PHAS2423 - Self-Study - Sturm-Liouville theory - Problems

(1) Demonstrate that functions

$$f_0(x) = 1, \qquad f_k(x) = \cos kx, \qquad g_k(x) = \sin kx,$$

where $k = 1, 2, 3, ...$ and $x \in [-\pi, \pi]$ are orthogonal to each other. Then find normalisation coefficients for these functions so as after normalisation they satisfy

$$\int_{-\pi}^{\pi} f_0 f_0 \, dx = \int_{-\pi}^{\pi} f_k f_k \, dx = \int_{-\pi}^{\pi} g_k g_k \, dx = 1.$$

Solution. Functions f_0, f_k, and g_k are orthogonal if their overlap integrals

$$\int_{-\pi}^{\pi} f_0 f_k \, dx \qquad \int_{-\pi}^{\pi} f_0 g_k \, dx \qquad \int_{-\pi}^{\pi} f_k f_n \, dx \qquad \int_{-\pi}^{\pi} f_k g_n \, dx \qquad \int_{-\pi}^{\pi} g_k g_n \, dx$$

are all equal to zero. Clearly,

$$\int_{-\pi}^{\pi} f_0 f_k \, dx = \int_{-\pi}^{\pi} \cos kx \, dx = \left[\frac{1}{k} \sin kx \right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} f_0 g_k \, dx = \int_{-\pi}^{\pi} \sin kx \, dx = \left[-\frac{1}{k} \cos kx \right]_{-\pi}^{\pi} = 0$$

To calculate other integrals, use the following trigonometric relations

$$\cos nx \cos kx = \frac{1}{2} \cos(n+k)x + \frac{1}{2} \cos(n-k)x,$$

$$\cos nx \sin kx = \frac{1}{2} \sin(n+k)x - \frac{1}{2} \sin(n-k)x,$$

$$\sin nx \sin kx = \frac{1}{2} \cos(n-k)x - \frac{1}{2} \cos(n+k)x.$$

For $n \neq k$:

$$\int_{-\pi}^{\pi} \cos nx \cos kx \, dx = \left[\frac{\sin(n+k)x}{2(n+k)} + \frac{\sin(n-k)x}{2(n-k)} \right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin nx \sin kx \, dx = \left[\frac{\sin(n-k)x}{2(n-k)} - \frac{\sin(n+k)x}{2(n+k)} \right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos nx \sin kx \, dx = \left[-\frac{\cos(n+k)x}{2(n+k)} + \frac{\cos(n-k)x}{2(n-k)} \right]_{-\pi}^{\pi} = 0$$

and for the overlap of f_n and g_n:

$$\int_{-\pi}^{\pi} \cos nx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2nx \, dx = \left[-\frac{\cos 2nx}{4n} \right]_{-\pi}^{\pi} = 0.$$

Thus, f_0, f_k, and g_k are orthogonal.

To calculate normalisation coefficients, consider the case of $n = k$:

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi \qquad\qquad \text{(for } f_0\text{)}.$$

Thus, a system of orthonormal basis set functions is given by:

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \qquad f_n(x) = \frac{\cos nx}{\sqrt{\pi}} \qquad g_n(x) = \frac{\sin nx}{\sqrt{\pi}} \qquad n{=}1{,}2{,}3{,}....$$

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(2) Use the method of Schmidt orthogonalisation in order to transform functions $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$ into functions $g_1(x)$, $g_2(x)$, and $g_3(x)$, which are orthogonal and normalised to 1 on the interval $0 \le x \le 1$.

Solution. First, normalise $f_1(x)$ to 1: calculate the norm of $f_1(x)$

$$C_1 = \int_0^1 f_1 \cdot f_1 \, dx = \int_0^1 1 \cdot 1 \, dx = [x]_0^1 = 1$$

and multiply f_1 by $1/\sqrt{C_1}$. Thus, $f_1(x)$ is normalised to 1 as it is. Hence, we can put

$$g_1(x) = 1.$$

According to the Schmidt method, to orthogonalise f_2 to g_1, we should write

$$h_2 = f_2 - g_1 \int_0^1 g_1 \cdot f_2 \, dx = x - \int_0^1 1 \cdot x \, dx = x - \left[\frac{x^2}{2} \right]_0^1 = x - \frac{1}{2}.$$

Function $h_2(x)$ is orthogonal to $g_1(x)$ but not normalised. To normalise it, calculate the norm of $h_2(x)$:

$$C_2 = \int_0^1 h_2 \cdot h_2\, dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

and multiply h_2 by $1/\sqrt{C_2}$:

$$g_2(x) = 2\sqrt{3}h_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right).$$

