Examination Paper Answer Sheet

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Candidate Number: DNHR9
Module Code: PHAS0026
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1) a) i) Let

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} \quad , \quad \frac{\partial^2 u}{\partial x \partial y} = u_{xy} \quad , \quad \frac{\partial^2 u}{\partial y^2} = u_{yy}$$

Our P.D.E can then be expressed as

$$A u_{xx} + B u_{xy} + C u_{yy} = 0 \quad , \quad B^2 = 4AC$$

One solution of this P.D.E. is

$$u(x,y) = f(x + \lambda y) \quad \text{where} \quad \lambda = -\sqrt{\frac{A}{C}}$$

Now $B^2 = 4AC \longrightarrow \dfrac{B^2}{4C^2} = \dfrac{A}{C} \longrightarrow \dfrac{B}{2C} = +\sqrt{\dfrac{A}{C}}$

$$\therefore \quad \lambda = \frac{-B}{2C}$$

In the same fashion as when solving a 1st Order PDE, we will try and find a second solution by considering a solution of the form

$$u(x,y) = h(x,y)\, g(x+\lambda y)$$

Letting $\rho = x + \lambda y$ we have

$$u(x,y) = h(x,y)\, g(\rho), \quad \rho = \rho(x,y)$$

Now calculating the derivatives

$$u_x = h_x \cdot g + h \cdot g' \cdot \rho_x$$

$$u_y = h_y \cdot g + h \cdot g' \cdot \rho_y$$

$$\rho_x = 1, \quad \rho_y = \lambda = -\frac{B}{2C}$$

$$\therefore u_x = h_x \cdot g + h \cdot g'$$

$$u_y = h_y \cdot g - \frac{B}{2C} \cdot h \cdot g'$$

Now, the second derivatives

$$u_{xx} = h_{xx} \cdot g + 2 h_x \cdot g' + h \cdot g''$$

$$u_{xy} = h_{xy} \cdot g + h_y \cdot g' - \frac{B}{2C} \cdot h_x \cdot g' - h \cdot g'' \cdot \frac{B}{2C}$$

$$u_{yy} = h_{yy} \cdot g - 2 h_y \cdot g' \cdot \frac{B}{2C} + h \cdot g'' \cdot \frac{B^2}{4C^2}$$

Using these derivatives in our original P.D.E. we get that

$$A(h_{xx}\cdot g + 2h_x\cdot g' + h\cdot g'')$$

$$+ B(h_{xy}\cdot g + h_y\cdot g' - \frac{B}{2C}\cdot h_x\cdot g' - h\cdot g''\cdot\frac{B}{2C})$$

$$+ C(h_{yy}\cdot g - 2h_y\cdot g'\cdot\frac{B}{2C} + h\cdot g''\cdot\frac{B^2}{4C^2})$$

$$= Ah_{xx}\cdot g + 2Ah_x g' + Ahg'' + Bh_{xy}g + Bh_y\cdot g'$$

$$-\frac{B^2}{2C}h_x\cdot g' - h\cdot g''\cdot\frac{B^2}{2C} + Ch_{yy}\cdot g - Bh_y g'$$

$$+ h\cdot g''\cdot\frac{B^2}{4C}$$

$$= Ah_{xx}g + 2Ah_x g' + Ahg'' + Bh_{xy}g + Bh_y\cdot g'$$

$$-2Ah_x g' - 2Ahg'' + Ch_{yy}\cdot g - Bh_y g'$$

$$+ hg''A$$

$$= Ah_{xx}g - Ahg'' + Bh_{xy}g + Ch_{yy}g - Bh_y g'$$

$$+ Ahg'' + Bh_y\cdot g'$$

$$= Ah_{xx}g + Bh_{xy}g + Ch_{yy}g = 0$$

$$\therefore\ U(x,y) = h(x,y)\,g(x+\lambda y),\quad \lambda = -\frac{B}{2C}$$

When $[Ah_{xx} + Bh_{xy} + Ch_{yy}]\cdot g = 0$

is satisfied, which implies that if either

- $Ah_{xx} + Bh_{xy} + Ch_{yy} = 0$, which is the same P.D.E. we started with.

- $g(x + \lambda y) = 0$, which is a trivial solution.

are met then we have another solution. Where g is an arbitrary and $h(x,y)$ is any solution of the differential equation.

