Introduction to the Minimal Model Program and Singularities

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1 Introduction, 9/4

Let X be an algebraic variety. We want to find a "good" representative in its birational class

$$\{X' \mid X \dashrightarrow X'\}\big/_{\text{isom.}},$$

where $X \dashrightarrow X'$ means that there are Zariski open subsets $U \subseteq X$, $U' \subseteq X'$ such that U is isomorphic to U'.

Recall. (MMP in dimension 2) We start from a smooth projective surface S. If there does not exist a -1 curve in S, then we end the process. If there is a -1 curve $E \subseteq S$, then we can contract E to a point and get a new smooth projective surface T. We then replace S by T and do this process until it ends.

Here, we need a theorem so called Castelnuovo's contraction theorem, it provides us the smoothness of T. Its higher dimensional analog is called the basepoint-free theorem.

The higher dimensional analog of -1 curve is the nefness of K_X, or we say, the Cone theorem.

Theorem 1.1 (Castelnuovo)**.** Let X be a d-dimensional smooth projective variety over an algebraic closed field $\mathbf{k}$, and let E be a divisor in X which is isomorphic to $\mathbb{P}^{d-1}$ with $N_{E/X} \cong \mathcal{O}_X(E)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$. Then there exists a smooth projective variety Y and a point $P \in Y$ such that $X \cong \mathrm{Bl}_P Y$ and E is the exceptional divisor under this isomorphism.

Proof. Choose a very ample divisor H on X such that $H^1(X, \mathcal{O}_X(H)) = 0$. Since E is isomorphic to $\mathbb{P}^{d-1}$, there exists $a \in \mathbb{N}$ such that $\mathcal{O}_X(H)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(a)$.

First, we show that $|H+aE|$ is basepoint-free. Since H is very ample, $|H+aE| \supseteq |H|$, so $|H+aE|$ separates points away E. So it suffices to show that $|H+aE|$ has no base points on E.

Fact. For $0 \le i \le a+1$, $H^1(X, \mathcal{O}_X(H+iE)) = 0$ by induction on i.

This shows that $\mathrm{H}^0(X, \mathcal{O}_X(H+aE)) \to \mathrm{H}^0(E, \mathcal{O}_X(H+aE)|_E)$ is surjective. By our assumption, $\mathcal{O}_X(H+aE)|_E \cong \mathcal{O}_{P^{d-1}}(a-a) = \mathcal{O}_{\mathbb{P}^{d-1}}$. Take $s \in \mathrm{H}^0(X, \mathcal{O}_X(H+aE))$ that maps to $1 \in \mathrm{H}^0(E, \mathcal{O}_E)$. Let $D = (s)_0$. Then $\mathrm{Supp}(D) \cap \mathrm{Supp}(E) = \varnothing$, i.e., there exists $D \in |H+aE|$ such that $q \notin \mathrm{Supp}(D)$ for all $q \in E$.

Hence, we get a morphism $\varphi = |H+aE| \colon X \to \mathbb{P}^N$ with $\varphi^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}(H+aE)$. Since $\mathcal{O}_X(H+aE)|_E \cong \mathcal{O}_E$, $\varphi(E)$ is a point $p_1 \in Y_1 := \varphi(X)$.

Since $|H+aE|$ also separates tangent vectors on $X \setminus E$, we see that $\varphi \colon X \setminus E \cong Y_1 \setminus \{p_1\}$. Applying Stein factorization, we get

$$\begin{tikzcd} X \arrow[r, "\pi"'] \arrow[rr, bend left, "\varphi"] & Y \arrow[r, "\text{fin.}"'] & Y_1 \end{tikzcd}$$

with π having connected fibers and $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Since E is irreducible, $\pi(E) = \{p\}$ and thus

$$X \setminus E \cong Y \setminus \{p\} \cong Y_1 \setminus \{p_1\}.$$

Now, we show that Y is smooth at p, i.e., $\mathcal{O}_{Y,p}$ is a regular local ring. This is equivalent to $\widehat{\mathcal{O}}_{Y,p}$ being local. Consider the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} \mathcal{O}_{Y,P}/\mathfrak{m}_P^n \arrow[r] & Y. \end{tikzcd}$$

By the theorem on formal functions,

$$\widehat{\mathcal{O}}_{Y,p} = \widehat{(\pi_*\mathcal{O}_X)}_p \cong \varprojlim_n \mathrm{H}^0(E_n, \mathcal{O}_{E_n}).$$

As a topological space, $E_n = E$ and $\mathcal{O}_{E_n} = \mathcal{O}_X/\mathfrak{m}_p^n\mathcal{O}_X$. Since $\pi^{-1}(p) = E$, $\mathfrak{m}_P\mathcal{O}_X \subseteq \mathscr{I}_E$ and using $\mathrm{Supp}(\mathcal{O}_X/\mathfrak{m}_P^n\mathcal{O}_X) = E$, $\mathscr{I}_E^m \subseteq \mathfrak{m}_P\mathcal{O}_X$ for some m. So the sequences

$(\mathcal{O}_X/\mathfrak{m}_P^n\mathcal{O}_X)$ and $(\mathcal{O}_X/\mathscr{I}_E^n)$ are cofinal. Hence,

$$\varprojlim_n \mathrm{H}^0(E_n, \mathcal{O}_{E_n}) \cong \varprojlim_n \mathrm{H}^0(E, \mathcal{O}_X/\mathscr{I}_E^n).$$

Claim. We have

$$\mathrm{H}^0(E, \mathcal{O}_X/\mathscr{I}_E^n) \cong {}^{\mathbf{k}[[x_0, \dots, x_{d-1}]]}\big/_{\langle x_0, \dots, x_{d-1}\rangle^n} =: A_n$$

for each n (and hence $\widehat{O}_{Y,p} \cong \mathbf{k}[[x_0, \dots, x_{d-1}]]$ is regular).

Proof of Claim. Consider the exact sequence

$$0 \longrightarrow \mathscr{I}_E^n/\mathscr{I}_E^{n+1} \longrightarrow \mathcal{O}_{E_{n+1}} \longrightarrow \mathcal{O}_{E_n} \longrightarrow 0.$$

Since $\mathscr{I}_E^n/\mathscr{I}_E^{n+1} \cong \mathrm{Sym}^n(\mathscr{I}_E/\mathscr{I}_E^2) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(n)$ (by $\mathscr{I}_E/\mathscr{I}_E^2 = N_{E/X}^\vee \cong \mathcal{O}_{P^{d-1}}(1)$), we get the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^{d-1}}(n)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{E_{n+1}}) \longrightarrow \mathrm{H}^0(\mathcal{O}_{E_n}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{d-1}}(n)) = 0.$$

For $n = 1$, $\mathrm{H}^0(\mathcal{O}_E) = \mathbf{k} = A_1$ and $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^{d-1}}(1)) = \langle x_0, \dots, x_{d-1}\rangle_{\mathbf{k}}$. So $\mathrm{H}^0(\mathcal{O}_{E_2}) \cong A_2$. We prove the statement by induction on n. Suppose $\mathrm{H}^0(\mathcal{O}_{E_n}) = A_n$. Lifting $x_0, \dots, x_{d-1}$ to $\mathrm{H}^0(\mathcal{O}_{E_{n+1}})$ and using $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^{d-1}}) = \langle x_0^{i_0} \cdots x_{d-1}^{i_{d-1}} \mid i_0 + \cdots + i_{d-1} = n\rangle_{\mathbf{k}}$, we get $\mathrm{H}^0(\mathcal{O}_{E_{n+1}}) = A_{n+1}$. □

Finally, we show that $X \cong \mathrm{Bl}_P Y$. We already have $\mathfrak{m}_P\mathcal{O}_X \subseteq \mathscr{I}_{\mathscr{E}}$. Since the images of $x_0, \dots, x_{d-1}$ generate $\mathscr{I}_E/\mathscr{I}_E^2 \cong \mathcal{O}_{\mathbb{P}^{d-1}}(1)$, $x_0, \dots, x_{d-1}$ generate $\mathscr{I}_E$, so $\mathfrak{m}_P\mathcal{O}_X = \mathscr{I}_E = \mathcal{O}_X(-E)$ is invertible.

By universal property of blow-up, we get the diagram

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr] & \mathrm{Bl}_P Y = Y' \arrow[d] \\ & Y. \end{tikzcd}$$

Let

$$\mathrm{Exc}(\rho) := \{x \in X \mid \rho^{-1} \text{ is not a morphism at } \rho(x)\}.$$

Fact. Let $\rho\colon X \to Y'$ be a birational map with $\mathrm{Exc}(\rho) \neq \varnothing$, X normal and Y' being $\mathbb{Q}$-factorial. Then $\mathrm{Exc}(\rho)$ is of pure codimension 1 in X and $\rho(\mathrm{Exc}(\rho))$ has codimension ≥ 2.

In our case, $X \setminus E \cong Y \setminus \{p\} \cong Y' \setminus E'$, so $\rho(E) = E'$, which has codimension 1. The fact then shows that $\mathrm{Exc}(\rho) = \varnothing$. ■

The category we work in the MMP.

- Objects: normal varieties, X, Y, ...
- Morphisms: $X \xrightarrow{\pi} Y$ with connected fibers, or $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ (contraction morphism)

Remark.

- The MMP (or Mori's theory) does not say much about finite morphisms.
- For any morphism $\varphi\colon X \to Y$ we can always take its Stein factorization $X \xrightarrow{\pi} Y' = \operatorname{Spec} \varphi_*\mathcal{O}_X \xrightarrow{g} Y$ where π has connected fibers and g is finite.
- In characteristic 0, if the fivers of φ are connected, Y' normal, then g is an isomorphism. (In positive characteristic, think of the Frobenius morphism.)
- In Mori's theory, we focus on curves (not divisors). The curves contracted by φ is same as the curves contracted by π.

Definition 1.2. Let X be a proper variety. Denote by $\mathrm{CDiv}(X)$ the group of Cartier divisors of X,

$$Z_1(X) := \{C = \textstyle\sum a_i C_i \mid a_i \in \mathbb{Z},\ C_i \text{ is an integral curve}\}$$

the group of 1-cycles. We say C is an effective 1-cycle if $a_i \geq 0$.

We say two Cartier divisors D and D' are numerically equivalent, denoted by $D \equiv D'$, if $C \cdot D = C \cdot D'$ for each $C \in Z_1(X)$. We say two 1-cycles C and C' are numerically equivalent, denoted by $C \equiv C'$, if $C \cdot D = C' \cdot D$ for each $D \in \mathrm{CDiv}(X)$.

We denote

$$N^1(X)_R = (\mathrm{CDiv}(X) \otimes_{\mathbb{Z}} R)/\equiv,$$
$$N_1(X)_R = (Z_1(X) \otimes_{\mathbb{Z}} R)/\equiv$$

for $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. $N^1(X)$ is sometimes denoted by $\mathrm{NS}(X)$, called the Néron-Severi group, and its rank $\rho(X) = \dim_{\mathbb{R}} N^1(X)_{\mathbb{R}}$ is finite (when X is smooth over $\mathbb{C}$, by Lefschetz theorem).

The pairing

$$N^1(X)_R \otimes N_1(X)_R \longrightarrow R$$

is nondegenerate. The groups N^1 and N_1 are functorial: if $\pi\colon X \to Y$ is a proper morphism with Y proper, then there are maps $\pi^*\colon N^1(Y)_{\mathbb{Z}} \to N^1(X)_{\mathbb{Z}}$ and $\pi_*\colon N_1(X)_{\mathbb{Z}} \to N_1(X)_{\mathbb{Z}}$. These maps are related to the pairing by the projection formula

$$\pi^*D \cdot C = D \cdot \pi_*C.$$

Definition 1.3 (Mori cone)**.** For $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, let

$$\mathrm{NE}(X)_R = \{\textstyle\sum a_i[C_i] \mid a_i \in R_{\geq 0},\, C_i \text{ is an integral curve}\}.$$

Let $\overline{\mathrm{NE}}(X)$ be the closure of $\mathrm{NE}(X)_{\mathbb{R}}$ in $N_1(X)_{\mathbb{R}}$.

Definition 1.4. Let V be an $\mathbb{R}$-vector space and let K be a cone in V, i.e., $r \cdot K \subseteq K$ for any $r \in \mathbb{R}_{\geq 0}$. A subcone $F \subseteq K$ is called extremal if for any u, $v \in K$ with $u + v \in F$, we have u, $v \in F$. F is called an extremal ray of K if $\dim F = 1$.

Fact. Let X, Y, Y' be projective varieties, $\pi\colon X \to Y$ be a morphism. Then

(a) $\mathrm{NE}(\pi) = \mathrm{NE}(X/Y) := \ker \pi_* \cap \mathrm{NE}(X)$ is an extremal face of $\mathrm{NE}(X)$.

(b) Assume π is a contraction morphism and let $\pi'\colon X \to Y'$ be another morphism. If $\mathrm{NE}(\pi) \subseteq \mathrm{NE}(\pi')$, then there exists a unique morphism $Y \to Y'$ satisfies the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"'] \arrow[rr, bend left, "\pi'"] & Y \arrow[r] & Y'. \end{tikzcd}$$

Theorem 1.5 (Kleiman's ampleness criterion)**.** Let X be a projective variety and D be a Cartier divisor on X. Then D is ample if and only if

$$D_{>0} = \{x \in N_1(X)_{\mathbb{R}} \mid D \cdot x > 0\} \supseteq \overline{\mathrm{NE}}(X) \setminus \{0\}.$$

2 Ampleness criterion, 9/7

In the following, a scheme always mean a separated scheme of finite type over a field $\mathbf{k}$.

Theorem 2.1 (Nakai–Moishezon criterion)**.** Let X be a projective scheme, D a Cartier divisor on X. Then D is ample if and only if $D^{\dim Y} \cdot Y > 0$ for every positive dimension closed subvariety $Y \subseteq X$.

Remark. The same result holds when X is proper and D is a $\mathbb{Q}$-Cartier divisor.

Definition 2.2. Let X be a proper scheme. A Cartier divisor D of X is **nef** (=numerically effective) if $D^{\dim Y} \cdot Y \geq 0$ for every subvariety $Y \subseteq X$.

Remark. If $D_1 \equiv D_2$, then D_1 is ample if and only if D_2 is ample; D_1 is nef if and only if D_2 is nef.

Lemma 2.3. Let X be a projective scheme of dimension n, H an ample Cartier divisor and D a Cartier divisor. Fix an integer $0 \leq r \leq n$. If $D^r \cdot Y \geq 0$ for every subvariety $Y \subseteq X$ of dimension r, then $D^r \cdot H^{n-r} \geq 0$.

Proof. We proceed by induction on $\dim X = n$. Without loss of generality, we may assume that X is integral and $0 < r < n$.

Since mH is very ample for some $m \gg 1$, there exists an effective divisor $Y \in |mH|$. Then

$$D^r \cdot H^{n-r} = \frac{1}{m} D^r \cdot H^{n-1-r} \cdot (mH) = \frac{1}{m} D^r \cdot H^{n-1-r} \cdot Y = \frac{1}{m} (D|_Y)^r \cdot (H|_Y)^{n-1-r},$$

which is nonnegative by induction. ∎

Now, for H ample Cartier, D nef, and every subvariety $Y \subseteq X$ of dimension r, we have

$$\begin{aligned}(H+D)^r \cdot Y &= H^r \cdot Y + \sum_{s=1}^{r} \binom{r}{s} D^s \cdot H^{r-s} \cdot Y \\ &= (H|_Y)^r + \sum_{s=1}^{r} \binom{r}{s} (D|_Y)^s \cdot (H|_Y)^{r-s} \geq (H|_Y)^r > 0.\end{aligned}$$

So $H + D$ is ample by (2.1).

We define the (open) ample cone

$$\mathrm{Amp}(X) = \{D \in N^1(X)_{\mathbb{Q}} \mid D \text{ is ample}\}$$

and the nef cone

$$\mathrm{Nef}(X) = \{D \in N^1(X)_{\mathbb{Q}} \mid D \text{ is nef}\}.$$

Corollary 2.4. The nef cone is the closure of the ample cone, the ample cone is the interior of the nef cone.

Proof. It is clear that $\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X)$ and $\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X)^\circ$.

Fix an ample divisor H. For each $D \in \mathrm{Nef}(X)$, $D + \varepsilon H$ is ample for all $\varepsilon > 0$, so D lies in $\overline{\mathrm{Amp}(X)}$. For each $D \in \mathrm{Nef}(X)^\circ$, $D - \varepsilon H$ is still nef for some $\varepsilon > 0$, so $D = (D - \varepsilon H) + \varepsilon H \in \mathrm{Amp}(X)$. ■

Theorem 2.5 (Kleiman)**.** Let X be a proper scheme. A Cartier divisor D is nef if and only if $D \cdot C \geq 0$ for every irreducible curve $C \subseteq X$.

Proof. The only if part is just the definition of nef. For the if part, we may assume that X is integral and projective by Chow's lemma: there exists a surjective birational morphism $\pi\colon X' \to X$ with X' projective.

We proceed by induction on $n = \dim X$. The statement is clearly true for $n = 1$. If $Y \subsetneq X$, then $D^{\dim Y} \cdot Y \geq 0$ by induction hypothesis, so it remains to prove $D^n \geq 0$ by (2.1).

Fix a very ample Cartier divisor H. Set $D_t = D + tH$. Consider the degree n polynomial

$$P(t) = D_t^n = D^n + \sum_{i=1}^{n-1} \binom{n}{i} (D^{n-i} \cdot H^i) t^i + H^n t^n.$$

Assume that $P(0) = D^n < 0$. Then it follows from $H^n > 0$ that there exists a largest $t_0 \in (0, \infty)$ such that $P(t_0) = 0$ and $P(t) > 0$ for $t > t_0$.

For $t \in (t_0, \infty) \cap \mathbb{Q}$, we see that D_t is ample: for every subvariety $Y \subseteq X$ of dimension r with $0 < r < n$, $D|_Y$ is nef by induction, so

$$\begin{aligned} D_t^r \cdot Y &= \sum_{s=1}^{r} \binom{r}{s} (D^s \cdot H^{r-s} \cdot Y) t^{r-s} + (H^r \cdot Y) t^r \\ &= \sum_{s=1}^{r} \binom{r}{s} ((D|_Y)^s \cdot (H|_Y)^{r-s}) t^{r-s} + (H^r \cdot Y) t^r > 0 \end{aligned}$$

for $t > 0$. Also, $D_t^n = P(t) > 0$, so D_t is ample for all rational $t > t_0$.

Note that

$$P(t) = D_t^n = D_t^{n-1} \cdot (D + tH) = D_t^{n-1} \cdot D + t(D_t^{n-1} \cdot H) =: Q(t) + R(t).$$

Since $D \cdot C \geq 0$ for all irreducible curve C, D_t is ample for rational $t > t_0$, $Q(t) = D \cdot D_t^{n-1} \geq 0$ for all rational $t > t_0$, and hence $D \cdot D_{t_0}^{n-1} \geq 0$. Also,

$$R(t) = t(D + tH)^{n-1} \cdot H = t(D|_H + tH|_H)^{n-1} > 0$$

for $t > 0$ ($D|_H$ is nef and $H|_H$ is ample). So

$$0 = P(t_0) = Q(t_0) + R(t_0) \geq R(t_0) > 0,$$

a contradiction. So $D^n \geq 0$. ■

Theorem 2.6 (Kleiman's Ampleness criterion)**.** Let X be a projective variety, D a Cartier divisor. Then D is ample if and only if

$$D_{>0} := \{x \in N_1(X)_{\mathbb{R}} \mid D \cdot x > 0\} \supseteq \overline{NE}(X) \setminus \{0\}.$$

Remark. If a divisor D satisfies

$$D_{>0} \supseteq NE(X) \setminus \{0\},$$

we say that D is strictly nef.

Proof. Suppose that D is ample. Clearly, $D \cdot z \geq 0$ for each $z \in \overline{NE}(X)$. Assume that $D \cdot z = 0$ for some $z \in \overline{NE}(X) \setminus \{0\}$. Since the intersection pairing is nondegenerate, there exists a divisor such that $E \cdot z < 0$. Then

$$(D + tE) \cdot z = t(E \cdot z) < 0$$

for all $t > 0$. So $D + tE$ can not be ample for all $t > 0$, this contradicts the fact that D is ample.

For the if part, choose a norm $\|-\|$ on $N_1(X)_{\mathbb{R}}$ and an ample divisor H. Define

$$K = \{z \in \overline{NE}(X) \mid \|z\| = 1\},$$

which is a compact. The linear functional $z \mapsto D \cdot z$ is positive on K by our assumption. The linear functional $z \mapsto H \cdot z$ is bounded from above on K. So there exists $a, b \in \mathbb{Q}_{>0}$ such that $D \cdot z \geq a$ and $H \cdot z \leq b$ for all $z \in K$.

Now, for each $z \in \overline{NE}(X) \setminus \{0\}$,

$$(D - \tfrac{a}{b}H) \cdot \frac{z}{\|z\|} \geq a - \frac{a}{b} \cdot b = 0,$$

so $D - \frac{a}{b}H$ is nef, and thus $D = (D - \frac{a}{b}H) + \frac{a}{b}H$ is ample. ■

Proof of (2.1). If D is ample, then mD is very ample for $m \gg 1$. This gives us an embedding $f = |mD| \colon X \hookrightarrow \mathbb{P}^N$ such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}(mD)$. Then

$$(mD)^{\dim Y} \cdot Y = (mD|_Y)^{\dim Y} = \deg_{\mathbb{P}^N} f(Y) > 0.$$

Conversely, we may assume that X is integral. We show by induction on $\dim X$ that D is ample. If $\dim X = 1$, then this is clearly true.

Claim. For $m \gg 1$,
$H^0(mD) > 0$.

Proof of Claim. Since X is projective, we can write $D \sim A - B$ with A, B very ample. Then there are exact sequences

$$0 \longrightarrow \mathcal{O}_X(mD - B) \longrightarrow \mathcal{O}_X((m+1)D) \longrightarrow \mathcal{O}_A((m+1)D) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_X(mD - B) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_B(mD) \longrightarrow 0.$$

By induction, $(m+1)D|_A$ and $mD|_B$ are ample, so h^i of these line bundles are 0 for $i > 0$ and $m \gg 1$. Hence, for $i \geq 2$, $m \gg 1$,

$$\mathrm{h}^i(mD) = \mathrm{h}^i(mD - B) = \mathrm{h}^i((m+1)D),$$

i.e., $\mathrm{h}^i(mD)$ is a constant for $m \gg 1$. Then

$$\chi(mD) = H^0(mD) - \mathrm{h}^1(mD) + \mathrm{const}.$$

for $m \gg 1$. On the other hand, by Riemann–Roch we have

$$\chi(mD) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

So

$$H^0(mD) > \frac{D^n}{n!} m^n + O(m^{n-1}) - O(1) = \frac{D^n}{n!} m^n + O(m^{n-1}) > 0$$

for $m \gg 1$. □

Since D is ample if and only if mD is ample, we may replace D by mD. The claim tells us that we may assume D is effective.

Since

$$\operatorname{Bs}|mD| = \bigcap_{0 \leq D' \sim mD} \operatorname{Supp} D' \subseteq \operatorname{Supp}(mD) = \operatorname{Supp} D,$$

$\mathcal{O}_X(mD)$ is g.g. away from $\operatorname{Supp} D$. To show $|mD|$ has no base points on $\operatorname{Supp} D$, consider

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0.$$

We get the long exact sequence

$$\mathrm{H}^0(mD) \longrightarrow \mathrm{H}^0(mD|_D) \longrightarrow H^1((m-1)|_D) \longrightarrow H^1(mD|_D) \longrightarrow H^1(mD|_D),$$

where $H^1(mD|_D) = 0$ for $m \gg 1$ since $D|_D$ is ample by induction hypothesis. This shows that $\mathrm{h}^1(mD)$ decreases with respect to m, and hence stable for $m \gg 1$. Then $\mathrm{H}^0(mD) \to \mathrm{H}^0(mD|_D)$ is surjective for $m \gg 1$. Since $mD|_D$ is g.g. for $m \gg 1$, $|mD|$ has no base point on $\operatorname{Supp} D$.

Now, for $m \gg 1$, we get a projective morphism

$$\varphi = |mD| \colon X \longrightarrow \mathbb{P}^N.$$

Since $D \cdot C > 0$ for any irreducible curve $C \subseteq X$, all fibers are finite set (φ is quasi-finite). This implies φ is a finite morphism. So $\mathcal{O}_X(mD) = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ is ample. ■

Corollary 2.7. Let X be a projective variety, H an ample divisor. Then

(1) $\overline{NE}(X)$ is a strongly convex cone, i.e., z, $-z \in \overline{NE}(X)$ implies $z = 0$;

(2) for each $a \in \mathbb{R}_{>0}$, $W_a := \{z \in \overline{NE}(X) \mid H \cdot z \leq a\}$ is compact. In particular, $W_a \cap \mathrm{NE}(X)_{\mathbb{Z}}$ is a finite set.

Proof. For (1), if z, $-z \in \overline{NE}(X) \setminus \{0\}$, then both $H \cdot z$, $H \cdot (-z)$ are positive, a contradiction.

For (2), fix a norm $\|-\|$ on $N_1(X)_{\mathbb{R}}$. Assume that W_a is not compact. Since W_a is closed, it is not bounded, i.e., there exists a sequence z_i in W_a such that $\|z_i\| \to \infty$. Since $(z_i/\|z_i\|)$ is a bounded sequence there is a convergent subsequence $z_{i_j}/\|z_{i_j}\| \to y \in \overline{NE}(X)$ with $\|y\| = 1$, but

$$H \cdot y = \lim_j \frac{H \cdot z_{i_j}}{\|z_{i_j}\|} \leq \lim_j \frac{a}{\|z_{i_j}\|} = 0,$$

a contradiction. ■

3 A rough introduction to Hilbert schemes and schemes of morphisms, 9/11

Let S be a scheme. We say a functor

$$F : \mathsf{Sch}_{/S}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$$

is **representable** if there exists a object $M \in \mathsf{Sch}_{/S}$ such that $\mathrm{Hom}_S(-, M) \xrightarrow{\sim} F(-)$. In particular, there exists $U \in F(M)$, the image of id_M under $\mathrm{Hom}_S(M, M) \xrightarrow{\sim} F(M)$, such that

$$\begin{aligned} \mathrm{Hom}_S(T, M) &\xrightarrow{\quad\sim\quad} F(T) \\ f &\longmapsto\quad\;\; f^*U. \end{aligned}$$

Remark. The morphism U is frequently called the universal element or universal family over M. Also, the pair (M, U) is unique up to isomorphisms.

Fact. Let S be a scheme over $\mathbf{k}$,

$$F\colon \mathsf{Sch}_{/S}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$$

be a representable functor, represented by $M \in \mathsf{Sch}_{/S}$, $x_0 \in M$. Assume that we have an **obstruction theory** "at the point x_0". Then knowing its tangent space t_0 and obstruction space Ob_0, we have

$$\dim t_0 \geq \dim_{x_0} M \geq \dim t_0 - \dim \mathrm{Ob}_0 \,.$$

Indeed, $\widehat{\mathcal{O}}_{M,x_0}$ can be represented by $\dim \mathrm{Ob}_0$ equations in $\widehat{\mathcal{O}}_{\mathbb{A}^{\dim t_0}}$.

Definition 3.1. Fix a closed subscheme (or a quasi-projective scheme over S) X of $\mathbb{P}^r_S$ and a polynomial $P \in \mathbb{Q}[m]$. We define the Hilbert functor

$$\mathbf{Hilb}^P_{X/S}\colon \mathsf{Sch}_{/S}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$$

as follows: we send an object $T \in \mathsf{Sch}_{/S}$ to

$$\left\{ \begin{array}{c} \text{closed subschemes } Y \in X \times_S T, \\ \text{proper and flat over } T \text{ with } \chi(\mathcal{O}_{Y_t}(m)) = P(m) \end{array} \right\}.$$

Theorem 3.2 (Grothendieck)**.** If S is a Noetherian scheme and X is a (quasi-)projective S-scheme, then the Hilbert functor $\mathbf{Hilb}^P_{X/S}$ is representable by a (quasi-)projective S-scheme $\mathrm{Hilb}^P_{X/S}$ (called the Hilbert scheme), and a universal family

$$\begin{tikzcd} \mathrm{Univ}^P_{X/S} \arrow[r, hook] \arrow[d] & X \times_S \mathrm{Hilb}^P_{X/S} \arrow[dl] \\ \mathrm{Hilb}^P_{X/S} & \end{tikzcd}$$

We then define the Hilbert scheme of X to be

$$\mathrm{Hilb}_{X/S} = \bigsqcup_{P \in \mathbb{Q}[m]} \mathrm{Hilb}^P_{X/S} \,.$$

Example 3.3. Consider $P(m) \equiv 1$. Then $\mathrm{Hilb}^P_{X/S} = X$ and the universal family is $\Delta \subseteq X \times X$.

Suppose that S is $\operatorname{Spec} \mathbf{k}$ and $X \subseteq \mathbb{P}^r_k$ is a hypersurface of degree d. Let $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(d))$,

$$P(m) = \binom{m+r}{r} - \binom{m+r-d}{r} = \frac{d}{(r-1)!}\, m^{r-1} + \text{l.o.t.}.$$

Then $\mathrm{Hilb}^P_X = \mathbb{P}(V^\vee)$ and the universal family is

$$\left(\sum_{i_0+\cdots+i_r=d} c_{i_0\cdots i_r} x_0^{i_0} \cdots x_r^{i_r} = 0 \right) \subseteq \mathbb{P}^r \times \mathbb{P}(V^\vee).$$

Fact. Let $S = \operatorname{Spec} \mathbf{k}$ with $\mathbf{k}$ algebraically closed. Assume that $Z \hookrightarrow X$ us a regular embedding. Then the tangent space of Hilb_X at $[Z]$ is $\mathrm{H}^0(N_{Z/X})$, and the obstruction of Hilb_X at $[Z]$ is $\mathrm{H}^1(N_{Z/X})$. Hence, we get

$$\mathrm{h}^0(N_{Z/X}) \geq \dim_{[Z]} \mathrm{Hilb}_X \geq \mathrm{h}^0(N_{Z/X}) - \mathrm{h}^1(N_{Z/X}).$$

Definition 3.4. Let X, Y be objects of $\mathsf{Sch}_{/S}$. Define the functor of morphisms from X to Y to be

$$\begin{aligned} \mathbf{Mor}_S(X,Y)\colon\ \mathsf{Sch}^{\mathrm{op}}_{/S} &\longrightarrow \mathsf{Sets} \\ T &\longmapsto \mathrm{Hom}_T(X \times_S T, Y \times_S T). \end{aligned}$$

Theorem 3.5 (FGA explained)**.** Let S be a Noetherian scheme, X a projective scheme over S, Y a quasi-projective scheme over S (so that $X \times_S Y$ is quasi-projective over S). Assume moreover that X is flat over S. Then $\mathbf{Mor}_S(X, Y)$ is represent by an open subscheme $\mathrm{Mor}_S(X, Y)$ in $\mathrm{Hilb}_{X \times_S Y/S}$.

Sketch of proof. For each $T \in \mathsf{Sch}_{/S}$, denote $(-)_T = - \times_S T$. Consider $f \in \mathrm{Hom}_T(X_T, Y_T)$ and

$$\begin{tikzcd}
X_T \arrow[drr, bend left, "f"] \arrow[ddr, bend right, equal] & & \\
& X_T \times_T Y_T \arrow[r] \arrow[d] & Y_T \arrow[d] \\
& X_T \arrow[r] & T.
\end{tikzcd}$$

We get a morphism $(\mathrm{id}_{X_T}, f) \colon X_T \to X_T \times_T Y_T = (X \times_S Y)_T$ and the graph $\Gamma_T(f) = \mathrm{Im}(\mathrm{id}_{X_T}, f)$.

Since $Y \to S$ is separated, $\Gamma_T(f)$ is closed in $X_T \times_T Y_T$, so X, and hence $\Gamma_T(f)$, are proper and flat over S. This gives us a well-defined set map

$$\begin{aligned}
\Gamma_T \colon \quad \mathbf{Mor}_S(X, T)(T) &\longrightarrow \mathbf{Hilb}_{X \times_S Y/S}(T) \\
f &\longmapsto \Gamma_T(f),
\end{aligned}$$

which is functorial in T, i.e., there is a natural transformation of functors

$$\Gamma \colon \mathbf{Mor}_S(X, T) \longrightarrow \mathbf{Hilb}_{X \times_S Y/S}.$$

If $G \subseteq X_T \times_T Y_T$ is a family of closed subschemes of $X \times_S Y$ which is proper and flat over T, then

$$\{t \in T \mid \pi_t \colon G_t \xrightarrow{\sim} (X_T)_t\}$$

is open, where π is the projection $G \to X_T$. Hence, there exists an open subscheme $\mathrm{Mor}_S(X, Y)$ of $\mathrm{Hilb}_{X \times_S Y/S}$ that represents $\mathbf{Mor}_S(X, Y)$. ■

Fact. Let $S = \operatorname{Spec} \mathbf{k}$ with $\mathbf{k}$ algebraically closed.

(♠) Let X a projective variety, Y a quasi-projective variety, $f \colon X \to Y$ be a morphism such that Y is smooth along $f(X)$.

Then the tangent space of $\mathrm{Mor}(X,Y)$ at $[f]$ is $\mathrm{H}^0(f^*T_Y)$, and the obstruction space of $\mathrm{Mor}(X,Y)$ at $[f]$ is $H^1(f^*T_Y)$. Hence,

$$\mathrm{h}^0(f^*T_Y) \geq \dim_{[f]} \mathrm{Mor}(X,Y) \geq \mathrm{h}^0(f^*T_Y) - \mathrm{h}^1(f^*T_Y).$$

Indeed, by the construction we have

$$\dim_{[f]} \mathrm{Mor}(X,Y) = \dim_{[\Gamma(f)]} \mathrm{Hilb}_{X\times Y} \geq \mathrm{h}^0(N_{\Gamma(f)/X\times Y}) - \mathrm{h}^1(N_{\Gamma(f)/X\times Y}),$$

and $N_{\Gamma(f)/X\times Y} \cong f^*T_Y$.

