PHAS2423 - Self Study - Partial Differential Equations - Problems and Solutions

(1) (a) Verify that any function of p, where $p = x^2 + 2y$, is a solution of

$$\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

Then determine whether $v(x, y)$ is a solution of this PDE if
(b) $v(x, y) = x^4 + 4x^2y + 4y^2$
(c) $v(x, y) = x^4 + 2x^2y + y^2$
(d) $v(x, y) = x^2(x^2 - 4) + 4y(x^2 - 2) + 4(y^2 - 1)$

Solution.

(a) Verify this by directly calculating the partial derivatives for any function $u(x, y) = u(x^2 + 2y) = u(p)$:

$$\frac{\partial u(p)}{\partial x} = \frac{du(p)}{dp} \cdot \frac{\partial p}{\partial x} = 2x\frac{du}{dp}.$$

$$\frac{\partial u(p)}{\partial y} = \frac{du(p)}{dp} \cdot \frac{\partial p}{\partial y} = 2\frac{du}{dp}.$$

Compare the calculated partial derivatives and observe that

$$\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

(b) Note that

$$v(x, y) = x^4 + 4x^2y + 4y^2 = (x^2 + 2y)^2 = p^2.$$

Thus, according to what we have demonstrated in **(a)**, $v(x, y)$ is a solution of the PDE. Check it by calculating both sides of the PDE explicitly:

$$\frac{\partial(x^4 + 4x^2y + 4y^2)}{\partial x} = 4x^3 + 8xy,$$

$$x\frac{\partial(x^4 + 4x^2y + 4y^2)}{\partial y} = x(4x^2 + 8y) = 4x^3 + 8xy.$$

(c) Calculate both sides of the PDE explicitly

$$\frac{\partial(x^4+2x^2y+y^2)}{\partial x} = 4x^3+4xy,$$

$$x\frac{\partial(x^4+2x^2y+y^2)}{\partial y} = x(2x^2+2y) = 2x^3+2xy$$

and observe that the RHS and LHS are different and, therefore, this $v(x,y)$ is not a solution of the PDE.

(d) Rearrange the terms in this $v(x,y)$:

$$x^2(x^2-4)+4y(x^2-2)+4(y^2-1) = x^4-4x^2+4x^2y-8y+4y^2-4 =$$

$$= (x^4+4x^2y+4y^2)-(4x^2+8y)-4 = (x^2+2y)^2-4(x^2+2y)-4 = p^2-4p-4.$$

Thus, according to what we have shown in **(a)**, this $v(x,y)$ is a solution of the PDE. Check this explicitly:

$$\frac{\partial v(x,y)}{\partial x} = 4x^3-8x+8xy,$$

$$x\frac{\partial v(x,y)}{\partial y} = x(4x^2-8+8y) = 4x^3-8x+8xy.$$

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(2) Find solutions of the PDE

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y} = 0,$$

for which

(a) $u(0,y) = y$ (one-dimensional boundary condition);

(b) $u(1,1) = 1$ (zero-dimensional boundary condition).

Consider cases (a) and (b) separately.

Solution. This is equation of the type

$$A(x,y)\frac{\partial u}{\partial x}+B(x,y)\frac{\partial u}{\partial y} = 0,$$

where $A(x,y) = 1/x$ and $B(x,y) = 1/y$. To solve it (see lecture notes) we need to construct and integrate

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \quad \text{which is} \quad \frac{dx}{1/x} = \frac{dy}{1/y} \quad \text{i.e.,} \quad x\,dx = y\,dy.$$

Integrating it gives

$$\frac{1}{2}x^2 = \frac{1}{2}y^2 + C,$$

where C is an arbitrary constant and we can set

$$p = x^2 - y^2.$$

Thus, any function $u(p)$, where $p = x^2 - y^2$, is a solution of this PDE.

Apply the boundary conditions.

(a) For $x = 0$ and arbitrary y, we have $p = 0 - y^2 = -y^2$. Then, the function at the boundary is:

$$u(0, y) = y = \sqrt{-p}.$$

Thus, for arbitrary x,

$$u(x, y) = \sqrt{-p} = \sqrt{-(x^2 - y^2)} = \sqrt{y^2 - x^2}.$$

(b) At the point x=1, y=1 the parameter p becomes $p = x^2 - y^2 = 0$. To satisfy the condition $u(1, 1) = 1$, one can set

$$u(x, y) = 1 + g(x^2 - y^2),$$

where $g(x^2 - y^2)$ is any function, such that $g(0) = 0$.

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(3) Find solutions of the PDE

$$\sin x \frac{\partial u}{\partial x} + \cos x \frac{\partial u}{\partial y} = \cos x,$$

for which
 (a) $u(\pi/2, y) = 0$;
 (b) $u(\pi/2, y) = y(y + 1)$.
Consider cases (a) and (b) separately.