Letting $h(x,y) = x$

$h_x = 1$, $h_y = 0$

$\therefore h_{xx} = 0$, $h_{xy} = 0$, $h_{yy} = 0$

$\therefore Ah_{xx} + Bh_{xy} + Ch_{yy} = 0$

Which implies that $u(x,y) = x \cdot g(x + \lambda y)$ is a solution and the general solution is therefore

$$u(x,y) = f(x + \lambda y) + x \cdot g(x + \lambda y)$$

ii) Letting $h(x,y) = y$

$h_x = 0$, $h_y = 1$

$\therefore h_{xx} = 0$, $h_{xy} = 0$, $h_{yy} = 0$

$\therefore A h_{xx} + B h_{xy} + C h_{yy} = 0$

Which implies that $u(x,y) = y \cdot \tilde{g}(x + \lambda y)$
is a solution and the general solution is
therefore also expressed as

$$u(x,y) = \tilde{f}(x + \lambda y) + y \cdot \tilde{g}(x + \lambda y)$$

Letting these two general solutions equal each other
we get that

$$\tilde{f} + y\tilde{g} = f + xg$$

we know that $\rho = x + \lambda y \longrightarrow x = \rho - \lambda y$

$$\therefore \tilde{f} + y\tilde{g} = f + (\rho - \lambda y) g$$
$$= f + \rho \cdot g + y(-\lambda) \cdot g$$

Comparing both sides we then see that

$$\tilde{f} = f + \rho \cdot g$$
$$\tilde{g} = -\lambda g$$

b) i) $$3u_{xx} - 5u_{xy} + 2u_{yy} = 0$$

Looking at our discriminant we see that the P.D.E. is hyperbolic

$$B^2 - 4AC = (-5)^2 - 4(3)\cdot 2 = 25 - 24 = 1$$

$$\therefore \quad B^2 > 4AC$$

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}$$

$$= \frac{-(-5) \pm 1}{2 \cdot 2} = \frac{5 \pm 1}{4}$$

$$\therefore \quad \lambda_1 = 1.5 \quad , \quad \lambda_2 = 1$$

$$\therefore \quad P_1 = x + 1.5y \quad , \quad P_2 = x + y$$

$$\therefore \quad u(x,y) = f(x + 1.5y) + g(x + y)$$

ii) $$u(x, y = x) = f(x + 1.5x) + g(x + x)$$

$$= f(2.5x) + g(2x)$$

$$= \cos(x)$$

We also know that $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$

Which allows us to enquire that the functions f and g are of the form $\frac{1}{2} e^{ikx}$, where one set of solutions would be

$$f(2.5x) = \frac{1}{2} \exp\left(i \frac{1}{2.5} x\right)$$

$$g(2x) = \frac{1}{2} \exp\left(-i \frac{1}{2} x\right)$$

Where we have replaced K with constants that will ensure $f(2.5x) + g(2x) = \cos(x)$. The second set of solutions being when negative sign is switched.

$$f(2.5x) = \frac{1}{2} \exp\left(-i \frac{1}{2.5} x\right)$$

$$g(2x) = \frac{1}{2} \exp\left(i \frac{1}{2} x\right)$$

h

iii) $u(x, y = -x) = f(x - 1.5x) + g(x - x)$

$$= f(-0.5x) + g(0)$$

$$= f(-0.5x) + K$$

Where K is a constant

2) a) $$\tilde{f}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,t)\, e^{ikx}\, dx$$

$$\left(\frac{\partial}{\partial t} + Dk^2 + iV_d k\right) \tilde{f}(k,t) = 0$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x,t)\, e^{ikx}\, dx \right.$$

$$\left. + k(Dk + iV_d) \int_{-\infty}^{\infty} f(x,t)\, e^{ikx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\, e^{ikx}\, dx \right.$$

$$\left. + k(Dk + iV_d) \int_{-\infty}^{\infty} f(x,t)\, e^{ikx}\, dx \right)$$

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\, e^{ikx}\, dx = \int_{-\infty} \left[D \frac{\partial^2 f}{\partial x^2} + V_d \frac{\partial f}{\partial x} \right] e^{ikx}\, dx$$

$$= D \int_{-\infty}^{\infty} \frac{\partial^2 f}{\partial x^2}\, e^{ikx}\, dx + V_d \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\, e^{ikx}\, dx$$

