GTYR3 - PHAS0026

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$$u(x,y) = f(p(x,y))$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial p^2}\left(\frac{\partial p}{\partial x}\right)^2 + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial x^2}$$

$$\rightarrow \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial p^2}\left(\frac{\partial p}{\partial y}\right)^2 + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial p^2}\frac{\partial p}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial y \partial x}$$

$$\rightarrow \quad A\left(\frac{\partial^2 f}{\partial p^2}\left(\frac{\partial p}{\partial x}\right)^2 + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial x^2}\right) + B\left(\frac{\partial^2 f}{\partial p^2}\frac{\partial p}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial y \partial x}\right) +$$

$$C\left(\frac{\partial^2 f}{\partial p^2}\left(\frac{\partial p}{\partial y}\right)^2 + \frac{\partial f}{\partial p}\frac{\partial^2 p}{\partial y^2}\right) = 0$$

Let $\frac{\partial p}{\partial x}$ = Constant

& $\frac{\partial p}{\partial y}$ = Constant.

$\Rightarrow$

$$C\lambda^2 + B\lambda + A = 0$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}$$

$$B^2 = 4AC$$

$$\rightarrow \lambda = \frac{-B}{2C} \;?$$

$$B = 2\sqrt{AC}$$

$$\lambda = -\frac{2\sqrt{AC}}{2C} = -\sqrt{\frac{A}{C}}$$

$$u(x,y) = f(x + \lambda y)$$

$$u(x,y) = h(x,y)\, g(x + \lambda y)$$

$$\rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} g + h\frac{\partial g}{\partial p}\frac{\partial p}{\partial x}$$

$$= \frac{\partial h}{\partial x} g + h\frac{\partial g}{\partial p}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 h}{\partial x^2} g + \frac{\partial h}{\partial x}\frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial g}{\partial p} + h\frac{\partial^2 g}{\partial p^2}$$

$$= \frac{\partial^2 h}{\partial x^2} g + 2\frac{\partial h}{\partial x}\frac{\partial g}{\partial p} + h\frac{\partial^2 g}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} g + h \frac{\partial g}{\partial p} \frac{\partial p}{\partial y}$$

$$= \frac{\partial h}{\partial y} g + \lambda h \frac{\partial g}{\partial p}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 h}{\partial y^2} g + \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} \cdot \lambda + \lambda \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} + \lambda^2 h \frac{\partial^2 g}{\partial p^2}$$

$$= \frac{\partial^2 h}{\partial y^2} g + 2\lambda \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} + \lambda^2 h \frac{\partial^2 g}{\partial p^2}$$

$$\frac{\partial^2 u}{\partial x y} = \frac{\partial^2 h}{\partial x \partial y} g + \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} + \lambda \frac{\partial h}{\partial x}\frac{\partial g}{\partial p} + h\lambda \frac{\partial^2 g}{\partial p^2}$$

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$A\left[\frac{\partial^2 h}{\partial x^2} g + 2 \frac{\partial h}{\partial x}\frac{\partial g}{\partial p} + h \frac{\partial^2 g}{\partial x^2}\right]$$

$$+ B\left[\frac{\partial^2 h}{\partial x \partial y} g + \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} + \lambda \frac{\partial h}{\partial x}\frac{\partial g}{\partial p} + h\lambda \frac{\partial^2 g}{\partial p^2}\right]$$

$$+ C\left[\frac{\partial^2 h}{\partial y^2} g + 2\lambda \frac{\partial h}{\partial y}\frac{\partial g}{\partial p} + \lambda^2 h \frac{\partial^2 g}{\partial p^2}\right] = 0$$

$$① \; h \frac{\partial^2 g}{\partial p^2}\left[A + \lambda B + \lambda^2 C\right] +$$

$$② \; \frac{\partial g}{\partial p}\left[2A \frac{\partial h}{\partial x} + B \frac{\partial h}{\partial y} + B\lambda \frac{\partial h}{\partial x} + 2\lambda C \frac{\partial h}{\partial y}\right] +$$

$$③ \; Ag \frac{\partial^2 h}{\partial x^2} + Bg \frac{\partial^2 h}{\partial x \partial y} + Cg \frac{\partial^2 h}{\partial y^2} = 0$$

$$\therefore \lambda = -\sqrt{\frac{A}{C}} \qquad B^2 = 4AC \Rightarrow A = \frac{B^2}{4C}$$