Solution. First, find solution of the homogeneous equation. For that, integrate

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)} \qquad \text{which is} \qquad \frac{dx}{\sin x} = \frac{dy}{\cos x}.$$

$$\frac{\cos x}{\sin x}\,dx = \frac{d(\sin x)}{\sin x}\,dx = dy,$$

which gives:

$$\ln(\sin x) = y + C.$$

Thus, any function $f(p)$, where $p = y - \ln(\sin x)$ is a solution of the homogeneous equation.

A particular solution of this inhomogeneous equation is a function $v(x, y)$, such that

$$\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 1.$$

One can take $v(x, y) = y$. Then, the general solution is

$$u(x, y) = y + f(y - \ln(\sin x)).$$

Apply the boundary conditions.

(a) At this boundary $x = \pi/2$, $\sin(\pi/2) = 1$, and $p = y - \ln 1 = y$. Thus, for the function $u(x, y)$ at the boundary we have (see the general solution above)

$$u(\pi/2, y) = 0 = y + f(y) = p + f(p).$$

From here, $f(p) = -p$, i.e.,

$$u(x, y) = y - (y - \ln(\sin x)) = \ln(\sin x).$$

(b) At this boundary $x = \pi/2$, $\sin(\pi/2) = 1$, and $p = y - \ln 1 = y$. Thus, for the function $u(x, y)$ at the boundary we have (see the general solution above)

$$u(\pi/2, y) = y(y + 1) = y + f(y),$$

which gives

$$y^2 + y = y + f(y) \qquad \text{i.e.} \qquad p^2 = f(p).$$

Thus,

$$u(x, y) = y + (y - \ln(\sin x))^2.$$

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(4) Find the most general solution of

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y^2} = 0,$$

which is consistent with

$$\frac{\partial u}{\partial y} = 1 \qquad \text{when } y = 0 \text{ for all } x.$$

and evaluate $u(0,1)$.

Solution. This equation has the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial u}{\partial y^2}=0$$

with A=1, B=–3, C=2. The corresponding characteristic equation is

$$2\lambda^2-3\lambda+1=0,$$

which has two solutions

$$\lambda_1=1 \qquad \text{and} \qquad \lambda_2=\frac{1}{2}.$$

Thus, the most general solution of this PDE is

$$u(x,y)=f(x+y)+g\left(x+\frac{y}{2}\right),$$

where $f(p_1)$ and $g(p_2)$ are arbitrary functions of $p_1=x+y$ and $p_2=x+y/2$, respectively.

Calculate $\partial u/\partial y$ (take into account $p_1=x+y$ and $p_2=x+y/2$):

$$\frac{\partial u}{\partial y}=\frac{df}{dp_1}\cdot\frac{\partial p_1}{\partial y}+\frac{dg}{dp_2}\cdot\frac{\partial p_2}{\partial y}=\frac{df}{dp_1}+\frac{dg}{dp_2}\cdot\frac{1}{2}.$$

For y=0, $p_1=p_2$ for all x and can be denoted as p. Therefore,

$$\frac{\partial u}{\partial y}=\frac{df}{dp}+\frac{1}{2}\frac{dg}{dp}=1,$$

which gives rise to an equation with respect to f' and g':

$$\frac{dg}{dp}=2\left(1-\frac{df}{dp}\right).$$

Integrating this equation with respect to p gives

$$g(p)=2(p-f(p))+2C=2(p-f(p)+C),$$

where C is an arbitrary constant. Therefore,

$$u(x,y)=f(p_1)+g(p_2)=f(x+y)+g\left(x+\frac{y}{2}\right)=f(x+y)+2\left[\left(x+\frac{y}{2}\right)-f\left(x+\frac{y}{2}\right)+C\right].$$

Clearly, for y=0 ($p_1=p_2$)

$$\frac{\partial u}{\partial y}=\frac{df}{dp_1}+1-2\frac{df}{dp_2}\frac{1}{2}=1.$$

Check the $u(x,y)$ satisfies the given equation:

$$\frac{\partial u}{\partial x}=\frac{df}{dp_1}+2-2\frac{df}{dp_2} \qquad\qquad \frac{\partial^2 u}{\partial x^2}=\frac{d^2 f}{dp_1^2}-2\frac{d^2 f}{dp_2^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dp_1^2} - 2\frac{d^2 f}{dp_2^2} \cdot \left(\frac{1}{2}\right)^2 \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{d^2 f}{dp_1^2} - 2\frac{d^2 f}{dp_2^2} \cdot \frac{1}{2}.$$

Combining these together gives

$$\left(\frac{d^2 f}{dp_1^2} - 2\frac{d^2 f}{dp_2^2}\right) - 3\left(\frac{d^2 f}{dp_1^2} - \frac{d^2 f}{dp_2^2}\right) + 2\left(\frac{d^2 f}{dp_1^2} - \frac{1}{2}\frac{d^2 f}{dp_2^2}\right) = 0.$$

Finally,

$$u(0,1) = f(1) + 1 - 2f(1/2) + C.$$