$$= D \left[\frac{\partial f}{\partial x}\, e^{ikx} - ik \int \frac{\partial f}{\partial x}\, e^{ikx}\, dx \right]_{-\infty}^{\infty}$$

$$+ V_d \int_{-\infty} \frac{\partial f}{\partial x}\, e^{ikx}\, dx$$

$$= D\left[\frac{\partial \rho}{\partial x} e^{ikx}\right]_{-\infty}^{\infty} + (V_d - ikD)\int_{-\infty}^{\infty} \frac{\partial \rho}{\partial x} e^{ikx}\, dx$$

$$= D\left[\frac{\partial \rho}{\partial x} e^{ikx}\right]_{-\infty}^{\infty} + (V_d - ikD)\left[\rho e^{ikx} - ik\int \rho e^{ikx}\, dx\right]_{-\infty}^{\infty}$$

$$D\left[\frac{\partial \rho}{\partial x} e^{ikx}\right]_{-\infty}^{\infty} = D\left[\left.\frac{\partial \rho}{\partial x}\right|_{x=\infty} \cdot e^{ik\infty} - \left.\frac{\partial \rho}{\partial x}\right|_{x=-\infty} e^{-ik\infty}\right]$$

$$= 0$$

$$(V_d - ikD)\left[\rho e^{ikx} - ik\int \rho e^{ikx}\, dx\right]_{-\infty}^{\infty}$$

$$= (V_d - ikD)\left(\left[\rho e^{ikx}\right]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} \rho e^{ikx}\, dx\right)$$

$$= (V_d - ikD)\left(0 - ik\tilde{\rho}\right)$$

$$= (-k^2 D - ikV_d)\tilde{\rho}$$

$$\therefore \frac{1}{\sqrt{2\pi}}\left((-k^2 D - ikV_d)\tilde{\rho} + k\left(Dk + iV_d\right)\tilde{\rho}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\tilde{\rho}\left(-k^2 D + k^2 D - ikV_d + ikV_d\right)$$

$$= 0$$

$\therefore \tilde{\rho}$ Satisfies the differential equation

b) Using our P.D.E.

$$\frac{\partial \tilde{\rho}}{\partial t} + DK^2 \tilde{\rho} + i V_d K \tilde{\rho} = 0$$

We get that

$$\frac{\partial \tilde{\rho}}{\partial t} = -(DK^2 + i V_d K)\tilde{\rho}$$

Which implies that

$$\tilde{\rho}(K,t) = \tilde{\rho}(k,0)\exp(-(DK^2 + i V_d K)t)$$

The function $\tilde{\rho}(k,0)$ is defined from the initial condition at time $t=0$ to be

$$\tilde{\rho}(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(x,0)\exp(+ikx)\,dx$$

Where $\rho(x,0) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-x_0)^2}{2\sigma^2}\right)$

$$\therefore \tilde{\rho}(k,0) = \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)\exp(+ikx)\,dx$$

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)\exp(+ikx)\,dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x^2 - 2x_0 x + x_0^2) + ikx\right) dx$$

$$= \int_{-\infty}^{\infty} \exp\left(\left(\frac{-1}{2\sigma^2}\right)x^2 + \left(\frac{x_0}{\sigma^2} + ik\right)x + \left(-\frac{x_0^2}{2\sigma^2}\right)\right) dx$$

Using the integral

$$\int_{-\infty}^{\infty} dx \exp(-ax^2 + bx + c)$$

$$= \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right)$$

We have that

$$a = \frac{1}{2\sigma^2}$$

$$b = \frac{x_0}{\sigma^2} + ik$$

$$c = \frac{-x_0^2}{2\sigma^2}$$

$$\therefore \quad I = \sqrt{\pi 2 \sigma^2} \exp\left(\frac{\left(\frac{x_0}{\sigma^2} + ik\right)^2}{4\left(\frac{1}{2\sigma^2}\right)} + \left(-\frac{x_0^2}{2\sigma^2}\right)\right)$$

$$= \sigma\sqrt{2\pi} \exp\left(\frac{\sigma^2}{2}\left(\frac{x_0^2}{\sigma^4} + 2ik\frac{x_0}{\sigma^2} + (ik)^2\right) - \frac{x_0^2}{2\sigma^2}\right)$$

$$= \sigma\sqrt{2\pi}\exp\left(\frac{x_0^2}{2\sigma^2} + ikx_0 - \frac{k^2\sigma^2}{2} - \frac{x_0^2}{2\sigma^2}\right)$$

