## Derived Category

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## 1 Categories

#### 1.1 Definitions

**Definition 1.1.** A locally small category  $\mathcal{C}$  consists of

- A class  $Ob \mathcal{C}$  of objects.
- For all  $X, Y \in \text{Ob } \mathcal{C}$ , a set of morphisms

$$\operatorname{Hom}(X,Y) = \{ \varphi : X \to Y \}.$$

• A collection of maps: for all  $X, Y, Z \in \text{Ob } \mathcal{C}$ ,

$$\operatorname{Hom}(X,Y) \times \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$$
  
 $(\varphi,\psi) \mapsto \psi \circ \varphi$ 

subject to the following conditions.

- The sets Hom(X, Y) are pairwise disjoint.
- For each  $X \in \text{Ob } \mathcal{C}$ , there exists  $\text{id}_X \in \text{Hom}(X, X)$  such that  $\text{id}_X \circ \varphi = \varphi$  and  $\psi \circ \text{id}_X = \psi$ .
- $(\varphi \circ \psi) \circ \chi = \varphi \circ (\psi \circ \chi)$

It follows from the definition that  $id_X$  is unique.

**Definition 1.2.** A morphism  $\varphi: X \to Y$  in a category  $\mathcal{C}$  is called isomorphism if there exists  $\psi: Y \to X$  such that  $\varphi \circ \psi = \mathrm{id}_Y$  and  $\psi \circ \varphi = \mathrm{id}_X$ . We say that X and Y are isomorphic.

**Definition 1.3.** A covariant functor  $F: \mathcal{C} \to \mathcal{D}$  between two categories  $\mathcal{C}$ ,  $\mathcal{D}$  consists of

• A map

$$Ob \mathcal{C} \to Ob \mathcal{D}$$
  
 $X \mapsto F(X).$ 

• A map

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y))$$
  
 $\varphi \mapsto F(\varphi)$ 

for all  $X, Y \in \text{Ob } \mathcal{C}$  such that  $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$  and  $F(\text{id}_X) = \text{id}_{F(X)}$ .

**Proposition 1.4.** A covariant functor sends isomorphisms to isomorphisms.

*Proof.* Let  $\varphi: X \to Y$  be an isomorphism. Then there exists a morphism  $\psi: Y \to X$  such that  $\psi \circ \varphi = \mathrm{id}_X$  and  $\varphi \circ \psi = \mathrm{id}_Y$ . Since

$$\operatorname{id}_{F(X)} = F(\operatorname{id}_X) = F(\psi \circ \varphi) = F(\psi) \circ F(\varphi),$$
  
 $\operatorname{id}_{F(Y)} = F(\operatorname{id}_Y) = F(\varphi \circ \psi) = F(\varphi) \circ F(\psi),$ 

 $F(\varphi)$  is an isomorphism.

**Definition 1.5.** A contravariant functor  $F: \mathcal{C} \to \mathcal{D}$  is defined similarly, with

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(F(Y),F(X))$$
  
 $\varphi \mapsto F(\varphi)$ 

We may view the contravariant functor F as a functor from the opposite category  $\mathcal{C}^{\text{op}}$  to  $\mathcal{D}$ .

**Definition 1.6.** A functor  $F: \mathcal{C} \to \mathcal{D}$  is called

- full if  $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(F(X),F(Y))$  is surjective for all  $X,Y \in \operatorname{Ob} \mathcal{C}$ ;
- faithful if  $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(F(X),F(Y))$  is injective for all  $X,Y \in \operatorname{Ob} \mathcal{C}$ .

**Proposition 1.7.** Given a fully faithful functor  $F: \mathcal{C} \to \mathcal{D}$ . Let  $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ . Then f is an isomorphism if and only if F(f) is an isomorphism.

Proof. The only if part follows from (1.4). Suppose that F(f) is an isomorphism. Then there exists  $\varphi: F(Y) \to F(X)$  such that  $\varphi \circ F(f) = \mathrm{id}_{F(X)}$  and  $F(f) \circ \varphi = \mathrm{id}_{F(Y)}$ . Since  $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$  is surjective, there exists  $g \in \mathrm{Hom}(X,Y)$  such that  $F(g) = \varphi$ . Then

$$F(g \circ f) = \varphi \circ F(f) = \mathrm{id}_{F(X)} = F(\mathrm{id}_X), \quad F(f \circ g) = F(f) \circ \varphi = \mathrm{id}_{F(Y)} = F(\mathrm{id}_Y).$$

It follows from the injectivity of  $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(F(X),F(Y))$  that  $g \circ f = \operatorname{id}_X$  and  $f \circ g = \operatorname{id}_Y$ , which shows that f is an isomorphism.

**Definition 1.8.** A subcategory  $\mathcal{D} \subseteq \mathcal{C}$  is a category  $\mathcal{D}$  such that

- $Ob \mathcal{D} \subseteq Ob \mathcal{C}$ ;
- $\operatorname{Hom}_{\mathcal{D}}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{C}}(X,Y)$  for all  $X, Y \in \operatorname{Ob} \mathcal{D}$  and is compatible with compositions and the identity.

We call  $\mathcal{D}$  a full subcategory if  $\operatorname{Hom}_{\mathcal{D}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$ 

### 1.2 Equivalence of categories

**Definition 1.9.** Two categories  $\mathcal{C}$  and  $\mathcal{D}$  are called isomorphic if there exists functors  $F: \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  such that  $F \circ G = \mathrm{id}_{\mathcal{D}}$  and  $G \circ F = \mathrm{id}_{\mathcal{C}}$ .

**Remark.** Equality of objects is a very restrictive notion. Even objects defined by universal properties (e.g.  $X \times Y$ ) are only unique up to unique isomorphism.

**Definition 1.10.** A functor  $F: \mathcal{C} \to \mathcal{D}$  is an equivalence of categories if

- F is fully faithful;
- F is essentially surjective, i.e., for each  $Y \in \mathcal{D}$ , there exists  $X \in \mathcal{C}$  such that  $F(X) \cong Y$ .

**Definition 1.11.** Given functors  $F, G : \mathcal{C} \to \mathcal{D}$ . A natural transformation  $\eta : F \to G$  is a collection of morphisms

$$\{\eta(X): F(X) \to G(X)\}_{X \in \text{Ob}\,\mathcal{C}}$$

such that for each  $\varphi: X \to Y$ , the following diagram commutes

$$F(X) \xrightarrow{F(\varphi)} F(Y)$$

$$\downarrow^{\eta(X)} \qquad \downarrow^{\eta(X)}$$

$$G(X) \xrightarrow{G(\varphi)} G(Y).$$

We define  $\operatorname{Funct}(\mathcal{C}, \mathcal{D})$  to be the category of functors  $F: \mathcal{C} \to \mathcal{D}$ , with natural transformations as morphisms. Then a natural isomorphism is an isomorphism in this category.

**Theorem 1.12.** Two category  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent if and only if there exists  $F: \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  such that  $F \circ G \cong \mathrm{id}_{\mathcal{D}}$  and  $G \circ F \circ \mathrm{id}_{\mathcal{C}}$ .

We say that G is a quasi-inverse of F.

**Theorem 1.13** (Yoneda's lemma). Let C be a category, and let  $C^{op}$  be its opposite category. The functor

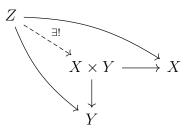
$$h_{\bullet}: \ \mathcal{C} \rightarrow \operatorname{Funct}(\mathcal{C}^{\operatorname{op}}, \operatorname{\mathsf{Sets}})$$

$$X \mapsto h_X = \operatorname{Hom}(-, X)$$

is fully faithful.

A contravariant functor  $F: \mathcal{C} \to \mathsf{Sets}$  is called representable if  $F \cong h_X$  for some  $X \in \mathsf{Ob}\,\mathcal{C}$ . Such X is unique up to unique isomorphism by Yoneda's lemma.

**Definition 1.14.** For all  $X, Y \in \text{Ob } \mathcal{C}$ . We define  $X \times Y$  to be the object satisfying the universal property:



**Proposition 1.15.** The object  $X \times Y$  is the unique object represents  $Z \mapsto \operatorname{Hom}(Z, X) \times \operatorname{Hom}(Z, Y)$ .

## 2 Additive and abelian categories

## 2.1 Additive category

Let  $\mathcal{C}$  be a category.

**Definition 2.1.** An object  $* \in \mathcal{C}$  is called

- initial if  $\# \operatorname{Hom}(*, X) = 1$  for each X;
- final if  $\# \operatorname{Hom}(X, *) = 1$  for each X.

If \* is both initial and final, we call \* a zero object.

All these objects are unique up to unique isomorphism.

#### **Definition 2.2.** A category $\mathcal{C}$ is called additive if

- 1)  $\operatorname{Hom}(X,Y)$  is an abelian group for all  $X,Y\in\operatorname{Ob}\mathcal{C}$  and compositions are bi-additive;
- 2) the zero object 0 exists;
- 3)  $X \times Y$  (or equivalently,  $X \oplus Y$ ) exists for all  $X, Y \in Ob \mathcal{C}$ .

A functor  $F: \mathcal{C} \to \mathcal{D}$  between additive category is called additive if  $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(F(X),F(Y))$  is a group homomorphism.

**Proposition 2.3.** Let  $F: \mathcal{C} \to \mathcal{D}$  be an additive functor. Then for all  $X, Y \in \mathcal{C}$ ,  $F(X \oplus Y) = F(X) \oplus F(Y)$ .

*Proof.* Let  $p_X: X \oplus Y \to X$ ,  $p_Y: X \oplus Y \to Y$  be the projections, and let  $i_X: X \to X \oplus Y$ ,  $i_Y: Y \to X \oplus Y$  be the inclusions. Then  $i_X \circ p_X + i_Y \circ p_Y = \mathrm{id}_{X \oplus Y}$ . Given any  $f: Z \to X$ ,  $g: Z \to Y$ , we see that  $h = f \circ F(p_X) + g \circ F(p_Y): Z \to F(X \oplus Y)$  satisfies

$$h \circ F(i_X) = f \circ F(\mathrm{id}_X) = f, \quad h \circ F(i_Y) = g \circ F(\mathrm{id}_Y) = g.$$

If  $h': Z \to F(X \oplus Y)$  is another morphism such that  $h' \circ F(i_X) = f$  and  $h' \circ F(i_Y) = g$ , then

$$h' = h' \circ F(i_X \circ p_X + i_Y \circ p_Y) = f \circ F(p_X) + g \circ F(p_Y) = h.$$

Hence, 
$$F(X \oplus Y) = F(X) \oplus F(Y)$$
.

## 2.2 Abelian category

Let  $\mathcal{C}$  be an additive catrgory.

**Definition 2.4.** For each  $f: X \to Y$ . Define the kernel of f to be the fiber product (if exists)

$$\ker f \longrightarrow X \\
\downarrow \qquad \qquad \downarrow_f \\
0 \longrightarrow Y.$$

Define the cokernel of f to be the fiber product (if exists)

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow^f & & \downarrow \\ Y & \longrightarrow & \operatorname{coker} f. \end{array}$$

**Proposition 2.5.** The kernel of  $f: X \to Y$  exists if and only if

$$\mathcal{C}^{\mathrm{op}} \to \operatorname{Sets}$$

$$Z \mapsto \ker(\operatorname{Hom}(Z,X) \xrightarrow{f \circ} \operatorname{Hom}(Z,Y))$$

is representable.

 $\underline{\wedge}$  The naive analogous statement for coker f is wrong. The correct statement is  $\operatorname{coker}(f)$  exists if and only if

$$\begin{array}{ccc} \mathcal{C} & \to & \mathsf{Sets} \\ Z & \mapsto & \ker(\mathrm{Hom}(Y,Z) \xrightarrow{\circ f} \mathrm{Hom}(X,Z)) \end{array}$$

is co-representable.

We define the image of f to be  $\operatorname{Im} f = \ker(Y \to \operatorname{coker} f)$ , and the coimage to be  $\operatorname{coIm} f = \operatorname{coker}(\ker f \to X)$ . The universal properties gives a unique factorization

$$X \to \operatorname{coIm} f \to \operatorname{Im} f \to Y$$
.

**Definition 2.6.** An abelian category is an additive category  $\mathcal{C}$  such that

- 4) for each morphism  $f: X \to Y$ , ker f and coker f exists;
- 5) the canonical map  $\operatorname{coIm} f \to \operatorname{Im} f$  is an isomorphism.

## 2.3 Exact sequences

Let  $\mathcal{C}$  be an abelian category. A sequence

$$\cdots \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \cdots$$

in C is called exact if ker  $f_i = \text{Im } f_{i-1}$  for each i. A short exact sequence is an exact sequence of the form

$$0 \to X \to Y \to Z \to 0.$$

A covariant additive functor  $F: \mathcal{C} \to \mathcal{D}$  between abelian categories is called

• left exact if for every short exact sequence

$$0 \to X \to Y \to Z \to 0$$
.

the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is exact;

• right exact if for every short exact sequence

$$0 \to X \to Y \to Z \to 0$$
,

the sequence

$$F(X) \to F(Y) \to F(Z) \to 0$$

is exact;

• exact if F is both left and right exact.

**Theorem 2.7** (Freyd-Mitchell Embedding Theorem). For every small abelian category  $\mathcal{C}$ , there exists a fully faithful exact functor  $F:\mathcal{C}\to R\text{-Mod}$  for some ring R.

This allows us to manipulate small abelian category as if they were category of Rmodules.

## 2.4 Adjoint functors

Let  $\mathcal{C}$ ,  $\mathcal{D}$  be arbitrary categories, and let  $F:\mathcal{C}\to\mathcal{D}$  and  $G:\mathcal{D}\to\mathcal{C}$  be functors.

**Definition 2.8.** We say that F is the left adjoint of G, and G is the right adjoint of F, if

$$\operatorname{Hom}_{\mathcal{D}}(F(-), -) \cong \operatorname{Hom}_{\mathcal{C}}(-, G(-))$$

in Funct( $\mathcal{C}^{\text{op}} \times \mathcal{D}$ , Sets). In this case, we write  $F \dashv G$ .

We may represent an adjoint pair as a diagram:

$$\mathcal{C}$$
 $G$ 
 $\mathcal{D}$ .

**Proposition 2.9.** Left adjoint and right adjoint are unique.

Suppose now that  $\mathcal{C}$  and  $\mathcal{D}$  are abelian categories, and  $F:\mathcal{C}\to\mathcal{D}$  and  $G:\mathcal{D}\to\mathcal{C}$  are additive functors.

**Proposition 2.10.** If  $F \dashv G$ , then F is right exact and G is left exact.

## 3 Derived categories: first definition

#### 3.1 Complexes

Let  $\mathcal{A}$  be an abelian category. A complex  $K^{\bullet}$  in  $\mathcal{A}$  is a sequence

$$\cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$$

of morphisms in  $\mathcal{A}$  such that  $d^i \circ d^{i-1}$  for all  $i \in \mathbb{Z}$ . A morphism of complexes  $f^{\bullet}: K^{\bullet} \to L^{\bullet}$  is a commutative diagram

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d} K^{i} \xrightarrow{d} K^{i+1} \longrightarrow \cdots$$

$$\downarrow^{f^{i-1}} \qquad \downarrow^{f^{i}} \qquad \downarrow^{f^{i+1}}$$

$$\cdots \longrightarrow L^{i-1} \xrightarrow{d} L^{i} \xrightarrow{d} L^{i+1} \longrightarrow \cdots$$

A morphism of complexes  $f^{\bullet}: K^{\bullet} \to L^{\bullet}$  is called a quasi-isomorphism if

$$H^n(f^{\bullet}): H^n(K^{\bullet}) \xrightarrow{\sim} H^n(L^{\bullet})$$

for all n. Define Kom(A) to be the complexes in A. Quite often, we will also consider various full subset of bounded complexes:

$$\operatorname{Kom}^{+}(\mathcal{A}) := \{ K^{\bullet} \mid K^{i} = 0, \ i \leq i_{0}(K^{\bullet}) \},$$

$$\operatorname{Kom}^{-}(\mathcal{A}) := \{ K^{\bullet} \mid K^{i} = 0, \ i \geq i_{0}(K^{\bullet}) \},$$

$$\operatorname{Kom}^{b}(\mathcal{A}) := \operatorname{Kom}^{+}(\mathcal{A}) \cap \operatorname{Kom}^{-}(\mathcal{A}).$$

Throughout this section, let ? be  $\emptyset$ , +, -, or b.

**Proposition 3.1.** If  $\mathcal{A}$  is abelian, then  $Kom^{?}(\mathcal{A})$  is also abelian.

We define the shift functor  $-[n]: \mathrm{Kom}^?(\mathcal{A}) \to \mathrm{Kom}^?(\mathcal{A})$  as follows: for  $K^{\bullet} \in \mathrm{Kom}^?(\mathcal{A})$ , we define  $K^{\bullet}[n]$  by  $K[n]^i = K^{n+i}$  and  $d_{K^{\bullet}[n]} = (-1)^n d_{K^{\bullet}}$ . Given  $f^{\bullet}: K^{\bullet} \to L^{\bullet}$ , we define  $f^{\bullet}[n]: K^{\bullet}[n] \to L^{\bullet}[n]$  by  $f[n]^i = f^{n+i}$ .

#### 3.2 Localization of a category

Let  $\mathcal{B}$  be a category and let S be a collection of morphisms (in  $\mathcal{B}$ ).

**Definition 3.2.** A strict localization of  $\mathcal{B}$  by S is

- a category  $S^{-1}\mathcal{B}$ ;
- a functor  $Q: \mathcal{B} \to S^{-1}\mathcal{B}$  that sends S to isomorphisms and satisfies the following universal property: for every functor  $F: \mathcal{B} \to \mathcal{D}$  sending S to isomorphisms, we have

$$\mathcal{B} \xrightarrow{Q} S^{-1}\mathcal{B}$$

$$\downarrow^{\exists !}$$

$$\mathcal{D}$$

**Definition 3.3.** If  $Q: \mathcal{B} \to S^{-1}\mathcal{B}$ . sends S to isomorphisms and the following weaker universal property:

for every functor  $F: \mathcal{B} \to \mathcal{D}$  sending S to isomorphisms, there exists  $\Phi: S^{-1}\mathcal{B} \to \mathcal{D}$ , unique up to (natural) isomorphisms, such that  $\Phi \circ Q = \cong F$ ,

we call  $S^{-1}\mathcal{B}$  a localization of  $\mathcal{B}$  by S.

**Theorem 3.4.** Strict localization exists as a large category.

**Definition 3.5.** The derived category  $D^{?}(A)$  of A is the localization of Kom $^{?}(A)$  by the quasi-isomorphisms.

The proof of theorem is easy but useless in practice: Simply set  $\mathrm{Ob}(S^{-1}\mathcal{B})=\mathrm{Ob}\,\mathcal{B}$  and

$$\operatorname{Hom}_{S^{-1}\mathcal{B}}(X,Y) = \{X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_n = Y\}/\sim.$$

Here, each  $X_i \leftrightarrow X_{i+1}$  is either  $X_i \to X_{i+1} \in \operatorname{Hom}_{\mathcal{B}}(X_i, X_{i+1})$  or  $X_i \leftarrow X_{i+1} \in S$ , and the equivalence relation  $\sim$  is defined by:  $\varphi \sim \psi$  if we can transform  $\varphi$  to  $\psi$  through the following:

$$(W_1 \xrightarrow{f} W_2 \xrightarrow{g} W_3) \sim (W_1 \xrightarrow{g \circ f} W_3)$$
$$(W_1 \xrightarrow{s} W_2 \xleftarrow{s} W_1) \sim (W_1 \xrightarrow{\mathrm{id}_{W_1}} W_1)$$
$$(W_1 \xleftarrow{s} W_2 \xrightarrow{s} W_1) \sim (W_1 \xrightarrow{\mathrm{id}_{W_1}} W_1)$$

It is difficult to tell whether  $\varphi = \psi$  in  $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y)$  and not clear whether  $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y)$  is a set. So we want good representative of  $\varphi$ , e.g.,  $\varphi = X \leftarrow X' \to Y$ .

To deal with these problems, instead of working with Kom(A), we work with the homotopy category K(A).

# 4 Derived categories: definition through homotopy categories

## 4.1 Homotopy category

Let  $K^{\bullet}$ ,  $L^{\bullet} \in \text{Kom}(\mathcal{A})$ . Let  $k^i : K^i \to L^{i-1}$  be morphisms. Define  $h^i = k^{i+1}d + dk^i : K^i \to L^i$ .

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d} K^{i} \xrightarrow{d} K^{i+1} \longrightarrow \cdots$$

$$\downarrow^{h^{i-1}} \downarrow^{h^{i}} \downarrow^{h^{i}} \downarrow^{h^{i+1}} \downarrow^{h^{i+1}}$$

$$\cdots \longrightarrow L^{i-1} \xrightarrow{d} L^{i} \xrightarrow{d} L^{i+1} \longrightarrow \cdots$$

Then  $h^{\bullet}: K^{\bullet} \to L^{\bullet}$  is a morphism of complexes.

**Definition 4.1.** The morphism  $h^{\bullet}$  is said to be homotopic to 0, written as  $h \sim 0$ .

**Proposition 4.2.** The collection  $\{h \sim 0\}$  forms an ideal, i.e., for all  $h_1^{\bullet}$ ,  $h_2^{\bullet} : K^{\bullet} \to L^{\bullet}$  homotopic to 0,

- $h_1^{\bullet} + h_2^{\bullet} \sim 0$ ;
- $f^{\bullet} \circ h_{1}^{\bullet} \sim 0$  for all  $f^{\bullet} : L^{\bullet} \to M^{\bullet}$

•  $h_1^{\bullet} \circ g^{\bullet} \sim 0$  for all  $g: N^{\bullet} \to K^{\bullet}$ .

We say that  $\varphi, \psi : K^{\bullet} \to L^{\bullet}$  are homotopic  $(\varphi \sim \psi)$  if  $\varphi - \psi \sim 0$ .

**Proposition 4.3.** If  $\varphi \sim \psi$ , then the morphisms

$$H^{\bullet}(\varphi), H^{\bullet}(\psi): H^{\bullet}(K^{\bullet}) \to H^{\bullet}(L^{\bullet})$$

are equal. In particular, if  $\varphi$  is a quasi-isomorphism and  $\varphi \sim \psi$ , then  $\psi$  is also a quasi-isomorphism.

**Definition 4.4.** The homotopy category  $K^{?}(A)$  is defined by

- $\operatorname{Ob} K^{?}(\mathcal{A}) = \operatorname{Ob} \operatorname{Kom}^{?}(\mathcal{A})$ ; and
- $\operatorname{Mor} K^{?}(A) = \operatorname{Mor} \operatorname{Kom}^{?}(A) / \sim$ .

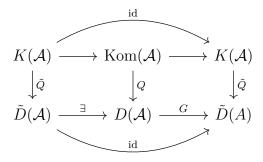
**Proposition 4.5.** The localization of  $K^{?}(A)$  by quasi-isomorphisms is canonically isomorphic to  $D^{?}(A)$ 

*Proof.* Let S be the collection of quasi-isomorphisms and let  $\tilde{D}(\mathcal{A}) = S^{-1}K(\mathcal{A})$ . Then  $\mathrm{Kom}(\mathcal{A}) \to K(\mathcal{A}) \to \tilde{D}(\mathcal{A})$  sends quasi-isomorphisms to isomorphisms, so that there exists a unique functor  $G: D(\mathcal{A}) \to \tilde{D}(\mathcal{A})$  such that the following diagram commute:

$$\operatorname{Kom}(\mathcal{A}) \xrightarrow{G} \tilde{D}(\mathcal{A})$$

$$C \cap D(\mathcal{A}).$$

It is clear that G is a bijection on objects. Choose a section  $K(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$  of  $\mathrm{Kom}(\mathcal{A}) \to K(\mathcal{A})$ . The universal property gives



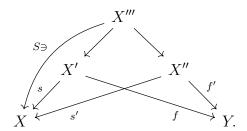
Thus G is surjective on morphisms. That G is injective on morphisms follows from

**Lemma 4.6.** If  $f^{\bullet} \sim g^{\bullet} : K^{\bullet} \to L^{\bullet}$ , then  $Q(f^{\bullet}) = Q(g^{\bullet})$ .

## 4.2 Morphisms in $D^{?}(A)$

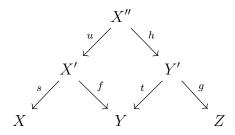
#### Proposition 4.7.

1)  $\operatorname{Hom}_{D^?(\mathcal{A})}(X,Y) = \{(s,f) \mid X \stackrel{s \in S}{\longleftrightarrow} X' \stackrel{f}{\to} Y\} / \sim$ , where  $(s,f) \sim (s',f')$  if we have the following commutative diagram in  $K^?(\mathcal{A})$ :



 $\sim$  is an equivalence relation.

2) Given  $X \stackrel{s}{\leftarrow} X' \stackrel{f}{\rightarrow} Y$  and  $Y \stackrel{t}{\leftarrow} Y' \stackrel{g}{\rightarrow} Z$  with s, t quasi-isomorphism. There exists a commutative diagram in  $K^{?}(\mathcal{A})$ :



such that u is a quasi-isomorphism.

**Definition 4.8.** Given a category  $\mathcal{C}$  and a collection of morphisms S, we say that S is a localizing system if

- (LS1)  $id_X \in S$  for all object  $S, S \circ S \subseteq S$ ;
- (LS2) (extension property) for all such diagrams

$$Z \qquad Y' \xrightarrow{f'} X'$$

$$\downarrow_{s \in S} \qquad \downarrow_{s' \in S}$$

$$X \xrightarrow{f} Y \qquad Z',$$

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there exists  $W \xrightarrow{t} X \in S$ ,  $W \xrightarrow{g} Z$ ,  $X' \xrightarrow{t'} W' \in S$  and  $Z' \xrightarrow{g'} W'$  such that the following diagrams commute:

$$\begin{array}{ccc} W \stackrel{g}{\longrightarrow} Z & Y' \stackrel{f'}{\longrightarrow} X' \\ \downarrow^t & \downarrow^s & \downarrow^{s'} & \downarrow^{t'} \\ X \stackrel{f}{\longrightarrow} Y & Z' \stackrel{g'}{\longrightarrow} W'; \end{array}$$

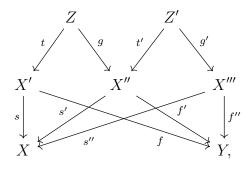
(LS3) for all  $f, g: X \to Y, f \circ s = g \circ s$  for some  $s \in S$  if and only if  $t \circ f = t \circ g$  for some  $t \in S$ .

**Lemma 4.9.** The collection of quasi-isomorphisms in  $K^{?}(A)$  forms a localizing system.

We show that the lemma implies the proposition:

Step 1. By (LS1) and (LS2), the elements in  $\operatorname{Hom}_{D^{?}(\mathcal{A})}(X,Y)$  can be represented by  $(X \xrightarrow{s} X', X' \xrightarrow{f} Y)$  and (LS2) also gives the existence of composition (we still need to check that it is well-defined).

Step 2. We prove that  $\sim$  is an equivalence relation. Transitivity is the least obvious. Assume  $(X \stackrel{s}{\leftarrow} X' \xrightarrow{f} Y) \sim (X \stackrel{s'}{\leftarrow} X'' \xrightarrow{f'} Y)$  and  $(X \stackrel{s'}{\leftarrow} X'' \xrightarrow{f'} Y) \sim (X \stackrel{s''}{\leftarrow} X''' \xrightarrow{f''} Y)$ . Then, by definition, we have the following diagram:



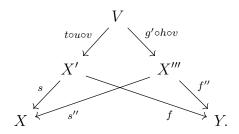
with  $s \circ t$  and  $s' \circ t'$  being quasi-isomorphisms. It follows from (LS2) that we may complete the diagram

$$\begin{array}{c} W \stackrel{h}{\longrightarrow} Z' \\ \downarrow^{u \in S} & \downarrow^{s' \circ t'} \\ Z \stackrel{s \circ t}{\longrightarrow} X. \end{array}$$

Since  $s' \circ (g \circ u) = s \circ t \circ u = s' \circ (t \circ h)$ , (LS3) gives a quasi-isomorphism  $v : V \to W$  such that  $(g \circ u) \circ v = (t \circ h) \circ v$ .

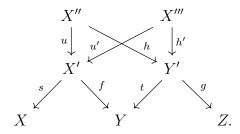
$$V \xrightarrow{v} W \xrightarrow{g \circ u} X'' \xrightarrow{s'} X$$

We verify that



defines the desired equivalence.

Step 3. We check the composition is well-defined. First, given



We need to show that  $(s \circ u, g \circ h) \sim (s \circ u', g \circ h')$ . Take W so that we may complete the diagram

$$W \xrightarrow{v \in S} X''$$

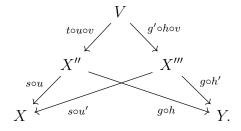
$$\downarrow^{k} \qquad \downarrow^{u}$$

$$X''' \xrightarrow{u'} X'.$$

Since  $t \circ (h \circ v) = f \circ u \circ v = f \circ u' \circ k = t \circ (h' \circ k)$ , (LS3) gives a quasi-isomorphism  $w: V \to W$  such that  $(h \circ v) \circ w = (h' \circ k) \circ w$ .

$$V \xrightarrow{w} W \xrightarrow{h \circ v} Y' \xrightarrow{t} Y$$

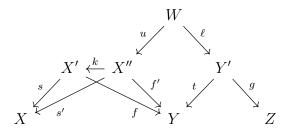
We check that



defines the desired equivalence. This shows that given (s, f) and (t, g), the composition  $(t, g) \circ (s, f)$  is well-defined.