$$\Rightarrow \lambda = -\sqrt{\frac{B^2}{4C} \times \frac{1}{C}} = \frac{-B}{2C}$$

$$① : \; A + \lambda B + \lambda^2 C = A - \frac{B}{2C} B + \frac{B^2}{4C^2} C$$

$$= A - \frac{B^2}{2C} + \frac{B^2}{4C}$$

$$= A - \frac{B^2}{4C}$$

$$= A - \frac{(4AC)}{4C} = A - A = 0$$

$$\therefore \; ① = 0$$

$$② \quad 2A \frac{\partial h}{\partial x} + B \frac{\partial h}{\partial y} + B\lambda \frac{\partial h}{\partial x} + 2\lambda C \frac{\partial h}{\partial y}$$

$$\Rightarrow (2A + \lambda B) \frac{\partial h}{\partial x} + (B + 2\lambda C) \frac{\partial h}{\partial y}$$

$$= \left(2A - \frac{B^2}{2C}\right) \frac{\partial h}{\partial x} + \left(B - \frac{2B}{2C} C\right) \frac{\partial h}{\partial y}$$

$$= \left(2A - \frac{B^2}{2C}\right)\frac{\partial h}{\partial x} + \left(B - \frac{2B}{2C}C\right)\frac{\partial h}{\partial y}$$

$$= \left(2A - \frac{4AC}{2C}\right)\frac{\partial h}{\partial x} + (B - B)\frac{\partial h}{\partial y}$$

$$= (2A - 2A)\frac{\partial h}{\partial x} + (B-B)\frac{\partial h}{\partial y}$$

$$= 0$$

① & ② $= 0$ $\therefore$

③ $$Ag\frac{\partial^2 h}{\partial x^2} + Bg\frac{\partial^2 h}{\partial x \partial y} + Cg\frac{\partial^2 h}{\partial y^2} = 0$$

$$\left(A\frac{\partial^2 h}{\partial x^2} + B\frac{\partial^2 h}{\partial x \partial y} + C\frac{\partial^2 h}{\partial y^2}\right)g = 0$$

$g \neq 0 \therefore$

$h(x,y) = x$

$\rightarrow \frac{\partial h}{\partial x} = 1 \rightarrow \frac{\partial^2 h}{\partial x^2} = 0$

All derivatives are 0 $\therefore$

$h(x,y) = x$

$\rightarrow u(x,y) = x\,g(x + \lambda y)$

ii) when $h(x,y) = x$

$\rightarrow u(x,y) = f(x + \lambda y) + x\,g(x + \lambda y)$

when $h(x,y) = y$

$u(x,y) = f(x + \lambda y) + y\,g(x,y)$

$p(x,y) = x + \lambda y$

$\rightarrow x = p - \lambda y \qquad y = \frac{p - x}{\lambda}$

b) $A = 3, \quad B = -5, \quad C = 2$

i) $u(x,y) = f(x + \lambda y) + g(x + \lambda y)$

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4ac}}{2C}$$

$\rightarrow \lambda_1 = \frac{3}{2}, \quad \lambda_2 = 1$

ii) $u(x, y=x) = \cos(x)$

$$\Rightarrow f\left(x + \tfrac{3}{2}x\right) + g(x+x) = \cos(x)$$

$$f\left(\tfrac{5}{2}x\right) + g(2x) = \cos(x)$$

$$= \tfrac{1}{2}\left[e^{ix} + e^{-ix}\right]$$

$$f(x, y=x) = \tfrac{1}{2} e^{i\frac{2}{5}x}$$

$$g(x, y=x) = \tfrac{1}{2} e^{-i\frac{1}{2}x}$$

$$f(x,y) = \tfrac{1}{2} e^{i\frac{2}{5}(x+\frac{3}{2}y)}$$

$$g(x,y) = \tfrac{1}{2} e^{-i\frac{1}{2}(x+y)}$$

CHECK SOLUTION:

$$u(x,y) = \tfrac{1}{2}\exp\left[\tfrac{2}{5}i\left(x+\tfrac{3}{2}y\right)\right] + \tfrac{1}{2}\exp\left[-\tfrac{1}{2}i(x+y)\right]$$

$$\frac{\partial u}{\partial x} = \tfrac{1}{2}\times\tfrac{2}{5}i\exp\left[\tfrac{2}{5}i\left(x+\tfrac{3}{2}y\right)\right] - \tfrac{1}{4}i\exp\left[-\tfrac{1}{2}i(x+y)\right]$$

