模式识别课程

Fisher线性判别

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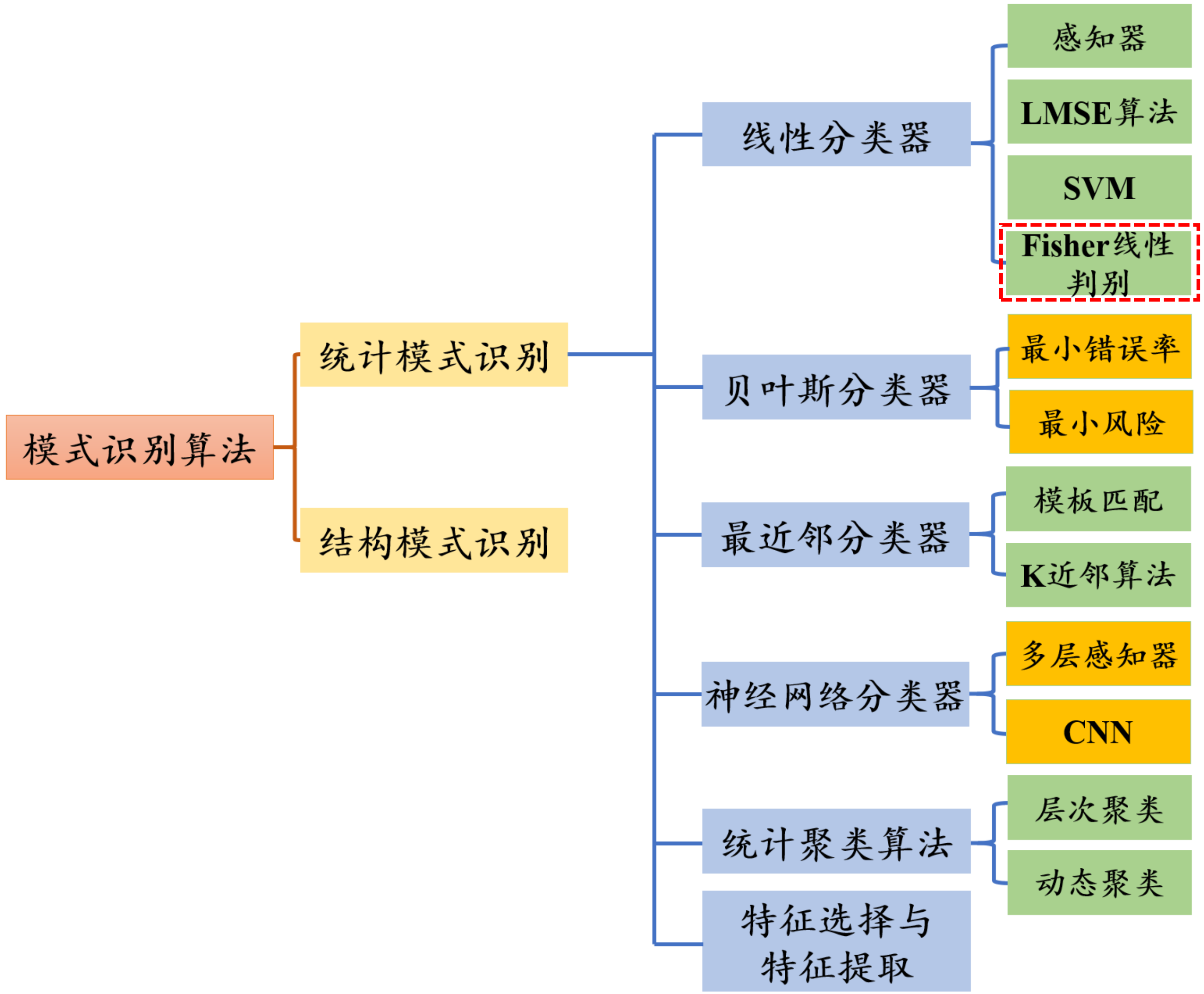
模式识别-线性分类器


