meng 21200 pset 1

init

In [1]:
```
import math
import matplotlib.pyplot as plt
import numpy as np
from scipy.optimize import brentq
import pandas as pd
```

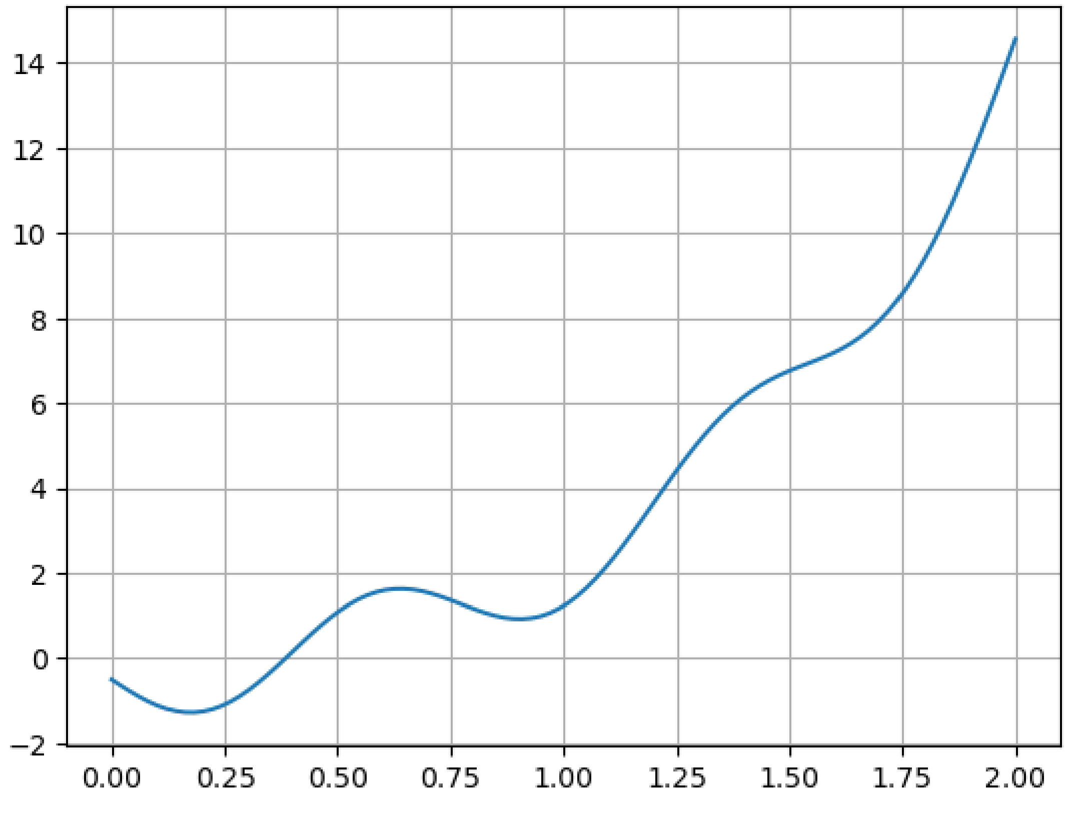
problem 1

a

In [2]:
```
# defining function
def f(x):
    return x * math.exp(x) - math.sin(8 * x) - 0.5
```

b

In [3]:
```
# plotting function
x = np.linspace(0, 2, num=100)
y = [f(i) for i in x]
plt.plot(x, y)
plt.grid()
```



In [4]:
```
# finding roots
# obvious from graph that only one root exists
trueValue = brentq(f, 0, 2)
trueValue
```

Out[4]:
```
0.3845760112195518
```

c

In [5]:
```
# defining bisection function
def bisection(f, x1, x2, tol):
    error = 100
    i = 0
    itLimit = 1000
    while error > tol and i < itLimit:
        i += 1
        approx = (x1 + x2) / 2
        if f(x1) * f(approx) < 0:
            x2 = approx
        else:
            x1 = approx
        error = (x2 - x1) / 2
    if i < itLimit:
        return approx
```

