PHAS2423 - Self-Study - Ordinary Differential Equations - Problems and Solutions

(1) ODE: Laplace transform. Use the Laplace transform to

(a) find a particular solution of the non-homogeneous equation:

$$y'' + 4y' + 4y = e^{-2x},$$

which satisfies the boundary conditions $y(0) = y'(0) = 4$.

(b) find particular solutions of the system of two homogeneous equations:

$$f'(x) + g'(x) - 3g(x) = 0 \qquad \text{and} \qquad f''(x) + g'(x) = 0,$$

which satisfy the boundary conditions

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad g(0) = \frac{4}{3}.$$

Solution.

(a) Let us introduce a notation for the Laplace transform of the function $y(x)$:

$$Y = L[y] = \int_0^\infty y(x)e^{-px}dx$$

Laplace transforms of $y'(x)$ and $y''(x)$ (see lecture notes) are:

$$L[y'] = -y(0) + pY \qquad \text{and} \qquad L[y''] = -y'(0) - py(0) + p^2Y.$$

Laplace transform is a linear operator. Hence, after applying to both parts of the differential equation, we obtain an algebraic equation

$$L[y''+4y'+4y] = L[y'']+4L[y']+4L[y] = -y'(0)-py(0)+p^2Y+4\left(-y(0) + pY\right)+4Y = L[e^{-2x}].$$

Calculate the Laplace transform in the right hand side:

$$L[e^{-2x}] = \int_0^\infty e^{-(p+2)x}dx = \left[-\frac{1}{(p+2)}e^{-2x}\right]_0^\infty = \frac{1}{p+2}.$$

Taking into account the boundary conditions, the algebraic equation becomes

$$-4 - 4p + p^2Y - 16 + 4pY + 4Y = (p^2 + 4p + 4)Y - 4p - 20 = \frac{1}{p+2}.$$

Hence,

$$(p+2)^2\, Y = \frac{1}{p+2} + 4p + 20$$

and, after dividing both parts by $(p+2)^2$,

$$Y = \frac{1}{(p+2)^3} + \frac{4p}{(p+2)^2} + \frac{20}{(p+2)^2}.$$

In order to simplify the second term, notice that

$$\frac{p}{(p+2)^2} = \frac{p+2-2}{(p+2)^2} = \frac{p+2}{(p+2)^2} - \frac{2}{(p+2)^2} = \frac{1}{p+2} - \frac{2}{(p+2)^2}$$

and substitute this into the expression for Y. Thus,

$$Y = \frac{1}{(p+2)^3} + \frac{4}{p+2} - \frac{8}{(p+2)^2} + \frac{20}{(p+2)^2} = \frac{1}{(p+2)^3} + \frac{4}{p+2} + \frac{12}{(p+2)^2}.$$

Refer to the table of Laplace transforms:

$$L\left[x^n e^{ax}\right] = \frac{n!}{(s-a)^{n+1}}.$$

Use this relation to find contributions to $y(x)$:

$$L[y_1] = \frac{4}{p+2} = 4\frac{0!}{(p-(-2))^{0+1}} \qquad \longrightarrow \qquad y_1(x) = 4e^{-2x}$$

$$L[y_2] = \frac{12}{(p+2)^2} = 12\frac{1!}{(p-(-2))^{1+1}} \qquad \longrightarrow \qquad y_2(x) = 12xe^{-2x}$$

$$L[y_3] = \frac{1}{(p+2)^3} = \frac{1}{2}\frac{2!}{(p-(-2))^{2+1}} \qquad \longrightarrow \qquad y_3(x) = \frac{1}{2}x^2e^{-2x}.$$

Thus,

$$y(x) = y_1(x)+y_2(x)+y_3(x) = \frac{1}{2}x^2e^{-2x} + 12xe^{-2x} + 4e^{-2x} = e^{-2x}\left(\frac{x^2}{2} + 12x + 4\right).$$