模式识别算法
统计模式识别
结构模式识别
线性分类器
贝叶斯分类器
最近邻分类器
神经网络分类器
统计聚类算法
特征选择与特征提取
感知器
LMSE算法
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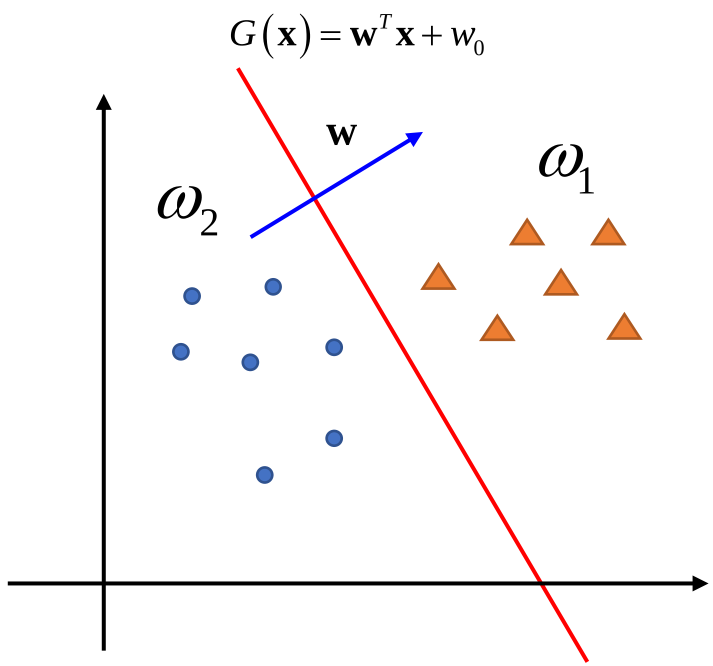
Fisher线性判别
最小错误率
最小风险
模板匹配
K近邻算法
多层感知器
CNN
层次聚类
动态聚类