Next, we need to show that if  $(X \stackrel{s}{\leftarrow} X' \stackrel{f}{\rightarrow} Y) \sim (X \stackrel{s'}{\leftarrow} X'' \stackrel{f'}{\rightarrow} Y)$ , then for all  $(Y \stackrel{t}{\leftarrow} Y' \stackrel{g}{\rightarrow} Z), (t,g) \circ (s,f) \sim (t,g) \circ (s',f')$ . Since  $(s,f) \sim (s',f')$ , there exists X''' and

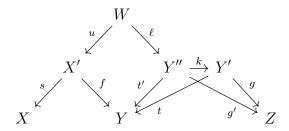
morphisms  $X''' \xrightarrow{h} X'$ ,  $X''' \xrightarrow{h'} X''$  such that  $s \circ h = s' \circ h'$  is a quasi-isomorphism. So we may replace X'' by X''' so that there exists a morphism  $X'' \xrightarrow{k} X'$  such that  $s' = s \circ k$ .



Take W that complete the above diagram with  $u \in S$ , then

$$(t \circ g) \circ (s, f) = (s \circ (k \circ u), g \circ \ell) = (s' \circ u, g \circ \ell) = (t \circ g) \circ (f', s').$$

Finally, we need to show that if  $(Y \stackrel{t}{\leftarrow} Y' \stackrel{g}{\rightarrow} Z) \sim (Y \stackrel{t'}{\leftarrow} Y'' \stackrel{g'}{\rightarrow} Z)$ , then for all  $(X \stackrel{s}{\leftarrow} X' \stackrel{f}{\rightarrow} Y)$ ,  $(t,g) \circ (s,f) \sim (t',g') \circ (s,f)$ . Since  $(t,g) \sim (t',g')$ , there exists Y''' and morphisms and morphisms  $Y''' \stackrel{h}{\rightarrow} Y'$ ,  $Y''' \stackrel{h'}{\rightarrow} X''$  such that  $t \circ h = t' \circ h'$  is a quasi-isomorphism. So we may replace Y'' by Y''' so that there exists a morphism  $Y'' \stackrel{k}{\rightarrow} Y'$  such that  $t' = t \circ k$ .



Take W that complete the above diagram with  $u \in S$ , then

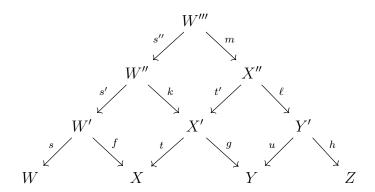
$$(t \circ g) \circ (s, f) = (s \circ u, g \circ k \circ \ell) = (s \circ u, g' \circ \ell) = (t' \circ g') \circ (f, s).$$

Step 4. Define the category Roof(A) by taking Ob Roof(A) = Ob(Kom(A)) and

$$\operatorname{Hom}_{\operatorname{Roof}(\mathcal{A})}(X,Y) = \{X \stackrel{s \in S}{\longleftrightarrow} Z \stackrel{f}{\to} Y\} / \sim .$$

We need to show that the composition is associative. Given

By (LS2) we can take W'', X'', and W''' that completes the diagram



with s', t', and s'' being quasi-isomorphisms. Then

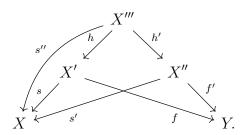
$$(u,h) \circ ((t,g) \circ (s,f)) = (u,h) \circ (s' \circ s, g \circ k) = (s \circ s' \circ s'', h \circ \ell \circ m)$$
$$= (t \circ t', h \circ \ell) \circ (s,f) = ((u,h) \circ (t,g)) \circ (s,f).$$

Let  $F: \operatorname{Kom}(\mathcal{A}) \to \operatorname{Roof}(\mathcal{A})$  be a functor defined by F(X) = X and  $F(f) = (X \stackrel{\operatorname{id}_X}{\longleftarrow} X \stackrel{f}{\to} Y)$ . It remains to show that  $D(\mathcal{A}) = \operatorname{Roof}(\mathcal{A})$ . We show that for each functor  $H: \operatorname{Kom}(\mathcal{A}) \to \mathcal{D}$  that sends quasi-isomorphisms to isomorphisms, there exists a unique functor  $G: \operatorname{Roof}(\mathcal{A}) \to \mathcal{D}$  such that  $H = G \circ F$ .

Step 5. (uniqueness) Assuming G exists. Since  $F = \operatorname{id}$  on objects,  $G \circ F = H$  gives  $G(X) = H \circ F^{-1}(X)$ . Let  $\varphi = (s, f)$  be a morphism in  $\operatorname{Roof}(A)$ . Then  $\varphi \circ F(s) = F(f)$ . Apply G to both sides gives  $G(\varphi) \circ H(s) = H(f)$ . Since H(s) is invertible,  $G(\varphi) = H(f) \circ H(s)^{-1}$ . So G is uniquely determined.

Step 6. (existence) The uniqueness of G suggest us to construct G as follows: for  $X \in \text{Roof}(\mathcal{A})$ , define  $G(X) = H \circ F^{-1}(X)$ ; for  $\varphi = (s, f) \in \text{Hom}_{\text{Roof}(\mathcal{A})}$ , define  $G(\varphi) = H(f) \circ H(s)^{-1}$ .

We check that G is a well-defined functor. If  $(X' \xrightarrow{s} X, f) \sim (X'' \xrightarrow{s'} X, f') \in \operatorname{Hom}_{\operatorname{Roof}(\mathcal{A})}(X, Y)$ , then there exists X''' and morphisms  $X''' \xrightarrow{h} X'$ ,  $X''' \xrightarrow{h'} X''$  such that  $s'' = s \circ h = s' \circ h'$  is a quasi-isomorphism.



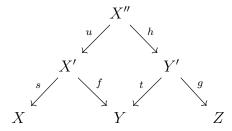
Hence,

$$H(f) \circ H(s)^{-1} = H(f) \circ H(h) \circ H(s'')^{-1}$$
$$= H(f') \circ H(h') \circ H(s'')^{-1} = H(f') \circ H(s')^{-1}.$$

So G is well-defined on the morphisms. It remains to check that  $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$  and  $G(\psi \circ \varphi) = G(\psi) \circ G(\varphi)$ . Since  $\mathrm{id}_X = (\mathrm{id}_X, \mathrm{id}_X)$ ,

$$G(\mathrm{id}_X) = H(\mathrm{id}_X) \circ H(\mathrm{id}_X)^{-1} = \mathrm{id}_{H(X)} \mathrm{id}_{H(X)}^{-1} = \mathrm{id}_{G(X)}$$
.

Let  $\varphi = (s, f)$  and let  $\psi = (t, g)$ . Then  $\psi \circ \varphi = (s \circ u, g \circ h)$  for some u, h such that  $f \circ u = t \circ h$  and u is a quasi-isomorphism.



Hence,

$$H(\psi \circ \varphi) = H(g \circ h) \circ H(s \circ u)^{-1} = H(g) \circ H(h) \circ H(u)^{-1} \circ H(s)^{-1}$$
$$= H(g) \circ H(t)^{-1} \circ H(f) \circ H(s)^{-1} = H(\psi) \circ H(\varphi).$$

This completes the proof of (4.7).

## 4.3 Mapping cones

Given  $f^{\bullet}: K^{\bullet} \to L^{\bullet}$ , the cone of f is the complex  $C(f)^{\bullet}$  defined by  $C(f)^{i} = K^{i+1} \oplus L^{i}$ , i.e.,  $C(f)^{\bullet} = K^{\bullet}[1] \oplus L^{\bullet}$ , and

$$d_{C(f)}^{i} \begin{pmatrix} k^{i+1} \\ \ell^{i} \end{pmatrix} = \begin{pmatrix} -d_{K}^{i+1} & 0 \\ f^{i+1} & d_{L}^{i} \end{pmatrix} \begin{pmatrix} k^{i+1} \\ \ell^{i} \end{pmatrix}.$$

We can easily check that

$$d_{C(f)}^{i+1}d_{C(f)}^{i} = \begin{pmatrix} -d_{K}^{i+2} & 0 \\ f^{i+2} & d_{L}^{i+1} \end{pmatrix} \begin{pmatrix} -d_{K}^{i+1} & 0 \\ f^{i+1} & d_{L}^{i} \end{pmatrix} = \begin{pmatrix} (-d_{K}^{i+2})(-d_{K}^{i+1}) & 0 \\ -f^{i+2}d_{K}^{i+1} + d_{L}^{i+1}f^{i+1} & d_{L}^{i+1}d_{L}^{i} \end{pmatrix} = 0.$$

We have natural maps

$$K^{\bullet} \xrightarrow{f} L^{\bullet} \xrightarrow{\tau} C(f)^{\bullet} \longrightarrow K^{\bullet}[1]$$

$$\ell^{i} \longmapsto \begin{pmatrix} 0 \\ \ell^{i} \end{pmatrix}, \begin{pmatrix} k^{i+1} \\ \ell^{i} \end{pmatrix} \longmapsto k^{i+1}.$$

The short exact sequence

$$0 \to L^{\bullet} \to C(f)^{\bullet} \to K^{\bullet}[1] \to 0$$

induces to a long exact sequence

$$\cdots \to H^i(K^{\bullet}) \to H^i(L^{\bullet}) \to H^i(C(f)^{\bullet}) \to H^{i+1}(K^{\bullet}) \to \cdots$$

Given morphisms  $f_1$ ,  $f_2$ ,  $\alpha$ ,  $\beta$  with  $\beta \circ f_1 = f_2 \circ \alpha$ . We can find  $\gamma : C(f_1)^{\bullet} \to C(f_2)^{\bullet}$  such that the following diagram commute.

$$K_{1}^{\bullet} \xrightarrow{f_{1}} L_{1}^{\bullet} \longrightarrow C(f_{1})^{\bullet} \longrightarrow K_{1}^{\bullet}[1]$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma} \qquad \downarrow^{\alpha[1]}$$

$$K_{2}^{\bullet} \xrightarrow{f_{2}} L_{2}^{\bullet} \longrightarrow C(f_{2})^{\bullet} \longrightarrow K_{2}^{\bullet}[1].$$
(TR3)

In fact, we can simply take  $\gamma = \alpha[1] \oplus \beta$ , so that

$$\begin{split} \gamma^{i+1}d^i_{C(f_1)} - d^i_{C(f_2)}\gamma^i &= \begin{pmatrix} \alpha^{i+2} & 0 \\ 0 & \beta^{i+1} \end{pmatrix} \begin{pmatrix} -d^{i+1}_{K_1} & 0 \\ f^{i+1}_1 & d^i_{L_1} \end{pmatrix} - \begin{pmatrix} -d^{i+1}_{K_2} & 0 \\ f^{i+1}_2 & d^i_{L_2} \end{pmatrix} \begin{pmatrix} \alpha^{i+1} & 0 \\ 0 & \beta^i \end{pmatrix} \\ &= \begin{pmatrix} -\alpha^{i+2}d^{i+1}_{K_1} + d^{i+1}_{K_2}\alpha^{i+1} & 0 \\ \beta^{i+1}f^{i+1}_1 - f^{i+1}_2\alpha^{i+1} & \beta^{i+1}d^i_{L_1} - d^i_{L_2}\beta^i \end{pmatrix} = 0. \end{split}$$

**Proposition 4.10** (TR2). Given  $f: K^{\bullet} \to L^{\bullet}$ , there is a morphism  $g: K^{\bullet}[1] \to C(\tau)$  such that in  $K(\mathcal{A})$ , g is an isomorphism and the following diagram commute.

$$L^{\bullet} \xrightarrow{\tau} C(f)^{\bullet} \longrightarrow K^{\bullet}[1] \xrightarrow{f[1]} L^{\bullet}[1]$$

$$\parallel \qquad \qquad \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$L^{\bullet} \xrightarrow{\tau} C(f)^{\bullet} \longrightarrow C(\tau)^{\bullet} \longrightarrow L^{\bullet}[1].$$

Proof. Note that  $C(\tau)^i = L^{i+1} \oplus C(f)^i = L^{i+1} \oplus K^{i+1} \oplus L^i$ . We define  $g^i(k^{i+1}) = (-f^{i+1}(k^{i+1}), k^{i+1}, 0)$ . It can be checked easily that g makes the above diagram commute. In  $K(\mathcal{A})$ , we can check that the projection  $C(\tau)^i = L^{i+1} \oplus K^{i+1} \oplus L^i \to K^{i+1}$  is the inverse of g.

Using the mapping cone, we can now prove (4.9).

Proof of (4.9). (LS1) is obvious. Given

$$Z \qquad Y' \xrightarrow{f'} X'$$

$$\downarrow_{s \in S} \qquad \downarrow_{s' \in S}$$

$$X \xrightarrow{f} Y \qquad Z',$$

We have the morphisms

$$Z \xrightarrow{s} Y \xrightarrow{\tau} C(s) \longrightarrow Z[1],$$

which is isomorphic to

$$C(\tau)[-1] \longrightarrow Y \xrightarrow{\tau} C(s) \longrightarrow C(\tau),$$

in  $K^{?}(a)$  by (4.10). (TR3) gives us a commutative diagram

$$C(\tau \circ f)[-1] \xrightarrow{t} X \xrightarrow{\tau \circ f} C(s) \longrightarrow C(\tau \circ f)$$

$$\downarrow \qquad \qquad \downarrow f \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$C(\tau)[-1] \longrightarrow Y \xrightarrow{\tau} C(s) \longrightarrow C(\tau),$$

Define  $W = C(\tau \circ f)[-1]$ . Since s is a quasi-isomorphism,  $H^{\bullet}(C(s)) = 0$ , thus  $t : W \to X$  is also a quasi-isomorphism. The other square can be complete similarly.

For (LS3), it is enough to show that for all  $f \in \operatorname{Hom}_{K^?(\mathcal{A})}(K^{\bullet}, L^{\bullet})$ ,  $sf \sim 0$  for some quasi-isomorphism s if and only if  $ft \sim 0$  for some quasi-isomorphism t. We only prove the only if part, the if part can be done similarly. Suppose  $s: L^{\bullet} \to M^{\bullet}$  and sf = hd + dh for some  $h: K^{\bullet} \to M^{\bullet}[-1]$ . Define  $g^{\bullet}: K^{\bullet} \to C(s)^{\bullet}[-1] = L^{\bullet} \oplus M^{\bullet}[-1]$  by letting  $g^i(k^i) = (f^i(k^i), -h^i(k^i))$ . We check that g is indeed a morphism:

$$g^{i+1}d_K^i - d_{C(s)[-1]}^i g^i = \begin{pmatrix} f^{i+1} \\ -h^{i+1} \end{pmatrix} d_K^i - \begin{pmatrix} d_L^i & 0 \\ -s^i & -d_M^{i-1} \end{pmatrix} \begin{pmatrix} f^i \\ -h^i \end{pmatrix}$$
$$= \begin{pmatrix} f^{i+1}d_K^i - d_L^i f^i \\ -h^{i+1}d_K^i + s^i f^i - d_M^{i-1} h^i \end{pmatrix} = 0.$$

It is clear that f is the composition of g and the projection map  $C(s)^{\bullet}[-1] \to L^{\bullet}$ .

$$C(g)^{\bullet}[-1] \xrightarrow{t} K^{\bullet} \xrightarrow{g} C(s)^{\bullet}[-1]$$

$$C(s)^{\bullet}[-1] \xrightarrow{f} L^{\bullet} \xrightarrow{s} M^{\bullet}$$

Since s is a quasi-isomorphism,  $H^{\bullet}(C(s)^{\bullet}) = 0$ , and thus the projection map  $t : C(g)[-1] \to K^{\bullet}$  is also a quasi-isomorphism. Define  $k^{\bullet}$  by

$$C(g)^{\bullet}[-1] \xrightarrow{k^{\bullet}} C(s)^{\bullet}[-2]$$

$$\parallel \qquad \qquad \parallel$$

$$K^{\bullet} \oplus L^{\bullet}[-1] \oplus M^{\bullet}[-2] \xrightarrow{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} L^{\bullet}[-1] \oplus M^{\bullet}[-2].$$

Since

$$\begin{split} d_{C(s)[-1]}^i &= \begin{pmatrix} d_L^i & 0 \\ -s^i & -d_{M[-1]}^{i-1} \end{pmatrix}, \\ d_{C(g)[-1]}^i &= \begin{pmatrix} d_K^i & 0 \\ -g^i & -d_{C(s)[-1]}^{i-1} \end{pmatrix} = \begin{pmatrix} d_L^i & 0 & 0 \\ -f^i & -d_L^{i-2} & 0 \\ h^i & s^{i-1} & d_{M[-1]}^{i-2}, \end{pmatrix} \end{split}$$

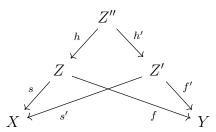
we see that

$$\begin{split} k^{i+1}d^i_{C(g)[-1]} + d^{i-1}_{C(s)[-1]}k^i &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d^i_L & 0 & 0 \\ -f^i & -d^{i-2}_L & 0 \\ h^i & s^{i-1} & d^{i-2}_{M[-1]}, \end{pmatrix} \\ &+ \begin{pmatrix} d^i_L & 0 \\ -s^i & -d^{i-1}_{M[-1]} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} f^i & 0 & 0 \\ -h^i & 0 & 0 \end{pmatrix} = gt, \end{split}$$

i.e.,  $gt \sim 0$ . Hence,  $ft \sim 0$ .

#### 4.4 Derived categories are additive

Let  $\mathcal{A}$  be an abelian category. Given  $\varphi$ ,  $\varphi' \in \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y)$ , which are represented by  $(Z \xrightarrow{s} X, f)$  and  $(Z' \xrightarrow{s'} X, f')$ , respectively. We construct  $\varphi + \varphi' \in \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y)$  as follows. By extension property (LS2), there exists Z'' and morphisms  $Z'' \xrightarrow{h} Z$ ,  $Z'' \xrightarrow{h'} Z'$  such that  $s \circ h = s' \circ h'$  is a quasi-isomorphism. Then we define  $\varphi + \varphi' = (s \circ h = s' \circ h', f \circ h + f' \circ h')$ .



With this definition, we verify that

**Proposition 4.11.**  $D^{?}(A)$  is additive.

## 4.5 Localization of subcategories

**Proposition 4.12.** We have  $D^{b}(X) \subseteq D^{\pm}(X) \subseteq D(X)$  as full subcategory.

Let  $\mathcal{C}$  be a category,  $S \subseteq \operatorname{Mor} \mathcal{C}$ . For  $\mathcal{B} \subseteq \mathcal{C}$  a full subcategory, define  $S_{\mathcal{B}} = S \cap \operatorname{Mor} \mathcal{C}$ . Question. When does the natural functor  $S_{\mathcal{B}}^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$  realize  $S_{\mathcal{B}}^{-1}$  as a full subcategory?

#### **Lemma 4.13.** Assume that S is a localizing system satisfying

- a)  $S_{\mathcal{B}}$  is a localizing system;
- b) For each  $X' \xrightarrow{s \in S} X$  with  $X \in \mathcal{B}$ , there exists  $X'' \to X'$  such that the composition  $(X'' \to X) \in S_{\mathcal{B}}$ ,

or b') with all arrows reversed.

Then  $S_{\mathcal{B}}^{-1}\mathcal{B} \hookrightarrow S^{-1}\mathcal{C}$  is fully faithful.

## 4.6 $\mathcal{A} \to \mathbf{D}^{?}(\mathcal{A})$ is a fully faithful embedding

Let  $K^{\bullet}$  be an object of Kom<sup>?</sup>( $\mathcal{A}$ ). We call  $K^{\bullet}$  an  $H^0$ -complex if  $H^i(K^{\bullet}) = 0$  for each  $i \neq 0$ .

**Proposition 4.14.** The functor  $\Phi : \mathcal{A} \to \mathrm{Kom}^{?}(\mathcal{A}) \to D^{?}(\mathcal{A})$  yields an equivalence of  $\mathcal{A}$  with the full subcategory of  $D^{?}(\mathcal{A})$  consisting of  $H^{0}$ -complexes.

Proof. Consider

$$H^0: D^?(\mathcal{A}) \to \mathcal{A}$$
  
 $X^{\bullet} \mapsto H^0(X^{\bullet}).$ 

For all  $X, Y \in \mathcal{A}$ , we have

$$\operatorname{Hom}_{\mathcal{A}}(X,Y) \xrightarrow{\Phi} \operatorname{Hom}_{D^{?}(\mathcal{A})}(X,Y) \xrightarrow{H^{0}} \operatorname{Hom}_{\mathcal{A}}(X,Y)$$

by definition. Now we prove that

$$\operatorname{Hom}_{D^{?}(\mathcal{A})}(X,Y) \xrightarrow{H^{0}} \operatorname{Hom}_{\mathcal{A}}(X,Y) \xrightarrow{\Phi} \operatorname{Hom}_{D^{?}(\mathcal{A})}(X,Y).$$

Given  $\varphi = (Z \xrightarrow{s} X, f) \in \operatorname{Hom}_{D^{?}(\mathcal{A})}(X, Y)$ .  $\Phi(H^{0}(\varphi))$  is represented by  $H^{0}(f) \circ H^{0}(s)^{-1}$ .

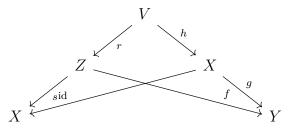
Let

$$V = \cdots \xrightarrow{d_Z^{-3}} Z^{-2} \xrightarrow{d_Z^{-2}} Z^{-1} \xrightarrow{d_Z^{-1}} \ker d_Z^0 \longrightarrow 0 \longrightarrow \cdots$$

$$\downarrow^r \qquad \qquad \downarrow^{\mathrm{id}} \qquad \downarrow^{\mathrm{id}} \qquad \downarrow \qquad \downarrow$$

$$Z = \cdots \xrightarrow{d_Z^{-3}} Z^{-2} \xrightarrow{d_Z^{-2}} Z^{-1} \xrightarrow{d_Z^{-1}} Z^0 \xrightarrow{d_Z^0} Z^1 \xrightarrow{d_Z^1} \cdots,$$

and let  $h: V \to X$  defined by  $h^0 = H^0(s) \circ (\ker d_Z^0 \to H^0(Z))$ , and  $h^{i\neq 0} = 0$ . We verify that



commutes in  $\mathrm{Kom}^?(\mathcal{A})$  and r is a quasi-isomorphism.

Given an  $H^0$ -complex Z, both morphisms  $r: V \to Z$  and  $V \to H^0(Z)$  are quasi-isomorphism. Hence Z lies in the essential image of  $\mathcal{A} \to D^?(\mathcal{A})$ .

Let  $\operatorname{Kom}_0^?(\mathcal{A}) \subseteq \operatorname{Kom}^?(\mathcal{A})$  be the full subcategory consisting of complexes with d=0. We regard as the full subcategory of  $\operatorname{Kom}^?(\mathcal{A})$ , by sending  $X \in \mathcal{A}$  to  $[X]^{\bullet}$  with  $[X]^0 = X$  and  $[X]^{i\neq 0} = 0$ .

**Proposition 4.15.** Assume that  $\mathcal{A}$  is semi-simple, i.e., every short exact sequence splits in  $\mathcal{A}$ . Show that

$$\operatorname{Kom}_0^?(\mathcal{A}) \xrightarrow{\sim} \operatorname{Kom}^?(\mathcal{A}) \xrightarrow{\sim} D^?(\mathcal{A}).$$

So 
$$K^{\bullet} \cong \bigoplus_{i} H^{i}(K^{\bullet})[-i].$$

**Proposition 4.16.** Show that the essential image of  $D^+(A) \to D(A)$  consists of complexes  $X^{\bullet}$  with  $H^i(X^{\bullet}) = 0$  for all  $i \ll 0$ . The similar statement hold for  $D^-(A)$  and  $D^{\mathrm{b}}(A)$ .

## 5 Triangulated categories

We've seen that D(A) is additive. Though D(A) is not abelian for most A, it is nevertheless a triangulated category.

#### 5.1 The data of triangulated categories

A triangulated category  $\mathcal{T}$  is an additive category together with:

• an additive automorphism [1]:  $\mathcal{T} \to \mathcal{T}$ , called the shift functor; from this, a triangle is a diagram of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

where a morphism of triangles is a commutative diagram

• and a collection of triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

called exact or distinguished triangles, denoted by \( \text{\( \omega\)}\) (we also use the notation

$$\begin{array}{ccc} X & & & Y \\ \swarrow & & & \\ Z & & & \end{array} \right)$$

subject to the axioms  $(TR1) \sim (TR4)$  below.

**Example 5.1.** For  $K^{?}(\mathcal{A})$  and  $D^{?}(\mathcal{A})$ , [1] is the shift functor induced by [1]: Kom<sup>?</sup>( $\mathcal{A}$ )  $\rightarrow$  Kom<sup>?</sup>( $\mathcal{A}$ ). The distinguished triangles are all triangles isomorphic to

$$X^{\bullet} \xrightarrow{f} Y^{\bullet} \longrightarrow \operatorname{Cone}(f) \longrightarrow X^{\bullet}[1]$$

for some morphism  $f: X^{\bullet} \to Y^{\bullet}$  of complexes. Recall that if  $X^{\bullet} \to Y^{\bullet} \to Z^{\bullet} \to X^{\bullet}[1]$  is a distinguished triangle, then we have a long exact sequence

$$\cdots \to H^i(X^{\bullet}) \to H^i(Y^{\bullet}) \to H^i(Z^{\bullet}) \to H^{i+1}(X^{\bullet}) \to \cdots$$

Proposition 5.2. Let

$$0 \to X^{\bullet} \to Y^{\bullet} \to Z^{\bullet} \to 0$$

be a short exact sequence in  $Kom^{?}(A)$ . Then

$$X \longrightarrow Y \longrightarrow Z \stackrel{0}{\longrightarrow} X[1].$$

is a distinguished triangle in  $D^{?}(A)$ .

Given  $X^{\bullet} \in D^{?}(\mathcal{A})$ , we define the canonical truncations

$$\tau_{\leq i} X^{\bullet} = (\cdots \to X^{i-2} \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} \ker d^{i} \to 0 \to \cdots)$$
$$\tau_{>i} X^{\bullet} = (\cdots \to 0 \to \operatorname{Im} d^{i} \to X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots).$$

(5.2) implies that the triangle

$$\tau_{\leq i} X^{\bullet} \stackrel{f}{\longrightarrow} X^{\bullet} \longrightarrow \tau_{>i} \longrightarrow \tau_{\leq i} X^{\bullet}[1]$$

is distinguished. We have

$$H^{j}(\tau_{\leq i}X^{\bullet}) = \begin{cases} H^{j}(X^{\bullet}), & \text{if } j \leq i \\ 0, \text{ else,} \end{cases} H^{j}(\tau_{>i}X^{\bullet}) = \begin{cases} H^{j}(X^{\bullet}), & \text{if } j > i \\ 0, \text{ else.} \end{cases}$$

**Remark.** Shouldn't confound  $\tau_{\leq i}$ ,  $\tau_{>i}$  with the naive truncation

$$\sigma_{\leq i} X^{\bullet} = (\cdots \to X^{i-1} \xrightarrow{d^{i-1}} X^{i} \to 0 \to \cdots)$$
  
$$\sigma_{>i} X^{\bullet} = (\cdots \to 0 \to X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots).$$

The triangle

$$\sigma_{\leq i} X^{\bullet} \xrightarrow{f} X^{\bullet} \longrightarrow \sigma_{>i} X^{\bullet} \longrightarrow \sigma_{\leq i} X^{\bullet}[1]$$

is also distinguished.

## 5.2 Axioms and properties of triangulated categories

(TR1)

• 
$$\triangle : X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1].$$

• Given an isomorphism of triangles

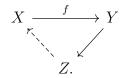
$$\Delta': \qquad X' \longrightarrow Y' \longrightarrow Z' \longrightarrow X'[1]$$

$$\downarrow^{\wr} \qquad \downarrow^{\wr} \qquad \downarrow^{\wr}$$

$$\Delta: \qquad X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

 $\triangle'$  is distinguished if and only if  $\triangle$  is distinguished.

• Any morphism  $f: X \to Y$  can be completed to a distinguished triangle



(TR2) A triangle

$$X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{v} Z^{\bullet} \xrightarrow{w} X^{\bullet}[1]$$

is distinguished if and only if

$$Y^{\bullet} \stackrel{v}{\longrightarrow} Z \stackrel{w}{\longrightarrow} X^{\bullet}[1] \stackrel{-u[1]}{\longrightarrow} Y^{\bullet}[1]$$

is distinguished.

(TR3) Given two distinguished triangles

$$\triangle = \begin{array}{c} X \xrightarrow{r} Y \\ Z, \end{array} \qquad \triangle = \begin{array}{c} X' \xrightarrow{r} Y' \\ Z' \end{array}$$

and a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow^f & & \downarrow^g \\ X & \longrightarrow & Y. \end{array}$$

there exists  $h: Z' \to Z$  such that the diagram

$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow X'[1]$$

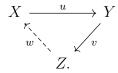
$$\downarrow^f \qquad \downarrow^g \qquad \downarrow^h \qquad \downarrow^{f[1]}$$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

commutes.