$$= \sigma\sqrt{2\pi}\exp\left(ikx_0 - \frac{k^2\sigma^2}{2}\right)$$

$$\therefore \tilde{f}(k,0) = \frac{1}{2\pi\sigma}\cdot I$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(ikx_0 - \frac{k^2\sigma^2}{2}\right)$$

$$\therefore \tilde{f}(k,t) = \frac{1}{\sqrt{2\pi}}\exp\left(ikx_0 - \frac{k^2\sigma^2}{2}\right)\exp\left(-(Dk^2 + iV_d k)t\right)$$

this therefore implies that

$$A(k) = \exp\left(ikx_0 - \frac{k^2\sigma^2}{2}\right)$$

c) If we define $\tilde{f}(k,t)$ to be the product of two functions, where

$$\tilde{f}(k,t) = \frac{1}{\sqrt{2\pi}} A(k)\, G(k,t)$$

Where $A(k) = \exp\left(ikx_0 - \dfrac{k^2\sigma^2}{2}\right)$

$$G(k,t) = \exp\left(-(Dk^2 + iV_d k)t\right)$$

to find $\rho(x,t)$ we need to evaluate the convolution integral

$$\rho(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-s,t) f(s)\, ds$$

Where s is a dummy variable,

$g(x,t)$ is the inverse fourier transform of $G(k,t)$, and

$f(x)$ is the inverse fourier transform of $A(k)$.

We already know $f(x) = \rho(x,0)$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

Therefore we need to find the inverse fourier transform of $G(k,t)$.

Which can be done by evaluating

$$g(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k,t) \exp(-ikx)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(Dk^2 + iV_d k)t) \exp(-ikx)\, dk$$

let $$I = \int_{-\infty}^{\infty} \exp(-(Dk^2 + iV_d k)t) \exp(-ikx)\, dk$$

$$= \int_{-\infty}^{\infty} \exp(-(Dk^2 + iV_d k)t - ikx)\, dk$$

$$= \int_{-\infty}^{\infty} \exp(-(DK^2 t + i V_d K t + iKx)\, dK$$

$$= \int_{-\infty}^{\infty} \exp(-(DK^2 t + i(V_d t + x)K\, dK$$

$$= \int_{-\infty}^{\infty} \exp(-tD(K^2 + \frac{i(V_d t + x)K}{tD})\, dK$$

$$= \int_{-\infty}^{\infty} \exp\left[-tD\left(K^2 + \frac{i(V_d t + x)K}{tD} + \left(\frac{i(V_d t + x)}{2tD}\right)^2\right) - \frac{(V_d t + x)^2}{4tD}\right] dK$$

$$= \exp\left[-\frac{(V_d t + x)^2}{4tD}\right] \int_{-\infty}^{\infty} \exp\left[-tD\left(K + \frac{i(V_d t + x)}{2tD}\right)^2\right] dK$$

let $\quad h = K + \dfrac{i(V_d t + x)}{2tD}$

$$\therefore \quad I = \exp\left[-\frac{(V_d t + x)^2}{4tD}\right] \cdot \int_{-\infty}^{\infty} \exp\left[-tDh^2\right] dh$$

Using the identity $\displaystyle\int_{-\infty}^{\infty} \exp(-ah^2)\, dh = \sqrt{\frac{\pi}{a}}$

$$\therefore I = \exp\left[-\frac{(V_d t + x)^2}{4tD}\right]\sqrt{\frac{\pi}{tD}}$$

$$\therefore \quad g(x,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(V_d t + x)^2}{4tD}\right]\sqrt{\frac{\pi}{tD}}$$

$$g(x,t) = \frac{1}{\sqrt{2tD}} \exp\left[-\frac{(V_d t + x)^2}{4tD}\right]$$

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-s,t)\, f(s)\, ds$$

$\therefore f(x,t)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2tD}} \exp\left[-\frac{(V_d t + (x-s))^2}{4tD}\right] \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(s-x_0)^2}{2\sigma^2}\right) ds$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sigma\sqrt{tD\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(V_d t + (x-s))^2}{4tD}\right] \exp\left(-\frac{(s-x_0)^2}{2\sigma^2}\right) ds$$

$$\text{let } I = \int_{-\infty}^{\infty} \exp\left[-\frac{(V_d t + (x-s))^2}{4tD}\right] \exp\left(-\frac{(s-x_0)^2}{2\sigma^2}\right) ds$$