线性分类器 –Fisher线性判别

Fisher线性判别：把样本**投影**到一个方向上，然后在这个**一维空间**确定分类的阈值。通过阈值点且与投影方向垂直的超平面就是分类面。



线性分类器－Fisher线性判别

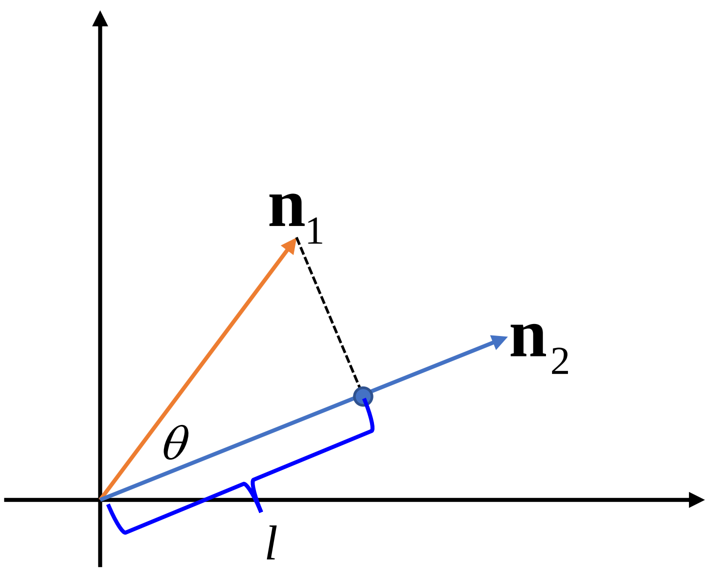
Q: 什么是投影？

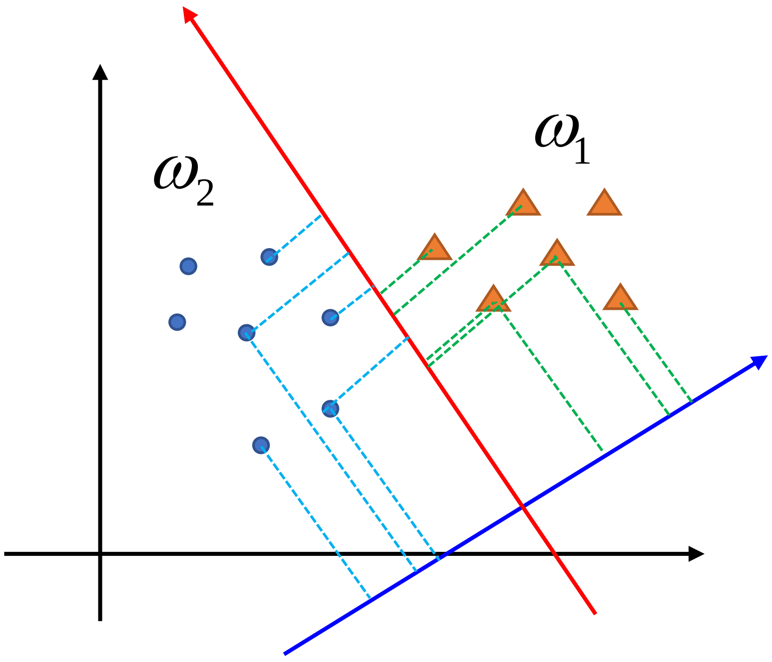
Q: 如何选择投影方向？

$\mathbf{n}_1$在$\mathbf{n}_2$上的投影是l

$$l = \|\mathbf{n}_1\|\cos\theta = \|\mathbf{n}_1\|\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_2\|}$$

Q: 哪一个投影方向最佳？



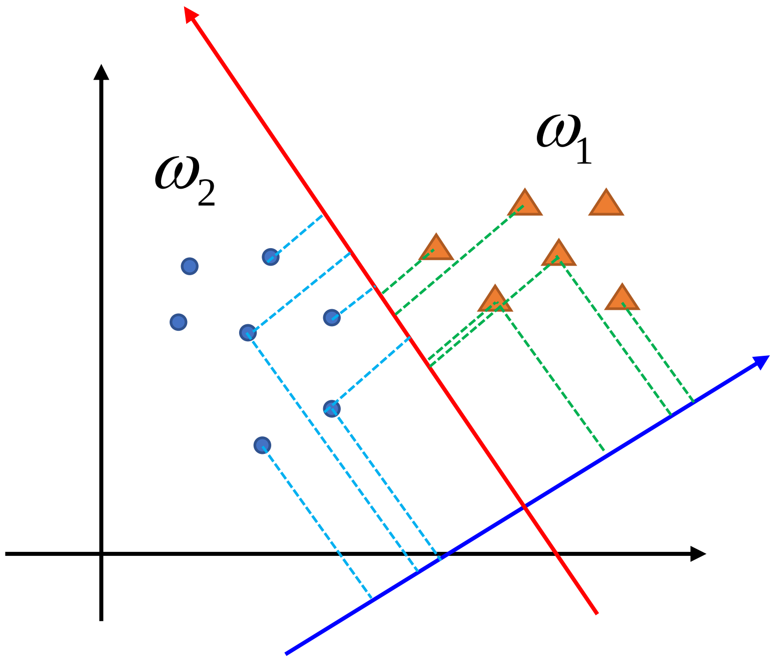


线性分类器－Fisher线性判别

Fisher线性判别思想：

选择投影方向，使投影后两类相隔尽可能远，而同一类内部的样本又尽可能的聚集。

Q: 如何判断聚类程度



线性分类器－Fisher线性判别

记号：

ω_1的样本为$\aleph_1=\{\mathbf{x}_1^1,\mathbf{x}_2^1,\ldots,\mathbf{x}_{N_1}^1\}$，$\omega_2$的样本为$\aleph_2=\{\mathbf{x}_1^2,\mathbf{x}_2^2,\ldots,\mathbf{x}_{N_2}^2\}$

投影方向为：$\mathbf{w}$，则投影以后的样本为：$\boxed{y_i=\mathbf{w}^T\mathbf{x}_i}$

定义：

原样本空间的【类内离散度矩阵】为：

$$\mathbf{S}_i=\sum_{\mathbf{x}_j^i\in\aleph_i}\left(\mathbf{x}_j^i-\mathbf{m}_i\right)\left(\mathbf{x}_j^i-\mathbf{m}_i\right)^T,\qquad i=1,2$$

$$\boxed{\mathbf{m}_i=\frac{1}{N_i}\sum_{\mathbf{x}_j^i\in\aleph_i}\mathbf{x}_j^i,\qquad i=1,2}$$

【总类内离散度矩阵】为：$\boxed{\mathbf{S}_w=\mathbf{S}_1+\mathbf{S}_2\qquad \text{对称矩阵}}$

【类间离散度矩阵】为：$\boxed{\mathbf{S}_b=\left(\mathbf{m}_1-\mathbf{m}_2\right)\left(\mathbf{m}_1-\mathbf{m}_2\right)^T\qquad \text{对称矩阵}}$