Fix a closed subscheme B of X and $g\colon B \to Y$. We want to study $f\colon X \to Y$ satisfying the assumption (♠) with $f|_B = g$. There is a restriction

$$\mathrm{Mor}(X,Y) \longrightarrow \mathrm{Mor}(B,Y)$$

with fiber $\mathrm{Mor}(X,Y;g)$ at $[g]$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_B \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

Tensoring it with f^*T_Y, we get

$$0 \longrightarrow f^*T_Y \otimes \mathscr{I}_B \longrightarrow f^*T_Y \longrightarrow g^*T_Y \longrightarrow 0,$$

and hence the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(f^*T_Y \otimes \mathscr{I}_B) \longrightarrow \mathrm{H}^0(f^*T_Y) \longrightarrow \mathrm{H}^0(g^*T_Y) \\ &\longrightarrow \mathrm{H}^1(f^*T_Y \otimes \mathscr{I}_B) \longrightarrow \mathrm{H}^1(f^*T_Y) \longrightarrow \mathrm{H}^1(g^*T_Y). \end{aligned}$$

Fact. We have

$$\mathrm{h}^0(f^*T_Y \otimes \mathscr{I}_B) \geq \dim_{[f]} \mathrm{Mor}(X,Y;g) \geq \mathrm{h}^0(f^*T_Y \otimes \mathscr{I}_B) - \mathrm{h}^1(f^*T_Y \otimes \mathscr{I}_B).$$

This is proved by Mori.

3.1 Mori's Bend-and-Break technique

Let X be a smooth projective variety, $f\colon C \to X$ a non-constant morphism, where C is a smooth projective curve. Fix a point $c_0 \in C$. Assume that

$$\dim_{[f]} \mathrm{Mor}(C,X; f|_{c_0}) \geq 1.$$

Then there exists a smooth affine pointed curve (T, t_0) and a nontrivial deformation family F of f fixing $\{c_0\}$:

$$F\colon C \times T \longrightarrow X$$

such that $F(c, t_0) = f(t)$ for each $c \in C$, $F(c_0, t) = f(c_0)$ for each $t \in T$, and $F|_{C\times\{t\}} \neq f$ for general $t \in T$. Let $\overline{T}$ be a smooth compactification of T. Then we get

$$\overline{F}\colon C \times \overline{T} \dashrightarrow X$$

that extends F.

Claim 3.6. The morphism $\overline{F}$ is not defined at (c_0, t_1) for some $t_1 \in \overline{T} \setminus T$.

If so, then we can resolve $C \times \overline{T}$ (note that $C \times \overline{T}$ is a surface) by blowing up points:

$$\begin{tikzcd} S \arrow[d, "\varepsilon"] \arrow[dr, "e"] & \\ C \times \overline{T} \arrow[r, dashed, "\overline{F}"'] \arrow[d, "\pi"] & X \\ \overline{T} & \end{tikzcd}$$

The fiber $S_{t_0} = \pi^{-1}(t_0)$ is $C \times \{t_0\}$. So $S_{t_1} = \pi^{-1}(t_1)$ is the proper transform $\widetilde{C \times \{t_1\}}$ of $C \times \{t_1\}$ in S union the exceptional divisor E for ε.

Since $\{c_0\} \times \overline{T}$ intersects $C \times \{t_1\}$ transversally,

$$\widetilde{\{c_0\} \times \overline{T}} \cap \widetilde{C \times \{t_1\}} = \varnothing.$$

It follows from $e(\widetilde{\{c_0\} \times \overline{T}}) = \{f(c_0)\}$ that $f(c_0) \in e(E) \subseteq X$.

Define

$$f'\colon C \xleftarrow{\ \varepsilon\ } \widetilde{C \times \{t_1\}} \xrightarrow{\ e\ } X$$

and $Z = e_*[E]$. We get:

Proposition 3.7 (Mori)**.** There exists a (possibly constant) morphism $f'\colon C \to X$ and a nonzero effective 1-cycle Z of rational curves with $f(c_0) \in \operatorname{Supp} Z$ such that

$$f_*[C] = e_*[S_{t_0}] \equiv e_*[S_{t_1}] = f'_*[C] + Z.$$

In particular, there exists a rational curve in X through $f(c_0)$.

Proof of (3.6). Assume that $\overline{F}\colon C \times \overline{T} \to X$ is a morphism. It contracts $\{c_0\} \times \overline{T}$ to a point $f(c_0) \in X$. By rigidity lemma, $\overline{F}$ can be decomposed into $C \times \overline{T} \xrightarrow{p_1} C \xrightarrow{g} X$. Then

$$g(c) = g(p_1(c, t_0)) = F(c, t_0) = f(c)$$

for each c and hence $F|_{C\times\{t\}} = f$ for each $t \in T$. But then F is trivial, a contradiction. ∎

4 Existence of rational curves, 9/14

Theorem 4.1 (MM 86)**.** Let X be a projective normal variety of dimension $n \geq 1$ over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Fix an ample Cartier divisor H on X. For a curve $C \subseteq X \setminus \operatorname{Sing} X$ with $K_X \cdot C < 0$ and a point $c \in C$, there exists a rational curve Γ on X through c with

$$0 \leq H \cdot \Gamma \leq 2n \, \frac{H \cdot C}{-K_X \cdot C}.$$

Remark. The curves C and Γ might have singularities and Γ might pass through $\operatorname{Sing} X$.

Proposition 4.2. Let X, H, and $\mathbf{k}$ be as above, C a smooth projective curve, and $f\colon C \to X$ a non-constant map. Let $B = \{c_1, \ldots, c_b\} \subseteq C$ be a finite subset. Assume that

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_B) \geq 1.$$

Then there exists a rational curve Γ on X such that $f(c_{i_0}) \in \Gamma$ for some $1 \leq i_0 \leq b$ with

$$H \cdot \Gamma \leq 2 \, \frac{H \cdot f_* C}{b}.$$

Proof. Let C' be the normalization of $f(C)$. Then there exists a unique morphism $f'\colon C \to C'$ that factors through f. We note that

$$\deg f = [K(C) : K(f(C))] = [K(C) : K(C')] = \deg f'.$$

If $C' \cong \mathbb{P}^1$ and $\deg(C \to f(C)) \geq b/2$, just take $\Gamma = C$ (so that

$$H \cdot [f(C)] = \frac{H \cdot f_*[C]}{\deg f} \leq 2 \, \frac{H \cdot f_* C}{b}.$$

From now on, we will assume that if $C' \cong \mathbb{P}^1$, then $\deg(C \to f(C)) < b/2$. By assumption, there exists a smooth affine pointed curve (T, t_0), a nontrivial deformation

$F\colon C\times T\to X$ such that $F|_{C\times\{t_0\}}=f$ and $F(\{c_i\}\times T)=\{f(c_i)\}$ for all $1\le i\le b$. We first prove that $F(C\times T)\not\subseteq f(C)$. Indeed, we have

$$\dim_{[f']}\operatorname{Mor}(C,C';f'|_B)\le \mathrm{h}^0(C,(f')^*T_{C'}\otimes\mathscr{I}_B),$$

and

$$\begin{aligned}\deg((f')^*T_{C'}\otimes\mathscr{I}_B)&=\deg f'\cdot\deg T_{C'}-b\\&\le(\deg f')(2-2g(C'))-b.\end{aligned}$$

Note that this is negative if $g(C')\ge 1$, and less than $b/2\cdot 2-b=0$ if $g(C')=0$. Hence, if $F(C\times T)\subseteq f(T)$, then there exists a unique map $F'\colon C\times T\to C'$ that factors through F, and thus

$$1=\dim T\le\dim_{[f']}\operatorname{Mor}(C,C';f'|_B)=0,$$

a contradiction. In particular, $F(C\times T)$ is a surface.

By rigidity lemma, T is not proper. Let $\overline{T}$ be a smooth compactification of T. Consider a resolution

$$\begin{tikzcd} S \arrow[d, "\varepsilon"] \arrow[dr, "e"] & \\ C\times\overline{T} \arrow[r, dashed, "F"] & X. \end{tikzcd}$$

For $i=1,\ldots,b$, wee denote $E_{i1},\ldots,E_{in}$ the total transforms on S of the exceptional (-1)-curves by blowing up an (infinitely near) point over $\{c_i\}\times\overline{T}$. We see that the intersection number

$$E_{ij}\cdot E_{i'j'}=-\delta_{ii'}\delta_{jj'}.$$

Let T_i be the proper transform of $\{c_i\}\times\overline{T}$, let $\varepsilon^*\overline{T}=\varepsilon^*(\{p\}\times\overline{T})$ for a general point $p\in C$, and let $\varepsilon^*C=\varepsilon^*(C\times\{t_0\})$. Write

$$T_i\equiv\varepsilon^*\overline{T}-\sum_{j=1}^{n_i}\epsilon_{ij}E_{i,j},$$

where

$$\epsilon_{ij}=T_i\cdot E_{i,j}=\begin{cases}1 & \text{if the blown up point is on the proper transform of } \{c_i\}\times\overline{T},\\ 0 & \text{else.}\end{cases}$$

Write also

$$e^*H=e^*(H|_{e(S)})\equiv a\varepsilon^*C+d\varepsilon^*\overline{T}-\sum_{i=1}^{b}\sum_{j=1}^{n_i}a_{ij}E_{ij}+G,$$

where $G \perp \langle \varepsilon^* C, \varepsilon^* T, E_{ij} \rangle$.

Since

$$\varepsilon^*\overline{T} \cdot \varepsilon^*\overline{T} = 0, \quad \varepsilon^* C \cdot \varepsilon^*\overline{T} = 1, \quad E_{ij} \cdot \varepsilon^*\overline{T} = 0,$$

and e^*H is nef,

$$a_{ij} = e^*H \cdot E_{ij} \geq 0, \quad a = e^*H \cdot \varepsilon^*\overline{T} \geq 0.$$

Since $e(T_i) = \{f(c_i)\}$,

$$0 = e^*H \cdot T_i = a - \sum_{j=1}^{n_i} \epsilon_{ij} a_{ij}.$$

Summing over i, we get

$$ba = \sum_{i=1}^{b} \sum_{j=1}^{n_i} \epsilon_{ij} a_{ij}.$$

Claim. The self intersection $G^2 \leq 0$.

Proof of Claim. Assume $G^2 > 0$. Since $\varepsilon^* C \cdot G = 0 = (\varepsilon^* C)^2$, so by Hodge index theorem, $\varepsilon^* C \equiv 0$. But $\varepsilon^* C \cdot \varepsilon^*\overline{T}$, a contradiction. □

This gives us

$$\begin{aligned} 0 < (e^*H)^2 = 2ad - \sum_{i,j} a_{ij}^2 + G^2 &\leq 2ad - \sum_{i,j} a_{ij}^2 \\ &= \frac{2d}{b} \sum_{i,j} \epsilon_{ij} a_{ij} - \sum_{i,j} a_{ij}^2 \\ &\leq \sum_{i,j} \varepsilon_{ij} a_{ij} \left(\frac{2d}{b} - a_{ij} \right). \end{aligned}$$

So there exists some (i_0, j_0) such that

$$\varepsilon_{i_0 j_0} > 0, \quad a_{i_0 j_0} > 0, \quad \frac{2d}{b} - a_{i_0 j_0} > 0.$$

Since

$$H \cdot f_* C = e^*H \cdot \varepsilon^* C = d, \quad H \cdot e_* E_{i_0 j_0} = e^*H \cdot E_{i_0 j_0} = a_{i_0 j_0},$$

it is clear that every irreducible component of $e_* E_{i_0 j_0}$ has degree at most $2d/b$.

The intersection $E_{i_0 j_0} \cdot T_{i_0} = \epsilon_{i_0 j_0} = 1$ tells us that the rational cycle $e_* E_{i_0 j_0}$ passes through $f(c_{i_0})$. Pick an irreducible Γ of $e_* E_{i_0 j_0}$ which passes through $f(c_{i_0})$ but is not contracted by e. ■

Proof of (4.1). First, suppose that $p > 0$. Define

$$\begin{tikzcd} C_m \arrow[r, "\mathrm{Fr}^m"] \arrow[rr, bend right, "f"'] & C' \arrow[r] & C \subseteq X, \end{tikzcd}$$

where $C' \to C$ is the normalization and Fr is the Frobenius morphism. Note that $g := g(C') = g(C_m)$.

Let $B_m \subseteq C_m$ be a nonempty finite subset and $b_m = |B_m|$. Then

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C_m, X; f|_{B_m}) &\geq h^0(C_m, f^*T_X \otimes \mathscr{I}_{B_m}) - h^1(C_m, f^*T_X \otimes \mathscr{I}_{B_m}) \\ &= \chi(f^*T_X \otimes \mathscr{I}_{B_m}) \\ &= \deg(f^*T_X \otimes \mathscr{I}_{B_m}) + 1 - g(C_m) \\ &= \deg(f^*T_X) - nb_m + n(1-g) \\ &= -p^m(K_X \cdot C) - nb_m + (1-g), \end{aligned}$$

which is positive if we take

$$b_m = \left\lfloor \frac{p^m(-K_X \cdot C) - 1}{n} \right\rfloor + 1 - g.$$

Note that this number is greater than 0 for $m \gg 1$ since $-K_X \cdot C > 0$.

By (4.2), there exists a rational curve $\Gamma_m \subseteq X$ through some point of $f(B_m)$ such that

$$0 \leq H \cdot \Gamma_m \leq \frac{2H \cdot f_*[C_m]}{b_m} = \frac{2p^m(H \cdot C)}{b_m}.$$

Since $\frac{p^m}{b_m}$ tends to $\frac{n}{-K_X \cdot C}$ as $m \to \infty$ and $H \cdot \Gamma_m$ is an integer,

$$H \cdot \Gamma_m \leq 2n\, \frac{H \cdot C}{-K_X \cdot C} =: r$$

for $m \gg 1$.

Let

$$M_r = \bigsqcup_{\deg_H P(m) \leq r} \operatorname{Mor}^{P(m)}(\mathbb{P}^1, X), \quad P(m) = \chi(\mathbb{P}^1, mg^*H),$$

be the quasi-projective scheme that parameterize $g\colon \mathbb{P}^1 \to X$ of degree at most r.

Fact. The image of the evaluation map $\mathrm{ev}_r\colon \mathbb{P}^1 \times M_r \to X$ is closed in X.

Then $C \cap \operatorname{Im}(\mathrm{ev}_r)$ is closed in C. But $C \cap \operatorname{Im}(\mathrm{ev}_r)$ can not be finite since we could then take B_m so that $f(B_m) \not\subseteq C \cap \operatorname{Im}(\mathrm{ev}_r)$. This shows that $C \subseteq \operatorname{Im}(\mathrm{ev}_r)$.

For $p = 0$, we prove this theorem via reduction modulo p. Consider a finitely generated $\mathbb{Z}$-algebra $R \subseteq \mathbf{k}$ over which X, C, c, H are defined. Note that all of these can be described by finite equations. Consider the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_R \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{k} & \longrightarrow & \operatorname{Spec} R, \end{array}$$

so that C_R, c_R, H_R pulls-back to C, c, H, respectively. By shrinking $\operatorname{Spec} R$, we may assume that C_R is smooth over $\operatorname{Spec} R$ (by generic smoothness), X_R is smooth along C_R, and that H_R is a relative ample Cartier divisor.

Let $g_R \colon \mathbb{P}^1_R \to X_R$ be the constant morphism that maps to $c_R \in C_R$. Consider the scheme

$$\bigsqcup_{0<\deg_{H_R} P \le r} \mathrm{Mor}_{\operatorname{Spec} R}(\mathbb{P}^1_R, X_R; g_R) \xrightarrow{\rho} \operatorname{Spec} R, \quad r = 2n\,\frac{H \cdot C}{-K_X \cdot C}.$$

Let $\mathfrak{m}$ be a maximal ideal of R. Then $\operatorname{char} R/\mathfrak{m}$ is positive. By the positive characteristic case, $\mathfrak{m} \in \operatorname{Im} \rho$. Now, since $\operatorname{Im} \rho$ is a constructable subset of $\operatorname{Spec} R$ (by Chavalley's theorem), and $\operatorname{Im} \rho$ contains a dense subset $\operatorname{Max} R$ of $\operatorname{Spec} R$. The generic point η of $\operatorname{Spec} R$ lies in $\operatorname{Im} \rho$. ∎

5 The covering trick, 9/18

To prove a vanishing for a certain $\mathbb{Q}$-divisor L, one could pull L back to a covering on which the problem simplifies in some ways.

Lemma 5.1 (Injectivity lemma)**.** Let $\pi \colon Y \to X$ be a finite, surjective morphism of varieties over $\mathbb{C}$ with X normal. Let $\mathscr{E}$ be a locally free sheaf on X. Then the natural morphism

$$\mathrm{H}^j(X, \mathscr{E}) \longrightarrow \mathrm{H}^j(Y, \pi^*\mathscr{E})$$

induced by $\mathscr{E} \to \pi_*\pi^*\mathscr{E}$ is injective. In particular, if $\mathrm{H}^j(\pi^*\mathscr{E}) = 0$ for some $j \ge 0$, then $\mathrm{H}^j(\mathscr{E}) = 0$.

Proof. Since π is finite, $K(Y)/K(X)$ is a finite field extension of degree $\deg \pi$. This gives us the trace map $\mathrm{tr}_{K(Y)/K(X)} \colon K(Y) \to K(X)$.

Now, we construct a map $\mathrm{tr}_{Y/X}\colon \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ induced by $\mathrm{tr}_{K(Y)/K(X)}$. This problem is local, so we may assume $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Note that $A \to B$ is injective since π is surjective. For each $\alpha \in K(Y)$, let

$$m_{\alpha,K(X)}(t) = t^d + \alpha_{d-1}t^{d-1} + \cdots + \alpha_0$$

be the minimal polynomial of α in $K(X)$. Then $d = [K(X)(\alpha)\colon K(X)]$ and

$$\mathrm{tr}_{K(Y)/K(X)}(\alpha) = -\frac{\deg \pi}{d}\,\alpha_{d-1}.$$

By our assumption, X, and hence A, are normal. Since π is finite, any $\beta \in B$ is integral over A, i.e., $m_{\beta,K(X)}(t) \in A[t]$. This shows that $\mathrm{tr}_{K(Y)/K(X)}(\beta) \in A$ and gives us the desired map $\mathrm{tr}_{Y/X}$.

Let $\mathrm{tr} = \frac{1}{\deg \pi}\,\mathrm{tr}_{Y/X}\colon \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Then $\mathrm{tr} \circ \pi^* = \mathrm{id}_{\mathcal{O}_X}$, i.e., $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathscr{F}$, where $\mathscr{F} = \operatorname{coker} \pi^*$. Then

$$\mathrm{H}^j(Y, \pi^*\mathscr{E}) = \mathrm{H}^j(X, \pi_*\pi^*\mathscr{E}) = \mathrm{H}^j(X, \mathscr{E} \otimes \pi_*\mathcal{O}_Y) = \mathrm{H}^j(X, \mathscr{E}) \oplus \mathrm{H}^j(X, \mathscr{E} \otimes \mathscr{F}). \qquad \blacksquare$$

Let X be an affine variety, $s \in \Gamma(X, \mathcal{O}_X)$ be a nonzero regular function. We define

$$Y = (t^m - s = 0) \subseteq X \times \mathbb{A}^1$$

so that $s^{1/m}$ make sense on Y. We see that $\pi\colon Y \to X$ is a cyclic covering branched along $D = \operatorname{div}(s)$. By gluing this construction, we get:

Proposition 5.2. Let X be a variety, $\mathscr{L}$ an invertible sheaf of X. Fix an positive integer m, $s \in \mathrm{H}^0(X, \mathscr{L}^{\otimes m})$ be a nonzero section, $D = \operatorname{div}(s)$. Then there exists a finite surjective morphism $\pi\colon Y \to X$ branched along D and a section $s' \in \mathrm{H}^0(Y, \pi^*\mathscr{L})$ such that $(s')^m = \pi^* s$. The effective divisor $D' = \operatorname{div}(s')$ maps isomorphically to D. Moreover, if X and D are smooth, then so are Y and D'.

Proof. We only prove the "moreover part". If X and D are smooth, then $Y \setminus D' \to X \setminus D$ are étale shows that $Y \setminus D$ is smooth. Since D' maps to D isomorphically, D' is also smooth. Finally, D' is a smooth effective Cartier divisor, so $\mathcal{O}_{Y,y}$ is a regular local ring for $y \in D'$. $\blacksquare$

Theorem 5.3 (Kodaira vanishing)**.** Let X be a smooth projective variety, A an ample divisor on X. Then $\mathrm{H}^i(X, K_X + A) = 0$ for all $i > 0$.

Proof. Equivalently, we prove $H^j(X, -A) = 0$ for $j < n := \dim X$. Since A is ample, there exists a smooth divisor $D \in |mA|$ for $m \gg 1$. Then (5.2) tells us that there exists an m-fold cyclic covering $\pi \colon Y \to X$ branched along D with $\pi^* D = mD'$. By construction, $D' \in |\pi^* A|$ is a smooth effective ample divisor. By (5.1), it suffices to prove that $H^j(Y, -D') = 0$ for $j < n$. Note that D' is ample since D is ample.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-D') \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{D'} \longrightarrow 0.$$

We get the long exact sequence

$$\mathrm{H}^{j-1}(\mathcal{O}_Y) \longrightarrow \mathrm{H}^{j-1}(\mathcal{O}_{D'}) \longrightarrow \mathrm{H}^j(-D') \longrightarrow \mathrm{H}^j(\mathcal{O}_Y) \longrightarrow \mathrm{H}^j(\mathcal{O}_{D'}).$$

Fact. The map $r^{p,q} \colon H^{p,q}(Y) \to H^{p,q}(D')$ is bijective for $p + q \leq n - 2$ and injective for $p + q = n - 1$. This follows from the Lefschetz hyperplane theorem that

$$\begin{tikzcd} \mathrm{H}^j(Y, \mathbb{C}) \arrow[r, "r"] \arrow[d, equal] & \mathrm{H}^j(D', \mathbb{C}) \arrow[d, equal] \\ \bigoplus_{p+q=j} \mathrm{H}^{p,q}(Y) \arrow[r, "\oplus r^{p,q}"] & \bigoplus_{p+q=j} \mathrm{H}^{p,q}(D') \end{tikzcd}$$

is bijective for $j \leq n - 2$ and injective for $j = n - 1$.

So $\mathrm{H}^j(-D') = 0$ for $j \leq n - 1$. ■

Definition 5.4. Let X be a smooth variety of dimension n. We say $D = \sum D_i$ has simple normal crossings (SNC) if each D_i is smooth and for each $p \in X$, $\mathcal{O}_{D,p} \cong \mathcal{O}_{X,p}/\langle z_1 \cdots z_k \rangle$ for some $k \leq n$ and some regular sequence $(z_1, \ldots, z_n)$ of $\mathcal{O}_{X,p}$.

A $\mathbb{Q}$-divisor $D = \sum d_i D_i$ has SNC support if $D_{\text{red}} = \sum D_i$ is a SNC divisor.

Theorem 5.5 (Kawamata–Viehweg vanishing)**.** *Let X be a smooth projective variety, D an ample $\mathbb{Q}$-divisor such that $\lceil D \rceil - D$ has SNC support. Then for each $i > 0$,*

$$H^i(X, K_X + \lceil D \rceil) = 0$$

Lemma 5.6 (Kawamata covering)**.** *Let X and D be as in the above theorem. Write*

$$\lceil D \rceil - D = \sum_i a_i D_i, \quad 0 \leq a_i < 1.$$

Then for sufficiently divisible $m \gg 1$, there exists a finite surjective morphism $\pi \colon Y \to X$ such that

(1) $K(Y)/K(X)$ is a Kummer extension;

(2) Y is smooth and projective;

(3) $\pi^* D$ is an integral divisor;

(4) the canonical divisor

$$K_Y = \pi^*\Big(K_X + \sum_i (1 - \tfrac{1}{m})D_i + \sum_j (1 - \tfrac{1}{m})H_j\Big)$$

and $\sum D_i + \sum H_j$ has SNC;

(5) $1 - \frac{1}{m} \geq a_i$ for all i.

Remark. The branch locus of π is $\bigcup D_i \cup \bigcup H_j$, and π preserves smoothness by adding branch locus artificially. Since X and Y are smooth, π is flat.

Lemma 5.7. Let X be a smooth variety, $|V|$ a base-point-free linear system, $\sum D_i$ a SNC divisor. If $H \in |V|$ is a general divisor, then $H + \sum D_i$ again has SNC.

Proof. This is just Bertini's theorem. ∎

Proof of (5.6). Let $n = \dim X$. Fix a very ample divisor A on X. Take a sufficiently divisible $m \gg 1$ so that $1 - \frac{1}{m} \geq a_i$, $ma_i \in \mathbb{N}$, and $mA - D_i$ is ample for each $i \in I$. By (5.7), we can take n general elements $H_1^{(i)}, \ldots, H_n^{(i)} \in |mA - D_i|$ such that $\sum D_i + \sum_{i,k} H_k^{(i)}$ has SNC.

Let $X = \bigcup U_\lambda$ be an affine open cover of X with transition functions of $\mathcal{O}_X(A)$

$$\{a_{\lambda\mu} \in \mathrm{H}^0(U_\lambda \cap U_\mu) \mid \mathcal{O}_X^\times\}$$

and local sections of $\mathcal{O}_X(mA)$

$$\{\varphi_{k\lambda}^{(i)} \in \mathrm{H}^0(U_\lambda, \mathcal{O}_X)\}$$

such that

$$(H_k^{(i)} + D_i)|_{U_\lambda} = \mathrm{div}(\varphi_{k\lambda}^{(i)})$$

and $\varphi^{(i)}_{k\lambda} = a^m_{\lambda\mu}\varphi^{(i)}_{k\mu}$ on $U_\lambda \cap U_\mu$.

Claim. The normalization Y of X in $K(X)[(\varphi^{(i)}_{k\lambda})^{1/m}]$ for some λ provides the desired cover.

(1), (3) and (4) are trivial by our construction. It remains to prove that Y is smooth and projective. Note that Y is projective since π is finite. For a closed point $x \in U_\lambda$, set $I_x = \{i \in I \mid x \in D_i\}$. Since $\bigcap_{k=1}^n H^{(i)}_k \cap D_i = \varnothing$, for each $i \in I_x$, there exists k_i such that $x \notin H^{(i)}_{k_i}$. Now the set

$$R_x := \{\varphi^{(i)}_{k\lambda} \mid i \in I_x\} \cup \{\varphi^{(i)}_{k\lambda} \mid i \notin I_x, x \in H^{(i)}_k\} \cup \{\varphi^{(i)}_{k\lambda}/\varphi^{(i)}_{k_i\lambda} \mid i \in I_x, x \in H^{(i)}_k\}$$

forms a part of a regular system of parameters of the regular local ring $\mathcal{O}_{X,x}$. The set

$$T_x = \{\varphi^{(i)}_{k\lambda}/\varphi^{(i)}_{k_i\lambda} \mid i \in I_x, x \notin H^{(i)}_k\} \cup \{\varphi^{(i)}_{k\lambda} \mid i \notin I_x, x \notin H^{(i)}_k\}$$

are all units in $\mathcal{O}_{X,x}$. Then Y is smooth at any $y \mapsto x$ by the following lemma:

Lemma 5.8. Let $(R, \mathfrak{m})$ be a regular local $\mathbb{C}$-algebra of dimension n with residue field $\mathbb{C}$, $\{z_1, \ldots, z_n\}$ a regular system of parameters, and $\{u_1, \ldots, u_s\} \subseteq R^\times$. Fix $m \in \mathbb{N}$, $1 \le \ell \le n$. Let

$$R' = R[z_1^{1/m}, \ldots, z_\ell^{1/m}, u_1^{1/m}, \ldots, u_s^{1/m}].$$

Then for any maximal ideal $\mathfrak{m}'$ of R', $R'_{\mathfrak{m}'}$ is a regular local ring with a regular system of parameter $z_1^{1/m}$, …, $z_\ell^{1/m}$, $z_{\ell+1}$, …, z_n.

Proof of (5.8). We check that

$$\mathfrak{m}' = \langle z_1^{1/m}, \ldots, z_\ell^{1/m}, z_{\ell+1}, \ldots, z_n, u_1^{1/m} - \alpha_1, \ldots, u_s^{1/m} - \alpha_s\rangle_{R'}$$

for some $\alpha_t \in \mathbb{C}^\times$. Indeed, since $\mathfrak{m}' \supseteq \mathfrak{m}' \cap R = \mathfrak{m} \ni z_i = (z_i^{1/m})^m$, z_j for $1 \le i \le \ell$, $\ell + 1 \le j \le n$, $\mathfrak{m}' \ni z_i^{1/m}$. On the other hand, since $R'/\mathfrak{m}' \cong \mathbb{C}$, $u_t^{1/m} + \mathfrak{m}'$ corresponds to some $\alpha_t \in C^\times$. So the RHS is contained in $\mathfrak{m}'$ with $R'/\text{RHS} \cong \mathbb{C}$. So $\mathfrak{m}'$ is equal to the RHS.

It suffices to show that

$$u_t^{1/m} - \alpha_t \in \langle z_1^{1/m}, \ldots, z_\ell^{1/m}, z_{\ell+1}, \ldots, z_n\rangle_{R'}$$

since then $\mathfrak{m}'$ is generated by n elements. This is just because

$$u_t - \alpha_t^m = (u_t^{1/m} - \alpha_t)(u_t^{(m-1)/m} + u_t^{(m-2)/m}\alpha_t + \cdots + \alpha_t^{m-1}) \in \mathfrak{m}' \cap R = \mathfrak{m}$$

and

$$u_t^{(m-1)/m} + u_t^{(m-2)/m}\alpha_t + \cdots + \alpha_t^{m-1} \equiv m\alpha_t^{m-1} \neq 0 \pmod{\mathfrak{m}'}. \qquad \blacksquare$$

$\blacksquare$

Proof of (5.5). Take a finite Galois covering $\pi\colon Y \to X$ as in the lemma. Let $G = \mathrm{Gal}(K(Y)/K(X))$. Then there exists a natural G-action of $\pi_*\mathcal{O}_Y(K_Y + \pi^*D)$, which compatible with the action of G on $K(Y)$.

Claim. $(\pi_*\mathcal{O}_Y(K_Y + \pi^*D))^G \cong \mathcal{O}_X(K_X + \lceil D \rceil)$.

Since the functors $\Gamma(X, -)$ and $(-)^G$ commute for G-sheaves, assuming this claim we get

$$\begin{aligned}\mathrm{H}^i(X, K_X + \lceil D \rceil) &= H^i(X, \pi_*\mathcal{O}_Y(K_Y + \pi^*D))^G \\ &= H^i(Y, K_Y + \pi^*D)^G = 0\end{aligned}$$

since π^*D is ample (π is finite).

To prove the claim, let U be a Zariski open subset of X, then

$$\begin{aligned}&\Gamma(U, (\pi_*\mathcal{O}_Y(K_Y + \pi^*D))^G) \\ &\quad = \{f \in K(Y)^G \mid (\mathrm{div}(f) + K_Y + \pi^*D)|_{\pi^{-1}(U)} \geq 0\} \\ &\quad = \{f \in K(X) \mid (\mathrm{div}(f) + K_X + (1 - \tfrac{1}{m})(\textstyle\sum_i D_i + \sum_j H_j) + D)|_U \geq 0\} \\ &\quad = \{f \in K(X) \mid (\mathrm{div}(f) + K_X + \lceil D \rceil)|_U \geq 0\}\end{aligned}$$

since $1 - \frac{1}{m} - a_i \in [0, 1)$. $\blacksquare$

6 Big divisors, 9/21

Definition 6.1. Let X be a projective variety, D a Cartier divisor. Define

$$\mathbb{N}(D) = \mathbb{N}(X, D) := \{m \in \mathbb{N} \mid \mathrm{h}^0(mD) \neq 0\}.$$

This is a semigroup if $\mathbb{N}(D)$ is nonempty, and there exists $e \in \mathbb{N}$ such that $e\mathbb{Z}$ is the subgroup of $\mathbb{Z}$ generated by $\mathbb{N}(D)$. This $e = e(D) = \gcd \mathbb{N}(D)$ is called the **exponent** of D.

We see that there exists $m_0 > 0$ such that for $m \geq m_0$, $me \in \mathbb{N}(D)$.

Lemma 6.2. Let X be a projective scheme over a field with dimension n. Fix a Cartier divisor B. Then

$$\mathrm{h}^0(mB) = O(m^n).$$

Proof. Let H be a very ample divisor on X such that $\mathrm{h}^i(mH) = 0$ for all i, $m \in \mathbb{N}$ and $H - B \sim D \geq 0$ is ample. Then

$$\mathrm{h}^0(mB) = \mathrm{h}^0(mH - mD) \leq \mathrm{h}^0(mH) = \chi(mH) = \frac{H^n}{n!}m^n + O(m^{n-1}) = O(m^n). \qquad \blacksquare$$

Definition 6.3. Let X be a proper variety of dimension n. A Cartier divisor D is **big** if there exists $c > 0$ such that $\mathrm{h}^0(mD) = c \cdot m^n$ for $m \in \mathbb{N}(X, D)$ large enough.

A $\mathbb{Q}$-divisor D is big if there exists $k \in \mathbb{N}$ such that kD is Cartier and big.

An $\mathbb{R}$-divisor $D = \sum a_i D_i$ is big if each D_i is big and $a_i \in \mathbb{R}_{>0}$.

