# Fluid Mechanics

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Why is fluid mechanics important?
A large number of physical systems can be described as a fluid flow, from atmospheric models to astrophysical ones.
The mechanics of a fluid is based on macroscopic properties such as viscosity and compressibility. These properties can be described at the microscopic level.
There are many mathematical issues when studying fluids, such as flow instability when a particular aspect of the system is changed.
We will not look too deeply into this chaos, and will instead look to establish the basic differential equations and associated conservation principles that are used to describe fluid mechanics at a macroscopic scale.

How do we define a fluid? If a material can generate a restoring force that acts against being twisted then it is not a fluid. Our main objective in fluid mechanics is to determine the vector field $\boldsymbol{v}(\boldsymbol{r}, t)$ describing its motion - we describe the velocity of any given small packet of fluid at a position $\boldsymbol{r}$ at a time $t$.

We are interested in the behaviour of a fluid contained within a region V. The boundary of this region might be open to fluid input or loss, or it may be impenetrable. We can also have moving boundaries - such as a paddle - that cause motion in the fluid. There are also forces, such as gravity of pressure gradient, that will effect the equations, as well as more exotic forces

Fluids can carry other quantities as well as suspended particles, such as the internal energy of the fluid - characterised by temperature. However, we will not look at deviations from a constant temperature. Other quantities carried by a fluid can include dissolved species such as salt, or more complicated thermodynamic quantities such as entropy
Also consider that with dissolved salt there will be a diffusion taking place, allowing the concentration of salt dissolved to reach a point of equilibrium regardless of fluid flow.
A less obvious quantity carried by fluid is momentum. Packets of fluid can acquire momentum by being in the proximity of other packets in motion, with the momentum transfer caused by a diffusive process. This is related to the viscosity.

We shall find that certain non-linear PDEs describe the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ and the associated scalar pressure field $p(\boldsymbol{r}, t)$. The solutions to these equations require boundary conditions provded by the specific nature of the problem.

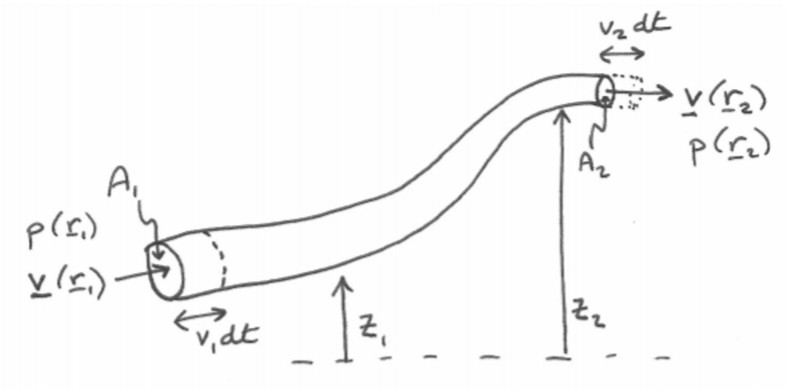
Bernouilli's Equation

This equation connects time-independent velocity and pressure fields along fluid streamlines.

A streamline is simply a trajectory of infinitesimal fluid packets carried along by the flow.

Bernouilli's principle is based on conservation of energy, but is only valid if diffusive transport of momentum and energy are neglected.
This means it is only valid for *inviscid* fluids - defined as fluids with zero viscosity - and for fluids where thermal conductivity is negligible. The fluid must also be incompressible, with a constant density $\rho$ indepdendent of local pressure, temperature, etc.

This is okay though, as most liquids are fairly incompressible, and gasses are mostly incompressible so long as the typical fluid flow are small compared to the speed of sound.




Let us derived the equation:

Imagine a flow through a tube as above. Our key assumption is that fluid, momentum, and energy are not exchanged across the sides of the tube, but only through the end. Let us look at a time independent velocity field $\boldsymbol{v}(\boldsymbol{r})$**.** In a time, $dt$, a mass $m = \rho\boldsymbol{v}(\boldsymbol{r}_1) \cdot d\boldsymbol{S}_1$ of fluid enters the tube through the area $A_1$. Here, $d\boldsymbol{S}_1$ is a vector of magnitude $A_1$ orientated parallel to the velocity.

