PHAS2423 - Problem Based Learning - Fluid Dynamics - Problems and Solutions

(1) A rectangular bath of cross-sectional area 1 m^2 is filled to a depth of 30 cm. Estimate the time it takes for the water to drain out if the area of the plug-hole is 10 cm^2. [4]

Solution:
If we can assume that the water is inviscid, then this is an application of Bernoulli's equation. We ought to check at the end to see whether the Reynolds number is always much greater than unity. Consider a streamline starting near the top of the bathwater and ending some way down the plug-hole. In both cases, the water is at atmospheric pressure, since in the drainpipe the water no longer has to support a column of water above it. So, Bernoulli's equation tells us that the speed of the water in the drainpipe is given by

$$\frac{1}{2}\rho v^2 = \rho g h$$

where ρ is the density of the fluid and h is the depth of the water in the bath. The problem has actually turned into a rather elementary application of the conservation of energy. The fastest flow is at the start of the draining process where h is $30cm$ and $v = (ugh)^{1/2} \approx 2.5m/s$. The diameter of the plug-hole is about $0.035m$, the density of water is $103kg/m^3$ and the viscosity at $40-50C$ is about $6\times 10^{-4}Ns/m^2$ (look it up on the web, such as http://www.engineeringtoolbox.com/water-dynamic-kinematic-viscosity-d_596.html). The Reynolds number based on plug-hole diameter is therefore about 145000 and so the regime is indeed that of inviscid flow, and it remains approximately so throughout the process. The remaining task is to set up and solve a differential equation for the height of the bathwater as a function of time. If there area of the bath is A_b, then the volume change in time dt is $A_b dh$. This is also equal (in magnitude, but not in sign) to $A_p v dt$, where A_p is the cross sectional area of the plug-hole. Thus

$$\frac{dh}{dt} = -\frac{A_p}{A_b}(2gh)^{1/2}$$

and hence

$$2h^{1/2} = -\frac{A_p}{A_b}(2g)^{1/2}t + 2h_0^{1/2}$$

using the initial condition $h = h_0$ at $t = 0$. An estimate of the time at which $h = 0$ is therefore

$$t_{\text{empty}} = \frac{A_b}{A_p}\left(\frac{2h_0}{g}\right)^{1/2} = \frac{1}{10^{-3}}\left(\frac{0.6}{10}\right)^{1/2} \approx 245s \approx 4\text{min}$$

which does not seem unreasonable.

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(2) A layer of fluid of depth d is in steady laminar flow down a wide plane inclined to the horizontal at an angle θ. The density and shear viscosity of the fluid are ρ and μ, respectively. Regarding this as a form of channel flow, show that the volumetric rate of flow per unit width of the plane is given by

$$Q = \frac{\rho g d^3 \sin\theta}{3\mu}$$

where g is the gravitational acceleration. You may neglect the spatial dependence of the pressure in the fluid and in the air above it. You may also neglect the viscosity of the air.

Solution:
From the description of the problem, it ought to be clear that we are talking about 2-d channel flow, *i.e.* the velocity has a non-zero component v_x in the direction of the maximum slope of the plane, and that it is a function of the distance y above the plane only. Furthermore, the pressure is taken to be the same everywhere, so the only forces in operation are viscous and gravitational. The time derivative and inertial terms in the Navier-Stokes equation vanish and we have for its x-component

$$0 = \mu\frac{d^2 v_x}{dy^2} + \rho g \sin\theta,$$

where the projection of the gravitational force per unit volume ρg along the direction of motion has been taken. Apart from the substitution of a body force for a pressure gradient force, this is the same equation as explored in the notes, so we expect the profile $v_x(y)$ to be parabolic. At $y = 0$, we expect a no-slip boundary condition, $v_x(0) = 0$. At the free surface, $y = d$, we expect a zero shear stress boundary condition (the fluid is adjacent to air which has zero viscosity and therefore no ability to match a non-zero shear stress at the surface of the fluid. This requires the condition $dv_x/dy = 0$ at $y = d$ and so the profile that satisfies these conditions is

$$v_x(y) = \frac{\rho g \sin\theta}{2\mu}(2dy - y^2).$$

The volumetric flow rate per unit width of the plane is given by

$$Q = \int_0^d dy v_x(y) = \frac{\rho g \sin\theta}{2\mu}\int_0^d dy(2dy - y^2) = \frac{\rho g d^3 \sin\theta}{3\mu}.$$

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(3) A fluid with shear viscosity μ lies between two long concentric cylinders of radius a_1 and a_2, respectively, with $a_1 < a_2$. The outer cylinder is held stationary, while the inner cylinder rotates with angular velocity Ω. Assuming that the flow is incompressible, steady and laminar, such that packets of fluid move in circular paths, and ignoring gravity and making use of the following form for the Laplacian of the velocity field in cylindrical polar coordinates:

$$\begin{aligned}
(\nabla^2 \mathbf{v})_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \\
(\nabla^2 \mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \\
(\nabla^2 \mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}
\end{aligned}$$

a) solve the Navier-Stokes equation in the limit $\mathrm{Re} \ll 1$, such that the advective term can be neglected, to show that the pressure is constant everywhere, that the velocity field between the cylinders satisfies

$$r\frac{d^2 v_\theta}{dr^2} + \frac{dv_\theta}{dr} - \frac{v_\theta}{r} = 0$$

and find the v_θ that satisfies no-slip boundary conditions [hint: try expanding in powers of the radius] [8]

b) calculate the torque, or moment, exerted by the fluid on the outer cylinder; [6]