**Remark.** The map  $h: Z' \to Z$  above is not required to be unique (in any sense). If  $\mathcal{T}$  satisfies (TR1), (TR2), (TR3), we call  $\mathcal{T}$  a pre-triangulated category.

**Proposition 5.3.** Given a distinguished triangle



- (i)  $v \circ u = 0$ .
- (ii) For each  $U \in \mathcal{T}$ , the sequence

$$\cdots \to \operatorname{Hom}(U,X[i]) \xrightarrow{u[i] \circ} \operatorname{Hom}(U,Y[i]) \xrightarrow{v[i] \circ} \operatorname{Hom}(U,Z[i]) \to \operatorname{Hom}(U,X[i+1]) \to \cdots$$

is exact (similar for Hom(X[i], U) etc.)

*Proof.* (i) Consider the morphism of distinguished triangles

$$X \longrightarrow X \longrightarrow 0 \longrightarrow X[1]$$

$$\downarrow u \qquad \downarrow f \qquad \qquad \downarrow \downarrow$$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

where f exists by (TR3). Hence  $v \circ u = f \circ 0 = 0$ .

(ii) By (TR2), it is enough to show that the sequence is exact at Hom(U, Y[i]). By (i),  $v[i] \circ u[i] \circ = (v \circ u)[i] \circ = 0$ , so  $\text{Im}(u[i] \circ) \subseteq \text{ker}(v[i] \circ)$ . For each  $f \in \text{ker}(v[i] \circ)$ , we have the morphism of distinguished triangles

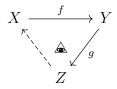
$$V \longrightarrow V \longrightarrow 0 \longrightarrow V[1]$$

$$\downarrow^{g} \qquad \downarrow^{f} \qquad \downarrow^{g[1]}$$

$$X[i] \xrightarrow{u[i]} Y[i] \xrightarrow{v[i]} Z[i] \xrightarrow{w[i]} X[i+1],$$

where g exists by (TR3). Hence  $f = u[i] \circ g \in \text{Im}(u[i] \circ)$ .

**Proposition 5.4.** In (TR3), if f and g are isomorphisms, so is h. In particular, for each  $f: X \to Y$ , the isomorphism class of the object Z in the distinguished triangle



completing f is unique.

We write Cone(f) := Z and call it the mapping cone of f.

*Proof.* For each  $U \in \mathcal{T}$ , apply  $\operatorname{Hom}_{\mathcal{T}}(U, -)$  to the diagram

$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow X'[1] \longrightarrow Y'[1]$$

$$\downarrow^{f} \qquad \downarrow^{g} \qquad \downarrow^{h} \qquad \downarrow^{f[1]} \qquad \downarrow^{g[1]}$$

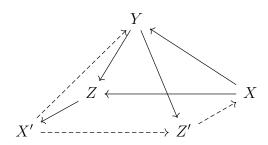
$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \longrightarrow Y[1].$$

It follows from (5.3) that the two rows of

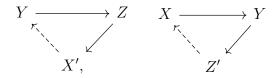
are both exact. Since  $f_*$ ,  $g_*$ ,  $f[1]_*$ ,  $g[1]_*$  are isomorphisms, the five lemma shows that  $h_*$  is also an isomorphism. Hence, by Yoneda's lemma,  $h: Z' \to Z$  is an isomorphism.

**Remark.** Given  $f: X \to Y$  in  $\mathcal{T}$ . Only the isomorphism class of  $\operatorname{Cone}(f)$  is well-defined. In general there is no functorial construction of  $\operatorname{Cone}(f)$ .

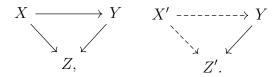
#### (TR4) Assume we have the following diagram



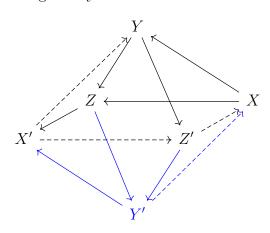
containing the distinguished triangles



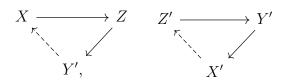
and the commutative diagrams



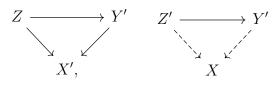
Then we can complete the diagram by



with

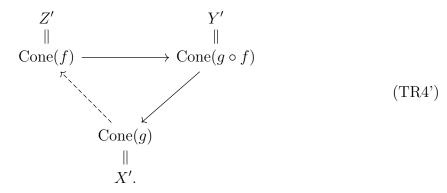


distinguished and



commute.

(TR4) implies that given  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathcal{T}$ , there exists a distinguished triangle



Unfolding the octahedron, we have

(TR4) implies the following

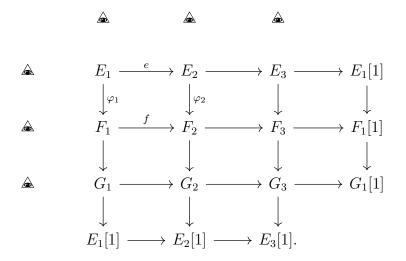
Lemma 5.5. Each commutative diagram

$$E_1 \xrightarrow{e} E_2$$

$$\downarrow^{\varphi_1} \qquad \downarrow^{\varphi_2}$$

$$F_1 \xrightarrow{f} F_2$$

can be completed to a commutative diagram

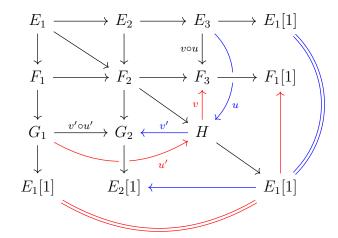


*Proof.* We already know that  $G_1$ ,  $G_2$ ,  $E_3$ ,  $F_3$  exist (mapping cone). Let  $H = \text{Cone}(E_1 \to F_2)$ . Applying (TR4') and ( $\Upsilon$ ) to both  $E_1 \xrightarrow{e} E_2 \xrightarrow{\varphi_2} F_2$  and  $E_1 \xrightarrow{\varphi_1} F_1 \xrightarrow{f} F_2$  gives

$$E_3 \xrightarrow{u} H \xrightarrow{v'} G_2 \longrightarrow E_3[1]$$

$$G_1 \xrightarrow{u'} H \xrightarrow{v} F_3 \longrightarrow G_1[1]$$

such that



is commutative. Define  $G_3 = \operatorname{Cone}(E_3 \xrightarrow{u} H \xrightarrow{v} F_3)$ . We finish the proof by applying (TR4') and (\Upsilon') to  $E_3 \xrightarrow{u} H \xrightarrow{v} F_3$ .

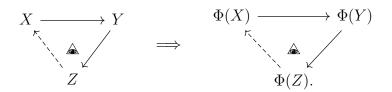
**Theorem 5.6.** The homotopy categories  $K^{?}(A)$  are triangulated.

We've already proven that  $K^{?}(A)$  satisfies (TR1) ~ (TR3).

**Remark.** The homotopy category  $K^{?}(\mathcal{A})$  of  $\mathcal{A}$  makes sense as long as  $\mathcal{A}$  is an additive category.  $K^{?}(\mathcal{A})$  is again a triangulated category with [1] and distinguished triangles are defined the same way.

#### 5.3 Exact functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be triangulated categories. A triangle functor (or exact functor) is an additive functor  $\Phi: \mathcal{C} \to \mathcal{D}$  such that  $\Phi \circ [1] = [1] \circ \Phi$  and  $\Phi$  preserves distinguished triangles, i.e.,



**Example 5.7.** For every additive functor  $F: \mathcal{A} \to \mathcal{B}$  between abelian categories, the induced functor  $K^{?}(\mathcal{A}) \to K^{?}(\mathcal{B})$  is exact.

Exactness is preserved under adjunction.

**Proposition 5.8.** Let  $F: \mathcal{C} \to \mathcal{D}$  be an exact functor. If  $G \dashv F$  or  $F \dashv G$ , then  $G: \mathcal{D} \to \mathcal{C}$  is exact. In particular, if F is an equivalence of category, then its quasi-inverse is also an exact functor.

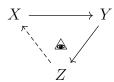
*Proof.* We prove the proposition when  $F \dashv G$ .

$$G \circ [1] = [1] \circ G$$
: Since  $F \circ [n] = [n] \circ F$ , we have

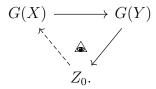
$$\operatorname{Hom}(A, G(B[1])) \cong \operatorname{Hom}(F(A), B[1]) \cong \operatorname{Hom}(F(A)[-1], B)$$
  
 $\cong \operatorname{Hom}(F(A[-1]), B) \cong \operatorname{Hom}(A[-1], G(B)) \cong \operatorname{Hom}(A, G(B)[1]),$ 

and all these isomorphisms are functorial in  $A \in \mathcal{C}$  and  $B \in \mathcal{D}$ . By Yoneda's lemma,  $G(B[1]) \cong G(B)[1]$ , functorial in B. Hence  $G \circ [1] = [1] \circ G$ .

G preserves distinguished triangles: Given distinguished triangle



in D. We complete  $G(X) \to G(Y)$  into a distinguished triangle



Apply F to this distinguished triangle, we get

$$FG(X) \longrightarrow FG(Y) \longrightarrow F(Z_0) \longrightarrow FG(X)[1]$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

where  $FG(X) \to X$  and  $FG(Y) \to Y$  are adjunction morphisms, and the blue arrow

 $F(Z_0) \to Z$  comes from (TR3). We have morphisms of triangles

where  $\triangle_1$  is distinguished and we want to show  $\triangle_3$  is also distinguished. For each  $A \in \mathcal{C}$ , as

is exact, (\*) and the five lemma implies

$$\operatorname{Hom}(A, G(Z)) \cong \operatorname{Hom}(A, Z_0),$$

functorial in A. So  $\triangle_1 \to \triangle_3$  implies  $G(Z) \cong Z_0$  by Yoneda's lemma. Hence  $\triangle_3 \cong \triangle_1$  and therefore  $\triangle_3$  is exact.

**Definition 5.9.** An equivalence of triangulated categories is an exact functor  $\Phi: \mathcal{C} \to \mathcal{D}$  which is an equivalence of categories. A triangulated subcategory of  $\mathcal{D}$  is a subcategory  $\mathcal{C} \subseteq \mathcal{D}$  carrying a structure of triangulated category such that  $\mathcal{C} \hookrightarrow \mathcal{D}$  is exact.

**Proposition 5.10.** Let  $\mathcal{C}$  be a full subcategory of a triangulated category  $\mathcal{D}$ . Then  $\mathcal{C} \subseteq \mathcal{D}$  is a triangulated subcategory if and only if

- (i) [1] restricts to an automorphism on C;
- (ii) for each distinguished triangle



in  $\mathcal{D}$ ,  $X, Y \in \mathcal{C}$  implies Z is isomorphic to an object in  $\mathcal{C}$ .

**Example 5.11.**  $K^{\mathrm{b}}(\mathcal{A}) \subseteq K^{\pm}(\mathcal{A}) \subseteq K(\mathcal{A})$  and  $D^{\mathrm{b}}(\mathcal{A}) \subseteq D^{\pm}(\mathcal{A}) \subseteq D(\mathcal{A})$  are inclusions of full triangulated subcategory.

**Remark.** The canonical truncations  $\tau_{\leq i}$ ,  $\tau_{>i}:D^{?}(\mathcal{A})\to D^{?}(\mathcal{A})$  are not exact functors.

#### 5.4 Localizations of triangulated categories

When does a localization of a triangulated category  $\mathcal{T}$  carry a natural structure of triangulated category?

Let S be a collection of morphisms in  $\mathcal{T}$ . We say that S is compatible with triangulation if

- $f \in S$  if and only if  $f[1] \in S$
- for each diagram

there exists  $h: Z' \to Z$  so that we can complete the diagram

$$\begin{array}{ccccc} & & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ & & \downarrow^f & & \downarrow^g & & \downarrow^h & & \downarrow^{f[1]} \\ & & & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1]. \end{array}$$

**Theorem 5.12.** Let  $\mathcal{T}$  be a triangulated category, and let S be a localized system compatible with triangulation. Then  $S^{-1}\mathcal{T}$  is a triangulated category, with respect to:

- The natural shift functor on  $S^{-1}\mathcal{T}$ .
- A triangle in  $S^{-1}\mathcal{T}$  is called distinguished if it is isomorphic to the image of a distinguished triangle under  $\mathcal{T} \to S^{-1}\mathcal{T}$ .

Corollary 5.13. The derived category  $D^{?}(A)$ , together with the shift functor and distinguished triangles defined before, is a triangulated category.

## Interlude: On diagram chasing

Let  $\mathcal{A}$  be an abelian category. As we mentioned before, in many situations we can't regard objects of  $\mathcal{A}$  as sets. Even if they are sets (e.g.,  $\mathcal{A} = \mathsf{Ab}$ ), in general we can't understand set-theoretically ker f and coker f. For instance,

- monomorphism  $\neq$  injection (e.g.,  $\mathcal{A} =$  the category of divisible abelian groups,  $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ );
- epimorphism  $\neq$  surjection (e.g.,  $\mathcal{A} = \text{Ring}, \mathbb{Z} \to \mathbb{Q}$ ).

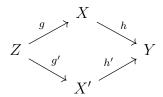
Fortunately, there exists a formalism allowing us to understand morphisms in  $\mathcal{A}$ , as if they were sets. Here we summarize the rules.

## **Elements**

Let Y be an object of  $\mathcal{A}$ . An element y of Y (denoted  $y \in Y$ ) is an equivalence class of pairs

$$(X \in \mathcal{A}, h : X \to Y)$$

by the equivalence relation  $(X,h) \sim (X',h')$  if there exists



with g, g' epic.

## Maps induced by morphisms

Given a morphism  $f: Y_1 \to Y_2$  in  $\mathcal{A}$ , we get

$$f: \{ \text{ elements of } Y_1 \} \rightarrow \{ \text{ elements of } Y_2 \}$$
  
 $(X, h) \mapsto (X, f \circ h).$ 

# Diagram chasing rules

The element  $0 \in Y$  is defined by  $0 \mapsto Y$ .

**Proposition.** Let  $f: Y_1 \to Y_2$  be a morphism in  $\mathcal{A}$ .

- (i) f is a monomorphism if and only if for each  $y \in Y_1$ , f(y) = 0 implies y = 0, or equivalently, for all  $y, y' \in Y_1$ , f(y) = f(y') implies y = y'.
- (ii) f is an epimorphism if and only if for each  $y_2 \in Y_2$ , there exists  $y_1 \in Y_1$  such that  $f(y_1) = y_2$ .

- (iii) f = 0 if and only if for each  $y \in Y$ , f(y) = 0.
- (iv)  $Y_1 \xrightarrow{f} Y \xrightarrow{g} Y_2$  is exact if and only if  $g \circ f = 0$  and for each  $y \in Y$  with g(y) = 0, there exists  $y_1 \in Y_1$  such that  $f(y_1) = y$ .

For each  $y = (X, h) \in Y$ , define  $-y = (X, -h) \in Y$ .

**Proposition.** Let  $g: Y_1 \to Y_2$  be a morphism in  $\mathcal{A}$ . Let  $y, y' \in Y_1$  such that g(y) = g(y'). Then exists  $z \in Y_1$  such that g(z) = 0 and

- for each  $f: Y_1 = Y$  such that f(y) = 0, we have f(z) = -f(y');
- for each  $f: Y_1 = Y$  such that f(y') = 0, we have f(z) = -f(y).

## 6 Derived functors

Let  $F: \mathcal{A} \to \mathcal{B}$  be a covariant additive functors between abelian categories. Assume F is left exact (e.g.,  $F = \Gamma(X, -) : \mathsf{QCoh}(X/k) \to \mathsf{Vect}_k$ ). We want to construct the right derived functor

$$RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

which is an exact functor such that  $R^iF = H^i(RF)$ . That RF is exact implies, e.g., for each short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in  $\mathcal{A}$ .

$$\mathsf{R}F(X) \to \mathsf{R}F(Y) \to \mathsf{R}F(Z) \to \mathsf{R}F(X)[1]$$

is an distinguished triangle in  $D^+(\mathcal{B})$ , and for each left exact sequence

$$0 \to X \to Y \to Z$$

in  $\mathcal{A}$ . We have the long exact sequence

$$\cdots \to R^i F(X) \to R^i F(Y) \to R^i F(Z) \to R^{i+1} F(X) \to \cdots$$

Rough plan of construction:

(1) RF maps complexes to complexes, but we're not able to give a general explicit construction of  $RF(X^{\bullet})$  for every  $X^{\bullet} \in Kom(\mathcal{A})$ .

- (2) Instead, we only define  $\mathsf{R}F(X^{\bullet})$  for complexes  $X^{\bullet}$  consisting of F-adapted objects  $\mathcal{I}_F \subseteq \mathcal{A}$ .
- (3) The full subcategory  $\mathcal{I}_F \subseteq \mathcal{A}$  satisfies

$$S^{-1}K^+(\mathcal{I}_F) \xrightarrow{\sim} D^+(\mathcal{A}),$$

so enough to define RF on  $K^+(\mathcal{I}_F)$ .

Sometimes we will only consider functors  $F: \mathcal{A} \to \mathcal{B}$  which are left exact. For right exact functors  $G: \mathcal{A} \to \mathcal{B}$ , the construction of left derived functor  $LG: D^-(\mathcal{A}) \to D^-(\mathcal{B})$  and other statements are similar.

## 6.1 *F*-adapted objects

A complex  $X^{\bullet} \in \text{Kom}(\mathcal{A})$  is called acyclic if  $H^{i}(X^{\bullet}) = 0$  for all i.

**Definition 6.1.** Let  $F : \mathcal{A} \to \mathcal{B}$  be a left (resp. right) exact functor. A class of objects  $\mathcal{I}_F \subseteq \mathcal{A}$  is called F-adapted if the following conditions hold.

- (i)  $\mathcal{I}_F$  is stable under finite direct sums.
- (ii) F sends acyclic complexes in  $\mathrm{Kom}^+(\mathcal{I}_F)$  (resp.  $\mathrm{Kom}^-(\mathcal{I}_F)$ ) to acyclic complexes.
- (iii) Any object in  $\mathcal{A}$  is a subobject (resp. quotient) of some object of  $\mathcal{I}_F$ .

If  $\mathcal{I}_F$  satisfies (i) and (iii) we also say that  $\mathcal{A}$  contains sufficiently many objects in  $\mathcal{I}_F$ , or  $\mathcal{I}_F$  is sufficiently large.

**Example 6.2.** An object  $I \in \mathcal{A}$  is called injective if for each monomorphism  $f: A \to B$  and for each morphism  $g: A \to I$ , there exists a lifting

$$A \xrightarrow{g} \mathring{|}_{\tilde{g}}$$

$$A \xrightarrow{f} B.$$

Let  $\mathcal{I} \subseteq \mathcal{A}$  be the full subcategory of injective objects.

**Proposition 6.3.** If  $\mathcal{A}$  contains enough injective objects, then  $\mathcal{I}$  is F-adapted for each left-exact functor F.

*Proof.* Given  $I, J \in \mathcal{I}$ . For any monomorphism  $f: A \to B$  and any morphism  $(g, h): A \to I \oplus J$ , we have liftings

We see that  $(\tilde{g}, \tilde{h})$  gives a lifting

$$A \xrightarrow{f} B.$$

$$I \oplus J$$

$$\uparrow_{(\tilde{g},\tilde{h})}$$

So  $\mathcal{I}$  is stable under finite direct sums.

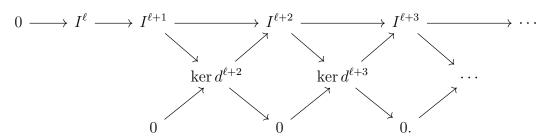
Let

$$I^{\bullet}: \quad 0 \to I^{\ell} \xrightarrow{d^{\ell}} I^{\ell+1} \xrightarrow{d^{\ell+1}} I^{\ell+2} \to \cdots$$

be an acyclic complex in  $\mathrm{Kom}^+(\mathcal{I}_F)$ . We want to show that

$$F(I^{\bullet}): 0 \to F(I^{\ell}) \xrightarrow{F(d^{\ell})} F(I^{\ell+1}) \xrightarrow{F(d^{\ell+1})} F(I^{\ell+2}) \to \cdots$$

is also acyclic. Since  $H^i(I^{\bullet}) = 0$  for each i, we may decompose  $I^{\bullet}$  into exact sequences



For each  $i \geq \ell + 1$ , since  $I^{i+1}$  is injective, the short exact sequence

$$0 \longrightarrow \ker d^{i+1} \xrightarrow{d^i} \ker d^{i+1} \longrightarrow \ker d^{i+2} \longrightarrow 0$$

$$\downarrow I^i$$

splits. Hence, the left-exactness of F gives a short exact sequence

$$0 \longrightarrow F(\ker d^{i+1}) \xrightarrow{F(d^i)} F(I^{i+1}) \longrightarrow F(\ker d^{i+2}) \longrightarrow 0$$

for each i. These exact sequences patches together into a long exact sequence  $F(I^{\bullet})$ .

**Proposition 6.4.** For each  $X^{\bullet}\mathcal{I}K^{+}(\mathcal{A})$  and for each  $I^{\bullet} \in K^{+}(\mathcal{I})$ , the natural map

$$\operatorname{Hom}_{K(\mathcal{A})}(X^{\bullet}, I^{\bullet}) \to \operatorname{Hom}_{D(\mathcal{A})}(X^{\bullet}, I^{\bullet})$$

is an isomorphism.

Proof.

Dually, an object  $P \in \mathcal{A}$  is called projective if for each epimorphism  $f: B \to A$  and for each morphism  $g: P \to A$ , there exists a lifting

$$P \downarrow g \downarrow g \downarrow A.$$

$$B \xrightarrow{\tilde{g}_f} A.$$

Let  $\mathcal{P} \subseteq \mathcal{A}$  be the full subcategory of projective objects. Again, if  $\mathcal{A}$  contains enough projective object then  $\mathcal{P}$  is F-adapted for each right exact functor G.

Let  $\mathcal{I}_F \subseteq \mathcal{A}$  be a full subcategory of F-adapted objects for some left exact functor  $F: \mathcal{A} \to \mathcal{B}$ . Then  $\mathcal{I}_F$  is an additive subcategory of  $\mathcal{A}$ . Recall that  $K^?(\mathcal{I}_F)$  is an triangulated category.

**Lemma 6.5.** Let S be the class of quasi-isomorphisms in  $K^+(\mathcal{I}_F)$ . Then S is a localizing system in  $K^+(\mathcal{I}_F)$  compatible with triangulation.

*Proof.* Adapt the proof that quasi-isomorphisms in  $K^{?}(A)$  form a localizing system compatible with triangulation.

**Proposition 6.6.** The natural functor  $\Psi: S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$  is an equivalence of triangulated categories.

Proof. First, we show that  $S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$  is essentially surjective. Namely, for each  $C^{\bullet} \in \mathrm{Kom}^+(\mathcal{A})$ , there exists  $I^{\bullet} \in \mathrm{Kom}^+(\mathcal{A})$  such that  $C^{\bullet}$  is quasi-isomorphic to  $I^{\bullet}$ . We may assume that  $C^i = 0$  for each i < 0. Construct  $C^{\bullet} \to I^{\bullet}$  by induction: For the initial step, since  $\mathcal{I}_F$  is F-adapted, we can find a monomorphism  $C^0 \hookrightarrow I^0 \in \mathcal{I}_F$ . Consider the fibered coproduct

$$0 \longrightarrow C^0 \longrightarrow C^1$$

$$\downarrow \qquad \qquad \downarrow$$

$$I^0 \longrightarrow C^1 \sqcup_{C_0} I^0,$$

We then take a monomorphism  $C^1 \sqcup_{C_0} I^0 \hookrightarrow I^1 \in \mathcal{I}_F$ . Then, we complete the diagram

$$0 \longrightarrow C^0 \longrightarrow C^1$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow I^0 \longrightarrow I^1.$$

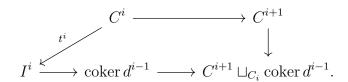
Now, assume that we have

$$0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^i$$

$$\downarrow^{s^0} \qquad \qquad \downarrow^{s^i}$$

$$0 \longrightarrow I^0 \longrightarrow \cdots \xrightarrow{d^{i-1}} I^i.$$

Consider



Then take a monomorphism  $C^{i+1} \sqcup_{C_i} \operatorname{coker} d^{i-1} \hookrightarrow I^{i+1} \in \mathcal{I}_F$  so that we can complete the diagram

$$0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^i \longrightarrow C^{i+1}$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^i \longrightarrow I^{i+1}.$$

This gives us a complex  $I^{\bullet} \in \text{Kom}^+(\mathcal{I}_F)$  and a morphism  $s^{\bullet} : C^{\bullet} \to I^{\bullet}$ . We still need to show that  $H^i(C^{\bullet}) \to H^i(I^{\bullet})$  is an isomorphism for each  $i \geq 0$ , which is a simple diagram chasing. (\*)

We show that  $S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$  is fully faithful. By (4.13), it is enough show for each quasi-isomorphism  $X \stackrel{s}{\to} X'$  with  $X \in K^+(\mathcal{I}_F)$ , there exists a morphism  $X' \to X''$  such that the composition  $X \to X'$  is a quasi-isomorphism in  $K^+(\mathcal{I}_F)$ . We just prove the existence of the quasi-isomorphism  $X' \to X''$ .

To show that the equivalence  $S^{-1}K^+(\mathcal{I}_F) \xrightarrow{\sim} D^+(\mathcal{A})$  is triangulated, it suffices to verify that

- [1] restricts to an automorphism on  $S^{-1}K^+(\mathcal{I}_F)$ ;
- for each distinguished triangle



with  $X, Y \in \text{Kom}^+(\mathcal{I}_F)$ , Z is quasi-isomorphic to some object in  $\text{Kom}^+(\mathcal{I}_F)$ .

Both are clear.

### 6.2 Construction of the derived functor

Let  $F: \mathcal{A} \to \mathcal{B}$  be a left exact functor, and let  $\mathcal{I}_F \subseteq \mathcal{A}$  be a full subcategory of F-adapted objects. We have seen that

$$\Psi: S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$$

defines an equivalence of categories.

### **Lemma 6.7.** We have a factorization

$$K^{+}(\mathcal{I}_{F}) \xrightarrow{I^{\bullet} \mapsto F(I^{\bullet})} D^{+}(\mathcal{B})$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$S^{-1}K^{+}\mathcal{I}_{F}.$$

*Proof.* By the universal property of localization, enough to show that F maps a quasi-isomorphism  $f: X^{\bullet} \to Y^{\bullet}$  in  $\mathrm{Kom}^+(\mathcal{I}_F)$  to a quasi-isomorphism in  $\mathrm{Kom}^+(\mathcal{B})$ .

Since f is a quasi-isomorphism, C(f) is acyclic, so C(F(f)) = F(C(f)) is also acyclic. Hence, F(f) is a quasi-isomorphism.

Choose a quasi-inverse

$$\Phi: D^+(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^+(\mathcal{I}_F)$$

of  $\Psi$ .

**Definition 6.8.** The right derived functor of F is

$$RF: D^{+}(\mathcal{A}) \xrightarrow{\Phi} S^{-1}K^{+}(\mathcal{I}_{F}) \xrightarrow{\overline{F}} D^{+}(\mathcal{B})$$

$$I^{\bullet} \mapsto F(I^{\bullet}).$$

Define  $R^i F(X^{\bullet}) = H^i(\mathsf{R} F(X^{\bullet}).$ 

**Proposition 6.9.** The functor RF is an exact functor.

*Proof.* It is clear that both  $\Phi$  and  $\overline{F}$  commute with [1]. Since  $\overline{F}$  sends distinguished triangles to distinguished triangles (because F(C(f)) = C(F(f))). So  $\overline{F}$  is exact. It remains to show that  $\Phi$  sends distinguished triangles to distinguished triangles; namely, for each triangle



in  $S^{-1}K^+(\mathcal{I}_F)$ , which is distinguished in  $D^+(\mathcal{A})$ , it is isomorphic in  $S^{-1}K^+(\mathcal{I}_F)$  to some distinguished triangle.

We represent  $\varphi: X \to Y$  by  $(T \xrightarrow{s} X, f)$ , where  $T \in \mathrm{Kom}^+(\mathcal{I}_F)$ . Then there is a commutative diagram

$$T \xrightarrow{g} Y \xrightarrow{C(s)} T[1] \qquad \triangleq \text{ in } S^{-1}K^{+}(\mathcal{I}_{F})$$

$$\downarrow^{s} \qquad \qquad \downarrow \qquad \qquad \downarrow^{s[1]}$$

$$X \xrightarrow{\varphi} Y \xrightarrow{S} Z \xrightarrow{X[1]} \qquad \triangleq \text{ in } D^{+}(\mathcal{A}),$$

where the blue arrow (a morphism in  $D^+(A)$ ) exists by (TR3). Since s is a quasi-isomorphism, C(s) is quasi-isomorphic to Z. As  $\Phi$  is fully faithful, it gives an isomorphism

$$T \longrightarrow Y \longrightarrow C(s) \longrightarrow T[1]$$

$$\downarrow^{\wr} \qquad \qquad \downarrow^{\wr} \qquad \qquad \downarrow^{\wr}$$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

of triangles in  $S^{-1}K^+(\mathcal{I}_F)$ .

