PHAS2423 - Problem Based Learning I - Problems and Solutions

(1) Determinant of a 3×3 matrix ($|A|$) can be expressed as

$$|A|\epsilon_{lmn} = A_{li}A_{mj}A_{nk}\epsilon_{ijk}.$$

(a) Demonstrate that determinant of the transpose matrix A^T is equal to determinant of the matrix A:

$$|A^T| = |A|.$$

(b) Show that if C is a product of two square matrices, $C = AB$, then

$$|C| = |AB| = |A|\,|B|.$$

---

(2) In a certain system of units the electromagnetic stress tensor is given by

$$M_{ij} = E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(E_kE_k + B_kB_k),$$

where E_i and B_i are components of the 1st-order tensors representing the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively.

(a) Demonstrate that components M_{ij} transform as components of a tensor.

(b–d) For $|\boldsymbol{E}|$=$|\boldsymbol{B}|$ (but $\boldsymbol{E} \neq \boldsymbol{B}$):
(b) show that $\boldsymbol{E} \pm \boldsymbol{B}$ are principal axes of the tensor $\boldsymbol{M}$;
(c) determine the third principal axis and
(d) find all principal values.

---

(3) A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate system the particles are at positions

$$\pm a(3,1,-1) \qquad \pm a(1,-1,3) \qquad \pm a(1,3,-1) \qquad \pm a(-1,1,3).$$

The body rotates about an axis passing through the origin. Show that, if the angular velocity and angular momentum vectors are parallel, then their ratio must be $40ma^2$, $64ma^2$, or $72ma^2$.

---

(4) Quantities $x(t)$ and $y(t)$ satisfy a system of equations

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0 \qquad \frac{d^2y}{dt^2} + 2n\frac{dy}{dt} + n^2y = \mu\frac{dx}{dt}$$

with the following boundary conditions at $t = 0$:

$$x(0) = y(0) = \frac{dy(t)}{dt} = 0 \qquad \text{and} \qquad \frac{dx}{dt} = \lambda.$$

Use the Laplace transform method to show that

$$y(t) = \frac{1}{2}\mu\lambda t^2\left(1 - \frac{1}{3}nt\right)e^{-nt}.$$