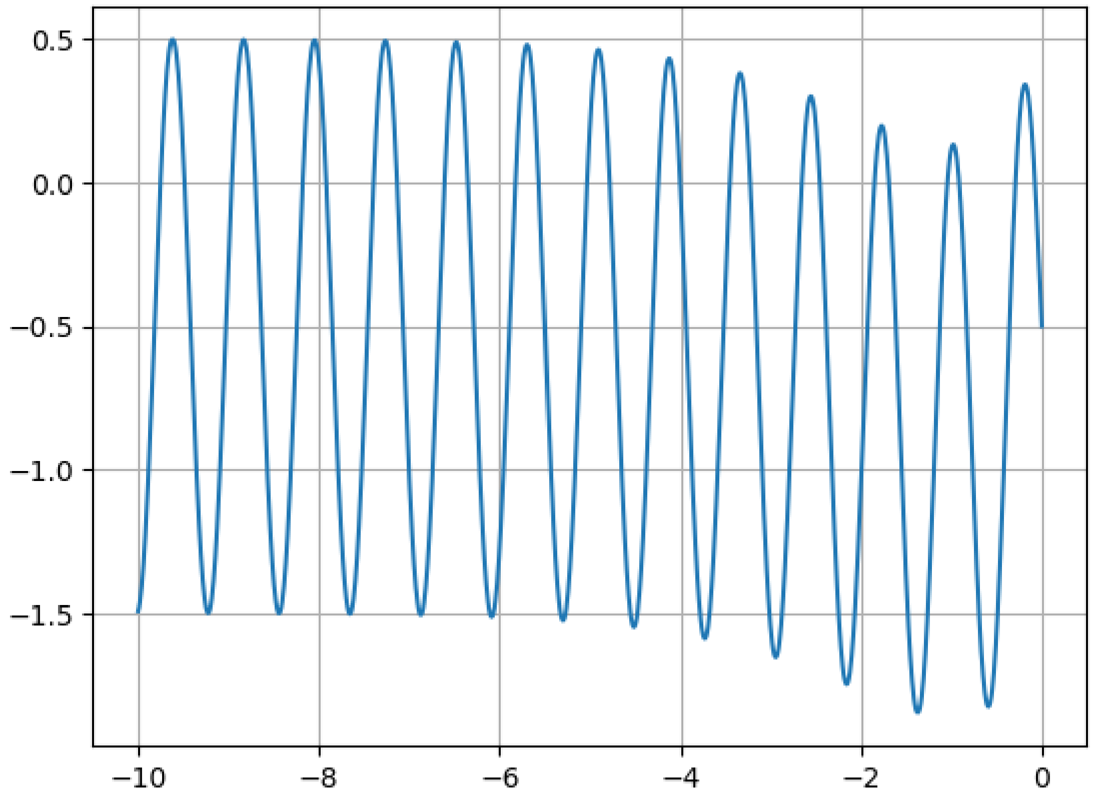
d

In [6]:
```
bisection(f, 0, 2, 0.0001)
```

Out[6]:
```
0.3846435546875
```

e

In [7]:
```
x = np.linspace(-10, 0, num=1000)
y = [f(i) for i in x]
plt.plot(x, y)
plt.grid()
```



```
In [8]: bisection(f, -10, 0, 0.0001)
```

```
Out[8]: -5.822601318359375
```

```
In [9]: bisection(f, -10, -8, 0.0001)
```

```
Out[9]: -8.1807861328125
```

```
In [10]: bisection(f, -8.8, -8.6, 0.0001)
```

```
Out[10]: -8.7048828125
```

Got different results, because the program was searching in a different interval each time. Distinct roots exist within these intervals.