To check that the solution is correct, calculate derivatives of $y(x)$:

$$y'(x) = -e^{-2x}(x^2+24x+8) + e^{-2x}(x+12) = e^{-2x}(-x^2-23x+4)$$

$$y''(x) = e^{-2x}(2x^2+46x-8) + e^{-2x}(-2x-23) = e^{-2x}(2x^2+44x-31),$$

confirm that the boundary conditions are satisfied:

$$y(0) = y'(0) = 4$$

and that the ODE is satisfied:

$$\begin{aligned} e^{-2x} &= e^{-2x}(2x^2+44x-31) + 4e^{-2x}(-x^2-23x+4) + 4e^{-2x}\left(\tfrac{x^2}{2} + 12x + 4\right) \\ &= e^{-2x}x^2(2-4+2) + e^{-2x}x(44-92+48) + e^{-2x}(-31+16+16). \end{aligned}$$

(b) Perform the Laplace transform of both equations. Use (see the lecture notes)

$$L[f'] = -f(0) + pL[f] \qquad \text{and} \qquad L[f''] = -f'(0) - pf(0) + p^2 L[f]$$

Let us introduce notations for the Laplace transforms of the functions $f(x)$ and $g(x)$:

$$F = L[f] = \int_0^\infty f(t)e^{-pt}dt$$
$$G = L[g] = \int_0^\infty g(t)e^{-pt}dt$$

In these notations, the Laplace transforms of the two given equations are:

$$-f(0) + pF - g(0) + pG - 3G = 0$$

and

$$-f'(0) - pf(0) + p^2F - g(0) + pG = 0.$$

Taking into account the boundary conditions, obtain a system of two algebraic equations:

$$pF + pG - 3G = \frac{4}{3} \qquad \text{and} \qquad p^2F + pG = \frac{4}{3}.$$

By equating the left hand sides we find

$$\begin{aligned} pF + pG - 3G &= p^2F + pG \\ -(p^2F - pF) &= 3G \\ G &= -\tfrac{1}{3}p(p-1)F \end{aligned}$$

Substitute G into the 2nd algebraic equation:

$$\begin{aligned} \tfrac{4}{3} &= p^2F - \tfrac{1}{3}p^2(p-1)F \\ &= F(p^2 - \tfrac{1}{3}p^3 + \tfrac{1}{3}p^2) = F(\tfrac{4}{3}p^2 - \tfrac{1}{3}p^3) \\ &= -\tfrac{1}{3}p^2(p-4)F. \end{aligned}$$

Hence

$$F = \frac{-4}{p^2(p-4)} \qquad \text{and} \qquad G = \frac{4}{3}\frac{p-1}{p(p-4)}.$$

Represent F as a sum of simple fractions (review PHAS1245):

$$\frac{-4}{p^2(p-4)} = \frac{A}{p-4} + \frac{Bp + C}{p^2} = \frac{Ap^2 + Bp^2 + Cp - 4Bp - 4C}{p^2(p-4)}.$$

From this:

$$C = 1 \qquad B = \frac{1}{4} \qquad A = -\frac{1}{4}$$

and, therefore,

$$F = \frac{-4}{p^2(p-4)} = \frac{-1}{4(p-4)} + \frac{1}{4p} + \frac{1}{p^2}.$$

Similarly, for G

$$G = \frac{4(p-1)}{3p(p-4)} = \frac{1}{p-4} + \frac{1}{3p}.$$

Use the table of Laplace transform to find functions $f(x)$ and $g(x)$:

$$f(x) = -\frac{1}{4}e^{4x} + \frac{1}{4} + x \qquad \text{and} \qquad g(x) = e^{4x} + \frac{1}{3}.$$

