Analytical Gradients of the Optimization

In this section, we introduce the key analytical gradients of the trajectory optimization problem.

A. Preliminaries

According to the theory of Lie groups and Lie algebra for robotics [1], [2], we denote the conversion from the axis-angle rotation vector $\boldsymbol{r} \in \mathfrak{so}(3)$ to the rotation matrix $\boldsymbol{R} \in \mathrm{SO}(3)$ as $\boldsymbol{R} = \exp(\boldsymbol{r}^\wedge)$, and the inverse conversion is denoted as $\boldsymbol{r} = \ln(\boldsymbol{R})^\vee$.

According to the linear approximation of the Baker-Campbell-Hausdorff (BCH) formula, we have

$$\ln(\exp(\boldsymbol{r}_1^\wedge)\exp(\boldsymbol{r}_2^\wedge))^\vee \approx \begin{cases} \boldsymbol{J}_l(\boldsymbol{r}_2)^{-1}\boldsymbol{r}_1 + \boldsymbol{r}_2, & \text{when}\, \boldsymbol{r}_1 \to \boldsymbol{0}, \\ \boldsymbol{J}_r(\boldsymbol{r}_1)^{-1}\boldsymbol{r}_2 + \boldsymbol{r}_1, & \text{when}\, \boldsymbol{r}_2 \to \boldsymbol{0}. \end{cases} \tag{1}$$

Here, $\boldsymbol{J}_r(\boldsymbol{r}) = \boldsymbol{J}_l(-\boldsymbol{r})$, and $\boldsymbol{J}_l(\boldsymbol{r})$ can be calculated as

$$\boldsymbol{J}_l(\boldsymbol{r}) = \frac{\sin\theta}{\theta}\boldsymbol{I} + \left(1 - \frac{\sin\theta}{\theta}\right)\boldsymbol{a}\boldsymbol{a}^\top + \frac{1-\cos\theta}{\theta}\boldsymbol{a}^\wedge, \tag{2}$$

$$\boldsymbol{J}_l(\boldsymbol{r})^{-1} = \frac{\theta}{2}\cot\frac{\theta}{2}\boldsymbol{I} + \left(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\right)\boldsymbol{a}\boldsymbol{a}^\top - \frac{\theta}{2}\boldsymbol{a}^\wedge, \tag{3}$$

where θ and $\boldsymbol{a}$ are the angle and axis of $\boldsymbol{r}$, respectively.

B. Gradients of Orientation Distances

Here we generally introduce the gradient regarding the orientation distance. We define the weighted scalar distance of orientations $\boldsymbol{R}$ and $\boldsymbol{R}_\mathrm{d}$ as

$$d_\mathrm{r}(\boldsymbol{R}, \boldsymbol{R}_\mathrm{d}, \boldsymbol{W}_\mathrm{r}) = \frac{1}{2}\boldsymbol{r}_\mathrm{e}^\top \boldsymbol{W}_\mathrm{r}\boldsymbol{r}_\mathrm{e}, \tag{4}$$

where $\boldsymbol{r}_\mathrm{e}$ is defined as $\ln\left(\boldsymbol{R}\boldsymbol{R}_\mathrm{d}^{-1}\right)^\vee$, and $\boldsymbol{W}_\mathrm{r}$ is a semi-positive definite matrix for weighting. Here, $\boldsymbol{R}$ is determined by a variable $\boldsymbol{x}$, and $\boldsymbol{R}_\mathrm{d}$ is a constant desired orientation. Note that $\boldsymbol{R} = \exp(\boldsymbol{r}^\wedge)$ and $\boldsymbol{R}_\mathrm{d} = \exp(\boldsymbol{r}_\mathrm{d}^\wedge)$.

The gradient of d_r w.r.t. the variable $\boldsymbol{x}$ is derived as

$$\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{x}} = \frac{\partial d_\mathrm{r}}{\partial \boldsymbol{r}_\mathrm{e}}\frac{\partial \boldsymbol{r}_\mathrm{e}}{\partial \boldsymbol{x}} = \frac{\partial d_\mathrm{r}}{\partial \boldsymbol{r}_\mathrm{e}}\frac{\partial \boldsymbol{r}_\mathrm{e}}{\partial \boldsymbol{\varphi}}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}} \tag{5}$$

For convenience of the following calculation, here we introduce a perturbation variable $\boldsymbol{\varphi} \in \mathfrak{so}(3)$, which means that we left perturb $\boldsymbol{R}$ by $\Delta\boldsymbol{R}$ (i.e., $(\Delta\boldsymbol{R})\boldsymbol{R}$), where $\Delta\boldsymbol{R} = \exp(\boldsymbol{\varphi}^\wedge)$.