Solution:
a) The Navier-Stokes equation is

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v}.$$

We should seek a solution with $\mathbf{v} = (0, v_\theta(r), 0)$ and $p(r)$. The viscous terms in the Navier-Stokes equation are

$$\begin{aligned}
(\nabla^2 \mathbf{v})_r &= 0 \\
(\nabla^2 \mathbf{v})_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} \\
(\nabla^2 \mathbf{v})_z &= 0.
\end{aligned}$$

After substituting thes and neglecting the inertial term (since we are told that $Re \ll 1$) the radial component of the Navier-Stokes equation is

$$0 = -\frac{dp}{dr}$$

so that the pressure is actually a constant everywhere, and the tangential component of the Navier-Stokes equation is

$$0 = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r^2}$$

We need to integrate the second of these equations. To do so, it is convenient to rearrange it into the form

$$\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) = r\frac{d^2v_\theta}{dr^2} + \frac{dv_\theta}{dr} = \frac{v_\theta}{r}.$$

One way to do this is to try a solution $v_\theta \propto r^\alpha$, which yields $\alpha(alpha-1)r^{\alpha-1} + \alpha r^{\alpha-1} = r^{\alpha-1}$. This implies that $\alpha^2 = 1$ and the general solution is

$$v_\theta = Ar + \frac{B}{r},$$

where A and B are constants. The boundary conditions are $v_\theta(a_1) = \Omega a_1$ and $v_\theta(a_2) = 0$ Hence $B = -Aa_2^2$ and $\Omega a_1 = A(a_1 - a_2^2/a_1)$. The solution is therefore

$$v_\theta(r) = \frac{\Omega(r - a_2^2/r)}{(1-(a_2/a_1)^2)}.$$

b) The velocity gradient at the outer cylinder has only a radial component:

$$\left.\frac{dv_\theta}{dr}\right|_{a_2} = \frac{\Omega(1+a_2^2/a_2^2)}{(1-(a_2/a_1)^2)} = \frac{2\Omega}{(1-(a_2/a_1)^2)}$$

and the shear stress acting on the outer cylinder is this expression times μ. The area upon which this stress acts is $2\pi a_2 L$, where L is the length of the cylinders and the torque $\mathcal{T}$ is the force multiplied by the perpendicular distance to the axis so that

$$\mathcal{T} = \frac{4\Omega\pi\mu a_2^2 L}{(1-(a_2/a_1)^2)}.$$

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(4) A sphere of radius a falls through a fluid at a speed U such that the flow pattern around it corresponds to a Reynolds number much greater than unity.

a) Verify that the flow potential $\Phi(r,\theta) = Ur\cos\theta\,(1 + a^3/(2r^3))$ defined in a frame moving with the sphere is a solution to Laplace's equation and satisfies the boundary conditions at infinity and at the surface of the sphere. [4]

b) Verify by evaluating the curl of the flow field in spherical polar coordinates (i.e not by simply stating that $\nabla \times \nabla\Phi = 0$) that the vorticity is zero everywhere. [5]

c) Show that the pressure on the surface of the sphere is given by

$$p_s(\theta) = p_0 + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta\right)$$

in terms of the density of the fluid ρ and the pressure p_0 far from the sphere. [hint: consider Bernouilli's equation] [4]

Solution:
a) We need to insert the potential flow into Laplace's equation in spherical polars, namely

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0$$

This is straightforwardly checked:

$$\begin{aligned} & -\frac{1}{r^2}\partial_r(r^2 U\cos\theta(1 - 2a^3/(2r^3))) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta U r\sin\theta(1 + a^3/(2r^3))) \\ = & -\frac{1}{r^2}(U\cos\theta(2r + a^3/r^2)) + \frac{U}{r\sin\theta}(1 + a^3/(2r^3))(2sin\theta\cos\theta) \\ = & \ 0 \end{aligned}$$

Next, we should check the boundary conditions. The radial velocity is proportional to

$$\frac{\partial\Phi}{\partial r} = U\cos\theta(1 - a^3/r^3)$$

and at the sphere surface, $r = q$, this vanishes as required. At $r \to \infty$, the potential reduces to $\Phi(r,\theta) = Ur\cos\theta = Uz$. This is a linear potential with a gradient in the z-direction only. In other words a uniform flow in the z-direction. This is again the correct boundary condition.

b) We should employ the velocity components

$$\begin{aligned} v_r &= -\frac{\partial\Phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \\ v_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \\ v_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\phi} = 0 \end{aligned}$$

and the components of the curl of $\mathbf{v}$ in spherical polars:

$$
\begin{aligned}
(\nabla \times \mathbf{v})_r &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial \phi}\right) \\
(\nabla \times \mathbf{v})_\theta &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right) \\
(\nabla \times \mathbf{v})_\theta &= \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)
\end{aligned}
$$

so clearly $(\nabla \times \mathbf{v})_r = 0$ and $(\nabla \times \mathbf{v})_\theta = 0$ and we only need to focus on the ϕ-component. This can be done by hand, but using Mathematica can avoid slips of the pen.

c) The fluid is inviscid, incompressible and the flow is irrotational. Therefore, we can apply Bernouilli's equation such that $\rho v^2/2 + p$ is a constant everywhere. Or, alternatively, we could consider a streamline running towards the central point on the leading face of the sphere, this then deviates around the sphere to the centre of the trailing face of the sphere; $\rho v^2/2 + p$ is a constant along this trajectory. Far from the sphere, the value of this invariant is $\rho U^2/2 + p_0$ and on the sphere it is $\rho v_\theta(a)^2 + p_s(\theta)$, where $p_s(\theta)$ is the pressure on the surface of the sphere as a function of angle. Hence

$$
p_s(\theta) = \frac{1}{2}\rho U^2 + p_0 - \frac{1}{2}\rho v_\theta(a)^2 = p_0 + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)
$$