The same mass of fluid exits through the outlet from the tube, such that:

$$\rho\boldsymbol{v}(\boldsymbol{r}_1) \cdot d\boldsymbol{S}_1 = \rho\boldsymbol{v}(\boldsymbol{r}_2) \cdot d\boldsymbol{S}_2$$

Let us now increment the time by $dt$. In this time, a packet of fluid has moved $v_2 dt$ out of the end of the tube, and a packet has moved $v_1 dt$ into the inlet. Any velocity $v_k$ is shorthand for:

$$v_k = |\boldsymbol{v}(\boldsymbol{r}_k)|$$

We treat this as the creation of a packet at the exit and the elimination of a packet at the inlet.

We can look at the kinetic energy.

Each packet has a kinetic energy density (kinetic energy per unit volume):

$$\frac{\rho v_k^2}{2}$$

And so the kinetic energy of an infinitesimal volume is:

$$dK_k = \frac{1}{2}\rho v_k^2 \cdot A_k v_k dt$$

Therefore, as we define our packet to be created at the exit, and to be elimiated at the entrance, our change of kinetic energy in the tube is:

$$dK = \frac{1}{2}\rho v_2^2 A_2 v_2 dt - \frac{1}{2}\rho v_1^2 A_1 v_1 dt$$

Next we can look at the potential energy:

Each packet has a gravitational potential energy density of:

$$\rho g z_k$$

And so our change in potential is:

$$dV = \rho g z_2 \cdot A_2 v_2 dt - \rho g\, z_1 \cdot A_1 v_1 dt$$

From these two energies we can see the total energy change in the tube is:

$$dE = dK + dV$$

We can imagine this as a piston pushing into the inlet of the tube, causing a piston to be pushed out the end.

The work done by the surrounding fluid at the inlet is the force times the displacement:

$$p(\boldsymbol{r}_1) \cdot A_1 v_1 dt$$

And so our net work done is:

$$dW = p(\boldsymbol{r}_1) \cdot A_1 v_1 dt - p(\boldsymbol{r}_2) \cdot A_2 v_2 dt$$

If we neglect energy loss through the tube walls, as well as any change in temperature in the fluid, conservation of energy dictates:

$$dW = dK + dV$$

$\therefore$

$$p(\boldsymbol{r}_1) \cdot A_1 v_1 dt - p(\boldsymbol{r}_2) \cdot A_2 v_2 dt$$
$$= \frac{1}{2}\rho v_2^2 A_2 v_2 dt - \frac{1}{2}\rho v_1^2 A_1 v_1 dt + \rho g z_2 \cdot A_2 v_2 dt - \rho g\, z_1 \cdot A_1 v_1 dt$$

$\rightarrow$

$$\frac{1}{2}\rho v_2^2 A_2 v_2 + \rho g z_2 \cdot A_2 v_2 + p(\boldsymbol{r_2}) \cdot A_2 v_2$$
$$= \frac{1}{2}\rho v_1^2 A_1 v_1 + \rho g\, z_1 \cdot A_1 v_1 + p(\boldsymbol{r_1}) \cdot A_1 v_1$$

We next use the conservation of mass, as the same mass must enter as exits out tube, and so:

$$v_1 A_1 = v_2 A_2$$

$\rightarrow$

$$\frac{1}{2}\rho v_2^2 A_2 v_2 + \rho g z_2 \cdot A_2 v_2 + p_2 \cdot A_2 v_2 = \frac{1}{2}\rho v_1^2 A_2 + \rho A_2 v_2 g z_1 + p_1 A_2 v_2$$

*Where* $p_k = p(\boldsymbol{r}_k)$

$\rightarrow$

$$\frac{1}{2}\rho v_2^2 + \rho g z_2 + p_2 = \frac{1}{2}\rho v_1^2 + \rho g z_1 + p_1$$

*Or:*

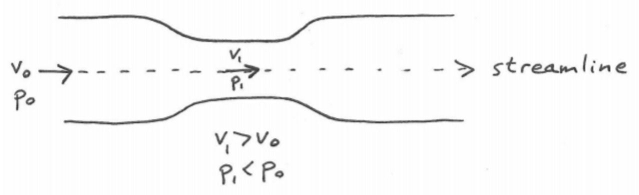
$$\frac{1}{2}\rho v^2 + \rho g z + p = constant$$

This is Bernouilli's equation.
It can be generalised to:

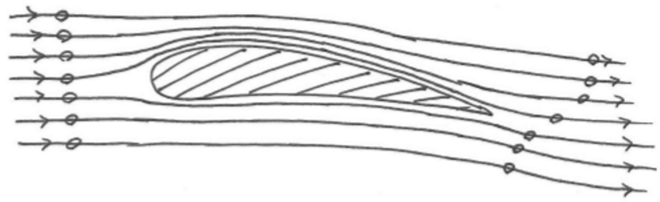
$$\frac{1}{2}\rho v^2 + \rho \phi + p = constant$$

Where $\phi$ is the scalar potential energy field per unit mass.