$$= \tfrac{1}{5}i\exp\left[\tfrac{2}{5}i\left(x+\tfrac{3}{2}y\right)\right] - \tfrac{1}{4}i\exp\left[-\tfrac{1}{2}i(x+y)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2}{25}\exp\left[\tfrac{2}{5}i\left(x+\tfrac{3}{2}y\right)\right] - \tfrac{1}{8}\exp\left[-\tfrac{1}{2}i(x+y)\right]$$

$$= \frac{-2}{25}f - \frac{1}{8}g$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\tfrac{1}{5}if - \tfrac{1}{4}ig\right)$$

$$= \tfrac{1}{5}i\times\tfrac{6}{10}if - \tfrac{1}{4}i\times -\tfrac{1}{2}ig$$

$$= -\frac{6}{50}f - \frac{1}{8}g$$

$$\frac{\partial u}{\partial y} = \frac{6}{20}if - \frac{1}{4}ig$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{36}{200}f - \frac{1}{8}g$$

$$\Rightarrow 3\times\left[-\tfrac{2}{25}f - \tfrac{1}{8}g\right] - 5\left[-\tfrac{6}{50}f - \tfrac{1}{8}g\right] + 2\left[-\tfrac{36}{200}f - \tfrac{1}{8}g\right]$$

$$= f\left[\frac{-6}{25} + \frac{6}{10} - \frac{36}{100}\right] + g\left[-\frac{3}{8} + \frac{5}{8} - \frac{2}{8}\right]$$

$$= f\times 0 + g\times 0 = 0 \quad \text{(As required.)}$$

iii)

2) a)

$$\left(\frac{\partial}{\partial t} + DK^2 + iV_d k\right)\frac{1}{\sqrt{2\pi}}\int p(t,x)\,e^{ikx}\,dx = 0$$

$\times \sqrt{2\pi}$ & expand:

$$\int \underset{①}{\frac{\partial}{\partial t} p(t,x)\, e^{ikx}\,dx} + \left[Dk^2 + iV_d k\right] \underset{②}{\int p(t,x)\, e^{ikx}\,dx} = 0$$

① $$\int \frac{\partial}{\partial t} p(t,x)\, e^{ikx}\,dx = \int \left(D\frac{\partial^2 p}{\partial x^2} + V_d \frac{\partial p}{\partial x}\right) e^{ikx}\,dx$$

$$= D\underset{(A)}{\int \frac{\partial^2 p}{\partial x^2} e^{ikx}\,dx} + V_d \underset{(B)}{\int \frac{\partial p}{\partial x} e^{ikx}\,dx}$$

(B): $u = e^{ikx}$ $\quad dv = \partial p$

$du = ike^{ikx}$ $\quad v = p$

$$(B) = ikV\left[pe^{ikx}\right]_{-\infty}^{\infty} - ikV\int p e^{ikx}\,dx$$

$0 \rightarrow$

$$= -ikV_d \int p\, e^{ikx}\,dx$$

(A): $u = e^{ikx}$ $\quad dv = \partial^2 p$

$du = ike^{ikx}$ $\quad v = \frac{\partial p}{\partial x}$

$$(A) = \left[e^{ikx}\frac{\partial p}{\partial x}\right]_{-\infty}^{\infty} - ikD\int \frac{\partial p}{\partial x} e^{ikx}\,dx$$

0

$$= -ikD\left[-ik \int p e^{ikx}\,dx\right] = -k^2 D \int pe^{ikx}\,dx$$

$$A + B + \left(Dk^2 + iV_d k\right)\int pe^{ikx}\,dx = 0$$

$$-k^2 D\int pe^{ikx}\,dx - ikV_d\int pe^{ikx}\,dx + \left(Dk^2 + iV_d k\right)\int pe^{ikx}\,dx = 0$$

$$\Rightarrow \left(-k^2 D - ikV_d + Dk^2 + ikV_d\right)\int pe^{ikx}\,dx = 0$$

$$0 \times \int pe^{ikx}\,dx = 0 \quad \checkmark$$

b)

$$p(x, t=0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

$$\tilde{p}(x, t=0) = \frac{1}{\sqrt{2\pi\sigma^2}} \times \frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)\exp(ikx)\,dx$$

$$= \frac{1}{2\pi\sigma}\int \exp\left(-\frac{(x^2 - 2xx_0 + x_0^2)}{2\sigma^2} + ikx\right)dx$$