To check the correctness of the solutions, calculate derivatives of $f(x)$ and $g(x)$

$$f'(x) = -e^{4x} + 1 \qquad f''(x) = -4e^{4x} \qquad g'(x) = 4e^{4x},$$

confirm that the boundary conditions are satisfied

$$f(0) = 0 \qquad f'(0) = 0 \qquad g(0) = \frac{4}{3},$$

and that the system of equations is satisfied:

$$f'(x) + g'(x) - 3g(x) = -e^{4x} + 1 + 4e^{4x} - 3(e^{4x} + \frac{1}{3}) = 0$$

$$f''(x) + g'(x) = -4e^{4x} + 4e^{4x} = 0.$$

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(2) ODE: variation of parameters method. Use the method of variation of parameters to

(a) find the general solution of the the non-homogeneous equation

$$y''(x) + \omega^2 y(x) = \sin(\omega x),$$

which satisfies the boundary conditions $y(0) = y'(0) = 0$.
(b) find the general solution of the non-homogeneous equation

$$x^2 y''(x) - 2xy'(x) + 2y = x\ln(x),$$

given that the solutions of the corresponding homogeneous equation are x and x^2.

Solution.
(a) Since $\cos\omega x$ and $\sin\omega x$ are linearly independent solutions of the corresponding homogeneous equation, the complementary function is

$$y_c(x) = A\cos\omega x + B\sin\omega x$$

and we look for a particular solution in the form

$$y_p(x) = k_1(x)\cos\omega x + k_2(x)\sin\omega x,$$

where the functions $k_1(x)$ and $k_2(x)$ are subject to the conditions (see the lecture notes)

$$k_1' \cos\omega x + k_2' \sin\omega x = 0$$
$$\omega(-k_1' \sin\omega x + k_2' \cos\omega x) = \sin\omega x.$$

From the first condition we have

$$k_1' = -k_2' \frac{\sin\omega x}{\cos\omega x}.$$

Thus, the second condition gives:

$$\omega k_2' \left(\frac{\sin^2\omega x}{\cos\omega x} + \cos\omega x \right) = \omega k_2' \frac{1}{\cos\omega x} = \sin\omega x.$$

Hence,

$$k_2' = \frac{\cos\omega x \sin\omega x}{\omega} \qquad \text{and} \qquad k_1' = -\frac{\sin^2\omega x}{\omega}.$$

Integrate k_1' and k_2':

$$k_1(x) = -\frac{1}{\omega}\int \sin^2\omega x\, dx = -\frac{1}{\omega}\int \left(\frac{1-\cos 2\omega x}{2}\right) dx = -\frac{1}{2\omega}x + \frac{1}{4\omega^2}\sin 2\omega x + c_1$$

$$k_2(x) = \frac{1}{\omega}\int \sin\omega x \cos\omega x\, dx = \frac{1}{2\omega^2}\sin^2\omega x + c_2.$$

Thus, a particular solution of the ODE can be written as

$$y_p(x) = \left(-\frac{1}{2\omega}x + \frac{1}{4\omega^2}\sin 2\omega x\right)\cos\omega x + \frac{1}{2\omega^2}\sin^3\omega x.$$

This expression for $y_p(x)$ can be simplified using trigonometric relations

$$\sin 2\omega x = 2\sin\omega x\cos\omega x \qquad\qquad \sin^2 x + \cos^2 x = 1.$$

$$y_p(x) = -\frac{1}{2\omega}x\cos\omega x + \frac{1}{2\omega^2}\sin\omega x\cos^2\omega x + \frac{1}{2\omega^2}\sin^3\omega x = \frac{1}{2\omega^2}\left(-\omega x\cos\omega x + \sin\omega x\right).$$