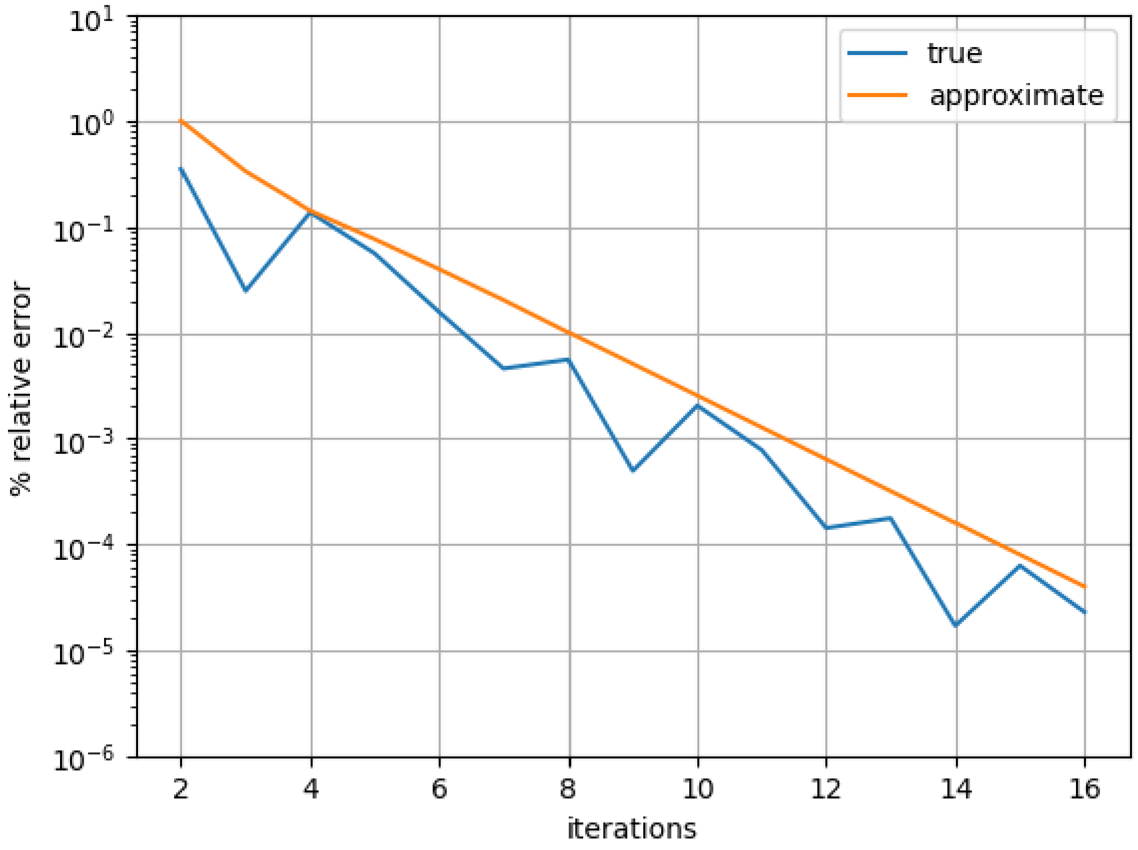
problem 2

```
In [11]: def bisectionErrorPlot(f, x1, x2, tol):
             error = 100
             tpre = []
             apre = []
             i = 0
             papprox = 0
             while error > tol:
                 i += 1
                 approx = (x1 + x2) / 2
                 if f(x1) * f(approx) < 0:
                     x2 = approx
                 else:
                     x1 = approx
```

```
        error = (x2 - x1) / 2
        tpre.append(abs((trueValue - approx) / trueValue))
        apre.append(abs((papprox - approx) / approx))
        papprox = approx

    itNum = [j for j in range(2, i)]
    plt.ylabel('% relative error')
    plt.xlabel('iterations')
    plt.yscale("log")
    plt.ylim([0.000001, 10])
    plt.plot(itNum, tpre[2:], label="true")
    plt.plot(itNum, apre[2:], label="approximate")
    plt.legend(loc="upper right")
    plt.grid()
    return approx
bisectionErrorPlot(f, 0, 2, 0.00001)
```

Out[11]:
```
0.3845672607421875
```



problem 3

a

In [12]:
```
# interpreting the problem statement as: subintervals is the size of each subir
def BisectMany(f, x1, x2, tol, subintervals):
    num = math.trunc((x2 - x1) / subintervals)
    roots = []
    for i in range(num + 1):
        lower = x1 + subintervals * i
        upper = x1 + subintervals * (i + 1)
        if f(lower) * f(upper) > 0:
            continue

        # the upper bound of an interval cannot exceed x2
        if upper > x2:
            upper = x2
        # if the last interval only contains x2, end the search
        if lower == x2:
            break
        roots.append(bisection(f, lower, upper, tol))
        # using pandas here to help format display
        df = pd.DataFrame(roots, columns=['roots'])
    return df
```

b

In [13]:
```
BisectMany(f, -10, 2, 0.00001, 2)
```

Out[13]:

	roots
0	-8.180893
1	-7.919846
2	0.384567

In [14]:
```
BisectMany(f, -10, 2, 0.00001, 1)
```

Out[14]:

	roots
0	-8.180893
1	-7.919846
2	-5.034866
3	-4.783707
4	-1.046829
5	-0.916428
6	0.384567

In [15]:
```
BisectMany(f, -10, 2, 0.00001, 0.1)
```

Out[15]:

	roots
0	-9.751941
1	-9.490320
2	-8.966467
3	-8.705042
4	-8.180896
5	-7.919836
6	-7.395178
7	-7.134851
8	-6.609143
9	-6.350232
10	-5.822522
11	-5.566321
12	-5.034851
13	-4.783704
14	-4.245276
15	-4.003284
16	-3.452380
17	-3.226477
18	-2.654260
19	-2.455017
20	-1.849451
21	-1.689221
22	-1.046838
23	-0.916418
24	-0.292517
25	-0.075867
26	0.384583

In [16]:

```
BisectMany(f, -10, 2, 0.00001, 0.01)
```

Out [16]:

	roots
0	-9.751934
1	-9.490332
2	-8.966465
3	-8.705020
4	-8.180918
5	-7.919863
6	-7.395176
7	-7.134863
8	-6.609160
9	-6.350254
10	-5.822520
11	-5.566348
12	-5.034863
13	-4.783691
14	-4.245254
15	-4.003262
16	-3.452402
17	-3.226465
18	-2.654277
19	-2.455020
20	-1.849434
21	-1.689238
22	-1.046816
23	-0.916426
24	-0.292520
25	-0.075879
26	0.384590