$$= \int_{-\infty}^{\infty} \exp\left[-\left(\frac{(V_d t + (x-s))^2}{4tD} + \frac{(s-x_0)^2}{2\sigma^2}\right)\right] ds$$

$$= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{4tD\sigma^2}\left(\sigma^2((V_d t + x) - s)^2 + 2tD(s-x_0)^2\right)\right] ds$$

$$\frac{-1}{4tD\sigma^2}\left(\sigma^2\left((V_d t + x) - s\right)^2 + 2tD\,(s - x_0)^2\right)$$

$$= \frac{-1}{4tD\sigma^2}\Big(\sigma^2 (V_d t + x)^2 - 2\sigma^2 s (V_d t + x) + \sigma^2 s^2$$

$$+ 2tD s^2 - 4tDsx_0 + 2tDx_0^2\Big)$$

$$= \frac{-1}{4tD\sigma^2}\Big((2tD + \sigma^2) s^2 - (2\sigma^2 (V_d t + x) + 4tDx_0) s$$

$$+ \left[2tDx_0^2 + \sigma^2 (V_d t + x)^2\right]\Big)$$

$$= \frac{-(2tD + \sigma^2)}{4tD\sigma^2} s^2 + \frac{(2\sigma^2 (V_d t + x) + 4tDx_0)}{4tD\sigma^2} s$$

$$- \frac{(2tDx_0^2 + \sigma^2 (V_d t + x)^2)}{4tD\sigma^2}$$

Using the integral

$$\int_{-\infty}^{\infty} dx \exp\left(-ax^2 + bx + c\right)$$

$$= \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right)$$

We have that

$$a = \frac{(2tD + \sigma^2)}{4tD\sigma^2}$$

$$b = \frac{(2\sigma^2 (V_d t + x) + 4tDx_0)}{4tD\sigma^2}$$

$$C = -\frac{\left(2tDx_0^2 + \sigma^2 (V_d t + x)^2\right)}{4tD\sigma^2}$$

$$\therefore I = \sqrt{\frac{\pi 4tD\sigma^2}{(2tD + \sigma^2)}} \cdot \exp\left(\frac{\left(2\sigma^2 (V_d t + x) + 4tDx_0\right)^2}{4(2tD + \sigma^2)} - \frac{\left(2tDx_0^2 + \sigma^2 (V_d t + x)^2\right)}{4tD\sigma^2}\right)$$

$$\therefore p(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2\sigma\sqrt{tD\pi}} \sqrt{\frac{\pi 4tD\sigma^2}{(2tD + \sigma^2)}} \cdot \exp(\cdots)$$

$$\frac{1}{\sqrt{2\pi}} \frac{1}{2\sigma\sqrt{tD\pi}} \sqrt{\frac{\pi 4tD\sigma^2}{(2tD + \sigma^2)}} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi 4tD\sigma^2}{4\sigma^2 tD\pi (2tD + \sigma^2)}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(2tD + \sigma^2)}}$$

As we know $p(x,t) = \frac{1}{r(t)} \exp\left(\frac{-x^2(t)}{2r^2(t)}\right)$

$\therefore$ from inspection we can say that

$$r(t) = \sqrt{(2tD + \sigma^2)}$$

Now looking at the exponential

$$\exp\left(\frac{\left(2\sigma^2 (V_d t + x) + 4tDx_0\right)^2}{4(2tD + \sigma^2)} - \frac{\left(2tDx_0^2 + \sigma^2 (V_d t + x)^2\right)}{4tD\sigma^2}\right)$$

we will take a factor of $(2r^2(t))^{-1}$ out

$$\therefore \Rightarrow \exp\left[\frac{1}{2(2tD+\sigma^2)}\left(\frac{(2\sigma^2(V_d t+x)+4tDx_0)^2}{2} - \frac{(2tDx_0^2+\sigma^2(V_d t+x)^2)\cdot(2tD+\sigma^2)}{2tD\sigma^2}\right)\right]$$

$$\therefore -x^2(t) = \frac{(2\sigma^2(V_d t+x)+4tDx_0)^2}{2} - \frac{(2tDx_0^2+\sigma^2(V_d t+x)^2)\cdot(2tD+\sigma^2)}{2tD\sigma^2})$$