**Remark.** More generally, given any exact functor  $F: K^+(A) \to K^+(B)$ . Assume that there exists a full triangulated subcategory  $\mathcal{I}_F \subseteq K^+(A)$  such that

- for each  $X^{\bullet} \in K^{+}(\mathcal{A})$ , there exists a quasi-isomorphism  $X^{\bullet} \to I^{\bullet} \in \mathcal{I}_{F}$ ;
- $I^{\bullet} \in \mathcal{I}_F$  is acyclic implies  $F(I^{\bullet})$  is acyclic.

Then  $RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$  exists.

# 6.3 How unique is RF?

A priori, RF depends on  $\mathcal{I}_F$  and  $\Phi: D^+(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^+(\mathcal{I}_F)$ . Given another  $\mathcal{I}_F'$  and  $\Phi': D^+(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^+(\mathcal{I}_F')$ , the two constructions of RF are related by a natural isomorphism

$$D^{+}(\mathcal{A}) \xrightarrow{\Phi} \int_{\overline{F}'}^{F} D^{+}(\mathcal{B})$$

$$S^{-1}K^{+}(\mathcal{I}_{F}')$$

$$S^{-1}K^{+}(\mathcal{I}_{F}')$$

defined as follows:

Recall that  $\Phi$  and  $\Psi$  are quasi-inverse. So there exists  $\beta: \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi \circ \Phi$ . Similarly, there exists  $\beta': \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi' \circ \Phi'$ . Given  $X^{\bullet} \in D^+(\mathcal{A})$ , and let

$$I^{\bullet} = \Phi(X^{\bullet}), \quad J^{\bullet} = \Phi'(X^{\bullet}).$$

 $\beta$  and  $\beta'$  yields quasi-isomorphisms  $X^{\bullet} \to I^{\bullet}$  and  $X^{\bullet} \to J^{\bullet}$ , which are functorial in  $X^{\bullet}$ . Since  $\overline{F}$  maps quasi-isomorphisms to quasi-isomorphisms, we get quasi-isomorphisms  $F(X^{\bullet}) \to F(I^{\bullet})$  and  $F(X^{\bullet}) \to F(J^{\bullet})$ , which gives a quasi-isomorphism

$$\overline{F}(\Phi(X^{\bullet})) = F(I^{\bullet}) \to F(J^{\bullet}) = \overline{F}'(\Phi'(X^{\bullet}))$$

in  $D^+(\mathcal{B})$ , functorial in  $X^{\bullet}$ . Hence,  $\overline{F}\Phi$  is natural isomorphic to  $\overline{F}'\Phi'$ .

**Proposition 6.10** (Universal property of RF). We have a morphism of functors

$$K^{+}(\mathcal{A}) \xrightarrow{Q_{\mathcal{A}}} D^{+}(\mathcal{A})$$

$$\downarrow^{\varepsilon} \qquad D^{+}(\mathcal{B})$$

$$K^{+}(\mathcal{B}).$$

$$K^{+}(\mathcal{B}).$$

Moreover, for each exact functor  $G: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ , and for each morphism  $Q_{\mathcal{B}} \circ K(F) \to G \circ Q_{\mathcal{A}}$ , there exists a  $\nu: \mathsf{R}F \to G$  such that the following diagram commute.

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}}$$

$$\downarrow^{\varepsilon} \qquad \qquad \downarrow^{\varepsilon}$$

$$\mathsf{R}F \circ Q_{\mathcal{A}}.$$

Here we only define  $\varepsilon: Q_{\mathcal{B}} \circ K(F) \to \mathsf{R}F \circ Q_{\mathcal{A}}$ . As before, choose  $\beta: \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi \circ \Phi$ . Then for each  $X \in K^+(\mathcal{A})$ , there is a functorial resolution  $Q_{\mathcal{A}}(X) \to I^{\bullet} = \Phi(Q_{\mathcal{A}}(X)) \in \mathrm{Kom}^+(\mathcal{I}_F)$  in  $D^+(\mathcal{A})$ , represented by  $(X \xrightarrow{s} C, I^{\bullet} \xrightarrow{t} C)$  in  $K^+(\mathcal{A})$ . Apply K(F), we get  $(K(F)(X) \xrightarrow{K(F)(s)} K(F)(C), K(F)(I^{\bullet}) \xrightarrow{K(F)(t)} K(F)(C))$ .

We can assume that  $C \in \mathrm{Kom}^+(\mathcal{I}_F)$ . As C,  $I^{\bullet}$ , t is a quasi-isomorphism implies F(C(t)) = C(F(t)) is acyclic, which means K(F)(t) is a quasi-isomorphism. We get  $Q_{\mathcal{B}}K(F)(X) \to Q_{\mathcal{B}}K(F)(I^{\bullet})$  functorial in X. As the diagram

commutes, we get

$$Q_{\mathcal{B}}K(F)(I^{\bullet}) = \overline{F}(I^{\bullet}) = \overline{F}(\Phi(Q_{\mathcal{A}}(X))) = \mathsf{R}F(Q_{\mathcal{A}}(X)).$$

This defines  $\varepsilon: Q_{\mathcal{B}} \circ K(F) \to \mathsf{R}F \circ Q_{\mathcal{A}}$ .

## 6.4 The largest F-adapted class

Let  $F : \mathcal{A} \to \mathcal{B}$  be a left exact functor. Given an F-adapted class of objects  $\mathcal{I}_F$ , sometimes there aren't so many objects in  $\mathcal{I}_F$  to compute RF in practice.

**Example 6.11.** Let  $(X, \mathcal{O}_X)$  be a ringed space, and let  $\mathcal{I}_F$  be the class of injective  $\mathcal{O}_X$ -modules.  $\mathcal{O}_X$ -mod has enough injective objects, but essentially the only injective resolution we know is the "Godement" resolution  $(\mathscr{F} \hookrightarrow \mathscr{I} \text{ with } \mathscr{I}(U) := \prod_{x \in U} I_x$ , where  $I_x$  is an injective abelian group that contains  $\mathscr{F}_X$ ), which is useless in practive, to compute e.g.,  $\mathsf{R}\Gamma$ , where

$$\Gamma: \mathcal{O}_X - \mathsf{mod} \to \mathsf{Ab}$$
  $\mathscr{F} \mapsto \Gamma(X, \mathscr{F}).$ 

We want to find  $\mathcal{I}_F$  containing as many objects as possible. Assume that  $\mathsf{R}F$  exists. An object  $X \in \mathcal{A}$  is called F-acyclic if  $R^i F(X) = 0$  for each  $i \neq 0$ .

**Theorem 6.12.** Let  $\mathcal{Z} \subseteq \mathcal{A}$  be the full subcategory of F-acyclic objects.

- (i)  $\mathcal{I}_F \subseteq \mathcal{Z}$  for any F-adapted class  $\mathcal{I}_F$ .
- (ii)  $\mathcal{I}_F$  exists if and only if  $\mathcal{Z}$  is sufficiently large.

In this case,  $\mathcal{Z}$ , and also all sufficiently large subclasses of  $\mathcal{Z}$ , are F-adapted.

*Proof.* (i) For each  $X \in \mathcal{I}_F$ , we have  $R^i F(X) = H^i(F(X)[0]) = 0$  for each  $i \neq 0$ .

(ii) The only if part follows from (i). It remains to show that every sufficiently large subclass  $R \subseteq \mathcal{Z}$  is F-adapted; namely, F sends acyclic complexes  $K^{\bullet} \in \mathrm{Kom}^+(R)$  to acyclic complexes. Note that if

$$0 \rightarrow K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow 0$$

is exact with  $K_1, K_2, K_3 \in R$ ,

$$0 \rightarrow F(K_1) \rightarrow F(K_2) \rightarrow F(K_3) \rightarrow 0$$

is also exact (because  $R^iF(K_j)=0$  for  $i\neq 0$  and j=1,2,3). Now given an acyclic complex

$$(\cdots \to 0 \to K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots) \in \mathrm{Kom}^+(R),$$

we decompose it into short exact sequences

$$0 \to K^0 \to K^1 \to \operatorname{Im} d^1 \to 0,$$

$$0 \to \operatorname{Im} d^1 \to K^2 \to \operatorname{Im} d^2 \to 0,$$

$$\vdots$$

$$0 \to \operatorname{Im} d^i \to K^{i+1} \to \operatorname{Im} d^{i+1} \to 0,$$

$$\vdots$$

Apply RF and induction on i, we get  $\operatorname{Im} d^i \in R$ . So

$$F(\operatorname{Im}(K^i \xrightarrow{d^i} K^{i+1})) = \operatorname{Im}(F(K^i) \xrightarrow{F(d^i)} F(K^{i+1}))$$

and

$$0 \to \operatorname{Im} F(d^{i}) \to F(K^{i+1}) \to \operatorname{Im} F(d^{i+1})$$

is exact. Hence  $\operatorname{Im} F(d^i) = \ker F(d^{i+1})$ , i.e.,  $F(K^{\bullet})$  is acyclic.

# 6.5 Composition of derived functors

Let  $F: \mathcal{A} \to \mathcal{B}$ ,  $G: \mathcal{B} \to \mathcal{C}$  be left exact functors between abelian categories such that  $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$ . Given  $X \in D^+(\mathcal{A})$ , it is easy to see that  $R(G \circ F)(X) \cong RG(RF(X))$ : choose  $I^{\bullet} \in \text{Kom}^+(\mathcal{I}_F)$  that is quasi-isomorphic to X, then

$$\mathsf{R}(G \circ F)(X) \cong G(F(I^{\bullet})) \cong \mathsf{R}G(F(I^{\bullet})) \cong \mathsf{R}G(\mathsf{R}F(I^{\bullet})).$$

Next proposition says that  $R(G \circ F)(X) \cong RG(RF(X))$  is functorial in  $X \in D^+(A)$ .

**Proposition 6.13.** Assume that  $\mathcal{I}_F \subseteq \subseteq \mathcal{A}$  (resp.  $\mathcal{I}_G \subseteq \mathcal{B}$ ) is an F-adapted (resp. G-adapted) class such that  $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$ . Then  $\mathsf{R}F$ ,  $\mathsf{R}G$ ,  $\mathsf{R}(G \circ F)$  exist and

$$R(G \circ F) \cong RG \circ RF.$$

# Interlude 2: Spectral sequences

Let  $F: \mathcal{A} \to \mathcal{B}$  and  $G: \mathcal{B} \to \mathcal{C}$  be left exact functors between abelian categories. Assume that

- there exists F-adapted class  $\mathcal{I}_F \subseteq \mathcal{A}$ ;
- there exists G-adapted class  $\mathcal{I}_G \subseteq \mathcal{B}$ ;
- $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$ ,

which implies that RF, RG,  $R(G \circ F)$  exist and  $R(G \circ F) \cong RG \circ RF$ . We can use spectral sequence to relate  $R^pG(R^qF(X))$  and  $R^{p+q}(G \circ F)(X)$ .

**Proposition 6.14.** For each  $X \in D^+(A)$ , there exists a spectral sequence

$$E_2^{p,q} = R^p G(R^q F(X)) \Rightarrow R^{p+q}(G \circ F)(X).$$

## Definition

Let  $\mathcal{A}$  be an abelian category. A spectral sequence in  $\mathcal{A}$  consists of

- $E^n \in \mathcal{A}$  for each  $n \in \mathbb{Z}$ ;
- for each  $r \in \mathbb{Z}_{\geq 0}$ ,  $\{E_r^{p,q} \in \mathcal{A} \mid p,q \in \mathbb{Z}\}$ , called the  $r^{\text{th}}$  page;
- $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ , such that  $d_r \circ d_r = 0$ ;
- isomorphisms

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{Im}(d_r : E_r^{p-r,q+r_1} \to E_r^{p,q})},$$

subject to the following conditions:

• For each (p,q), there exists  $r_0$  such that for all  $r \geq r_0$ ,

$$E_r^{p-r,q+r-1} \xrightarrow{d_r=0} E_r^{p,q} \xrightarrow{d_r=0} E_r^{p+r,q-r+1}.$$

This implies  $E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong \cdots =: E_{\infty}^{p,q}$ .

• Each  $E^n$  admits a decreasing filtration

$$\cdots \subseteq F^{p+1}E^n \subseteq F^pE^n \subseteq F^{p-1}E^n \subseteq \cdots,$$
 with  $\bigcap_p F^pE^n = 0$  and  $\bigcup_p F^pE^n = E^n$  such that 
$$E^{p,q}_\infty \cong F^pE^{p+q}/_{F^{p+1}E^{p+q}} =: \operatorname{Gr}_F^pE^{p+q}.$$

We say that the spectral sequence  $\{E_r^{p,q}\}$  converges to  $\{E^n\}$ , and write  $E_r^{p,q} \Rightarrow E^{p+q}$ . We say that  $\{E_r^{p,q}\}$  degenerate at page  $r_0$  if  $d_r^{p,q} = 0$  for each  $r \geq r_0$  and for each p, q. In this case,  $E_{\infty}^{p,q} = E_{r_0}^{p,q}$ . If  $\mathcal{A}$  is semisimple, then  $E^n \cong \bigoplus_{p+q=n} E_{r_0}^{p,q}$  (non-canonical).

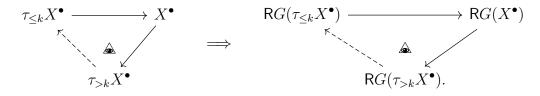
## An example

Let  $G: \mathcal{A} \to \mathcal{B}$  be left exact. Assume there exists a G-adapted class  $\mathcal{I}_G$ . Then we get the right derived functor  $RG: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ .

Corollary 6.15. If  $RG(A) \in D^{b}(\mathbb{B})$  for each  $A \in \mathcal{A}$ , then RG induces

$$RG: D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{B}).$$

*Proof.* Let  $X^{\bullet} \in D^{\mathrm{b}}(\mathcal{A})$ . Then for each  $k \in \mathbb{Z}$ ,



Then induction on the number of integers j such that  $H^j(X^{\bullet}) \neq 0$  completes the proof.

We present a proof using spectral sequences.

*Proof.* Let  $X^{\bullet} \in D^{b}(\mathcal{A})$ . Apply (6.14) to G and  $F = \mathrm{id}_{A}$ , we get

$$E_2^{p,q} = R^pG(R^qF(X^\bullet)) = R^pG(H^q(X^\bullet)) \Rightarrow R^{p+q}(G)(X^\bullet).$$

Since  $X^{\bullet} \in D^{\mathrm{b}}(\mathcal{A})$  and  $\mathsf{R}G(H^q(X^{\bullet})) \in D^{\mathrm{b}}(X)$ , there exists C > 0 such that  $E_2^{p,q} = R^pG(H^q(X^{\bullet})) = 0$  for all |p|, |q| > C. As  $E_{\infty}^{p,q}$  is a sub-quotient of  $E_2^{p,q}$ , we have  $E_{\infty}^{p,q} = 0$  for all |p|, |q| > C. Hence  $R^{p+q}G(X^{\bullet}) = E^{p+q} = 0$  whenever |p+q| > 2C, so  $\mathsf{R}G(X^{\bullet}) \in D^{\mathrm{b}}(\mathcal{B})$ .

# 7 Examples of derived functors

## **7.1** Ext

Let  $\mathcal{A}$  be an abelian category.

**Definition 7.1.** For each  $X, Y \in \mathcal{A}$ , we define

$$\operatorname{Ext}_{A}^{i}(X,Y) := \operatorname{Hom}_{D(A)}(X[0], Y[i]).$$

Some immediate properties of Ext:

- $\operatorname{Ext}^0(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y).$
- $\operatorname{Ext}^{i}(X,Y) = \operatorname{Hom}_{D(\mathcal{A})}(X[k],Y[k+i])$  for each  $k \in \mathbb{Z}$ .
- For each  $X, Y, Z \in \mathcal{A}$ , we have a bilinear map

$$\operatorname{Ext}^{i}(X,Y) \times \operatorname{Ext}^{j}(Y,Z) \to \operatorname{Ext}^{i+j}(X,Z)$$

defined by composition. It is called the Yoneda product.

• We have seen that every short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

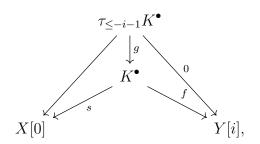
induces a long exact sequence

$$\cdots \to \operatorname{Ext}^i(X'',Y) \to \operatorname{Ext}^i(X,Y) \to \operatorname{Ext}^i(X',Y) \to \operatorname{Ext}^{i+1}(X'',Y) \to \cdots$$

and similar for  $\operatorname{Ext}^{i}(Y, -)$ .

**Proposition 7.2.** For each i < 0,  $\operatorname{Ext}^{i}(X, Y) = 0$ .

*Proof.* Given  $X[0] \to Y[i]$ , represented by  $(K^{\bullet} \xrightarrow{s} X[0], f)$ . Since i < 0, we have



where the composition  $s \circ g$  is a quasi-isomorphism. Hence  $\operatorname{Ext}^i(X,Y) = \operatorname{Hom}(X,Y[i]) = 0$ .

**Proposition 7.3.** Assume that  $\mathcal{A}$  has enough injectives (resp. projectives). Then  $\operatorname{Ext}^i(X,-) \cong R^i \operatorname{Hom}(X,-)$  (resp.  $\operatorname{Ext}^i(-,X) \cong R^i \operatorname{Hom}(-,X)$ ).

*Proof.* For each  $A^{\bullet}$ ,  $B^{\bullet} \in \text{Kom}(\mathcal{A})$ , define

$$\operatorname{Hom}^n(A^{\bullet}, B^{\bullet}) = \prod_{i \in \mathbb{Z}} \operatorname{Hom}(A^i, B^{n+i})$$

and

$$d^n: \operatorname{Hom}^n(A^{\bullet}, B^{\bullet}) \to \operatorname{Hom}^{n+1}(A^{\bullet}, B^{\bullet})$$

$$f \mapsto d_B \circ f - (-1)^n f \circ d_A.$$

Then  $(\operatorname{Hom}^n(A^{\bullet}, B^{\bullet}), d^n)$  is a complex. We have

$$\ker d^i = \operatorname{Hom}_{\operatorname{Kom}(\mathcal{A})}(A^{\bullet}, B^{\bullet}[i]), \quad H^i(\operatorname{Hom}^{\bullet}(A^{\bullet}, B^{\bullet})) = \operatorname{Hom}_{K(\mathcal{A})}(A^{\bullet}, B^{\bullet}[i]).$$

Now given  $Y \in \mathcal{A}$ , let  $Y \to I^{\bullet}$  be an injective resolution. Then by (6.4)

$$\operatorname{Ext}^{i}(X,Y) = \operatorname{Hom}_{D(\mathcal{A})}(X,I^{\bullet}[i]) \cong \operatorname{Hom}_{K(\mathcal{A})}(X,I^{\bullet}[i])$$
$$= H^{i}(\operatorname{Hom}^{\bullet}(X,I^{\bullet})) = H^{i}(\operatorname{Hom}(X,I^{\bullet}[i])) = R^{i}\operatorname{Hom}(X,Y).$$

The second statement can be proved similarly.

## 7.2 Tensor product

### **7.2.1** *R*-modules

Let R be a ring with 1, and let N be a left R-module. Then  $F = - \otimes_R N : \mathsf{mod}\text{-}R \to \mathsf{Ab}$  is a right exact functor. Flat modules form a class of F-adapted objects. Then we get

$$\mathsf{L} F = - \otimes_R^{\mathsf{L}} N : D^-(R\operatorname{\mathsf{-mod}}) \to D^-(\mathsf{Ab})$$

We define  $\operatorname{Tor}_i^R(M,N) = H^{-i}(M \otimes_R^{\mathsf{L}} N)$ .

### 7.2.2 Coherent sheaves

Let X be a variety over a field k. Let  $\mathscr{F} \in \mathsf{Coh}\,X$ . Then

$$-\otimes \mathscr{F}: \mathsf{Coh}\, X \to \mathsf{Coh}\, X$$

is right exact.

Assume X is quasi-projective. Then for each  $\mathscr{E} \in \mathsf{Coh}\, X,\, \mathscr{E}$  has a resolution

$$\cdots \to \mathscr{L}^1 \to \mathscr{L}^0 \to \mathscr{E}$$

by locally free sheaves of finite rank. Moreover if  $\mathscr{E}^{\bullet} \in \mathrm{Kom}^{-}(\mathsf{Coh}\,X)$  is an acyclic complex of locally free sheaves, then  $\mathscr{E}^{\bullet} \otimes \mathscr{F}$  is also acyclic. So locally free sheaves on X form a class of objects adapted for  $-\otimes \mathscr{F}$ . Hence, we get

$$-\otimes^{\mathsf{L}}\mathscr{F}:D^{-}(X)\to D^{-}(X)=:D^{-}(\mathsf{Coh}\,X),$$

and  $\mathcal{T}or_1(\mathscr{E},\mathscr{F}) := H^{-i}(\mathscr{E} \otimes^{\mathsf{L}} \mathscr{F}).$ 

Assume X is smooth quasi-projective. Then every  $\mathscr{E} \in \mathsf{Coh}\,X$  has a finite locally free resolution. Hence,

$$-\otimes^{\mathsf{L}}\mathscr{F}:D^{\mathrm{b}}(X)\to D^{\mathrm{b}}(X):=D^{\mathrm{b}}(\mathsf{Coh}\,X).$$

### 7.2.3 Complexes of coherent sheaves

Let X be a quasi-projective variety over a field k. Given  $\mathscr{E}^{\bullet}$ ,  $\mathscr{F}^{\bullet}$ . Define

$$(\mathscr{E}^{\bullet} \otimes \mathscr{F}^{\bullet})^{i} = \bigoplus_{p+q=i} \mathscr{E}^{p} \oplus \mathscr{F}^{q},$$

with  $d^i = d_{\mathscr{E}} \otimes \operatorname{id} + (-1)^i \operatorname{id} \otimes d_{\mathscr{F}}$ . Then

$$-\otimes \mathscr{F}^{\bullet}: K^{-}(\operatorname{Coh} X) \to K^{-}(\operatorname{Coh} X)$$

is right exact, and complexes of locally free sheaves in  $K^-(\mathsf{Coh}\,X)$  are again adapted for  $-\otimes\mathscr{F}^{\bullet}$ . This gives the derived functor  $-\otimes^{\mathsf{L}}\mathscr{F}^{\bullet}:D^-(X)\to D^-(X)$ . Furthermore, we get

$$-\otimes^{\mathsf{L}} -: D^{-}(X) \times D^{-}(X) \to D^{-}(X)$$

with  $D^{\mathrm{b}}(X) \times D^{\mathrm{b}}(X)$  if X is smooth.

## 7.3 Pullback and pushforward

Let  $f: X \to Y$  be a morphism of ringed spaces. Then  $f^*$  is the left adjoint of  $f_*$ . Hence,  $f^*$  is right exact and  $f_*$  is left exact.

Now, let X, Y be quasi-projective varieties over a field k. Then  $f^* : \operatorname{\mathsf{Coh}} Y \to \operatorname{\mathsf{Coh}} X$  is right exact. Since locally free sheaves are  $f^*$ -adapted, we get the left derived functor  $\mathsf{L} f^* : D^-(Y) \to D^-(X)$ , and  $\mathsf{L} f^*$  maps  $D^{\mathrm{b}}(Y)$  to  $D^{\mathrm{b}}(X)$  if Y is smooth.

If X and Y are noetherian schemes, we have  $f_*: \mathsf{QCoh} \to \mathsf{QCoh}$ , which is left exact. Since  $\mathsf{QCoh}$  has enough injectives, we have the right derived functor  $\mathsf{R}f_*: D^+(\mathsf{QCoh}\,X) \to D^+(\mathsf{QCoh}\,X)$ . We define the higher direct image  $R^if_*(\mathscr{F}^{\bullet})$  to be  $H^i(\mathsf{R}f_*\mathscr{F}^{\bullet})$ .

In particular, when X is defined over a field k,

$$\Gamma: \mathsf{QCoh}\,X \to \mathsf{Vect}_k$$

is the pushforward of  $p: X \to \operatorname{Spec} k$ . So we define

$$\mathsf{R}\Gamma = \mathsf{R}p_* : D^+(\mathsf{QCoh}\,X) \to D^+(\mathsf{QCoh}\,X),$$

and  $H^i(X, \mathscr{F}^{\bullet}) = R^i\Gamma(\mathscr{F}^{\bullet})$  the hypercohomology.

For each  $\mathscr{F} \in \mathsf{QCoh}\,X$ , as  $R^i f_* \mathscr{F} = 0$  for  $|i| \gg 1$ ,  $\mathsf{R} f_*$  induces

$$Rf_*: D^{\mathrm{b}}(\operatorname{\mathsf{QCoh}} X) \to D^{\mathrm{b}}(\operatorname{\mathsf{QCoh}} Y).$$

**Theorem 7.4.** Assume  $f: X \to Y$  is a proper morphism. Then  $f_*: \mathsf{Coh}\, X \to \mathsf{Coh}\, Y$ .

Assume  $f: X \to Y$  is proper. In general,  $\operatorname{\mathsf{Coh}} X$  does not have enough injectives so we can't define  $\mathsf{R} f_*: D^?(\operatorname{\mathsf{Coh}} X) \to D^?(\operatorname{\mathsf{Coh}} Y)$  as a derived functor of  $f_*: \operatorname{\mathsf{Coh}} X \to \operatorname{\mathsf{Coh}} Y$ . To define  $\mathsf{R} f_*: D^\mathrm{b}(\operatorname{\mathsf{Coh}} X) \to D^\mathrm{b}(\operatorname{\mathsf{Coh}} Y)$ , we need to proceed differently.

## 7.4 Derived category of coherent sheaves

**Definition 7.5.** A thick subcategory  $\mathcal{A}$  of an abelian category  $\mathcal{B}$  is a full abelian subcategory such that for each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with M',  $M'' \in \mathcal{A}$ , we have  $M \in \mathcal{A}$ .

Let  $\mathcal{A} \subseteq \mathcal{B}$  be a thick subcategory. Define  $D^?_{\mathcal{A}}(\mathcal{B}) \subseteq D^?(\mathcal{B})$  as the full subcategory of complexes  $X^{\bullet}$  with  $H^i(X^{\bullet}) \in \mathcal{A}$  for each i.

**Lemma 7.6.** Let X be a noetherian scheme and let  $\varphi : \mathscr{E} \to \mathscr{F}$  be a surjective morphism of quasi-coherent sheaves. Assume that  $\mathscr{F}$  is coherent. Then there exists a coherent subsheaf  $\mathscr{G} \subseteq \mathscr{E}$  such that  $\varphi|_{\mathscr{G}}$  is surjective.

**Proposition 7.7.** For any noetherian scheme X, and for ? = - or b,

$$D^{?}(X) := D^{?}(\mathsf{Coh}\,X) \xrightarrow{\sim} D^{?}_{\mathsf{Coh}\,X}(\mathsf{QCoh}\,X).$$

Proof. Let ? = b, the other case is similar. Let  $\mathscr{F}^{\bullet} \in D^{b}_{\mathsf{Coh}\,X}(\mathsf{QCoh}\,X)$ , and assume that  $\mathscr{F}^{i}$  is coherent for all  $i \geq i_{0}$  for some  $i_{0}$ . Since  $\mathscr{F}^{i_{0}} \to \mathrm{Im}\,d^{i_{0}}$  and  $\ker d^{i_{0}} \to \frac{\ker d^{i_{0}}}{\mathrm{Im}\,d^{i_{0}-1}}$  are surjections and the codomains are coherent, by (7.6) there exists coherent subsheaves  $\mathscr{F}' \subseteq \mathscr{F}^{i_{0}}$  and  $\mathscr{F}'' \subseteq \ker d^{i_{0}}$  such that  $\mathscr{F}' \to \mathrm{Im}\,d^{i_{0}}$  and  $\mathscr{F}'' \to \frac{\ker d^{i_{0}}}{\mathrm{Im}\,d^{i_{0}-1}}$  are also surjections. Replace  $\mathscr{F}^{i_{0}}$  by the coherent subsheaf  $\mathscr{F}^{i_{0}}_{\mathrm{new}}$  generated by  $\mathscr{F}'$ ,  $\mathscr{F}'' \subseteq \mathscr{F}^{i_{0}}$  and  $\mathscr{F}^{i_{0}-1}$  by  $(d^{i_{0}-1})^{-1}(\mathscr{F}^{i_{0}}_{\mathrm{new}})$ . Do this inductively, we finally get a complex of coherent sheaves that is quasi-isomorphic to the original complex.

**Proposition 7.8.** Let  $f: X \to Y$  be a proper morphism between noetherian schemes. Then

$$Rf_*: D^+(\operatorname{QCoh} X) \to D^+(\operatorname{QCoh} Y)$$

induces  $Rf_*: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ .

*Proof.* Enough to show that for each  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}_{\mathsf{Coh}\,X}(\mathsf{QCoh}\,X)$ , we have

$$\mathsf{R} f_* \mathscr{F}^{\bullet} \in D^{\mathrm{b}}_{\mathsf{Coh}\, Y}(\mathsf{QCoh}\, Y),$$

namely  $R^i f_* \mathscr{F}^{\bullet} \in \mathsf{Coh}\, Y$ . We have a spectral sequence

$$E_2^{p,q} = R^p f_*(H^q(\mathscr{F}^{\bullet})) \Rightarrow R^{p+q} f_* \mathscr{F}^{\bullet}.$$

Since  $R^p f_*(H^q(\mathscr{F}^{\bullet}))$  are coherent,  $E^{p,q}_{\infty}$  are also coherent. Since  $E^{p,q}_2 = 0$  for all |p|,  $|q| \gg 1$ ,  $E^{p,q}_{\infty} = 0$  for all |p|,  $|q| \gg 1$ . Thus  $R^{p+q} f_* \mathscr{F}^{\bullet}$  is an extension of finitely many coherent sheaves  $E^{p,q}_{\infty}$ , showing that  $R^{p,q} f_* \mathscr{F}^{\bullet}$  is coherent.

