MENG 21200 - Principles of Engineering Analysis II
Winter 2023

Molecular Engineering
University of Chicago

Problem Set 5

ASSIGNED: February 3, 2023
DUE: **Sunday, February 12, 2023** at 1 pm.

Problem set guidelines:

1. Each solution must be your own work.
2. All problems that involve python must be completed in Jupyter Notebooks**.** Some problems may not require python and may be better completed with pen and paper.
3. Highlight your final answer when providing numerical results. Provide plots, graphs, and tables of your results when appropriate.
4. **Submission Instructions:** This quarter we will use Gradescope to collect your submissions and grade them. We will only accept a single PDF file with your compiled solutions. Please follow the instructions below to obtain a PDF file from a Jupyter Notebook:
   i. Make sure all your code runs without error, and all figures (if any) show up where intended. We will not be running your code, therefore it is essential that your solutions output and highlight your results. Please be mindful of your line length so that it fits into the PDF layout and your results are clearly shown.
   ii. Select `File->Download as->HTML (.html)`. This will download an HTML version `your_homework.html` of your notebook to your computer, typically in your `Download` folder.
   iii. Open `your_homework.html` in your web browser (just double-click on it). ***Use the `File->Print` command of your browser to produce a PDF file.*** *(Do not produce the PDF in other ways, such as "Export to PDF" or other. Alternative methods will usually produce poor page margins, format headers differently, fail to incorporate figures, and so forth.)*
   iv. **Submit your PDF file** `your_homework.pdf` to Gradescope. Do *not* submit your HTML file.
5. Problem sets are due at the beginning of the class period on the due date listed. Late problem sets will not be accepted.

Note that you can wrap lines of python code using the "\" operator to ensure that all your code is visible within the width of the page. See the following examples:

In [1]:
```
print("demonstra\
tion")
```
demonstration

In [2]:
```
def factorial(n):
    if n==1:
        return(1)
    return(n*\
        factorial(n-1))

factorial(4)
```
Out[2]: 24

In [3]:
```
factorial\
(4)
```
Out[3]: 24

Problem 1 – Dynamic equilibration of liquid levels

A U-tube manometer is initially filled with water, but is exposed to a pressure difference such that the water level on the left side of the U-tube is 0.05 m higher than the water level on the water level on the right. At $t = 0$ the pressure difference is suddenly removed. When friction is neglected, the height of the water level on the left side, y, measured from the mid-plane between the two initial water levels is given by the solution of the equation:

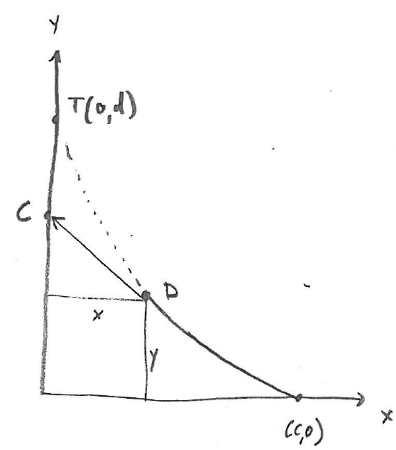
$$L\frac{d^2y}{dt^2} = -2gy$$

where $L = 0.2$ m is the total length of the U-tube.

Use the 4th-order Runge-Kutta method to solve the ODE. Provide plots of y and $\frac{dy}{dt}$ as a function of t for the first 10 seconds.

Problem 2 – Pet and Prey Models

Suppose that a cat starts at the origin and runs with speed a straight towards a tree located at the point $T(0, d)$ on the y-axis. At the same time a dog starts at the point $(c, 0)$ on the x-axis, running with speed b, and pursuing the cat by always running directly toward it.



It is possible to derive an equation that describes the chase path taken by the dog in pursuing the cat. Let $D = (x, y)$ be the dog's position as a function of time and $C = (0, at)$ be the cat's position. Since the dog directly chases the cat

$$\frac{dy}{dx} = -\frac{at - y}{x}$$

$$xy' = -at + y$$

Differentiating both sides with respect to x

$$xy'' + y' = -a\frac{dt}{dx} + y'$$

$$xy'' = -a\frac{dt}{dx}$$

If s is the path of the dog and $\frac{ds}{dt}$ is the speed of the dog, then $\frac{dt}{dx}$ is related to the speed of the dog and position of the dog as follows:

$$\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx} = \frac{1}{b}\frac{ds}{dx}$$

$$ds = (dx^2 + dy^2)^{1/2} = -dx\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = -dx(1 + (y')^2)^{1/2}$$

Note that the negative sign arises because s increases as x decreases. Thus,

$$\frac{dt}{dx} = -\frac{1}{b}\,(1 + (y')^2)^{1/2}$$

and

$$xy'' = \frac{a}{b}(1 + (y')^2)^{1/2}$$

(a) What are the initial condition(s) that fully define the system?
(b) Integrate this differential equation to prepare a plot of the dog's position y as a function of its x position. Assume that the cat's speed is $a = 20\,ft/s$, the dog's speed is $b = 30\,ft/s$, and $c = 150\,ft$. Feel free to use <u>any</u> of the tools at your disposal to integrate.
(c) What is the maximum distance that the cat can start from the tree to ensure that it can reach safety?
(d) Plot the cat's maximum safe distance from the tree (on the y-axis) as a function of the dog's speed (on the x-axis). Assume again that the cat's speed is $a = 20\,ft/s$ and $c = 150\,ft$.
(e) Clearly the dog could be more successful in catching the cat if it always ran toward a position 10 feet in front of the cat rather than directly at it. Where will the dog catch the cat under these conditions?
(f) The dog could do even better if it anticipated that the cat was headed to the tree and therefore chose itself to run straight toward the tree rather than the cat. This type of pursuit path is the approach taken frequently by humans and relies upon our intelligence and predictive/integrative prowess. Doing some research in the literature and thinking about natural systems, do predator/prey systems in nature exhibit pursuit paths described by these limiting behaviors?

Problems 3 and 4
Please see attached Jupyter notebook template.

Problem 5 – Solving boundary value problems
Adapted from Chapra and Canale, Problem 27.6
We are given the differential equation

$$\frac{d^2T}{dx^2} - 10^{-7}(T+273)^4 + 4(150 - T) = 0$$

(a) Recast this differential equation as a system of two first-order differential equations.
(b) Implement the shooting method to solve the equation given the boundary conditions of $T(0) = 200$ and $T(0.5) = 100$. Provide a plot of your solution and discuss whether it satisfies the boundary conditions.

Problem 6
Given the boundary value problem:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 6x \quad \text{with} \quad \left.\frac{dy}{dx}\right|_{x=0} = -1 \text{ and } y(1) = 1$$

(a) Solve using a finite difference method based on the central difference approximation for both the first and second derivatives (both have a truncation error of $O(h^2)$). Adjust the number of sub-intervals (or your mesh-size) into which you subdivide the domain to balance solution accuracy and computational efficiency. Show your work in setting up the problem, and then use python to solve for $y(x)$ for points between $0 \le x \le 1$.
(b) Solve using the `scipy.integrate.solve_bvp` method in the python libraries. Plot y versus x <u>and</u> dy/dx versus x.