$$= \frac{1}{2\pi\sigma} \int \exp\left(\frac{-x^2 + 2xx_0 - x_0^2}{2\sigma^2} + ikx \right) dx$$

$$= \frac{1}{2\pi\sigma} \int \exp\left(\frac{1}{2\sigma^2} \left(-x^2 + x(2x_0 + 2ik\sigma^2) - x_0^2 \right) \right) dx$$

$$\Rightarrow a = \frac{1}{2\sigma^2} \quad b = \frac{x_0}{\sigma^2} + ik \quad , \quad c = -\frac{x_0^2}{2\sigma^2}$$

$$= \frac{1}{2\pi\sigma} \times \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c \right)$$

$$\frac{b^2}{4a} + c = \left(\frac{x_0}{\sigma^2} + ik \right)^2 \times \frac{\sigma^2}{2} - \frac{x_0^2}{2\sigma^2}$$

$$= \left(\frac{x_0^2}{\sigma^4} + \frac{2ikx_0}{\sigma^2} - k^2 \right) \frac{\sigma^2}{2} - \frac{x_0^2}{2\sigma^2}$$

$$= \cancel{\frac{x_0^2}{2\sigma^2}} + ikx_0 - \frac{k^2\sigma^2}{2} - \cancel{\frac{x_0^2}{2\sigma^2}}$$

$$= ikx_0 - \frac{k^2\sigma^2}{2}$$

$$\Rightarrow = \frac{1}{2\pi\sigma} \sqrt{\pi \times 2\sigma^2} \exp\left(ikx_0 - \frac{k^2\sigma^2}{2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\left(\frac{\sigma^2 k^2}{2} - ikx_0 \right) \right]$$

$$A(k) = \exp\left[-\left(\frac{\sigma^2 k^2}{2} - ikx_0 \right) \right]$$

c) $$\hat{p}(k,t) = A(k)\, e^{-(Dk^2 + iVk)t}$$

$$= A(k)\, T(k,t)$$

$$p(x,t) = \int_{-\infty}^{\infty} t(x-s,t)\, a(s)\, ds$$

$$t(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-(Dk^2 + iVk)t \right] \exp(ikx)\, dk$$

$$= \frac{1}{2\pi} \int \exp\left[-Dk^2 t + ik(x - vt) \right] dk$$

$$-Dk^2 + ik(x-v) :$$

$$a = -Dt \quad b = i(x - vt) \quad c = 0$$

$a = -Dt \quad b = i(x-vt) \quad c = 0$

$$= -Dt\left(k - \frac{i(x-vt)}{2D}\right)^2 - \frac{i^2(x-vt)^2}{-4Dt}$$

$$= -Dt\left(k - \frac{i(x-vt)}{2D}\right)^2 - \frac{(x-vt)^2}{4Dt}$$

$$\Rightarrow \quad t(x,t) = \frac{1}{2\pi} \times \exp\left(-\frac{(x-vt)^2}{4Dt}\right) \int_{-\infty}^{\infty} \exp\left[-Dt\left(k - \frac{i(x-vt)}{2D}\right)^2\right] dK$$

Let $s = K - \frac{i(x-vt)}{2D}$

$$= \frac{1}{2\pi} \times \exp\left(\frac{-(x-vt)^2}{4Dt}\right) \int \exp(-Dt\, s^2)\, ds$$

$$= \frac{1}{2\pi} \times \exp\left(\frac{-(x-vt)^2}{4Dt}\right) \times \sqrt{\frac{\pi}{Dt}}$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{Dt}} \exp\left(\frac{-(x-vt)^2}{4Dt}\right)$$

$$p(x,t) = \int_{-\infty}^{\infty} t(x-s,t)\, q(s)\, ds = \int_{-\infty}^{\infty} t(x,t)\, q(x-s)\, ds$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{Dt}} \times \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{(s-vt)^2}{4Dt}\right) \times \exp\left(-\frac{(x-s-x_0)^2}{2\sigma^2}\right) ds$$

$$= -\left[\frac{(s-vt)^2}{4Dt} + \frac{(-s+(x-x_0))^2}{2\sigma^2}\right]$$

Let $a = vt$, $b = (x-x_0)$,

$$= -\left[\frac{s^2 - 2sa + a^2}{4Dt} + \frac{s^2 - 2sb + b^2}{2\sigma^2}\right]$$

$$= -\left[\frac{2\sigma^2(s^2 - 2sa + a^2) + 4Dts^2 - 8Dtbs + 4b^2Dt}{8Dt\sigma^2}\right]$$

$$= -\left[\frac{s^2[2\sigma^2 + 4Dt] + s[-2a - 8Dtb] + \boxed{[2\sigma^2a^2 + 4b^2Dt]}}{8Dt\sigma^2}\right]$$