线性分类器－Fisher线性判别

投影后(一维空间)

【均值】$\overline{m}_i = \dfrac{1}{N_i} \sum\limits_{\mathbf{x}_j^i \in \aleph_i} \mathbf{w}^T \mathbf{x}_j^i = \mathbf{w}^T \mathbf{m}_i, \qquad i = 1, 2$

【类内离散度】$\overline{s_i^2} = \sum\limits_{\mathbf{x}_j^i \in \aleph_i} \left(y_j^i - \overline{m_i} \right)^2, \qquad i = 1, 2$

【总类内离散度】$\overline{s_w} = \overline{s_1^2} + \overline{s_2^2}$

【类间离散度】$\overline{s_b} = \left(\overline{m_1} - \overline{m_2} \right)^2$

Fisher线性判别准则：

$$\max J_F(\mathbf{w}) = \frac{\overline{s_b}}{\overline{s_w}} = \frac{\left(\overline{m_1} - \overline{m_2} \right)^2}{\overline{s_1^2} + \overline{s_2^2}} = \frac{\mathbf{w}^T \mathbf{S}_b \mathbf{w}}{\mathbf{w}^T \mathbf{S}_w \mathbf{w}}$$

$$maxJ_F(w) = \frac{\overline{s_b}}{\overline{s_w}} = \frac{(\overline{m_1} - \overline{m_2})^2}{\overline{s_1^2} + \overline{s_2^2}}$$

$$\begin{aligned}
(\overline{m_1} - \overline{m_2})^2 &= (\overline{m_1} - \overline{m_2})(\overline{m_1} - \overline{m_2}) \\
&= (w^T m_1 - w^T m_2)(w^T m_1 - w^T m_2) \\
&= w^T (m_1 - m_2) w^T (m_1 - m_2) \\
&= w^T (m_1 - m_2)(m_1 - m_2)^T w \\
&= w^T S_b w
\end{aligned}$$

$$\because w^T (m_1 - m_2) = (m_1 - m_2)^T w$$

$$\overline{s_1^2} + \overline{s_2^2} =$$

$$\begin{aligned}
\overline{s_1^2} &= \sum_{x_j^i \in X_i} (y_j^1 - \overline{m_1})^2 \\
&= \sum_{x_j^i \in X_i} (w^T x_j - w^T m_i)(w^T x_j - w^T m_i) \\
&= \sum_{x_j^i \in X_i} w^T (x_j - m_1)(x_j - m_1)^T w \\
&= w^T \sum_{x_j^i \in X_i} (x_j - m_1)(x_j - m_1)^T w \\
&= w^T S_1 w
\end{aligned}$$

同理，有

$$\overline{s_2^2} = \sum_{x_j^i \in X_i} (y_j^2 - \overline{m_2})^2 = w^T S_2 w$$

线性分类器－Fisher线性判别

$$\max J_F(\mathbf{w}) = \frac{\overline{s_b}}{\overline{\overline{s_w}}} = \frac{\left(\overline{m_1} - \overline{m_2}\right)^2}{\overline{s_1^2} + \overline{s_2^2}} = \frac{\mathbf{w}^T \mathbf{S}_b \mathbf{w}}{\mathbf{w}^T \mathbf{S}_w \mathbf{w}}$$

$$\Downarrow$$

$$\max \left\{ \mathbf{w}^T \mathbf{S}_b \mathbf{w} \right\}, s.t. \mathbf{w}^T \mathbf{S}_w \mathbf{w} = c \neq 0$$ 固定分母、最大化分子

$$\nabla_{\mathbf{x}}(\mathbf{x}^T \mathbf{A} \mathbf{x}) = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$$

$$\Downarrow$$

$$L(\mathbf{w}, \lambda) = \mathbf{w}^T \mathbf{S}_b \mathbf{w} - \lambda \left(\mathbf{w}^T \mathbf{S}_w \mathbf{w} - c \right)$$

$$\Downarrow \frac{\partial L}{\partial \mathbf{w}} = 0$$

$$\mathbf{S}_b \mathbf{w}^* - \lambda \mathbf{S}_w \mathbf{w}^* = 0$$

若$\mathbf{S}_w$是非奇异的，$\boxed{\text{则}\mathbf{S}_w^{-1} \mathbf{S}_b \mathbf{w}^* = \lambda \mathbf{w}^*}$