First, it is easy to obtain

$$\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{r}_\mathrm{e}} = \boldsymbol{r}_\mathrm{e}^\top \boldsymbol{W}_\mathrm{r} \tag{6}$$

Second, regarding $\frac{\partial \boldsymbol{r}_\mathrm{e}}{\partial \boldsymbol{\varphi}}$, we have

$$\begin{aligned}\frac{\partial \boldsymbol{r}_\mathrm{e}}{\partial \boldsymbol{\varphi}} &= \lim_{\boldsymbol{\varphi}\to\boldsymbol{0}} \frac{\ln\left(\exp(\boldsymbol{\varphi}^\wedge)\exp(\boldsymbol{r}^\wedge)\left(\exp(\boldsymbol{r}_\mathrm{d}^\wedge)\right)^{-1}\right)^\vee}{\boldsymbol{\varphi}} \\ &\quad - \frac{\ln\left(\exp(\boldsymbol{r}^\wedge)\left(\exp(\boldsymbol{r}_\mathrm{d}^\wedge)\right)^{-1}\right)^\vee}{\boldsymbol{\varphi}} \\ &= \lim_{\boldsymbol{\varphi}\to\boldsymbol{0}} \frac{\ln\left(\exp(\boldsymbol{\varphi}^\wedge)\exp(\boldsymbol{r}_\mathrm{e}^\wedge)\right)^\vee - \ln\left(\exp(\boldsymbol{r}_\mathrm{e}^\wedge)\right)^\vee}{\boldsymbol{\varphi}}\end{aligned} \tag{7}$$

It follows from the BCH formula (1) that

$$\frac{\partial \boldsymbol{r}_\mathrm{e}}{\partial \boldsymbol{\varphi}} = \lim_{\boldsymbol{\varphi}\to\boldsymbol{0}} \frac{\boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1}\boldsymbol{\varphi} + \boldsymbol{r}_\mathrm{e} - \boldsymbol{r}_\mathrm{e}}{\boldsymbol{\varphi}} = \boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1} \tag{8}$$

We thus obtain $\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{\varphi}} = \boldsymbol{r}_\mathrm{e}^\top \boldsymbol{W}_\mathrm{r}\boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1}$. Moreover, using (3), we can obtain $\boldsymbol{r}_\mathrm{e}^\top \boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1} = \boldsymbol{r}_\mathrm{e}^\top$. Thus, in special cases where the weights for each orientation dimension are the same (i.e., $\boldsymbol{W}_\mathrm{r} = w\boldsymbol{I}$), we further have $\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{\varphi}} = w\boldsymbol{r}_\mathrm{e}^\top \boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1} = w\boldsymbol{r}_\mathrm{e}^\top$.

Third, as the perturbation variable $\boldsymbol{\varphi} \to \boldsymbol{0}$, we have $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}} = \boldsymbol{J}_\mathrm{a}(\boldsymbol{x})$, where $\boldsymbol{J}_\mathrm{a}(\boldsymbol{x})$ is the space Jacobian that relates the spatial angular velocity to $\dot{\boldsymbol{x}}$.

C. Gradients of Pose Distances

Similar to the derivation of the gradients of orientation distances, the general formula of the gradients of pose distances is derived as follows.

The weighted scalar distance between poses $\boldsymbol{T}$ and $\boldsymbol{T}_\mathrm{d}$ is defined as

$$d(\boldsymbol{T}, \boldsymbol{T}_\mathrm{d}, \boldsymbol{W}) = \frac{1}{2}\boldsymbol{e}^\top \boldsymbol{W}\boldsymbol{e} \tag{9}$$

where $\boldsymbol{e} = [\boldsymbol{p}_\mathrm{e}; \boldsymbol{r}_\mathrm{e}]$, in which $\boldsymbol{p}_\mathrm{e} = \boldsymbol{p} - \boldsymbol{p}_\mathrm{d}$ and $\boldsymbol{r}_\mathrm{e} = \ln\left(\exp(\boldsymbol{r}^\wedge)\left(\exp(\boldsymbol{r}_\mathrm{d}^\wedge)\right)^{-1}\right)^\vee$. Here, $\boldsymbol{T}$ is determined by a variable $\boldsymbol{x}$, and $\boldsymbol{T}_\mathrm{d}$ is a constant desired pose. Similar to (5), we introduce a perturbation variable $\boldsymbol{\phi} \in \mathfrak{se}(3)$. Then, the gradient of d w.r.t. $\boldsymbol{x}$ is derived as

$$\frac{\partial d}{\partial \boldsymbol{x}} = \frac{\partial d}{\partial \boldsymbol{e}}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\phi}}\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}} \tag{10}$$

where $\frac{\partial d}{\partial \boldsymbol{e}} = \boldsymbol{e}^\top \boldsymbol{W}$ and

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\phi}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1} \end{bmatrix} \tag{11}$$

Additionally, we have $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x})$, where $\boldsymbol{J}(\boldsymbol{x})$ is the space Jacobian that relates the spatial twist to $\dot{\boldsymbol{x}}$.

D. Gradients of $\mathcal{J}_\mathrm{object}$

It is easy to see that $\mathcal{J}_\mathrm{object}$ is only relevant to the object pose at time T. The gradient between the position distance cost and the object position variable is easy to derive. Here, we introduce the gradient regarding the orientation distance (i.e., $\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{r}_{\mathrm{o},T}}$). For brevity, we omit the subscripts o and T.