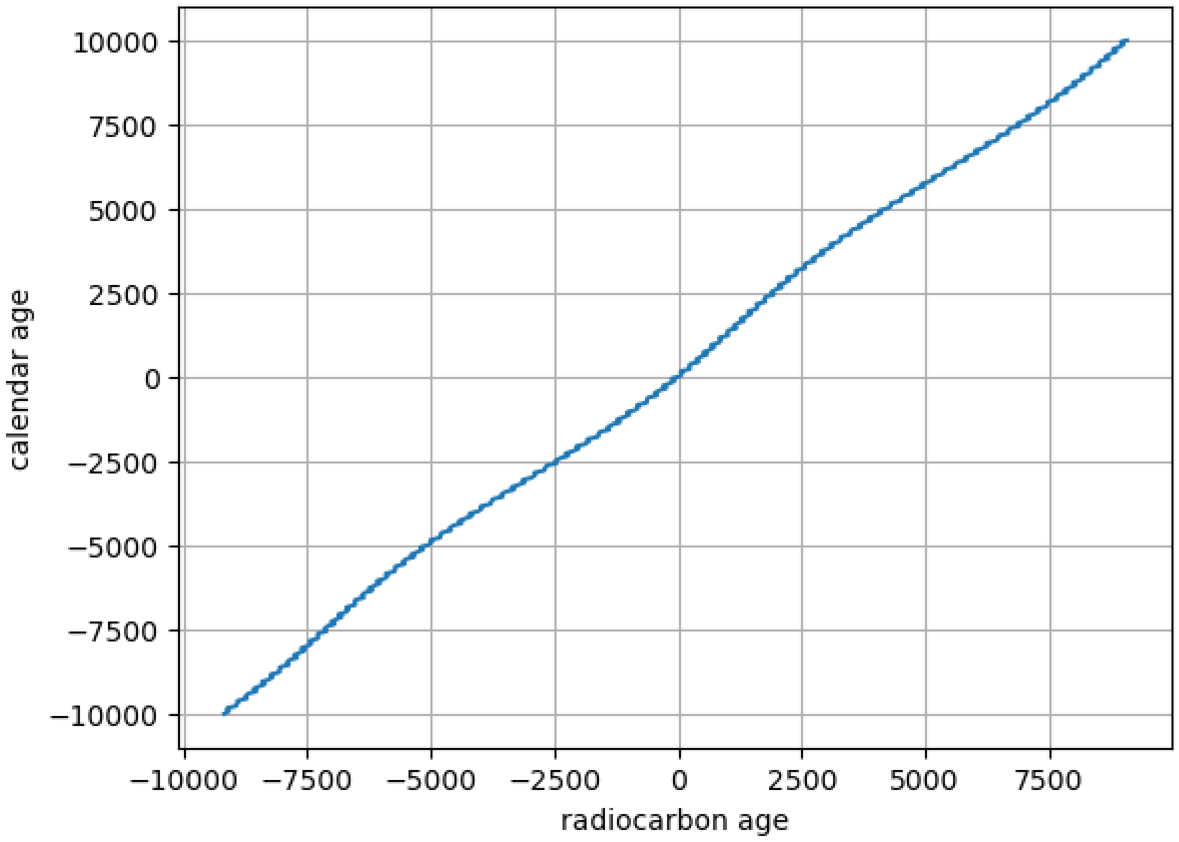
Decreasing the size of the subintervals increases the number of "roots" found, but from the graph, it is obvious that some roots found are far off from actual roots of the function.

problem 4

In [17]:

```
# g is a function representing the different between calendar age, x, and radic
# radiocarbon age is subtracted from both sides of the given equation
def g(x):
    return 0.92 * x - 4758 + \
           250 * math.cos((x + 700) * math.pi / 4500) + \
           50 * math.sin(x * math.pi / 100) + \
           20 * math.cos(x * 2 * math.pi / 11)

r = np.linspace(-10000, 10000, num=1000)
c = [g(i) + 4518 for i in r]
plt.xlabel('radiocarbon age')
plt.ylabel('calendar age')
plt.plot(c, r)
plt.grid()
```



from the graph, we can see that for a radiocarbon age of 4518 years, the calendar age will be between 5000 and 7500 years. we will search for roots in this interval.