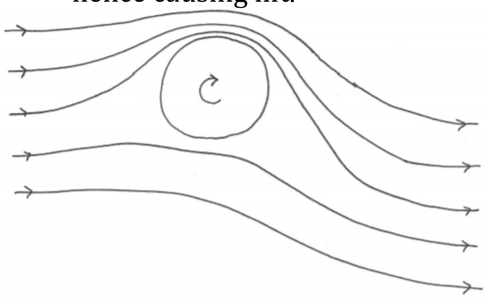
Let us look at some examples to illustrate this principle:




In this example, we have a fluid flow that constricts to a smaller diameter and then grows again
As a result, the velocity increases here, and so looking at the Bernouilli equation we see that the pressure must drop.



Next we can look at an aerofoil. The shape of the aerofoil forces the air to increase in velocity above, which causes a decrease in pressure.
This decrease in pressure results in a net upwards force on the aerofoil, and hence causing lift.



A third example is the Magnus effect.
Imagine a spinning cylinder. Its boundaries drag on the fluid, causing it to increase in velocity in the direction of spin, and decrease in velocity at the opposing end.
This, similarly to the aerofoil, results in a pressure difference, and so a net force.

Channel flow:

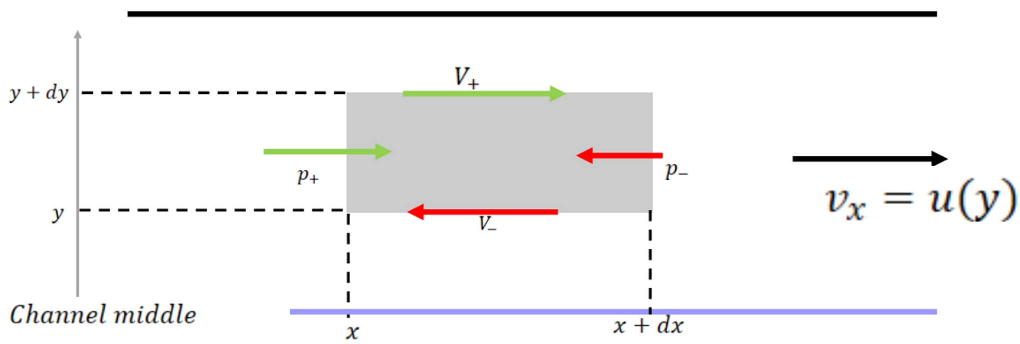
Bernouilli's equation is based on conservation of energy, and is rather elementary.
Let us now establish a differential equation to describe channel flow.
We consider the motion of an incompressible fluid, passing down a parallel sided channel with a rectangular cross-section, driven by a pressure difference between the inlet and outlet.
The velocity, $u$, is independent of the distance along the channel we are.
The velocity is also independent of the $z$ coordinate
Let us imagine that the velocity is not independent of $y$, the distance above the middle of the channel. This means that at a given boundary we will have a difference in flow velocity above or below:

$$\frac{du}{dy}$$

We now introduce viscosity, the constant that relates the sheer stress at a boundary:

$$\tau = \mu \frac{du}{dy}$$

This is called Newton's law of viscosity, and it is true for many fluids.




Let us for our example imagine that the velocity increases as y increases:

$$\frac{du}{dy} > 0$$

In this example we have 4 forces acting on our fluid segment.

$V_+$ is a force acting in the direction of flow, coming from the viscocity (we have assumed that the velocity increases as a function of $y$). This force therefore is:

$$V_+ = \mu\, u'(y + dy) \cdot dx\, dz$$

$V_-$ is a force acting against the flow, coming from the viscocity:

$$V_- = \mu\, u'(y) \cdot dx\, dz$$

$p_+$ is a force acting in direction of flow, coming from the pressure behind the segment.

$$p_+ = p(x) dy\, dz$$

$p_-$ is a force acting against the flow, coming from the pressure ahead of it:

$$p_- = p(x + dx) dy\, dz$$

As we know our velocity is independent of the x coordinate, the net forces acting on our segment must be 0:

$$\mu\, u'(y + dy) dx dz + p(x) dy\, dz = \mu\, u'(y) dx\, dz\ + p(x + dx) dy\, dz$$