Therefore, the general solution is

$$y(x) = \frac{1}{2\omega^2}\left(-\omega x\cos\omega x + \sin\omega x\right) + A\cos\omega x + B\sin\omega x$$

and its first derivative is

$$y'(x) = \frac{1}{2\omega^2}\left(-\omega\cos\omega x + \omega^2 x\sin\omega x + \omega\cos\omega x\right) - A\omega\sin\omega x + B\omega\cos\omega x,$$

where A and B are constants to be determined using the boundary conditions. From $y(0) = 0$ we find that $A = 0$ and from $y'(0) = 0$ we find that $B = 0$. Thus, the solution is

$$y(x) = \frac{1}{2\omega^2}\left(-\omega x\cos\omega x + \sin\omega x\right).$$

(b) Check that $y_1(x) = x$ and $y_2(x) = x^2$ are solutions of the homogeneous equation:

$$x^2 y_1'' - 2x y_1' + 2y_1 = 0 - 2x + 2x = 0$$
$$x^2 y_2'' - 2x y_2' + 2y_2 = 2x^2 - 4x^2 + 2x^2 = 0$$

A particular solution of the non-homogeneous equation is represented as

$$y_p(x) = k_1(x)x + k_2(x)x^2,$$

where $k_1(x)$ and $k_2(x)$ need to be determined from the conditions (see lecture notes)

$$k_1'\, y_1 + k_2'\, y_2 = 0 \qquad \text{i.e.,} \qquad k_1'\, x + k_2'\, x^2 = 0$$
$$k_1'\, y_1' + k_2'\, y_2' = \frac{f(x)}{a_2(x)} \qquad \text{i.e.,} \qquad k_1' + k_2'\, 2x = \frac{x \ln x}{x^2} = \frac{\ln x}{x}.$$

From the first condition we have

$$k_1' = -k_2'\, x$$

and, therefore, the second condition becomes

$$-k_2'\, x + k_2'\, 2x = k_2'\, x = \frac{\ln x}{x}.$$

Thus,

$$k_2' = \frac{\ln x}{x^2} \qquad \text{and} \qquad k_1' = -\frac{\ln x}{x}.$$

Integrate k_1':

$$-k_1 = \int \frac{\ln x}{x}\, dx = \int \ln x\, d(\ln x) = \frac{1}{2} \ln^2 x + c_1.$$

Integrate k_2':

$$k_2 = \int \frac{1}{x^2} \ln x\, dx = -\frac{1}{x} \ln x + \int \frac{1}{x^2}\, dx = -\frac{1}{x} \ln x - \frac{1}{x} + c_2 = -\frac{1}{x} (\ln x + 1) + c_2$$

Thus, a particular solution of the non-homogeneous ODE is

$$y_p(x) = k_1\, x + k_2\, x^2 = -\frac{x}{2} \ln^2 x - x(\ln x + 1) = -\frac{x}{2} \ln^2 x - x \ln x - x$$

and the general solution is (use the fact that x is a solution of the homogeneous ODE)

$$y(x) = -\frac{x}{2} \ln^2 x - x \ln x + Ax + Bx^2,$$

where A and B are arbitrary constants.

(3) Properties of the δ function.

(a) Evaluate

$$\int_0^3 (5x-2)\delta(2-x)dx.$$

(b) Generalised function $\theta(x)$ is equal to zero for $x < a$ and 1 for $x \geq a$, where $a > 0$. Express the first derivative of the function θ using the Dirac δ function.

(c) Show that for $m \leq n$ (m and n are non-negative and integer), the generalised function $x^m \delta^{(n)}(x)$, where $\delta^{(n)}(x)$ is the n^{th} derivative of the δ function satisfy

$$x^m \delta^{(n)}(x) = (-1)^m \frac{n!}{(n-m)!} \delta^{(n-m)}(x).$$

Solution.
(a)

$$\int_0^3 (5x-2)\,\delta(2-x)\,dx = \int_0^3 f(x)\,\delta(2-x)\,dx = f(2) = 5 \cdot 2 - 2 = 8$$

(b) Consider integral of the first derivative of the function $\theta(x)$ with the test function $f(x)$ and integrate it by parts:

$$\int_{-\infty}^{+\infty} f(x)\,\theta'(x)\,dx = f(x)\theta(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x)\,\theta(x)\,dx = (0-0) - \int_a^{+\infty} f'(x)\,dx = -f(x)|_a^{\infty} = f(a).$$

The right hand side is also given by

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a).$$

Compare the left hand side integrals and observe that

$$\theta'(x) = \delta(x-a).$$

(c) We need to demonstrate that for any well-behaved (see lecture notes) test function $f(x)$ the following relation holds

$$\int_a^b f(x) x^m \delta^{(n)}(x)\,dx = (-1)^m \frac{n!}{(n-m)!} \int_a^b f(x) \delta^{(n-m)}(x)\,dx,$$

where lower and upper limits a and b are such that the test function and all of its derivatives are zero at least at these points and everywhere outside domain (a, b).