In [18]:
```
a = BisectMany(g, 5000, 7500, 0.001, 0.1)
roots = a['roots'].values.tolist()
# filter out roots found that aren't actually roots, i.e. produce an absolute e
# note that the answer output will change depending on the parameters (tol and
ans = []
for i in roots:
    if abs(g(i)) < 0.01:
        ans.append(i)
print(f"the range is {round(min(ans), 2)} to {round(max(ans), 2)}")
```

```
the range is 5248.62 to 5363.65
```

problem 5

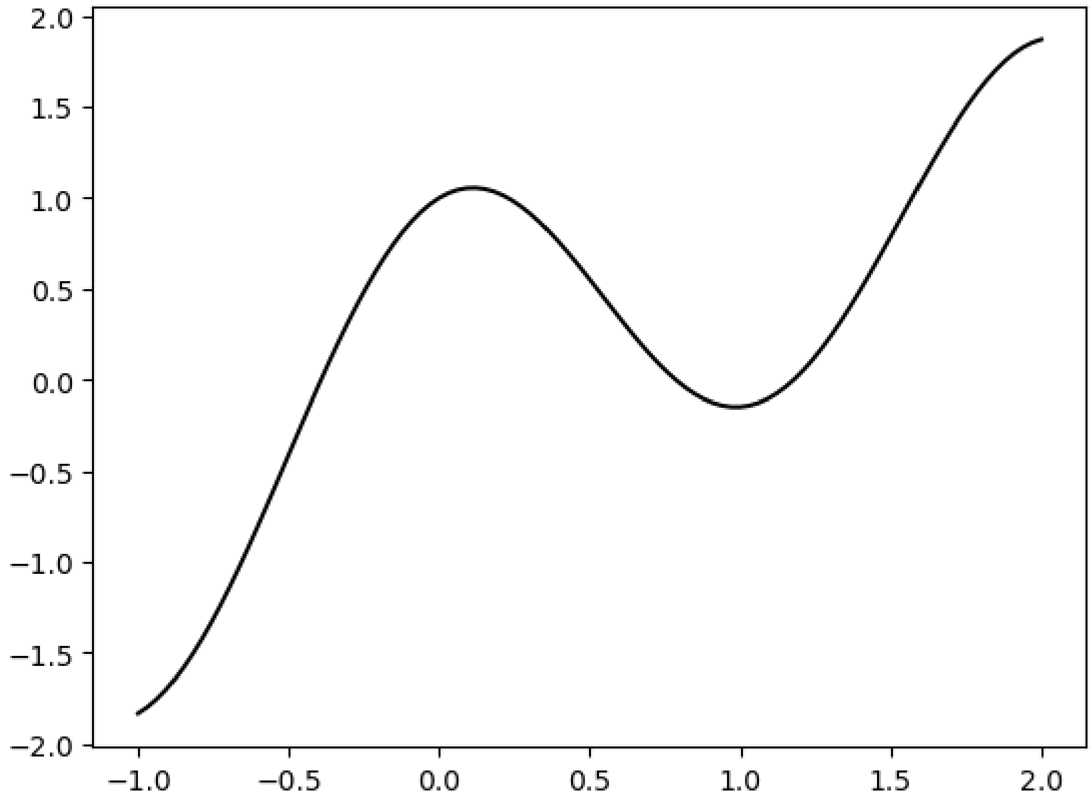
note: parts a and b are from the discussion notebook

a

In [19]:
```
def g(x):
    return np.sin(x) + np.cos(3 * x)

x = np.linspace(-1, 2, 100)
plt.plot(x, g(x),'k')
```