线性分类器 –Fisher线性判别

$$\mathbf{S}_w^{-1}\mathbf{S}_b\mathbf{w}^* = \lambda\mathbf{w}^*$$

$$\Downarrow$$

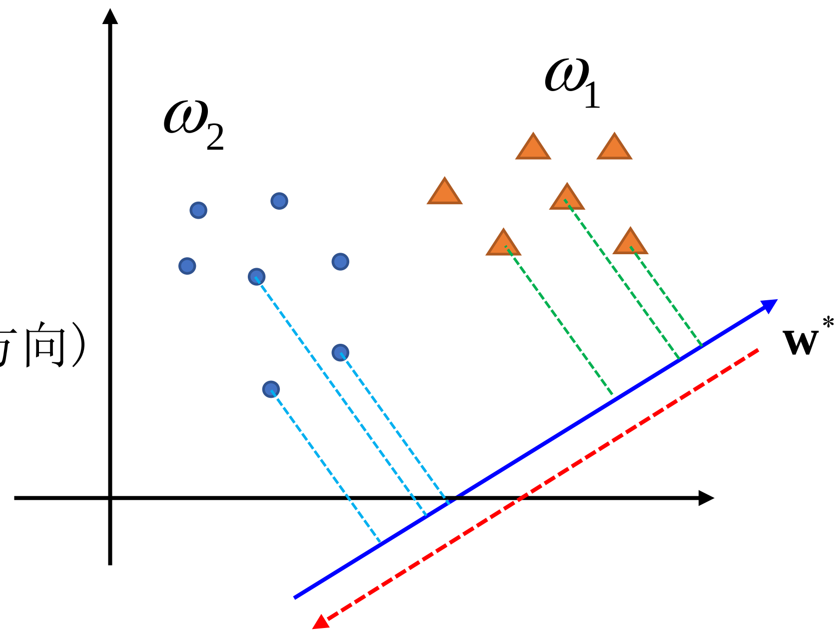
$$\lambda\mathbf{w}^* = \mathbf{S}_w^{-1}(\mathbf{m}_1-\mathbf{m}_2)\boxed{(\mathbf{m}_1-\mathbf{m}_2)^T\mathbf{w}^*}$$

$$\Downarrow$$

$$\lambda\mathbf{w}^* = \mathbf{S}_w^{-1}(\mathbf{m}_1-\mathbf{m}_2)\cdot r(\text{标量，不影响}\mathbf{w}^*\text{的方向})$$

$$\Downarrow$$

$$\mathbf{w}^* = \frac{r}{\lambda}\mathbf{S}_w^{-1}(\mathbf{m}_1-\mathbf{m}_2)$$



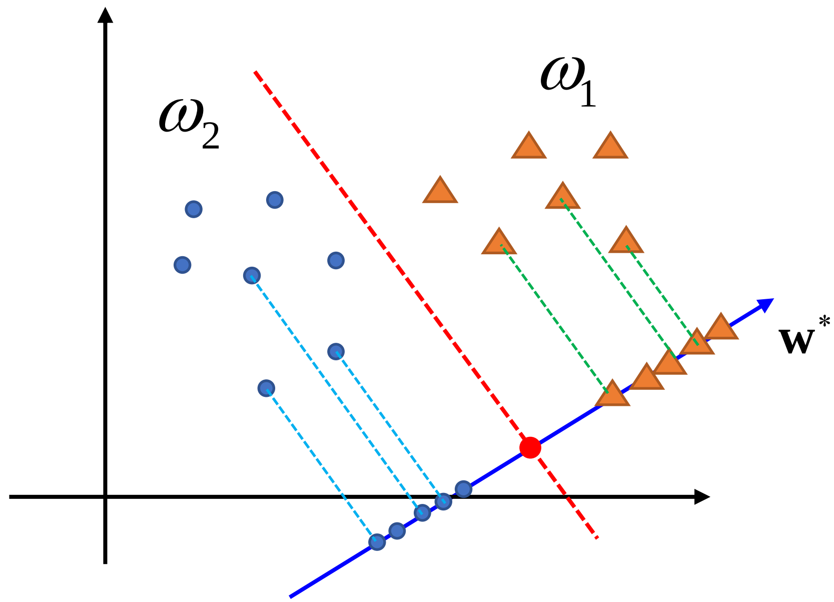
忽略常数的影响，常取$\mathbf{w}^* = \mathbf{S}_w^{-1}(\mathbf{m}_1-\mathbf{m}_2)$