Definition 6.4. Let X be a normal variety. The **Iitaka dimension** of a Cartier divisor D is

$$\kappa(D) = \kappa(X, D) := \sup_{m \in \mathbb{N}(D)} \{\dim \operatorname{Im} \phi_{|mD|}\} \in \{-\infty\} \cup [0, \dim X],$$

where $\phi_{|mD|} \colon X \dashrightarrow \mathbb{P}H^0(mD)^\vee$ is the rational map induced by $|mD|$.

If X is non-normal, pass to its normalization $\nu \colon \widetilde{X} \to X$ and set

$$\kappa(X, D) = \kappa(\widetilde{X}, \nu^* D).$$

Remark. Let $f \colon X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let F be a general fiber of X. Iitaka conjectured that

$$\kappa(X) \geq \kappa(Y) + \kappa(F),$$

where $\kappa(-) = \kappa(-, K_-)$ is the Kodaira dimension.

Example 6.5.

1. Choose a projective variety T of dimension $d \geq 1$ such that there exists non-trivial $\eta \in \operatorname{Pic} T$ with $\eta^{\otimes e} = \mathcal{O}_T$. Let Y be any projective variety of dimension k, B a

very ample divisor on Y. Let $X = Y \times T$, $\mathcal{O}_X(D) = \mathcal{O}_Y(B) \boxtimes \eta$. Then $e(D) = e$, $\mathbb{N}(D) = \mathbb{N}e$, and mD is base-point-free if $m \in \mathbb{N}(D)$. In this case, $\kappa(D) = k$.

2. Let $T \subseteq \mathbb{P}^2$ be a nodal plane cubic curve. Take a non-torsion line bundle $\eta \in \operatorname{Pic}^\circ(X) \cong \mathbb{G}_m$. Then $\mathrm{h}^0(T, \eta^{\otimes m}) = 0$ for all $m > 0$ but

$$\mathrm{h}^0(\widetilde{T}, m\nu^*\eta) = \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1.$$

By taking product as in 1., one get examples with $\mathrm{h}^0(X, mD) = 0$ for each $m \in N$ and $\kappa(X', D') \in \mathbb{N}$ arbitrary.

Proposition 6.6 (Characterization of big divisors)**.** Let X be a projective variety of dimension n, D a Cartier divisor on X. Then the followings are equivalent:

(1) D is big;

(2) for any ample Cartier divisor A, there exists $m \in \mathbb{N}$, an effective divisor E such that $mD \sim A + E$;

(2') for some ample Cartier divisor A, there exists $m \in \mathbb{N}$, an effective divisor E such that $mD \sim A + E$;

(2") for some ample Cartier divisor A, there exists $m \in \mathbb{N}$, an effective divisor E such that $mD \equiv A + E$;

(3) $\kappa(X, D) = \dim X$, i.e., for some $m > 0$, the rational map $\phi_{|m\nu^*D|} \colon \widetilde{X} \dashrightarrow \mathbb{P}\,\mathrm{H}^0(m\nu^*D)^\vee$ is birational onto its image.

Lemma 6.7 (Kodaira's lemma)**.** Let X be a projective (proper) variety, D a big Cartier divisor. Then for any effective Cartier divisor F on X, we have

$$\mathrm{h}^0(X, mD - F) > 0$$

for large $m \in \mathbb{N}(X, D)$.

Proof. Let $n = \dim X$. Then

$$\mathrm{h}^0(X, mD - F) \geq \mathrm{h}^0(X, mD) - \mathrm{h}^0(F, mD|_F) = c \cdot m^n - O(m^{n-1}) > 0$$

for some $c > 0$ and large enough $m \in \mathbb{N}(D)$. ■

Proof of (6.6). (1) $\Rightarrow$ (2). Suppose D is big. Take $r \gg 1$ such that $rA \sim H_r \geq 0$ and $(r+1)A \sim H_{r+1} \geq 0$. Apply Kodaira's lemma with $F = H_{r+1}$ to find $m \in \mathbb{N}(D)$ and $F' \in |mD - H_{r+1}|$ with $mD \sim H_{r+1} + F' \sim A + H_r + F'$. Take $E = H_r + F'$ we get what we want in (2).

The implications (2) $\Rightarrow$ (2') $\Rightarrow$ (2") are trivial.

(2") $\Rightarrow$ (3). If $mD \equiv A + E$, then $mD - E$ is ample. Take m larger, we may assume that $H \sim mD - E$ is very ample. We see that

$$\kappa(X, D) \geq \kappa(X, H) = \dim X,$$

as desired.

(3) $\Rightarrow$ (1). First, we assume that X is normal and $Y = \operatorname{Im} \phi_{|D|} \subseteq \mathbb{P}\,\mathrm{H}^0(D)^\vee$ has dimension n. Then

$$\mathrm{h}^0(\mathcal{O}_Y(m)) + \sum_{i>0} (-1)^i\, \mathrm{h}^i(\mathcal{O}_Y(m)) = \chi(\mathcal{O}(m)) = \frac{\deg Y}{n!}\, m^n + O(m^{n-1}).$$

Since the second term in the LHS is $O(m^{n-1})$, we see that $\mathrm{h}^0(\mathcal{O}_Y(m)) = \Omega(m^n)$. Let U be the largest Zariski open subset of X on which $\phi = \phi_{|D|}$ is defined. Since X is normal and Y is proper, it follows from the valuative criterion for properness that $\operatorname{codim}_X(X \setminus U) \geq 2$. $\phi\colon U \to Y$ is dominant implies that $\mathcal{O}_Y \to \phi_* \mathcal{O}_U$ is injective, and hence

$$\mathcal{O}_Y(m) \longrightarrow \phi_* \mathcal{O}_U \otimes \mathcal{O}_Y(m) = \phi_* \phi^* \mathcal{O}_Y(m) = \phi_* \mathcal{O}_U(mD)$$

is injective. This shows that

$$\phi_{|D|}^\sharp \colon\ \mathrm{H}^0(Y, \mathcal{O}_Y(m)) \longrightarrow \mathrm{H}^0(U, \mathcal{O}_X(mD)) = \mathrm{H}^0(X, \mathcal{O}_X(mD))$$

by Hartogs's extension theorem.

In general, let $\nu\colon \widetilde{X} \to X$ be the normalization. Then there exists a exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \nu_* \mathcal{O}_{\widetilde{X}} \longrightarrow \eta \longrightarrow 0,$$

where η is supported on a scheme of dimension at most $n-1$. Tensoring this sequence by $\mathcal{O}(mD)$, we get

$$\begin{aligned} \mathrm{h}^0(\mathcal{O}_X(mD)) &\leq \mathrm{h}^0(\nu_* \mathcal{O}_X \otimes \mathcal{O}_X(mD)) \\ &\leq \mathrm{h}^0(\mathcal{O}_X(mD)) + \mathrm{h}^0(\eta \otimes \mathcal{O}_X(mD)) = \mathrm{h}^0(\mathcal{O}_X(mD)) + O(m^{n-1}). \end{aligned}$$

Since $\nu_*\mathcal{O}_{\widetilde{X}} \otimes \mathcal{O}_X(mD) = \nu_*\mathcal{O}_{\widetilde{X}}(m\nu^*D)$, $\kappa(\widetilde{X}, \mu^*D) = n$ if and only if $\mathrm{h}^0(mD) > c \cdot m^n$ for some $c > 0$ and large enough $m \in \mathbb{N}(D)$. ■

Proposition 6.8. Let X be a projective variety of dimension n, D a Cartier divisor. Then the followings are equivalent:

(1) D is big and nef;

(2) D is nef with $D^n > 0$;

(3) there exists an effective divisor E such that $D - \frac{1}{k}E$ is ample for $k \gg 1$.

Proof. (1) $\Leftrightarrow$ (2). Since D is nef,

$$\mathrm{h}^0(mD) = \chi(mD) - \sum_{i>0}(-1)^i\,\mathrm{h}^i(mD) = \frac{D^n}{n!}\,m^n + O(m^{n-1}).$$

So D is big if and only if $D^n > 0$.

(1) $\Rightarrow$ (3). Fix an ample divisor A, there exists $m \gg 1$ such that $mD \sim A + E$. For $k \geq m$,

$$kD = (k-m)D + mD \sim (k-m)D + A + E,$$

i.e., $D - \frac{1}{k}E = \frac{1}{k}((k-m)D + A)$ is ample.

(3) $\Rightarrow$ (1). Since $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, $D \in \mathrm{Nef}(X)$. Since $kD = (kD - E) + E$, D is big. ■

Example 6.9. Choose X a smooth projective surface, E a (-1)-curve, A a very ample divisor on X. Then $D_\ell = A + \ell E$ is big for $\ell > 0$ but

$$D_\ell^2 = A^2 + 2\ell\, A\cdot E + \ell^2 E^2 < 0$$

for $\ell \gg 1$.

7 Discrepancies, 9/25

Definition 7.1. Let $f\colon Y \dashrightarrow X$ be a birational map of varieties, $Z = \overline{\{\eta\}}$ be a closed subvariety. The **birational (strict, or proper) transform** of Z is

$$f_*Z := \begin{cases} \overline{f(\eta)} \subseteq X & \text{if } \eta \in \operatorname{dom} f, \\ 0 & \text{if } \eta \notin \operatorname{dom} f. \end{cases}$$

If $g\colon X \to Y$ is a birational morphism, we define $g_*^{-1}Z = (g^{-1})_*Z$.

Remark. If f is not a morphism, then f_* need not preserve linear or algebraic equivalence.

In the following, we assume that X is a normal variety over an algebraically closed field $\mathbf{k}$ with $\operatorname{char}\mathbf{k} = 0$.

A Weil divisor $D = \overline{\{\eta\}}$ on X is a prime divisor that defines uniquely a map $v_D = v(D, X)\colon K(X)^\times \to \mathbb{Z}$ by using the DVR $\mathcal{O}_{X,\eta}$.

Definition 7.2. Let $f\colon Y \to X$ be a birational morphism from a normal variety. Any prime divisor $E \subseteq Y$ is called a **divisor over** X. The center of E is $\operatorname{center}_X(E) := \overline{f(E)} \subseteq X$. A rank 1 valuation $v\colon K(X)^\times \to \mathbb{Z}$ is **geometric** (or algebraic) if there exists a divisor E over X such that $v = v_E$.

Remark. If $E \subseteq Y \to X$ and $E' \subseteq Y' \to X$ are divisors over X, then $\operatorname{center}_X(E) = \operatorname{center}_X(E')$ and $v(E, Y) = v(E', Y')$ if and only if $Y \to X \dashrightarrow Y'$ is an isomorphism at the generic points $e \in E$ and $e' \in E'$, i.e., $\mathcal{O}_{Y,e} = \mathcal{O}_{Y',e'}$.

Let X' be a normal variety that is birational to X. Then $K := K(X) = K(X')$. A geometric valuation $v(D = \overline{\{\eta\}}, X)\colon K^\times \to \mathbb{Z}$ corresponds to

$$\{f_*D \mid f\colon X \dashrightarrow Y \text{ bir.}, \eta \in \operatorname{dom} f\}.$$

Recall that the set of Weil divisors (with R-coefficients) is denoted by

$$\operatorname{WDiv}(X)_R = Z^1(X) = \{D = \textstyle\sum a_i D_i \mid D_i \text{ prime divisors}, a_i \in R\}.$$

Definition 7.3. Let $f\colon Y \dashrightarrow X$ be a birational map. We define $f_*\colon \operatorname{WDiv}(Y) \to \operatorname{WDiv}(X)$ by extending the coefficients. The map f is said to be **isomorphic in codimension** 1 if f_* is bijective.

Let $D = \sum a_i D_i \in \operatorname{WDiv}(X)$. Define a subsheaf $\mathcal{O}_X(D) \subseteq \mathscr{K}$ of the constant sheaf corresponds to $K(X)$ by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X) \mid v_{D_i}(f) \geq -a_i \ \forall D_i \cap U \neq \varnothing\}.$$

Facts.

- The sheaf $\mathcal{O}_X(D)$ is divisorial sheaf, i.e., a reflexive sheaf of rank 1. Recall that a coherent sheaf $\mathscr{F}$ is reflexive if $\mathscr{F} \xrightarrow{\sim} \mathscr{F}^{\vee\vee}$, $(-)^\vee = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$.
- $D \sim D'$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.
- There is a one-to-one correspondence between $\mathrm{WDiv}(X)$ and the set of divisorial sheaves. The class group $\mathrm{Cl}(X)$ then corresponds to the set of divisorial sheaves modulo isomorphisms.
- $\mathcal{O}_X(D + D') = (\mathcal{O}_X(D) \otimes \mathcal{O}_X(D'))^{\vee\vee}$. We define $\mathcal{O}_X(D)^{[m]} = \mathcal{O}_X(mD)$.
- If $D \geq 0$, then $\mathcal{O}_X(-D) \cong \mathscr{I}_D$.

Let U be the nonsingular part of X. Then X is normal implies that $\operatorname{codim}_X(X \backslash U) \geq 2$. Since $\omega_U = \det(\Omega^1_{X/\mathbf{k}}|_U) \in \operatorname{Pic} U$, $j_*\omega_U$ is a divisorial sheaf, called the **canonical sheaf** of X, and hence equal to $\mathcal{O}_X(K_X)$ for some $K_X \in \mathrm{WDiv}(X)$. $j_*\omega_U$, the divisor is called the **canonical divisor** of X. Note that by our construction, K_X is defined up to linear equivalences. Usually, we will fix a divisor K_X.

Remark. Let X be a normal projective variety. Then $\mathcal{O}_X(K_X)$ is isomorphic to the dualizing sheaf ω_X of X.

Definition 7.4. A divisor $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ is **$\mathbb{Q}$-Cartier** if $mD \in \mathrm{CDiv}(X)$ for some $m \in \mathbb{Z}$.

The variety X is **$\mathbb{Q}$-Gorenstein** if K_X is $\mathbb{Q}$-Cartier, **$\mathbb{Q}$-factorial** if every Weil divisor is $\mathbb{Q}$-Cartier.

Definition 7.5. Let X be a normal variety, D a $\mathbb{Q}$-divisor. A **log resolution** of (X, D) is a proper birational morphism $f : Y \to X$ such that

- Y is smooth,

- $\mathrm{Exc}(f)$ is a divisor,
- $\mathrm{Exc}(f) + f_*^{-1}D$ has SNC support.

From Hironaka's theorem, log resolutions always exist for varieties over characteristic zero field.

Definition 7.6. Let X be a normal variety $\Delta = \sum a_i D_i$ be a $\mathbb{Q}$-divisor. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e., $m(K_X + \Delta) \in \mathrm{CDiv}(X)$ for some m.

Suppose $f\colon Y \to X$ is a birational map with Y normal. Set $V = Y \setminus \mathrm{Exc}(f)$. Since

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta))|_V \cong f^*\mathcal{O}_X(m(K_X + \Delta))|_V,$$

for each prime divisor $E \subseteq \mathrm{Exc}(f)$, there exists $a(E, X, \Delta) \in \mathbb{Q}$, called the **discrepancy** of E, such that $m \cdot a(E, X, \Delta) \in \mathbb{Z}$ and

$$m(K_Y + f_*^{-1}\Delta) \sim mf^*(K_X + \Delta) + \sum_{E \subseteq \mathrm{Exc}(f)} m \cdot a(E, X, \Delta)E.$$

We set $a(D_i, X, \Delta) = -a_i$ and $a(D, X, \Delta) = 0$ for any prime divisor $D \subseteq X$ with $D \neq D_i$ for each i.

Remark. The number $a(E, X, \Delta)$ depends only on the valuation v_E but not on the particular choice of f and Y. Some authors use log discrepancies, defined as $1 + a(E, X, \Delta)$.

If we fix canonical divisors such that $f_*K_Y = K_X$, then the exact sequence

$$\bigoplus_{E \subseteq \mathrm{Exc}(f)} \mathbb{Q}E \longrightarrow \mathrm{WDiv}(Y)_{\mathbb{Q}} \longrightarrow \mathrm{WDiv}(X)_{\mathbb{Q}} \longrightarrow 0$$

shows that there exists $\Delta_Y \in \mathrm{WDiv}(Y)_{\mathbb{Q}}$ such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Here, Δ_Y is uniquely determined as the sum of $f_*^{-1}\Delta$ and a $\mathbb{Q}$-divisor supported on $\mathrm{Exc}(f)$, i.e., $f_*\Delta_Y = \Delta$.

Definition 7.7. We define the discrepancy of Δ to be

$$\mathrm{discrep}(X, \Delta) := \inf \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\},$$

the total discrepancy of Δ to be

$$\mathrm{totaldiscrep}(X, \Delta) := \inf \{a(E, X, \Delta) \mid E \text{ is a divisor over } X\}.$$

Definition 7.8. Let X be a variety, $x \in X$ a (not necessarily closed) point, $D = \{(U_i, f_i)\}$ an effective Cartier divisor. The multiplicity of D at x is

$$\operatorname{mult}_x D := \operatorname{ord}(f_i) = \max\{d \in \mathbb{N} \mid x \in \mathfrak{m}_{X,x}^d\},$$

where we choose i such that $x \in U_i$. If X is normal and $E = \overline{\{\eta\}}$ is a prime divisor, then

$$\operatorname{mult}_E D = \operatorname{mult}_\eta D = \operatorname{coeff}_E D.$$

Lemma 7.9. Let Z be a closed subvariety of a smooth variety X, $\Delta = \sum a_i D_i$ a $\mathbb{Q}$-divisor on X. Let E be the irreducible component of $p^{-1}(Z)$ which dominates Z, where $p\colon \operatorname{Bl}_Z X \to X$ is the blow-up. Then

$$a(E, X, \Delta) = (\operatorname{codim}_X Z - 1) - \sum a_i \operatorname{mult}_Z D_i.$$

Proof. Replacing X by $X \setminus \operatorname{Sing} Z$, we may assume Z is smooth. Then

$$K_{\operatorname{Bl}_Z X} - p^* K_X = (\operatorname{codim}_X Z - 1)E,$$
$$p^* D_i - p_*^{-1} D_i = (\operatorname{mult}_Z D_i)E$$

do the job. ∎

Lemma 7.10. Let Y be a normal variety, $f\colon Y \to X$ a proper birational map, $\Delta_X \in \operatorname{WDiv}(X)_{\mathbb{Q}}$. Write

$$K_Y + \Delta_Y = f^*(K_X + \Delta_X)$$

where $f_* K_Y = K_X$, $\Delta_Y \in \operatorname{WDiv}(Y)_{\mathbb{Q}}$ with $f_*\Delta_Y = \Delta_X$. Then for any divisor F over X,

$$a(F, Y, \Delta_Y) = a(F, X, \Delta_X).$$

In particular,

$$\operatorname{discrep}(X, \Delta_X) = \min_{E \subseteq \operatorname{Exc}(f)} \{\operatorname{discrep}(Y, \Delta_Y), a(E, X, \Delta_X)\},$$
$$\operatorname{totaldiscrep}(X, \Delta_X) = \operatorname{totaldiscrep}(Y, \Delta_Y).$$

Proof. If $F \subseteq Y$, the equality follows from the definition of Δ_Y.

Assume that F does not appear as a divisor on Y. Without loss of generality, let $F \subseteq Z$, with Z normal and $g\colon Z \to Y$ birational. Indeed, if $F \subseteq Y'$, then we take Z to be the normalization of the graph Γ_h of the birational map $h\colon Y \dashrightarrow Y'$.

To compute $a(F, Y, \Delta_Y)$, we find $\Delta_Z \in \operatorname{WDiv}(Z)_{\mathbb{Q}}$ such that

$$K_Z + \Delta_Z = g^*(K_Y + \Delta_Y)$$

and $g_* K_Z = K_Y$, $g_* \Delta_Z = \Delta_Y$. Then

$$K_Z + \Delta_Z = (f \circ g)^*(K_X + \Delta_X)$$

and $(f \circ g)_* K_Z = K_X$, $(f \circ g)_* \Delta_Z = \Delta_X$. The multiplicity of Δ_Z at F is

$$-a(F, X, \Delta_X) = \operatorname{mult}_F \Delta_Z = -a(F, Y, \Delta_Y).$$

Hence, $a(F, Y, \Delta_Y) = a(F, X, \Delta_X)$. ■

8 Singularities, 9/28

Proposition 8.1. Either $\operatorname{discrep}(X, \Delta) = -\infty$ or

$$-1 \leq \operatorname{totaldiscrep}(X, \Delta) \leq \operatorname{discrep}(X, \Delta) \leq 1.$$

Proof. Let $U = X \setminus \operatorname{Sing} X$. Then $\operatorname{discrep}(X, \Delta) \leq \operatorname{discrep}(U, \Delta|_U)$. So we may assume that X is smooth. Now let $Z = \overline{\{\eta\}}$ be any codimension 2 subvariety of X such that $\eta \notin D_i$ for each i, where $\Delta = \sum a_i D_i$. Consider the blow-up $\operatorname{Bl}_Z X \to X$ with $\operatorname{center}_X E = Z$. Then (7.9) tells us that

$$a(E, X, \Delta) = 1 - \sum a_i \operatorname{mult}_Z D_i = 1.$$

Hence, $\operatorname{discrep}(X, \Delta) \leq 1$.

Now, suppose that $\operatorname{totaldiscrep}(X, \Delta) < -1$. Then there exists a divisor $E \subseteq Y$ over X such that $a(E, X, \Delta) = -1 - c$ with $c > 0$. Again, we may replace Y with $Y \setminus \operatorname{Sing} Y$, so that Y is smooth.

Write

$$K_Y + \Delta_Y = f^*(K_X + \Delta), \quad f_* K_Y = K_X, \quad f_* \Delta_Y = \Delta.$$

Then

$$\Delta_Y = (1 + c)E + f_*^{-1}\Delta + (\text{other exceptional divisors}).$$

Pick $Z_0 \subseteq E$, $\operatorname{codim}_Y Z_0 = 2$ and not in the support of the last two terms of the above equation. Then $\operatorname{mult}_{Z_0} E = 1 + c$. Define $Y_1 = \operatorname{Bl}_{Z_0} Y$ and E_1 be an exceptional divisor that dominates Z. Then (7.9) together with (7.10) tell us

$$a(E_1, X, \Delta) = a(E_1, Y, \Delta_Y) = 1 - (1 + c) = -c.$$

Let $Z_1 = E_1 \cap (g_1)_*^{-1} E$. Then for $E_2 \subseteq Y_2 = \operatorname{Bl}_{Z_1} Y_1$ that dominates Z, same computation gives

$$a(E_2, X, \Delta) = a(E_2, Y, \Delta_Y) = -2c.$$

By induction, we can construct

$$E_m \subseteq Y_m = \operatorname{Bl}_{Z_{m-1}} Y_{m-1} \xrightarrow{g_m} Y_{m-1}, \quad Z_{m-1} = E_m \cap (g_{m-1})_*^{-1} E$$

such that $a(E_m, X, \Delta) = -mc$. ∎

Definition 8.2. If $a_i \in [0,1]$ for each i, we call $D = \sum a_i D_i$ a **boundary divisor**. If $a_i \in (-\infty, 1]$ for each i, we call D a **subboundary divisor**.

Proposition 8.3. Let X be a smooth variety, $\Delta = \sum a_i D_i$ be a subboundary divisor having SNC support. Then

$$\operatorname{discrep}(X, \Delta) = \min \left\{ 1, \min\{1 - a_i\}, \min_{D_i \cap D_j \neq \varnothing, i \neq j} \{1 - a_i - a_j\} \right\}.$$

Proof. Let $r(X, \Delta)$ be the RHS of the equation in the proposition.

Consider $\operatorname{Bl}_Z X \to X$ and E an exceptional divisor with $\operatorname{center}_X E = Z$, where $\operatorname{codim}_X Z = 2$. We see from (7.9) that

$$a(E, X, \Delta) = \begin{cases} 1 & \text{if } Z \not\subseteq \operatorname{Supp} \Delta, \\ 1 - a_i & \text{if } Z \subseteq D_i, Z \not\subseteq D_j \ \forall j \neq i, \\ 1 - a_i - a_j & \text{if } Z \subseteq D_i \cap D_j, i \neq j. \end{cases}$$

This shows that $\operatorname{discrep}(X, \Delta) \leq r(X, \Delta)$.

It remains to prove $\operatorname{discrep}(X, \Delta) \geq r(X, \Delta)$. For an exceptional divisor $E \subseteq Y$ over X, say $f \colon Y \to X$, we want to prove $a(E, X, \Delta) \geq r(X, \Delta)$.

Fact. After possibly shrinking X, there exists a sequence of blow-ups along smooth centers that factors f:

$$Y = X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{g_1} X_0 \subseteq X.$$

We see that $r(X_0, \Delta|_{X_0}) \geq r(X, \Delta)$, so we may assume $X = X_0$. Now, we induction on m. For $m = 1$, $f\colon Y = \operatorname{Bl}_Z X \to X$, $\operatorname{codim}_X Z \geq 2$ such that $E + f_*^{-1}\Delta$ has SNC support (by shrinking X around a general point z of Z). Set $I = \{i \mid Z \subseteq D_i\}$. Since $\sum D_i$ has SNC, we see that $\#I \leq \operatorname{codim}_X Z$. Also,

$$\begin{aligned}
a(E, X, \Delta) &= (\operatorname{codim}_X Z - 1) - \sum_{i \in I} a_i \\
&= (\operatorname{codim}_X Z - \#I) - 1 + \sum_{i \in I} (1 - a_i) \\
&\geq \begin{cases} 1 & \text{if } I = \varnothing, \\ 1 - a_{i_0} & \text{if } I = \{i_0\} \\ -1 + \sum_{i \in I_0} (1 - a_i) & \text{if } I_0 \subseteq I, \#I_0 = 2. \end{cases}
\end{aligned}$$

Hence, $a(E, X, \Delta) \geq r(X, \Delta)$.

For $m > 1$, write $Y \xrightarrow{f_1} X_1 = \operatorname{Bl}_Z X \xrightarrow{g_1} X$, $K_{X_1} + \Delta_1 = g_1^*(K_X + \Delta)$. WLOG,

$$\Delta_1 = -a(E_1, X, \Delta)E + (g_1)_*^{-1}\Delta$$

has SNC support. Note that we also have $-a(E_1, X, \Delta) \leq 1$ (since X is smooth and Δ has SNC support) so that Δ_1 is subboundary. So

$$\begin{aligned}
r(X_1, \Delta_1) &\geq \min\left\{ r(X, \Delta), 1 + a(E_1, X, \Delta) - \max_{E_1 \cap (g_1)_*^{-1} D_i = \varnothing} a_i \right\} \\
&\geq \min\{r(X, \Delta), a(E_1, X, \Delta)\} \geq r(X, \Delta)
\end{aligned}$$

by the $m = 1$ case. By induction hypothesis on f_1, one has

$$a(E, X, \Delta) = a(E, X_1, \Delta_1) \geq r(X_1, \Delta_1),$$

as desired. ∎

Definition 8.4. • A pair (X, Δ) is called a pair if X is a normal variety, Δ is a boundary divisor, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

- A pair $(X, \Delta = \sum a_i D_i)$ has SNC at a (not necessarily closed) point $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring and there exists an open neighbourhood $U \subseteq X$ with local coordinate $z_1, \ldots, z_n \in \mathfrak{m}_{X,x}$ such that for each D_i, there exists $c(i)$ such that $D_i = (z_{c(i)} = 0)$ near x.

- We define $\operatorname{snc}(X, \Delta)$ to be the largest open subset U of X such that $(U, \Delta|_U)$ has SNC, and $\text{non-snc}(X, \Delta) = X \setminus \operatorname{snc}(X, \Delta)$.

Next we define the 6 classes of singularities that are most important for the MMP. Recall that

$$1 \geq \operatorname{discrep}(X, \Delta) \geq \operatorname{totaldiscrep}(X, \Delta) \geq -1.$$

if $\operatorname{discrep}(X, \Delta) \neq -\infty$. So we say a pair (X, Δ) is

- terminal if $\operatorname{discrep}(X, \Delta) > 0$,
- canonical if $\operatorname{discrep}(X, \Delta) \geq 0$,
- purely log terminal (plt for short) if $\operatorname{discrep}(X, \Delta) > -1$,
- log canonical (lc for short) if $\operatorname{discrep}(X, \Delta) \geq -1$,
- Kawamata log terminal (klt for short) if $\operatorname{totaldiscrep}(X, \Delta) > -1$,
- divisorial log terminal (dlt for short) if $a(E, X, \Delta) > -1$ whenever $\operatorname{center}_X E \subseteq \text{non-snc}(X, \Delta)$.

Remark. We see that (X, Δ) is klt if and only if $\operatorname{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor = 0$.

If Δ is only a subboundary divisor, then (X, Δ) is called sub-plt, sub-lc if $\operatorname{discrep}(X, \Delta) > -1, \geq -1$, sub-klt if $\operatorname{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor \leq 0$.

If $\Delta = 0$, we say X has $\star$ singularity if $(X, 0)$ is $\star$. Note that in this case, plt = dlt = klt, so we simply called X lt.

If $\dim X = 1$, then $(X, \sum a_i D_i)$ is termial = klt if $a_i < 1$ for each i, is canonical = plt = dlt = lc if $a_i \leq 1$ for each i.

There are some trivial implications terminal $\Rightarrow$ canonical, klt $\Rightarrow$ plt $\Rightarrow$ dlt $\Rightarrow$ lc, terminal $\Rightarrow$ klt, and canonical $\Rightarrow$ plt.

Lemma 8.5 (monotonicity property)**.** Let X be a normal variety, Δ, $\Delta' \in \mathrm{WDiv}_{\mathbb{Q}}$ so that (X,Δ), $(X,\Delta+\Delta')$ are pairs. Let $E \subseteq Y \xrightarrow{f} X$ be a morphism with f birational. Then

$$a(E,X,\delta) = a(E,X,\Delta+\Delta') + \operatorname{coeff}_E f^*\Delta'.$$

In particular, if $\Delta' \geq 0$, then $a(E,X,\Delta) \geq a(E,X,\Delta+\Delta')$ for any divisor E over X and the inequality is strict if and only if $\operatorname{center}_X E \subseteq \operatorname{Supp}\Delta'$. Hence,

$$\operatorname{discrep}(X,\Delta) \geq \operatorname{discrep}(X,\Delta+\Delta'), \quad \operatorname{totaldiscrep}(X,\Delta) \geq \operatorname{totaldiscrep}(X,\Delta+\Delta').$$

Proof. Write $K_Y + \Delta_Y = f^*(K_X+\Delta)$. Then

$$K_Y + \Delta_Y + f^*\Delta' = f^*(K_X + \Delta + \Delta').$$

So

$$a(E,X,\Delta+\Delta') = -\operatorname{coeff}_E(\Delta_1 + f^*\Delta') = a(E,X,\Delta) - \operatorname{coeff}_E f^*\Delta',$$

as desired. ■

Corollary 8.6. Let $\Delta = \sum a_i D_i$ be a sub-boundary divisor. Then

(1) there exists a log resolution f for (X,Δ) such that $\sum f_*^{-1} D_i$ is smooth;

(2) for any such log resolution f with

$$a := \min_{E \subseteq \operatorname{Exc}(f)} \{a(E,X,\Delta)\} \geq -1,$$

we have

$$\operatorname{discrep}(X,\Delta) = \min\left\{a, \min_i\{1-a_i\}, 1\right\}.$$

This implies that the infimum in the definitions of $\operatorname{discrep}(X,\Delta)$ and $\operatorname{totalcrep}(X,\Delta)$ is actually a finite infimum, and hence minimum.

Corollary 8.7. Given normal variety X. Let $f\colon Y \to X$ be any resolution of singularities. Assume that

$$a := \min_{E \subseteq \operatorname{Exc}(f)} \{a(E,X)\} \geq 0.$$

Then

$$\operatorname{discrep}(X) = \min\{a, 1\}.$$

In particular, X has canonical (resp. terminal) singularities if and only if there exists a resolution $f\colon Y \to X$ with $K_Y = f^*K_X + \sum a_i E_i$ such that $a_i \geq 0$ (resp. $a_i > 0$) for each i.

Proof. Write $K_Y + \Delta_Y = f^*K_X$ with $f_*K_Y = K_X$, $f_*\Delta_Y = 0$. We have

$$\Delta_Y = -\sum_{E \subseteq \mathrm{Exc}(f)} a(E, X, \Delta)E \leq 0.$$

So

$$\mathrm{discrep}(Y, \Delta_Y) \geq \mathrm{discrep}(Y, 0) = 1$$

since Y is smooth. This is an equality since $\mathrm{discrep}(Y, \Delta_Y) \leq 1$. Hence,

$$\mathrm{discrep}(X) = \min_{E \subseteq \mathrm{Exc}(f)} \{\mathrm{discrep}(Y, \Delta_Y), a(E, X)\} = \min_{E \subseteq \mathrm{Exc}(f)} \{1, a(E, X)\}. \qquad \blacksquare$$

Proof of (8.6). (1) By Hironaka's theorem, there exists a log resolution $g\colon Z \to X$ for (X, Δ) so that $\sum g_*^{-1}D_i$ has SNC. Then, by induction, a sequence of blow-ups over the mutual intersection of $g_*^{-1}D_i$'s gives f.

(2) Write $K_Y + \Delta_Y = f^*(K_X + \Delta)$ with $f_*K_Y = K_X$, $f_*\Delta_Y = \Delta$ as usual. Then

$$\Delta_Y = f_*^{-1}\Delta - \sum_{E \subseteq \mathrm{Exc}(f)} a(E, X, \Delta)E$$

is a subboundary divisor.