## 7.5 Local Hom

Let X be a quasi-projective variety over a field k. For  $\mathscr{E}^{\bullet} \in \mathrm{Kom}^{-}(\mathsf{QCoh}\,X)$  and  $\mathscr{F}^{\bullet} \in \mathrm{Kom}^{+}(\mathsf{QCoh}\,X)$ , we define

$$\mathscr{H}om^{i}(\mathscr{E}^{\bullet},\mathscr{F}^{\bullet}) = \prod_{j \in \mathbb{Z}} \mathscr{H}om(\mathscr{E}^{j},\mathscr{F}^{j+i})$$

with  $d^i(\{\mathscr{E}^j \xrightarrow{f^j} \mathscr{F}^{j+i}\}_j) = \{f^{j+1\circ d-(-1)^i d\circ f^j}\}_j$ . We get a left exact functor

$$\mathscr{H}om^{\bullet}(\mathscr{E}^{\bullet}, -) : K^{+}(\operatorname{QCoh} X) \to K^{+}(\operatorname{QCoh} X).$$

If  $\mathscr{E}^{\bullet} \in \operatorname{Kom^b}(\operatorname{\mathsf{Coh}} X)$ , it maps  $K^{\operatorname{b}}(\operatorname{\mathsf{Coh}} X)$  to itself.

Using the same argument constructing  $\mathsf{R} f_*: D^\mathrm{b}(X) \to D^\mathrm{b}(X)$ , we get the right derived functor

$$\mathsf{R}\mathscr{H}om(\mathscr{E}^{\bullet},-):D^+(\mathsf{QCoh}\,X)\to D^+(\mathsf{QCoh}\,X).$$

Again, it maps  $D^{\mathrm{b}}(X)$  to itself if  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$ .

Likewise, using the same argument constructing  $Lf^*$ , we get the right derived functor

$$R\mathscr{H}om(-,\mathscr{F}^{\bullet}): D^{-}(QCoh X) \to D^{+}(QCoh X),$$

which maps  $D^{\mathrm{b}}(X)$  to itself if  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$  and X is smooth. Using these derived functors, we get

$$\mathsf{R}\mathscr{H}\!\mathit{om}(-,-):D^-(\operatorname{\mathsf{QCoh}}\nolimits X)\times D^+(\operatorname{\mathsf{QCoh}}\nolimits X)\to D^+(\operatorname{\mathsf{QCoh}}\nolimits X),$$

which maps  $D^{\mathrm{b}}(X) \times D^{\mathrm{b}}(X)$  to  $D^{\mathrm{b}}(X)$ .

# 8 Some properties of derived functors in algebraic geometry

## 8.1 Composition

Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be morphisms of smooth quasi-projective varieties.

Proposition 8.1. We have

(1) 
$$R(g \circ f)_* \xrightarrow{\sim} Rg_* \circ Rf_*$$
;

(2) 
$$L(g \circ f)^* \xrightarrow{\sim} Rf^* \circ Rg^*$$
.

*Proof.* (1) Given an injective object  $\mathcal{I} \in \mathsf{QCoh}\,X$ . Then  $\mathcal{I}$  is flasque, so  $f_*\mathcal{I}$  is also flasque. Thus  $R^j g_*(f_*\mathcal{I}) = 0$  for each  $j \neq 0$ , i.e.,  $f_*$  sends injective objects to  $g_*$ -acyclic objects. This proves (1).

(2) If  $\mathscr{F}$  is a locally free sheaf on Z, then  $g^*\mathscr{F}$  is also locally free. This proves (2).

## 8.2 Projection formula

**Proposition 8.2.** Let  $f: X \to Y$  be a proper morphism between smooth quasiprojective varieties. For all  $\mathscr{E} \in D^{\mathrm{b}}(X)$  and  $\mathscr{F} \in D^{\mathrm{b}}(Y)$ , we have functorial isomorphisms

$$\mathsf{R}f_*\mathscr{E}\otimes^{\mathsf{L}}\mathscr{F}\xrightarrow{\sim}\mathsf{R}f_*(\mathscr{E}\otimes^{\mathsf{L}}\mathsf{L}f^*\mathscr{F}).$$

*Proof.* Let  $\mathcal{I}^{\bullet}$  be a complex of injective objects that is quasi-isomorphic to  $\mathscr{E}$ , and let  $\mathscr{L}^{\bullet}$  be a complex of locally free sheaves of finite length that is quasi-isomorphic to  $\mathscr{F}$ . Then it follows from the classical projection formula that

$$\mathsf{R} f_* \mathscr{E} \otimes^{\mathsf{L}} \mathscr{F} \cong (f_* \mathcal{I}^{\bullet}) \otimes \mathscr{L}^{\bullet} \cong f_* (\mathcal{I}^{\bullet} \otimes f^* \mathscr{L}^{\bullet}).$$

Since

$$\mathsf{R}f_*(\mathscr{E} \otimes^{\mathsf{L}} \mathsf{L}f^*\mathscr{F}) \cong \mathsf{R}f_*(\mathcal{I}^{\bullet} \otimes f^*\mathscr{L}^{\bullet}),$$

it suffices to show that the morphism

$$\alpha: f_*(\mathcal{I}^{\bullet} \otimes f^* \mathscr{L}^{\bullet}) \to \mathsf{R} f_*(\mathcal{I}^{\bullet} \otimes f^* \mathscr{L}^{\bullet}),$$

which comes from the natural transformation (see (6.10))

$$K^+(\operatorname{QCoh} X) \xrightarrow{\operatorname{R} f_*} D^+(\operatorname{QCoh} Y)$$

$$K^+(\operatorname{QCoh} Y),$$

is an isomorphism. The statement is local, so up to shrinking Y, we can assume that  $\mathscr{L}^i = \bigoplus_{J_i} \mathcal{O}_Y$ . Then  $f^*\mathscr{L}^i = \bigoplus_{J_i} \mathcal{O}_X$  and thus  $\mathcal{I}^j \otimes f^*\mathscr{L}^i \cong \bigoplus_{J_i} \mathcal{I}^j$  is injective. It follows from  $\mathcal{I}^{\bullet} \otimes f^*\mathscr{L}^{\bullet}$  that  $\alpha$  is an isomorphism.

Based on similar arguments, we can also show the following.

**Proposition 8.3.** Let  $f: X \to Y$  be a morphism of smooth projective varieties. Given  $\mathscr{E}^{\bullet}$ ,  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(Y)$ , we have

$$\mathsf{L} f^* \mathscr{E}^{\bullet} \otimes^{\mathsf{L}} \mathsf{L} f^* \mathscr{F}^{\bullet} \xrightarrow{\sim} \mathsf{L} f^* (\mathscr{E}^{\bullet} \otimes^{\mathsf{L}} f^{\bullet})$$

and the isomorphism is functorial.

## 8.3 Adjunction

**Proposition 8.4.** Let  $f: X \to Y$  be a morphism of smooth projective varieties. For all  $\mathscr{E} \in D^{\mathrm{b}}(X)$  and  $\mathscr{F} \in D^{\mathrm{b}}(Y)$ , we have functorial isomorphisms

$$\mathsf{R}\mathscr{H}om(\mathscr{F},\mathsf{R}f_*\mathscr{E})\xrightarrow{\sim}\mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathsf{L}f^*\mathscr{F},\mathscr{E}).$$

In particular,

$$\mathsf{R}\,\mathrm{Hom}(\mathscr{F},\mathsf{R}f_*\mathscr{E})\xrightarrow{\sim}\mathsf{R}\,\mathrm{Hom}(\mathsf{L}f^*\mathscr{F},\mathscr{E}),$$
 
$$\mathrm{Hom}(\mathscr{F},\mathsf{R}f_*\mathscr{E})\xrightarrow{\sim}\mathrm{Hom}(\mathsf{L}f^*\mathscr{F},\mathscr{E}).$$

*Proof.* The proof is similar to the projection formula. With the same notation in that proof, we have

$$\mathsf{R}\mathscr{H}om(\mathscr{F},\mathsf{R}f_*\mathscr{E})\cong\mathscr{H}om(\mathscr{L}^{\bullet},f_*\mathcal{I}^{\bullet})\cong f_*\mathscr{H}om(f^*\mathcal{L}^{\bullet},\mathcal{I}^{\bullet})$$

and

$$\mathsf{R} f_* \mathsf{R} \mathscr{H} om(\mathsf{L} f^* \mathscr{F}, \mathscr{E}) \cong \mathsf{R} f_* \mathscr{H} om)(f^* \mathscr{L}^{\bullet}, \mathcal{I}^{\bullet}).$$

Similar argument shows that  $f_*\mathscr{H}om(f^*\mathcal{L}^{\bullet},\mathcal{I}^{\bullet}) \to \mathsf{R}f_*\mathscr{H}om)(f^*\mathcal{L}^{\bullet},\mathcal{I}^{\bullet})$  is an isomorphism.

Let X be a smooth projective variety. We can also show the following in  $D^{\mathrm{b}}(X)$ : for all  $\mathscr{E}^{\bullet}$ ,  $\mathscr{F}^{\bullet}$ ,  $\mathscr{G}^{\bullet} \in D^{\mathrm{b}}(X)$ , we have

$$\mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet},\mathscr{E}^{\bullet})\otimes^{\mathsf{L}}\mathscr{G}^{\bullet}\cong\mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet},\mathscr{E}^{\bullet}\otimes^{\mathsf{L}}\mathscr{G}^{\bullet})\cong\mathsf{R}\mathscr{H}om(\mathsf{R}\mathscr{H}om(\mathscr{E}^{\bullet},\mathscr{F}^{\bullet}),\mathscr{G}^{\bullet}),$$
 
$$\mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet},\mathsf{R}\mathscr{H}om(\mathscr{E}^{\bullet},\mathscr{G}^{\bullet})\cong\mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet}\otimes^{\mathsf{L}}\mathscr{E}^{\bullet},\mathscr{G}^{\bullet}).$$

In particular, if  $\mathscr{G}^{\bullet} = \mathcal{O}_X$ , then

$$\mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet},\mathscr{E}^{\bullet})\cong (\mathscr{F}^{\bullet})^{\vee}\otimes^{\mathsf{L}}\mathscr{E}^{\bullet},$$

where  $(\mathscr{F}^{\bullet})^{\vee} = \mathsf{R}\mathscr{H}om(\mathscr{F}^{\bullet}, \mathcal{O}_X).$ 

## 8.4 Grothendieck-Verdier duality

Here is a particular case of the GV duality

**Theorem 8.5.** Let  $f: X \to Y$  be a proper morphism between smooth quasi-projective varieties over some field k. Then for all  $\mathscr{E} \in D^{\mathrm{b}}(\mathscr{E})$  and  $\mathscr{F} \in D^{\mathrm{b}}(\mathscr{F})$ , there exists a functorial isomorphism

$$\mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathscr{E},f^!\mathscr{F})\cong\mathsf{R}\mathscr{H}om(\mathsf{R}f_*\mathscr{E},\mathscr{F}),$$

where  $f^!\mathscr{F} = \mathsf{L} f^*\mathscr{F} \otimes \omega_X \otimes f^*\omega_Y^{\vee}[\dim X - \dim Y]$  and  $\omega_X$ ,  $\omega_Y$  are the canonical line bundles on X, Y, respectively.

In particular,

$$\operatorname{Hom}(\mathscr{E}, f^!\mathscr{F}) \cong \operatorname{Hom}(\mathsf{R}f_*\mathscr{E}, \mathscr{F}),$$

namely, f' is a right adjoint of  $f_*$ .

When  $f: X \to \operatorname{Spec} k$ , we obtain Serre duality

$$\operatorname{Hom}(\mathscr{E}, \omega_X[n]) \cong \operatorname{Hom}(\mathsf{R}\Gamma(\mathscr{E}^{\bullet}), k).$$

### 8.5 Serre functor

Let  $\mathcal{D}$  be a k-linear triangulated category. Assume

$$\sum_i \dim \operatorname{Hom}^i(\mathscr{E},\mathscr{F}) < \infty$$

for all  $\mathscr{E}$ ,  $\mathscr{F} \in \mathcal{D}$ .

**Definition 8.6.** A Serre functor is an autoequivalence

$$S: \mathcal{D} \to \mathcal{D}$$

such that for all  $\mathscr{E}, \mathscr{F} \in \mathcal{D}$ , there exists an isomorphism

$$\operatorname{Hom}(\mathscr{E},\mathscr{F}) \cong \operatorname{Hom}(\mathscr{F},S(\mathscr{E}))^{\vee},$$

which is functorial in  $\mathscr{E}$  and  $\mathscr{F}$ .

**Example 8.7.** Let X be a smooth projective variety over k. Then  $S = -\otimes(\omega_X[\dim X])$ :  $D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$  is a Serre functor. In particular, when  $\omega_X \cong \mathcal{O}_X$ , then  $[\dim X]$ :  $D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$  is a Serre functor.

For a triangulated category  $\mathcal{D}$ , if  $[n]: \mathcal{D} \to \mathcal{D}$  is a Serre functor, we call  $\mathcal{D}$  a Calabi-Yau category of dimension n.

**Proposition 8.8.** Given a k-linear exact functor

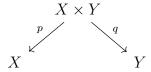
$$F: \mathcal{C} \to \mathcal{D}$$

between k-linear triangulated category, admitting Serre functors  $S_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$  and  $S_{\mathcal{D}}: \mathcal{D} \to \mathcal{D}$ . Then

- F has a left adjoint if and only if F has a right adjoint;
- F is an equivalence implies that  $F \circ S_{\mathcal{C}} \cong S_{\mathcal{D}} \circ F$ .

### 8.6 Fourier-Mukai transforms

Let X, Y be smooth projective varieties. Let



be the projections.

**Definition 8.9.** A Fourier-Mukai transform is a functor of the form

$$\Phi_{X \to Y}^{\mathscr{P}}: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$$

$$\mathscr{F}^{\bullet} \mapsto \mathsf{R}q_{*}(p^{*}\mathscr{F}^{\bullet} \otimes^{\mathsf{L}} \mathscr{P})$$

for some  $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$ . We call  $\mathscr{P}$  the Fourier-Mukai kernel of  $\Phi^{\mathscr{P}}$ .

We see that  $\Phi_{X\to Y}^{\mathscr{P}}$  is an exact functor. Given smooth projective varieties  $X,\,Y,\,Z$  and

$$\mathscr{P} \in D^{\mathrm{b}}(X \times Y), \quad \mathscr{Q} \in D^{\mathrm{b}}(Y \times Z),$$

the composition of the Fourier-Mukai transforms  $\Phi^{\mathcal{Q}} \circ \Phi^{\mathcal{P}}$  is also a Fourier-Mukai transform with kernel

$$\mathscr{R} = \mathsf{R}(p_{ZX})_*(p_{XY}^*\mathscr{P} \otimes^{\mathsf{L}} p_{YZ}^*\mathscr{Q}),$$

where  $p_{YZ}$ ,  $p_{ZX}$ ,  $p_{XY}$  are the projections

We call  $\mathscr{R}$  the convolution of  $\mathscr{P}$  and  $\mathscr{Q}$  and write  $\mathscr{R} = \mathscr{P} * \mathscr{Q}$ .

**Proposition 8.10.** Let  $f: X \to Y$  be a morphism of smooth projective varieties and let  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$ . Then  $\mathsf{R} f_*, \mathsf{L} f^*, -\otimes \mathscr{F}^{\bullet}$ , [1] are Fourier-Mukai transforms.

Theorem 8.11 (Orlov). Given an exact functor

$$F: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y).$$

If F is fully faithful, then  $F = \Phi_{X \to Y}^{\mathscr{P}}$  for some  $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$ . Moreover,  $\mathscr{P}$  is unique up to isomorphisms.

**Remark.** There exist examples of exact functor  $F: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$  which are not Fourier-Mukai transforms.

**Proposition 8.12.** Let  $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$ . We have

$$\Phi_{Y \to X}^{\mathscr{P}^{\vee}} \circ S_Y \dashv \Phi_{X \to Y}^{\mathscr{P}}, \quad \Phi_{X \to Y}^{\mathscr{P}} \dashv S_X \circ \Phi^{\mathscr{P}^{\vee}},$$

where  $S_X$ ,  $S_Y$  are the Serre functors and

$$\mathscr{P}^{\vee} = \mathsf{R}\mathscr{H}om(\mathscr{P}, \mathcal{O}_{Y\times X}) \in D^{\mathrm{b}}(Y\times X).$$

*Proof.* We only prove  $\Phi_{Y\to X}^{\mathscr{P}^{\vee}} \circ S_Y \dashv \Phi_{X\to Y}^{\mathscr{P}}$ . Given  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$  and  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(Y)$ , we have

$$\operatorname{Hom}(\Phi^{\mathscr{P}^{\vee}} \circ S_{Y}(\mathscr{F}^{\bullet}), \mathscr{E}^{\bullet}) = \operatorname{Hom}(\operatorname{R}p_{*}(\Phi^{\mathscr{P}^{\vee}} \otimes^{\mathsf{L}} q^{*}\mathscr{F}^{\bullet} \otimes q^{\bullet}\omega_{Y}[\operatorname{dim}Y], \mathscr{E}^{\bullet})$$

$$\cong \operatorname{Hom}(\operatorname{R}p * (\mathscr{P}^{\vee} \otimes^{\mathsf{L}} q^{*}\mathscr{F}^{\bullet} \otimes q^{*}\omega_{Y}), \mathscr{E}^{\bullet}[-\operatorname{dim}Y])$$

$$\cong \operatorname{Hom}(\mathscr{P}^{\vee} \otimes^{\mathsf{L}} q^{*}\mathscr{F}^{\bullet} \otimes q^{*}\omega_{Y}, p^{!}\mathscr{E}^{\bullet}[-\operatorname{dim}Y])$$

$$= \operatorname{Hom}(\mathscr{P}^{\vee} \otimes^{\mathsf{L}} q^{*}\mathscr{F}^{\bullet} \otimes q^{*}\omega_{Y}, p^{*}\mathscr{E}^{\bullet} \otimes p^{*}\omega_{X} \otimes q^{*}\omega_{Y} \otimes p^{*}\omega_{X}^{\vee})$$

$$\cong \operatorname{Hom}(\mathscr{P}^{\vee} \otimes^{\mathsf{L}} q^{*}\mathscr{F}^{\bullet}, p^{*}\mathscr{E}^{\bullet}) \cong \operatorname{Hom}(q^{*}\mathscr{F}^{\bullet}, \mathscr{P}^{\otimes}p^{*}\mathscr{E}^{\bullet})$$

$$\cong \operatorname{Hom}(\mathscr{F}^{\bullet}, q_{*}(\mathscr{P} \otimes p^{*}\mathscr{E}^{\bullet})) = \operatorname{Hom}(\mathscr{F}^{\bullet}, \Phi^{\mathscr{P}}(\mathscr{E}^{\bullet})).$$

# 9 Semi-orthogonal decomposition

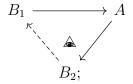
# 9.1 X is connected if and only if $D^{\mathbf{b}}(X)$ is indecomposable

### 9.1.1 Decomposition of triangulated category

Let  $\mathcal{T}$  be a triangulated category.

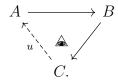
**Definition 9.1.** We say that  $\mathcal{T}$  is decomposed into triangulated subcategories  $\mathcal{C}_1$ ,  $\mathcal{C}_2 \subseteq \mathcal{T}$  if

i) for each  $A \in \mathcal{T}$ , there exists  $B_i \in \mathcal{C}_i$  such that



ii)  $\operatorname{Hom}(B_1, B_2) = \operatorname{Hom}(B_2, B_1) = 0$  for all  $B_i \in \mathcal{C}_i$ .

Proposition 9.2. Given a distinguished triangle



- If u = 0, then  $B = A \oplus C$ .
- If  $\mathcal{T}$  decomposed into  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ , then for each  $A \in \mathcal{T}$ , there exists  $B_1 \in \mathcal{C}_1$ ,  $B_2 \in \mathcal{C}_2$  such that  $A = B_1 \oplus B_2$ .

We say that  $\mathcal{T}$  is indecomposable if for each decomposition of  $\mathcal{T}$  into  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ , either  $\mathcal{C}_1 = 0$  or  $\mathcal{C}_2 = 0$ .

**Proposition 9.3.** Let X be a noetherian scheme. Then  $D^{\mathrm{b}}(X)$  is indecomposable if and only if X is connected.

### **9.1.2** Support

**Definition 9.4.** Let  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$ . The support of  $\mathscr{F}^{\bullet}$  is defined as

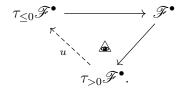
$$\operatorname{Supp} \mathscr{F}^{\bullet} = \bigcup_{i} \operatorname{Supp}(H^{i}(\mathscr{F}^{\bullet})).$$

**Lemma 9.5.** Suppose  $\mathscr{F}^{\bullet}$  is an object in  $D^{\mathrm{b}}(X)$  such that Supp  $\mathscr{F}^{\bullet} = Z_1 \sqcup Z_2$  with  $Z_1$ ,  $Z_2$  closed. Then  $\mathscr{F}^{\bullet} = \mathscr{F}_1^{\bullet} \oplus \mathscr{F}_2^{\bullet}$  with Supp  $\mathscr{F}_i^{\bullet} \subseteq Z_i$ .

*Proof.* We can assume that  $H^k(\mathscr{F}^{\bullet}) = 0$  for all k < 0 and  $H^0(\mathscr{F}^{\bullet}) \neq 0$ . We induction on the length of  $\mathscr{F}^{\bullet}$ , i.e., the maximal number  $\ell$  such that  $H^{\ell}(\mathscr{F}^{\bullet}) \neq 0$ .

If  $\ell = 0$ , then  $\mathscr{F}^{\bullet} \cong \mathscr{H}[0]$  with  $\mathscr{H} = H^0(\mathscr{F}^{\bullet}) \in \mathsf{Coh}\,X$ . We have  $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$  with  $\mathsf{Supp}\,\mathscr{H}_i \subseteq Z_i$ .

For general case, consider the distinguished triangle



We have  $\ell(\tau_{>0}\mathscr{F}^{\bullet}[1]) = \ell(\mathscr{F}^{\bullet}) - 1$ , so  $\tau_{>0}\mathscr{F}^{\bullet} \cong \mathscr{G}_{1}^{\bullet} \oplus \mathscr{G}_{2}^{\bullet}$  with  $\operatorname{Supp} \mathscr{G}_{i}^{\bullet} \subseteq Z_{i}$ . We have  $\tau_{\leq 0}\mathscr{F}^{\bullet} \cong \mathscr{H}[0]$  for  $\mathscr{H} = H^{0}(\mathscr{F}^{\bullet})$ . Write  $\mathscr{H} = \mathscr{H}_{1} \oplus \mathscr{H}_{2}$  with  $\operatorname{Supp} \mathscr{H}_{i} \subseteq Z_{i}$ .

Claim.  $\operatorname{Hom}(\mathscr{G}_{1}^{\bullet}, \mathscr{H}_{2}[1]) = 0$ .

*Proof of Claim.* Consider the spectral sequence (by taking the left exact functors  $\text{Hom}(-, \mathcal{H}_2)$  and id)

$$E_2^{p,q} = \operatorname{Ext}^p(H^{-q}(\mathscr{G}_1^{\bullet}), \mathscr{H}_2) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{G}_1^{\bullet}, \mathscr{H}_2).$$

For each q, we have the spectral sequence (by taking the left exact functors id and  $\mathscr{H}om(-,\mathscr{H}_2)$ )

$$(E^q)_2^{s,t} = H^s(X, \mathcal{E}xt^t(\mathcal{H}^{-q}(\mathscr{G}_1^{\bullet}), \mathscr{H}_2)) \Rightarrow \operatorname{Ext}^{s+t}(\mathcal{H}^{-q}(\mathscr{G}_1^{\bullet}), \mathscr{H}_2).$$

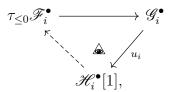
Since  $\mathscr{E}xt^t(\mathscr{H}^{-q}(\mathscr{G}_1^{\bullet}),\mathscr{H}_2)=0$  as  $\operatorname{Supp}(\mathscr{H}^{-q}(\mathscr{G}_1^{\bullet}))\cap\operatorname{Supp}\mathscr{H}_2=\varnothing,$ 

$$\operatorname{Ext}^p(\mathscr{H}^{-q}(\mathscr{G}_1^{\bullet}),\mathscr{H}_2) = 0$$

for each p. Hence,

$$\operatorname{Hom}(\mathscr{G}_1^{\bullet},\mathscr{H}_2[1]) = \operatorname{Ext}^1(\mathscr{G}_1^{\bullet},\mathscr{H}_2) = 0. \qquad \Box$$

Similarly,  $\operatorname{Hom}(\mathscr{G}_2, \mathscr{H}_1[1]) = 0$ . So  $u : \tau_{>0} \mathscr{F}^{\bullet} \to \tau_{\leq 0} \mathscr{F}^{\bullet}[1]$  can be decomposed into  $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$ , where  $u_1 : \mathscr{G}_1^{\bullet} \to \mathscr{H}_1[1]$  and  $u_2 : \mathscr{G}_2^{\bullet} \to \mathscr{H}_2[1]$ . Take  $\mathscr{F}_i^{\bullet}$  such that complete the distinguished triangle



then Supp  $\mathscr{F}_i^{\bullet} \subseteq Z_i$ . By (TR3) and (\*),

### 9.1.3 Proof of (9.3)

Assume that  $X = X_1 \sqcup X_2$ . Let  $j_i : X_i \hookrightarrow X$  be the inclusion.

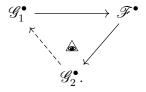
Lemma 9.6. The functor

$$R(j_i)_*: D^{\mathrm{b}}(X_i) \to D^{\mathrm{b}}(X)$$

is fully faithful, whose essential image consists of  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$  with Supp  $\mathscr{F}^{\bullet} \subseteq X_i$ .

Regarding  $D^{b}(X_{1})$ ,  $D^{b}(X_{2})$  as full subcategories of  $D^{b}(X)$ , we show that  $D^{b}(X)$  decomposed into  $D^{b}(X_{1})$ ,  $D^{b}(X_{2})$ :

• For each  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$ , (9.5) gives a decomposition  $\mathscr{F}^{\bullet} \cong \mathscr{G}_{1}^{\bullet} \oplus \mathscr{G}_{2}^{\bullet}$ , where  $\mathrm{Supp} \mathscr{G}_{i}^{\bullet} \subseteq X_{i}$ , So we have the distinguished triangle



• For each  $\mathscr{G}_1^{\bullet} \in D^{\mathrm{b}}(X_1)$  and each  $\mathscr{G}_2^{\bullet} \in D^{\mathrm{b}}(X_2)$ , we claim that

$$\operatorname{Hom}_{D^{\mathrm{b}}(X)}(\mathscr{G}_{1}^{\bullet},\mathscr{G}_{2}^{\bullet})=0.$$

Indeed, let  $f: \mathscr{G}_1^{\bullet} \to \mathscr{G}_2^{\bullet}$  in  $D^{\mathrm{b}}(X)$  represented by  $(\mathscr{G}_1^{\bullet} \to \mathcal{I}^{\bullet}, \mathscr{G}_2^{\bullet} \to \mathcal{I}^{\bullet})$  with  $\mathscr{G}_2^{\bullet} \to \mathcal{I}^{\bullet}$  quasi-isomorphism. We can assume that  $\operatorname{Supp} \mathscr{G}_1^j \subseteq X_1$  and  $\operatorname{Supp} \mathcal{I}^j \subseteq X_2$  for each j, so  $\mathscr{G}_1^{\bullet} \to \mathcal{I}^{\bullet}$  is zero.

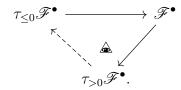
Same argument shows that  $\operatorname{Hom}_{D^{b}(X)}(\mathscr{G}_{2}^{\bullet},\mathscr{G}_{1}^{\bullet})=0.$ 

Now assume that X is connected. Suppose that  $D^{\mathrm{b}}(X)$  decomposes into  $D_1$ ,  $D_2 \subseteq D^{\mathrm{b}}(X)$ . We have  $\mathcal{O}_X \cong \mathscr{G}_1^{\bullet} \oplus \mathscr{G}_2^{\bullet}$  with  $\mathscr{G}_i^{\bullet} \in D_i$ . As  $H^j(\mathscr{G}_1^{\bullet}) = H^j(\mathscr{G}_2^{\bullet}) = 0$  for each  $j \neq 0$ , we can assume that  $\mathscr{G}_i^{\bullet} = \mathscr{G}_i \in \mathsf{Coh}\, X$ . Hence, both  $\mathscr{G}_1$  and  $\mathscr{G}_2$  are ideal sheaves  $\mathcal{I}_{X_1}$ ,

 $\mathcal{I}_{X_2}$ . Since X is connected and  $\mathcal{O}_X \cong \mathcal{I}_{X_1} \oplus \mathcal{I}_{X_2}$ , either  $X_1 = \emptyset$  or  $X_2 = \emptyset$ . With loss of generality, we assume that  $X_1 = \emptyset$ , then  $X_2 = X$ , so  $\mathscr{G}_2 = 0$  and thus  $\mathcal{O}_X \in D_1$ .