Q: Fisher判别给出了最优投影方向，唯一吗？

根据特征向量的求解，应该有多个

Q: Fisher判别给出了最优投影方向，怎么求分类面呢？

线性分类器 –Fisher线性判别



分类准则：判断位于d 的哪一侧

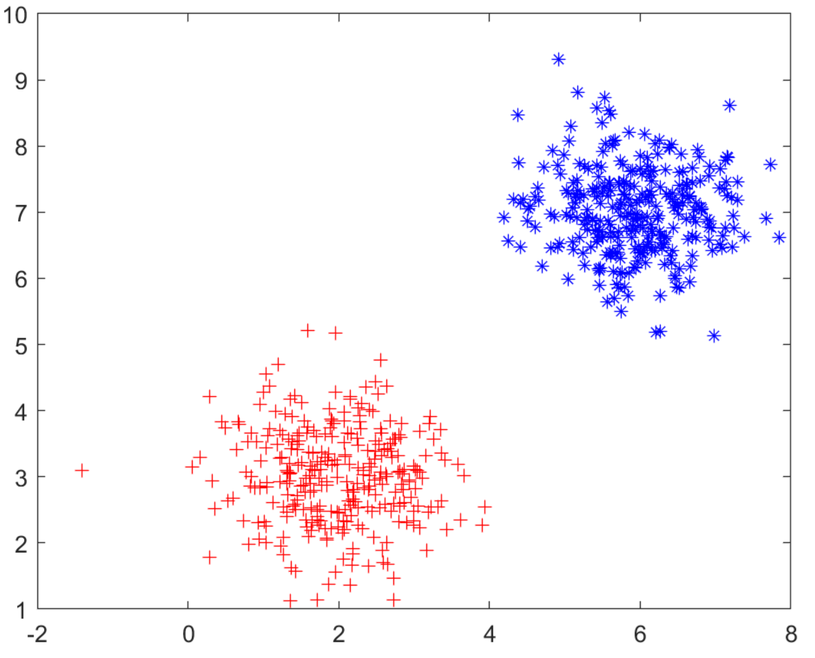
$$d = \frac{\overline{m_1} + \overline{m_2}}{2}$$

⇒ 分界面方程: $\left(\mathbf{w}^*\right)^T \mathbf{x} - d = 0$

线性分类器 –Fisher线性判别

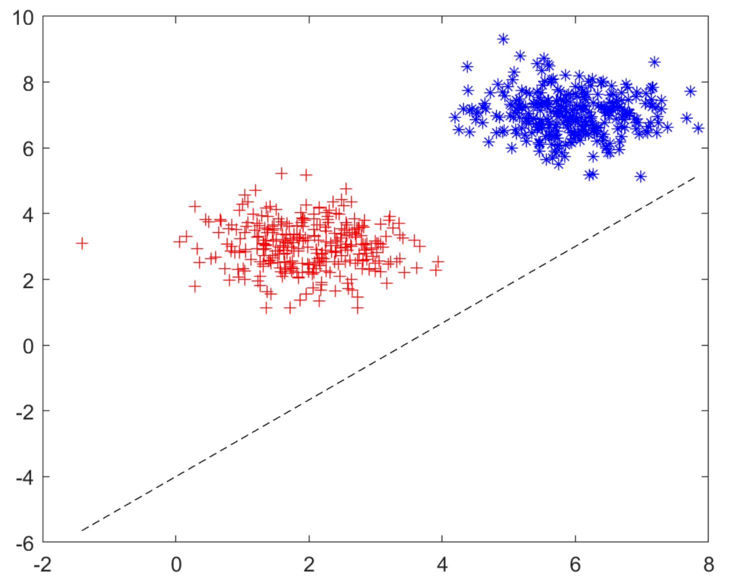
```
clc;
close all;
clear all;
%% 生成数据
randn('seed',2020);
mu1 = [2 3];
sigma1 = [0.5 0;
          0 0.5];
data1 = mvnrnd(mu1,sigma1,300);

randn('seed',2021);
mu2 = [6 7];
sigma2 = [0.5 0;
          0 0.5];
data2 = mvnrnd(mu2,sigma2,300);
```