$\rightarrow$

$$\frac{p(x + dx) - p(x)}{dx} = \mu \frac{u'(y + dy) - u'(y)}{dy}$$

$\rightarrow$

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}$$

Next, we assume that the pressure gradient along the channel is negative - i.e. The pressure decreases the further along the channel we are.
As the flow pattern is independent of x, then $dp/dx$ must be a constant:

$$\frac{dp}{dx} = -\alpha$$

We can then find our solution for $u(y)$

$$u(y) = u(0) + Ay - \left(\frac{\alpha}{2\mu}\right) y^2$$

It is logical that our velocity profile to be symmetric about the channel centre, $y = 0$, and so $A = 0$

$$u(y) = u(0) - \left(\frac{\alpha}{2\mu}\right) y^2$$

Let the channel have a height of $d$. We now impose a boundary condition, known as a no-slip boundary condition, that for $y = \pm d/2$ the velocity is 0:

$$0 = u(0) - \frac{\alpha}{2\mu}\frac{d^2}{4}$$

$$\rightarrow u(0) = \frac{\alpha d^2}{8\mu}$$

And so:

$$u(y) = \frac{\alpha d^2}{8\mu}\left(1 - \frac{4y^2}{d^2}\right)$$

The volumetric flow rate down the channel can then be found be integrating this over the cross sectional area:

$$Q = \int dz \int dy\ \frac{\alpha d^2}{8\mu}\left(1 - \frac{4y^2}{d^2}\right)$$

For a channel with a length $D$ in the z direction, this evaluates to:

$$Q = D\frac{\alpha d^3}{12\mu}$$

We can instead consider a pipe with circular cross section of radius $a$. This is called Poiseuille flow, and its volumetric flow rate evaluates to

$$Q = \frac{\pi \alpha a^4}{8\mu}$$

Basic equations of fluid motion

The next step is to derive differential equations to describe more general fluid flows.

We will need 4 partial differential equations, along with appropriate boundary conditions, to determine $p(\boldsymbol{r}, t)$& $\boldsymbol{v}(\boldsymbol{r}, t)$.

The first of these is the continuity equation

This arises from the conservation of mass.

Let us consider a compressible fluid:

$$\rho(\boldsymbol{r}, t) \neq constant$$

Imagine a small cuboidal packet of fluid at position $\boldsymbol{r}$, with volume $dV = dxdydz$

If we imagine a flow only in the x direction, we can see that a mass per unit time of:

$$\rho(x, y, z, t)v_x(x, y, z, t)dydz$$

Enters the packet, and a mass per unit time of:

$$\rho(x + dx, y, z, t)v_x(x + dx, y, z, t)dydz$$

Leaves. The difference between these two values is then the rate of change of mass inside the volume in the x direction:

$$[\rho(x, y, z, t)v_x(x, y, z, t) - \rho(x + dx, y, z, t)v_x(x + dx, y, z, t)]dydz = -\left.\frac{\partial \rho v_x}{\partial x}\right|_{y,z} dxdydz$$

We can generalise this to allow for all 3 dimensions by summing the above equation but for $y$ & $z$.

This rate of change of density in all 3 directions is then the rate of change of density with respect to time, and so we can write our general equation as:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v})$$

If our fluid behaved like an ideal gas, we would substitute $p = \rho kT/m$ to get our equation in terms of pressure, velocity, and a temperature field.

If our fluid is incompressible, then its density is constant, and so our equation becomes:

$$\nabla \cdot \boldsymbol{v} = 0$$

*This is sometimes called a condition for solenoidal, or divergence-free flow*

Our next step is to apply Newton's second law to determine the other 3 PDEs describing the flow.

We wish to look at the rate of change of momentum in the fluid.

Let us introduce $\mathbf{v}(x, y, z, t)$ - The momentum per unit mass (I guess just velocity ?).

If we imagine a volume element with fluid entering from the negative $x$ direction, we can see a total momentum of:

$$\rho(x, y, z, t)v_x(x, y, z, t)\mathbf{v}(x, y, z, t)$$

Enters the volume element. This momentum can then be carried into all three dimensions. The inwards and outward flows therefore contribute an overall momentum:

$$-\partial_i(\rho v_i \mathbf{v})dxdydz$$

With $i$ representing the 3 dimensions to sum over. The rate of change of momentum of the volume element can be written as:

$$\frac{\partial(\rho \mathbf{v})}{\partial t}\, dV$$

And so, whilst also throwing some random fucking term for the gradient of pressure???, we get:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} = -\partial_i(\rho v_i \mathbf{v}) - \nabla p$$

For an inviscid fluid, we can write:

$$\partial_i(\rho v_i \mathbf{v}) = \rho v_i \partial_i \mathbf{v} + \mathbf{v}\partial_i(\rho v_i)$$