Let  $x \in X$  be a closed point. As  $\mathcal{O}_{X,x}$  is simple in  $\mathsf{Coh}\,X$ , either  $\mathcal{O}_{X,x} \in D_1$  or  $\mathcal{O}_{X,x} \in D_2$ . As  $\mathsf{Hom}(\mathcal{O}_X, \mathcal{O}_{X,x}) \neq 0$ , necessarily  $\mathcal{O}_{X,x} \in D_1$ .

Assume that  $\mathscr{F}^{\bullet}$  is a nonzero object in  $D_2$ . We can assume that  $H^0(\mathscr{F}^{\bullet} \neq 0)$  and  $H^i(\mathscr{F}^{\bullet}) = 0$  for all i > 0. Then we have a distinguished triangle



Choose  $x \in \text{Supp}(H^0(\mathscr{F}^{\bullet}))$  and a surjection  $H^0(\mathscr{F}^{\bullet}) \to \mathcal{O}_{X,x}$ . We see that

$$\tau_{\leq 0}\mathscr{F}^{\bullet} \to H^0(\mathscr{F}^{\bullet}) \to \mathcal{O}_{X,x}$$

is nonzero, contradicting  $\tau_{\leq 0} \mathscr{F}^{\bullet} \in D_2$  and  $\mathcal{O}_{X,x} \in D_1$ . Hence  $D^{\mathrm{b}}(X)$  is indecomposable.

## 9.2 Semi-orthogonal decomposition

Analogy: Decomposition of triangulated category corresponds to  $A = B \oplus C$  in an abelian category  $\mathcal{A}$ , while semi-orthogonal decomposition corresponds to a short exact sequence

$$0 \to B \to A \to C \to 0$$

in  $\mathcal{A}$ .

### 9.2.1 Definition

Let  $\mathcal{T}$  be a triangulated category.

**Definition 9.7.** A semi-orthogonal decomposition of  $\mathcal{T}$  is a sequence of strictly full triangulated subcategory  $\mathcal{C}_1, \ldots, \mathcal{C}_m$  such that

- i)  $\operatorname{Hom}_{\mathcal{T}}(C_i, C_j) = 0$  for all  $C_i \in \mathcal{C}_i$ ,  $C_j \in \mathcal{C}_j$  whenever i > j;
- ii) for each  $X \in \mathcal{T}$ , there exists a decomposition

$$0 \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 = X$$

such that  $Cone(f_i) \in \mathcal{C}_i$ .

We write  $\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ .

Here, a strictly full subcategory  $\mathcal{B} \subseteq \mathcal{A}$  is a full subcategory such that for each  $X \in \mathcal{B}$ , we have

$$Y \cong X \text{ in } \mathcal{A} \implies Y \in \mathcal{B}.$$

An semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$  is called **maximal** if  $\mathcal{C}_i$  does not admit any nontrivial semi-orthogonal decomposition for each i.

Given some additive full subcategories  $D_1, \ldots, D_m \subseteq \mathcal{T}$ . The **thick closure** of  $D_1, \ldots, D_m$  is the smallest strictly full triangulated subcategory  $\mathcal{D} \subseteq \mathcal{T}$  such that  $\mathcal{D} \supseteq \mathcal{D}_i$  for each i and  $\mathcal{D}$  is thick, i.e., closed under taking direct summands.

If  $\mathcal{D} = \mathcal{T}$ , we say that  $\mathcal{D}_1, \ldots, \mathcal{D}_m$  classically generate  $\mathcal{T}$ .

**Proposition 9.8.** Given strictly full thick triangulated subcategories  $C_1, \ldots, C_m \subseteq \mathcal{T}$  which satisfying  $\text{Hom}(C_i, C_j) = 0$  for all i > j. Then the followings are equivalent:

- $C_1, \ldots, C_m$  classically generate  $\mathcal{T}$ ;
- $\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$  is an semi-orthogonal decomposition.

**Proposition 9.9.** Given an semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ . For each  $X \in \mathcal{T}$ , the  $X_i$ 's and the  $C_i$ 's in the decomposition

$$0 \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 = X$$

are unique up to isomorphisms.

So we get projection functors

$$P_i: \mathcal{T} \to \mathcal{C}_i \quad P_{i,m}: \mathcal{T} \to \langle \mathcal{C}_i, \dots, \mathcal{C}_m \rangle$$
  
 $X \mapsto \mathcal{C}_i, \qquad X \mapsto X_i,$ 

One way of obtaining semi-orthogonal decomposition is by taking the orthogonal complement of an admissible subcategory.

### 9.2.2 Admissible subcategory

Let  $\mathcal{D}$  be a triangulated category, and let  $\mathcal{D}' \subseteq \mathcal{D}$  be a full triangulated subcategory. The right orthogonal complement of  $\mathcal{D}'$  (with respect to  $\mathcal{D}$ ) is the full subcategory  $(\mathcal{D}')^{\perp} \subseteq \mathcal{D}$ 

with

$$\mathrm{Ob}(\mathcal{D}')^{\perp} = \{ X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \ \forall Y \in \mathcal{D}' \}.$$

The left orthogonal complement  $^{\perp}D'$  is defined similarly.

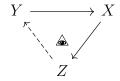
**Proposition 9.10.** If  $\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ , then

$$\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_{k-1}, \langle \mathcal{C}_k, \dots, \mathcal{C}_{\ell} \rangle, \mathcal{C}_{\ell+1}, \dots, \mathcal{C}_m \rangle$$

and

$$\langle \mathcal{C}_k, \dots, \mathcal{C}_\ell \rangle = {}^{\perp} \langle \mathcal{C}_1, \dots, \mathcal{C}_{k-1} \rangle \cap \langle \mathcal{C}_{\ell+1}, \dots, \mathcal{C}_m \rangle^{\perp}.$$

**Definition 9.11.** A subcategory  $\mathcal{D}'$  of  $\mathcal{D}$  is called right (resp. left) admissible if for each  $X \in \mathcal{D}$ , there exists a distinguished triangle

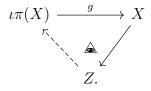


such that  $Y \in \mathcal{D}'$  and  $Z \in (\mathcal{D}')^{\perp}$  (resp.  $Y \in {}^{\perp}\mathcal{D}'$  and  $Z \in \mathcal{D}'$ ).

We say that  $\mathcal{D}'$  is admissible if  $\mathcal{D}'$  is left and right admissible. By definition, if  $\mathcal{D}'$  is right (resp. left) admissible, then  $\mathcal{D} = \langle (\mathcal{D}')^{\perp}, \mathcal{D}' \rangle$  (resp.  $\mathcal{D} = \langle \mathcal{D}', {}^{\perp}\mathcal{D}' \rangle$ ). The following proposition is useful.

**Proposition 9.12.** Let  $\mathcal{D}' \subseteq \mathcal{D}$  be a full triangulated subcategory. Then  $\mathcal{D}' \subseteq \mathcal{D}$  is right (resp. left) admissible if and only if the inclusion  $\iota : \mathcal{D}' \to \mathcal{D}$  has a right (resp. left) adjoint  $\pi : \mathcal{D} \to \mathcal{D}'$ .

*Proof.* Suppose  $\mathcal{D}' \to \mathcal{D}$  has a right adjoint. Then the element  $g : \iota \pi(X) \to X$  corresponding to id:  $\pi(X) \to \pi(X)$  gives a distinguished triangle



For each  $Y' \in \mathcal{D}'$ , by the naturality of adjunction, the following diagram commutes:

$$\operatorname{Hom}_{\mathcal{D}}(\iota(Y'), \iota\pi(X)[j]) \xrightarrow{g\circ} \operatorname{Hom}_{\mathcal{D}}(\iota(Y'), X[j])$$

$$\uparrow^{\natural}$$

$$\operatorname{Hom}_{\mathcal{D}'}(Y', \pi(X)[j]).$$

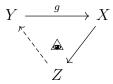
As  $\iota : \mathcal{D}' \to \mathcal{D}$  is fully faithful,  $\operatorname{Hom}_{\mathcal{D}'}(Y', \pi(X)[j]) \to \operatorname{Hom}_{\mathcal{D}}(\iota(Y'), \iota\pi(X)[j])$  is an isomorphism. If follows from the exact sequence

$$\operatorname{Hom}_D(\iota(Y'), \iota\pi(X)) \stackrel{\sim}{\longrightarrow} \operatorname{Hom}_D(\iota(Y'), X) \stackrel{\sim}{\longrightarrow} \operatorname{Hom}_D(Y', Z) \stackrel{\sim}{\longrightarrow}$$

$$\longrightarrow \operatorname{Hom}_D(\iota(Y'), \iota \pi(X)[1]) \stackrel{\sim}{\longrightarrow} \operatorname{Hom}_D(\iota(Y'), X[1])$$

gives  $\operatorname{Hom}_D(Y', Z) = 0$ .

Suppose  $\mathcal{D}' \subseteq \mathcal{D}$  is right admissible. For each  $X \in \mathcal{D}$ , choose  $Y \in \mathcal{D}'$ ,  $Z \in (\mathcal{D}')^{\perp}$  such that the triangle



is distinguished. Define  $\pi(X) = Y$ . Then for each  $A \in \mathcal{D}'$ ,

$$g \circ : \operatorname{Hom}_{\mathcal{D}'}(A, \pi(X)) \to \operatorname{Hom}_{\mathcal{D}}(\iota(A), X)$$
 (8)

is an isomorphism. The isomorphism is clearly functorial in A. It remains to show that it is functorial in X.

Given a morphism  $X' \xrightarrow{f} X$ . We get two distinguished triangles

$$Y' \longrightarrow X' \longrightarrow Z' \longrightarrow Y'[1]$$

$$\downarrow^f$$

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1].$$

Since  $\operatorname{Hom}(Y', Z) = \operatorname{Hom}(Y', Z[-1]) = 0$ , we have unique morphisms

$$Y' \longrightarrow X' \longrightarrow Z' \longrightarrow Y'[1]$$

$$\downarrow^{\exists !} \qquad \downarrow^{f} \qquad \downarrow^{\exists !} \qquad \downarrow$$

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1],$$

which shows that  $(\delta)$  is functorial in X.

**Remark.** In many references, when we define semi-orthogonal decomposition

$$\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$$
,

we require that each  $C_i$  is admissible. When  $T = D^b(X)$  where X is smooth and projective:

**Theorem 9.13.** If  $D^{b}(X) = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ , then each  $\mathcal{C}_i$  is admissible.

### 9.2.3 Examples of admissible subcategories

Let X, Y be smooth projective varieties. Assume that we have  $F: D^{\rm b}(X) \to D^{\rm b}(Y)$ , which is fully faithful and exact. Orlov's theorem shows that F is Fourier-Mukai. So F has left and right adjoints. Thus  $F: D^{\rm b}(X) \to D^{\rm b}(Y)$  embeds  $D^{\rm b}(X)$  as an admissible subcategory of  $D^{\rm b}(Y)$ .

**Proposition 9.14.** Let  $f: X \to B$  be a projective morphism between smooth projective varieties. Assume that  $Rf_*\mathcal{O}_X = \mathcal{O}_B$ , e.g., f has connected fiber. Then  $Lf^*: D^b(B) \to D^b(X)$  is fully faithful, thus realizing  $D^b(B)$  as an admissible subcategory of  $D^b(X)$ .

*Proof.* By adjunction and projection formula, we have morphisms

$$\mathscr{F}^{\bullet} \to \mathsf{R} f_* \mathsf{L} f^* \mathscr{F}^{\bullet} \cong \mathscr{F}^{\bullet} \otimes^{\mathsf{L}} \mathsf{R} f_* \mathcal{O}_X \cong \mathscr{F}^{\bullet}.$$

The composition is an isomorphism by checking on a bounded locally free sheaf resolution  $\mathscr{L}^{\bullet} \to \mathscr{F}^{\bullet}$  locally. So id  $\cong \mathsf{R} f_* \mathsf{L} f^*$ . It follows from the diagram

$$\operatorname{Hom}(\mathscr{E}^{\bullet},\mathscr{F}^{\bullet}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{E}^{\bullet}, \mathsf{R} f_{*} \mathsf{L} f^{*} \mathscr{F}^{\bullet})$$

$$\downarrow^{\circ} \qquad \qquad \downarrow^{\circ} \qquad \qquad \downarrow^$$

that  $Lf^*$  is fully faithful.

Let  $\mathcal{D}$  be a k-linear triangulated category.

**Definition 9.15.** An object  $E \in \mathcal{D}$  is called exceptional if

$$\operatorname{Hom}(E, E[\ell]) = \begin{cases} k, & \text{if } \ell = 0\\ 0, & \text{else.} \end{cases}$$

**Proposition 9.16.** The thick closure of an exceptional object  $E \in \mathcal{D}$  consists of all object isomorphic to  $\bigoplus_{i \in I} E[i]^{j_i}$ , where  $I \subseteq \mathbb{Z}$  is a finite set.

**Proposition 9.17.** Assume that  $\sum_{i} \dim \operatorname{Hom}^{i}(A, B) < \infty$  for all  $A, B \in \mathcal{D}$ . Then the thick closure  $\langle E \rangle \subseteq \mathcal{D}$  of an exceptional object E is an admissible subcategory.

*Proof.* Given  $A \in \mathcal{D}$ . We have the distinguished triangle

where  $\operatorname{Hom}(E, A[i]) \otimes E[-i]$  is in fact  $E[-i]^{\oplus \dim \operatorname{Hom}(E, A[i])}$ . As E exceptional, applying  $\operatorname{Hom}(E[-i], -)$  to f gives

$$\operatorname{Hom}(E, A[i]) \xrightarrow[[i]]{\sim} \operatorname{Hom}(E[-i], A).$$

Thus  $\operatorname{Hom}(E[-i], B) = 0$  for each i, so  $B \in \langle E \rangle^{\perp}$  and  $\langle E \rangle$  is right admissible.

The proof of left-admissibility is similar.

### 9.2.4 Exceptional collection

**Definition 9.18.** Let  $E_1, \ldots, E_m \in \mathcal{D}$  be exceptional objects.

- If  $\operatorname{Hom}(E_i, E_j[\ell]) = 0$  for all i > j and for all  $\ell$ , we can  $(E_1, \dots, E_m)$  an exceptional collection.
- An exceptional collection is full if  $E_1, \ldots, E_m$  classically generate  $\mathcal{D}$ .

If  $E_1, \ldots, E_m$  is a full exceptional collection of  $\mathcal{D}$ , then  $\langle E_1, \ldots, E_m \rangle := \langle \langle E_1 \rangle, \ldots, \langle E_1 \rangle \rangle$  is an semi-orthogonal decomposition of D. More generally, if  $E_1, \ldots, E_m$  is an exceptional collection, then  $\langle \mathcal{C}^{\perp}, E_1, \ldots, E_m \rangle$  is an semi-orthogonal decomposition of  $\mathcal{D}$ , where  $\mathcal{C}^{\perp} := \langle E_1, \ldots, E_m \rangle$ .

## Interlude 3: Yoneda extensions

Let  $\mathcal{A}$  be an abelian category.

**Definition 9.19.** Let  $A, B \in \mathcal{A}$ . A degree i Yoneda extension of B is an exact sequence of the form

$$E: 0 \to A \to Z^{-(i-1)} \to Z^{-(i-2)} \to \cdots \to Z^0 \to B \to 0.$$

Two Yoneda extension E, E' are equivalent if there exists a commutative diagram

$$E: \qquad 0 \longrightarrow A \longrightarrow Z^{-(i-1)} \longrightarrow \cdots \longrightarrow Z^0 \longrightarrow B \longrightarrow 0$$

$$\parallel \qquad \uparrow \qquad \qquad \uparrow \qquad \parallel$$

$$E'': \qquad 0 \longrightarrow A \longrightarrow (Z^{-(i-1)})'' \longrightarrow \cdots \longrightarrow (Z^0)'' \longrightarrow B \longrightarrow 0$$

$$\parallel \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$E': \qquad 0 \longrightarrow A \longrightarrow (Z^{i-1})' \longrightarrow \cdots \longrightarrow (Z^0)' \longrightarrow B \longrightarrow 0,$$

where E'' is also a Yoneda extension.

**Proposition 9.20.** The above definition defines an equivalence relation.

Let

$$\operatorname{Ex}^i(A,B) = \{ \text{ Yoneda extension of } B \text{ by } A \} / \{ \text{ the equivalence} \} \cdot$$

Consider the map

$$\delta : \operatorname{Ex}^{i}(B, A) \to \operatorname{Hom}_{\mathcal{D}(A)}(B, A[i]) = \operatorname{Ext}^{i}(B, A)$$

by sending  $E: A \to Z^{\bullet} \to B$  to the roof

$$\begin{array}{c}
B \\
\uparrow \\
\cdots \longrightarrow 0 \longrightarrow A \longrightarrow Z^{i-1} \longrightarrow \cdots \longrightarrow Z^{0} \longrightarrow 0 \longrightarrow \cdots \\
\downarrow^{id} \\
\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

**Lemma 9.21.** The map  $\delta : \operatorname{Ex}^i(B,A) \to \operatorname{Ext}^i(B,A)$  is a bijection.

In particular if  $\delta(A \to Z^{\bullet} \to B) = 0$ , then  $Z^{\bullet} \cong A[i] \oplus B$  in  $\mathcal{D}^{b}(\mathcal{A})$ .

# 10 Full exceptional collection

10.1 
$$\mathcal{D}^{\mathbf{b}}(\mathbb{P}^n)$$

**Theorem 10.1** (Beilinson). The line bundles

$$\mathcal{O}(a), \ \mathcal{O}(a+1), \ \dots, \mathcal{O}(a+n)$$

form a full exceptional collection in  $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ .

*Proof.* Since  $\text{Hom}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) \cong H^{\ell}(\mathbb{P}^n, \mathcal{O}(j-i))$ , which is 0 if -n < j - i < 0. These line bundle form an exceptional collection.

It remains to show that  $\mathcal{O}(a), \ldots, \mathcal{O}(a+n)$  classically generate  $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ . As  $-\otimes \mathcal{O}(j)$  defines an equivalence of category from  $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$  to itself, it is enough to prove this for some a.

For each full additive subcategory of  $\mathcal{C} \subseteq \mathcal{D}$ , we say that  $\mathcal{C}$  generates  $\mathcal{D}$  if  $\langle \mathcal{C} \rangle^{\perp} = 0$ .

**Proposition 10.2.** The subcategory  $\langle \mathcal{C} \rangle^{\perp} = 0$  if and only if for all  $C \in \mathcal{C}$ ,  $X \in \mathcal{D}$  and for all i,  $\text{Hom}_{\mathcal{D}}(C, X[i]) = 0$ .

It is enough to prove the following

**Lemma 10.3.** Let X be a projective variety over a field k of dimension n. If  $\mathscr{L}$  is an globally generated (base-point-free) ample line bundle, then  $\bigoplus_{i=0}^{n} \mathscr{L}^{-i}$  generates  $\mathcal{D}^{\mathrm{b}}(X)$ .

*Proof of Lemma*. Let  $\varphi = |\mathcal{L}| : X \to \mathbb{P}^N$ , which is a finite morphism. Consider the Koszul resolution

$$\cdots \longrightarrow \bigwedge^{\ell+1} H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}(-\ell-1)$$

$$\xrightarrow{f_{\ell}} \bigwedge^{\ell} H^{0}(\mathbb{P}^{N}, \mathcal{O}(1)) \otimes \mathcal{O}(-\ell) \longrightarrow \cdots \longrightarrow \mathcal{O} \longrightarrow 0,$$

where  $f_{\ell}$  is the composition  $(H^0 := H^0(\mathbb{P}^N, \mathcal{O}(1)))$ 

$$\bigwedge^{\ell+1} H^0 \otimes \mathcal{O}(-\ell-1) \to \bigwedge^{\ell} H^0 \otimes H^0 \otimes \mathcal{O}(-\ell-1) \to \bigwedge^{\ell} H^0 \otimes \mathcal{O}(-\ell)$$

Pulling back the Koszul complex by the finite morphism  $\varphi$ , we obtain an exact sequence

$$0 \to \mathcal{L}^{-N-1} \to \cdots \to (\mathcal{L}^{-k})^{\oplus \binom{N+1}{k}} \to \cdots \to \mathcal{O}_X \to 0.$$

Let  $K := \ker((\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to (\mathscr{L}^{-n})^{\oplus \binom{N+1}{n}})$ . We get an exact sequence (i.e., a Yoneda extension of  $\mathcal{O}_X$  by K)

$$0 \to K \to (\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathscr{L}^{-1})^{\oplus N+1} \to \cdots \to \mathcal{O}_X \to 0.$$

Since dim X = n, we have  $\operatorname{Ext}^{n+1}(\mathcal{O}_X, K) = 0$ , so  $\mathcal{O}_X$  is a direct summand of

$$0 \to (\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathscr{L}^{-1})^{\oplus N+1} \to \cdots \to 0$$

in  $\mathcal{D}^{\mathrm{b}}(X)$ . Applying the exact functor  $(-\otimes \mathcal{L}^{-n-j-1}) \circ \mathsf{R} \operatorname{Hom}(-, \mathcal{O}_X)$  to this result shows that  $\mathcal{L}^{-n-j-1}$  is a direct summand of

$$0 \to (\mathscr{L}^{-n-j})^{\oplus N+1} \to \cdots \to (\mathscr{L}^{-j})^{\oplus \binom{N+1}{n+1}} \to \cdots \to 0.$$

This for each  $j \geq 0$ ,  $\mathcal{L}^{-j} \in \langle \mathcal{L}^{-i} \mid 0 \leq i \leq n \rangle$ .

Given  $E^{\bullet} \in \mathcal{D}^{\mathrm{b}}(X)$  such that

$$\mathsf{R}\operatorname{Hom}\Bigl(\bigoplus_{i=0}^n\mathscr{L}^{-i},E^{\bullet}\Bigr)=0.$$

Then  $\mathsf{R}\,\mathsf{Hom}(\mathscr{L}^{-j},E^{\bullet}=0$  for each  $j\geq 0$ . Namely,  $\mathsf{R}\Gamma(X,E^{\bullet}\otimes\mathscr{L}^{j})=0$  for each  $j\geq 0$ . Up to shifting, we can assume that  $H^{i}(E^{\bullet})=0$  for each i>0. We show that  $H^{0}(E^{\bullet})=0$ ; this proves the lemma by induction.

Recall that  $\dim X - n$ . We have

$$\mathsf{R}\Gamma(X, \tau_{\leq -n-1}(E^{\bullet}\otimes \mathscr{L}^{j}) \in \mathcal{D}^{<0}(\mathsf{Vect}_{k}).$$

This is because

$$E_2^{p,q} = R^p \Gamma(X, H^q(\tau_{\leq -n-1}(E^{\bullet} \otimes \mathscr{L}^j)) \Rightarrow R^{p+q} \Gamma(X, \tau_{\leq -n-1}(E^{\bullet} \otimes \mathscr{L}^j)),$$

and 
$$R^p\Gamma(X, H^q(\tau_{\leq -n-1}(E^{\bullet}\otimes \mathcal{L}^j)) = 0$$
 if  $p \geq n+1$  or  $q \geq -n$ .

Consider the distinguished triangle

$$\mathsf{R}\Gamma(X, \tau_{\leq -n-1}(E^{\bullet} \otimes \mathscr{L}^{j})) \to \mathsf{R}\Gamma(X, E^{\bullet} \otimes \mathscr{L}^{j}) \to \mathsf{R}\Gamma(X, \tau_{>-n-1}(E^{\bullet} \otimes \mathscr{L}^{j})) \xrightarrow{[1]} .$$

Since  $R\Gamma(X, E^{\bullet} \otimes \mathcal{L}^{j}) = 0$ , we have

$$R^0\Gamma(X, \tau_{>-n-1}(E^{\bullet}\otimes \mathscr{L}^j))=0.$$

Since  $H^i(E^{\bullet}) = 0$  for i > 0 and  $\mathscr{L}$  is ample, we can choose  $j_0 \in \mathbb{Z}$  such that for each  $j \geq j_0$ ,  $R^p\Gamma(X, H^q(\tau_{>-n-1}(E^{\bullet} \otimes \mathscr{L}^j))) = 0$  if  $p \geq 1$ . Consider

$$E_2^{p,q} = R^p \Gamma(X, H^q(\tau_{>-n-1}(E^{\bullet} \otimes \mathscr{L}^j))) \Rightarrow R^{p+q} \Gamma(X, \tau_{>-n-1}(E^{\bullet} \otimes \mathscr{L}^j)).$$

Since  $E_2^{p,q} \neq 0$  only if  $p \leq 0$  and  $q \leq 0$ ,

$$0=R^0\Gamma(X,\Gamma(X,\tau_{>-n-1}(E^\bullet\otimes\mathscr{L}^j))=E_2^{0,0}=H^0(X,H^0(E^\bullet)\otimes\mathscr{L}^j)$$

for each  $j \geq j_0$ . As  $\mathscr{L}$  is ample, necessarily  $H^0(E^{\bullet}) = 0$ .

**Remark.** Let  $\mathcal{C} \subseteq \mathcal{D}$  be a full additive subcategory.

- If  $\mathcal{C}$  classically generates  $\mathcal{D}$ , then  $\mathcal{C}$  generates  $\mathcal{D}$ .
- If  $\mathcal{C}$  is a right admissible full triangulated subcategory, then the converse also holds.

## 10.2 Strong exceptional collection

Let  $\mathcal{D}$  be a triangulated category.

**Definition 10.4.** An exceptional collection  $E_1, \ldots, E_m$  of  $\mathcal{D}$  is called strong if  $\text{Hom}(E_i, E_j[k])$  for all i, j and for all  $k \neq 0$ .

The full exceptional collection

$$\mathcal{O}(a), \ \mathcal{O}(a+1), \ \ldots, \ \mathcal{O}(a+n)$$

of  $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$  is strong.

**Theorem 10.5** (Bondal). Let X be a smooth projective variety. If  $(E_1, \ldots, E_m)$  is a strong full exceptional collection of  $D^{\mathrm{b}}(X)$ , then

$$\mathsf{R}\operatorname{Hom}_{\mathcal{D}^\mathrm{b}(X)}\Bigl(\bigoplus_{i=1}^m E_i,-\Bigr):\mathcal{D}^\mathrm{b}(X)\stackrel{\sim}{ o} \mathcal{D}^\mathrm{b}(\mathsf{mod}\text{-}A_\mathrm{fin}),$$

where A is the endomorphism ring  $\operatorname{End}\Bigl(\bigoplus_{i=1}^m E_i\Bigr)$  and  $\operatorname{\mathsf{mod-}} A_{\operatorname{fin}}$  is the category of right A-modules of finite type.

Given a triangulated category  $\mathcal{D}$ . If  $\mathcal{D} \cong \mathcal{D}^{?}(\mathcal{A})$  for some abelian category  $\mathcal{A}$ , we call  $\mathcal{A}$  the heart (of a triangulated structure) of  $\mathcal{D}$ . In bondal's theorem, the statement

$$\mathcal{D}^{\mathrm{b}}(\mathsf{Coh}\,X) \cong \mathcal{D}^{\mathrm{b}}(\mathsf{mod}\text{-}A_{\mathrm{fin}}$$

provides two hearts of  $\mathcal{D}^{\mathrm{b}}(X)$  of different nature.

**Remark.** It is rare that  $\mathcal{D}^{\mathrm{b}}(X)$  admits of full exceptional collection

$$(E_1,\ldots,E_m).$$

For instance, we will see that this implies

- $H^{p,q}(X) = 0$  for all  $p \neq q$ ;
- $K(\mathcal{D}^{b}(X)) = \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_m].$

Conjecture 10.6. If  $\mathcal{D}^{b}(X)$  has a full exceptional collection, then X is rational.

## 10.3 Quiver representations

### 10.3.1 Quiver

A quiver Q is an oriented graph. Formally,  $Q = (Q_0, Q_1, s, t)$ , where

- $Q_0$  is a set of vertices;
- $Q_1$  is a set of edges;
- $s, t: Q_1 \to Q_0$  source and target map, i.e., if  $a \xrightarrow{\alpha} b \in Q_1$ ,  $s(\alpha) = a$  and  $t(\alpha) = b$ .

#### 10.3.2 Path algebra

Given  $a, b \in Q_0$ . A **path** from a to b is a sequence  $\alpha_1, \ldots, \alpha_n \in Q_1$  such that  $s(\alpha_1) = a$ ,  $t(\alpha_i) = s(\alpha_{i+1})$ , and  $t(\alpha_n) = b$ . We write  $p = (a \mid \alpha_1, \ldots, \alpha_n \mid b)$ , and define s(p) = a, t(p) = b. The length of the path  $\ell(p) := n$ . n = 0 is allowed:  $e_a := (a \mid |a|)$ .