$$= 2(\sigma^2(V_d t+x)+2tDx_0)^2 - (2tDx_0^2+\sigma^2(V_d t+x)^2)\cdot\left(\frac{1}{\sigma^2}+\frac{1}{2tD}\right)$$

$$= 2(\sigma^4(V_d t+x)^2+4tDx_0\sigma^2(V_d t+x)+4t^2D^2x_0^2) - \frac{2tDx_0^2}{\sigma^2} - x_0^2 - (V_d t+x)^2 - \frac{\sigma^2(V_d t+x)^2}{2tD}$$

$$= 2\sigma^4(V_d t+x)^2 + 8tDx_0\sigma^2(V_d t+x) + 8t^2D^2x_0^2 - (V_d t+x)^2 - \frac{2tDx_0^2}{\sigma^2} - \frac{\sigma^2(V_d t+x)^2}{2tD} - x_0^2$$

$$= \left(2\sigma^4 - \frac{\sigma^2}{2tD} - 1\right)(V_d t+x)^2 + 8tDx_0\sigma^2(V_d t+x) - \left(-8t^2D^2 + \frac{2tD}{\sigma^2} + 1\right)x_0^2$$

=

3) a) i)

This equation also satisfies the law of conservation energy.

A steady state implies $\longrightarrow \frac{\partial \vec{V}}{\partial t} = 0$

$\mu = 0$

$$\therefore \rho \frac{\partial \bar{V}}{\partial t} + \rho(\bar{V} \cdot \nabla \bar{V}) = -\nabla P + \mu \nabla^2 \bar{V}$$

$$\longrightarrow \rho(\bar{V} \cdot \nabla \bar{V}) = -\nabla P$$

$$\bar{V} \times (\nabla \times \bar{V}) = \frac{\nabla(\bar{V} \cdot \bar{V})}{2} - \bar{V} \cdot \nabla \bar{V}$$

$$\bar{V} \cdot \nabla \bar{V} = \frac{\nabla(\bar{V} \cdot \bar{V})}{2} - \bar{V} \times (\nabla \times \bar{V})$$

$$\therefore \rho\left(\frac{\nabla(\bar{V} \cdot \bar{V})}{2} - \bar{V} \times (\nabla \times \bar{V})\right)$$

$$= -\nabla P$$

$$\therefore \bar{V} \cdot \rho\left(\frac{\nabla(\bar{V} \cdot \bar{V})}{2} - \bar{V} \times (\nabla \times \bar{V})\right)$$

$$= -\bar{V} \cdot \nabla P$$

$$\longrightarrow \rho \cdot \left(\frac{\bar{V} \cdot \nabla(\bar{V} \cdot \bar{V})}{2} - \bar{V} \cdot \bar{V} \times (\nabla \times \bar{V})\right)$$

$$= -\bar{V} \cdot \nabla P$$

$$\bar{V} \cdot \bar{V} \times (\nabla \times \bar{V}) = 0$$

$$\therefore \quad \rho \left(\frac{\bar{V} \cdot \nabla (V^2)}{2} \right) = -\bar{V} \cdot \nabla p$$

$$\bar{V} \cdot \nabla \left(\frac{1}{2} \rho V^2 + p \right) = 0$$

ii)

$$\rho \frac{\partial \bar{V}}{\partial t} + \rho (\bar{V} \cdot \nabla \bar{V}) = -\nabla p + \mu \nabla^2 \bar{V}$$

$$\nabla \times \left[\rho \frac{\partial \bar{V}}{\partial t} + \rho (\bar{V} \cdot \nabla \bar{V}) \right] =$$

$$\nabla \times \left[-\nabla p + \mu \nabla^2 \bar{V} \right]$$

$$\rho \left[\frac{\partial (\nabla \times \bar{V})}{\partial t} + \frac{\nabla (\bar{V} \cdot V)}{2} - \bar{V} \times (\nabla \times \bar{V}) \right.$$

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Query/Assumption Made

In this section note any assumptions, if any, that were made to enable you to answer the question

*please note that assumptions will only be taken into account subsequent to an investigation into whether or not there was an error on the question paper. Assumptions made to re-write or re-phrase the question to a more favourable question will NOT be taken into account.

A copy of this form should be submitted with your examination script and uploaded along with your answer page(s):

Once your online submission has been completed via Turnitin this form will be used by the department to be forwarded to the Chair of the Examination Board for consideration.

You do not need to take any further action.

UCL Examinations Office

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