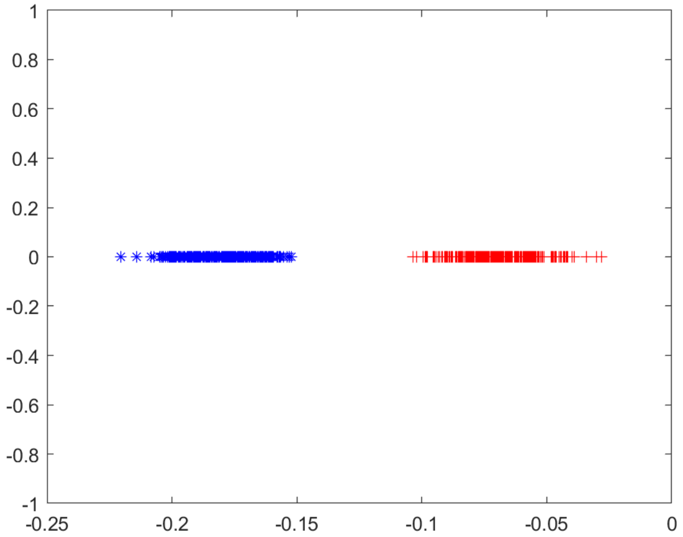
线性分类器 –Fisher线性判别

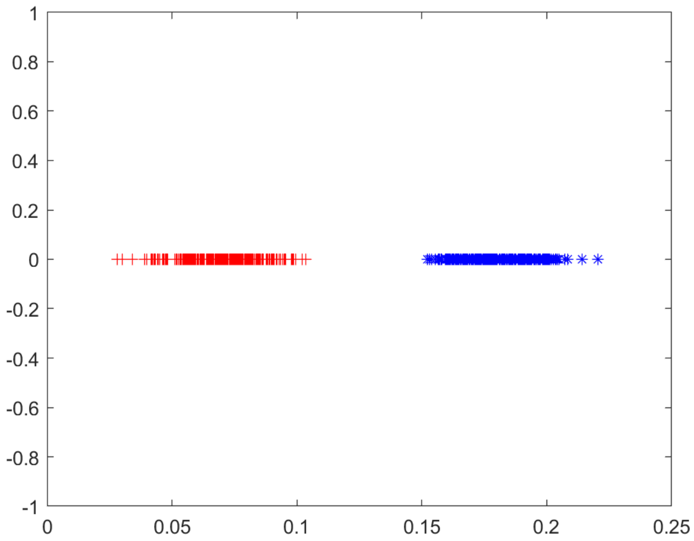
```
mu_1 = mean(data1,1);
mu_2 = mean(data2,1);
tmp = data1 - repmat(mu_1,[size(data1,1),1]);
S1 = tmp'*tmp;
tmp = data2 - repmat(mu_2,[size(data2,1),1]);
S2 = tmp'*tmp;
Sw = S1 + S2;
% w_star = inv(Sw)*(mu_1-mu_2)';
w_star = Sw\(mu_1-mu_2)';
%% Plot w_star
Data = [data1;data2];
[xmin, ymin] = min(Data,[],1);
[xmax, ymax] = max(Data,[],1);
X = xmin:0.1:xmax;
k = w_star(2)/(w_star(1)+eps);
plot(X,k*X-4,'k--');
```