The **path algebra** of Q over a field k is the graded associative k-algebra kQ defined as

- the paths in Q form the basis of kQ;
- grading  $(kQ)_n$  is defined by the length of the path;
- given two paths  $p_1$ ,  $p_2$ , define

$$p_1 p_2 = \begin{cases} \text{concatenation of } p_1 \text{ and } p_2, \text{ if } t(p_1) = s(p_2) \\ 0, \text{ else.} \end{cases}$$

A **cycle** is a path p with  $\ell(p) \geq 1$  such that s(p) = t(p)

**Proposition 10.7.** The dimension of the algebra kQ is finite if and only if Q is acyclic, i.e., without any cycle.

### 10.3.3 Quiver with relations

A **relation**  $\rho$  in a quiver Q is an element

$$\rho = \sum_{i} a_i p_i \in kQ$$

such that  $\ell(p_i) \geq 2$  and  $s(p_i) = s(p_i)$ ,  $t(p_i) = t(p_i)$  for all i, j.

A quiver with relations  $(Q, \rho)$  is a quiver Q endowed with a set of relations  $\rho = \{\rho_j\}$ .

The **path algebra** of  $(Q, \rho)$  is  $A_Q := {^kQ}/{I}$ , where I is the two-sided ideal generated by  $\rho_i$ .

### 10.3.4 Quiver representations

A representation of the quiver Q is the data

$$W = ((W_i)_{i \in Q_0}, (w_\alpha)_{\alpha \in Q_1})$$

where each  $W_i$  is a k-vector space and each  $w_{\alpha}: W_{s(\alpha)} \to W_{t(\alpha)}$  is a k-linear map.

Assume that  $(Q, \rho)$  is a quiver with relations. A representation of  $(Q, \rho)$  is a representation W of Q such that for each  $\rho_i \in \rho$ , the corresponding linear map

$$\rho_i: W_{s(rho_i)} \to W_{t(\rho_i)}$$

is zero. Morphisms of quiver representations are defined in the obvious way. We set  $\dim W = \sum_{i \in Q_0} \dim W_i$ , the dimension of the quiver representation.

**Proposition 10.8.** Let  $(Q, \rho)$  be a quiver with relations. Then mod- $({}^{kQ}/{}_{I})_{\text{fin}}$  is equivalent to the category of finite dimensional representation of  $(Q, \rho)$ .

# 10.4 Full exceptional collection and quiver with relations

Let X be a smooth projective variety. Assume that  $\mathcal{D}^{\mathrm{b}}(X)$  admits a strongly full exceptional collection

$$(E_1,\ldots,E_m).$$

Recall that they satisfy

$$\operatorname{Hom}(E_i, E_j) = \begin{cases} k, & \text{if } i = j \\ 0, & \text{if } i > j. \end{cases}$$

So

$$A = \operatorname{End}\left(\bigoplus_{i=1}^{m} E_i\right) = \left(\bigoplus_{i=1}^{m} k e_i\right) \oplus \left(\bigoplus_{i < j} \operatorname{Hom}(E_i, E_j)\right),$$

where  $e_i$  is the generator of  $\operatorname{Hom}(E_i, E_i)$ . We now construct acyclic  $(Q, \rho)$  such that  $A \cong {}^{kQ}/_{I}$ . Let

- $Q_0 = \{1, \ldots, m\};$
- $e_i \in A$  is the path of length 0 in Q at the vertex i;
- for all i < j, consider the linear map

$$\varphi_{i,j}: \prod_{i < k < j} \operatorname{Hom}(E_i, E_k) \times \operatorname{Hom}(E_k, E_j) \to \operatorname{Hom}(E_i, E_j)$$

defined by composition. Choose a basis  $\alpha_1, \ldots, \alpha_{n_{ij}}$  of  $\text{Hom}(E_i, E_j)/\text{Im }\varphi_{ij}$ . These  $\alpha_1, \ldots, \alpha_{n_{ij}}$  define the edges  $i \to j$  in  $Q_1$ .

• An element  $p = \sum a_i p_i \in kQ$  with  $s(p_i) = s(p_j)$ ,  $t(p_i) = t(p)j$  is in I if and only if the corresponding map  $E_{s(p)} \to E_{t(p)}$  is zero.

As an example, for the exceptional collection

$$\mathcal{O}, \ \mathcal{O}(1), \ \ldots, \ \mathcal{O}(n)$$

of  $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ , the associated quiver is



**Proposition 10.9.** The ideal I is generated by the relations

$$x_k x'_\ell - x_\ell x'_k$$

for all  $k \neq \ell \in \{1, ..., n+1\}$ , where  $\{x_1, ..., x_{n+1}\}$  (resp.  $\{x'_1, ..., x'_{n+1}\}$ ) is the set of arrows  $i \to (i+1)$  (resp.  $(i+1) \to (i+2)$ ).

# 11 Grothendieck-Riemann-Roch

In this section, we work over  $\mathbb{C}$ . Let X be a smooth quasi-projective variety.

### 11.1 Chern classes of a vector bundle

For each vector bundle  $\mathscr{E}$  over X, we have the **Chern classes** of  $\mathscr{E}$ ,

$$c_i(\mathscr{E}) \in H^{2i}(X,\mathbb{Z}).$$

They satisfy the following properties

- 1)  $c_0(\mathscr{E}) = 1$ ;
- 2)  $c_1(\mathcal{O}_X(D)) = [D]$  for any divisor D on X;
- 3) given short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

we have 
$$c(\mathscr{E}) = c(\mathscr{E}') c(\mathscr{E}'')$$
, where  $c(\mathscr{E}) = \sum_{i \geq 0} c_i(\mathscr{E})$ ;

4)  $c_i(\mathscr{E}) = 9$  for each  $i > \operatorname{rk} \mathscr{E}$ .

### 11.2 Chern classes of a coherent sheaf

Let  $\mathscr{F}$  be a coherent sheaf on X. As X is smooth we can choose a locally free resolution

$$0 \to \mathcal{L}_{\ell} \to \cdots \to \mathcal{L}_{0} \to \mathscr{F} \to 0.$$

Define 
$$c(\mathscr{F}) = \prod_{i} c(\mathscr{L}_i)^{(-1)^i}$$
. Here,

$$c(\mathcal{L})^{-1} = 1 + (1 - c(\mathcal{L})) + (1 - c(\mathcal{L}))^2 + \cdots$$

**Lemma 11.1.** The Chern class  $c(\mathcal{F})$  is independent of the choice of resolution and still satisfy  $1) \sim 4$ ).

### 11.3 Chern character

Let X be a smooth quasi-projective variety.  $\mathscr E$  a vector bundle on X.

#### 11.3.1 A particular example

Assume that  $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$ . Then we define

$$\operatorname{ch}(\mathscr{E}) = \sum_{i} e^{\operatorname{c}_{1}(\mathscr{L}_{i})}.$$

### 11.3.2 General definition

In general,  $\mathscr{E}$  does not split into line bundles. Let  $c_t(\mathscr{E}) = \sum t^i c_i(\mathscr{E})$ , called the **Chern polynomial**. Write formally

$$c_t(\mathscr{E}) = \prod_{i=1}^{\operatorname{rk}\mathscr{E}} (1 + \alpha_i t).$$

The formal variables  $\alpha_i$  are called **Chern roots**. (When  $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$ , we can take  $\alpha_i = c_1(\mathscr{L}_i)$ .)

We define the **Chern character**  $\operatorname{ch}(\mathscr{E})$  to be  $\sum e^{\alpha_i}$ .  $\operatorname{ch}(\mathscr{E})$  is actually a  $\mathbb{Q}$ -linear combination of product of Chern classes. Explicitly, the first few terms are

$$Ch(\mathscr{E}) = rk(\mathscr{E}) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \dots \in H^*(X, \mathbb{Q}).$$

**Lemma 11.2.** Let  $\mathscr{E}$ ,  $\mathscr{E}'$ ,  $\mathscr{E}''$  be vector bundles.

1) If there is a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

then 
$$Ch(\mathscr{E}) = Ch(\mathscr{E}') + Ch(\mathscr{E}'')$$
.

- 2)  $\operatorname{Ch}(\mathscr{E} \otimes \mathscr{E}') = \operatorname{Ch}(\mathscr{E}) \operatorname{Ch}(\mathscr{E}').$
- 3) The definition of Chern character can be extended to coherent sheaves in a unique way, subject to 1).

## 11.4 Grothendieck group

#### 11.4.1 Abelian category

Let  $\mathcal{A}$  be an abelian category and let  $\mathcal{B} \subseteq \mathcal{A}$  a additive subcategory. We define

$$K(\mathcal{B}) = \frac{\bigoplus_{E \in \mathcal{B}} \mathbb{Z} \cdot [E]}{\langle [E] - [E'] | 0 \to E' \to E \to E'' \to 0 \text{ exact in } \mathcal{A} \rangle}.$$

For example,  $K(\mathsf{Vect}_{k, \mathrm{fin}} \cong \mathbb{Z}.$ 

### **11.4.2** K(Coh X)

Let X be a smooth quasi-projective variety. Define  $K(X) = K(\mathsf{Coh}\,X)$ . Since X is smooth, we actually have

$$K(\operatorname{Vect} X) \cong K(X),$$

where  $\mathsf{Vect}\,X$  is the category of vector bundles over X of finite rank. As tensoring with a vector bundle preserves exact sequence, given  $\mathscr{E},\,\mathscr{E}'\in\mathsf{Vect}\,X$ , we can define

$$[\mathscr{E}] \cdot [\mathscr{E}'] = [\mathscr{E} \otimes \mathscr{E}'].$$

This defines  $K(X) \cong K(\text{Vect } X)$  as a ring. Chern character extends to a ring homomorphism

$$\text{Ch}: \ \mathrm{K}(X) \ \to \ H^*(X,\mathbb{Q})$$
 
$$[\mathscr{F}] \ \mapsto \ \mathrm{Ch}(\mathscr{F}).$$

## **11.4.3** $K(D^{\mathbf{b}}(X))$

Let  $\mathcal{D}$  be a triangulated category. We have a similar definition:

$$K(\mathcal{D}) := \frac{\bigoplus_{E \in \mathcal{D}} \mathbb{Z} \cdot [E]}{\langle [E] - [E'] - [E''] \mid F \to E \to G \to F[1] \text{ distinguished} \rangle}$$

We call  $K(\mathcal{D})$  the Grothendieck group of  $\mathcal{D}$ .

**Proposition 11.3.** Let  $\mathcal{A}$  be an abelian category.

• For each  $E^{\bullet} \in D^{\mathrm{b}}(\mathcal{A})$ , we have

$$[E^{\bullet}] = \sum_{i} (-1)^{i} [H^{i}(E^{\bullet})] = \sum_{i} (-1)^{i} [E^{i}]$$

in  $K(D^b(\mathcal{A}))$ .

• We have an equivalence of category

$$K(\mathcal{A}) \xrightarrow{\sim} K(D^{b}(\mathcal{A})).$$

In particular, if X is a smooth quasi-projective variety,

$$K(D^{b}(X)) \cong K(X).$$

**Proposition 11.4.** Given  $\mathscr{F}_1^{\bullet}$ ,  $\mathscr{F}_2^{\bullet} \in D^{\mathrm{b}}(X)$ , we have

$$[\mathscr{F}_1^{\bullet}]\cdot [\mathscr{F}_2^{\bullet}] = [\mathscr{F}_1^{\bullet}\otimes^{\mathsf{L}}\mathscr{F}_2^{\bullet}].$$

By construction, all exact functor between triangulated category  $F:\mathcal{D}_1\to\mathcal{D}_2$  induces  $F:\mathrm{K}(\mathcal{D}_1)\to\mathrm{K}(\mathcal{D}_2)$ . For example, let  $f:X\to Y$  be a proper morphism between smooth quasi-projective varieties. Then the functor  $\mathsf{R} f_*:D^\mathrm{b}(X)\to D^\mathrm{b}(Y)$  induces  $\mathsf{R} f_*:\mathrm{K}(X)\to\mathrm{K}(Y)$ .

### 11.5 Grothendieck-Riemann-Roch

#### 11.5.1 Todd class

Let X be a smooth quasi-projective variety,  $\mathscr{E}$  a vector bundle of rank r on X,  $\alpha_1$ , ...,  $\alpha_r$  the Chern roots of  $\mathscr{E}$ . We define the Todd class  $\mathrm{Td}(\mathscr{E})$  to be

$$\prod_{i=1}^r Q(\alpha_i),$$

where  $Q(x) = \frac{x}{1 - e^{-x}}$ . Td( $\mathscr{E}$ ) is again a  $\mathbb{Q}$ -linear combination of products of Chern classes:

$$Td(\mathscr{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{2}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \dots \in H^*(X, \mathbb{Q}).$$

#### 11.5.2 GRR

**Theorem 11.5.** Let  $f: X \to Y$  be a proper morphism of smooth quasi-projective varieties. Then we have the following commutative diagram:

$$D^{b}(X) \xrightarrow{Rf_{*}} D^{b}(Y)$$

$$\downarrow \qquad \qquad \downarrow$$

$$K(X) \xrightarrow{Rf_{*}} K(Y)$$

$$\downarrow^{\operatorname{ch}(-)\operatorname{Td}_{X}} \qquad \downarrow^{\operatorname{ch}(-)\operatorname{Td}_{Y}}$$

$$H^{\bullet}(X,\mathbb{Q}) \xrightarrow{f_{*}} H^{\bullet}(Y,\mathbb{Q}),$$

where  $\mathrm{Td}_X = \mathrm{Td}(T_X)$ ,  $\mathrm{Td}_Y = \mathrm{Td}(T_Y)$ .

### 11.5.3 Hirzebruch-Riemann-Roch

When  $f: X \to \{ \mathrm{pt} \}$ , for each  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$  we have

$$\mathsf{R} f_*[\mathscr{E}^\bullet] = [\mathsf{R} \Gamma(\mathscr{E}^\bullet)] = \sum (-1)^i \dim H^i(X,\mathscr{E}^\bullet) =: \chi(\mathscr{E}^\bullet).$$

Corollary 11.6 (HRR). For each  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$ ,

$$\chi(\mathscr{E}^{\bullet}) = \int_X \operatorname{ch}(\mathscr{E}^{\bullet}) \operatorname{Td}_X.$$

#### 11.5.4 Fourier-Mukai transforms and GRR

Let X, Y be smooth projective varieties. For each  $\alpha \in D^b(X)$  (or  $\alpha \in K(X)$ ), define

$$v(\alpha) = \operatorname{ch}(\alpha) \sqrt{\operatorname{Td}_X} \in H^{\bullet}(X, \mathbb{Q}).$$

We call  $v(\alpha)$  the Mukai vector. Let  $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$ , and let  $p: X \times Y \to X$  and  $q: X \times Y \to Y$  be the projections.

**Proposition 11.7.** We have the commutative diagram:

$$D^{b}(X) \xrightarrow{\Phi^{\mathscr{P}}} D^{b}(Y)$$

$$\downarrow^{v} \qquad \qquad \downarrow^{v}$$

$$H^{\bullet}(X,\mathbb{Q}) \xrightarrow{} H^{\bullet}(Y,\mathbb{Q})$$

$$\beta \longmapsto q_{*}(v(\mathscr{P}) \cup p^{*}\beta).$$

**Remark.** This induced map  $H^{\bullet}(X,\mathbb{Q}) \to H^{\bullet}(Y,\mathbb{Q})$  is just  $\mathbb{Q}$ -linear. In general it does not preserve the grading, nor the cup-product.

## 11.6 Grothendieck group and semi-orthogonal decomposition

Let  $\mathcal{D}$  be a triangulated category. Assume that  $\mathcal{D}$  admits an semi-orthogonal decomposition  $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ . Recall that we have the projection functors  $p_i : \mathcal{D} \to \mathcal{C}_i$ .

Proposition 11.8. The morphism

$$K(\mathcal{D}) \stackrel{\sim}{\to} K(\mathcal{C}_1) \oplus \cdots \oplus K(\mathcal{C}_m)$$

$$[F] \mapsto ([p_1(F)], \dots, [p_m(F)])$$

is well-defined and s a group isomorphism.

In particular, if  $E_1, \ldots, E_m$  is a full exceptional collection, then  $K(\mathcal{D})$  is isomorphic to  $\mathbb{Z}^m$ . When  $\mathcal{D} = D^b(X)$ , this is very rare, because usually  $K(\mathcal{D}) = K(X)$  is usually infinitely dimensional.

# 12 Invariants under *D*-equivalence

Let X, Y be smooth projective varieties. Assume that  $D^{b}(X) \cong D^{b}(Y)$  as triangulated categories. Then X and Y share same common invariants. We will see some examples of

such invariants.

### 12.1 Dimension

**Proposition 12.1.** Suppose there is an equivalence  $\Phi: D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$ . Then  $\dim X = \dim Y$ .

*Proof.* Let  $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$  be the Fourier-Mukai kernel such that  $\Phi = \Phi^{\mathscr{P}}$ . As  $\Phi$  is an equivalence, we have

$$\Phi^{-1} \dashv \Phi \dashv \Phi^{-1}$$
.

Since the Fourier-Mukai kernel of left adjoint of  $\Phi$  is  $\mathscr{P}^{\vee} \otimes p_Y^* \omega_Y[\dim Y]$ , and the Fourier-Mukai kernel of right adjoint of  $\Phi$  is  $\mathscr{P}^{\vee} \otimes p_X^* \omega_X[\dim X]$ , where  $p_X: X \times Y \to X$  and  $p_Y: X \times Y \to Y$  are the projections, the uniqueness of Fourier-Mukai kernel gives an isomorphism

$$\mathscr{P}^{\vee} \otimes p_Y^* \omega_Y[\dim Y] \cong \mathscr{P}^{\vee} \otimes \pi_X^* \omega_X[\dim X].$$

As they are nonzero in  $D^{\mathrm{b}}(X \times Y)$  and  $H^{i}(\mathscr{P}^{\vee} \otimes p_{Y}^{*}\omega_{Y}) = H^{i}(\mathscr{P}^{\vee}) \otimes p_{Y}^{*}\omega_{Y}$  is non-zero if and only if  $H^{i}(\mathscr{P}^{\vee} \otimes p_{X}^{*}\omega_{X}) = H^{i}(\mathscr{P}^{\vee}) \otimes p_{X}^{*}\omega_{X}$  is non-zero (note that  $p_{Y}^{*}\omega_{Y}$  and  $p_{X}^{*}\omega_{X}$  are locally free), necessarily dim  $X = \dim Y$ .

**Question.** Let X, Y be smooth projective varieties. Assume that there is an embedding  $D^{\mathrm{b}}(X) \hookrightarrow D^{\mathrm{b}}(Y)$ , do we have  $\dim X \leq \dim Y$ ?

## 12.2 Grothendieck group

Let  $\Phi: D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$ . We have

$$K(X)\cong K(D^{\mathrm{b}}(X))\cong K(D^{\mathrm{b}}(Y))\cong K(Y),$$

where all  $\cong$  are group isomorphisms. It may happen that  $\Phi(\mathcal{O}_X) \neq \mathcal{O}_Y$ . In this case,  $K(X) \cong K(Y)$  is not a ring isomorphism.

## 12.3 Cohomology, Euler characteristic

Let X, Y be smooth projective varieties over  $\mathbb{C}$ .

**Proposition 12.2.** The equivalence  $\Phi: D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$  implies

- $H^{\bullet}(X, \mathbb{Q}) \cong H^{\bullet}(Y, \mathbb{Q})$  as  $\mathbb{Q}$ -vector spaces;
- e(X) = e(Y), where  $e(-) = \sum_{i=1}^{n} (-1)^i \dim H^i(-, \mathbb{Q})$  is the Euler characteristic.

*Proof.* Let  $\mathscr{P}, \mathscr{Q} \in D^{\mathrm{b}}(X \times Y)$  such that  $\Phi = \Phi^{\mathscr{P}}$  and  $\Phi^{-1} = \Phi^{\mathscr{Q}}$ . We have

$$D^{b}(X) \xrightarrow{\Phi^{\mathscr{P}}} D^{b}(Y) \xrightarrow{\Phi^{\mathscr{Q}}} D^{b}(X)$$

$$\downarrow^{v} \qquad \qquad \downarrow^{v} \qquad \qquad \downarrow^{v}$$

$$H^{\bullet}(X,\mathbb{Q}) \xrightarrow{\varphi_{\mathscr{P}}} H^{\bullet}(Y,\mathbb{Q}) \xrightarrow{\varphi_{\mathscr{Q}}} H^{\bullet}(X,\mathbb{Q}),$$

where  $\varphi_{\mathscr{P}}(\beta) = q_*(v(\mathscr{P}) \cup p^*\beta)$  and  $\varphi_{\mathscr{Q}}$  is defined similarly.

As  $\varphi_{\mathscr{Q}} \circ \varphi_{\mathscr{P}}(\beta) = p_*(v(\mathscr{P} * \mathscr{Q}) \cup p^*\beta)$  and  $\mathscr{P} * \mathscr{Q} \cong \mathcal{O}_{\Delta_X} \in D^{\mathrm{b}}(X \times X)$ , we have  $\varphi_{\mathscr{Q}} \circ \varphi_{\mathscr{P}} = \mathrm{id}$ , and similarly  $\varphi_{\mathscr{P}} \circ \varphi_{\mathscr{Q}} = \mathrm{id}$ . Hence  $H^{\bullet}(X, \mathbb{Q}) \cong H^{\bullet}(Y, \mathbb{Q})$ .

Since  $v(\mathscr{P}) = \operatorname{ch}(\mathscr{P}) \cup \sqrt{\operatorname{Td}_{X \times Y}} \in H^{\operatorname{even}}(X \times Y, \mathbb{Q})$ . The morphism

$$\varphi_{\mathscr{P}}: H^{\mathrm{even}}(X, \mathbb{Q}) \oplus H^{\mathrm{odd}}(X, \mathbb{Q}) \xrightarrow{\sim} H^{\mathrm{even}}(Y, \mathbb{Q}) \oplus H^{\mathrm{odd}}(Y, \mathbb{Q})$$

preserves the  $\mathbb{Z}/2\mathbb{Z}$ -grading. Hence  $e(X) = \dim H^{\text{even}} - \dim H^{\text{odd}} = e(Y)$ .

Conjecture 12.3. If  $D^{b}(X) \cong D^{b}(Y)$ , then  $h^{p,q}(X) = h^{p,q}(Y)$  for all p, q.

The conjecture is known to hold when dim  $X \leq 3$  (Popa-Schnell).

**Proposition 12.4.** Under the assumptaion  $D^{b}(X) \cong D^{b}(Y)$ , the sums of vertical Hodge number are preserved, i.e., for each k,

$$\sum_{p-q=k}h^{p,q}(X)=\sum_{p-q=k}h^{p,q}(Y).$$

## 12.4 Canonical rings

Let X, Y, be smooth projective varieties over a field k. The **canonical ring** of X is defined to be

$$R(X) = \bigoplus_{i>0} H^0(X, \omega_X^{\otimes i}).$$

**Proposition 12.5** (Orlov). If  $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$ , then  $R(X) \cong R(Y)$  as graded k-algebra.

*Proof.* As before, let  $\mathscr{P}$ ,  $\mathscr{Q} \in D^{\mathrm{b}}(X \times Y)$  such that  $\Phi = \Phi_{X \to Y}^{\mathscr{P}} : D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$  and  $\Phi^{-1} = \Phi_{X \to X}^{\mathscr{Q}}$ .

First, we prove that  $\Phi_{X\to Y}^{\mathcal{Q}}: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$  is also an equivalence. Indeed,  $\mathscr{P} * \mathscr{Q} \cong \mathcal{O}_{\Delta_X}$ ,  $\mathscr{Q} * \mathscr{P} \cong \mathcal{O}_{\Delta_Y}$ . Let  $\tau: X \times Y \to Y \times X$  be the isomorphism that sends (x,y) to (y,x). Then  $(\tau^*\mathscr{Q}) * (\tau^*\mathscr{P}) \cong \mathcal{O}_{\Delta_X}$ ,  $(\tau^*\mathscr{P}) * (\tau^*\mathscr{Q}) \cong \mathcal{O}_{\Delta_Y}$ . Thus  $\Phi_{X\to Y}^{\mathscr{Q}} = \Phi_{X\to Y}^{\tau^*\mathscr{Q}}$  is an equivalence.

Let  $\iota: X \to X \times X$ ,  $\iota: Y \to Y \times Y$  be the diagonal maps. We have

$$H^{0}(X, \omega_{X}^{\otimes k}) \cong \operatorname{Hom}_{X \times X}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X}^{\otimes k}),$$
  
$$H^{0}(Y, \omega_{Y}^{\otimes k}) \cong \operatorname{Hom}_{Y \times Y}(\iota_{*}\mathcal{O}_{Y}, \iota_{*}\omega_{Y}^{\otimes k}).$$

Consider  $(\tau^* \mathscr{Q}) \boxtimes \mathscr{P} \in D^{\mathrm{b}}((X \times X) \times (Y \times Y))$ . We will show that  $\Phi^{\tau^* \mathscr{Q} \boxtimes \mathscr{P}}(\iota_* \omega_X^{\otimes k}) \cong \iota_* \omega_Y^{\otimes k}$ , so that  $R(X) \cong R(Y)$  as graded k-vector spaces.

Let  $\mathscr{S} = \Phi^{\tau^* \mathscr{Q} \boxtimes \mathscr{P}}(\iota_* \omega_X^k)$ . Then  $\Phi^{\mathscr{S}}$  can be factorized as

$$D^{\mathrm{b}}(Y) \xrightarrow{\Phi^{\mathscr{Q}}} D^{\mathrm{b}}(X) \xrightarrow{\Phi^{\iota_* \omega_X^{\otimes k}}} D^{\mathrm{b}}(X) \xrightarrow{\Phi^{\mathscr{P}}} D^{\mathrm{b}}(Y).$$

Since  $\Phi^{\iota_*\omega_X^{\otimes k}} = (-\otimes \omega_X)^k = S_X^k[-kn]$ , where  $n = \dim X = \dim Y$ , (8.8) gives

$$\Phi^{\mathscr{S}} = \Phi^{\mathscr{P}} \circ \Phi^{\iota_* \omega_X^{\otimes k}} \circ \Phi^{\mathscr{Q}} \cong \Phi^{\mathscr{P}} \circ S_X^k[-kn] \circ \Phi^{\mathscr{Q}} \cong S_Y^k[-kn] = \Phi^{\iota_* \omega_Y^{\otimes k}}.$$

Hence,  $\iota_*\omega_Y^{\otimes k} \cong \mathscr{S} = \Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}(\iota_*\omega_X^k).$ 

Finally, we show that  $R(X) \xrightarrow{\sim} R(Y)$  is a ring homomorphism. In fact, this follows from the following commutative diagram:

$$H^{0}(X, \omega_{X}^{\otimes k}) \otimes H^{0}(X, \omega_{X}^{\otimes \ell}) \xrightarrow{\cdot} H^{0}(X, \omega_{X}^{\otimes (k+\ell)})$$

$$\downarrow^{\natural} \qquad \qquad \downarrow^{\natural}$$

$$\operatorname{Hom}(\mathcal{O}_{X}, \omega_{X}^{\otimes k}) \otimes \operatorname{Hom}(\omega_{X}^{\otimes \ell}, \omega_{X}^{\otimes (k+\ell)}) \xrightarrow{\circ} \operatorname{Hom}(\mathcal{O}_{X}, \omega_{X}^{\otimes (k+\ell)})$$

$$\downarrow^{\natural} \qquad \qquad \downarrow^{\natural}$$

$$\operatorname{Hom}_{X \times X}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X}^{\otimes k}) \otimes \operatorname{Hom}_{X \times X}(\iota_{*}\omega_{X}^{\otimes \ell}, \iota_{*}\omega_{X}^{\otimes (k+\ell)}) \xrightarrow{\circ} \operatorname{Hom}_{X \times X}(\iota_{*}\mathcal{O}_{X}, \omega_{X}^{\otimes (k+\ell)})$$

$$\downarrow^{\Phi^{\tau^{*} \mathscr{D} \boxtimes \mathscr{P}}} \qquad \qquad \downarrow^{\Phi^{\tau^{*} \mathscr{D} \boxtimes \mathscr{P}}$$

$$\operatorname{Hom}_{Y \times Y}(\iota_{*}\mathcal{O}_{Y}, \iota_{*}\omega_{Y}^{\otimes k}) \otimes \operatorname{Hom}_{Y \times Y}(\iota_{*}\omega_{Y}^{\otimes \ell}, \iota_{*}\omega_{Y}^{\otimes (k+\ell)}) \xrightarrow{\circ} \operatorname{Hom}_{Y \times Y}(\iota_{*}\mathcal{O}_{Y}, \omega_{Y}^{\otimes (k+\ell)}).$$

**Remark.** One can show that  $\omega_X$  (resp.  $\omega_X^{\vee}$ ) is ample if and only if  $\omega_Y$  (resp.  $\omega_Y^{\vee}$ ) is ample. Hence if  $\omega_X$  or  $\omega_X^{\vee}$  is ample,

$$D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y) \iff X \cong Y.$$

Corollary 12.6. We have  $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$  if and only if  $\mathrm{K}(X) \cong \mathrm{K}(Y)$ .

## 12.5 Hochschild (co)homology

Let X be a smooth projective variety over a field k. Define the bigraded k-algebra

$$HH(X) = \bigoplus_{i,\ell \in \mathbb{Z}} HA_{i,\ell}(X),$$

where  $HA_{i,\ell}(X) = \operatorname{Ext}_{X\times X}^i(\iota_*\mathcal{O}_X, \iota_*\omega_X^{\ell})$  with product defined by the Yoneda product.

