Below are some examples using Einstein summation convention. For each example the equivalent vector or matrix version of that operation will be provided.

1 Dot/Scalar product

The scalar product between two vectors **a** and **b** is defined as:

$$\mathbf{a}\cdot\mathbf{b} \Rightarrow a_i b_i = \sum_i a_i b_i,$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + c_1 c_2,$$

$$\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (4\times 1) + (3\times 3) + (-1\times 2) = 11.$$

2 Matrix multiplication

For two matrices **A** and **B**.

$$\mathbf{AB} \Rightarrow a_{ij} b_{jk} = \sum_j a_{ij} b_{jk}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} (4\times 0) + (0\times 3) & (4\times 1) + (0\times 2) \\ (0\times 0) + (2\times 3) & (0\times 1) + (2\times 2) \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 6 & 4 \end{pmatrix}$$

3 Tensor/Outer product

The tensor/outer product between two vectors is defined below:

$$\mathbf{a}\otimes\mathbf{b} \Rightarrow a_i b_j = c_{ij}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\times 5 & 0\times 0 & 0\times 1 \\ 2\times 5 & 2\times 0 & 2\times 1 \\ 1\times 5 & 1\times 0 & 1\times 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

In some ways it may be easier to think of this new operation in terms of the index notation above. Here we multiply the components of two vectors together represented by a_i and b_j respectively. We do this for every possible value of i and j (usually up to 3 for 3 dimensions). Since there is no sum we do not add any of these together instead they form the components of a matrix c_{ij}.

4 Defuq is a Tensor?

We can use are index notation to represent objects called tensors. What is a tensor you ask. Well the answer is something that transforms as a tensor. Below we shall define what this means with an example. We shall look back at some maths from MM2. We shall work here in two dimensions because I'm too lazy to give you a 3D example. When we talk about tensors we are interested in their transformation properties since these are well defined. This is useful to know when moving between coordinate systems or different reference frames. Suppose I have two sets of coordinates; one set we denote with a prime $x'_i = \{r, \phi\}$ and another set $x_i = \{x, y\}$. This means that for the polar coordinates we have:

$$x'_1 = r,$$

$$x'_2 = \phi,$$

while for the Cartesian we have:

$$x_1 = x,$$

$$x_2 = y.$$

These are just polar coordinates and Cartesian coordinates. As you know the Cartesian coordinates can be written as functions of the polar ones as follows:

$$x = rcos(\phi),$$

$$y = rsin(\phi).$$

From these relations we can find dx and dy in terms of the polar coordinates. These are found to be:

$$dx = cos(\phi)dr - rsin(\phi)d\phi,$$

$$dy = sin(\phi)dr + rcos(\phi)d\phi.$$

Here it can be seen that:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \phi}d\phi,$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \phi}d\phi.$$

We can rewrite the two expressions above as one matrix equation:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} dr \\ d\phi \end{pmatrix}.$$

Alternatively, we could rewrite this using the Einstein summation:

$$dx_i = \frac{\partial x_i}{\partial x'_j}dx_j,$$

where the repeated index j denotes a summation. The partial derivatives here refer to the components of the orthogonal Jacobian transformation matrix from polar to Cartesian coordinates. In the lecture notes this is defined:

$$S_{ij} = \frac{\partial x_i}{\partial x'_j}.$$

The inverse of the original Jacobian matrix will give us the transformation to Cartesian coordinates to polar coordinates. These transformation matrices are always orthogonal at least within the context of Maths for Theoretical. We define the components of the inverse matrix as:

$$(S_{ij})^{-1} = S_{ji} = L_{ij} = \frac{\partial x'_j}{\partial x_i}.$$

. As we know from linear algebra multiplying a matrix with its inverse gives the identity matrix. In index notation the components of the identity matrix are represented with the Kronecker delta symbol δ_{ij}. From the above we can determine the following result.

$$L_{ij}L_{ik} = S_{ji}L_{ik} = \delta_{jk}.$$

Initially what we had above were the components of two matrices being summed over down the rows of each which you would never do in linear algebra. This would be equivalent to summing along the columns in the transpose of the first matrix and down the rows of the second. It just so happens that this matrix is orthogonal so the result is the identity.

Now that we've got our transformation matrices we can define a tensor. The components of a tensor of rank one otherwise known as a vector transform as follows:

$$V'_i = L_{ij}V_j.$$

Going back to our coordinate systems above if we wanted to transform a vector from Cartesian to polar coordinates we take the inverse of the original matrix we found and apply the transformation to the vector we want to express in the polar coordinate system. That's all there is to it for a tensor of rank one at least but the for higher rank tensors the definition it follows on naturally. For tensors of rank two and three we get the following:

$$U'_{ij} = L_{im}L_{jn}U_{mn},$$

$$W'_{ijk} = L_{il}L_{jm}L_{kn}W_{lmn}.$$

Essentially, for each index a tensor component has we need to apply the transformation matrix once for each index. An n^{th} order tensor requires n transformation matrices.

5 Levi-Civita

The three index Levi-Civita symbol is defined:

$$\epsilon_{ijk} = \begin{cases} 0 & if \ i = j \ or \ j = k \ or \ i = k \\ 1 & for \ even \ permutations \\ -1 & for \ odd \ permutations \end{cases} .$$

Here even permutations simply refers to the indices being in numerical order meaning that if you wrote them out infinitely many times you'd get 123123123123123... (this can start with 2 or 3 depending on the first index. For instance $\epsilon_{123} = \epsilon_{231} = 1$. An odd pass is where this is not the case and none of the indices are equal.

There are two important identies/properties of Levi-Civita. There are more but I don't know where that shit comes from. I don't even know where the second of these comes from but you just have to learn these. Firstly, if we swap an the positions of the indices around as follows we get:

$$\epsilon_{ijk} = -\epsilon_{jik},$$

which makes sense because even passes become odd passes and vice versa since we would have changed the order of the indices. The second important identity is:

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}.$$

Together these two identities help us make what would have been tedious proofs involving vectors much faster.

The cross product can be expressed in terms of Levi-Civita as follows:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

You can convince yourself that this is true by writing out the resulting non-zero terms for $i = 1$ which should be:

$$\epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2,$$

which for Cartesian coordinates just becomes:

$$a_y b_z - a_z b_y,$$

which is just the first component of the cross product of the two vectors!

The divergence of the curl of a vector can be shown to be zero using the Levi-Civita symbol. First, we rewrite this in index notation as follows:

$$\nabla \cdot (\nabla \times \mathbf{a}) = \partial_i \epsilon_{ijk} \partial_j a_k$$

Now our Levi-Civita guy can be thought of as a constant since its just ones, zeros and minus ones. We can therefore move it to the front. Equally, the differentiation commutes with itself the order of ∂_i and ∂_j with respect to themselves doesn't matter provided. Then we can utilise the first property we noted for our Levi-Civta by swapping the i with the j. Then we move our partials and move the Levi-Civita back to it's original position between them. Doing this yields:

$$\partial_i \epsilon_{ijk} \partial_j a_k = \epsilon_{ijk} \partial_i \partial_j a_k = -\epsilon_{jik} \partial_i \partial_j a_k = -\epsilon_{jik} \partial_j \partial_i a_k = -\partial_j \epsilon_{jik} \partial_i a_k.$$

In the end we are left with:

$$\nabla \cdot (\nabla \times \mathbf{a}) = -\nabla \cdot (\nabla \times \mathbf{a}),$$

which can only be true if both sides are equal to zero. There is another example for a proof of a commonly used vector identity in the notes.