We can also notice that we can re-write the term:

$$\partial_i(\rho v_i \mathbf{v}) = \rho v_i \partial_i \mathbf{v} \ - \mathbf{v}\frac{\partial p}{\partial t}$$

$\rightarrow$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} = -\rho v_i \partial_i \mathbf{v} + \mathbf{v}\frac{\partial p}{\partial t} - \nabla p$$

*Take the continuity equation:*

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v}) = -\partial_i(\rho v_i)$$

$$\therefore \rho v_i \partial_i \mathbf{v} = \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

$\rightarrow$

$$\rho\frac{\partial(\mathbf{v})}{\partial t} = \mathbf{v}\frac{\partial p}{\partial t} - \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p$$

What's the next step? Lets just completely ignore and throw out a term and rearrange:

$$\mathbf{v}\frac{\partial p}{\partial t} = 0\,???$$

$$\rightarrow \rho\frac{\partial(\mathbf{v})}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p$$

*This is the Euler Equation*

It is non-linear, which means its hard to solve :(

If we look at the LHS of this equation, we can divide through by $\rho$ and write it instead as:

$$\frac{\partial(\mathbf{v})}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{D\mathbf{v}}{Dt}$$

This value is known as the material derivative, or the substantive derivative.

The Bernouilli and Euler equations:

We've already looked at Bernouilli's equation, which is valid for steady incompressible inviscid flows. We then also derived Euler's equation (but really badly...)

Let us relate these two. We start with the following vector calculus identity:

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{\mathbf{1}}{\mathbf{2}}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{v}$$

Let us take Eulers equation:

$$\rho\frac{\partial(\mathbf{v})}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p$$

If we imagine a stead incompressible flow, then $\frac{\partial \mathbf{v}}{\partial t} = 0$ & $\rho = constant$, and so we can re-write it as:

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p$$

However, as we have proven:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \frac{\mathbf{1}}{\mathbf{2}} \nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

And so:

$$\rho \left( \frac{\mathbf{1}}{\mathbf{2}} \nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v}) \right) = -\nabla p$$

If we take the dot product of both sides with $\mathbf{v}$, we can rearrange to get:

$$\mathbf{v} \cdot \nabla \left( \frac{1}{2} \rho v^2 + p \right) = 0$$

What does this show? It implies that the gradient of $\frac{1}{2} \rho v^2 + p$ is perpendicular to the flow direction. I.e. This scalar quantity does not change its value along a streamline. This is a generalisation of Bernouilli's result. If we did not have the dot product:

$$\nabla \left( \frac{1}{2} \rho v^2 + p \right) = 0$$

Then this states that Bernouilli's equation holds at every point in the region, not just along the streamlines.

The condition $\nabla \times \mathbf{v} = 0$ is referred to as curl-free flow.

## Vorticity and Irrotational flow

Something cool about fluids is the way vortices are formed and how they evolve.

A vortex line is the axis of a rotating region of fluid. One of the most useful mathematical tools to describe vortex behaviour is called the vorticity, and is defined as the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

For a constant unidirectional flow its easy to see that $\boldsymbol{\omega} = 0$.

Next, imagine a sheer flow $\mathbf{v} = (u(y), 0,0)$.

We can represent this as:

$$\omega_i = \epsilon_{ijk} \partial_j v_k$$

For our given flow, the only non-zero term is:

$$\omega_3 = \epsilon_{321} \, \partial_2 v_1$$
$$\rightarrow \omega_z = -\partial_y v_x$$
$$= -\partial_y \, u(y)$$

What does this mean? Shear flows possess a velocity perpendicular to the shear gradient and direction of flow.

Let us consider a flow consisting of packets of fluid moving in a circular path about the origin.

We can write the velocity field as:

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$$

*Where $\boldsymbol{\Omega}$ is the angular velocity of rotation.*

We can then write the vorticity as:

$$\boldsymbol{\omega} = \nabla \times (\boldsymbol{\Omega} \times \mathbf{r})$$

$\rightarrow$

$$\omega_i = \epsilon_{ijk} \partial_j \, \epsilon_{klm} \Omega_l r_m$$
$$= \epsilon_{kij} \epsilon_{klm} \Omega_l \partial_j r_m$$
$$= (\delta_{il} \partial_{jm} - \partial_{im} \partial_{jl}) \Omega_l \partial_j r_m$$
$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \Omega_l \delta_{jm} = 3\Omega_i - \Omega_i = 2\Omega_i$$

So the vorticity at all points in a rigid body rotational flow is equal to twice the angular velocity.