The canonical ring  $R(X) = \bigoplus_{\ell} HA_{0,\ell}(X) \subseteq HH(X)$  as subalgebra. We could have proven directly that  $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$  implies  $HH(X) \cong HH(Y)$  as bigraded k-algebra.

We have another  $\operatorname{sub-}k$ -algebra

$$HH^{\bullet}(X) := \bigoplus_{i} HA_{i,0}(X) \cong HH(X),$$

called the **Hochschild cohomology** of X. The **Hochschild homology** of X is defined as

$$HH_{\bullet}(X) = \bigoplus_{i} HA_{i+\dim X,1}(X) = \bigoplus_{i} \operatorname{Ext}^{i+\dim X}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X}).$$

 $HH_{\bullet}(X)$  is a graded  $HH^{\bullet}(X)$ -module. So  $HH(X) \cong HH(Y)$  implies  $HH^{\bullet}(X) \cong HH^{\bullet}(Y)$  as k-algebra and  $HH_{\bullet}(X) \cong HH_{\bullet}(Y)$  as  $HH^{\bullet}$ -modules.

# 12.6 Hochschild-Kostant-Rosenberg isomorphism

**Theorem 12.7.** Let X be a smooth quasi-projective variety over a field k. Then

$$HH^{i}(X) \cong \operatorname{Ext}_{X \times X}^{i}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\mathcal{O}_{X}) \cong \bigoplus_{p+q=i} H^{q}(X, \bigwedge^{p} T_{X}),$$
$$HH_{i}(X) \cong \operatorname{Ext}_{X \times X}^{i}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X}) \cong \bigoplus_{q-p=i} H^{q}(X, \Omega_{X}^{p}).$$

When X is smooth over  $\mathbb{C}$ , this implies again that the sums of vertical Hodge numbers are invariant under D-equivalence.

**Remark.** In general, these isomorphism don't preserve product.

*Proof.* (sketch) Consider the local-to-global spectral sequences

$$H^{p}(X \times X, \mathscr{E}xt^{q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\mathcal{O}_{X})) \Rightarrow \operatorname{Ext}_{X \times X}^{p+q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\mathcal{O}_{X}) = HH^{p+q}(X),$$

$$H^{p}(X \times X, \mathscr{E}xt^{q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X})) \Rightarrow \operatorname{Ext}_{X \times X}^{p+q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X}) = HH_{p+q-\dim X}(X).$$

One way to prove this theorem is to show that these spectral sequence degenerate at  $E_2$ , so that

$$HH^{\ell}(X) = \bigoplus_{p+q=\ell} H^{p}(X \times X, \mathscr{E}xt^{q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\mathcal{O}_{X}))$$

$$HH_{\ell-\dim X}(X) = \bigoplus_{p+q=\ell} H^{p}(X \times X, \mathscr{E}xt^{q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\omega_{X})).$$

Assuming this, we finish the proof as follows: Choose an embedding  $j: X \hookrightarrow \mathbb{P}^N = \mathbb{P}(V)$ . We have the Euler sequence

$$0 \to \Omega_{\mathbb{P}^N}(1) \to V^{\vee} \to \mathcal{O}(1) \to 0.$$

Let  $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$  be the diagonal. Then

$$\mathcal{O}_{\mathbb{P}^N}(-1) \boxtimes \Omega_{\mathbb{P}^N}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^N} \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

is exact (because over  $(\ell, \ell') \in \mathbb{P}^N \times \mathbb{P}^N$ ,  $\mathcal{O}_{\mathbb{P}^N}(-1) \boxtimes \Omega_{\mathbb{P}^N}(1)|_{(\ell, \ell')} = \ell \otimes {\ell'}^{\perp}$ ). Apply  $(j \times j)^*$ , we get the exact sequence

$$\mathscr{E} = \mathcal{O}_X(-1) \boxtimes \Omega_X(1) \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0,$$

where  $\mathscr{E}$  is a locally free sheaf of rank  $d = \dim X$  over  $X \times X$ . Consider the Koszul resolution

$$0 \to \bigwedge^d \mathscr{E} \to \bigwedge^{d-1} \mathscr{E} \to \cdots \to \mathscr{E} \to \mathcal{O}_{X \times X} \to \iota_* \mathcal{O}_X \to 0.$$

We see that

$$\mathcal{E}xt^{q}(\iota_{*}\mathcal{O}_{X}, \iota_{*}\mathcal{O}_{X}) \cong \mathcal{H}om_{X \times X}(\iota_{*}\mathcal{O}_{X}[-q], \iota_{*}\mathcal{O}_{X})$$

$$\cong \mathcal{H}om_{X \times X} \left(\bigwedge^{q} \mathcal{E}, \iota_{*}\mathcal{O}_{X}\right)$$

$$\cong \iota_{*}\mathcal{H}om_{X} \left(\iota^{*} \bigwedge^{q} \mathcal{E}, \mathcal{O}_{X}\right)$$

$$\cong \iota_{*}\mathcal{H}om_{X} \left(\bigwedge^{q} (\iota^{*}\mathcal{E}), \mathcal{O}_{X}\right).$$

Since X is the zero locus of some section of  $\mathscr{E}^{\vee}$ ,  $\iota^*e \cong N_{X/X \times X}^{\vee} \cong \Omega_X$ . So  $\mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) \cong \iota_* \bigwedge^q T_X$ . Hence

$$H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X)) \cong H^p(X \times X, \iota_*\bigwedge^q T_X) \cong H^p(X, \bigwedge^q T_X).$$

Similarly, we have  $\mathscr{E}xt^q(\iota_*\mathcal{O}_X,\iota_*\omega_X)\cong\iota_*\Omega_X^{d-q}$ , thus

$$H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\omega_X)) \cong H^p(X, \Omega_X^{d-q}).$$

# 13 Spanning class

### 13.1 Definition

Let  $\mathcal{D}$  be a triangulated category.

**Definition 13.1.** A spanning class of  $\mathcal{D}$  is a collection  $\Omega$  of objects of  $\mathcal{D}$  such that  $\langle \Omega \rangle^{\perp} = {}^{\perp} \langle \Omega \rangle = 0$ . Explicitly, this means that the following are all equivalent:

- $E \cong 0$ ;
- $\operatorname{Hom}(F, E[i]) = 0$  for each  $F \in \Omega$  and for each  $i \in \mathbb{Z}$ ;
- $\operatorname{Hom}(E[i], F) = 0$  for each  $F \in \Omega$  and for each  $i \in \mathbb{Z}$ .

Spanning classes are like generators of groups, rings, etc., which are useful for practical reason.

**Proposition 13.2.** Assume that  $\mathcal{D}$  has a Serre functor. Then for any collection  $\Omega$ ,  $\langle \Omega \rangle^{\perp} = 0$  if and only if  $^{\perp}\langle \Omega \rangle = 0$ . Thus,  $\Omega$  generates  $\mathcal{D}$  if and only if  $\Omega$  is a spanning class.

## 13.2 Examples

Let X be a smooth projective variety over a field k.

#### 13.2.1 Closed points

Lemma 13.3. The collection

$$\Omega = \{ \mathcal{O}_x \mid x \in X \text{ is a closed point, } \}$$

is a spanning class of  $D^{\mathrm{b}}(X)$ .

*Proof.* Let  $E^{\bullet} \in D^{b}(X)$  be a nonzero object. We may assume that  $H^{0}(E^{\bullet}) \neq 0$  and  $H^{i}(E^{\bullet}) = 0$  for each i > 0. Choose  $x \in \text{Supp}(H^{0}(E^{\bullet}))$ . Consider

$$E_2^{p,q} = \operatorname{Ext}^p(H^{-q}(E^{\bullet}), \mathcal{O}_x) \Rightarrow \operatorname{Ext}^{p+q}(E^{\bullet}, \mathcal{O}_x).$$

We have  $E_2^{p,q} = 0$  if p < 0 or q < 0. So

$$E_2^{0,0} \cong E_\infty^{0,0} \cong \operatorname{Hom}(E^{\bullet}, \mathcal{O}_x) \cong \operatorname{Hom}(H^0(E^{\bullet}), \mathcal{O}_x) \neq 0$$

as 
$$x \in \text{Supp}(H^0(E^{\bullet}))$$
.

### 13.2.2 Ample line bundles

Let  $\mathscr{L}$  be an ample line bundle on X with  $\dim X = n$ . Assume that  $\mathscr{L}$  is globally generated. We have seen in the proof of Beilinson's theorem (10.1) that

$$\bigoplus_{i=0}^{n} \mathscr{L}^{-i}$$

spans  $D^{\mathrm{b}}(X)$ . In particular, for every ample line bundle  $\mathscr{L}$  on X,

$$\Omega = \{\mathcal{O}_X, \mathcal{L}^k, \dots, \mathcal{L}^{nk}\}$$

spans  $D^{\mathrm{b}}(X)$  whenever  $k \gg 0$  (such that  $\mathcal{L}^k$  is globally generated).

## 13.3 Some applications

**Theorem 13.4.** Let  $F: \mathcal{C} \to \mathcal{D}$  be an exact functor between triangulated categories. Assume that  $G \dashv F \dashv H$ . Let  $\Omega$  be a spanning class of  $\mathcal{C}$ . Assume that for any  $A, B \in \Omega$  and for any  $i \in \mathbb{Z}$ , the map

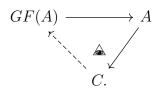
$$F: \operatorname{Hom}(A, B[i]) \to \operatorname{Hom}(F(A), F(B)[i])$$

is an isomorphism. Then  $F: \mathcal{C} \to \mathcal{D}$  is fully faithful.

*Proof.* As  $G \dashv F \dashv H$  and F is exact, both G and H are exact. For any  $A, B \in \mathcal{C}$  and for any  $i \in \mathbb{Z}$  we have the commutative diagram

$$\operatorname{Hom}(A,B[i]) \xrightarrow{\hspace{1cm}} \operatorname{Hom}(A,HF(B)[i])$$
 
$$\downarrow^{\natural}$$
 
$$\operatorname{Hom}(GF(A),B[i]) \xrightarrow{\hspace{1cm}} \operatorname{Hom}(F(A),F(B)[i]).$$

Complete  $GF(A) \to A$  to a distinguished triangle



Apply Hom(-, B[i]) to the triangle, we get

$$\cdots \longrightarrow \operatorname{Hom}(C, B[i]) \longrightarrow \operatorname{Hom}(A, B[i]) \longrightarrow \operatorname{Hom}(GF(A), B[i]) \longrightarrow \cdots$$

$$\downarrow^{\wr}$$

$$\operatorname{Hom}(F(A), F(B)[i]).$$

Now assume  $A, B \in \Omega$ , then the F in the above diagram becomes an isomorphism. So Hom(C, B[i]) = 0, and thus C = 0. Hence  $GF(A) \xrightarrow{\sim} A$  if  $A \in \Omega$ .

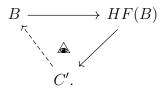
Still assuming  $A \in \Omega$ . For each  $B \in \mathcal{C}$ , we now have

$$\operatorname{Hom}(A,B[i]) \xrightarrow{\sim} \operatorname{Hom}(A,HF(B)[i])$$

$$\downarrow^{\natural} \qquad \qquad \downarrow^{\natural}$$

$$\operatorname{Hom}(GF(A),B[i]) \xrightarrow{\sim} \operatorname{Hom}(F(A),F(B)[i])$$

Consider the distinguished triangle



We get

$$\cdots \longrightarrow \operatorname{Hom}(A, B[i]) \stackrel{\sim}{\longrightarrow} \operatorname{Hom}(A, HF(B)[i]) \longrightarrow \operatorname{Hom}(A, C'[i]) \longrightarrow \cdots$$

So  $\operatorname{Hom}(A, C') = 0$ , and hence C' = 0. This gives  $B \xrightarrow{\sim} HF(B)$ . Hence, for any  $A, B \in \mathcal{C}$ ,

$$F: \operatorname{Hom}(A,B) \xrightarrow{\sim} \operatorname{Hom}(A,HF(B)) \xrightarrow{\sim} \operatorname{Hom}(F(A),F(B))$$

is an isomorphism.

Recall that an equivalence of triangulated categories  $F: \mathcal{C} \to \mathcal{D}$  commutes with Serre functors whenever they exist (8.8). We show that the converse is also true, and that verifying the commutativity for a spanning class is enough.

**Theorem 13.5.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be triangulated categories with Serre functors  $S_{\mathcal{C}}$  and  $S_{\mathcal{D}}$ , respectively. Let  $F: \mathcal{C} \to \mathcal{D}$  be an exact functor and assume that  $G \dashv F$ . Let  $\Omega$  be a spanning class of  $\mathcal{C}$ . Assume that  $\mathcal{C} \not\cong 0$  and  $\mathcal{D}$  is indecomposable. If  $F \circ S_{\mathcal{C}}(A) \cong S_{\mathcal{D}} \circ F(A)$  for all  $A \in \Omega$ , then  $F: \mathcal{C} \to \mathcal{D}$  is an equivalence of category.

We first prove another criterion:

**Theorem 13.6.** Let  $F: \mathcal{C} \to \mathcal{D}$  be a fully faithful exact functor of triangulated categories. Assume that  $\mathcal{D}$  is indecomposable and  $\mathcal{C} \ncong 0$ . Then F defines an equivalence of categories if and only if  $G \dashv F \dashv H$  and  $H(B) \cong 0$  implies  $G(B) \cong 0$  for each  $B \in \mathcal{D}$ .

*Proof.* Assume that F defines an equivalence of categories. Then  $F^{-1} \dashv F \dashv F^{-1}$  and the only if part is now clear.

**Lemma 13.7.** If  $F \dashv H$  and F is fully faithful, then  $HF \cong id_{\mathcal{C}}$ .

*Proof of Lemma.* For all  $A, B \in \mathcal{C}$ , we have

$$\operatorname{Hom}(B, HF(A)) \cong \operatorname{Hom}(F(B), F(B)) \cong \operatorname{Hom}(B, A)$$

and these isomorphisms are functorial in A, B. Thus  $HF \cong id_{\mathcal{C}}$  by Yoneda's lemma.

Assume that  $G \dashv F \dashv H$  and  $H(B) \cong 0$  implies  $G(B) \cong 0$  for each  $B \in \mathcal{D}$ . Let  $B \in \mathcal{D}$ . Let  $C \in D$  such that

$$FH(B) \to B \to C \to FH(B)[1]$$

is a distinguished triangle. Apply H to this triangle, we get the distinguished triangle

$$HFH(B) \to H(B) \to H(C) \to HFH(B)[1],$$

where  $HFH(B) \to H(B)$  is an isomorphism by the lemma. Hence  $H(C) \cong 0$ . We also have  $FHFH(B) \cong FH(B)$ .

Consider the full subcategories

$$\mathcal{D}_1 = \{ B \in \mathcal{D} \mid FH(B) \cong B \} \subseteq \mathcal{D}$$

$$\mathcal{D}_2 = \{ B \in \mathcal{D} \mid H(B) \cong 0 \} \subseteq \mathcal{D}$$

We just showed that for each  $B \in \mathcal{D}$ , there exists a distinguished triangle

$$B_1 \rightarrow B \rightarrow B_2 \rightarrow B_1[1]$$

with  $B_1 \in \mathcal{D}_1$  and  $B_2 \in \mathcal{D}_2$ . But for any  $B_1 \in \mathcal{D}_1$  and any  $B_2 \in \mathcal{D}_2$ , we have

$$\operatorname{Hom}(B_1, B_2) \cong \operatorname{Hom}(FH(B_1), B_2) \cong \operatorname{Hom}(H(B_1), H(B_2)) \cong 0.$$

$$\operatorname{Hom}(B_2, B_1) \cong \operatorname{Hom}(B_2, FH(B_1)) \cong \operatorname{Hom}(G(B_2), H(B_1)) \cong 0$$

since  $H(B_2) = 0$  and  $G(B_2) = 0$  by assumption. Hence  $\mathcal{D}$  decompses into  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . As  $\mathcal{C} \neq 0$ ,  $HF \cong \mathrm{id}_{\mathcal{C}}$  gives  $\mathcal{D}_2 \ncong \mathcal{D}$ . Since  $\mathcal{D}$  is indecomposable,  $\mathcal{D}_1 \cong \mathcal{D}$ . So  $F : \mathcal{C} \to \mathcal{D}$  is essentially surjective.

We continue the proof of (13.5).

Proof. We want to show that: If  $FS_{\mathcal{C}}(A) \cong S_{\mathcal{D}}F(A)$  for all  $A \in \Omega$ , then  $F : \mathcal{C} \to \mathcal{D}$ . We have  $G \dashv F \dashv H := S_{\mathcal{C}} \circ G \circ S_{\mathcal{D}}^{-1}$ . By the previous theorem, it is enough to show that for each  $B \in \mathcal{D}$ ,  $H(B) \cong 0$  implies  $G(B) \cong 0$ . Suppose  $H(B) \cong 0$ , then for each  $A \in \Omega$ ,

$$\operatorname{Hom}(A, G(B)[i]) \cong \operatorname{Hom}(G(B)[i], S_{\mathcal{C}}(A))^{\vee} \cong \operatorname{Hom}(B[i], FS_{\mathcal{C}}(A))^{\vee}$$
$$\cong \operatorname{Hom}(B[i], S_{\mathcal{D}}F(A))^{\vee} \cong \operatorname{Hom}(F(A), B[i])$$
$$\cong \operatorname{Hom}(A, H(B)[i]) \cong 0.$$

Hence 
$$G(B) \cong 0$$
.

# 14 Autoequivalence

## 14.1 Definition, first examples

Let  $\mathcal{D}$  be a triangulated category.

**Definition 14.1.** The group of autoequivalence is defined by

$$Aut(\mathcal{D}) := \{ \text{ isomorphism classes of equivalences } \mathcal{D} \to \mathcal{D} \}.$$

Examples of autoequivalences:

- $[1]: \mathcal{D} \xrightarrow{\sim} \mathcal{D}$
- $f^*: \mathcal{D}^?(X) \xrightarrow{\sim} \mathcal{D}^?(X)$ , where X is a noetherian scheme and  $f: X \xrightarrow{\sim} X$  is a morphism.
- $\otimes L : \mathcal{D}^{?}(X) \xrightarrow{\sim} \mathcal{D}^{?}(X)$ , where X is a noetherian scheme and L is a line bundle over X.

**Lemma 14.2.** Let X be a variety over a field k. We have an injective group homomorphism

$$\mathbb{Z} \times (\operatorname{Aut}(X) \ltimes \operatorname{Pic}(X)) \hookrightarrow \operatorname{Aut}(D^{?}(X))$$
$$(i, f, \mathscr{L}) \longmapsto [\mathscr{F}^{\bullet} \mapsto f^{*}(\mathscr{F}^{\bullet} \otimes \mathscr{L})[i]].$$

*Proof.* The group homomorphism part is clear. Assume that  $\Phi: D^{?}(X) \to D^{?}(X)$  defined by  $\mathscr{F}^{\bullet} \mapsto f^{*}(\mathscr{F}^{\bullet} \otimes \mathscr{L})[i]$  satisfies  $\Phi \cong \mathrm{id}$ . Then we have

$$\mathcal{O}_X \longrightarrow f^* \mathscr{L}[i] \longrightarrow \mathcal{O}_X.$$

This implies  $\operatorname{Ext}^i(\mathcal{O}_X, f^*L) \neq 0$ ,  $\operatorname{Ext}^{-i}(f^*\mathcal{L}, \mathcal{O}_X) \neq 0$ . This implies i = 0. So we get  $H^0(X, f^*\mathcal{L}) \neq 0$  and  $H^0(X, f^*\mathcal{L}^{\vee}) \neq 0$ , which gives  $f^*L \cong \mathcal{O}_X$ , and thus  $L \cong \mathcal{O}_X$ . Finally, for each closed point  $x \in X$ , we have

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,f(x)}[i] \longrightarrow \mathcal{O}_{X,x}.$$

So x = f(x). Hence  $f = id_X$ .

**Proposition 14.3** (Bondal-Orlov). Let X be a smooth projective variety with  $\omega_X$  or  $\omega_X^{\vee}$ . Then

$$\operatorname{Aut}(D^{\operatorname{b}}(X)) \cong \mathbb{Z} \times (\operatorname{Aut}(X) \ltimes \operatorname{Pic}(X)).$$

### 14.2 Spherical twists

### 14.2.1 Historical origin

Let X and  $X^{\vee}$  be mirror pairs of Calabi-Yau manifolds. The homological mirror symmetry conjecture asserts that there should be an equivalence

$$D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(\mathrm{Fuk}(X^{\vee})).$$

The Dehn twists along a Lagrangian sphere S on  $D^{\mathrm{b}}(\mathrm{Fuk}(X^{\vee}))$  should give us spherical twists on  $D^{\mathrm{b}}(X)$  associated to spherical objects  $\mathscr{E}^{\bullet}$ , and  $\mathscr{E}^{\bullet}$  corresponds to S.

### 14.2.2 Spherical objects

Let X be a smooth projective variety over a field k.

**Definition 14.4.** An object  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$  is called spherical if

- (i)  $\mathscr{E}^{\bullet} \otimes \omega_X \cong \mathscr{E}^{\bullet}$ ,
- (ii)  $\operatorname{Hom}(\mathscr{E}^{\bullet}, \mathscr{E}^{\bullet}[i]) = k$  if i = 0 or  $\dim X$  and equals to 0 otherwise.

Let  $\mathscr{E}^{\bullet} \in D^{\mathrm{b}}(X)$  be a spherical object. For each  $\mathscr{F}^{\bullet} \in D^{\mathrm{b}}(X)$ , choose  $T_{\mathscr{E}^{\bullet}}(\mathscr{F}^{\bullet})$  such that

$$\mathsf{R}\mathscr{H}om(\mathscr{E}^{\bullet},\mathscr{F}^{\bullet})\otimes\mathscr{E}^{\bullet}\to\mathscr{F}^{\bullet}\to T_{\mathscr{E}^{\bullet}}(\mathscr{F}^{\bullet})\to\mathsf{R}\mathscr{H}om(\mathscr{E}^{\bullet},\mathscr{F}^{\bullet})\otimes\mathscr{E}^{\bullet}[1]$$

is a distinguished triangle.

**Theorem 14.5.** The functor

$$T_{\mathscr{E}^{\bullet}}: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$$

defines an equivalence.

We call  $T_{\mathscr{E}^{\bullet}}$  the spherical twist associated to  $\mathscr{E}^{\bullet}$ . Let's first look at some example before proving the theorem.

### 14.2.3 Examples

(a) Let X be Calabi-Yau:  $\omega_X \cong \mathcal{O}_X$  and  $H^i(X, \mathcal{O}_X) = 0$  for all  $0 < i < \dim X$ . Every line bundle  $\mathscr{L}$  on X is spherical.

# Interlude 4: T-structures and torsion pairs

### 14.3 T-structures

Let  $\mathcal{D}$  be a triangulated category. We want to find some abelian category  $\mathcal{A}$  in  $\mathcal{D}$ .

**Definition 14.6.** A *t*-structure on  $\mathcal{D}$  is a pair  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  of full additive subcategory such that we have:

- Define  $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ ,  $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ .  $D^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$  and  $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$ .
- $\operatorname{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0.$
- For each  $E \in \mathcal{D}$ , there exists a distinguished triangle

$$t_{\leq 0}E \to E \to t_{\geq 1}E \to t_{\leq 0}E[1]$$

with  $t_{\leq 0}E \in \mathcal{D}^{\leq 0}$  and  $t_{\geq 1}E \in \mathcal{D}^{\geq 1}$ .

**Example 14.7** (standard t-structure). Assume that  $\mathcal{D} = D^{?}(\mathcal{A})$  for some abelian category  $\mathcal{A}$ . Define

$$\mathcal{D}^{\leq 0} = \{ E \in \mathcal{D} \mid H^{i}(E) = 0 \ \forall i > 0 \},$$
$$\mathcal{D}^{\geq 0} = \{ E \in \mathcal{D} \mid H^{i}(E) = 0 \ \forall i < 0 \}.$$

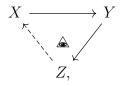
Then  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a t-structure of  $\mathcal{D}$  and  $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \cong \mathcal{A}$ .

**Theorem 14.8.** Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a t-structure of  $\mathcal{D}$ . Then  $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$  is an abelian category. We call  $\mathcal{A}$  the **heart** of the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ .

Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a t-structure of  $\mathcal{D}$  and let  $\mathcal{A}$  be its heart. Define

$$H^i_{\mathcal{A}}: \mathcal{D} \to \mathcal{A}$$
 $E \mapsto t_{\geq i}t_{\leq i}E,$ 

where  $t_{\leq i}E = (t_{\leq 0}(E[i]))[-i]$  and  $t_{\geq i}E = (t_{\geq 1}(E[1-i]))[i-1]$ . Then  $H_{\mathcal{A}}^i$  is a cohomological functor, namely for each distinguished triangle



the induced sequence

$$\cdots \longrightarrow H^{i}_{\mathcal{A}}(X) \longrightarrow H^{i}_{\mathcal{A}}(Y) \longrightarrow H^{i}_{\mathcal{A}}(Z) \longrightarrow H^{i+1}_{\mathcal{A}}(X) \longrightarrow \cdots$$

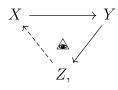
is exact. Assume  $X, Y \in \mathcal{A}$ , then  $H^i_{\mathcal{A}}(X) = H^i_{\mathcal{A}}(Y) = 0$  for each  $i \neq 0$ , so

$$0 \longrightarrow H_{\mathcal{A}}^{-1}(Z) \longrightarrow H_{\mathcal{A}}^{0}(X) \longrightarrow H_{\mathcal{A}}^{0}(Y) \longrightarrow H_{\mathcal{A}}^{0}(Z) \longrightarrow 0$$
$$0 \longrightarrow \ker f \longrightarrow X \xrightarrow{f} Y \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

In particular, for any  $X, Y, Z \in \mathcal{A}$ ,

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is exact in  $\mathcal{A}$  if and only if



in  $\mathcal{D}$ .

**Remark.** We have  $\operatorname{Ext}^1_{\mathcal{A}}(X,Y) \cong \operatorname{Ext}^1_{\mathcal{D}}(X,Y)$ , but in general  $\operatorname{Ext}^i_{\mathcal{A}}(X,Y) \not\cong \operatorname{Ext}^i_{\mathcal{D}}(X,Y)$ . So  $D^?(\mathcal{A}) \not\cong \mathcal{D}$  in general.

**Definition 14.9.** A t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  of  $\mathcal{D}$  is called bounded if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j}).$$

**Proposition 14.10.** Let  $A \subseteq \mathcal{D}$  be a full additive subcategory. Then A is the heart of a bounded t-structure if and only if the following conditions are satisfied:

- for each  $k \in \mathbb{Z}_{<0}$  and for any  $A, B \in \mathcal{A}$ ,  $\operatorname{Hom}_{\mathcal{D}}(A, B[k]) = 0$ ;
- for each  $E \in \mathcal{D}$  with  $E \neq 0$ , there exists integers  $k_1 > \cdots > k_n$  and

$$0 = E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow E$$

such that  $Cone(E_{i-1} \to E_i) \in \mathcal{A}[k_i]$ .

In this case, we have  $K(\mathcal{D}) \cong K(\mathcal{A})$ .

# 14.4 Torsion pairs

Let  $\mathcal{A}$  be an abelian category.

**Definition 14.11.** A torsion pair is a pair  $(\mathcal{T}, \mathcal{F})$  of full subcategories of  $\mathcal{A}$  such that:

- $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0;$
- for each  $E \in \mathcal{A}$ , there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \rightarrow 0$$

with  $T \in \mathcal{T}$  and  $F \in \mathcal{F}$ .

**Proposition 14.12.** Let  $(\mathcal{T}, \mathcal{F})$  be a torsion pair of  $\mathcal{A}$ . Then  $\mathcal{T} = {}^{\perp}\mathcal{F}$  and  $\mathcal{F} = \mathcal{T}^{\perp}$ . For each  $E \in \mathcal{A}$ , the objects  $T \in \mathcal{T}$  and  $F \in \mathcal{F}$  such that

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

is a short exact sequence are unique up to isomorphisms.

**Example 14.13.** Let X be a variety. In Coh(X),

$$\mathcal{T} = \{ \text{ torsion sheaf} \}, \quad \mathcal{F} = \{ \text{ torsion free sheaf} \},$$

form a torsion pair.

## 14.5 Tilting

Let  $\mathcal{D}$  be a triangulated category,  $\mathcal{A}$  the heart of a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . Suppose that  $\mathcal{A}$  admits a torsion pair  $(\mathcal{T}, \mathcal{F})$ . Define

$$^{\dagger}\mathcal{D}^{\leq 0} = \{ E \in \mathcal{D}^{\leq 0} \mid H^0(E) \in \mathcal{T} \}$$
$$^{\dagger}\mathcal{D}^{\geq 0} = \{ E \in \mathcal{D}^{\geq -1} \mid H^{-1}(E) \in \mathcal{F} \}$$

**Theorem 14.14** (Happel-Reiten-Smal $\varnothing$ ). The pair  $({}^{\dagger}\mathcal{D}^{\leq 0}, {}^{\dagger}\mathcal{D}^{\geq 0})$  is a t-structure on  $\mathcal{D}$ . We call it the **tilted** t-structure with respect to  $(\mathcal{T}, \mathcal{F})$ .