Finally, for f_3 we need to consider

$$h_3 = f_3 - g_1 \int_0^1 g_1 \cdot f_3\, dx - g_2 \int_0^1 g_2 \cdot f_3\, dx = x^2 - \int_0^1 x^2\, dx - 12\left(x - \frac{1}{2}\right)\int_0^1 \left(x - \frac{1}{2}\right) \cdot x^2\, dx$$

Since

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

and

$$\int_0^1 \left(x - \frac{1}{2}\right) \cdot x^2\, dx = \left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12},$$

we have

$$h_3(x) = x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}.$$

To normalise h_3, calculate

$$C_3 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{x^2}{3} - \frac{x}{3}\right) dx =$$

$$= \left[\frac{x^5}{5} + \frac{x^3}{3} + \frac{x}{36} - \frac{2x^4}{4} + \frac{x^3}{9} - \frac{x^2}{6}\right]_0^1 = \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} + \frac{1}{9} - \frac{1}{6} =$$

$$= \frac{1}{5} + \frac{12}{36} + \frac{1}{36} - \frac{18}{36} + \frac{4}{36} - \frac{6}{36} = \frac{1}{5} - \frac{7}{36} = \frac{36-35}{180} = \frac{1}{180} = \left(\frac{1}{6\sqrt{5}}\right)^2.$$

Thus,

$$g_3(x) = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right).$$

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(3) Use the method of Schmidt orthogonalisation in order to transform functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ into functions $g_1(x)$ and $g_2(x)$, which are orthogonal and normalised to 1 on the interval $0 \leq x \leq \pi/2$.

Solution. First, let us normalise $f_1(x)$. Calculate normalisation coefficient:

$$C_1 = \int_0^{\pi/2} f_1{\cdot}f_1\, dx = \int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2}\, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Thus,

$$g_1(x) = \frac{2}{\sqrt{\pi}} \sin x.$$

Then, following the Schmidt method, find

$$h(x) = f_2(x) - g_1(x) \int_0^{\pi/2} g_1 \cdot f_2\, dx = \cos x - \frac{4}{\pi} \int_0^{\pi/2} \sin x \cos x\, dx.$$

Since

$$\int_0^{\pi/2} \sin x \cos x\, dx = \left[\frac{1}{2} \sin^2 x\right]_0^{\pi/2} = \frac{1}{2},$$

for $h(x)$ we have

$$h(x) = \cos x - \frac{2}{\pi} \sin x.$$

Function $h(x)$ is orthogonal to $g_1(x)$. To normalise it to 1, calculate normalisation coefficient:

$$C_2 = \int_0^{\pi/2} h \cdot h\, dx = \int_0^{\pi/2} \left(\cos x - \frac{2}{\pi} \sin x\right)^2 dx =$$

$$= \int_0^{\pi/2} \left(\cos^2 x - \frac{4}{\pi} \cos x \sin x + \frac{4}{\pi^2} \sin^2 x\right) dx = \frac{\pi}{4} - \frac{4}{\pi}\frac{1}{2} + \frac{4}{\pi^2}\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\pi} = \frac{\pi^2 - 4}{4\pi}.$$

Thus,

$$g_2(x) = 2\sqrt{\frac{\pi}{\pi^2 - 4}} \left(\cos x - \frac{2}{\pi} \sin x\right).$$

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(4) Find the eigenfunction expansion for a solution of the inhomogeneous equation

$$\frac{d^2 y}{dx^2} + \omega^2 y = \sin^2 x,$$

where $y(x)$ satisfies the boundary conditions $y(0) = y(\pi) = 0$.

Solution. The boundary problem for this ODE is

$$\mathcal{L} y_n(x) = \lambda_n y_n(x),$$

where

$$\mathcal{L} = \frac{d^2 y}{dx^2} + \omega^2.$$

Clearly, eigenfunctions of $\mathcal{L}$ are

$$y_n(x) = A_n \sin nx + B_n \cos nx$$

and the corresponding eigenvalues $\lambda_n = \omega^2 - n^2$. Taking into account the boundary conditions gives

$$y_n(0) = A_n \cdot 0 + B_n \cos nx = 0,$$

i.e., $B_n = 0$ and

$$y_n(\pi) = A_n \sin n\pi = 0,$$

i.e., n is integer. Since $\sin(-nx) = -\sin nx$, we consider only positive values of n. To normalise y_n, require that

$$\int_0^\pi y_n(x) y_n(x)\,dx = \int_0^\pi A_n^2 \sin^2 nx\,dx = \frac{A_n^2}{2}\int_0^\pi (1-\cos 2nx)dx = 1,$$

which gives $A_n = \sqrt{2/\pi}$. Thus, normalised eigenfunctions of $\mathcal{L}$, which satisfy the boundary conditions, are

$$y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \qquad\qquad n = 1, 2, 3, ...$$

