WATCHING THE RIVER FLOW[1]

Introduction

The Navier-Stokes equations, the equations describing the flow of fluids are, amongst the equations of mathematical physics, probably the most interesting mathematically and the most important industrially. Their mathematical interest stems mainly from their nonlinearity, whilst their commercial importance impacts fields as diverse as chemical engineering and aeronautics. The aim of this project is to solve the equations for a few simple geometries, and at the same time to explore some of the richness of behaviour they show. This project draws on Chapter 19 of Landau *et al.*[3].

Figure 1: Using a 35mm camera, a telephoto lens and ASA 400 film, Pat Maloney, an engineering planner, photographed an F-4 Phantom II at the moment it broke the sound barrier at the Annual Point Magu Naval Air Station Air Show[2]. ”The photograph of the visible shock is rare,” stated Maloney. ”It required a humid day, split second timing, and no small measure of luck.”

Theory

Velocity field equations

If we confine ourselves to steady-state two-dimensional incompressible flow problems, in which all the variation takes place in the xy-plane, there is no time variation, and the density ρ of the flowing fluid is constant, we can express the relationship between the pressure p and the velocity $\mathbf{v} = (v_x, v_y)$ in terms of the continuity equation (representing the incompressibility)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0,$$

[1] Bob Dylan 1971

and the momentum equations

$$\begin{aligned}
\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right) &= v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\nu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right) &= v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y},
\end{aligned}$$

ν being the fluid viscosity.

An analytic solution

For some simple situations, these equations are soluble analytically. Consider flow between two parallel plates, each perpendicular to the y-axis, with no y component of velocity. The continuity equation then reduces to

$$\frac{\partial v_x}{\partial x}=0.$$

Then the momentum equations reduce to

$$\begin{aligned}
\rho\nu\frac{\partial^2 v_x}{\partial y^2} &= \frac{\partial p}{\partial x} \\
0 &= \frac{\partial p}{\partial y}.
\end{aligned}$$

For flow between plates we may assume that $\frac{\partial p}{\partial x}$ is constant. If we also require the x-component of velocity to be zero at $y=0$ and $y=H$ we find

$$v_x=\frac{1}{2\nu\rho}\frac{\partial p}{\partial x}\left(y^2-yH\right).$$

Vorticity

In vector form, the equations may be written as

$$\begin{aligned}
\nabla.\mathbf{v} &= 0 \\
(\mathbf{v}.\nabla)\mathbf{v} &= \nu\nabla^2\mathbf{v}-\frac{1}{\rho}\nabla p.
\end{aligned}$$

We now use the trick of relating the velocity field to a potential (rather as in electrostatics the vector electric field is related to the scalar potential – except that here we use a curl rather than a gradient), so that the stream function $\mathbf{u}$ gives the velocity through

$$\mathbf{v}=\nabla\times\mathbf{u},$$

and because of the identity

$$\nabla.\nabla\times\mathbf{w}=0$$

we can see that $\mathbf{v}$ defined this way automatically satisfies the continuity equation. Also, as $\mathbf{v}$ in our 2-D problem only has x and y components it is sufficient for $\mathbf{u}$ to have only a z component, so that

$$\mathbf{u} = (0, 0, u), \quad v_x = \frac{\partial u}{\partial y}, \quad v_y = -\frac{\partial u}{\partial x}.$$

We also define the vorticity,

$$\mathbf{w} = \nabla \times \mathbf{v},$$

which, for our 2-D problem, reduces to

$$\mathbf{w} = (0, 0, w) = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$

Why have we done all this? A little calculation will convince you that it results in a new pair of equations:

$$\begin{aligned} \nabla^2 \mathbf{u} &= -w, \\ \nu \nabla^2 \mathbf{w} &= [(\nabla \times \mathbf{u}).\nabla]\mathbf{w}, \end{aligned}$$

or, in 2D,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= -w, \\ \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) &= \frac{\partial u}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial y}. \end{aligned}$$

The project

Velocity form

If we discretise the velocity equations on a grid with a spacing h in the x and y directions, and take

$$x = ih, \quad i = 0, 1, ..., N_x; \qquad y = jh, \quad j = 0, 1, ..., N_y,$$

then in finite difference form we obtain equations of the form (for simplicity, we take $\rho = \nu = 1$ – you should form the equations with these parameters restored)

$$\begin{aligned} v_x(i+1,j) - v_x(i-1,j) + v_y(i,j+1) - v_y(i,j-1) &= 0 \\ v_x(i+1,j) + v_x(i-1,j) + v_x(i,j+1) + v_x(i,j-1) - 4v_x(i,j) &= \\ \frac{h}{2}v_x(i,j)\left(v_x(i+1,j) - v_x(i-1,j)\right) + \frac{h}{2}v_y(i,j)\left(v_x(i,j+1) - v_x(i,j-1)\right) & \\ +\frac{h}{2}\left(p(i+1,j) - p(i-1,j)\right) & \\ v_y(i+1,j) + v_y(i-1,j) + v_y(i,j+1) + v_y(i,j-1) - 4v_y(i,j) &= \\ \frac{h}{2}v_x(i,j)\left(v_y(i+1,j) - v_y(i-1,j)\right) + \frac{h}{2}v_y(i,j)\left(v_y(i,j+1) - v_y(i,j-1)\right) & \\ +\frac{h}{2}\left(p(i,j+1) - p(i,j-1)\right) & \end{aligned}$$