Similar to (7) and (8), we derive that

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{r}} = \left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\varphi}}\right)^{-1} = \boldsymbol{J}_l(\boldsymbol{r}) \tag{12}$$

We then have

$$\frac{\partial d_\mathrm{r}}{\partial \boldsymbol{r}} = \boldsymbol{r}_\mathrm{e}^\top \boldsymbol{W}_\mathrm{r}\boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{r}} = \boldsymbol{r}_\mathrm{e}^\top \boldsymbol{W}_\mathrm{r}\boldsymbol{J}_l(\boldsymbol{r}_\mathrm{e})^{-1}\boldsymbol{J}_l(\boldsymbol{r}) \tag{13}$$

E. Gradients of $\mathcal{J}_\mathrm{finger}$

We denote the Lie algebra corresponding to the object pose $\boldsymbol{T}_{\mathrm{o},t} \in SE(3)$ as $\boldsymbol{\xi}_{\mathrm{o},t} = [\boldsymbol{p}_{\mathrm{o},t}; \boldsymbol{r}_{\mathrm{o},t}] \in \mathfrak{se}(3)$, which is defined in $\mathcal{W}$. The optimization variable related to $d({}^{\mathcal{O}}\boldsymbol{T}_{i,t}, {}^{\mathcal{O}}\boldsymbol{T}_{i,0}, \boldsymbol{W}_\mathrm{f})$ contains the object pose $\boldsymbol{\xi}_{\mathrm{o},t}$ and finger joint angle $\boldsymbol{q}_{i,t}$. For brevity, we omit the subscripts o, i, and t. We further denote $\boldsymbol{\vartheta} = [\boldsymbol{\xi}; \boldsymbol{q}]$.

We can use (10) to calculate the gradient, but we still need to know the space Jacobian that relates the fingertip twist in $\mathcal{O}$ to $\dot{\boldsymbol{\vartheta}}$. As the object frame $\mathcal{O}$ is moving, this Jacobian is a relative Jacobian between the finger and object. We can calculate this relative Jacobian using individual manipulator Jacobians defined in $\mathcal{W}$ [3], in which we regard the object as a virtual manipulator. According to (2) in [3], the relative Jacobian between the fingertip twist in $\mathcal{O}$ and $\dot{\boldsymbol{\vartheta}}$ can be expressed as

$$ {}^{\mathcal{O}}\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{\vartheta}) = \begin{bmatrix} -{}^{\mathcal{O}}\boldsymbol{\Psi}_{\mathrm{f}}{}^{\mathcal{O}}\boldsymbol{\Omega}_{\mathrm{w}}\boldsymbol{J}_{\mathrm{o}}(\boldsymbol{\xi}) & {}^{\mathcal{O}}\boldsymbol{\Omega}_{\mathrm{w}}\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{q}) \end{bmatrix}, \tag{14} $$

where $\boldsymbol{J}_{\mathrm{o}}(\boldsymbol{\xi})$ is the space Jacobian that relates the object's twist in $\mathcal{W}$ to $\dot{\boldsymbol{\xi}}$, and $\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{q})$ is the space Jacobian that relates the fingertip's twist in $\mathcal{W}$ to $\dot{\boldsymbol{q}}$. Similar to (12), it can be obtained that

$$ \boldsymbol{J}_{\mathrm{o}}(\boldsymbol{\xi}) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_l(\boldsymbol{r}) \end{bmatrix}, \tag{15} $$

where $\boldsymbol{r}$ refers to the object orientation in $\mathcal{W}$. The finger Jacobian $\boldsymbol{J}_{\mathrm{f}}(\cdot)$ can be obtained from the finger's kinematics. The transformation matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$ are defined as

$$ {}^{a}\boldsymbol{\Psi}_{b} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{S}({}^{a}\boldsymbol{p}_{b}) \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \quad {}^{a}\boldsymbol{\Omega}_{b} = \begin{bmatrix} {}^{a}\boldsymbol{R}_{b} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{a}\boldsymbol{R}_{b} \end{bmatrix}, \tag{16} $$

where $\boldsymbol{S}(\boldsymbol{p})$ refers to the skew-symmetric matrix of vector $\boldsymbol{p}$.

F. Other Gradients

Other gradients, including the gradients of $\mathcal{J}_{\mathrm{joint}}$ and those of the constraints, can be easily derived. The details are omitted here for brevity.

References

[1] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[2] X. Gao, T. Zhang, Y. Liu, and Q. Yan, *14 Lectures on Visual SLAM: From Theory to Practice*. Publishing House of Electronics Industry, 2017.

[3] R. S. Jamisola Jr and R. G. Roberts, "A more compact expression of relative jacobian based on individual manipulator jacobians," *Robotics Auton. Syst.*, vol. 63, pp. 158–164, 2015.