Next, consider $\mathbf{v} = (0, v_\theta(r), 0)$ in cylindrical polar coordinates.

The curl of this velocity is the vorticity:

$$\nabla \times (0, v_\theta(r), 0) = \left(0, 0, \frac{v_\theta(r)}{r} + \partial_r v_\theta\right)$$

What does this mean? A flow with velocity only in the $\theta$ component results in a vorticity with components only in the z plane, and none in the radial one.
This is called a free vortex. Packets of fluid move such that they do not spin, with each facing the same way throughout their motion.
This can be demonstrated with Stokes' theorem:

$$\iint_S \nabla \times \mathbf{v}\, d\boldsymbol{S} = \oint_l \mathbf{v}\, dl$$

$$\iint \boldsymbol{\omega} \cdot d\boldsymbol{S} = \oint_0^{2\pi} v_\theta\, r\, d\theta = 2\pi K$$

And so, $\boldsymbol{\omega} = 2\pi K \delta(\boldsymbol{r}) \hat{\boldsymbol{e}}_{\boldsymbol{z}}$
A free vortex is a rough model of spinning fluid structures

The above indicates that vorticity is associated with circular motion, local spinning behaviour, and with shear flows.
Let us now establish an equation of motion for vorticity for an inviscid fluid.
We take the curl of Euler's equation:

$$\rho \frac{\partial(\mathbf{v})}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p \rightarrow$$

$$\rho \frac{\partial \boldsymbol{\omega}}{\partial t} + \rho \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla \times \nabla p$$

The curl of a gradient is 0, which we can prove:

$$\begin{aligned}
(\nabla \times \nabla p)_i &= \epsilon_{ijk} \partial_j \partial_k p \\
&= \epsilon_{ijk} \partial_k \partial_j p \\
&= \epsilon_{ikj} \partial_j \partial_k p \\
&= -\epsilon_{ijk} \partial_j \partial_k p \\
\rightarrow (\nabla \times \nabla p)_i &= -(\nabla \times \nabla p)_i = 0
\end{aligned}$$

*In line 2, we swap the partial derivatives $\partial_k$ and $\partial_j$ around.*
*In line 3, we swap the dummy variables themselves*

$$\rightarrow \rho \frac{\partial \boldsymbol{\omega}}{\partial t} + \rho \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = 0$$

Let us next consider the second term here:

$$\rho \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v})$$

*We can use the following identity, and then take its cross product*

$$\mathbf{v} \cdot \nabla \mathbf{v} = \frac{\mathbf{1}}{\mathbf{2}} \nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

$\rightarrow$

$$\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \times \left(\frac{1}{2} \nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})\right)$$

*Again we use the fact that the curl of a gradient is 0 to remove the first term of the RHS*

$$\begin{aligned}
\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) &= -\nabla \times \mathbf{v} \times (\nabla \times \mathbf{v}) \\
&\quad -\nabla \times (\mathbf{v} \times \boldsymbol{\omega})
\end{aligned}$$

*We can next evaluate our RHS term here:*

$$\begin{aligned}
-\big(\nabla \times (\mathbf{v} \times \boldsymbol{\omega})\big)_i &= -\epsilon_{ijk} \partial_j \epsilon_{kml} v_l \omega_m \\
&= \epsilon_{kij} \epsilon_{kml} \partial_j v_l \omega_m \\
&= \big(\delta_{im} \delta_{jl} - \delta_{il} \delta_{jm}\big) \partial_j v_l \omega_m \\
&= \partial_j v_j \omega_i - \partial_j v_i \omega_j \\
&\quad \textit{Product rule} \\
&= v_j \partial_j \omega_i + \omega_i \partial_j v_j - v_i\, \partial_j \omega_j - \omega_j \partial_j v_i \\
&\qquad \omega_j \partial_j v_i = 0 \\
&= \mathbf{v} \cdot \nabla \omega_i + (\nabla \cdot \mathbf{v}) \omega_i - \boldsymbol{\omega} \cdot \nabla v_i
\end{aligned}$$

With all of this we can produce the equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} + (\nabla \cdot \mathbf{v})\boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{v} = 0$$

$\rightarrow$

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{v} - (\nabla \cdot \mathbf{v})$$

What does this show? A flow with zero vorticity remains curl-free (irrotational) at later times so long as there is no viscosity.

Packets with zero vorticity move with no change in vorticity

If the initial $\boldsymbol{\omega}$ of a packet is 0, it will remain 0

This result has some consequences.