You should set up these equations for the special case of $v_y = 0$, and set up a successive overrelaxation procedure in which, at each step the mesh is scanned and for each i, j:

- $v_x(i+1, j)$ is set to $v_x(i-1, j)$.
- the residual $r(i, j)$ is formed as

$$\begin{aligned} r(i,j) &= \frac{1}{4}\Big[v_x(i+1,j) + v_x(i-1,j) + v_x(i,j+1) + v_x(i,j-1) \\ &\quad -\frac{h}{2}v_x(i,j)\left(v_x(i+1,j) - v_x(i-1,j)\right) \\ &\quad -\frac{h}{2}\left(p(i+1,j) - p(i-1,j)\right)\Big] - v_x(i,j) \end{aligned}$$

- v is updated with

$$v_x(i,j) \leftarrow v_x(i,j) + \omega r(i,j)$$

  where ω is less than zero for underrelaxation, 1 for standard relaxation, greater than 1 for overrelaxation.

- r is used to keep a measure of the maximum change to monitor convergence.

Use a code based on this scheme to investigate a situation in a rectangular channel in which $v_x = 0$ on the top and bottom surfaces, there is a constant pressure gradient, the inlet velocity is constant V_0 across the channel, and the outlet velocity has stabilised so that $\frac{\partial v_x}{\partial x} = 0$. A suitable set of parameters is $\rho = 1$, $\nu = 1$, $N_x = 400$, $N_y = 40$, $h = 1$, $\frac{\partial p}{\partial x} = -12$ and $V_0 = 1$. Compare the result with the analytic result. Experiment with different overrelaxation parameters, ω, to see how the convergence is affected.

Vorticity form

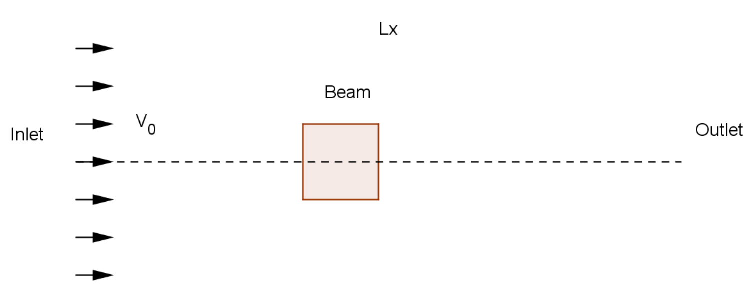

Figure 2: A beam in a confined flow.

You should set up the finite difference form of the vorticity equations. The scheme here is similar to the velocity case, except that there are now two update schemes, one for u and one for w, and at each grid point the two updates are applied one after the other. The problem to be solved here is a little more complicated: it is a square blockage in the middle of a flowing river: at the inlet and along the top surface there is a velocity V_0. If the mesh is characterised by a spacing h then the speed of the flow is described by a quantity $R = V_0 h/\nu$, which is related to the Reynolds number.

One can choose to perform the calculation on the whole region, or to assume symmetry about the centreline (dashed in the diagram). If we take the second option, the boundary conditions are as follows (you should convince yourself that these are correct).

$$
\begin{array}{lll}
u = 0; & w = 0; & \text{centreline} \\
u = 0; & w(i,j) = -2(u(i+1,j) - u(i,j)/h^2; & \text{beam back} \\
u = 0; & w(i,j) = -2(u(i,j+1) - u(i,j)/h^2; & \text{beam top} \\
u = 0; & w(i,j) = -2(u(i-1,j) - u(i,j)/h^2; & \text{beam front} \\
\dfrac{\partial u}{\partial x} = 0; & w = 0; & \text{inlet} \\
\dfrac{\partial u}{\partial y} = V_0; & w = 0; & \text{top} \\
\dfrac{\partial u}{\partial x} = 0; & \dfrac{\partial w}{\partial x} = 0; & \text{outlet.}
\end{array}
$$

Suitable conditions to start with are $V_0 = 1$, $R = 0.1$, a beam $8h$ on each side, and a smallish grid with $N_x = 24$, $N_y = 70$. Explore the convergence of the procedure, and find an optimal overrelaxation parameter. Explore the effects of changing the size of the mesh to give more space upstream and downstream of the beam. See what happens to the flow pattern as the R parameter is increased.

References

[1] Batchelor G.K. *An Introduction to Fluid Dynamics*, Cambridge University Press; 1967.

[2] `http://www.efluids.com/efluids/gallery/gallery_pages/breaking_sound_page.jsp`

[3] Landau R.H., Páez M.J. and Bordeianu C.C. *A Survey of Computational Physics*, Princeton University Press; 2008.

[4] Lautrup B. *Physics of Continuous Matter* Institute of Physics; 2005.

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January 2009