PHAS2423 - Self-Study - Sturm-Liouville theory - Problems

(1) Demonstrate that functions

$$f_0(x) = 1, \qquad f_k(x) = \cos kx, \qquad g_k(x) = \sin kx,$$

where $k = 1, 2, 3, ...$ and $x \in [-\pi, \pi]$ are orthogonal to each other. Then find normalisation coefficients for these functions so as after normalisation they satisfy

$$\int_{-\pi}^{\pi} f_0 f_0 \, dx = \int_{-\pi}^{\pi} f_k f_k \, dx = \int_{-\pi}^{\pi} g_k g_k \, dx = 1.$$

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(2) Use the method of Schmidt orthogonalisation in order to transform functions $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$ into functions $g_1(x)$, $g_2(x)$, and $g_3(x)$, which are orthogonal and normalised to 1 on the interval $0 \leq x \leq 1$.

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(3) Use the method of Schmidt orthogonalisation in order to transform functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ into functions $g_1(x)$ and $g_2(x)$, which are orthogonal and normalised to 1 on the interval $0 \leq x \leq \pi/2$.

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(4) Find the eigenfunction expansion for a solution of the inhomogeneous equation

$$\frac{d^2y}{dx^2} + \omega^2 y = \sin^2 x,$$

where $y(x)$ satisfies the boundary conditions $y(0) = y(\pi) = 0$.