Take out as a factor of exp

$$= \frac{-1}{8Dt\sigma^2}\left[[2\sigma^2 + 4Dt]s^2 + [-2a - 8Dtb]s\right]$$

$a = 2\sigma^2 + 4Dt \quad b = -2a - 8Dtb$

$$= \frac{-1}{8Dt\sigma^2}\left[(2\sigma^2+4Dt)\left(s + \frac{-2a-8Dtb}{4\sigma^2+8Dt}\right)^2 - \frac{(-2a-8Dtb)^2}{8\sigma^2+16Dt}\right]$$

Let $T = s + \frac{-2a-8Dtb}{4\sigma^2+8Dt}$

$$= \int \exp\left[\frac{-(2\sigma^2+4Dt)T^2}{8Dt\sigma^2}\right] dT$$

$$= \sqrt{\frac{8Dt\sigma^2\pi}{2\sigma^2+4Dt}} \times \frac{1}{2\sqrt{\pi Dt}}\frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left[\frac{-(2\sigma^2v^2t^2+4(x-x_0)^2Dt)}{8Dt\sigma^2} + \frac{(-2vt-8Dt(x-x_0)^2)}{(8Dt\sigma^2)(8\sigma^2+16Dt)}\right]$$

$$= \sqrt{\frac{8Dt\sigma^2\pi}{2\sigma^2+4Dt}} \times \frac{1}{\pi\sqrt{8Dt\sigma^2}} \times \exp\left[\frac{-1}{8Dt\sigma^2}\left(2\sigma^2v^2t^2+4(x-x_0)^2Dt + \right.\right.$$

3) a) $\rho\frac{\partial \underline{V}}{\partial t} + \rho\underline{V}\cdot\nabla\underline{V} = -\nabla P + \mu\nabla^2\underline{V}$

i) $\mu = 0$

$\rightarrow \rho\frac{\partial \underline{V}}{\partial t} + \rho\underline{V}\cdot\nabla\underline{V} = -\nabla P$

Euler equation.

Steady state $\rightarrow \frac{\partial \underline{V}}{\partial t} = 0$

$\rightarrow \rho\underline{V}\cdot\nabla\underline{V} = -\nabla P$

$$V\times(\nabla\times V) = \frac{1}{2}\nabla(V\cdot V) - V\cdot\nabla V$$

$$\rightarrow V\cdot\nabla V = \frac{1}{2}\nabla(V\cdot V) - V\times(\nabla\times V)$$

$$\rightarrow \rho\left(\frac{1}{2}\nabla(V\cdot V) - V\times(\nabla\times V)\right) = -\nabla P$$

$$V\cdot\rho\left(\frac{1}{2}\nabla(V\cdot V) - V\times(\nabla\times V)\right) = -V\cdot\nabla P$$

$$\rho\left[\frac{1}{2}\underline{V}\cdot\nabla(\underline{V}\cdot\underline{V}) - \underline{V}\cdot(\underline{V}\times(\nabla\times\underline{V}))\right] = -\underline{V}\cdot\nabla P$$

$$\underline{V}\cdot\underline{V} = V^2$$

$$\underline{V}\cdot(\underline{V}\times(\nabla\times\underline{V})) = 0$$

$$\rightarrow \rho\left[\frac{1}{2}\underline{V}\cdot\nabla V^2\right] = -\underline{V}\cdot\nabla P$$

$$\underline{V}\cdot\frac{\rho}{2}\nabla V^2 + \underline{V}\cdot\nabla P = 0$$

$$V\cdot\nabla\left[\frac{1}{2}\rho V^2 + P\right] = 0$$

As required.

