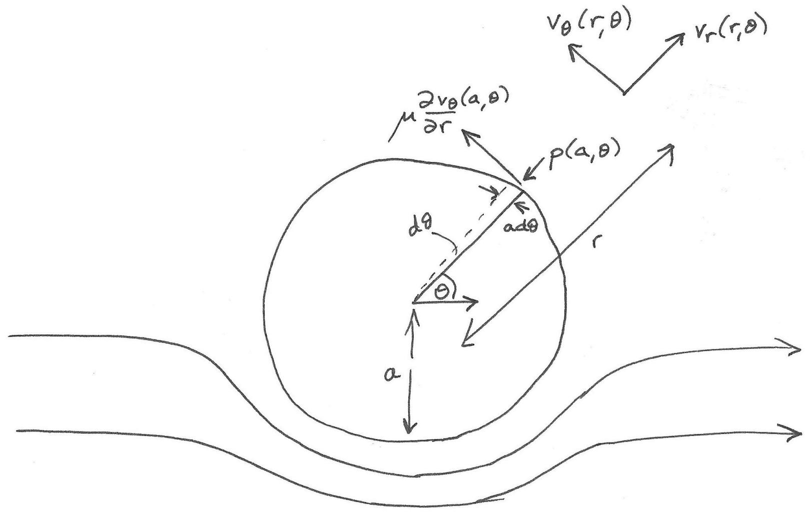
(4) The figure illustrates the flow of an inviscid fluid around a sphere. The drag force on the sphere may be written as an integral involving the local pressure on the surface. What is the magnitude of the drag force? [Hint: you might think that you need to substitute the derived potential flow solution into Euler's equation to obtain the pressure at the surface and then perform the integral, but the answer (whig might not be what you expect) can be obtained by a simpler route based upon imagining a reversal of all velocities.]



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(5) Show that the flow field $\mathbf{v} = (0, K(2\pi)^{-1}(1 - \exp[-r^2\rho/(4\mu t)]), 0)$ in cylindrical polar coordinates satisfies the Navier-Stokes equation. Assume the fluid is incompressible. You probably need to know that ∇^2 in cylindrical polars takes the form:

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\begin{aligned}
(\nabla^2 \mathbf{v})_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \\
(\nabla^2 \mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \\
(\nabla^2 \mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}
\end{aligned}
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Calculate the radial pressure gradient. Calculate the vorticity. Don't be afraid to use Mathematica! Describe what is going on physically in this flow field.