Let us first evaluate the integral in the right hand side:

$$\int f(x)\delta^{(n-m)}(x)\,dx = f(x)\delta^{(n-m+1)}(x)|_a^b - \int f'(x)\delta^{(n-m+1)}(x)\,dx = (-1)^1 \int f'(x)\delta^{(n-m+1)}(x)\,dx.$$

Performing this procedure $n-m$ times gives

$$\int f(x)\delta^{(n-m)}(x)\,dx = (-1)^{n-m} \int [f(x)]^{(n-m)}\delta(x)\,dx = (-1)^{n-m} f^{(n-m)}(0).$$

To evaluate the integral on the left, first define

$$g(x) = x^m f(x)$$

and perform the integration by parts n times:

$$\int_a^b f(x)x^m\delta^{(n)}(x)\,dx = \int_a^b g(x)\delta^{(n)}(x)\,dx = (-1)^n \int_a^b [g(x)]^{(n)}\delta(x)\,dx = (-1)^n g^{(n)}(0).$$

Now let us calculate the n^{th} derivative of the function $g(x)$:

$$[g(x)]^{(n)} = [x^m\, f(x)]^{(n)} = \sum_{k=0}^{k=n} \frac{n!}{k!(n-k)!}[x^m]^{(k)}[f(x)]^{(n-k)}$$

and evaluate its value at the point $x=0$ for each of the terms of this sum. Consider three cases:

1) Terms with $k<m$ have a factor of x^{m-k}, which turns to zero when $x=0$.

2) The only term with $k=m$ is

$$\frac{n!}{m!(n-m)!}[x^m]^{(m)}[f(x)]^{(n-m)}.$$

Since

$$[x^m]^{(m)} = m[x^{m-1}]^{(m-1)} = m(m-1)[x^{m-2}]^{(m-2)} = \ldots = m\cdot(m-1)\cdot(m-2)\cdot\ldots\cdot 2\cdot 1 = m!,$$

we have

$$\frac{n!}{m!(n-m)!}[x^m]^{(m)}[f(x)]^{(n-m)} = \frac{n!}{(n-m)!}[f(x)]^{(n-m)}$$

3) Terms with $k>m$ turn to zero by the differentiation rule of x^m.

Thus, at $x=0$

$$g^{(n)}(0) = \frac{n!}{(n-m)!}f^{(n-m)}(0).$$

Finally, compare the LHS and RHS of the original equality:

$$\text{LHS} = \int_a^b f(x)x^m\delta^{(n)}(x)\,dx = (-1)^n g^{(n)}(0) = (-1)^n\frac{n!}{(n-m)!}f^{(n-m)}(0)$$

$$\text{RHS} = (-1)^m\frac{n!}{(n-m)!}\int_a^b f(x)\delta^{(n-m)}(x)\,dx = (-1)^m\frac{n!}{(n-m)!}(-1)^{n-m}f^{(n-m)}(0)$$

and find LHS=RHS.

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(4) ODE: Green's functions. Use the method of the Green's function to solve

(a) the non-homogeneous equation

$$y''(x) + \omega^2 y(x) = e^{-x} \qquad \text{where } y(0) = y'(0) = 0 \text{ and } 0 \le x < \infty;$$

(b) the non-homogeneous equation

$$(x^2 + 1)y''(x) - 2xy'(x) + 2y = (x^2 + 1)^2,$$

for $0 \le x \le 1$ and the boundary conditions $y(0) = y(1) = 0$, given that the solutions of the corresponding homogeneous equation are x and $1 - x^2$.

