PHAS2423 - Self-Study - Ordinary Differential Equations - Problems

(1) ODE: Laplace transform. Use the Laplace transform to

(a) find a particular solution of the non-homogeneous equation:

$$y'' + 4y' + 4y = e^{-2x},$$

which satisfies the boundary conditions $y(0) = y'(0) = 4$.

(b) find particular solutions of the system of two homogeneous equations:

$$f'(x) + g'(x) - 3g(x) = 0 \qquad \text{and} \qquad f''(x) + g'(x) = 0,$$

which satisfy the boundary conditions

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad g(0) = \frac{4}{3}.$$

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(2) ODE: variation of parameters method. Use the method of variation of parameters to

(a) find the general solution of the the non-homogeneous equation

$$y''(x) + \omega^2 y(x) = \sin(\omega x),$$

which satisfies the boundary conditions $y(0) = y'(0) = 0$.
(b) find the general solution of the non-homogeneous equation

$$x^2 y''(x) - 2xy'(x) + 2y = x\ln(x),$$

given that the solutions of the corresponding homogeneous equation are x and x^2.

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(3) Properties of the δ function.

(a) Evaluate

$$\int_0^3 (5x-2)\delta(2-x)dx.$$

(b) Generalised function $\theta(x)$ is equal to zero for $x < a$ and 1 for $x \geq a$, where $a > 0$. Express the first derivative of the function θ using the Dirac δ function.

(c) Show that for $m \leq n$ (m and n are non-negative and integer), the generalised function $x^m \delta^{(n)}(x)$, where $\delta^{(n)}(x)$ is the n^{th} derivative of the δ function satisfy

$$x^m \delta^{(n)}(x) = (-1)^m \frac{n!}{(n-m)!} \delta^{(n-m)}(x).$$

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(4) ODE: Green's functions. Use the method of the Green's function to solve

(a) the non-homogeneous equation

$$y''(x) + \omega^2 y(x) = e^{-x} \qquad \text{where } y(0) = y'(0) = 0 \text{ and } 0 \leq x < \infty;$$

(b) the non-homogeneous equation

$$(x^2+1)y''(x) - 2xy'(x) + 2y = (x^2+1)^2,$$

for $0 \leq x \leq 1$ and the boundary conditions $y(0) = y(1) = 0$, given that the solutions of the corresponding homogeneous equation are x and $1 - x^2$.