Set $b_E = -a(E, X, \Delta)$. Since Y is smooth and Δ_Y is a subboundary divisor having SNC, it follows from (8.3) that

$$\mathrm{discrep}(Y, \Delta_Y) = \min_{i, E, E' \subseteq \mathrm{Exc}(f), E \cap E' = \varnothing} \{1 - a_i - b_E, 1 - b_E - b_{E'}, 1 - a_i, 1 - b_E, 1\}.$$

Since $1 - a_i - b_E$, $1 - b_E - b_{E'}$, $1 - b_E \geq -b_E$, this is at least

$$\min_{E \subseteq \mathrm{Exc}(f)} \{-b_E, 1 - a_i, 1\}.$$

Thus,

$$\begin{aligned} \mathrm{discrep}(X, \Delta) &= \min_{E \subseteq \mathrm{Exc}(f)} \{\mathrm{discrep}(Y, \Delta_Y), -b_E\} \\ &= \min_{E \subseteq \mathrm{Exc}(f)} \{1 - a_i, -b_E, 1\}, \end{aligned}$$

as desired. $\blacksquare$

Proposition 8.8. Let (X, Δ) be a sub-klt pair. Then there exists a log resolution $f\colon Y \to X$ for (X, Δ) such that if we write

$$K_Y + A_Y - B_Y = f^*(K_X + \Delta), \quad A_Y, B_Y \geq 0$$

and A_Y and B_Y have no common components, then $\operatorname{Supp} A_Y$ is smooth, i.e., disjoint union of smooth prime divisors.

Proof. For any log resolution $f\colon Y \to X$ for (X, Δ), write

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

with $f_* K_Y = K_X$, $f_* \Delta_Y = \Delta$, $\Delta = \sum d_i D_i$.

Since (X, Δ) is klt, there exists $m \in \mathbb{N}$ such that

$$\operatorname{totaldiscrep}(X, \Delta) > -1 + \frac{1}{m}.$$

So $d_i \leq 1 - \frac{1}{m}$ and thus $d_{i_1} + d_{i_2} \leq 2 - \frac{2}{m}$ for any i, i_1, i_2. Consider the partition

$$\left(0, 2 - \tfrac{2}{m}\right] = \bigsqcup_{k=1}^{2m-2} \left(\tfrac{k-1}{m}, \tfrac{k}{m}\right] = \bigsqcup_{k=1}^{2m-2} I_k.$$

We define

$$r(f) = (r_1, \ldots, r_{2m-2}(f)) \in \mathbb{Z}_{\geq 0}^{2m-2}$$

by the formula

$$r_k(f) = \# \left\{ (i_1, i_2) \mid D_{i_1} \cap D_{i_2} \neq \varnothing, i_1 < j_2, d_{i_1} + d_{i_2} \in I_k \right\}.$$

Consider the inverse lexicographic order $\succeq_{\text{invlex}}$ on $\mathbb{Z}_{\geq 0}^{m-2}$. Then $(\mathbb{Z}_{\geq 0}^{2m-2}, \succeq_{\text{invlex}})$ satisfies descending chain condition.

For a given f, if $r(f) = 0$, then we are done. Otherwise, take the maximal k such that $r_k(f) \neq 0$ and (i_1, i_2) realizing it, i.e., $Z = D_{i_1} \cap D_{i_2}$ and $\operatorname{mult}_Z \Delta_Y = d_{i_1} + d_{i_2} \in I_k$. Consider $g\colon Y' = \operatorname{Bl}_Z Y \to Y$ and let $f' = f \circ g$, E the excepsional divisor g. Write

$$K_{Y'} + \Delta_{Y'} = (f')^*(K_X + \Delta) = f^*(K_Y + \Delta_Y)$$

with $g_* K_{Y'} = K_Y$, $g_* \Delta_{Y'} = \Delta_Y$. Then

$$e = \operatorname{coeff}_E \Delta_{Y'} = -a(E, Y, \Delta_Y) = -1 + (d_{i_1} + d_{i_2}) \in -1 + I_k$$

by (7.9).

Since $d_{i_\ell} \le 1 - \frac{1}{m}$,

$$e + d_{i_\ell} \le -1 + \frac{k}{m} + 1 - \frac{1}{m} = \frac{k-1}{m},$$

i.e., $e + d_{i_\ell} \in I_{k'}$ for some $k' < k$. This means that the intersections of E and the strict transforms of D_{i_1}, D_{i_2} does not contribute to $r_{k'}(f')$ for $k' \ge k$. Hence, $r_{k'}(f') = r_{k'}(f) = 0$ for $k' > k$ and $r_k(f') = r_k(f) - 1$, i.e., $r(f) \succ_{\text{invlex}} r(f')$. By the DCC condition, we get a log resolution f such that $r_k(f) = 0$ for all k. ■

Corollary 8.9. Let (X, Δ) be a sub-klt, $f\colon (Y, A_Y - B_Y) \to (X, \Delta)$ a special log resolution as in (8.8). If

$$a(E, X, \Delta) < 1 + \operatorname{totaldiscrep}(X, \Delta) \qquad (\spadesuit)$$

for some divisor E over X, then $\operatorname{center}_Y E$ is a divisor. In particular,

$$\begin{aligned}
&\#\{\text{exceptional divisors } E \text{ over } X \text{ with } a(E, X, \Delta) = 0\}\\
&\quad \le \#\{\text{exceptional divisors } E \text{ over } X \text{ with } (\spadesuit)\}\\
&\quad \le \#\{f\text{-exceptional divisors}\} < \infty
\end{aligned}$$

Exceptional divisors E over X with $a(E, X, \Delta) = 0$ is called **crepant** (since it has zero discrepancy).

Proof. For an exceptional divisor E over Y, it follows from (8.5) that

$$a(E, X, \Delta) = a(E, Y, A_Y - B_Y) \ge a(E, Y, A_Y) \ge \operatorname{discrep}(Y, A_Y).$$

Write $A_Y = \sum a_i A_i \ge 0$. Then (8.3) and (7.10) tells us that

$$\begin{aligned}
\operatorname{discrep}(Y, A_Y) &= \min\left\{1, \min\{1 - a_i\}, \min_{A_i \cap A_j \ne \varnothing}\{1 - a_i - a_j\}\right\}\\
&\ge 1 + \operatorname{totaldiscrep}(Y, A_Y - B_Y) = 1 + \operatorname{totaldiscrep}(X, \Delta).
\end{aligned}$$ ■

9 Basepoint-free theorem, 10/2

Theorem 9.1 (Basepoint-free theorem). Let (X, Δ) be a projective klt pair, D a nef Cartier divisor on X such that there exists $a \in \mathbb{Q}_{>0}$ with $aD - (K_X + \Delta)$ is big and nef. Then $|mD|$ is basepoint-free for large enough $m \in \mathbb{N}(X, D)$, i.e., D is semi-ample.

Remark. If mD is bpf for all $m \gg 1$, then $D = (m+1)D - mD$ is Cartier and D is nef, so these two condition in the bpf theorem are necessary.

Theorem 9.2 (Non-vanishing theorem, Shokurov)**.** Let (X, Δ) be a projective klt pair, D a nef Cartier divisor such that there exists $a \in \mathbb{Q}_{>0}$ and an effective Cartier divisor A with $aD + A - (K_X + \Delta)$ is big and nef. Then

$$\mathrm{H}^0(X, mD + A) \neq 0$$

for $m \gg 1$.

Lemma 9.3. Let X be a projective variety, M a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then there exists a resolution $f \colon Y \to X$, $\sum F_i$ a SNC divisor on Y with $\bigcup F_i \supseteq \mathrm{Exc}(f)$ such that for each $\eta > 0$, there exists $p_i \in (0, \eta) \cap \mathbb{Q}$ with

$$f^*M - \sum p_i F_i$$

ample.

Proof of (9.1) assuming (9.2). By the non-vanishing theorem, $|mD| \neq \varnothing$ for $m \gg 1$. For any integer $b \geq 2$, let $B(b)$ be the reduced base locus of $|bD|$. Since X is noetherian, the decreasing sequence $\{B(b^r)\}$ stabilizes to a subset $B_\infty(b)$.

If all $B_\infty(b) = \varnothing$, then $B(2^r) = B(3^r) = \varnothing$ for $r \gg 1$. Then for $m \gg 1$, say $m = a2^r + b3^r$, we get

$$B(m) \subseteq B(2^r) \cup B(3^r) = \varnothing.$$

From now on, we assume that $B_\infty(b) = B(b^r) \neq \varnothing$ for some $b \geq 2$. There exists a log resolution $f \colon Y \to X$ and a SNC divisor $\sum F_i$ on Y (not necessarily f-exceptional) such that

(a) $|b^r f^*D - \sum r_i F_i|$ is basepoint-free, $r_i \geq 0$ for each i, and $\bigcup_{r_i > 0} f(F_i) = B_\infty(b) = B(b^r)$;

(b) $K_Y \equiv f^*(K_X + \Delta) + \sum a_i F_i$ with $a_i > -1$ for each i (since (X, Δ) is klt);

(c) $f^*(aD - (K_X + \Delta)) - \sum p_i F_i$ is ample for some $p_i \in (0, 1 + a_i) \cap \mathbb{Q}$ for each i.

Note that $a_i > 0$ only when F_i is f-exceptional since $\Delta \geq 0$.

Since $f^{-1}(B(b^r)) = \operatorname{Bs}|b^r f^*D|$, we want to find F_i with $r_i > 0$ such that $F_i \not\subseteq \operatorname{Bs}|b^r f^*D|$ for all $b \gg 1$.

For each $m \in \mathbb{N}$, $c \in \mathbb{Q}_{>0}$, we define $N_{m,c}$ by the formula

$$K_Y + N_{m,c} = mf^*D + \sum_i (-cr_i + a_i - p_i)F_i$$

so that

$$N_{m,c} = (m - cb^r - a)f^*D + c\left(b^r f^*D - \sum r_iF_i\right) + f^*(aD - (K_X + \Delta)) - \sum p_iF_i.$$

By our construction, $b^r f^*D - \sum r_iF_i$ is basepoint-free (and hence nef) and $f^*(aD-(K_X+\Delta)) - \sum p_iF_i$ is ample. Since f^*D is nef, $N_{m,c}$ is ample if $m \geq cb^r + a$. Since $B_\infty(b) \neq \varnothing$, not all r_i are zero. Take

$$c = \min_{r_i>0} \frac{1 + a_i - p_i}{r_i} \in \mathbb{Q}_{>0}$$

so that $\min_i(-cr_i + a_i - p_i) = -1$.

By perturbing the p_i a little (so that the divisor is still ample), we may assume that there is only one $i = i_0$ such that $c = \frac{1+a_{i_0}-p_{i_0}}{r_{i_0}}$, $r_{i_0} > 0$. Let $F = F_{i_0}$. We have then for $m \geq cb^r + a$,

$$K_Y + N_{m,c} = mf^*D + \sum_{i \neq i_0} (-cr_i + a_i - p_i)F_i - F = mf^*D + B - F.$$

Notice that $\lceil B \rceil$ is effective and f-exceptional (since F_i appears in $\lceil B \rceil$ implies $a_i > cr_i + p_i > 0$).

Pick m to be a power of b that is at least $\max\{b^r, cb^r + a\}$. We get

$$\lceil N_{m,c} \rceil = f^*D - K_Y + \lceil B \rceil - F,$$

and hence

$$\mathrm{H}^0(Y, mf^*D + \lceil B \rceil) \longrightarrow \mathrm{H}^0(F, mf^*D + \lceil B \rceil) \longrightarrow \mathrm{H}^1(Y, K_Y + \lceil N_{m,c} \rceil) = 0$$

by (5.5). Since $\mathrm{H}^0(Y, mf^*D + \lceil B \rceil) \cong \mathrm{H}^0(X, mD)$ and $f(F) \subseteq B_\infty(b) = B(m)$, all elements in $\mathrm{H}^0(Y, mf^*D + \lceil B \rceil)$ vanish on $F = F_{i_0}$. This implies $\mathrm{H}^0(F, mf^*D + \lceil B \rceil) = 0$. However, consider the pair $(F, (\lceil B \rceil - B)|_F)$. Since $f^*D|_F$ is nef Cartier and $\lceil B \rceil$ is effective Cartier,

$$N_{m,c}|_F = mf^*D|_F + \lceil B \rceil|_F - (K_F + (\lceil B \rceil - B)|_F)$$

is ample (and hence big and nef). So (9.2) shows that

$$\mathrm{H}^0(F, mf^*D + \lceil B \rceil) \neq 0,$$

a contradiction. ∎

10 Rationality theorem, 10/5

Theorem 10.1. Let (X, Δ) be a projective klt pair,

$$a = \min\{e \in \mathbb{N} \mid e(K_X + \Delta) \in \mathrm{CDiv}(X)\}$$

the index of (X, Δ), H a big and nef Cartier divisor on X. If $K_X + \Delta$ is not nef, then the nef value

$$r = r(H) := \sup\{t \in \mathbb{R} \mid H_t := H + t(K_X + \Delta) \text{ is nef}\} \in \mathbb{Q}$$

and $r/a = u/v$ with u, $v \in \mathbb{Z}$, $\gcd(u, v) = 1$, $0 < v \leq a(\dim X + 1)$.

First, we observe that

$$[0, r] = \{t \in \mathbb{R} \mid H_t \text{ is nef}\}$$

since H is nef, $K_X + \Delta$ is not nef, and nef is a closed condition. Also, for $t \in [0, r) \cap \mathbb{Q}$,

$$H_t = \frac{t}{r}\, H_t + \frac{r-t}{r}\, H$$

is a sum of a nef divisor and a big and nef divisor, hence big and nef.

If r is not rational, we shall consider

$$\{(p, q) \in \mathbb{N}^2 \mid \tfrac{q-1}{p} < r < \tfrac{q}{p}\},$$

which has infinite size. Since $pH + (q-1)(K_X + \Delta) = pH_{(q-1)/p}$ is big and nef, we can apply vanishing theorem on $pH_{q/p} = pH_{(q-1)/p} + K_X + \Delta$, which is not nef, hence has base points.

Proof. Without loss of generality, we may assume that $r > 0$. Let $n = \dim X$ and $D(x, y) = xH + y(K_X + \Delta)$. We see from the above observation that $D(x, y)$ is big and nef if $\frac{y}{x} \in [0, r) \cap \mathbb{Q}$.

First, we may assume that $D(1,0)$ and $D(1,a)$ are basepoint-free by (9.1). Indeed, take $t \in \mathbb{N}$ such that

$$D(t,a), \quad D(t,a)-(K_X+\Delta)=D(t,a-1)$$

are big and nef. Then $H' \in mD(t,a)$ is basepoint-free for $m \gg 1$. Note that

$$H'+(K_X+\Delta)=D(mt,(m+1)a)=\frac{m-1}{3}D(2t,a)+\frac{m+2}{3}D(t,2a).$$

So if we take t larger such that

$$D(2t,a-1), \quad D(t,2a-1)$$

are big and nef, then for $m \gg 0$ with $m \equiv 1 \pmod 3$, H' and $H'+a(K_X+\Delta)$ are basepoint-free. Now, since

$$H'+r'(K_X+\Delta)=H+\frac{ma+r'}{mt}(K_X+\Delta),$$

we get $r=\frac{ma+r'}{mt}$, $r'=mtr-ma$, i.e., $r \in \mathbb{Q}$ if and only if $r' \in \mathbb{Q}$. Once we know $r/a=u/v$, $r'/a=u'/v'$, we may choose m, t such that $\gcd(m,v)=\gcd(t,v)=1$. Then $v \leq v' \leq a(n+1)$. Hence, we may replace H by H'.

For $\eta \in \mathbb{Q}_{>0}$, let

$$\Lambda_\eta=\{(p,q)\in\mathbb{N}^2 \mid \tfrac{aq-\eta}{p}<r<\tfrac{aq}{p}\}=\{(p,q)\in\mathbb{N}^2 \mid 0<aq-rp<\eta\}.$$

It is clear that $\eta \leq \varepsilon$ implies $\Lambda_\eta \subseteq \Lambda_\varepsilon$.

If $r \notin \mathbb{Q}$, then Λ_η is an infinite set for all $\eta \in \mathbb{Q}_{>0}$ since $0<q-\frac{r}{a}p<\frac{\eta}{a}$ has infinitely many solutions. If $r \in \mathbb{Q}$ and exists $(p_0,q_0)\in\Lambda_\eta$. Then $(p=p_0+da, q=q_0+dr)$ also lies in Λ_η for any $dr \in \mathbb{N}$. In each case, Λ_η is an infinite set whenever it is nonempty.

If $(p,q)\in\Lambda_\eta$, then $D(p,aq-\eta)$ is big and nef, whereas $D(p,aq)$ is not nef, and hence not basepoint-free. Let $B(p,q)\neq\varnothing$ be the reduced base locus of $D(p,aq)$. Notice that $B(p,q)=X$ if and only if $|D(p,aq)|=\varnothing$.

If there exists $(p_0,q_0)\in\Lambda_1$, then $B(p,q)\subseteq B(p_0,q_0)$ for all $(p,q)\in\Lambda_1$ with $q \gg 1$. Indeed, for $(p,q)\in\Lambda_1$, let $q=\alpha q_0+\beta$ with $0\leq\beta<q_0$ with α, $\beta\in\mathbb{N}$. Then

$$D(p,aq)=\alpha D(p_0,aq_0)+\beta D(1,a)+(p-\alpha p_0-\beta)D(1,0).$$

Since

$$p - \alpha p_0 - \beta > p - \alpha p_0 - q_0 \geq p - \frac{q}{q_0} p_0 - q_0$$
$$> \frac{aq-1}{r} - \frac{q}{q_0} p_0 - q_0 = q\left(\frac{a}{r} - \frac{p_0}{q_0}\right) - \frac{1}{r} - q_0 > 0$$

for q large enough, we see that $B(p, aq) \subseteq B(p_0, aq_0)$ for q large enough (since $D(1, a)$ and $D(1, a)$ are basepoint-free).

Since X is noetherian, there is a closed subset B_∞ of X such that $B_\infty = B(p, q)$ for all $(p, q) \in \Lambda_1$ with $q \gg 1$.

We proceed by contradiction assuming that either $r \notin \mathbb{Q}$ or $r/a = u/v$ but $v > a(n+1)$. Then $\Lambda_{1/(n+1)}$ is nonempty (and hence infinite). The first statement clear holds when $r \notin \mathbb{Q}$. For the other case, there exists p, $q \in \mathbb{N}$ such that $0 < vq - up = 1 < \frac{v}{a(n+1)}$ so that $0 < aq - rp < \frac{1}{n+1}$, i.e., $(p, q) \in \Lambda_{1/(n+1)}$.

We claim that there exists sufficiently large $(p, q) \in \Lambda_1$ such that $|D(p, aq)| \neq \varnothing$ (thus $B_\infty \subsetneq X$). Indeed, since $D(p, aq) - (K_X + \Delta) = D(p, aq - 1)$ is big and nef, $\mathrm{H}^i(X, D(p, aq)) = 0$ for all $i > 0$. It is therefore enough to say $P(x, y) = \chi(X, D(x, ay))$ does not vanish at some point of Λ_1.

Lemma 10.2. Assume $\Lambda_{\eta/(n+1)}$ is infinite. If a polynomial $R(x, y)$ of degree at most n vanishes on all sufficiently large elements of Λ_η, then $R(x, y) \equiv 0$.

Proof of lemma. For $(p, q) \in \Lambda_{\eta/(n+1)}$ large enough , $(jp, jq) \in \Lambda_\eta$ for $1 \leq j \leq n+1$, and hence, $R(jp, jq) = 0$. Since $\deg R \leq n$, $R|_{xq=yp} \equiv 0$. Thus, R vanishes on infinitely many such lines, so $R \equiv 0$. $\square$

In our case, since $\Lambda_{1/(n+1)}$ is infinite and

$$P(x, 0) = \chi(X, xH) = \frac{H^n}{n!} x^n + O(x^{n-1}) \neq 0,$$

there exists $(p, q)_{\Lambda_1}$ with $\mathrm{h}^0(X, D(p, aq)) = P(p, q) \neq 0$, as desired.

Define

$$\Lambda_1^\infty = \{(p, q) \in \Lambda_1 \mid B(p, q) = B_\infty\}.$$

Fix $(p_0, q_0) \in \Lambda_1^\infty$ such that $(p, q) \in \Lambda_1^\infty$ if $q \geq q_0$. As in the proof of (9.1), there exists a log resolution $f : Y \to X$ and a SNC divisor $\sum F_i$ on Y (not necessarily exceptional) such that

(a) $|f^*D(p_0, aq_0) - \sum r_i F_i|$ is basepoint-free, $r_i \geq 0$ for each i, and

$$\bigcup_{r_i > 0} f(F_i) = B(p_0, q_0) = B_\infty.$$

(b) $K_Y \equiv f^*(K_X + \Delta) + \sum a_i F_i$ with $a_i > -1$ for each i. Note that $a_i > 0$ only when F_i is f-exceptional since $\Delta \geq 0$.

(c) $f^*D(p_0, aq_0 - 1) - \sum p_i F_i$ is ample for some $p_i \in (0, 1 + a_i) \cap \mathbb{Q}$ for each i.

Since $B_\infty \neq \varnothing$, r_i are not all zero. As in the proof of (9.1), consider the log canonical threshold

$$c = \min_{r_i > 0} \frac{1 + a_i - p_i}{r_i}, \quad \min_i \{-cr_i + a_i - p_i\} = -1$$

and may assume there exists a unique i_0 such that $c = \frac{1 + a_{i_0} - p_{i_0}}{r_{i_0}}$. Let $F = F_{i_0}$ and

$$B = \sum_{i \neq i_0} (-cr_i + a_i - p_i) F_i.$$

Notice that $\lceil B \rceil$ is effective, f-exceptional, and $F \not\subseteq \operatorname{Supp} B$.

Consider

$$\begin{aligned} N_{p,q} &= f^*D(p, aq) - K_Y + \sum_i (-cr_i + a_i - p_i) F_i \\ &= f^*D(p - (c+1)p_0, a(q - (c+1)q_0)) + (c+1) f^*D(p_0, q_0) \\ &\quad - f^*(K_X + \Delta) + \sum (-cr_i - p_i) F_i \\ &= f^*D(p - (c+1)p_0, a(q - (c+1)q_0)) + c\left(f^*D(p_0, q_0) - \sum r_i F_i\right) \\ &\quad + \left(f^*D(p_0, aq_0 - 1) - \sum p_i F_i\right). \end{aligned}$$

The second term is basepoint-free, hence nef; the third term is ample by our assumption.

Set

$$Q(x, y) = \chi(F, (f^*D(x, ay) + \lceil B \rceil)|_F), \quad \eta_0 = \min\{1, (c+1)(aq_0 - rp_0)\}.$$

Lemma 10.3. If $q > (c+1)q_0$ and $aq - rp \in \Lambda_{\eta_0}$, then $N_{p,q}$ is ample and $Q(p, q) = 0$.

Proof of lemma. Since $q > (c+1)q_0$ and $aq - rp < (c+1)(aq_0 - rp_0)$ implies $f^*D(p - (c+1)p_0, a(q-(c+1)q_0))$ is nef, it follows from the above calculation that $N_{p,q}$ is ample.

Since

$$K_Y + N_{p,q} = f^*D(p, aq) + B - F,$$

we have

$$\lceil N_{p,q} \rceil = f^*D(p, aq) - K_Y + \lceil B \rceil - F,$$

so

$$\begin{aligned} \mathrm{H}^0(Y, f^*D(p, aq) + \lceil B \rceil) &\longrightarrow \mathrm{H}^0(F, (f^*D(p, aq) + \lceil B \rceil)|_F) \\ &\longrightarrow \mathrm{H}^1(Y, f^*D(p, aq) + \lceil B \rceil - F) = 0 \end{aligned}$$

by (5.5) and $\mathrm{H}^0(Y, f^*D(p, aq) + \lceil B \rceil) \cong \mathrm{H}^0(X, f^*D(p, aq))$ since $\lceil B \rceil$ is f-exceptional. So all elements in $\mathrm{H}^0(Y, f^*D(p, aq) + \lceil B \rceil)$ vanishes on $F = F_{i_0}$ $(f(F) \subseteq B_\infty)$ implies $\mathrm{H}^0(F, (f^*D(p, aq) + \lceil B \rceil|_F) = 0$.

Since $(f^*D(p, aq) + B)|_F - K_F = N_{p,q}|_F$ is ample, we get

$$Q(p, q) = \chi((f^*D(p, aq) + \lceil B \rceil)|_F) = 0. \qquad \square$$

For $q > (c+1)q_0$ and $aq/p < r$, we have $f^*D(p, aq)|_F$ is nef and $N_{p,q}|_F = (f^*D(p, aq) + B)|_F - K_F$ is ample by our assumption in (10.3).

Since for $m \gg 1$, $(f^*D(mp, amq) + B)|_F - K_F$ is still ample,

$$Q(mp, mq) = \mathrm{h}^0((f^*D(mp, amq) + \lceil B \rceil)|_F) \neq 0$$

by (9.2). Then (10.2) implies $\Lambda_{\eta 0/(n+1)} = \varnothing$, and hence $r = u/v \in \mathbb{Q}$. When $(p, q) \in \Lambda_1$, $aq - rp = \frac{aqv - vp}{v}$ can take at most v values. So we may choose $(p_0, q_0) \in \Lambda_1^\infty$ such that $aq_0 - rp_0$ is maximal among elements of Λ_1^∞. If $(p, q) \in \Lambda_1^\infty$, $q > (c+1)q_0$,

$$aq - rp \leq aq - rp_0 < (c+1)(aq_0 - rp_0),$$

so $(p, q) \in \Lambda_{\eta_0}$, and hence $Q(p, q) = 0$ by the above calculation. Since $\Lambda_{1/(n+1)}$ is infinite, (10.2) then shows that $Q \equiv 0$ a contradiction. ■

11 Cone theorem and length of extramal ray, 10/12

Theorem 11.1 (Cone theorem)**.** Let (X, Δ) be a projective klt pair. Then

$$\mathscr{R} = \{\text{all } (K_X + \Delta)\text{-negative extremal rays in } \overline{\mathrm{NE}}(X)\}$$

is a countable set, and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + \sum_{R\in\mathscr{R}} R.$$

These rays are locally discrete in the half-space $\overline{\mathrm{NE}}(X)_{K_X+\Delta<0}$.

For each $R \in \mathscr{R}$, there exists a rational curve Γ spanning R such that

$$0 < -(K_X + \Delta) \cdot \Gamma \leq 2 \dim X.$$

Remark. For X smooth and $\Delta = 0$, the cone theorem was proved by Mori, and the bound was sharpened by $K_X \cdot \Gamma \leq \dim X + 1$.

Let $K_\Delta = K_X + \Delta$. One can perturb H such that $K_\Delta + rH$ meets $\overline{\mathrm{NE}}(X)$ in an extremal ray. Since $r(H) \in \mathbb{Q}$, there are at most countably many such rays.

Theorem 11.2 (Contraction theorem)**.** Let (X, Δ) be a projective klt pair, $\overline{\mathrm{NE}}(X) \supseteq F$ a $(K_X + \Delta)$-negative extremal face. Then there exists a unique (up to isomorphism) contraction $\mathrm{cont}_F \colon X \to Z$, called the contraction of F, to a projective variety Z such that $\mathrm{cont}_{F*} \mathcal{O}_X = \mathcal{O}_Z$ and for any irreducible curve $C \subseteq X$, $\mathrm{cont}_F(C)$ is a point if and only if $[C] \in F$. In particular, $-(K_X + \Delta)$ is cont_F-ample.

The image of the injective (by projection formula) map $\mathrm{cont}_F^* \colon \operatorname{Pic} Z \to \operatorname{Pic} X$ is

$$\{L \in \operatorname{Pic} X \mid L \cdot C = 0,\ \forall [C] \in F\}.$$

In particular, we have the exact sequence

$$0 \longrightarrow N^1(Z)_{\mathbb{R}} \xrightarrow{\mathrm{cont}_F^*} N^1(X)_{\mathbb{R}} \longrightarrow \langle F\rangle_{\mathbb{R}}^{\vee} \longrightarrow 0,$$

its dual exact sequence

$$0 \longrightarrow \langle F\rangle_{\mathbb{R}} \longrightarrow N_1(X)_{\mathbb{R}} \xrightarrow{\mathrm{cont}_{F*}} N_1(Z)_{\mathbb{R}} \longrightarrow 0,$$

and $\rho(Z) = \rho(X) - \dim F$.

Remark. Using this theorem, we can prove that $F = \overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(\mathrm{cont}_F)$.

The complexes in the theorem is not exact in general, e.g., $X = E \times E$ is a abelian surface, where $(E, 0)$ is an elliptic curve, $N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}} \ni f_1 = [\{0\} \times E]$, $f_2 = [E \times \{0\}]$, $\delta = [\Delta]$, the complex

$$0 \longrightarrow \langle f_2 \rangle_{\mathbb{R}} \longrightarrow N_1(X)_{\mathbb{R}} \xrightarrow{\mathrm{pr}_{2*}} N_1(E)_{\mathbb{R}} \longrightarrow 0$$

is not exact since $\mathrm{pr}_{2*}(\delta - f_1) = 0$ implies $0 = \delta^2 = (f_1 + af_2)^2 = 2a$ for some $a \neq 0$. Note that

$$\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \left\{ \alpha = xf_1 + yf_2 + z\delta \,\middle|\, \sum yz = 0, \sum x \geq 0 \right\}$$

is not polyhedral. If E is very general in the moduli, then $\{f_1, f_2, \delta\}$ is a basis of $N_1(X)_{\mathbb{R}}$.

Definition 11.3. Let $f \colon X \to Y$ be a proper morphism of varieties over a characteristic 0 algebraically closed field k, D a Cartier divisor on X.

- We say D is f-big if $\operatorname{rk} f_* \mathcal{O}_X(mD) > c \cdot m^n$ for some $c > 0$, and $m \gg 1$, where n is the dimension of the general fiber of f.
- We say D is f-free if $f^* f_* \mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective.

Theorem 11.4. Let X, Y be normal varieties over $\mathbb{C}$, $g \colon X \to Y$ be a projective morphism. Assume that (X, Δ) is a klt pair and $-(K_X + \Delta)$ is g-ample. Then every irreducible component $E \subseteq \mathrm{Exc}(f)$ is covered by rational curves Γ with $g(\Gamma)$ a point and such that

$$0 < -(K_X + \Delta) \cdot \Gamma \leq 2(\dim E - \dim g(E)). \tag{$\spadesuit$}$$

Proof. Without loss of generality, we may assume that g has connected fibers (by replacing g with its Stein factorization). Since Y is normal, $E = g^{-1}(g(E))$.

Since for a flat family of curves whose general fibers are rational curves, any irreducible component of its special fiber is again a rational curve and a rational curve can only degenerate into a union of rational curves of lower degree, it suffices to show that passing through a general point of E, there exists a rational curve contracted by g and satisfies ($\spadesuit$).

We may replace Y with the intersection of $\dim g(E)$ general hyperplane sections so that $g(E)$ is a point (note that the conditions are preserved).

Let H be a very ample divisor on X, $e = \dim E$. Let $\nu\colon \widetilde{E} \to E$ be the normalization of E, C a smooth curve in $\widetilde{E} \setminus \operatorname{Sing} \widetilde{E}$, given by intersection of $e-1$ general hyperplanes $D_1, \ldots, D_{e-1} \in |\nu^* H|$.

Claim. We have $(K_X + \Delta) \cdot H^{e-1} \cdot E \geq K_{\widetilde{E}} \cdot (\nu^* H)^{e-1}$, and the inequality is strict if $E \neq X$.

Proof of claim. If $E = X$, then

$$(K_X + \Delta) \cdot H^{\dim X - 1} \geq K_X \cdot H^{\dim X - 1}$$

since H is ample.

If $E \neq X$, then g is birational. We prove this by induction on e. If $e \geq 2$, then take a general normal hyperplane $X_1 \in |H|$. Let Y_1 be the normalization of $g(X_1)$, $E_1 = X_1 \cap E$ an irreducible component of dimension $e-1$ of $\operatorname{Exc}(g_1\colon X_1 \to Y_1)$. Note that $\widetilde{E}_1 = \nu^{-1}(X_1) \in |\nu^* H|_{E_1}|$ is normal. Since $K_{X_1} \sim (K_X + H)|_{X_1}$, $K_{\widetilde{E}_1} \sim (K_{\widetilde{E}} + \nu^* H)|_{\widetilde{E}_1}$. Then

$$\begin{aligned}
(K_{X_1} + \Delta_1) \cdot H^{e-2} \cdot E_1 &= (K_X + H + \Delta)|_H \cdot H^{e-2}|_H \cdot H|_H \\
&= (K_X + \Delta) \cdot H^{e-1} \cdot E + H^e \cdot E \\
K_{\widetilde{E}_1} \cdot (\nu^* H)^{e-2} &= (K_{\widetilde{E}} + \nu^* H)|_{\widetilde{E}_1} \cdot (\nu^* H)^{e-2} \\
&= K_{\widetilde{E}} \cdot (\nu^* H)^{e-1} + H^e \cdot E.
\end{aligned}$$

So it suffices to prove the case $e = 1$, i.e., $(K_X + \Delta) \cdot E > \deg K_{\widetilde{E}}$.

Assume to the contrary that $(K_X + \Delta) \cdot E \leq \deg K_{\widetilde{E}}$. Since $\operatorname{Pic}(E) \xrightarrow{\nu^*} \operatorname{Pic}(\widetilde{E})$ is surjective, there exists a Cartier divisor D_E on E such that $\nu^* \mathcal{O}(D_E) \cong \mathcal{O}(K_{\widetilde{E}}) = \omega_{\widetilde{E}}$. There exists an injective trace map $\nu_* \omega_{\widetilde{E}} \to \omega_E$ (since E is proper together with $e = 1$ implies E is projective, and hence ω_E is defined). We get

$$\mathrm{h}^0(\omega_E \otimes \mathcal{O}(-D_E)) \geq \mathrm{h}^0(\nu_* \omega_{\widetilde{E}} \otimes \mathcal{O}_X(-D_E)) = \mathrm{h}^0(\omega_{\widetilde{E}} \otimes \nu^* \mathcal{O}(-D_E)) = 1.$$

Since $\mathrm{h}^0(\omega_E \otimes \mathcal{O}(-D_E)) = \mathrm{h}^1(E, D_E)$ by Serre duality (note that E has no embedded point, hence E is a Cohen–Macaulay curve).