Consider the flow of an incompressible, inviscid fluid, through some complicated geometry, but with the boundary condition that the velocity field is uniform, and so has no-rotation.

If we ignore viscosity, then it will remain irrotational as it passes through the geometry and becomes non-uniform.

The development of vorticity is only due to viscous effects, which are relevant at low flow speeds near boundaries.

It is useful to look for solutions to Euler's equation based on a velocity potential:

$\mathbf{v} = -\nabla\Phi(\mathbf{r}, \mathrm{t})$

This representation ensures the flow is irrotational, as its curl is 0.

If we insert this form into the continuity equation (assuming incompressibility):

$\nabla \cdot \boldsymbol{v} = 0$

$\rightarrow \nabla^2\Phi = 0$

Oh look, we need to solve the Laplace equation. Fucking bastard

Let us consider the flow of an inviscid fluid around a sphere of radius $a$. Far from the sphere, the flow field is $\mathbf{U}$ , directed along the z axis of spherical polar cordinates centred on the stationary sphere. We will assume that the velocity potential has no dependence of $\phi$, and so we must solve:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) = 0$$

We can solve this through separation of variables. We are given boundary conditions:

$$v_r(r = a) = -\frac{\partial\Phi}{\partial \mathrm{r}} = 0$$

$$\lim_{r\to\infty} \nabla\Phi = -\mathbf{U}$$

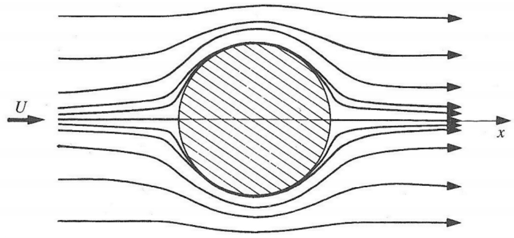
We end up with solution:

$$\Phi = -Ur\left(1 + \frac{a^3}{2r^3}\right)\cos(\theta)$$

$$v_r = U\left(1 - \frac{a^3}{r^3}\right)\cos(\theta)$$

$$v_\theta = -U\left(1 + \frac{a^3}{2r^3}\right)\sin(\theta)$$

This result looks very similar to electrostatic potential field around a dipole




Viscous Flow

Navier-Stokes equation

We must now introduce viscous effects properly.

Consider the viscous stress tensor, $\tau_{ij}$, defined to be the viscous force per unit area acting in the direction $i$ on an interface in the fluid with a normal vector pointing in the direction $j$. For a Newtonian fluid, we have the relationship:

$$\tau_{ij} = \Lambda_{ijkl}\dot{\epsilon}_{kl} = \Lambda_{ijkl}\frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right)$$

It is useful to use $\dot{\epsilon}_{kl}$, known as the rate of strain tensor, as it symmetric under exchange of suffices.

The tensor $\Lambda_{ijkl}$ is a set of 81 constants relating viscous stresses to rates of strain.

For our case, we will assume $\Lambda_{ijkl}$ is an isotropic tensor.

For a rank for isotropic tensor, we can write it in the form of products of isotropic rank two tensors:

$$\Lambda_{ijkl} = A\ \delta_{ij}\delta_{kl} + B\ \delta_{ik}\delta_{jl} + C\ \delta_{il}\delta_{jk}$$

Where A, B, and C are scalar fluid properties. With this form, we can write:

$$\tau_{ij} = A\ \delta_{ij}\ \nabla\cdot\mathbf{v} + (B + C)\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$= \lambda\delta_{ij}\nabla\cdot\mathbf{v} + \mu(\partial_j v_i + \partial_i v_j)$$

*Where $\mu$ & $\lambda$ are called the shear and bulk viscosities.*

The force in the $i$th direction on the fluid packet is proportional to a gradient in the stress.

For viscous forces, this is given by:

$$\partial_j\tau_{ij}\ dV$$

If we look at Euler's equation:

$$\rho\frac{\partial(\mathbf{v})}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p$$

We can see that we equate the LHS to the gradient of the pressure. Our viscous force tensor is a measure of force per unit area, just like pressure, and so it goes on the RHS (lol)

$$\rho\frac{\partial v_i}{\partial_t} + \rho\mathbf{v}\cdot\nabla v_i = -\partial_i p + \partial_j\tau_{ij}$$

$$= -\partial_i p + \lambda\partial_i(\nabla\cdot\mathbf{v}) + \mu\partial_j\partial_j v_i + \mu\partial_i(\nabla\cdot\mathbf{v})$$

If we assume that $\mu$ & $\lambda$ are constants, we can write this as:

$$\rho\frac{\partial(\mathbf{v})}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\nabla^2\mathbf{v}$$

For an incompressible fluid, $\nabla\cdot\mathbf{v} = 0$, and so we get:

$$\rho\frac{\partial(\mathbf{v})}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{F}$$

Here we have added a further force per unit volume, $\mathbf{F}$, normally representing gravity $\mathbf{F} = -\rho g\ \hat{\boldsymbol{e}}_{\mathbf{z}}$

This is the Navier-Stokes equation. Solve it and get a \$$1\times 10^6$

Reynolds Number:

We have looked at both inviscid fluids, and ones with viscous effects.