Out[19]:
```
[<matplotlib.lines.Line2D at 0x7fbbb0c63bb0>]
```



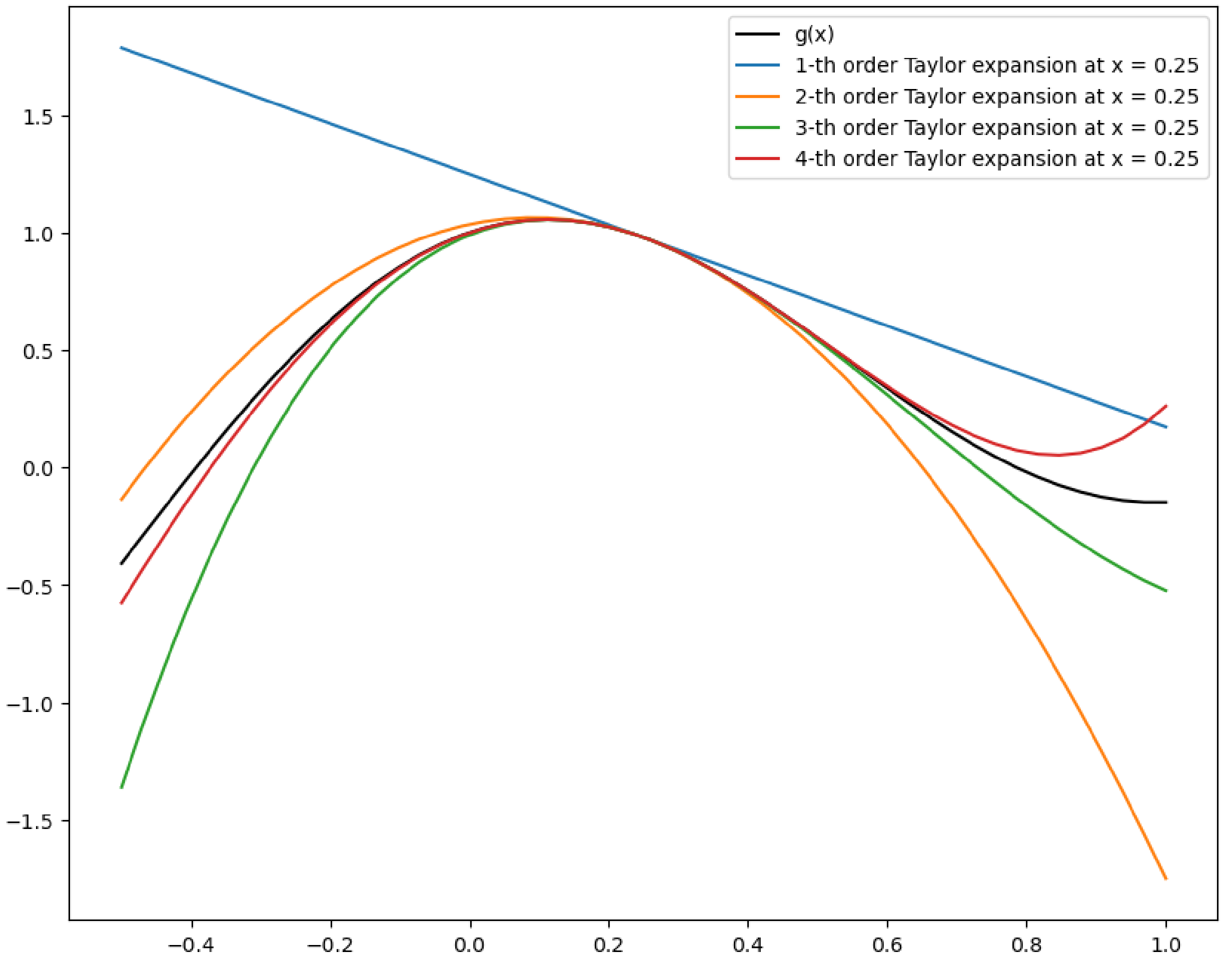
b

In [20]:
```
def g_prime_n(x,n):
    """
    n-th order derivative of g(x) = sin(x)+cos(3x)
    the derivative of sin(x) or cos(x) is the same every 4 order.
    """
    n = int(n)
    assert n >= 0
    if n%4 == 0 :
        return np.sin(x)+3**n*np.cos(3*x)
    if n%4 == 1:
        return np.cos(x)+3**n*(-np.sin(3*x))
    if n%4 == 2:
        return -np.sin(x)+3**n*(-np.cos(3*x))
    if n%4 == 3:
        return -np.cos(x)+3**n*(np.sin(3*x))

def taylor_g(x, x0, n):
    """ n is the order, x0 is the center of the taylor expansion."""
    result = 0
    for k in range(n+1):
        result += (x-x0)**k/np.math.factorial(k)*g_prime_n(x0,k)
    return result
```

In [21]:
```
x  = np.linspace(-0.5, 1)
x0 = 0.25
fig, ax = plt.subplots(figsize = (10, 8))
ax.plot(x, g(x), 'k', label = 'g(x)')
for n in range(1, 4 + 1):
    ax.plot(x, taylor_g(x, x0, n), label = f'{n}-th order Taylor expansion at x
ax.legend(loc='best')
```

Out[21]:
```
<matplotlib.legend.Legend at 0x7fbbb0cbe6d0>
```



c

In [22]:
```
print("true percent relative errors of Taylor approximations:")
for n in range(1, 4 + 1):
    error = abs((g(0.1) - taylor_g(0.1, x0, n)) / g(0.1))
    print(f"{n}-th order: {round(error * 100, 3)}%")
```

```
true percent relative errors of Taylor approximations:
1-th order: 8.086%
2-th order: 0.801%
3-th order: 0.128%
4-th order: 0.009%
```

order 4 approximation is required to acheive true relative error < 0.1%

d

In [23]:
```
error = 1
n = 1
best = taylor_g(0.1, x0, 1)
while error >= 0.01:
    n += 1
    current = taylor_g(0.1, x0, n)
    error = abs((best - current) / best)
    best = current
print(f"order {n} approximation is required to acheive true relative error < 1%
```

```
order 3 approximation is required to acheive true relative error < 1%
```