By our assumption, $D_E - (K_X + \Delta)|_E$ has nonnegative degree. After shrinking Y, we may assume that Y is contractible and Stein so that the higher cohomologies of $\mathbb{Z}$ and any coherent sheaf vanish. Moreover, we may assume that g induces an isomorphism over $Y \setminus g(E) = Y \setminus \{0\}$. Set $X_0 = g^{-1}(0) \supseteq E$. Since

$$\operatorname{Supp}(R^i g_* \mathbb{Z}) = \operatorname{Supp}(R^i g_* \mathscr{F}) = \{0\}$$

for any coherent sheaf $\mathscr{F}$ on X and $i > 0$. Using Leray spectral sequence, we get

$$\mathrm{H}^i(X, \mathbb{Z}) \cong \mathrm{H}^0(Y, R^i g_* \mathbb{Z}) \cong \mathrm{H}^i(X_0, \mathbb{Z}), \quad \mathrm{H}^i(X, \mathscr{F}) \cong \mathrm{H}^0(Y, R^i g_* \mathscr{F}).$$

We can extend D_E to a Cartier divisor D on X if we replace X by a small analytic neighbourhood of E (see [Debarre, Higher-dimensional algebraic geometry]). Then $D - (K_X + \Delta)$ is g-nef by assumption and g-big since g is birational. Since (X, Δ) is klt, we get $R^i g_* \mathcal{O}_X(D) = 0$ for $i > 0$ by (5.5) (for complex analytic variety by N. Nakayama), and hence $\mathrm{H}^i(X, D) = 0$ for $i > 0$. It follows that

$$0 = \mathrm{H}^1(X, D) \longrightarrow \mathrm{H}^1(E, D_E) \longrightarrow \mathrm{H}^2(X, \mathscr{I}_E(D)) \cong \mathrm{H}^0(Y, R^2 g_* \mathscr{I}_E(D)) = 0,$$

a contradiction. $\square$

Now, since $-(K_X + \Delta)$ is g-ample, $K_{\widetilde{E}} \cdot C < 0$ by our claim. Since $-\nu^*(K_X + \Delta)$ is ample on $\widetilde{E}$, it follows from (4.1) that for each $x \in C$, there exists a rational curve $\widetilde{\Gamma} \subseteq \widetilde{E}$ containing x with

$$-\nu^*(K_X + \Delta) \cdot \widetilde{\Gamma} \leq 2e \, \frac{-\nu^*(K_X + \Delta) \cdot C}{-K_{\widetilde{E}} \cdot C} \leq 2e$$

again by our claim. Therefore, E is covered by a rational curve $\Gamma = \nu_* \widetilde{\Gamma}$ with

$$0 < -(K_X + \Delta) \cdot \Gamma \leq 2(\dim E - \dim g(E)). \qquad \blacksquare$$

Lemma 11.5. Let X be a normal variety, Y a proper variety. Suppose there is a proper birational morphism $f\colon Y \to X$. Let D be a Cartier divisor on X, F an effective exceptional divisor on Y. Then

$$\mathcal{O}_X(D) \cong f_* \mathcal{O}_Y(f^* D + F).$$

In particular, $\mathrm{H}^0(X, D) \cong \mathrm{H}^0(Y, f^* D + F)$ and $f_* \mathcal{O}_X(F) \cong \mathcal{O}_X$.

Proof. Since f is dominant, there is an injection

$$\mathcal{O}_X(D) \longrightarrow f_*\mathcal{O}_Y \otimes \mathcal{O}_X(D) = f_*f^*\mathcal{O}_X(D).$$

Also, since F is effective, there is an injection

$$f^*\mathcal{O}_X(D) \to f^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(F) = \mathcal{O}_Y(f^*D + F).$$

This gives us an injection $f_*f^*\mathcal{O}_X(D) \to f_*\mathcal{O}_X(f^*D + F)$. Hence, there exists a natural injection $\varphi\colon \mathcal{O}_X(D) \to f_*\mathcal{O}_Y(f^*D + F)$.

Since X is normal, $\operatorname{codim}_X \operatorname{dom} f^{-1} \geq 2$ and φ is isomorphic over $X \setminus \operatorname{dom} f^{-1}$, φ is an isomorphism. ∎

Corollary 11.6. Let X be a normal variety, Y a proper factorial variety. Suppose there is a proper birational morphism $f: Y \to X$. Then

(1) for an exceptional divisor E, E is effective if and only if $f_*\mathcal{O}_Y(E) \cong \mathcal{O}_X$;

(2) if D, D' are Cartier divisors on X, F, F' are exceptional divisors on Y such that $f^*D + F \sim f^*D' + F'$, then $D \sim D'$ and $F = F'$.

Proof. (1) If E is effective, it follows from the lemma that $f_*\mathcal{O}_Y(E) \cong \mathcal{O}_X$. Conversely, we get

$$\mathcal{O}_Y = f^*\mathcal{O}_X \cong f^*f_*\mathcal{O}_Y(E) \longrightarrow \mathcal{O}_Y(E),$$

which gives a nonzero section in $\mathrm{H}^0(\mathcal{O}_Y(E))$, i.e., E is effective.

(2) Write $F = F_1 - F_2$, $F' = F_1' - F_2'$ with F_1, F_2, F_1', $F_2' \geq 0$. Then

$$f^*(D - D') + F_1 + F_2' \sim F_2 + F_1'.$$

It follows from the lemma that $D - D' \sim 0$. so $F_1 + F_2' \sim F_2 + F_1'$. By lemma,

$$\mathrm{H}^0(Y, F_1 + F_2') = \mathrm{H}^0(X, \mathcal{O}_X) \cong \mathbf{k}.$$

Hence, $F_1 + F_2' = F_2 + F_1'$, i.e., $F_1 = F_1'$, $F_2 = F_2'$, $F = F'$. ∎

12 Surface singularities

Theorem 12.1 (Mumford)**.** Let X be a normal surface, $f: Y \to X$ a proper birational map from a smooth surface Y with exceptional curves E_i. Then $(E_i \cdot E_j)$ is negative definite.

Proof. We only prove the case when X is projective. Let H be the pullback of an ample Cartier divisor on X. Then $H^2 > 0$ and $H \cdot E_i = 0$. Let $D = e^i E_i \neq 0$. Then $H \cdot D = 0$. By Hodge index theorem, we get $D^2 \leq 0$ and the equality holds if and only if $D \equiv 0$.

If D is effective, then $D \not\equiv 0$ since Y is a complete smooth surface, hence projective. In general, write $D = D^+ - D^-$, where D^+ is the positive part and D^- is the negative part. Then

$$D^2 = (D^+)^2 - 2\, D^+ \cdot D^- + (D^-)^2 \leq (D^+)^2 + (D^-)^2 < 0. \qquad \blacksquare$$

Lemma 12.2. Let S be a smooth surface, $C = \sum C_i \subseteq S$ be a sum of irreducible curves with $(C_i \cdot C_j)$ negative definite. Let $A = a^i C_i$, $a^i \in \mathbb{R}$ and assume that $A \cdot C_i \geq 0$ for each i. Then

(1) $a^i \leq 0$ for each i;

(2) if C is connected and $A \neq 0$, then $a^i < 0$ for each i.

Proof. Write $A = A^+ - A^-$ as usual. Assume that $A^+ \neq 0$. Then $(A^+)^2 < 0$ by our assumption. Hence, there is a component C_i of A^+ such that $C_i \cdot A^+ < 0$. Since C_i is not a component of A^-, $C_i \cdot A^- \geq 0$. Then $C_i \cdot A = C_i \cdot (A^+ - A^-) < 0$, a contradiction. This proves (1).

For (2), assume that C is connected and $\varnothing \neq \operatorname{Supp} A^- \subsetneq \operatorname{Supp} C$. Then there exists C_i such that $C_i \not\subseteq \operatorname{Supp} A^-$ but $C_i \cap \operatorname{Supp} A^- \neq \varnothing$. So

$$C_i \cdot A = -C_i \cdot A^- < 0,$$

a contradiction. $\blacksquare$

Corollary 12.3. Under the same condition in the above lemma. Assume C is connected. Then

$$\mathcal{Z}_{\text{num}} = \left\{ Z \in \bigoplus \mathbb{Z} C_i \setminus \{0\} \,\middle|\, Z \cdot C_i \leq 0,\ \forall i \right\}$$

has a unique minimal element.

Note that $\mathcal{Z}_{\text{num}}$ is a semigroup, and all $Z \in \mathcal{Z}_{\text{num}}$ is effective by our lemma.

Proof. Given $\alpha_j \in \mathbb{Z}_{<0}$ for each j. Then there exists $m^i/m \in \mathbb{Q}$, m^i, $m \in \mathbb{Z}$ such that

$$\big((m^i/m)C_i\big) \cdot C_j = \alpha_j < 0.$$

Our lemma implies that $m^i > 0$ for each i. So $m^iC_i \in \mathcal{Z}_{\mathrm{num}}$. Take any two elements $Z = m^iC_i$, $Z' = (m^i)'C_i$. Define $n^i = \min\{m^i, (m^i)'\}$ and put $Z \wedge Z' = n^iC_i$.

Fix any i, we show that $(Z \wedge Z') \cdot C_i \leq 0$. Without loss of generality, we may assume that $m^i \leq (m^i)'$. Then

$$(Z \wedge Z') \cdot C_i = m^iC_i^2 + \sum_{j \neq i} n^jC_j \cdot C_i \leq m^iC_i^2 + \sum_{j \neq i} m^jC_j \cdot C_i = Z \cdot C_i \leq 0.$$

So $Z \wedge Z' \in \mathcal{Z}_{\mathrm{num}}$. Hence, $\bigwedge_{Z \in \mathcal{Z}_{\mathrm{num}}} Z$ is the unique minimal element of $\mathcal{Z}_{\mathrm{num}}$. ■

Definition 12.4. Let $x \in X$ be a normal surface singularity, $f \colon Y \to X$ be any resolution. Write $f^{-1}(x)_{\mathrm{red}} = \sum E_i$. The **fundamental cycle** $Z_f = Z_{\mathrm{num}}$ of $f^{-1}(x)_{\mathrm{red}}$ is the unique minimal element of the set

$$\left\{ Z \in \bigoplus \mathbb{Z}E_i \setminus \{0\} \,\middle|\, Z \cdot E_i \leq 0,\ \forall i \right\}.$$

Proposition 12.5 (Laufer's algorithm)**.** We define Z_i recursively:

1. Choose any E_{i_1} and define $Z_1 = E_{i_1}$.
2. Assume that $Z_1, \ldots, Z_{\nu-1}$ are defined. If there exists i_ν such that $Z_{\nu-1} \cdot E_{i_\nu} > 0$, then define $Z_\nu = Z_{\nu+1} + E_{i_\nu}$.
3. If $Z_{\nu_0} \cdot E_i \leq 0$ for each i, then stop, and $Z_{\mathrm{num}} = Z_{\nu_0}$.

This procedure will stop at a finite stage. The sequence $Z_1, \ldots, Z_{\nu_0}$ reaching Z_{num} is called a computation sequence.

Proof. We can prove by induction that every $Z_i \leq Z_{\mathrm{num}}$, hence stop in finite steps. ■

Definition 12.6. Let S be a smooth surface, D a divisor on S. We define the **arithmetic genus** of D to be

$$p_a(D) = \frac{D^2 + K_S \cdot D}{2} + 1.$$

Proposition 12.7. If D is effective, then $p_a(D) = 1 - \chi(D, \mathcal{O}_D)$. If D is irreducible, then $p_a(D) \geq 0$, and $p_a(D) = 0$ if and only if $D \cong \mathbb{P}^1$.

Proof. By Nagata's compactificaition theorem, we may assume that S is a complete smooth surface. We get

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_D) = \frac{(-D)\cdot(-D-K_S)}{2} + \chi(\mathcal{O}_S) + \chi(\mathcal{O}_D)$$

by Riemann–Roch, i.e., $p_a(D) = 1 - \chi(\mathcal{O}_D)$.

We have $p_a(D) = 1 - (\mathrm{h}^0(\mathcal{O}_D) - \mathrm{h}^1(\mathcal{O}_D)) = \mathrm{h}^1(\mathcal{O}_D)$. If $D \cong \mathbb{P}^1$, then $p_a(D) = \mathrm{h}^1(\mathcal{O}_D) = 0$. Conversely, if $p_a(D) = 0$, take a normalization $\nu\colon \widetilde{D} \to D$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \nu_*\mathcal{O}_{\widetilde{D}} \longrightarrow \nu_*\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D \longrightarrow 0.$$

We get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathcal{O}_D) & \longrightarrow & \mathrm{H}^0(\nu_*\mathcal{O}_{\widetilde{D}}) & \longrightarrow & \mathrm{H}^0(\nu_*\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_D) \\ & & \| & & \| & & & & \| \\ & & \mathbf{k} & & \mathbf{k} & & & & 0 \end{array}$$

So $\mathrm{H}^0(\nu_*\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D) = 0$ and ν is an isomorphism. Hence, D is a smooth proper curve with $\mathrm{H}^1(\mathcal{O}_D) = 0$, i.e., $D \cong \mathbb{P}^1$. ■

Corollary 12.8. Let Z_{num} be the fundamental cycle of $f^{-1}(x)_{\mathrm{red}} = \sum E_i$, where $f\colon Y \to X$ and $x \in X$ is a normal surface singularity. Then $p_a(Z_{\mathrm{num}}) \geq 0$.

Proof. Let $Z_1, \ldots, Z_{\nu_0} = Z_{\mathrm{num}}$ be a computational sequence. By construction, $Z_\nu = Z_{\nu-1} + E_{i_\nu}$ and $Z_{\nu-1}\cdot E_{i_\nu} > 0$. Then

$$p_a(Z_\nu) = p_a(Z_{\nu-1} + E_{i_\nu}) = p_a(Z_{\nu-1}) + p_a(E_{i_\nu}) + Z_{\nu-1}\cdot E_{i_\nu} - 1 \geq p_a(Z_{\nu-1})$$

since $p_a(E_{i_\nu}) \geq 0$. Hence, $p_a(Z_{\mathrm{num}}) \geq p_a(Z_{i_1}) \geq 0$. ■

Definition 12.9. For a resolution $f\colon Y \to (X, x)$, we define

$$p_a(f) = \sup\{p_a(Z) \mid Z > 0,\ \operatorname{Supp} Z \subseteq f^{-1}(x)\} \geq p_a(Z_{\mathrm{num}}).$$

Remark. There are three kinds of genus: $\dim(R^1 f_*\mathcal{O}_Y)_x$, $p_a(f)$, and $p_a(Z_{\mathrm{num}})$. These are independent of choices of the resolution f by taking common resolutions.

Definition 12.10. We define

- the **geometric genus** of (X, x) to be $p_g(X, x) = \dim(R^1 f_* \mathcal{O}_Y)_x$;
- the **arithmetic genus** of (X, x) to be $p_a(X, x) = p_a(f)$;
- the **fundamental genus** of (X, x) to be $p_f(X, x) = p_a(Z_{\text{num}})$.

Proposition 12.11. For a normal surface singularity (X, x),

$$p_g(X, x) \geq p_a(X, x) \geq p_f(X, x) \geq 0.$$

Proof. It suffices to show that $p_g(X, x) \geq p_a(X, x)$. Let Z be a divisor on Y with $Z > 0$, $\operatorname{Supp} Z \subseteq f^{-1}(x)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We get

$$R^1 f_* \mathcal{O}_Y \longrightarrow R^1 f_* \mathcal{O}_Z \longrightarrow R^2 f_* \mathcal{O}_Y(-Z) = 0$$

since the fiber dimension of f is at most 1. Since $R^1 f_* \mathcal{O}_Z$ has support $\{x\}$, this is equal to $\mathrm{H}^1(Z, \mathcal{O}_Z)$. Then

$$p_a(Z) = 1 - \mathrm{h}^0(\mathcal{O}_Z) + \mathrm{h}^1(\mathcal{O}_Z) \leq \mathrm{h}^1(\mathcal{O}_Z) \leq \dim(R^1 f_* \mathcal{O}_Y)_x = p_g(X, x). \qquad \blacksquare$$

Definition 12.12. Let (X, x) be a normal surface singularity. It is called **rational** if $p_g(X, x) = 0$. It is called **strongly elliptic** if $p_g(X, x) = 1$, **weakly elliptic** if $p_a(X, x) = 1$.

Theorem 12.13 (Artin)**.** Let (X, x) be a normal surface singularity. Then the followings are equivalent:

(i) $p_g(X, x) = 0$;

(ii) $p_a(X, x) = 0$;

(iii) $p_f(X, x) = 0$.

Proof. See [Ishii, Theorem 7.3.1]. ■

Theorem 12.14. Let (X, x) be a normal surface singularity. Then the followings are equivalent:

1. $p_a(X,x) = 1$;

2. $p_f(X,x) = 1$.

Proof. See [Laufer, On minimally elliptic singularities, Corollary 4.2]. ■

13 Du Val singularities

Proposition 13.1. Let (X,x) be a rational surface singularity, $Z = Z_{\text{num}}$ a fundamental cycle of a resolution. Then $\operatorname{mult}_x X = -Z^2$ and the embedding dimension $\operatorname{e.dim}(X,x) = -Z^2 + 1$.

Definition 13.2. A normal surface singularity (X,x) is called a rational double point (RDP) if (X,x) is rational and $\operatorname{mult}_x X = 2$.

Remark. If (X,x) is an RDP, then $-Z^2 = \operatorname{mult}_x X = 3$, $\operatorname{e.dim}(X,x) = 3$, and hence (X,x) is a rational hypersurface singularity in $(\mathbb{C}^3, 0)$.

Proposition 13.3. A surface singularity (X,x) is an RDP if and only if it is ADE.

Proof. Write $(X,x) \cong (F = 0, 0) \subseteq (\mathbb{C}^3, 0)$. Then $\operatorname{mult}_0 F = 3$. So we can write

$$F = x^2 + a(y,z)x + b(y,z) \sim x^2 + f(y,z).$$

If $\operatorname{mult}_0 f = 2$, then

$$F = x^2 + f \sim x^2 + y^2 + z^m$$

for some $m \geq 2$. This gives us the A-types.

Claim. $(X,0) = (x^3 + uy^3 + u_a z^a y + u_b z^b = 0, 0)$ is canonical if and only if either $a \leq 3$ and $u_a(0) \neq 0$ or $b \leq 5$ and $u_b(0) \neq 0$.

Now, there are three cases:

- $a \geq 3$ and $b = 4$
- $a = 3$ and $b \geq 5$
- $a \geq 4$ and $b = 5$

For the first case and the third case, $a \geq b-1$ and $b = 4$ or 5. So

$$f = uy^3 + u_a z^a y + u_b z^b = \mathcal{O}^\times y^3 + v_1 z^a y + \mathcal{O}^\times z^b = \mathcal{O}^\times y^3 + (v_1 z^{a-b} y + \mathcal{O}^\times) z^b.$$

If $a \geq b$, we get

$$F = x^2 + y^3 + z^b, \quad b = 4, 5,$$

i.e., (E_6) and (E_8). Assume $a = b-1$. Then

$$\begin{aligned} f &= \mathcal{O}^\times y^3 + v_1 z^{b-1} y + \mathcal{O}^\times z^b \\ &= \mathcal{O}^\times y^3 + \mathcal{O}^\times (z + \mathcal{O}^\times v_1 y/b)^b - * z^{b-2} y^2 - \sum_{i \geq 3} * z^{b-i} y^i \\ &\sim \mathcal{O}^\times y^3 + v_2 z^{b-2} y^2 + \mathcal{O}^\times z^b \\ &\sim \mathcal{O}^\times y^3 + \mathcal{O}^\times z^b \end{aligned}$$

because $z(b-2) \geq b$, which gives us, again, (E_6) and (E_8).

For the second case,

$$f = \mathcal{O}^\times y^3 + \mathcal{O}^\times y z^3 + v_3 z^b, \quad b \geq 5.$$

By blowing-up the origin, we see in $y = y_1 z_1$, $z = z_1$, we get

$$\overline{f} = z_1^3 (\mathcal{O}^\times y_1^3 + \mathcal{O}^\times y_1 z_1 + v_3 z_1^{b-3}).$$

Since $\mathrm{mult}_0 \overline{f} = 2$ and $\overline{f}_2$ is not a square (if $b = 5$, we get $z_1(\mathcal{O}^\times y + v_3 z_1)$; if $b > 5$ then we get $\mathcal{O}^\times y_1 z_1$), $\overline{f}$ is reducible and so is f. Since $f_3 = c \cdot y^5$, one of the factor of f is of the form $(y + *)$. Hence,

$$f \sim y(\mathcal{O}^\times y^2 + v_4 y z^2 + \mathcal{O}^\times z^3) \sim y(\mathcal{O}^\times y^2 + \mathcal{O}^\times z^3) \sim y^3 + y z^3,$$

i.e., (E_7). ∎

14 Elliptic surface singularities

Let $f \colon Y \to X$ be a minimal resolution of a normal surface singularity. Write $f^{-1}(0)_{\mathrm{red}} = \sum E_i$. We may assume that X and Y are affine.

Definition 14.1. Write

$$K_Y = f^*K_X + \sum a^i E_i.$$

Since $f^{-1}(0)$ is connected, either $a_i < 0$ for each i or $a_i = 0$ for each i. We define the **anti-canonical cycle** Z_K to be $-\sum a^i E_i$, and write $Z_K = \lfloor Z_k \rfloor + (Z_k - \lfloor Z_k \rfloor) = Z + \Delta_Y$.

We say $(X, 0)$ is **numerically Gorenstein** if Z_k is an integral divisor, i.e., $\Delta_Y = 0$.

Remark. It follows from the definition that the anti-canonical cycle $Z_k = 0$ if and only if $(X, 0)$ is Du Val.

The singularity $(X, 0)$ is numerically Gorenstein if and only if the complex line bundle $\Omega^2_{X\setminus\{0\}}$ is topologically trivial.

If $(X, 0)$ is numerically Gorenstein, then $Z_{\text{num}} \leq Z_k$.

Proposition 14.2. Let L be an f-nef line bundle on Y, $Z = \lfloor Z_k \rfloor$. Then

$$\mathrm{H}^0(Y, L) \longrightarrow \mathrm{H}^0(Z, L|_Z)$$

is surjective and

$$\mathrm{H}^1(Y, L) \longrightarrow \mathrm{H}^1(Z, L|_Z)$$

is an isomorphism.

Proof. Pushing-forward the exact sequence

$$0 \longrightarrow L(-Z) \longrightarrow L \longrightarrow L|_Z \longrightarrow 0,$$

we get the long exact sequence

$$\begin{array}{ccccccccccc} f_*L & \longrightarrow & f_*(L|_Z) & \longrightarrow & R^1f_*L(-Z) & \longrightarrow & R^1f_*L & \longrightarrow & R^1f_*(L|_Z) & \longrightarrow & R^2f_*L(-Z) \\ & & & & \| & & & & & & \| \\ & & & & 0 & & & & & & 0, \end{array}$$

where $R^1f_*L(-Z) = 0$ follows from the following facts: $K_Y + \Delta_Y + Z \equiv_f 0$, $L(-Z) - (K_Y + \Delta_Y) \equiv_f L$ is f-nef and f-big (by definition), and (Y, Δ_Y) is klt. Hence, $\mathrm{H}^0(L) \to \mathrm{H}^0(L|_Z)$ is surjective and $\mathrm{H}^1(L) \xrightarrow{\sim} \mathrm{H}^1(L|_Z)$ since X and Y are affine. ∎

Proposition 14.3. The sheaf $\omega_X / f_*\omega_Y$ is dual to $R^1f_*\mathcal{O}_Y$. In particular, $R^1f_*\mathcal{O}_Y \cong \mathbb{C} = \mathcal{O}_{X,0}/\mathfrak{m}_{X,0}$ if and only if $f_*\omega_Y = \mathfrak{m}_{X,0}\omega_X$.

Proof. For an open neighbourhood V of $0 \in X$, take $U = f^{-1}V$, $E = \operatorname{Exc} f$. We get

$$\mathrm{H}^0_E(U,\omega_Y) \longrightarrow \mathrm{H}^0(U,\omega_Y) \longrightarrow \mathrm{H}^0(U \setminus E,\omega_Y) \longrightarrow \mathrm{H}^1_E(U,\omega_Y) \longrightarrow \mathrm{H}^1(U,\omega_Y).$$

Note that

$$\mathrm{H}^0_E(U,\omega_Y) = \{s \in \mathrm{H}^0(U,\omega_Y) \mid \operatorname{Supp} s \subseteq E\} = 0$$

and $\mathrm{H}^1_E(U,\omega_Y) = \mathrm{H}^1_E(U',\omega_Y) = \mathrm{H}^1_E(Y,\omega_Y) = \mathrm{H}^1_E(\omega_Y)$ for any $U' \supseteq E$. So we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\omega_Y & \longrightarrow & \omega_X|_{X\setminus\{0\}} & \longrightarrow & \mathrm{H}^1_E(\omega_Y) & \longrightarrow & R^1 f_*\omega_Y \\ & & & & \omega_X & & & & 0 \end{array}$$

by X normal and Grauert–Riemenshneider Vanishing theorem. Hence,

$$\omega_X / f_*\omega_Y \cong \mathrm{H}^1_E(\omega_Y),$$

which is dual to $R^1 f_*(\omega_Y \otimes \omega_Y^{-1})$ by local duality theorem, see [Ishii, Cor 3.5.15]. ■

Definition 14.4 (Ried)**.** We say $(X,0)$ is an elliptic Gorenstein surface singularity if K_X os Cartier and $R^1 f_*\mathcal{O}_Y \cong \mathbb{C}$ (strongly elliptic).

Theorem 14.5 (Laufer 77)**.** Working with the minimal resolution f, the followings are equivalent:

(i) $p_a(Z_{\mathrm{num}}) = 1$ and any connected proper subdivisor of $\sum E_i = f^{-1}(0)_{\mathrm{red}}$ contracts to a rational singularity.

(ii) $p_a(Z_{\mathrm{num}}) = 1$ and $p_a(D) < 1$ for all $0 < D < Z_{\mathrm{num}}$.

(iii) $Z_{\mathrm{num}} = Z_K$.

(iv) $(X,0)$ is elliptic Gorenstein.

Lemma 14.6. In the minimal resolution of a Gorenstein surface singularity $(X,0)$, we have $\mathrm{h}^1(\mathcal{O}_D) < \mathrm{h}^1(\mathcal{O}_{Z_K})$ for each $0 < D < Z_K$.

Remark (Reid)**.** For numerically Gorenstein $(X,0)$, the cohomological cycle Z_{coh} is equal to Z_K if and only if $(X,0)$ is Gorenstein.

Proof. Notice that $K_Y = -Z_K$ (since X is Gorenstein). By Serre duality,

$$\mathrm{H}^1(\mathcal{O}_{Z_K})^\vee \cong \mathrm{H}^0(\omega_{Z_K}) = \mathrm{H}^0(\mathcal{O}_{Z_K}(K_Y + Z_K)) = \mathrm{H}^0(Z_K, \mathcal{O}_{Z_K}),$$
$$\mathrm{H}^1(\mathcal{O}_D)^\vee \cong \mathrm{H}^0(\omega_D) = \mathrm{H}^0(\mathcal{O}_D(K_Y + Z_K)) = \mathrm{H}^0(D, \mathcal{O}_D(D - Z_K)).$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_K}(D - Z_K) \longrightarrow \mathcal{O}_{Z_K} \longrightarrow \mathcal{O}_{Z_K - D} \longrightarrow 0.$$

Since $\mathrm{H}^0(\mathcal{O}_{Z_K}) \to \mathrm{H}^0(\mathcal{O}_{Z_K - D})$ is nontrivial because both contain the constant sections,

$$\mathrm{h}^1(\mathcal{O}_D) = \mathrm{h}^0(\mathcal{O}_D(D - Z_K)) < \mathrm{h}^0(\mathcal{O}_{Z_K}) = \mathrm{h}^1(\mathcal{O}_{Z_K}),$$

as desired. ■

Corollary 14.7. Assume that $(X, 0)$ is an elliptic Gorenstein surface singularity. Set $Z = Z_{\text{num}} = Z_K$ for the minimal resolution $f\colon Y \to X$. Then either

- Z is an irreducible and reduced curve of $p_a(Z) = 1$, or
- for each irreducible component $E_i \subseteq f^{-1}(0)_{\text{red}}$ is a smooth rational curve with $E_i \cdot (-Z + E_i) = -2$.

Proof. Since $\mathcal{O}_Y$ is f-nef, (14.2) tells us that $\mathrm{H}^1(Y, \mathcal{O}_Y) \cong \mathrm{H}^1(Z, \mathcal{O}_Z)$. So $\mathrm{h}^1(\mathcal{O}_Z) = \dim R^1 f_* \mathcal{O}_Y = 1$. If Z is irreducible and reduced, then $p_a(Z) = \mathrm{h}^1(\mathcal{O}_Z) = 1$. Otherwise, $E_i < Z$, and hence $p_a(E_i) = \mathrm{h}^1(\mathcal{O}_{E_i}) < \mathrm{h}^1(Z) = 1$. So $p_a(E_i) = 0$, i.e., $E_i \cong \mathbb{P}^1$. Since $K_Y = -Z$, the last formula is just adjunction formula. ■

Example 14.8. We say $(X, 0)$ is a **simple elliptic singularity** if Z is a smooth elliptic curve E. For example, $(X, 0) = (x^3 + y^3 + z^3 = 0, 0) \subseteq (\mathbb{C}^3, 0)$. Then $Z_{\text{num}} = E = Z_K$ and $E^2 = -3$.

We say $(X, 0)$ is a **cusp surface singularity** if $E = \sum E_i$ is a nodal rational curve (I_1) or E_i forms a cycle (I_r), $r \geq 3$.

We say $(X, 0)$ is a Brieskorn–Pham singularity if $(X, 0) = (x^a + y^b + z^c = 0, 0)$. It is known that

$$p_g(X, 0) = \#\{(i, j, k) \mid i, j, k > 0, \tfrac{i}{a} + \tfrac{j}{b} + \tfrac{k}{c} \leq 1\}.$$

For example, when $(a, b, c) = (2, 3, 18k)$, $p_a = p_f = 1$ but $p_g = 3k > 1$.

15 Elliptic surface singularities II

Let $f\colon Y \to X$ be a minimal resolution of an elliptic surface singularity, $Z = Z_{\min} = Z_K$ be the fundamental cycle. We may assume that $\mathcal{O}(K_X) \cong \mathcal{O}_X$. Then $K_Y = -Z$. Set $L = \mathcal{O}_Y(-Z) \cong \omega_Y$. We understand

$$R(Y, L) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(Y, L^{\otimes n})$$

by reducing the problem first to Z and then to a 0-dimensional subscheme of Z.

Lemma 15.1. Let V be a proper (possibly non-reduced) curve such that $\mathrm{H}^1(\mathcal{O}_V) = 0$, L a nef line bundle on V. Then

(1) L is globally generated, and

(2) $\mathrm{H}^1(V, L) = 0$.

Proof. Let $V_{\mathrm{red}} = \bigcup V_i$. Pick general points $p_i \in V_i$, and Cartier divisors $D_i \subseteq V$ such that $D_i \cap V_i = \{p_i\}$. Set $m_i = \deg_{V_i}(L|_{V_i})$ and $L' = \mathcal{O}_V(\sum m_i D_i)$. Notice that the exponential sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V^{\times} \longrightarrow 1$$

is exact even if $\mathcal{O}_V$ has nilpotent elements [see, compact complex surfaces, p.63]. So we get

$$\begin{array}{rcl} 0 = \mathrm{H}^1(\mathcal{O}_V) \longrightarrow \mathrm{H}^1(\mathcal{O}_V^{\times}) & \xrightarrow{\mathrm{c}_1} & \mathrm{H}^2(V, \mathbb{Z}) = \bigoplus \mathbb{Z}[V_i] \\ L & \longmapsto & (\deg_{V_i} L|_{V_i})_i. \end{array}$$

Since $\mathrm{c}_1(L) = \mathrm{c}_1(L')$, we see that $L \cong L'$ is globally generated, except possibly at the points p_i. By varying p_i, we get (1).

For (2), it follows from (1) that $\mathrm{H}^0(V, L) \otimes \mathcal{O}_V \xrightarrow{\mathrm{ev}} L$ is surjective. Let $d = \mathrm{h}^0(V, L)$. Then we get

$$0 = \mathrm{H}^1(\mathcal{O}_V)^{\otimes d} \longrightarrow \mathrm{H}^1(V, L) \longrightarrow \mathrm{H}^2(\ker \mathrm{ev}) = 0,$$

i.e., $\mathrm{H}^1(V, L) = 0$. ■

Proposition 15.2. Let L be a nef line bundle on Z such that $\deg_Z L > 0$. Then

(1) $\mathrm{H}^1(Z, L) = 0$;

(2) there exists a section $s \in \mathrm{H}^0(Z, L)$ such that $(s = 0)$ is a 0-dimensional subscheme that does not intersect $\operatorname{Sing} Z_{\mathrm{red}}$ and $s|_{Z_{\mathrm{red}}}$ has no multiple zeroes;

(3) it there exists an irreducible component $C \subsetneq Z$ such that $\deg_C(L|_C) > 0$, set $Z' = Z - C$, then $\mathrm{H}^0(Z, L) \to \mathrm{H}^0(Z', L|_{Z'})$ is surjective.