These functions are also orthogonal (for $m \neq n$):

$$\int_0^\pi \sin mn \sin nx\,dx = \int_0^\pi \frac{\cos(m-n)x - \cos(m+n)x}{2} dx = \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right]_0^\pi = 0.$$

Represent the solution in the form

$$y(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sqrt{\frac{2}{\pi}} \sum_n^\infty c_n \sin nx.$$

To find coefficients c_n, substitute $y(x)$ into the ODE, which gives

$$\sum_n^\infty \left(\omega^2 - n^2\right) c_n y_n(x) = \sin^2 x,$$

and integrate both parts of this equation with $y_m(x)$:

$$\sum_n^\infty \left(\omega^2 - n^2\right) c_n \int_0^\pi y_m(x) y_n(x)\,dx = \int_0^\pi y_m(x) \sin^2 x\,dx.$$

Since functions $y_n(x)$ are orthonormal, we have

$$\sum_n^\infty \left(\omega^2 - n^2\right) c_n \delta_{mn} = \left(\omega^2 - m^2\right) c_m = \int_0^\pi y_m(x) \sin^2 x\,dx.$$

Thus,

$$c_m = \frac{1}{\omega^2 - m^2} \int_0^\pi y_m(x) \sin^2 x\,dx = \frac{1}{\omega^2 - m^2} \sqrt{\frac{2}{\pi}} \int_0^\pi \sin mx \sin^2 x\,dx.$$

Simplify $\sin mx \sin^2 x$:

$$\sin mx \sin^2 x = \frac{1}{2}\sin mx(1-\cos 2x) = \frac{\sin mx}{2} - \frac{\sin(m+2)x}{4} - \frac{\sin(m-2)x}{4}.$$

Thus, for $m \neq 2$:

$$I_m = \int_0^\pi \sin mx \sin^2 x\, dx = \left[-\frac{\cos mx}{2m} + \frac{\cos(m+2)x}{4(m+2)} + \frac{\cos(m-2)x}{4(m-2)}\right]_0^\pi.$$

Since $\cos n\pi = (-1)^n$ for integer n,

$$I_m = -\frac{(-1)^m}{2m} + \frac{1}{2m} + \frac{(-1)^{m+2}}{4(m+2)} - \frac{1}{4(m+2)} + \frac{(-1)^{m-2}}{4(m-2)} - \frac{1}{4(m-2)}.$$

Simplify this expression taking into account that $(-1)^m = (-1)^{m+2} = (-1)^{m-2}$ and

$$\frac{1}{m+2} + \frac{1}{m-2} = \frac{2m}{m^2-4}.$$

Obtain

$$I_m = \frac{1}{2m}\left[1-(-1)^m\right] + \frac{2m\,(-1)^m}{4(m^2-4)} - \frac{2m}{4(m^2-4)} = \frac{1-(-1)^m}{2}\left[\frac{1}{m} - \frac{m}{m^2-4}\right]$$

Thus,

$$I_m = \frac{1-(-1)^m}{2}\cdot\frac{-4}{m(m^2-4)} = 2\frac{(-1)^m-1}{m(m^2-4)},$$

i.e., coefficients c_m, where m is even, are zero.

Now consider the special case of $m=2$. Since

$$\sin 2x \sin^2 x = 2\sin^3 x \cos x,$$

$$I_2 = \int_0^\pi \sin 2x \sin^2 x\, dx = \left[\frac{\sin^4 x}{2}\right]_0^\pi = 0.$$

To summarise,

$$c_m = 0 \quad \text{if } m = 2n \qquad \text{and} \qquad c_m = -\frac{1}{\omega^2-m^2}\sqrt{\frac{2}{\pi}}\frac{4}{m(m^2-4)} \quad \text{if } m = 2n+1,$$

where $n = 0, 1, 2, ...$

Finally, the eigenfunction expansion of the solution of the ODE is $y(x) = \sum c_m y_m$:

$$y(x) = -4\sqrt{\frac{2}{\pi}}\sum_{m=1}^{\infty}{}' \frac{1}{\omega^2-m^2}\cdot\frac{1}{m(m^2-4)}\sqrt{\frac{2}{\pi}}\sin mx = \frac{8}{\pi}\sum_{m=1}^{\infty}{}' \frac{\sin mx}{m(m^2-\omega^2)(m^2-4)},$$

where the prime symbol indicates that the summation goes over odd values of m only.