problem 6

a

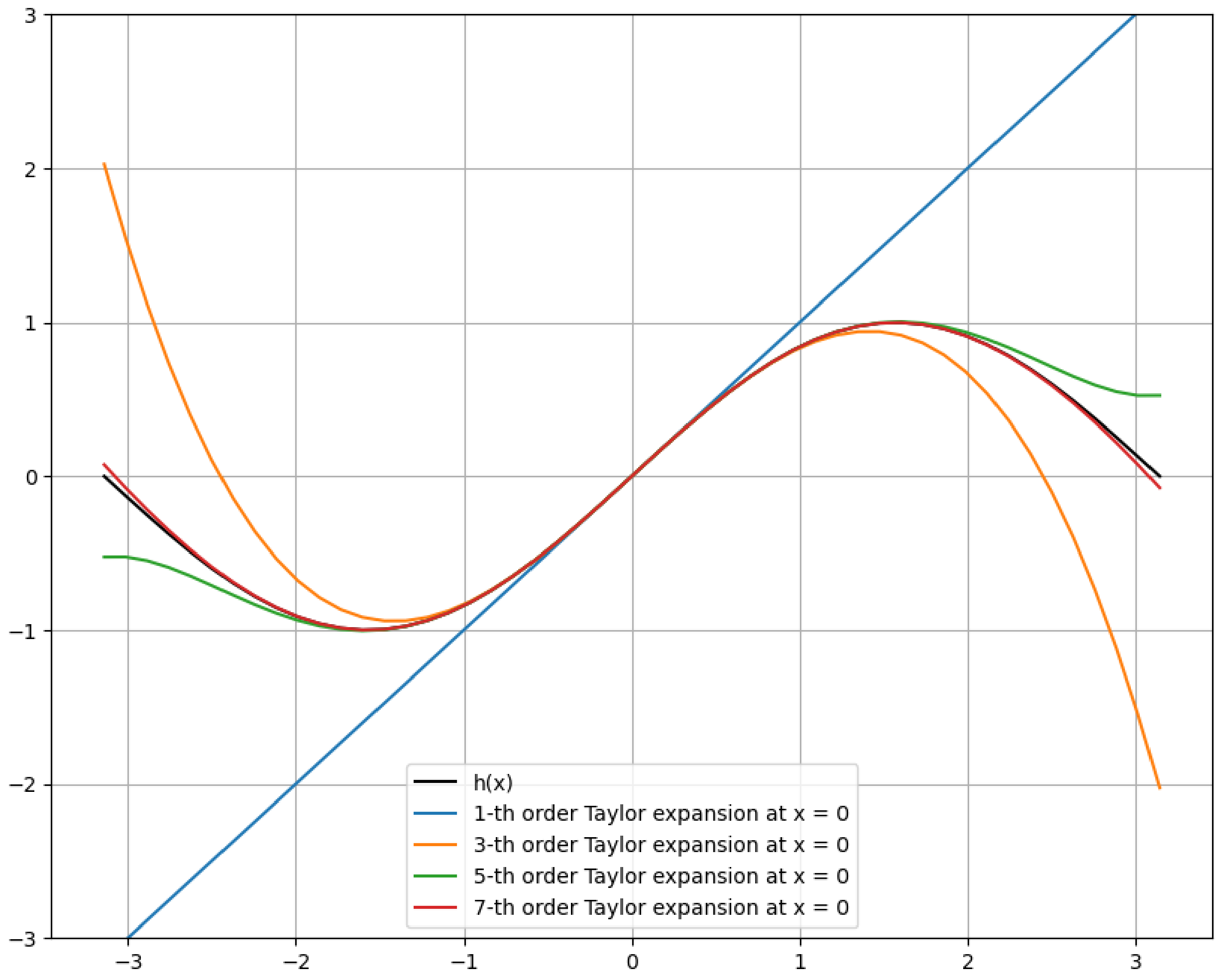
In [24]:
```
def h(x):
    return np.sin(x)
def h_prime_n(x, n):
    """
    n-th order derivative of h(x) = sin(x)
    the derivative of sin(x) is the same every 4 order.
    """
    n = int(n)
    assert n >= 0
    if n%4 == 0 :
        return np.sin(x)
    if n%4 == 1:
        return np.cos(x)
    if n%4 == 2:
        return -np.sin(x)
    if n%4 == 3:
        return -np.cos(x)

def taylor_h(x, x0, n):
    """ n is the order, x0 is the center of the taylor expansion."""
    result = 0
    for k in range(n+1):
        result += (x-x0)**k/np.math.factorial(k)*h_prime_n(x0,k)
    return result
```