Proof. Recall that by (14.2), $\mathrm{h}^1(\mathcal{O}_Z) = \mathrm{h}^1(\mathcal{O}_Y) = \dim R^1 f_* \mathcal{O}_Y = 1$. Note that L is ample (consider the normalization $\nu\colon \widetilde{Z} \to Z$, which is finite surjective, we see that $\deg_Z \nu^* L = \deg \nu \cdot \deg_Z L > 0$, so $\nu^* L$, and hence L is ample). If Z is irreducible and reduced, then $p_a(Z) = 1$. This gives (1), and (2) is just Bertini's theorem. Otherwise, there exists C as in (3). Then (14.6) gives us $\mathrm{h}^1(\mathcal{O}_{Z'}) < \mathrm{h}^1(\mathcal{O}_Z) = 1$, or $\mathrm{h}^1(\mathcal{O}_{Z'}) = 0$. Since $L|_{Z'}$ is nef on Z', (15.1) shows that $L|_{Z'}$ is globally generated and $\mathrm{H}^1(Z', L|_{Z'}) = 0$. By (14.7), $C \cong \mathbb{P}^1$ and $-C \cdot Z' = C \cdot (-Z + C) = -2$. Consider the exact sequence

$$0 \longrightarrow L(-Z')|_C \longrightarrow L \longrightarrow L|_{Z'} \longrightarrow 0.$$

We get the long exact sequence

$$\mathrm{H}^0(L) \longrightarrow \mathrm{H}^0(L|_{Z'}) \longrightarrow \mathrm{H}^1(L(-Z')|_C) \longrightarrow \mathrm{H}^1(L) \longrightarrow \mathrm{H}^1(L|_{Z'}) = 0.$$

Since $\deg_C L(-Z')|_C = \deg_C(L|_C) - 2 \geq -1$, we get $\mathrm{H}^1(L(-Z')|_C) = 0$. So $\mathrm{H}^1(Z, L) = 0$, that is, (1). The left part of the long exact sequence gives (3), and (2) is just by Bertini's theorem by lifting a general section in $\mathrm{H}^0(L|_{Z'})$. ■

We fix some notations. Assume that L is a nef line bundle on Z with $s \in \mathrm{H}^0(Z, L)$ satisfying the conditions in (15.2). Set $V = (s = 0)$. Then $A = \mathcal{O}_V$ is a semi-local ring with Jacobson radical $\mathfrak{m}$. Write

$$A = \bigoplus_{i=1}^r A_i, \quad \mathfrak{m} = \bigoplus_{i=1}^r \mathfrak{m}_i,$$

where $(A_i, \mathfrak{m}_i)$ are local Artinian $\mathbb{C}$-algebras. Set $V_i = \operatorname{Spc} A_i \subseteq Z$, which are Cartier divisors. Then $V = \sum V_i$. We define the **socle** of $\mathfrak{m}$ to be

$$\operatorname{socle}(\mathfrak{m}) = \{a \in \mathfrak{m} \mid \mathfrak{m} a = 0\}.$$

Note that if $A_i = \mathbb{C}[t]/(t^a)$, then $\operatorname{socal}((t)) = (t^{a-1})$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-V) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_V \longrightarrow 0.$$

Tensoring this with L, we get

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow L \longrightarrow L|_V \longrightarrow 0.$$

Set

$$W_L = \mathrm{Im}(\mathrm{H}^0(Z, L) \to \mathrm{H}^0(V, L|_V)) \subseteq \mathrm{H}^0(V, L|_V) = A \otimes L.$$

Remark. The subset W_L is a vector subspace of $A \otimes L$, which is in general not an A-submodule.

Lemma 15.3. We have

(1) if $A \neq 0$, then $\mathrm{codim}_{A \otimes L} W_L = 1$;

(2) for each j, the composition $W_L \to A \otimes L \to A/A_j \otimes L$ is surjective;

(3) if $\mathfrak{m} \neq 0$, the composition $W_L \to A \otimes L \to A/\operatorname{socle}(\mathfrak{m}) \otimes L$ is surjective;

(4) if $\dim_{\mathbb{C}} A = \deg_Z L \geq 2$, then W_L generates $A \otimes L$ as an A-module.

Proof. For (1), consider the exact sequence

$$\mathrm{H}^0(Z, L) \longrightarrow A \otimes L \longrightarrow \mathrm{H}^1(\mathcal{O}_Z) \longrightarrow \mathrm{H}^1(L).$$

Since $\deg_Z L = \dim_{\mathbb{C}} A > 0$, $\mathrm{H}^1(L) = 0$. (1) now follows from the fact that $\mathrm{H}^1(\mathcal{O}_Z) = \mathbb{C}$.

For each j, $\mathcal{O}_{V \setminus V_j}(V) = A/A_j \otimes L$. Since

$$0 \longrightarrow \mathcal{O}_Z(V_j) \longrightarrow L \longrightarrow L|_{V \setminus V_j} \longrightarrow 0$$

Since $\mathcal{O}_Z(V_j)$ has positive degree on Z, $\mathrm{H}^1(Z, \mathcal{O}_Z(V_j)) = 0$. Thus, $\mathrm{H}^0(Z, L) \to A/A_j \otimes L$ and hence $W_L \to A/A_j \otimes L$ is surjective, i.e., (2).

Assume $\mathfrak{m} \neq 0$, thus $\operatorname{socle}(\mathfrak{m}) \neq 0$. Let $C \subseteq Z$ be an irreducible component such that V has a non-reduced point (comes from $\operatorname{socle}(\mathfrak{m})$) on C. Then $\deg_C(L|_C) > 0$ and Z is not reduced along C. Set $Z' = Z - C > 0$. Let s' be the restriction of s to $L|_{Z'}$, $V' = (s' = 0) \subseteq Z'$. Then we get the long exact sequence

$$\mathrm{H}^0(L|_{Z'}) \longrightarrow \mathrm{H}^0(L|_{V'}) \longrightarrow \mathrm{H}^1(\mathcal{O}_{Z'}).$$

Since $\mathrm{H}^0(L) \to \mathrm{H}^0(L|_{Z'})$ is surjective by (15.2) and $\mathrm{H}^1(\mathcal{O}_{Z'}) = 0$ by (14.6), we get the surjection $\mathrm{H}^0(L) \to \mathrm{H}^0(L|_{V'})$. Since $\mathcal{O}_{V'}$ surjects $A/\operatorname{socle}(\mathfrak{m})$, $A/\operatorname{socle}(\mathfrak{m}) \otimes L$ is a quotient of $\mathrm{H}^0(L|_{V'})$, this proves (3).

If (4) fails, then all elements of W_L vanish at a point $V = \operatorname{Spec} A$, but (2) and (3) show that this cannot happen. ■

Proposition 15.4 (Laufer 77, Reid 76)**.** Let $(X, 0)$ be an elliptic Gorenstein surface singularity, Z the fundamental cycle for the minimal resolution $f\colon Y \to X$, L a nef line bundle on Z. Let $k = \deg_Z L \geq 1$ and

$$R(Z, L) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(Z, L^{\otimes n}).$$

(i) If $k \geq 2$, then L is globally generated.

(ii) If $k \geq 3$, then $R(Z, L)$ is generated by its elements of deg 1. If $k = 2$, then $R(Z, L)$ is generated by its elements of deg 1 and 2. If $k = 2$, then $R(Z, L)$ is generated by its elements of deg 1, 2, and 3. More precisely,

$$R(Z, L) = \mathbb{C}[x_1, \ldots, x_k]/I, \quad \deg x_i = 1, \; I = \langle \deg 2, 3 \text{ elements} \rangle$$

for $k \geq 3$;

$$R(Z, L) = \mathbb{C}[x, y, z]/(z^2 + q_4(x, y)), \quad \deg(x, y, z) = (1, 1, 2), \; \deg q_4 = 4$$

for $k = 2$;

$$R(Z, L) = \mathbb{C}[x, y, z]/(z^2 + y^3 + ayx^4 + bx^6), \quad \deg(x, y, z) = (1, 2, 3), \; a, b, \in \mathbb{C}.$$

Proof. We have $\dim_{\mathbb{C}} A = \deg_Z L = k \geq 1$. Assume $k \geq 2$, then we have the diagra

$$\begin{tikzcd}
 & \mathcal{O}_Z \arrow[d] \\
\mathrm{H}^0(Z, L) \otimes \mathcal{O}_Z \arrow[r] \arrow[d, two heads] & L \arrow[d] \\
W_L \otimes \mathcal{O}_V \arrow[r] & L|_V = A \otimes L
\end{tikzcd}$$

By (15.3), W_L generates $A \otimes L$ as an A-module. So $\mathrm{H}^0(Z, L) \otimes \mathcal{O}_Z \to L$ is surjective, i.e., L is globally generated, which proves (i).

For (ii), let T be any section of $L|_V$ generating $L|_V$. Note that $\mathrm{H}^0(V, L|_V^{\otimes n}) = A \cdot T^n$ and $R(V, L|_V) = A[T]$. Consider the long exact sequences

$$0 \longrightarrow \mathrm{H}^0(L^{n-1}) \longrightarrow \mathrm{H}^0(L^n) \longrightarrow \mathrm{H}^0(L|_V^n) \longrightarrow \mathrm{H}^1(L^{n-1}).$$

When $n \geq 2$, $\deg_Z L^{n-1} > 0$ and L^{n-1} is nef on Z, so $\mathrm{H}^1(L^{n-1}) = 0$. Let $R_Z = R(Z, L)$. Then

$$R_Z / s \cdot R_Z(n) = \begin{cases} A \cdot T^n & n \geq 2, \\ W_L & n = 1, \\ \mathbb{C} & n = 0. \end{cases}$$

For the proof of (ii), see [KM, p.141-143]. ■

Definition 15.5. Given $d \in \mathbb{N}$ and $w_1, \ldots, w_d \in \mathbb{N}$. Let $R = \mathbf{k}[x_1, \ldots, x_d]$. We define the weight of a monomial $x^M = \prod x_i^{m_i} \in R$ to be $w(x^M) = \sum m_i w_i$. For $f = \sum a_M x^M \in R$, defined the weight of f to be $w(f) = \min_{a_M \neq 0}\{w(x^M)\}$. We get the ideal

$$\mathfrak{m}^w(n) = \{f \in R \mid w(f) \geq n\}.$$

The weighted blow-up of $\mathbb{A}^d$ at 0 with weight w is

$$\mathrm{Bl}_0^w \mathbb{A}^d = \mathrm{Proj}_R \left(\bigoplus \mathfrak{m}^w(n) \right).$$

For any $X \subseteq \mathbb{A}^d$, this define $\mathrm{Bl}_0^w X$ as the strict transform of X in $\mathrm{Bl}_0^w \mathbb{A}^d$.

Theorem 15.6 (Laufer 77, Reid 76)**.** Under the same notations as in (15.4), define $k = -Z^2 = \deg_Z L$.

(i) If $k \geq 3$, then $\mathrm{mult}_0 X = k = \mathrm{e.dim}(X, 0)$. Choose any embedding $(X, 0) \subseteq (\mathbb{C}^k, 0)$. Let x_i be the coordinate on $\mathbb{C}^k$, $w(x_i) = 1$.

(ii) If $k = 1$ or 2, then $\mathrm{mult}_0 X = 2$ and $\mathrm{e.dim}(X, 0) = 3$. After an analytic coordinate change, it can be given by the equations

$$z^2 + q(x, y) = 0, \quad \mathrm{mult}_0 q = 4, \ w(x, y, z) = (1, 1, 2)$$

for $k = 2$;

$$z^2 + y^3 + y q_4(x) + q_6(x) = 0, \quad \mathrm{mult}_0 q_i \geq i, \ w(x, y, z) = (1, 2, 3).$$

(iii) Consider the weighted blow-up $g\colon \overline{Y} = \operatorname{Bl}^w_0 X \to X$. Then g factors through f and $\overline{Y}$ has only Du Val singularity.

Remark. For $k = 2$, we can choose $x = x_1$, $y = x_1 y_1$, $z = x_1^2 z_1$. For $k = 1$, we can choose $x = x_1$, $y = x_1^2 y_1$, $z = x_1^3 z_1$.

If $k = 2$, $\overline{Y}$ is the normalization of the standard blow up $\operatorname{Bl}_0 X$.

16 Cohen–Macaulay and duality

Definition 16.1. Let $(R, \mathfrak{m})$ be a noetherian local ring, M a finite R-module. We say

- M is Cohen–Macaulay (C–M) if $\dim M := \dim\{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$ is equal to $\operatorname{depth} M$, i.e., the maximal length of the M-regular sequences;
- M satisfies Serre's condition S_k $(k \geq 0)$ if $\operatorname{depth} M \geq \min\{k, \dim M\}$.

Definition 16.2. Let X be a noetherian scheme, $\mathscr{F}$ a coherent sheaf on M. We say

- $\mathscr{F}$ is C–M at a closed point $x \in X$ if $\mathscr{F}_x$ is a C–M module over $\mathcal{O}_{X,x}$;
- $\mathscr{F}$ is C–M if $\mathscr{F}$ is C–M at all closed point $x \in \operatorname{Supp} \mathscr{F}$;
- X is C–M if $\mathcal{O}_X$ is C–M;
- $\mathscr{F}$ is S_k if $\mathscr{F}_x$ is S_k for every $x \in X$.

Remark. A scheme is normal is just R_1 (regular in codimension 1) and S_2. A coherent sheaf $\mathscr{F}$ with $\dim \operatorname{Supp} \mathscr{F} = \dim X$ is C–M if and only if it is $\mathsf{S}_{\dim X}$. Hence, an isolated surface singularity is C–M if and only if it is normal.

Proposition 16.3. Let $h \in \mathfrak{m}$ be a non-zero divisor on M. Then M is C–M (resp. S_k) if and only if M/hM is C–M (resp. S_{k-1}).

Remark. For $R = M = \mathcal{O}_{X,x}$, $x \in X$, M is C–M if and only if there exists $h \in \mathfrak{m}_{X,x}$ which is a non-zero divisor on $\mathcal{O}_{X,x}$ such that $x \in H = (h = 0)$ is C–M. So a 3-fold singluarity (X, x) which is regular in codimension 1 is C–M if and only if there exists a normal surface $x \in H \subseteq X$.

Proposition 16.4. Let $f\colon X \to Y$ be a finite surjective morphism of varieties over an algebraically closed field $\mathbf{k}$ of characteristic 0. If X is C–M and Y is normal, then Y is also C–M.

Proof. Since Y is normal, we have the map $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective and there exists a section $\mathrm{tr} = \mathrm{tr}_{X/Y} / \deg f$ such that $\mathrm{tr} \circ f^* = \mathrm{id}_{\mathcal{O}_Y}$. So $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \operatorname{coker} f^*$. Hence, a sequence from $\mathcal{O}_Y$ is $\mathcal{O}_Y$-regular if it is $f_*\mathcal{O}_X$-regular. ■

Theorem 16.5 (Serre duality for C–M sheaves)**.** Let X be a projective scheme of pure dimension n over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $\mathscr{F}$ be a C–M sheaf on X such that $\operatorname{Supp} \mathscr{F}$ is of pure dimension n. Then there exists a dualizing sheaf ω_X such that $\mathrm{H}^i(X, \mathscr{F})$ is dual to $\mathrm{H}^{n-i}(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \omega_X))$.

Sketch of proof. Take a finite morphism $f\colon X \to P = \mathbb{P}^n_k$ (which exists by Noether normalization). Then $\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i(P, f_*\mathscr{F})$ and $\mathscr{F}$ is C–M if and only if $f_*\mathscr{F}$ is locally free.

Let $\omega_X = f^!\omega_P := \mathscr{H}om_{\mathcal{O}_P}(f_*\mathcal{O}_X, \omega_P)$. Then $\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i(P, f_*\mathscr{F})$ is dual to $\mathrm{H}^{n-i}(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \omega_X)) = \mathrm{H}^{n-i}(P, \omega_P \otimes (f_*\mathscr{F})^\vee)$. Here, we note that

$$\omega_P \otimes (f_*\mathscr{F})^\vee = \mathscr{H}om(f_*\mathscr{F}, \omega_P) = \mathscr{H}om(\mathscr{F}, f^!\omega_P) = \mathscr{H}om(\mathscr{F}, \omega_X).$$

■

Corollary 16.6. Let X and $\mathscr{F}$ be as in (16.5). Let D be an ample Cartier divisor on X. Then $\mathscr{F}$ is C–M if and only if $\mathrm{H}^i(X, \mathscr{F}(-rD)) = 0$ for every $i < n$ and $r \gg 1$.

Proof. If $\mathscr{F}$ is C–M, then

$$\mathrm{H}^i(X, \mathscr{F}(-rD)) = \mathrm{H}^{n-i}(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \omega_X)(rD))^\vee = 0$$

for $n - i > 0$ and $r \gg 1$.

Conversely, we induction on n. The $n = 0$ case is trivial. For $n > 0$, take any $x \in X$. Since $\mathrm{h}^0(\mathscr{F}(-rD)) = 0$, there does not exist any subsheaf $\mathscr{F}'$ of $\mathscr{F}$ with support

$\{x\}$ (otherwise $\mathrm{h}^0(\mathscr{F}'(-rD)) = 0$). For $r' \gg 0$, there exists $s \in \mathrm{H}^0(\mathcal{O}_X(r'D))$ such that $s(x) = 0$ and s does not vanish at any associated point of $\mathscr{F}$ (this is a finite set). Then $s\colon \mathscr{F} \to \mathscr{F}(r'D)$ is injective. Set $Y = (s = 0)$. Then it is easy to see that $\mathrm{H}^i(Y, \mathscr{F}_Y(-rD)) = 0$ for $i < n-1$, $r \gg 1$. Since $\mathscr{F}_Y$ is C–M by induction hypothesis and thus $\mathscr{F}$ is C–M at x by (16.3). ■

17 Rational singularities

Definition 17.1. Let $f\colon X \to Y$ be a resolution of singularity (over characteristic 0 field $\mathbf{k}$). We say $f\colon X \to Y$ is a **rational resolution** if $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = \mathcal{O}_Y$ for $i > 0$. We say that Y has rational singularities if every resolution is rational.

Remark. For $\operatorname{char} \mathbf{k} = p > 0$, one need to assume also that $R^i f_*\omega_X = 0$ for $i > 0$. In $\operatorname{char} \mathbf{k} = 0$, this holds by Grauert–Riemenschneider vanishing.

Theorem 17.2 (Kempf)**.** Let Y be a variety over an algebraically closed field $\mathbf{k}$ of characteristic 0. Then the followings are equivalent:

(1) Y has rational singularities;

(2) there exists a rational resolution of Y;

(3) Y is C–M and there exists a resolution $f\colon X \to Y$ such that $f_*\omega_X = \omega_Y$.

Remark. Let X, Y e projective scheme of pure dimension n, $f\colon X \to Y$ a generically finite morphism.

For each coherent sheaf $\mathscr{F}$ on X, we have $0 \le \dim \operatorname{Supp} R^j f_*\mathscr{F} \le n - j - 1$ for $j > 0$, so $\mathrm{H}^i(X, R^j f_*\mathscr{F}) = 0$ for $i + j \ge n$, $j > 0$ by Grothendieck vanishing.

Consider the Leray spectral sequence $E_2^{ij} = H^i(Y, R^j f_*\mathscr{F}) \Rightarrow H^{i+j}(X, \mathscr{F})$, we then get a surjection

$$\mathrm{H}^n(Y, f_*\mathscr{F}) \longrightarrow \mathrm{H}^n(X, \mathscr{F}).$$

Apply this to $\mathscr{F} = \omega_X$ and using the duality, we get the injection

$$\mathrm{H}^0(X, \mathscr{H}om(\omega_X, \omega_X)) \longrightarrow \mathrm{H}^0(X, \mathscr{H}om(f_*\omega_X, \omega_Y)).$$

The relative trace map $\mathrm{tr}_{X/Y}\colon f_*\omega_X \to \omega_Y$ is defined to be the image of id_{ω_X}.

The higher direct images $R^p f_*\mathcal{O}_X$ and $f_*\omega_X$ are in fact independent of the resolution $f\colon X \to Y$.

Proof of (17.2). We prove only the case when Y is projective. By the remark above, we see that (1) is equivalent to (2). Let $f\colon X \to Y$ be a resolution and D an ample Cartier divisor on Y. Then (5.5) implies that

$$\mathrm{H}^{n-i}(X, \mathcal{O}_X(-rf^*D))^\vee = \mathrm{H}^i(X, \omega_X(rf^*D)) = 0$$

for $i > 0$ and $r > 0$.

Consider the spectral sequence

$$E_2^{ij} = \mathrm{H}^i(X, R^j f_*\mathcal{O}_X \otimes \mathcal{O}_Y(-rD)) \Rightarrow \mathrm{H}^{i+j}(X, \mathcal{O}_X(-rf^*D)).$$

Suppose (2) and let f be a rational resolution. Then $R^j f_*\mathcal{O}_X$, and hence E_2^{ij}, is 0 for $j > 0$. This shows that the spectral sequence degenerates at E_2. Hence,

$$\mathrm{H}^i(Y, \mathcal{O}_Y(-rD)) = \mathrm{H}^i(X, \mathcal{O}_X(-rf^*D)) = 0.$$

By (16.6), Y is C–M.

When $i = n$, we get by duality that

$$\begin{aligned}
\mathrm{H}^0(Y, \omega_Y \otimes \mathcal{O}_Y(rD))^\vee &= \mathrm{H}^n(Y, \mathcal{O}_Y(-rD)) \\
&\cong \mathrm{H}^n(X, \mathcal{O}_X(-rf^*D)) \\
&= \mathrm{H}^0(X, \omega_X \otimes \mathcal{O}_X(rf^*D))^\vee \\
&= \mathrm{H}^0(Y, f_*(\omega_X \otimes \mathcal{O}_X(rf^*D)))^\vee = \mathrm{H}^0(Y, f_*\omega_X \otimes \mathcal{O}_X(rD))^\vee
\end{aligned}$$

So, if we choose r such that $\omega_Y \otimes \mathcal{O}_Y(rD)$ and $f_*\omega_X \otimes \mathcal{O}_X(rD)$ are globally generated, then we see that $\mathrm{tr}_{X/Y}\colon f_*\omega_X \to \omega_Y$ is an isomorphism. This proves that (2) implies (3).

For the converse, we induction on $n = \dim Y$. Note that $R^n f_*\mathcal{O}_X = 0$. We claim that $R^i f_*\mathcal{O}_X = 0$ outside a zero dimensional subset for all $i > 0$. Let H be a general hyperplane, which is a C–M scheme, and let $H' = H \times_Y X$. Then

$$f_*\omega_{H'} = f_*(\omega_X(H') \otimes \mathcal{O}_{H'}) = f_*\omega_X \otimes \mathcal{O}_H(H) = \omega_Y \otimes \mathcal{O}_H(H) = \omega_H.$$

By induction, $\mathcal{O}_H \otimes R^i f_* \mathcal{O}_X = R^i f_* \mathcal{O}_{H'} = 0$ for all $i > 0$. So

$$R^i f_* \mathcal{O}_Y \otimes \mathcal{O}_X(-H) \longrightarrow R^i f_* \mathcal{O}_X$$

is surjective. Then by Nakayama lemma, $\operatorname{Supp} R^i f_* \mathcal{O}_X \cap H = \varnothing$. This proves our claim.

This shows that $E_2^{ij} = 0$ for i, $j > 0$. For $j = 0$, $E_2^{i0} = \mathrm{H}^i(Y, f_*\mathcal{O}_X \otimes \mathcal{O}_Y(-rD)) = 0$ (by $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, we will prove this later) for $i < n$ and $r \gg 1$ since Y is C–M. Leray spectral sequence then implies that for $j < n-1$,

$$\mathrm{H}^0(Y, R^j f_* \mathcal{O}_X \otimes \mathcal{O}_Y(-rD)) = E_2^{0j} \cong H^j = H^j(X, \mathcal{O}_X(-rf^*D)) = 0$$

by (5.5). Since $\dim \operatorname{Supp} R^j f_* \mathcal{O}_X = 0$, we see that $R^j f_* \mathcal{O}_X = 0$ for $0 < j < n-1$.

Also, we have

$$\begin{array}{cccc} E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} & E_n^{n,0} \xrightarrow{\alpha} & \operatorname{coker}(d_n^{0,n-1}) \longrightarrow 0 \\ & & \\ E_2^{0,n-1} & E_2^{n,0} & \end{array}$$

■

18 Terminalization of canonical threefolds

Let X be an algebraic (resp. analytic) threefold with at worst canonical singularities. Then

$$e(X) := \#\{\text{exceptional divisor } E \text{ over } X \text{ with } a(E, X) = 0\}$$

is a finite number by [KM, Prop 2.36].

Theorem 18.1 (Reid)**.** There exists a crepant projective partial resolution $\pi_X \colon X^{\mathrm{ter}} \to X$ such that X^{ter} has only terminal singularities.

Remark. The Reid's terminalization construction is functorial for open embedding, and copatible with $(-)^{\mathrm{an}}$, i.e., $(\pi_X)^{\mathrm{an}} = \pi_{X^{\mathrm{an}}}$.

Sketch of proof. We use induction on $e(X)$. The base case $e(X) = 0$ is trivial since this implies X is already terminal.

For general case, let $p \in \operatorname{Sing} X$ and take an index 1 cover $(\widetilde{X}, \widetilde{p}) \to (X, p)$. There are three cases:

(a) $(\widetilde{X}, \widetilde{p})$ is NOT a cDV point for some $p \in \operatorname{Sing} X$;

(b) $(\widetilde{X}, \widetilde{p})$ is a cDV point for all $p \in \operatorname{Sing} X$ and $\dim \operatorname{Sing} X = 1$;

(c) $(\widetilde{X}, \widetilde{p})$ is a cDV point for all $p \in \operatorname{Sing} X$ and $\dim \operatorname{Sing} X = 0$.

For (a) and (b), we will find a crepant projective birational map $f \colon Y \to X$ such that $e(Y) < e(X)$, and we are done by induction. For (c), this implies X is terminal.

For (a), consider the Galois group $G = \operatorname{Gal}(\widetilde{X}/X)$. Let $\widetilde{f} \colon \widetilde{Y} \to \widetilde{X}$ be the weighted blow-up constructed in (**??**), where

$$w = \operatorname{wt}(x, y, z, t) = \begin{cases} (3,2,1,1) & \text{if } k = 1, \\ (2,1,1,1) & \text{if } k = 2, \\ (1,1,1,1) & \text{if } k \geq 3. \end{cases}$$

The ideas $\mathfrak{m}^w(n)$ are G-invariant, so we can descend the crepant resolution $\widetilde{f}$ to $f \colon Y = \widetilde{Y}/G \to X$, which is also crepant.

For (b), let C be an 1-dimensional irreducible component of $\operatorname{Sing} X$ with its reduced structure. Let I be the defining ideal of C. For $\nu \in \mathbb{Z}_{\geq 0}$, let $I^{(\nu)}$ be the ν^{th} symbolic power of I, i.e., the ideal sheaf consisting of germs of functions that have multiplicity at least ν at a general point of C. Consider $f \colon Y = \operatorname{Proj}_X(\bigoplus I^{(\nu)}) \to X$. Then Y is canonical, $K_Y = f^* K_X$ and every fiber $f^{-1}(x)$ are of dimension at most 1, and equals to 1 if $x \in C$.

For (c), we see that if $(\widetilde{X}, \widetilde{p})$ is terminal, then (X, p) is terminal. ■

19 Quotient singularities over $\mathbb{C}$

Recall that a singularity (X, x) is a quotient singularity if there exists a smooth germ $(Y, 0)$ and a finite group G acting on $(Y, 0)$ such that $(X, x) \cong (Y, 0)/G$.

Let (X, x) be a terminal threefold singularity of index r. Take an index 1 cover $(Y, 0) \to (X, x)$, where $\operatorname{Gal}(Y/X) \cong \mu_r$. Then $(Y, 0)$ is an isolated cDV point or a smooth point.

A group G giving a quotient singularity can be a linear group:

Theorem 19.1. Let (X,x) be a quotient singularity of dimension n. Then there exists a finite subgroup $G' \leq \mathrm{GL}(n,\mathbb{C})$ such that $(X,x) \cong (\mathbb{A}^n, 0)/G'$ as germs. In particular, a quotient singularity is algebraic.

Proof. By definition, there exists a smooth germ (Y,y) and a finite group G such that $(X,x) \cong (Y,y)/G$. Without loss of generality, we may assume the stabilizer G_y is G (otherwise, we take an analytic neighbourhood Y' of y such that $Y' \cap G_y = \{y\}$, then $(X,x) \cong (Y',y)/G_y$).

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{Y,y}$. Since $G = G_y$, $\mathfrak{m}$ is invariant under the action of G. (Algebraically, the stabilizer $G_y = (G \times \{y\}) \times_Y \operatorname{Spec} \kappa(y)$.) This defines a representation $\rho\colon G \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$. Take a regular system of parameters $z^1, \ldots, z^n$ of $\mathfrak{m}$ so that $z^1, \ldots, z^n$ forms a basis of $\mathfrak{m}/\mathfrak{m}^2$. Then $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ is isomorphic to $\mathrm{GL}(n,\mathbb{C})$. Write

$$\rho(g) \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} = \begin{pmatrix} g(z^1) \\ \vdots \\ g(z^n) \end{pmatrix}.$$

Define $y^i \in \mathcal{O}_{Y,y}$ by

$$Y = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = \frac{1}{\#G} \sum_{g \in G} \rho(g^{-1}) \begin{pmatrix} g(z^1) \\ \vdots \\ g(z^n) \end{pmatrix}.$$

Then $y^i - z^i \in \mathfrak{m}^2$ and hence $y^1, \ldots, y^n$ form a regular system of parameters of $\mathfrak{m}$. Write

$$Z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}.$$

For each $h \in G$,

$$\begin{aligned} (\#G) \cdot h \cdot Y &= \sum_{g \in G} \rho(g^{-1}) h \circ g \circ Z = \sum_{g \in G} \rho(h(gh)^{-1})(gh) \circ Z \\ &= \sum_{g \in G} \rho(h)\rho(g^{-1}) g \circ Z = (\#G) \cdot \rho(h) \cdot Y. \end{aligned}$$

So G acts on Y is linear with respect to $y^1, \ldots, y^n$, and we set $G' = \rho(G)$. ■

Remark. Even if (Y,y) is singular with embedding dimension e, there exists a finite subgroup $\rho(G) \leq \mathrm{GL}(e,\mathbb{C})$ such that $(X,x) \cong (Y,0)/\rho(G) \subseteq (\mathbb{A}^e, 0)/\rho(G)$.

Definition 19.2. An element $g \in \mathrm{GL}(n, \mathbb{C}) \setminus \{\mathrm{id}\}$ is a **pseudo-reflection** (p-rf) if $\operatorname{ord} g < \infty$ and $\operatorname{Fix} g = \{x \in \mathbb{C}^n \mid gx = x\}$ has codimension 1.

A subgroup of $\mathrm{GL}(n, \mathbb{C})$ generated by p-rfs is called a p-rf group. A finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ is **small** if it does not contain p-rfs.

Theorem 19.3 (Chevalley–Shephard–Todd)**.** If $H \leq \mathrm{GL}(n, \mathbb{C})$ is a finite p-rf group, then $\mathbb{A}^n/H \cong \mathbb{A}^n$.

Corollary 19.4. If (X, x) is a quotient singularity of dimension n, then there exists a small finite subgroup $G \leq \mathrm{GL}(n, \mathbb{C})$ such that $(X, x) \cong (\mathbb{A}^n, 0)/G$.

Proof. Let $G' \leq \mathrm{GL}(n, \mathbb{C})$ be a finite group such that $(X, x) \cong (\mathbb{A}^n, 0)/G'$. Let H be the subgroup of G' generated by p-rfs in G'. Then H is a normal subgroup: for $g \in G'$ and $h \in H$ a p-rf, we have

$$\operatorname{Fix}(ghg^{-1}) = g \cdot \operatorname{Fix}(h).$$

Consider $G = G'/H$ that acts on $\mathbb{A}^n/H \cong \mathbb{A}^n$. Then

$$\mathbb{A}^n/G \cong \frac{\mathbb{A}^n/H}{G'/H} \cong \mathbb{A}^n/G'.$$

Now, we need to check that G is small. For $g \in G' \setminus H$, we have

$$\operatorname{codim} \operatorname{Fix}(hg) \geq 2$$

for all $h \in H$. Let $\phi\colon \mathbb{A}^n \to \mathbb{A}^n/H$ be the projection. Then $gH \cdot \phi(x) = \phi(x)$ if and only if $gx = hx$ for some $h \in H$. So

$$\operatorname{Fix}(gH) = \bigcup_{h \in H} \phi(\operatorname{Fix}(hg))$$

has codimension at least 2 since ϕ is a finite morphism. ■

Corollary 19.5. Let $(Y, 0) \subseteq (\mathbb{A}^{n+1}, 0)$ be an n-dimensional hypersurface singularity. Let G be a finite group acting on Y. Then there exists a small, finite group $G' \leq \mathrm{GL}(n+1, \mathbb{C})$ acting on Y such that

$$(X, 0) \cong (Y, 0)/G \cong (Y, 0)/G' \subseteq (\mathbb{A}^n, 0)/G$$

Corollary 19.6. A quotient singularity is log terminal.

Proof. Let (X, x) be a quotient singularity. There exists a small finite group $G \leq \mathrm{GL}(n, \mathbb{C})$ such that $(X, x) \cong (\mathbb{A}^n, 0)/G$. Since G is small, G acts on $\mathbb{A}^n$ freely in codimension 1, so the projection $\mathbb{A}^n \to \mathbb{A}^n/G$ is crepant. Since $\mathbb{A}^n$ is smooth, $\mathbb{A}^n$, and hence $\mathbb{A}^n/G$, is log terminal. ■

Definition 19.7. Let $(Y, 0)$ be a smooth germ admitting a cyclic action μ_r which fixes 0. We fix a primitive character $\chi \colon \mu_r \to \mathbb{C}^\times$. We say $f \in \mathcal{O}_{Y,0}$ is **semi-invariant** with respect to χ if there exists $\mathrm{wt}(f) \in \mathbb{Z}/r\mathbb{Z}$, called the weight of f with respect to χ, such that $g(f) = \chi(g)^{\mathrm{wt}(f)} f$ for all $g \in \mu_r$.