Solution.
(a) The ODE for the Green's function is

$$G''(x,t) + \omega^2 G(x,t) = \delta(x - t).$$

Find its solution in the form of $f(t)y_1(x) + g(t)y_2(x)$, where y_1 and y_2 are the solutions of the homogeneous equation $y'' + \omega^2 y = 0$. We will use $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$.
Then, the Green's functions for $x < t$ and $x > t$ are

$$G_{x<t}(x,t) = A(t) \cos \omega x + B(t) \sin \omega x$$

$$G_{x>t}(x,t) = C(t) \cos \omega x + D(t) \sin \omega x$$

Use the boundary conditions and the continuity/discontinuity conditions to find coefficients $A(t)$, $B(t)$, $C(t)$, $D(t)$. The boundary conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0$$

are satisfied if

$$G(0,t) = 0 \qquad \text{and} \qquad G'(0,t) = 0.$$

Hence

$$G_{x<t}(0,t) = A \cdot 1 + B \cdot 0 = 0 \qquad \text{and} \qquad G'_{x<t}(0,t) = -\omega A \cdot 0 + \omega B \cdot 1 = 0,$$

which means that $A(t) = B(t) = 0$.
The continuity condition at $x = t$ is

$$C \cos \omega t + D \sin \omega t = 0,$$

which gives

$$C = -D \frac{\sin \omega t}{\cos \omega t}.$$

The discontinuity condition at $x = t$ is

$$-C\omega \sin \omega t + D\omega \cos \omega t = \omega D \left(\frac{\sin^2 \omega t}{\cos \omega t} + \cos \omega t \right) = \omega D \frac{1}{\cos \omega t} = 1.$$

Thus,

$$D = \frac{\cos\omega t}{\omega} \qquad \text{and} \qquad C = -\frac{\sin\omega t}{\omega}$$

and the only non-zero Green's function is

$$G_{x>t}(x,t) = \frac{1}{\omega}\left(\sin\omega x\cos\omega t - \cos\omega x\sin\omega t\right).$$

Therefore, the solution of the original ODE is

$$y(x) = \frac{1}{\omega}\left(\sin\omega x\int_0^x \cos\omega t\, e^{-t}\, dt - \cos\omega x\int_0^x \sin\omega t\, e^{-t}\, dt\right)$$

To calculate the first integral, integrate it by parts twice. This gives

$$\int_0^x \cos\omega t\, e^{-t}\, dt = \left[\frac{1}{\omega}e^{-t}\sin\omega t\right]_0^x - \left[\frac{1}{\omega^2}e^{-t}\cos\omega t\right]_0^x - \int_0^x \cos\omega t\, e^{-t}\, dt.$$

The integrals on the left and one the right are identical, hence, we can write

$$\left(1+\frac{1}{\omega^2}\right)\int_0^x \cos\omega t\, e^{-t}\, dt = \frac{1}{\omega}e^{-x}\sin\omega x - \frac{1}{\omega^2}e^{-x}\cos\omega x + \frac{1}{\omega^2},$$

which gives

$$\int_0^x \cos\omega t\, e^{-t}\, dt = \frac{1}{\omega^2+1}\left(\omega e^{-x}\sin\omega x - e^{-x}\cos\omega x + 1\right).$$

Exactly the same strategy is used to calculate the second integral:

$$\int_0^x \sin\omega t\, e^{-t}\, dt = \frac{1}{\omega^2+1}\left(-\omega e^{-x}\cos\omega x - e^{-x}\sin\omega x + \omega\right).$$