Satisfys Conservation of energy.

ii) $$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\frac{1}{\rho}\nabla P + \mu\nabla^2\underline{V}$$

LHS: $$\underline{V}\cdot\nabla\underline{V} = \frac{1}{2}\nabla(V\cdot V) - V\times(\nabla\times V)$$

$$\nabla\times\left[\frac{\partial V}{\partial t} + \frac{1}{2}\nabla(V\cdot V) - (V\times(\nabla\times V))\right]$$

$$= \frac{\partial(\nabla\times V)}{\partial t} + \frac{1}{2}\underbrace{(\nabla\times\nabla)}_{\nabla\times\nabla=0}(V\cdot V) - \nabla\times(V\times(\omega))]$$

$$= \frac{\partial\omega}{\partial t} - \nabla\times(V\times\omega)$$

$$= \frac{\partial\omega}{\partial t} - (\omega\cdot\nabla)V + (V\cdot\nabla)\omega$$

$$= \left[\frac{\partial}{\partial t} + V\cdot\nabla\right]\omega - (\omega\cdot\nabla)V$$

RHS:

$$-\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2 V$$

$$\nabla\times RHS = -\frac{1}{\rho}(\nabla\times\nabla)P + \frac{\mu}{\rho}\nabla^2(\nabla\times V)$$

$$= 0 + \frac{\mu}{\rho}\nabla^2\omega$$

$\rightarrow$ LHS = RHS.

$$\left[\frac{\partial}{\partial t} + V\cdot\nabla\right]\omega - (\omega\cdot\nabla)V = \frac{\mu}{\rho}\nabla^2\omega$$

$$\left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\omega - (\omega\cdot\nabla)v = \frac{\mu}{\rho}\nabla^2\omega$$

$$\left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\omega = (\omega\cdot\nabla)v + \frac{\mu}{\rho}\nabla^2\omega$$

$$(\omega\cdot\nabla)v = \omega\cdot\nabla\underline{v} - \omega(\nabla\cdot v)$$

$$\Rightarrow \frac{D\omega}{Dt} = \left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\omega = -\omega(\nabla\cdot v) + \omega\cdot\nabla v + \frac{\mu}{\rho}\nabla^2\omega$$

$$\omega = \hat{z}\,\omega_z(r) \qquad v = \hat{\phi}\,v(r)$$

$$\Rightarrow \frac{\partial\omega}{\partial t} = 0 \qquad v\cdot(\nabla\omega) = \hat{\phi}\cdot\hat{z}\ \ldots$$
$$= 0$$

$LHS = 0$

$\therefore\ RHS = 0$

b) $v = (0, 0, \omega(r))$

$$\omega = \nabla\times V$$
$$= \left(\frac{1}{r}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial V_r}{\partial\phi} - \frac{\partial V_z}{\partial r}\right)\hat{\phi}$$
$$+ \frac{1}{r}\left(\frac{\partial(rV_\phi)}{\partial r} - \frac{\partial V_r}{\partial\phi}\right)\hat{z}$$

$$\Rightarrow V = (0, V_\phi(r), 0)$$

ii) $\nabla\times \underline{V}(0, V_\phi(r), 0) = \underline{\omega}\,(0, 0, \omega(r))$

$$\nabla\times\underline{V}(0, V_\phi(r), 0) = \left(\frac{1}{r}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial V_r}{\partial\phi} - \frac{\partial V_z}{\partial r}\right)\hat{\phi}$$
$$+ \frac{1}{r}\left(\frac{\partial(rV_\phi)}{\partial r} - \frac{\partial V_r}{\partial\phi}\right)\hat{z}$$
$$= \left(\frac{1}{r}0 - 0\right)\hat{r} + (0 - 0)\hat{\phi}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rV_\phi(r)\right) - 0\right)\hat{z}$$

$$\omega(0,0,\omega(r)) = \frac{1}{r}\frac{d}{dr}\left(rV_\phi(r)\right)$$
$$= \frac{1}{r}\left(V_\phi(r) + r\frac{dV_\phi}{dr}\right)$$

$$\Rightarrow \frac{1}{r}V_\phi(r) + \frac{dV_\phi}{dr} = \begin{cases}\Omega & r\le a\\ 0 & r>a\end{cases}$$

$$\frac{dv}{dr} + \frac{1}{r} v = 0$$

$$\Rightarrow \frac{dv}{dr} = -\frac{1}{r} v$$

$$\Rightarrow v = e^{-\frac{a}{r} + c}$$

$$\Rightarrow \frac{dv}{dr} = -\frac{a}{r} e^{-\frac{a}{r} + c}$$

$$\Rightarrow a = 1$$

$$\Rightarrow V_\phi = \exp\left(-\frac{1}{r} + c\right).$$

$$V_\phi = A e^{-\frac{1}{r}}$$

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