We write $\mathbb{A}^n/\frac{1}{r}(a^1, \ldots, a^n)$ for the quotient $\mathbb{A}^n/\mu_r$ in which every coordinate x^i of $\mathbb{A}^n$ is semi-invariant with $\mathrm{wt}(x^i) = a^i$. Explicitly,

$$\mu_r = \left\langle \begin{pmatrix} \zeta^{a^1} & & \\ & \ddots & \\ & & \zeta^{a^n} \end{pmatrix} \right\rangle \leq \mathrm{GL}(n, \mathbb{C}),$$

where ζ is a primitive r^{th} roots of unity.

A (small) cyclic quotient singularity of type $\frac{1}{r}(a^1, \ldots, a^n)$ is a singularity (analytically) isomorphic to $(\mathbb{A}^n/\frac{1}{r}(a^1, \ldots, a^n), 0)$ (with μ_r small).

Remark. If $\gcd(b, r) = 1$, then $\mathbb{A}^n/\frac{1}{r}(a^1, \ldots, a^n) \cong \mathbb{A}^n/\frac{1}{r}(ba^1, \ldots, ba^n)$ by taking χ^b instead of χ.

Lemma 19.8. Let $\overline{M} = \mathbb{Z}_n$ with dual lattice $\overline{N} = \mathbb{Z}^n$ and let $N = \overline{N} + \frac{1}{r}(a^1, \ldots, a^n) \cdot \mathbb{Z}$. Let $e_1, \ldots, e_n$ be the standard basis of $N_\mathbb{R}$, Σ the fan consisting faces of the cone $\sigma = \mathrm{Cone}(e_1, \ldots, e_n)$. Then $\mathbb{A}^n/\frac{1}{r}(a^1, \ldots, a^n)$ is isomorphic to X_Σ.

Proof. An element $\alpha \in N_\mathbb{R}$ lies in N if and only if $\alpha \equiv \frac{1}{r}(ka^1, \ldots, ka^n) \pmod{\mathbb{Z})^n}$ for some k. So an element $m = (m_1, \ldots, m_n) \in \overline{M}_\mathbb{R}$ lies in $M = N^\vee$ if and only if $\frac{1}{r} \sum a^i m_i \in \mathbb{Z}$.

Since $\mathbb{A}^n = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \overline{M}]$ and the affine coordinate ring of $\mathbb{A}^n/\frac{1}{r}(a^1, \ldots, a^n)$ is

$\mathbb{C}[x^1, \ldots, x^n]^G$ where

$$G = \left\langle g = \begin{pmatrix} \zeta^{a^1} & & \\ & \ddots & \\ & & \zeta^{a^n} \end{pmatrix} \right\rangle,$$

we only need to check $\mathbb{C}[\sigma^\vee \cap \overline{M}]^G = \mathbb{C}[\sigma^\vee \cap M]$. For each $m \in \overline{M}$, $g \cdot x^m = \zeta^{\sum a^i m_i} x^m$. So x^m is G-invariant if and only if $\frac{1}{r} \sum a^i m_i \in \mathbb{Z}$, i.e., $m \in M$, as desired. ■

Definition 19.9. An element $e \in N \setminus \{0\}$ is primitive if $\mathbb{Z} \cdot e = N \cap R \cdot e$ in $N_\mathbb{R}$.

Fix a primitive element $e = \frac{1}{r}(w^1, \ldots, w^n) \in N \cap \sigma$. Let $\Sigma^*(e)$ be the star subdivision of Σ at e, i.e., the fan in $N_\mathbb{R}$ consisting of faces of $\mathrm{Cone}(e_1, \ldots, e_{i-1}, e, e_i, \ldots, e_n)$ for $1 \leq i \leq n$. The weighted blow-up of $X = \mathbb{A}^n / \frac{1}{r}(a^1, \ldots, a^n)$ with weights $\mathrm{wt}(x^1, \ldots, x^n) = e$ is

$$\pi \colon B = X_{\Sigma^*(e)} \longrightarrow X = X_\Sigma.$$

It is isomorphic outside $\bigcap_{w^i \neq 0} (x^i = 0)$ in X.

Let E be the exceptional divisor of π.

Proposition 19.10. Suppose that $\mu_r \subseteq \mathrm{GL}(n, \mathbb{C})$ is small. Then

$$K_B = \pi^* K_X + \left(\frac{1}{r} \sum w^i - 1 \right) E,$$

i.e., $a(E, X) = \frac{1}{r} \sum w^i - 1$.

Fix $r \in \mathbb{N}$. For $k \in \mathbb{Z}$, let $\overline{k} = k - \lfloor \frac{k}{r} \rfloor r$.

Theorem 19.11 (Reid–Tai criterion)**.** A small cyclic quotient singularity of type $\frac{1}{r}(a^1, \ldots, a^n)$ is terminal (resp. canonical) if and only if $\frac{1}{r} \sum \overline{ka^i} > 1$ (resp. ≥ 1) for all $0 < k < r$.

Proof. Use the description of $X = \mathbb{A}^n / \frac{1}{r}(a^1, \ldots, a^n) = X_\Sigma$.

Suppose that X is terminal (resp. canonical). Then for $0 < k < r$, take a primitive element $\frac{1}{r}(b^1, \ldots, b^n)$ in N from the ray $\mathbb{R}^+ \cdot \frac{1}{r}(\overline{ka^1}, \ldots, \overline{ka^n})$. Let E_k be the excepsional divisor obtained by the weighted blow-up of X_Σ with $\mathrm{wt}(x^1, \ldots, x^n) = \frac{1}{r}(b^1, \ldots, b^n)$. Since

$$a(E_k, X) = \frac{1}{r} \sum b^i - 1 > 0 \quad (\text{resp. } \geq 0),$$

we get $\frac{1}{r} \sum \overline{ka^i} \geq \frac{1}{r} \sum \overline{b^i} > 1$ (resp. ≥ 1).

Conversely, take a unimodular subdivision of X_Σ which provides a log resolution of X_Σ [CLS, Thm 11.2.2]. Every exceptional prime divisor E corresponds to a ray in $N_\mathbb{R}$ generated by a primitive element $\alpha = \frac{1}{r}(c^1, \dots, c^n)$ in which at least two of $c^i > 0$.

Recall that $\alpha \in \sigma \cap N \subseteq N$ if and only if $\alpha \equiv \frac{1}{r}(ka^1, \dots, ka^n) \pmod{\mathbb{Z}^n}$ for some $0 \le k < r$, i.e., $(c^1, \dots, c^n) \equiv (ka^1, \dots, ka^n) \pmod{r}$. This implies $c^i \ge \overline{ka^i}$ for all i.

Now,

$$a(E, X) = \frac{1}{r}\sum c^i - 1 \ge \begin{cases} 1 & \text{if } k = 0, \\ \frac{1}{r}\sum \overline{ka^i} - 1 & \text{if } 0 < k < r. \end{cases}$$

So $\frac{1}{r}\sum \overline{ka^i} > 1$ (resp. ≥ 1) implies $a(E, X) > 0$ (resp. ≥ 0) for any exceptional divisor E, i.e., X is terminal (resp. canonical). ■

In 3 dimension, the condition of Reid–Tai criterion is well-understood:

Theorem 19.12 (White)**.** Let $r \in \mathbb{N}$, a^1, a^2, $a^3 \in \mathbb{Z}$. If $\overline{ka^1} + \overline{ka^2} + \overline{ka^3} > r$ for all $0 < k < r$, then $r \mid a^i + a^j$ for some $1 \le i < j \le 3$.

Remark. When $r = 1$, 2, this is trivial. Without loss of generality, we may assume $r \ge 3$. We explain an abstract approach due to [Morrison and Stevens, Terminal quotient singularities in dimension three and four]. The proof by [White, Lattice tetrahedra] is in 1964.

Definition 19.13. A **Dirichlet character** of $(\mathbb{Z}/r\mathbb{Z})^\times$ is a group homomorphism

$$\chi\colon (\mathbb{Z}/r\mathbb{Z})^\times \to \mathbb{C}^\times.$$

We sometimes extend it to $\mathbb{Z} \to \mathbb{C}$ by letting $\chi(n) = 0$ if $\gcd(n, r) > 1$.

The **conductor** f of χ is the minimal number $r' \mid r$ such that χ factors through $(\mathbb{Z}/r\mathbb{Z})^\times \to (\mathbb{Z}/r'\mathbb{Z})^\times$.

We say χ is even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Definition 19.14. We define $B_1 : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ by letting

$$B_1(q) = \begin{cases} q - \lfloor q \rfloor - \frac{1}{2} & \text{if } q \notin \mathbb{Z} \\ 0 & \text{if } q \in \mathbb{Z}. \end{cases}$$

Note that B_1 is an odd function.

We define the generalized Bernoulli number

$$B_{1,\chi} = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^\times} \chi(a) B_1\left(\frac{a}{f}\right) \in \mathbb{C}.$$

Proposition 19.15 (Dirichlet)**.** If χ is odd, then $B_{1,\chi} \neq 0$.

Sketch of proof. Consider the Dirichlet function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \operatorname{Re} s > 1.$$

It may be continued analytically to $\mathbb{C}$ except for a simple pole at $s = 1$ when $\chi = 1$.

We see that $L(1,\chi) \neq 0$ when $\chi \neq 1$: regarding

$$\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) \cong (\mathbb{Z}/r\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times,$$

we let $K = \mathbb{Q}(\zeta_r)^{\ker \chi}$. Consider the Dedekind zeta function

$$\zeta_K(s) = \prod_{a=0}^{b-1} L(s,\chi^a),$$

where b is the order of χ. This function has a simple pole at $s = 1$, so none of the factors $L(s,\chi^a)$, $a > 0$, can vanish at $s = 1$. In particular, $L(1,\chi) \neq 0$.

From functional equation, we have

$$L(1-n,\chi) \neq 0$$

when $n \in \mathbb{N}$ and n is even (resp. odd) if χ is even (resp. odd).

One can define

$$\sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

When $\chi = 1$, $B_{n,1}$ is the ordinary Bernoulli number B_n. Since

$$B_{n,\chi} = f^{n-1} \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^\times} \chi(a) B_n\left(\frac{a}{f}\right),$$

where

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}.$$

By a contour integration, one can see that

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}.$$

Hence if χ is odd, by taking $n = 1$,

$$B_{1,\chi} = -L(0,\chi) \neq 0. \qquad \blacksquare$$

Definition 19.16. Let $V = \mathbb{C}[(\mathbb{Z}/r\mathbb{Z})^\times]$, the group algebra of $(\mathbb{Z}/r\mathbb{Z})^\times$ over $\mathbb{C}$ generated by σ_a, $a \in (\mathbb{Z}/r\mathbb{Z})^\times$. For $q \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$, define

$$S(q) = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^\times} B_1(aq)\sigma_a \in V.$$

Let W be the vector subspace of V generate by

$$\{S(q) \mid q \in \tfrac{1}{r}\mathbb{Z}/\mathbb{Z}\}.$$

Let $W^\perp \subseteq V^\vee$ be the orthogonal complement with respect to the natural pairing $V \otimes V^\vee \to \mathbb{C}$.

For each $a \in (\mathbb{Z}/r\mathbb{Z})^\times$, let $\lambda_a = \sigma_a^\vee + \sigma_{-a}^\vee \in V^\vee$ so that

$$\lambda_a(S(q)) = B_1(aq) + B_1(-aq) = 0.$$

Hence $\lambda_a \in W^\perp$.

Theorem 19.17. The $\mathbb{C}$-vector space $W^\perp$ is generated by $\{\lambda_a \mid a \in (\mathbb{Z}/r\mathbb{Z})^\times\}$. In particular, $\dim W^\perp = \dim W = \phi(r)/2$.

Proof. Let $\chi\colon (\mathbb{Z}/r\mathbb{Z})^\times \to \mathbb{C}^\times$ be an arbitrary character with conductor f. Using

$$\mathbb{Z}/f\mathbb{Z} \cong \tfrac{1}{f}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z},$$

we define

$$w_\chi = \sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times} \chi(a) S\left(\frac{a}{f}\right) = \sum_{b \in (\mathbb{Z}/r\mathbb{Z})^\times} \left(\sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times} \chi(a) B_1\left(\frac{ba}{f}\right)\right) \sigma_b \in W.$$

The coefficient of σ_1 is $B_{1,\chi} \neq 0$ if χ is odd.

Notice that there is an representation $\rho\colon (\mathbb{Z}/r\mathbb{Z})^\times \to \mathrm{GL}(V)$ by letting $\rho(a)(\sigma_b) = \sigma_{ab}$. We see that

$$\begin{aligned} \rho(c)(\omega_\chi) &= \sum_{b \in (\mathbb{Z}/r\mathbb{Z})^\times} \left(\sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times} \chi(a) B_1\left(\frac{ba}{f}\right)\right) \sigma_{cb} \\ &= \sum_{b \in (\mathbb{Z}/r\mathbb{Z})^\times} \left(\sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times} \chi(a) B_1\left(\frac{c^{-1}ba}{f}\right)\right) \sigma_b \\ &= \sum_{b \in (\mathbb{Z}/r\mathbb{Z})^\times} \left(\sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times} \chi(ca) B_1\left(\frac{ba}{f}\right)\right) \sigma_b = \chi(c) w_\chi. \end{aligned}$$

So w_χ lies in the χ-eigenspace of the $(\mathbb{Z}/r\mathbb{Z})^\times$-action. So all non-zero w_χ are linearly independent. Hence,

$$\{w_\chi \mid \chi \text{ is odd}\}$$

is a linearly independent subset of W, and hence $\dim W \geq \phi(r)/2$.

On the other hand,

$$\operatorname{codim}_V W = \dim W^\perp \geq \dim\langle \lambda_a \mid a \in (\mathbb{Z}/r\mathbb{Z})^\times \rangle = \frac{\phi(r)}{2}.$$

Since $\dim V = \phi(r)$, we get the result. ■

Proof of (19.12). Write $a(k) = \overline{ka^1} + \overline{ka^2} + \overline{ka^3}$. Observe that

$$\overline{kb} + \overline{(r-k)b} = \begin{cases} r & \text{if } \overline{kb} \neq 0, \\ 0 & \text{if } \overline{kb} = 0. \end{cases}$$

So $a(k) + a(r-k) = r \cdot \#\{i \mid \overline{ka^i} \neq 0\}$.

By the assumption, $a(k) > r$ for all $0 < k < r$, so

$$a(k) + a(r-k) > 2r,$$

and hence equal to $3r$. This implies $\overline{ka^i} + \overline{(r-k)a^i} = r$ for all $0 < k < r$. Hence, $\gcd(a^i, r) = 1$ for all i.

Furthermore, since $r < a(k) < 2r$,

$$a(k) = \overline{k(a^1 + a^2 + a^3)} + r$$

and thus $\gcd(a^1 + a^2 + a^3, r) = 1$. This equation can be rewritten as

$$\sum \left(\overline{ka^i} - \frac{r}{2} \right) = \overline{k(a^1 + a^2 + a^3)} - \frac{r}{2}.$$

So

$$\sum B_1(a^i q) = B_1((a^1 + a^2 + a^3)q)$$

for each $q \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$. Consider

$$\mu = \sum \sigma^\vee_{a^i} - \sigma^\vee_{a^1+a^2+a^3} \in V.$$

Since

$$\mu(S(q)) = \sum B_1(a^i q) - B_1((a^1 + a^2 + a^3)q) = 0,$$

we get $\mu \in W^\perp = \langle \sigma^\vee_a + \sigma^\vee_{-a} \mid a \in (\mathbb{Z}/r\mathbb{Z})^\times \rangle$. Hence, $a^1 + a^2 + a^3 \equiv a^k$ for some k, as desired. ■

Theorem 19.18 (Terminal Lemma)**.** A cyclic quotient threefold singularity $(X, 0)$ is terminal if and only if it is of type $\frac{1}{r}(1, -1, b)$ for some $\gcd(r, b) = 1$ and $\mu_r \subseteq \mathrm{GL}(3)$ is small.

Proof. If $(X, 0)$ is of type $\frac{1}{r}(1, -1, b)$, then

$$\sum \overline{ka^i} = k + (r - k) + \overline{kb} > r$$

for all $0 < k < r$. Hence, by (19.11), $(X, 0)$ is terminal.

Conversely, if $(X, 0) = \mathbb{A}^3/\frac{1}{r}(a^1, a^2, a^3)$ is terminal, then $(X, 0)$ is isolated. So μ_r acts on X freely outside 0, i.e., the action is small. This shows that $\gcd(r, a^i) = 1$. By (19.11) and (19.12), we may assume that $a^1 + a^2 = r$ after permutation. Take $a \in \mathbb{Z}$ such that $\overline{aa^1} = 1$. Then $(X, 0)$ becomes of type $\frac{1}{r}(\overline{aa^1}, \overline{aa^2}, \overline{aa^3}) = \frac{1}{r}(1, -1, b)$, in which $\gcd(b, r) = 1$. ■

20 Terminal singularities of higher index

Let $\mathfrak{m} \trianglelefteq \mathcal{O}_{\mathbb{A}^4,0}$ be the maximal ideal. Write $f = f_2 + f_3 + \cdots \in \mathfrak{m}$, where f_2 is the quadratic part of f, f_3 is the cubic part of f, and so on. Write, for example, $x_1x_2 \in f$ if the coefficient of x_1x_2 in f is nonzero.

Theorem 20.1 (Mori). Let $(Y,0) \subseteq (\mathbb{A}^4,0)$ be a cDV singularity (not smooth) defined by $f=0$ with an action μ_r with weight $\mathrm{wt}(x^1,\ldots,x^4) = (a^1,\ldots,a^4)$.

If $(X,0) = (Y,0)/\mu_r \subseteq \mathbb{A}^4/\frac{1}{r}(a^1,\ldots,a^4)$ is a terminal singularity, then one of the followings holds after changing expression of type $\frac{1}{r}(a^1,\ldots,a^4)$ and orbifold coordinates x^1, …, x^4:

Name/ind.	type of action	f	condition
cA$/r$	$\frac{1}{r}(1,-1,0,b)$	$x^1x^2 + g(x^3,(x^4)^n)$	$g \in \mathfrak{m}^2$, $\gcd(b,r)=1$
cA$x/4$	$\frac{1}{4}(1,3,2,1)$	$(x^1)^2+(x^2)^2+g(x^3,(x^4)^2)$	$g \in \mathfrak{m}^2$
cA$x/2$	$\frac{1}{2}(1,0,1,1)$	$(x^1)^2+(x^2)^2+g(x^3,x^4)$	$g \in \mathfrak{m}^4$
cD$/3$	$\frac{1}{3}(0,1,2,2)$	$(x^1)^2+g(x^2,x^3,x^4)$	$g \in \mathfrak{m}^3$, $g_3 = \begin{cases} (x^2)^3+(x^3)^3+(x^4)^3 \\ (x^2)^3+x^3(x^4)^2 \\ (x^2)^3+(x^3)^3 \end{cases}$
cD$/2$	$\frac{1}{2}(1,1,0,1)$	$(x^1)^2+g(x^2,x^3,x^4)$	$g \in \mathfrak{m}^3$, $x^2x^3x^4$ or $(x^2)^2x^3 \in g$
cE$/2$	$\frac{1}{2}(1,0,1,1)$	$(x^1)^2+(x^2)^2+x^2g(x^3,x^4)+h(x^3,x^4)$	$g \in \mathfrak{m}^4$, $h_4 \neq 0$.

Remark. cA$/r$ should be considered as the main series and the other cases are the exceptional case.

Proposition 20.2 (RI). Let $A = \mathbb{A}^n/\frac{1}{r}(a^1,\ldots,a^n)$ with orbifold coordinate x^1, …, x^n, $(X,0)$ be the analytic subspace of A defined by a semi-invariant function $f \in \mathcal{O}_{\mathbb{A}^n,0}$. Suppose that μ_r acts on $(f=0)$ freely in codimension 1. Let $N = \mathbb{Z}^n + \frac{1}{r}(a^1,\ldots,a^n)\cdot\mathbb{Z}$, $\sigma = \mathrm{Cone}(e_1,\ldots,e_n) \subseteq N_{\mathbb{R}}$.

If $(X,0)$ is terminal (resp. canonical), then for each $\frac{1}{r}(b^1,\ldots,b^n) \in N \cap \sigma$ with $\#\{i \mid b^i > 0\} \geq 3$, the weighted order $\mathrm{ord}(f)$ of f with respect to $\mathrm{wt}(x^1,\ldots,x^n) = (b^1,\ldots,b^n)$ satisfies

$$\mathrm{ord}(f) + r < (\text{resp.} \leq) \sum b^i.$$

Remark. In Reid's notation ([YPG, p. 372] or [Ishii, Def. 8.3.10]), set $\alpha = \frac{1}{r}(b^1, \ldots, b^n)$ and

$$\alpha\left(\prod x^i\right) = \frac{1}{r}\sum b^i, \quad \alpha(f) = \frac{1}{r}\operatorname{ord}(f)$$

where $\alpha(g) = \min\{\langle \alpha, m\rangle \mid \chi^m \in g\}$ for $g \in \mathbb{C}[\sigma^\vee \cap M]$.

We shall demonstrate Mori's theorem in the case when $(Y, 0)$ is (an isolated) cA.

(Ri) If $d = \gcd(a^i, r) > 1$, then some power of $x^i \in f$. Indeed, the action of some element of μ_r fixes the x^i-action pointwisely, i.e., let $a^i = d(a^i)'$, $r = dr'$, then $\zeta^{r'} \cdot x^i = (\zeta^{r'})^{a^i} x^i = x^i$.

(Rii) We have $\gcd(a^i, a^j, r) = 1$ for distinct $1 \le i < j \le 4$. Otherwise the action is not free on $(f = 0) \cap (x^i x^j\text{-plane})$, which has positive dimension at the origin.

Remark. If σ is a coordinate change of $(\mathbb{A}^n, 0)$ i.e., $\sigma \in \operatorname{Aut}(\mathcal{O}_{\mathbb{A}^n,0})$, then

$$\frac{1}{r}\sum_{i=0}^{r-1} \tau^{-i}\sigma\tau^i$$

is an orbifold coordinate change of μ_r acting on $\mathbb{A}^n$, where τ is given by a generator of μ_r.

Since $(Y, 0)$ is singular, cA, so $f \in \mathfrak{m}^2$ and f_2 has rank ≥ 2.

Lemma 20.3. After orbifold coordinate change, either

(i) $f = x^1x^2 + g(x^3, x^4)$, or

(ii) $f = (x^1)^2 + (x^2)^2 + g(x^3, x^4)$ with $x^3x^4 \notin g_2$.

Proof. We claim that $f_2 = x^1x^2 + g_2(x^3, x^4)$ or $(x^1)^2 + (x^2)^2 + h_2(x^3, x^4)$ with $x^3x^4 \notin h_2$. The lemma follows from the claim and Tougeron's implicit function theorem.

If $(x^i)^2 \in f_2$, then we can eliminate the linear term in x^i. After orbifold coordinate change, write $f_2 = p_2(x^1, \ldots, x^i) + (x^{i+1})^2 + \cdots (x^4)^2$ for some $0 \le i \le 4$ so that $(x^\ell)^2 \notin p_2$ for all $\ell = 1, \ldots, i$.

- If $p_2 = 0$, then $f_2 = \sum (x^i)^2$, and we get (ii).
- If $p_2 \neq 0$, we may assume $x^1x^2 \in p_2$ and thus $i \ge 2$. Write

$$f_2 \sim x^1x^2 + a(x^3, x^4)x^1 + b(x^3, x^4)x^2, c(x^3, x^4) = x^1x^2 + g(x^3, x^4),$$

i.e., (i). $\blacksquare$

When $f = x^1x^2 + g(x^3, x^4)$, we shall derive cA$/r$ or cA$x/4$: for $0 < k < r$, let

$$\mathrm{ord}_k(f)$$

be the weighted order of f with respect to $\mathrm{wt}(x^1, \dots, x^4) = (\overline{ka^1}, \dots, \overline{ka^4})$. By (Rii), $\#\{i \mid \overline{ka^i} > 0\} \geq 3$, so it follows from (20.2) that

$$\mathrm{ord}_k(f) + r < \sum \overline{ka^i} \qquad (\spadesuit)$$

for all $0 < k < r$. Let $p = \gcd(a^1 + a^2, r)$. Since $(x^1)^\ell$, $(x^2)^\ell \notin f$, using (Ri) we get $\gcd(a^i, r) = 1$ for $i = 1, 2$.

We claim that $p \mid a^3$ or $p \mid a^4$. Otherwise $0 < s = \frac{r}{p} < r$ and all $\overline{sa^i} > 0$. Without loss of generality, we may assume that $\overline{sa^3} + \overline{sa^4} \leq r$ (otherwise replace each a^i by $\overline{-a^i}$). Since $p \mid a^1 + a^2$, $\overline{sa^1} + \overline{sa^2} = r$. By ($\spadesuit$),

$$\mathrm{ord}_s(f) + r < \sum \overline{sa^i} \leq r + r = 2r.$$

However,

$$\mathrm{ord}_s(f) \equiv \mathrm{ord}_s(x^1x^2) \equiv s(a^1 + a^2) \equiv 0 \pmod{r},$$

a contradiction. So we may assume that $p \mid a^3$.

Next, we show that $q = \gcd(a^4, r) = 1$. Since $p \mid a^3$, $p \mid r$, and $\gcd(a^3, a^4, r) = 1$ by (Rii), we see that $\gcd(a^4, p) = 1$. If $q > 1$, then there exists $\ell > 0$ such that $(x^4)^\ell \in f$ by (Ri). Then

$$\ell a^4 = \mathrm{wt}((x^4)^\ell) \equiv \mathrm{wt}(x^1x^2) = a^1 + a^2 \pmod{r},$$

and hence $q \mid a^1 + a^2$. But then $q \mid \gcd(a^1 + a^2, p)$, a contradiction.

Let $\gcd(a^3, r) = pg$. If $g > 1$, then some power of $x^3 \in f$, so $pg \mid a^1 + a^2$ (as before). This implies $pg \mid p$, a contradiction. Hence, $\gcd(a^3, r) = p = \gcd(a^1 + a^2, r)$.

One can now write

$$f = x^1x^2 + g(x^3, (x^4)^p), \quad g \in \mathfrak{m}^2.$$

If $p = r$, then

$$(a^1, a^2, a^3, a^4) \equiv (a^1, -a^1, 0, a^4) \pmod{r},$$

where a^1, a^4 are coprime to r. We get the cA$/r$ case.

Now we shall assume that $p < r$. If $k(a^1 + a^2) \equiv \pm p \pmod r$ for some $0 < k < r$, then $\overline{ka^i}$ for each $i = 1, 2, 3, 4$. Indeed, since

$$k(a^1 + a^2) \equiv \pm p \not\equiv 0 \pmod r,$$

$r \nmid k$ and hence $\gcd(k, r/p) = 1$. So $r \nmid ka^i$ for $i = 1, 2, 3, 4$ (note that $p = \gcd(a^3, r)$.

Let

$$\mathscr{S} = \{k \in \{1, \dots, r-1\} \mid k(a^1 + a^2) \equiv \pm p \pmod r,\ \overline{ka^3} + \overline{ka^4} \le r\}.$$

We see that

$$\#\mathscr{S} = \begin{cases} p & \text{if } r > 2p, \\ \frac{p}{2} & \text{if } r = 2p, \end{cases}$$

since the first condition gives $2p$, p solutions, respectively, and the second condition cuts the solutions into half (note that $\gcd(a^3, a^4, r) = 1$.

For $k \in \mathscr{S}$,

$$\mathrm{ord}_k(f) + r < \sum \overline{ka^i} \le \overline{ka^1} + \overline{ka^2} + r,$$

and

$$\mathrm{ord}_k(f) \equiv \mathrm{ord}_k(x^1x^2) \equiv \overline{ka^1} + \overline{ka^2} \pmod r.$$

So $\overline{ka^1} + \overline{ka^2} - \mathrm{ord}_k(f) = r$ and $\mathrm{ord}_k(f) \equiv \pm p \pmod r$. This implies $\mathrm{ord}_k(f) = p$ or $r - p$.

Let

$$\mathscr{S}_1 = \{k \in \mathscr{S} \mid \mathrm{ord}_k(f) = p\},$$
$$\mathscr{S}_2 = \{k \in \mathscr{S} \mid \mathrm{ord}_k(f) = r - p,\ r > 2p\}.$$

Then $\mathscr{S} = \mathscr{S}_1 \sqcup \mathscr{S}_2$ and $\mathscr{S}_2 = \varnothing$ if $r = 2p$.

If $k \in \mathscr{S}_1$, then

$$\mathrm{ord}_k(x^1x^2 + g(x^3, (x^4)^p)) = \mathrm{ord}_k(f) = p \ge 2.$$

Since $\mathrm{ord}_k(x^1x^2) > p$, $\mathrm{ord}_k(g) = p$. Since $g \in \mathfrak{m}^2$, we must have $(x^4)^p \in g$ and hence $\overline{ka^4} = 1$, i.e., there is at most one $0 < k < r$ such that $\overline{ka^4} = 1$.

If $k \in \mathscr{S}_2$, then $\overline{ka^1} + \overline{ka^2} = 2r - p$. So $r - p < \overline{ka^i} < r$ for $i = 1, 2$. Hence, by $\gcd(r, a^1) = 1$,

$$\#\mathscr{S}_2 \le \#((r-p, r) \cap \mathbb{N}) = p - 1.$$

Now, when $r = 2p$,

$$1 \le \frac{p}{2} = \#\mathscr{S} = \#\mathscr{S}_1 \le 1.$$

So $p = 2$ and $r = 4$. One can set

$$\mathrm{wt}(x^1, x^2, x^3, x^4) = (1, 1, 2, a^4)$$

with $a^4 = 1$ or 3. By (♠) with $k = 1$, we get

$$\mathrm{ord}_1(f) + 4 < 4 + a^4.$$

So $a^4 = 3$.

Since $\mathscr{S}_1 \ne \varnothing$, $(x^4)^2 \in g$. Thus after orbifold coordinate change, one can write

$$f = 4x^1x^2 + (x^3)^n + (x^4)^2 = (x^1 + x^2)^2 + (x^4)^2 + (x^3)^n - (x^1 - x^2)^2$$

with $n \ge 3$ odd. This gives cA$x/4$.

Assume $r > 2p$, we will derive a contradiction. Since

$$p = \#\mathscr{S} = \#\mathscr{S}_1 + \#\mathscr{S}_2 \le 1 + (p - 1) = p,$$

$\mathscr{S}_2 = \{0 < k < r \mid r - p < \overline{ka^1} < r\}$. Since $\gcd(a^i, r) = 1$, $i = 1, 2$, there exists $0 < k_i < r$ such that $k_i a^1 \equiv r - i \pmod{r}$. Then $2k_1 \equiv k_2 \pmod{r}$ and k_1, $k_2 \in \mathscr{S}_2$ when $p > 2$. We see that

$$\overline{k_i a^2} = 2r - p - (r - i) = r - p + i$$

for $i = 1, 2$. But then

$$2(r - p + 1) \equiv 2k_1a^2 \equiv k_2a^2 \equiv r - p + 2 \pmod{r},$$

i.e., $r \mid p$, a contradiction.

This shows that p must be 2, and thus r is even. In this case, $\#\mathscr{S}_1 = \#\mathscr{S}_2 = 1$, say $\mathscr{S}_i = \{k_i\}$, $i = 1, 2$. Then $\overline{k_2a^1} = \overline{k_2a^2} = r - 1$, and hence $a^1 = a^2$. Since $k_1 \in \mathscr{S}_1$, we get

$$2 \cdot \overline{k_1a^1} = \overline{k_1a^1} + \overline{k_1a^2} = r + 2.$$

So $2k_1a^1 \equiv 2 \pmod r$ and $(x^4)^2 \in g$ with $\overline{k_1a^4} = 1$ as above. This gives

$$(\overline{2k_1a^1}, \overline{2k_1a^2}, \overline{2k_1a^4}) = (2, 2, 2).$$

By (♠) with $k = 2k_1$, we get

$$4 + r \le \mathrm{ord}_{2k_1}(f) + r < 6 + \overline{2k_1a^3},$$

or $r - 2 < \overline{2k_1a^3} < r$. But $\overline{2k_1a^3}$ is even since r is even, a contradiction.

Remark. By the above proof, we have seen in the case (i), $(X, 0)$ is cA$/r$ or cA$x/4$ and $(x^4)^2 \in f$ if $(X, 0)$ is cA$x/4$.

For case (ii), i.e., $f = (x^1)^2 + (x^2)^2 + g(x^3, x^4)$, we shall derive cA$/2$, cA$x/4$, cA$x/2$ with $x^3x^4 \notin g$.

We claim that r is a power of 2. Otherwise there exists odd $s \mid r$, $s > 2$. Consider the (crepant) finite morphism $Y/\mu_s \to Y/\mu_r$. We see that Y/μ_s is also terminal. Without loss of generality, we may assume that $r > 2$ is odd.

Since $(x^1)^2 + (x^2)^2$ is semi-invariant,

$$2a^1 = \mathrm{wt}((x^1)^2) \equiv \mathrm{wt}((x^2)^2) = 2a^2 \pmod r.$$

Then $a^1 = a^2$, and hence reduces to case (i) by changing the coordinate $(x^1)^2 + (x^2)^2 = (x^1 + \sqrt{-1}x^2)(x^1 - \sqrt{-1}x^2)$. Since r is odd, we must get cA$/r$, which is of type $\frac{1}{r}(1, -1, 0, b) = \frac{1}{r}(a^1, -a^1, 0, 1)$, i.e., $\gcd(a^1, r) = 1$ and $a^1 = a^2 \equiv -a^1 \pmod r$. But then $r \mid 2a^1$ and hence $r \mid 2$, a contradiction.