Now we have everything to write $y(x)$:

$$\begin{aligned} y(x) \;=\; & \tfrac{1}{\omega}\tfrac{1}{\omega^2+1}\sin\omega x\left(\omega e^{-x}\sin\omega x - e^{-x}\cos\omega x + 1\right) \\ & -\tfrac{1}{\omega}\tfrac{1}{\omega^2+1}\cos\omega x\left(-\omega e^{-x}\cos\omega x - e^{-x}\sin\omega x + \omega\right) \end{aligned}$$

This expression is easily simplified into

$$y(x) = \frac{1}{\omega^2+1}\left(e^{-x} - \cos\omega x + \frac{1}{\omega}\sin\omega x\right).$$

(b) Check that functions $y_1 = x$ and $y_2 = 1 - x^2$ are solutions of the homogeneous equation:

$$(x^2+1)\cdot 0 - 2x + 2x = 0$$

$$(x^2+1)(-2) - 2x(-2x) + 2(1-x^2) = 0$$

Find $G(x,t)$ by solving inhomogeneous equation

$$(x^2+1)G''(x,t) - 2xG'(x,t) + 2G(x,t) = \delta(x-t),$$

where function $G(x,t)$ has the form of $f(t)y_1(x)+g(t)y_2(x)$. For this, let

$$G_{x<t}(x,t) = A(t)x + B(t)(1-x^2)$$
$$G_{x>t}(x,t) = C(t)x + D(t)(1-x^2)$$

and find coefficients $A(t)$, $B(t)$, $C(t)$, $D(t)$ using the boundary conditions and the continuity/discontinuity conditions.
From the boundary conditions we obtain:

$$G_{x<t}(x=0,t) = A\cdot 0 + B\cdot 1 = 0$$
$$G_{x>t}(x=1,t) = C\cdot 1 + D\cdot 0 = 0.$$

Hence, $B=C=0$.
From the continuity condition for $G(x,t)$ at $x=t$:

$$G_{x>t}(x=t,t) - G_{x<t}(x=t,t) = D(1-t^2) - At = 0.$$

Hence,

$$A = D\frac{1-t^2}{t}.$$

From the discontinuity condition for $G(x,t)$ at $x=t$:

$$G'_{x>t}(x=t,t) - G'_{x<t}(x=t,t) = -D2t - A = \frac{1}{t^2+1}.$$

After substituting for A:

$$-D\left(2t+\frac{1-t^2}{t}\right) = -D\left(\frac{2t^2+1-t^2}{t}\right) = -D\frac{t^2+1}{t} = \frac{1}{t^2+1},$$

from which we find

$$D = -\frac{t}{(t^2+1)^2}$$

and, therefore,

$$A = \frac{t^2-1}{(t^2+1)^2}.$$

Thus, the Green's function is

$$G_{x<t}(x,t) = \frac{t^2-1}{(t^2+1)^2}x \qquad \text{for } x<t$$

$$G_{x>t}(x,t) = \frac{t}{(t^2+1)^2}(x^2-1) \qquad \text{for } x>t.$$

Now, find the solution $y(x)$:

$$y(x) = \int_0^x G_{x>t}(x,t)f(t)\,dt + \int_x^1 G_{x<t}(x,t)f(t)\,dt,$$

where $f(t)$ is the right hand side of the original non-homogeneous equation. Thus,

$$y(x) = (x^2-1)\int_0^x \frac{t}{(t^2+1)^2}(t^2+1)^2\,dt + x\int_x^1 \frac{t^2-1}{(t^2+1)^2}(t^2+1)^2\,dt,$$

which simplifies into

$$y(x) = (x^2-1)\int_0^x t\,dt + x\int_x^1 t^2\,dt - x\int_x^1 dt = \left[(x^2-1)\frac{1}{2}t^2\right]_0^x + \left[x\frac{1}{3}t^3\right]_x^1 - [xt]_x^1\,,$$

and gives

$$y(x) = \frac{x^4}{2} - \frac{x^2}{2} + \frac{x}{3} - \frac{x^4}{3} - x + x^2 = \frac{1}{6}x^4 + \frac{1}{2}x^2 - \frac{2}{3}x.$$