# CP 312
## Assignment 1 Solutions

E. Zima

October 6, 2018

1. [2 marks] Using the definition of $\Theta$-notation, prove that $(13n+3)(9n+1)(\log(4n^2+100)) \in \Theta(n^2 \log n)$.

   (Solution 1)

   We have

$$\begin{array}{ccccccccl} 0 & \leq & 13n & \leq & 13n+3 & \leq & 14n & & \text{for } n \geq 3 \\ 0 & \leq & 9n & \leq & 9n+1 & \leq & 10n & & \text{for } n \geq 1. \end{array}$$

   Multiplying these two inequalities together,

$$0 \leq 117n^2 \leq (13n+3)(9n+1) \leq 140n^2 \qquad \text{for } n \geq 3,$$

   (this is valid as all parts of inequalities are non-negative).

   Now, $2\log n = \log n^2 < (\log(4n^2+100)) \leq \log(5n^2)$ (for $n \geq 10$) $= 2\log n + \log 5 \leq 3\log n$ (for $n \geq 5$).

   Thus,

$$2\log n \leq \log(4n^2+100) \leq 3\log n, \quad \text{for } n \geq 10. \qquad (1)$$

   Define $c_1 = 234$, $c_2 = 420$ and $n_0 = 10$; then

$$0 \leq c_1 n^2 \log n \leq (13n+3)(9n+1)\log(4n^2+100) \leq c_2 n^2 \log n \qquad \text{for } n \geq n_0.$$

   This proves the desired result.

   (Solution 2)

   Expand $(13n+3)(9n+1)$ as $117n^2+40n+3$. Note, that $117n^2+40n+3 \leq 117n^2+40n^2+3n^2 = 160n^2$ for $n \geq 1$. Note also, that $117n^2+40n+3 \geq 117n^2$ for $n \geq 0$. Thus

$$117n^2 \leq (13n+3)(9n+1) \leq 160n^2, \text{ for } n \geq 1.$$

   Combining with (1) define $c_1 = 234$, $c_2 = 480$ and $n_0 = 10$; this gives the desired result.

2. (a) We compute $\lim_{n\to\infty} \frac{f(n)}{g(n)}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{1052n^3 + 10n^2 + 1001}{n^4/5000000 + 2n} \\
&= \lim_{n\to\infty} \frac{1052/n + 10/n^2 + 1001/n^4}{1/5000000 + 2/n^3} \\
&= \lim_{n\to\infty} \frac{0}{1/5000000} \\
&= 0.
\end{aligned}$$

**Hence,** $f(n) \in o(g(n))$.

(b) $g(n) = \sqrt{\sqrt{n}} = n^{\frac{1}{4}}$.

Observe that $f(n) = \log^3(n^{10}) \leq (10^3)\log^3 n = h(n)$.

Now, $h(n) \in o(g(n))$ (see basic facts in the assignment description), and $f(n) \leq h(n)$ for all $n \geq 2$.

**Hence,** $f(n) \in o(g(n))$.

(c) Observe that $g(n) = 2n^4 \log n^{2004} = 4008n^4 \log n = 4008f(n)$.

**Hence,** $f(n) \in \Theta(g(n))$.

(d) Note that

$$f(n) = 16^{\log\sqrt{n}} = 2^{4\log\sqrt{n}} = 2^{\log n^2} = n^2 = g(n).$$

**Hence,** $f(n) \in \Theta(g(n))$.

(e) Note that

$$\sin\frac{n\pi}{2} = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence,

$$f(n) = \begin{cases} n^3 & \text{if } n \equiv 1 \pmod 4 \\ n & \text{if } n \equiv 3 \pmod 4 \\ n^2 & \text{if } n \text{ is even.} \end{cases}$$

Since $3 > 5/3$, $f(n)$ is not $O(g(n))$. Since $5/3 > 1$, $f(n)$ is not $\Omega(g(n))$.

None of the symbols can be used here as remaining symbols are stronger (for example, $f(n)$ is not $o(g(n))$ because if it is then $f(n) \in O(g(n))$ which we have show is not possible).

