PHAS2423 - Self-Study - Cartesian Tensors - Problems

(1) The summation convention.
(a) Express the following using the summation convention.

$$\text{(a.1) } x'_i = \sum_{j=1}^{3} a_{ij}x_j; \qquad \text{(a.2) } T'_{kl} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ki}a_{lj}T_{ij}; \qquad \text{(a.3) } B'_{pqr} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} L_{pi}L_{qj}L_{rk}B_{ijk}.$$

(b) Write the following using explicit summation.

$$\text{(b.1) } C_{jknm}A_nB_{jk}; \qquad \text{(b.2) } (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})A_{ik}.$$

(c) Show that

$$\text{(c.1) } \delta_{ij}\delta_{jk} = \delta_{ik}; \qquad \text{(c.2) } \delta_{ii} = N.$$

(d) Evaluate

$$\text{(d.1) } \delta_{ij}\delta_{jk}\delta_{km}\delta_{im}; \qquad \text{(d.2) } \epsilon_{jk2}\epsilon_{k2j}; \qquad \text{(d.3) } \epsilon_{23i}\epsilon_{2i3}.$$

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(2) Rotation.
(a) Orthogonality. An orthogonal matrix L has components L_{ij}. Evaluate the following:

$$\text{(a.1) } L_{ij}L_{jk}; \qquad \text{(a.2) } L_{ji}L_{kj}; \qquad \text{(a.3) } L_{ij}L_{ik}; \qquad \text{(a.4)} L_{ij}L_{kj}.$$

(b) Rotation. Show that the transformation matrix L for a rotation of the coordinate system by an angle θ about $\boldsymbol{e}_3$ axis is

$$L = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Consecutive rotations. Show that two consecutive rotations of the coordinate system by an angle θ about $\boldsymbol{e}_3$ axis is also a rotation about the same axis with the value of the rotation angle of 2θ.

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(3) Transformation of tensors.

(a) Contraction. Given that T_{ijk} and V_n are components of the 3rd order and 1st order tensors, respectively,
(a.1) Show that T_{iij} is a 1st-order tensor.
(a.2) Show that $T_{ijk}V_k$ is a 2nd-order tensor.

(b) Outer product. If quantities A_{ij} and B_{kl} are components of 2nd order tensors, show that quantities T_{ijkl} formed by $T_{ijkl} = A_{ij}B_{kl}$ is a 4th-order tensor.

(c) Vectors. For the case of a two-dimensional space
(c.1) Show that $\boldsymbol{v} = (x_2, -x_1)$ transforms as a vector under rotation of the coordinate system.
(c.2) Show that $\boldsymbol{v} = (x_2, x_1)$ is not a vector.

(d) Scalars.
(d.1) Show that the scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is, indeed, a scalar.
(d.2) Show that $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ is a scalar (assume that $\boldsymbol{v}$ is a vector).

(e) Higher order tensors. Demonstrate that matrix T represents a 2^{nd} order tensor:

$$\boldsymbol{T} = \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}.$$

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(4) Quotient theorem. Given that $\boldsymbol{A}$ is an arbitrary tensor and $\boldsymbol{B}$ is a non-zero tensor, prove the quotient theorem for the following cases:

(a) $X_iA_{ij} = B_j$ **(b)** $X_{ij}A_k = B_{ijk}$

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(5) Application of tensors ϵ_{ijk} and δ_{ij} Use properties of the Levi-Civita and Kronecker tensors to prove the following identities for vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$:

(a)

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

(b)

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d}]\, \boldsymbol{c} - [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}]\, \boldsymbol{d}$$

(c) Find an explicit expression for the i^{th} component of vector $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$.

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(6) A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate system the particles are at positions

$$\pm a(3, 1, -1) \qquad \pm\, a(1, -1, 3) \qquad \pm\, a(1, 3, -1) \qquad \pm\, a(-1, 1, 3).$$

The body rotates about an axis passing through the origin. Show that, if the angular velocity and angular momentum vectors are parallel, then their ratio must be $40ma^2$, $64ma^2$, or $72ma^2$.