线性分类器 –Fisher线性判别

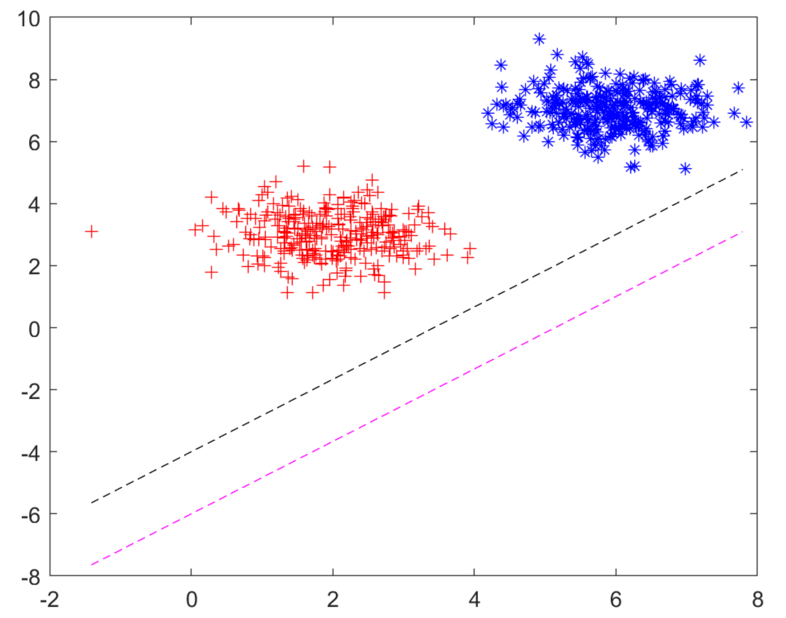
```
%%
w_star = -w_star;
y1 = data1*w_star;
y2 = data2*w_star;
% plot((y1+4)/k,y1,'r+');hold on;
% plot((y2+4)/k,y2,'b*');
figure,plot(y1,zeros(length(y1)),'r+');hold on;
plot(y2,zeros(length(y2)),'b*');hold on;
```





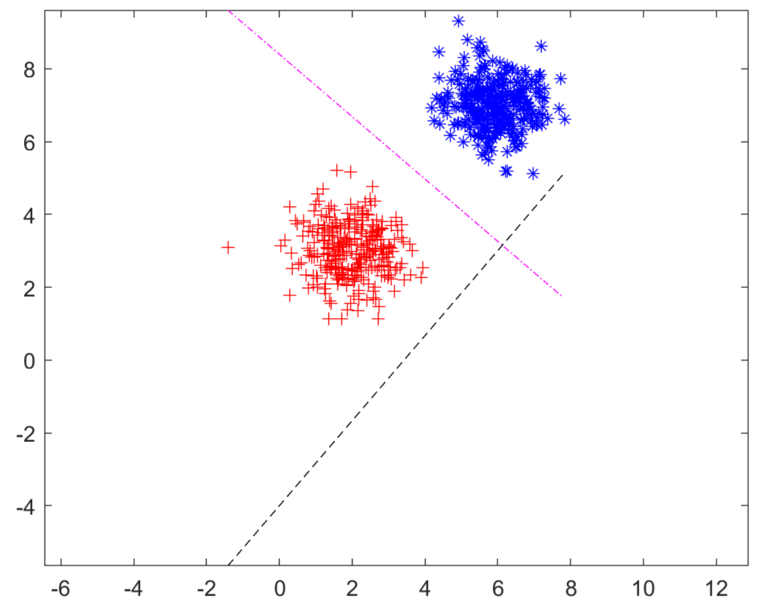
线性分类器 –Fisher线性判别

```
%% Sw-1Sb特征值分解
Sb = (mu_1-mu_2)'*(mu_1-mu_2);
Tmp = inv(Sw)*Sb;
[V,D] = eig(Tmp);
D = diag(D);
[~,ind] = max(D);
v = V(:,ind);
k1 = v(2)/(v(1)+eps);
figure,plot(data1(:,1),data1(:,2),'r+');hold on;
plot(data2(:,1),data2(:,2),'b*');hold on;
plot(X,k*X-4,'k--');
plot(X,k1*X-6,'m--');
```



线性分类器 –Fisher线性判别

```
%% G(x)=w'x+w0
mu_y1 = mean(y1);
mu_y2 = mean(y2);
d = (mu_y1+mu_y2)/2;
w0 = -d;
figure,plot(data1(:,1),data1(:,2),'r+');hold on;
plot(data2(:,1),data2(:,2),'b*');hold on;
plot(X,k*X-4,'k--');hold on;
Y = (-w_star(1)*X-w0)/(w_star(2));
plot(X,Y,'m-.');
axis equal;
```



线性分类器－Fisher线性判别

线性分类器 –Fisher线性判别