3. (a) No, it is not true.
   Consider $f_1(n) = 3n \in \Theta(n)$ and $f_2(n) = 2n \in \Theta(n)$. However $f_1(n) - f_2(n) = n \in \Omega(n) \notin O(1)$.

   (b) No, it is not true.
   Consider $f_1(n) = n$, $g_1(n) = n^2$, $f_2(n) = n$, $g_2(n) = n^2$. Then $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, however $f_1(n)f_2(n) = n^2 \in O(n^4)$ but $\notin \Theta(n^4)$.

   (c) It is true
   $\exists\ c_1 > 0, c_2 > 0, n_1 \geq 0$ such that $c_1 g(n) \leq f_1(n) \leq c_2 g(n), \forall n \geq n_1$;
   $\exists\ d_1 > 0, d_2 > 0, n_2 \geq 0$ such that $d_1 g(n) \leq f_2(n) \leq d_2 g(n), \forall n \geq n_2$;
   From the last inequality it follows (as all parts of it are positive), that

   $$\frac{1}{d_2}\frac{1}{g(n)} \leq \frac{1}{f_2(n)} \leq \frac{1}{d_1}\frac{1}{g(n)}, \forall n \geq n_2$$

   Multiplying left-hand side, middle and right-hand side parts of the last and first inequalities:
   $\frac{c_1}{d_2} \leq \frac{f_1(n)}{f_2(n)} \leq \frac{c_2}{d_1}$, $\forall n \geq \max\{n_1, n_2\}$.
   From here it follows (by definition) that $f_1(n)/f_2(n) \in \Theta(1)$.

   (d) No, it is not true.
   Consider $f(n) = n^3$ and $g(n) = n^2$. Then $\log(f(n)) \in O(\log(g(n)))$, as $\log(f(n)) = 3\log n$ and $\log(g(n)) = 2\log n$. However $n^3 \notin O(n^2)$.

4. Give a tight $\Theta$-bound on the running time ...

   The time required to execute the inner `for` loop (on $j$) is $ni - i + 1 \in \Theta(n \cdot i)$. The time to execute the outer `for` loop (on $i$) is

   $$\sum_{i=1}^{n} \Theta(n \cdot i) = \Theta\left(\sum_{i=1}^{n} n \cdot i\right) = \Theta\left(n \sum_{i=1}^{n} i\right) = \Theta\left(n \cdot 1/2 \cdot n \cdot (n+1)\right) = \Theta(n^3).$$

   Running time of the algorithm is $\Theta(n^3)$.

5. Give a tight $\Theta$-bound on the running time ...

   Complexity of the first line is $\Theta(1)$.

   The time required to execute the inner `for` loop (on $j$) is $\Theta(\lceil \log i \rceil)$. The time to execute the outer `for` loop (on $i$) is

   $$\sum_{i=1}^{k} \Theta(\lceil \log i \rceil) = \Theta\left(\sum_{i=1}^{k} \lceil \log i \rceil\right).$$

   $$\sum_{i=1}^{k} \lceil \log i \rceil \leq \sum_{i=1}^{k} (\log i + 1) \leq k \log k + k =$$
   $$= k \log k + k \in O(k \log k).$$

$$\sum_{i=1}^{k} \lceil \log i \rceil \geq \sum_{i=1}^{k} \log i \geq \sum_{i=k/2+1}^{k} \log i \geq$$

$$\geq \sum_{i=k/2+1}^{k} \log \frac{k}{2} \geq \frac{k}{2} \log \frac{k}{2} \in \Omega(k \log k).$$

Thus, the time to execute the double loop is $\Theta(k \log k)$.

The number of iterations of the second loop on $i$ is $22k$. Complexity of this loop is $\Theta(k)$.

Since $1 \in o(k \log k)$, $k \in o(k \log k)$, and $k = n$ the running time of the algorithm is $\Theta(n \log n)$.

Another possible analysis of the double loop is: using $\lceil \log i \rceil \in \Theta(\log i)$ and $\Theta(\Theta(f(i))) = \Theta(f(i))$, write

$$\sum_{i=1}^{k} \Theta(\lceil \log i \rceil) = \Theta\left(\sum_{i=1}^{k} \log i\right) = \Theta(\log k!) = \Theta(k \log k).$$

FYI only!

not for distribution!