Its important to know when viscous forces will be significant.

We can do this by comparing expected contributions to the Navier-Stokes equation.

The best way to do this is to contrast the viscous force term $\mu\nabla^2\mathbf{v}$ with the term $\rho\ \mathbf{v}\cdot\nabla\mathbf{v}$.

Using this, we should be able to neglect the viscous term if the fluid speed is large or the viscosity is low.

To do this, we cast solutions to the Navier-Stokes equation into dimensionless quantities:

$$\boldsymbol{r} = L\boldsymbol{r}' \rightarrow \nabla' = L\nabla$$

$$\mathbf{v} = u\mathbf{v}'$$

$\rightarrow$

$$\mu\nabla^2\mathbf{v} \rightarrow \mu\cdot\frac{\nabla'^2}{L^2}\cdot u\mathbf{v}' = \frac{\mu u}{L^2}\ \nabla'^2\mathbf{v}'$$

$$\rho\ \mathbf{v}\cdot\nabla\mathbf{v} \rightarrow \rho\ u\mathbf{v}'\cdot\frac{\nabla'}{L}u\mathbf{v}' = \frac{\rho u^2}{L}\ \mathbf{v}'\cdot\nabla\mathbf{v}'$$

We can ignore the viscous terms if our dimensionless viscous term coefficient $(\frac{\mu u}{L^2})$ is much smaller than the other term:

$$\frac{\mu u}{L^2} \ll \frac{\rho u^2}{L}$$
$$\rightarrow \frac{\mu}{Lu\rho} = \frac{1}{Re} \ll 1$$

Here we define Reynolds number, $Re = \frac{Lu\rho}{\mu}$. For $Re \gg 1$ flows are inviscid

Stokes flow around a sphere

Consider the flow of a fluid around a sphere of radius $a$ moving through an incompressible fluid at a (low) speed $U$.

For simplicity, we will take our coordinate system to have its origin at the spheres centre. In this frame, the fluid will have a constant speed $-U$ at infinity.

We will use spherical polar coordinates, with the direction of the fluid motion defined by $\theta = 0$.

We seek a solution to the Navier-Stokes equation in the form:

$$\mathbf{v} = (v_r(r,\theta), v_\theta(r,\theta), 0)$$

We assume that $v_\phi = 0$ by symmetry.

We have two boundary conditions at the radius.

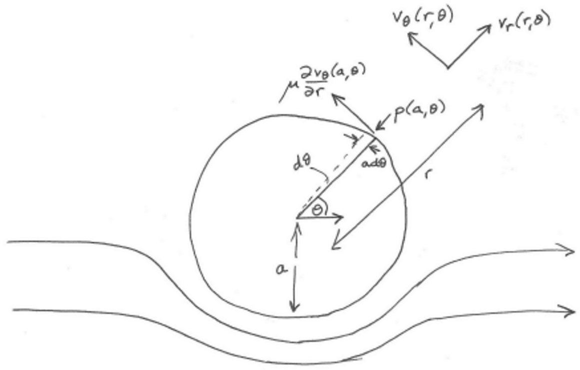
$$v_r(a,\theta) = 0$$
$$v_\theta(a,\theta) = 0$$

The first of these represents the impenetrability of the sphere, whilst the second represents a no-slip boundary condition.

We have two more boundary conditions for the velocity, based on conditions far from the sphere:

$$v_r \rightarrow U cos(\theta)$$
$$v_\theta \rightarrow -U sin(\theta)$$
$$For\, r \rightarrow \infty$$

We also know our pressure field will tend towards a constant $p_0$ for $r \rightarrow \infty$.




If we neglect the advective term, and in the absence of body forces, the Navier-Stokes equation reduces to:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{v} = 0$$

Time to solve this beast

The Laplacian in spherical polar coordinates is:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) +$$

*????*

*In notes, it is stated that we don't need to memorise, or know how the various terms are generated, and so I assume we can't be examined on this section ?*