b

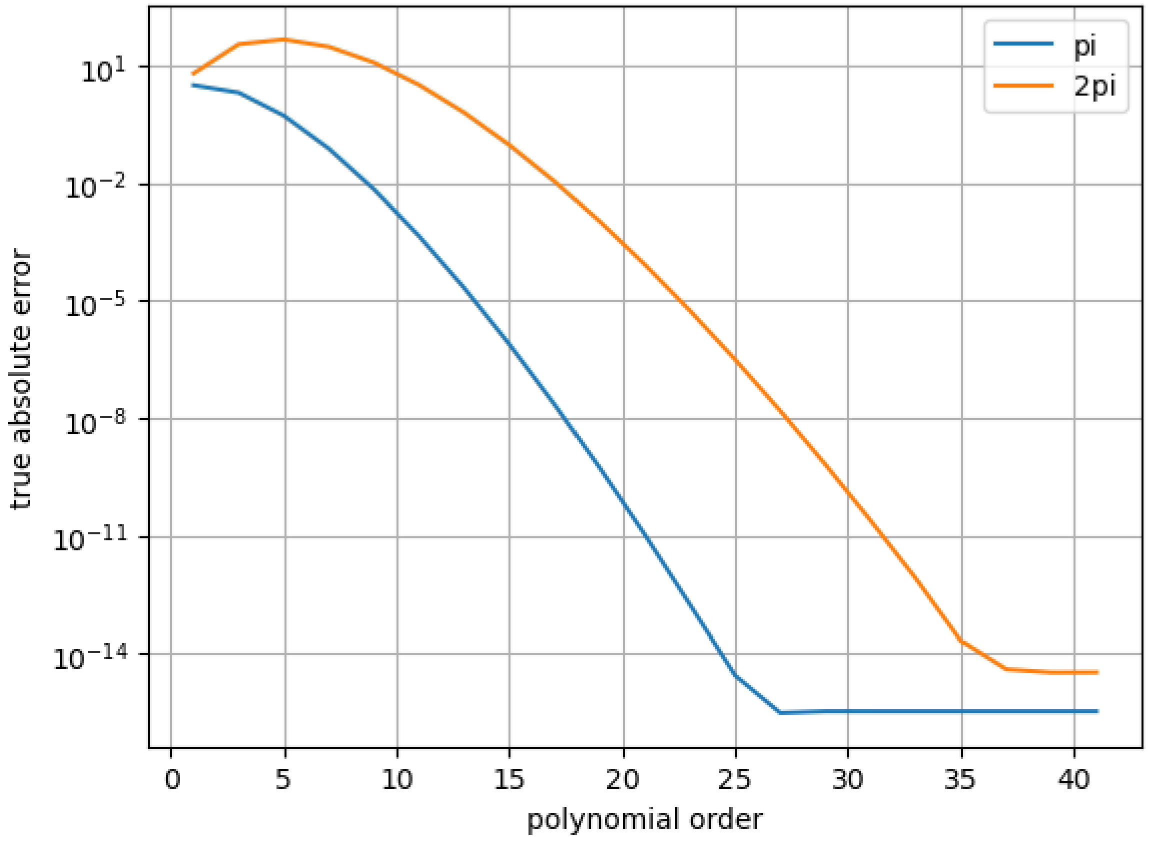
In [25]:
```
x  = np.linspace(-math.pi, math.pi)
# by convention, the Taylor series for sine is centered at the origin
x0 = 0
fig, ax = plt.subplots(figsize = (10, 8))
plt.ylim([-3, 3])
plt.grid()
ax.plot(x, h(x), 'k', label = 'h(x)')
for m in range(4):
    n = 2 * m + 1
    ax.plot(x, taylor_h(x, x0, n), label = f'{n}-th order Taylor expansion at x
ax.legend(loc='best')
```

Out[25]:
```
<matplotlib.legend.Legend at 0x7fbbb0b2bbe0>
```



c, d

```
In [26]: # note that sin(pi) = 0
order = []
error1 = []
error2 = []
for m in range(0, 21):
    n = 2 * m + 1
    order.append(n)
    error1.append(abs(taylor_h(math.pi, x0, n)))
    error2.append(abs(taylor_h(2 * math.pi, x0, n)))
plt.ylabel('true absolute error')
plt.xlabel('polynomial order')
plt.yscale("log")
plt.plot(order, error1, label='pi')
plt.plot(order, error2, label='2pi')
plt.legend(loc='best')
plt.grid()
```



analytically, the radius of convergence of the Taylor series of sin is infinite, and there should be no limit to the precision to which we can approximate sin(x) at pi. however, python's double float class can only hold ~15 decimal places, which limits the precision we can achieve computationally because of truncation error. this is shown in the graph, which begins to flatten around 10^-15.

the Taylor polynomials are more accurate at pi. this is because the closer to the center of the approximation, or the origin in our case, the faster the Taylor series converges.