Next, we show that $r = 2$ or 4. Otherwise, $8 \mid r$ and we may assume that $r = 8$. Similarly, $2a^1 = 2a^2 \pmod 8$, i.e., $a^1 \equiv a^2 \pmod 4$. One can apply to Y/μ_4 the result of the case (i) by the same method above and get either cA$/4$ or cA$x/4$. The case cA$/4$ never occurs since $\gcd(a^1, 4) = 1$ and $a^1 = a^2 \equiv -a^1 \pmod 4$, a contradiction. For the case cA$x/4$,

$$\mathrm{rk}\, f_2 = 3,$$

say $(x^1)^2$, $(x^2)^2$, $(x^4)^2 \in f$. Then

$$2a^1 \equiv 2a^2 \equiv 2a^4 \pmod 8,$$

or $a^1 \equiv a^2 \equiv a^4 \pmod 4$. Then we may assume $a^1 = a^2$ after permutation. This contradicts to the cA/8 case.

Suppose that $r = 4$. If $a^1 = a^2$, one can derive the case c$\mathrm{A}x/4$. This condition is satisfied after permutation whenever $\operatorname{rk} f_2 \geq 3$. We only have to deal with the case $f_2 = (x^1)^2 + (x^2)^2$ with $a^1 \neq a^2$. Since $2a^1 \equiv 2a^2 \pmod 4$ and $\gcd(a^1, a^2, 4) = 1$, we may assume that $a^1 = 1$, $a^2 = 3$.

Apply to Y/μ_2 the result of the case (i), we see that it is of type $\frac{1}{2}(1, 1, 0, b)$ and thus exactly one of a^3, a^4 is even, say $2 \mid a^3$. Since $\gcd(a^3, r) > 1$, it follows from (Ri) that $(x^3)^\ell \in f$ for some ℓ. So

$$\ell a^3 = \mathrm{wt}((x^3)^\ell) \equiv \mathrm{wt}((x^1)^2) = 2 \pmod 4.$$

This shows that $a^3 = 2$. Possibly permuting x_1 and x_2 and changing a_i with $4 - a_i$, one can take $f_2 = (x^1)^2 + (x^2)^2$ and $(a^1, a^2, a^3, a^4) = (1, 3, 2, 1)$, which is c$\mathrm{A}x/4$.

For $r = 2$, if $a^1 = a^2$, then $cA/2$ as above. For $f_2 = (x^1)^2 + (x^2)^2$ with $a^1 = 1$, $a^2 = 0$. Then $\gcd(a^2, a^j, r) = 1$ for $j = 3$, 4 implies $a^3 = a^4 = 1$, which is c$\mathrm{A}x/2$.

21

22 Threefold flips after Shokurov

Recall that for a klt (or plt) pair (X, Δ), a flipping contraction for (X, Δ) is a small projective birational morphism $f\colon X \to Z$ with $-(K_X + \Delta)$ being f-ample and $\rho(X/Z) = 1$.

Proposition 22.1. For a dlt pair (X, Δ), TFAE:

(1) (X, Δ) is plt;

(2) $\lfloor \Delta \rfloor$ is normal;

(3) $\lfloor \Delta \rfloor$ is the disjoint union of its irreducible components.

Reduction of klt flips to pl (prelimiting) flips

Definition 22.2. Let $(X, S+B)$ be a klt pair with $\lfloor S+B \rfloor = S$ irreducible and $S \not\subseteq \operatorname{Supp} B$.

A pl flipping contraction is a flipping contraction $f \colon X \to Z$ for $K_X + S + B$ such that X is $\mathbb{Q}$-factorial and S is f-negative (or F-anti ample).

Remark. This definition is slightly more restrictive than [K+92, Def. 18.6], [C+07, Def. 4.3.1], [Sho03, 1.1].

Theorem 22.3. klt flips exist in dimension n provided that:

$(\mathrm{PL})_n$ pl flips exists in dimension n, and

$(\mathrm{ST})_n$ special termination holds in dimension n.

In general, $(\mathrm{MMP})_{n-1} \implies (\mathrm{ST})_n$, so we know that $(\mathrm{ST})_n$ holds for $n \leq 4$. Our goal is to prove that flips of threefold pl flipping contractions exist.

Let $f \colon X \to Z$ be a flipping contraction for $K_X + S + B$. We note that existence of flips is local on Z in the Zariski topology, we always assume that Z is affine.

Set $A = \mathrm{H}^0(Z, \mathcal{O}_Z) = \mathrm{H}^0(X, \mathcal{O}_X)$ be the affine coordinate ring. So the pl flip exists if and only if

$$R(X, K_X + S + B) := \bigoplus_{i \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(i(K_X + S + B))$$

is a finitely generated A-algebra.

Definition 22.4. A **function algebra** on X is a graded A-subalgebra $V = \bigoplus_{i \geq 0} V_i$ of the polynomial algebra $\mathbb{C}(X)[T]$ where $V_0 = A$ and each $V_i \subseteq \mathbb{C}(X)$ is a coherent A-module.

We say V is **bounded** (by D) if there exists $D \in \mathrm{WDiv}(X)$ such that

$$V \subseteq \mathrm{H}^0(X, \mathcal{O}_X(iD)) \quad \forall i.$$

Let $S = \overline{\{\eta\}} \subseteq X$ be an irreducible subvariety of codimension 1. We say V is regular along S if

(1) $V \subseteq \mathcal{O}_{X_0,S} = \mathcal{O}_{X,\eta} \subseteq \mathbb{C}(X)$,

(2) $V_1 \not\subseteq \mathfrak{m}_{X,S} = \{\varphi \in \mathbb{C}(X) \mid \nu_S(\varphi) \geq 1\}$.

If V is regular along S, the restricted algebra $V^0 = \operatorname{res}_S V$ is

$$V^0 = \bigoplus V_i^0,$$

where V_i^0 is the image of V_i under $\mathcal{O}_{X,S} \to \mathcal{O}_{X,S}/\mathfrak{m}_{X,S} = \mathbb{C}(S)$ (so $V_1^0 \neq 0$).

Remark. If V is bounded by D, regular along S, and $S \not\subseteq \operatorname{Supp} F$, then $V^0 = \operatorname{res}_S V$ is also bounded.

Fix any f-negative $D \in \operatorname{WDiv}(X)$. Then $\rho(X/Z) = 1$ implies that $D \sim r(K_X + S + B)$for some $r \in \mathbb{Q}^+$. So $R(X, K_X + S + B)$ is finitely generated if and only if $R(X, D)$ is finitely generated by [KM, Cor. 6.14 (4)].

Lemma 22.5. Let $f\colon X \to Z$ be a pl flipping contraction for $K_X + S + B$, $0 \leq D \in \operatorname{WDiv}(X)$ be f-negative such that $S \subseteq \operatorname{Supp} D$. Then the flip of f exists if and only if $R^0 = \operatorname{res}_S R(X, D)$ is finitely generated.

Proof. If the flip of f exists, then $R = R(X, D)$, and hence R^0, is finitely generated.

Conversely, since $\rho(X/Z) = 1$ and Z is affine, we may assume that $D \sim S$. Then there exists $t \in \mathbb{C}(X)$ such that $\operatorname{div}(t) + D = S \geq 0$, so $t \in \mathrm{H}^0(X, \mathcal{O}_X(D))$.

Claim. The kernel of $R \to R^0$ is generated by t.

Proof of Claim. For $\varphi \in R_n$, we have $\operatorname{div}(\varphi) + nD \geq 0$ and φ has a zero along S. So

$$0 \leq \operatorname{div}(\varphi) + nD - S = \operatorname{div}(\varphi/t) + (n-1)D,$$

and hence, $\varphi/t \in R_{n-1}$ and $\varphi = t \cdot \varphi/t \in \langle t \rangle$. □

Hence, R is finitely generated. ■

Remark. Let S be an f-negative irreducible divisor. Then $f_*\mathcal{O}_X(S)$ is globally generated. Let $\varphi \in \mathrm{H}^0(Z, f_*\mathcal{O}_X(S)) = \mathrm{H}^0(X, \mathcal{O}_X(S))$ be a general section and $D := \operatorname{div}(\varphi) + S \geq 0$. Then $0 \leq D \sim S$ and $S \not\subseteq \operatorname{Supp} D$.

Now, we need to prove that the restricted algebra R^0 is finitely generated. We will see that this is equivalent to R_S, a *pbd* algebra associated with R^0, is finitely generated. We will show that R_S is a Shokurov algebra for all dimension, and a Shakurov algebra with mobile system on a surface is finitely generated.

Definition 22.6. Let X be a normal variety. Consider a category with objects of the form $Y \to X$ proper birational from a normal variety Y (called a model of X). Morphisms between the objects $Y \to X$ and $Y' \to X$ are just morphisms $Y \to Y'$ so that the diagram commutes.

An (integral) Weil **b-R-divisor** (b for birational) on X is an element

$$\mathbf{D} \in \mathbf{WDiv}(X)_R := \varprojlim \mathrm{WDiv}(Y)_R,$$

where the projective limit is take over all models $f\colon Y \to X$ of X under $f_*\colon \mathrm{WDiv}(Y)_R \to \mathrm{WDiv}(X)_R$.

Remark. If $f\colon Y \to X$ is a model of X, then $f_*\colon \mathbf{WDiv}(Y) \to \mathbf{WDiv}(X)$ is an isomorphism.

A Zariski–Riemann space if a subring of a field K is a locally ringed space whose points are valuation rings $k \subseteq R \subseteq K$. Then $\mathfrak{X} := \varprojlim Y$ is a Zariski–Riemann space. Note that $\mathfrak{X}$ is nor a scheme anymore for $\dim X \geq 2$. We have

$$\mathrm{WDiv}(\mathfrak{X}) = \varprojlim \mathrm{WDiv}(Y) = \mathbf{WDiv}(X).$$

Definition 22.7. Let $\mathbf{D} = \sum d_\Gamma \Gamma$ be a b-divisor on X. For each $U \subseteq X$, define

$$\mathrm{H}^0(U, \mathcal{O}_X(\mathbf{D})) = \{\varphi \in \mathbb{C}(X) \mid \nu_\Gamma(\varphi) + d_\Gamma \geq 0 \text{ for } \Gamma \text{ with center on } U\}.$$

In general, it is often not quasi-coherent. However, it is a coherent sheaf in cases of interest to us.

For $Y \to X$ a model of X, the trace of $\mathbf{D}$ on Y is $\mathrm{tr}_Y \mathbf{D} = \mathbf{D}_Y = \sum_{\Gamma \in \mathrm{WDiv}(Y)} d_\Gamma \Gamma$.

Remark. The linear system of $\mathbf{D}$ is defined to be

$$|\mathbf{D}| = \mathbb{P}\,\mathrm{H}^0(X, \mathcal{O}_X(\mathbf{D})).$$

Note that in general, $\mathbb{P}\,\mathrm{H}^0(X, \mathbf{D})$ is a proper subspace of $\mathbb{P}\,\mathrm{H}^0(X, \mathbf{D}_X)$.

Example 22.8. For $\varphi \in \mathbb{C}(X)^\times$, $\mathbf{div}(\varphi) = \sum \nu_\Gamma(\varphi)\Gamma$, where we sum over all geometric valuation Γ with center on X. b-divisors of this form are called **principal** b-divisors. As before, we say $\mathbf{D}_1 \sim \mathbf{D}_2$ if $\mathbf{D}_1 - \mathbf{D}_2 = \mathbf{div}_X(\varphi)$ for some $\varphi \in \mathbb{C}(X)^\times$.

Let $B \in \mathrm{WDiv}(X)_\mathbb{Q}$ with $K_X + B$ $\mathbb{Q}$-Cartier. The discrepancy b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ is the b-divisor with trace $\mathbf{A}_Y$ defined by

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y$$

on models $f\colon Y \to X$ of X.

Let $D \in \mathrm{WDiv}(X)$, $\widehat{D}$ (in an abuse of notation, write D instead of $\widehat{D}$) is the b-divisor with trace $\widehat{D}_Y = f_*^{-1}D$ on models $f\colon Y \to X$ of X.

Remark. Let $\mathbf{D}$ be a b-divisor on X. If there exists a model $f\colon Y|toX$ of X and $0 \leq D_Y \in \mathrm{CDiv}(Y)_\mathbb{Q}$ such that $\mathbf{D} = \overline{D}_Y$, then $\mathcal{O}_X(\mathbf{D}) = f_*\mathcal{O}_X(D_Y)$ is coherent. In general, for any $0 \leq \mathbf{D} \leq \overline{D}_Y$, $\mathcal{O}_X(\mathbf{D})$ is coherent.

Lemma 22.9 (C+07, Lem. 2.3.14)**.** Let X be a smooth variety, $D \in \mathrm{WDiv}_\mathbb{Q}$ has SNC support, $\mathbf{A} = \mathbf{A}(X, D)$ the discrepancy b-divisor of (X, D). If $f\colon Y \to X$ is a model of X, then

$$\lceil \mathbf{A}_Y \rceil = f^*\lceil \mathbf{A}_X \rceil + \sum \delta^i E_i,$$

whree the E_i's are f-exceptional divisors and $\delta^i > 0$.

Proof. By definition, $D = -\mathbf{A}_X$. This gives

$$\begin{aligned} K_Y &= f^*(K_X + D) + \mathbf{A}_Y \\ &= f^*(K_X + \{-\mathbf{A}_X\}) + \mathbf{A}_Y - \lceil f^*\mathbf{A}_X \rceil. \end{aligned}$$

By assumption, $(X, \{-\mathbf{A}_X\})$ is a klt pair, so we get

$$\sum \delta^i E_i = \lceil \mathbf{A}_Y - \lceil f^*\mathbf{A}_X \rceil \rceil \geq 0,$$

as desired. ∎

Using this lemma, one can show that for a normal variety X and a $\mathbb{Q}$-divisor D such that $K_X + D$ Cartier, the sheaf $\mathcal{O}_X(\lceil \mathbf{A}(X, D) \rceil)$ is coherent. If $D \geq 0$, this is a multiplier ideal sheaf $\mathfrak{I}(D)$.

Definition 22.10. For $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ and a vector subspace $0 \neq V \subseteq \mathrm{H}^0(X, \mathcal{O}_X(D))$, the mobile (movable) part of D with respect to V is

$$\mathrm{Mob}_V\, D := \sum_{\Gamma} \left(- \inf_{0 \neq \varphi \in V} \nu_{\Gamma}(\varphi) \right) \Gamma.$$

When $V = \mathrm{H}^0(\mathcal{O}_X(D))$, we simply write $\mathrm{Mob}\, D$.

For $C \in \mathrm{WDiv}(X)_{\mathbb{Q}}$, we say D is C**-saturated** if $\mathrm{Mob}\lceil D + C \rceil \leq D$.

Remark. If D is not integral, the definition says $\mathrm{Mob}_V\, D = \mathrm{Mob}_V \lfloor D \rfloor$. If D is integral, $\mathrm{Mob}\, D = D - \mathrm{Fix}\,|D|$, where $\mathrm{Fix}\,|D|$ is the biggest divisor $F \geq 0$ such that $F \leq D'$ for all $D' \in |D|$. The support $\mathrm{Supp}(\mathrm{Fix}\,|D|)$ is the divisorial part of $\mathrm{Bs}\,|D|$. If $|\lceil D + C \rceil| = \varnothing$, then D is always C-saturated. Therefore only $|D| \neq \varnothing$, $|\lceil D + C \rceil| = \varnothing$ is useful.

We say that a property $\mathscr{P}$ holds on high models ($Y \to X$ of X) if $\mathscr{P}$ holds on a particular model $Y \to X$ of X and on every higher models $Y' \to Y \to X$.

Definition 22.11. A b-divisor $\mathbf{D}$ on X is $\mathbf{C}$-saturated if $\mathbf{D}_Y$ is $\mathbf{C}_Y$-saturated on high model $Y \to X$ of X. If $Y \to X$ is a model such that $\mathbf{D}_Y$ is $\mathbf{C}_Y$-saturated, we say that saturation holds on Y. (For a klt pair (X, B) and $\mathbf{C} = \mathbf{A}(X, B)$, we sometimes replace $\mathbf{C}_Y$- by canonical.)

We say $\mathbf{D}$ is exceptionally saturated over X if it is $\widehat{E}$-saturated for all E effective and exceptional over X.

Proposition 22.12. Let D be a $\mathbb{Q}$-Cartier integral divisor. Then the $\mathbb{Q}$-Cartier closure $\overline{D}$ is exceptionally saturated over X.

Proof. For each model $f \colon Y \to X$ of X,

$$f_* \mathcal{O}_Y(\lceil f^* D + \textstyle\sum a^i E_i \rceil) = \mathcal{O}_Y(D)$$

if all E_i are f-exceptional and all $a^i \geq 0$. So $\operatorname{Mob}\lceil f^*D + \sum a^i E_i \rceil = \operatorname{Mob} f^*D \leq f^*D$, as desired. ■

Lemma 22.13. Let X be a normal variety, $B \in \operatorname{WDiv}(X)_{\mathbb{Q}}$ such that $K_X + B$ is $\mathbb{Q}$-Cartier, $\mathbf{D}$ a b-divisor on X, $\mathbf{A} = \mathbf{A}(X, B)$. Let $Y \to X$ be a model of X satisfying

(1) Y is smooth and $\mathbf{D}_Y + \mathbf{A}_Y$ has SNC support,

(2) $\overline{\mathbf{D}}_Y = \mathbf{D}$, i.e., it descends to Y.

Then canoincal saturation holds on Y if and only if it holds on any higher model $f\colon Y' \to Y$, i.e.,

$$\operatorname{Mob}\lceil \mathbf{D}_Y + \mathbf{A}_Y \rceil \leq \mathbf{D}_Y \quad \iff \quad \operatorname{Mob}\lceil \mathbf{D}_{Y'} + \mathbf{A}_{Y'} \rceil \leq \mathbf{D}_{Y'}.$$

Proof. By (2),

$$K_{Y'} = f^*(K_X + \{-\mathbf{D}_Y - \mathbf{A}_Y\}) + \mathbf{D}_{Y'} + \mathbf{A}_{Y'} - f^*\lceil \mathbf{D}_Y + \mathbf{A}_Y \rceil.$$

By (1), $(Y, \{-\mathbf{D}_Y - \mathbf{A}_Y\})$ is klt. So

$$\lceil \mathbf{D}_{Y'} + \mathbf{A}_{Y'} \rceil = f^*\lceil \mathbf{D}_Y + \mathbf{A}_Y \rceil + E,$$

where E is f-exceptional and effective. This gives us the result by using the previous lemma. ■

From now on, we always tacitly assume that the sheaf $\mathcal{O}_X(\mathbf{D})$ of b-divisor is coherent.

Definition 22.14. A sequence $\mathbf{D}_\bullet = \{\mathbf{D}_i\}_{i=1}^{\infty}$ of b-divisors is convex if $\mathbf{D}_1 > 0$ and

$$\mathbf{D}_{i+j} \geq \frac{i}{i+j}\mathbf{D}_i + \frac{j}{i+j}\mathbf{D}_j \quad \forall i, j.$$

(Note that this is slightly different with the usual convexity.)

We say $\mathbf{D}_\bullet$ is bounded if there exists $D \in \operatorname{CDiv}(X)_{\mathbb{Q}}$ such that $\mathbf{D}_i \leq \overline{D}$ for all i.

Remark. If $\mathbf{D}_\bullet$ is convex, then it is increasing in the sense that $\mathbf{D}_i \leq \mathbf{D}_j$ if $i \mid j$.

If moreover $\mathbf{D}_\bullet$ is bounded, then we define

$$\lim_{i\to\infty} \mathbf{D}_i = \sup \mathbf{D} \in \operatorname{WDiv}(X)_{\mathbb{R}}.$$

Definition 22.15. A pseudo-b-divisorial (pbd) algebra is the function algebra

$$R = R(X, \mathbf{D}_\bullet) = \bigoplus_{i \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(i\mathbf{D}_i))$$

naturally associated to a convex sequence $\mathbf{D}_\bullet$ (called the characteristic sequence of the pbd algebra) of b-divisors.

Note that R_i is a subalgebra of $\mathbb{C}(X)$ and the convexity of $\mathbf{D}_\bullet$ implies that $R_i \cdot R_j \subseteq R_{i+j}$.

We say that $R(X, \mathbf{D}_\bullet)$ is bounded if it is bounded sa a function algebra, equivalently, $\mathbf{D}_\bullet$ is bounded.

Definition 22.16. A convex sequence $\mathbf{D}_\bullet$ of effective b-divisors is $\mathbf{C}$-a-saturated (a for asymptotically) if

$$\mathrm{Mod}\lceil j\mathbf{D}_{iY} + \mathbf{C}_Y \rceil \leq j\mathbf{D}_{jY}$$

for all i, j on higher models $Y \to X$ of $Y(i,j) \to X$. We say this saturation is uniform if $Y(i,j)$ is independent of i, j.

A pbd algebra $R(X, \mathbf{D}_\bullet)$ is canonically a-saturated if $\mathbf{D}_\bullet$ is canonically a-saturated.

Definition 22.17. A Shokurov algebra is a bounded canonically a-saturated pbd-algebra.

The finite generation conjecture states that for a klt pair (X, B) and birational contraction $X \to Z$ to an affine space Z such that $-(K_X + B)$ is big and nef over Z (called a weak Fano contraction), all Shokurov algebras on X are finitely generated.

Remark. In general, the restricted algebra $R^0 = \mathrm{res}_S R$ is NOT a pbd algebra.

Definition 22.18. An integral b-divisor $\mathbf{D}$ is mobile (b-free) if there exists a model $Y \to X$ of X such that $|\mathbf{D}_Y|$ is free and $\overline{\mathbf{D}}_Y = \mathbf{D}$.

For $D \in \mathrm{WDiv}(X)_\mathbb{R}$, $V \subseteq \mathrm{H}^0(X, \mathcal{O}_X(D))$ a vector subspace, the mobile b-part of D with respect to V is

$$\mathbf{Mob}_V\, D = \sum_{\Gamma \subseteq Y \to X} \left(- \inf_{0 \neq \varphi \in V} \nu_\Gamma(\varphi) \right) \Gamma$$

Remark. If D is integral $\mathbb{Q}$-Cartier and $f\colon Y \to X$ is a model of X, then

$$(\mathbf{Mob}_V\, D)_Y = f^*D - \operatorname{Fix} f^*|V|.$$

If, in addition, $V = \mathrm{H}^0(\mathcal{O}_X(D))$,

$$(\mathbf{Mob}\, D)_Y = \operatorname{Mob} f^*D = \operatorname{Mob}\lceil f^*D\rceil.$$

Lemma 22.19. The mobile b-part $\mathbf{Mob}\, D$ of a $\mathbb{Q}$-Cartier $D \in \mathrm{WDiv}(X)$ is exceptional saturated over X.

Proof. Let $f\colon Y \to X$ be a model of X and E an effective exceptional divisor. We have

$$\operatorname{Mob}\lceil(\mathbf{Mob}\, D)_Y + E\rceil \leq \operatorname{Mob}\lceil f^*D + E\rceil = \operatorname{Mob}\lceil f^*D\rceil = (\mathbf{Mob}\, D)_Y,$$

as desired. ■

Lemma 22.20. Let $V = \bigoplus_{i\geq 0} V_i$ be a function algebra on X. We define the b-divisors

$$\mathbf{M}_i = \sum_{\Gamma\subseteq Y\to X} \left(-\inf_{\varphi\in V_i} \nu_\Gamma(s)\right)\Gamma$$

Then $\mathbf{M}_i$ has the properties:

(1) $V_i \subseteq \mathrm{H}^0(X, \mathbf{M}_i)$;

(2) $\mathbf{M}_i$ is mobile;

(3) $\mathbf{M}_i + \mathbf{M}_j \leq \mathbf{M}_{i+j}$.

Proof. For each $s \in V_i$,

$$\mathbf{div}(s) + \mathbf{M}_i = \sum_{\Gamma} \left(\nu_\Gamma(s) + \left(-\inf_{\varphi\in V_i} \nu_\Gamma(s)\right)\right)\Gamma \geq 0$$

This gives (1). (3) simply follows from $V_i \cdot V_j \subseteq V_{i+j}$. For (2), fix any resolution $g\colon Y \to X$. By definition,

$$(\mathbf{M}_i)_Y = \sum_{\Gamma\subseteq Y} \left(-\inf_{\varphi\in V_i} \nu_\Gamma(s)\right)\Gamma.$$

Consider the linear system

$$\Lambda_i = \{(\mathbf{M}_i)_Y + \mathbf{div}(\varphi)_Y \mid \varphi \in V_i\} \subseteq |(\mathbf{M}_i)_Y|$$

and the blow-up $h\colon Y' \to Y$ along $\operatorname{Bs}\Lambda_i$. Then $h^*\Lambda_i$, and hence $h^*|(\mathbf{M}_i)_Y|$, is free. By definition,

$$(\mathbf{M}_i)_{Y'} = h^*(\mathbf{M}_i)_Y - \inf\{h^*(\mathbf{M}_i)_Y + \mathbf{div}(\varphi)_{Y'} \mid \varphi \in V_i\} = h^*(\mathbf{M}_i)_Y.$$

Hence, $\mathbf{M}_i$ is mobile (b-free). ■

Lemma 22.21. Let $V = \bigoplus_{i\geq 0} V_i$ be a function algebra on X. There exists a pbd algebra $R^V = R(X, \mathbf{D}_\bullet)$ such that

(0) R^V is integral over V;

(1) V is bounded if and only if R^V is bounded;

(2) V is finitely generated if and only if R^V is finitely generated.

Proof. By the above lemma, there exists a sequence of mobile b-divisors $\mathbf{M}_\bullet$. Multiplying by a suitable rational rational function, WLOG, $\mathcal{O}_X \subseteq V_1$, i.e., the b-divisor $\mathbf{M}_1 > 0$. We take $\mathbf{D}_i = \mathbf{M}_i/i$ so that $\mathbf{D}_\bullet$ is convex. We get a pbd algebra

$$R^V = R(X, \mathbf{D}_\bullet) = \bigoplus \mathrm{H}^0(X, \mathbf{M}_i) \supseteq V.$$

If R^V is bounded, then there exists a divisor D such that $V_i \subseteq \mathrm{H}^0(X, \mathbf{M}_i) \subseteq \mathrm{H}^0(X, iD)$, and hence V is bounded. Conversely, if $V_i \subseteq \mathrm{H}^0(X, iD)$, then using

$$\mathbf{M}_i = \sum \left(- \inf_{\varphi \in V_i} \nu_\Gamma(\varphi) \right) \Gamma,$$

we see that $\mathbf{M}_i \leq i\overline{D}$. This gives (1)

For a proof of (0), see [Sho03, Prop. 4.15(6)]. Moreover,

$$\bigoplus_{j\geq 0} V_i^j \subseteq \bigoplus_{j\geq 0} \mathrm{H}^0(X, j\mathbf{M}_i)$$

is an integral extension.

Now, if R^V is finitely generated, then by a truncation, we may assume that R^V is generated by $(R^V)_1$. Since R^V is integral over $V' := \bigoplus V_1^j$, R^V is a finitely generated V'-module. Then V is also a finitely generated V'-module, hence a finite algebra.

Conversely, it follows from the construction that V and R^V are function algebra with the same quotient field. By Noether's theorem on the finiteness of the integral closure, if V is a finitely generated algebra, so is R^V. ■

Lemma 22.22 (Limiting criterion)**.** Assume that $R = R(X, \mathbf{D}_\bullet)$ is a pbd algebra such that $\mathbf{M}_i = i\mathbf{D}_i$ is mobile. Then R is finitely generated if and only if there exists $i_0 \in \mathbb{N}$ such that $\mathbf{D}_{i_0} = \mathbf{D}_{ii_0}$ for all i.

Remark. Assume there exists a proper birational map $X \to Z$, where Z is affine. Then each pbd algebra arises from a mobile sequence.

Proof. Suppose such i_0 exists. By passing to a truncation, we may assume that $i_0 = 1$. Then $R = \bigoplus_{i\geq 0} \mathrm{H}^0(X, i\mathbf{M}_1)$. Let $Y \to X$ be a model of X such that $|(\mathbf{M}_1)_Y|$ is free and $(\overline{\mathbf{M}}_1)_Y = \mathbf{M}_1$. Then $R = R(Y, (\mathbf{M}_1)_Y)$ is finitely generated.

Conversely, since R is finitely generated, there exists $i_0 \in \mathbb{N}$ such that $R^{(i_0)} = \bigoplus \mathrm{H}^0(X, \mathbf{M}_{ii_0})$ is finitely generated by degree 1 elements. Then

$$\mathrm{H}^0(X, \mathbf{M}_{ii_0} = \mathrm{H}^0(X, \mathbf{M}_{ii_0})^i \subseteq \mathrm{H}^0(X, i\mathbf{M}_{i_0}).$$

Since $\mathbf{D}_0$ is convex, $\mathbf{M}_{ii_0} = ii_0\mathbf{D}_{ii_0} \geq ii_0\mathbf{D}_{i_0} = i\mathbf{M}_{i_0}$. So we get another side of inclusion, i.e., $\mathbf{D}_{ii_0} = \mathbf{D}_{i_0}$. ■

Definition 22.23. Let $\mathbf{M}$ be a mobile b-divisor on X. Let $X \supseteq S$ be an irreducible normal subvariety of codimension 1 with $S \not\subseteq \operatorname{Supp} \mathbf{M}_X$. We define mobile restriction of $\mathbf{M}$ to S as follows: pick a model $Y \to X$ such that $|\mathbf{M}_Y|$ is free and $\overline{\mathbf{M}}_Y = \mathbf{M}$ and the strict transform $S' \subseteq Y$ of S is normal. We define

$$\mathbf{M}^0 = \operatorname{res}_S \mathbf{M} = \overline{\mathbf{M}}_Y|_{S'} \in \mathbf{WDiv}(S') \cong \mathbf{WDiv}(S).$$

We can prove that $(\mathbf{M}_1 + \mathbf{M}_2)^0 = \mathbf{M}_1^0 + \mathbf{M}_2^0$ and $\mathbf{M}_1 \geq \mathbf{M}_2 \implies \mathbf{M}_1^0 \geq \mathbf{M}_2^0$.

Recall that $R = R(X, D)$ is finitely generated if and only if $R^0 = \operatorname{res}_S R$ is finitely generated. Note that $R \subseteq R(X, \mathbf{D}_\bullet) = \bigoplus \mathrm{H}^0(X, \mathbf{M}_i)$, where $\mathbf{M}_i = \mathbf{Mob}(iD)$, and $R^0 \subseteq R(S, \mathbf{D}'_\bullet) = \bigoplus \mathrm{H}^0(S, M'_i)$ (by restriction).

Lemma 22.24. We have

$$R_S := \bigoplus_{i\geq 0} \mathrm{H}^0(S, \mathbf{M}_i^0) = R(S, \mathbf{D}'_\bullet).$$

Proof. By definition, for each $i \geq 0$, pick a model $Y \supseteq S'$ such that

$$\begin{aligned}(\mathbf{M}_i^0)_{S'} &= (\mathbf{M}_i)_Y|_S \\ &= \sum_{\Gamma \subseteq Y} \left(-\inf_{\varphi \in \mathrm{H}^0(X, iD)} \nu_\Gamma(\varphi) \right) \Gamma|_S \\ &= \sum_{\Gamma' \subseteq S'} \left(-\inf_{\varphi \in \mathrm{Im}(\mathrm{H}^0(X, iD) \to \mathbb{C}(S))} \nu_{\Gamma'}(\varphi) \right) \Gamma' = (\mathbf{M}_i')_{S'}. \end{aligned}$$ ■

Lemma 22.25 (C+07, Lem. 2.4.3)**.** The algebra R_S is a Shokurov algebra, i.e., a bounded a-saturated pbd-algebra.

Proof. Since $(R^\circ)_i \subseteq \mathrm{H}^0(S, iD|_S)$, R_S is a bounded pbd algebra. It suffices to show that R_S is a-saturated. By constuction, $\mathbf{M}_i = \mathbf{Mob}(iD)$ and thus $(\mathbf{M}_i)_Y = \mathbf{Mob}(f^*iD)$ on models $f\colon Y \to X$ of X.

Fix i, $j > 0$ and choose a model $f\colon Y \to X$ (depends on i, j) such that

(i) f is a log resolution $(S + B + (\mathbf{M}_i)_X + (\mathbf{M}_j)_X)$;

(ii) $|(\mathbf{M}_i)_Y|$, $|(\mathbf{M}_j)_Y|$ are free and $\mathbf{M}_i = \overline{(\mathbf{M}_i)_Y}$, $\mathbf{M}_j = \overline{(\mathbf{M}_j)_Y}$.

Write

$$K_Y = f^*(K_X + S + B) - f_*^{-1}(S + B) + F = f^*(K_X + S + B) - \mathbf{A}_Y.$$

Let $\mathbf{A}' = \mathbf{A} + \widehat{S}$, where $\widehat{S}$ is the strict transform b-divisor of S. Since $\mathbf{A}'_Y = -f_*^{-1}B + F$, it follows by the adjunction that

$$\mathbf{A}'_Y|_S = \mathbf{A}(S, \mathrm{Diff}_S(B))_{S'},$$

where $S' = \widehat{S}_Y$. Recall that $R_S = R(S, \mathbf{D}_\bullet^0)$, where $\mathbf{D}_i^0 = \mathbf{M}_i^0/i$.

Claim. $\mathrm{Mob}(\lceil j(\mathbf{D}_i^0)_{S'} + \mathbf{A}(S, \mathrm{Diff}_S(B))_{S'} \rceil) \leq j(\mathbf{D}_j^0)_{S'}$ for each i, j with $i \geq j$.

Proof of claim. Consider

$$0 \longrightarrow \mathcal{O}_Y(-S') \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

We have

$$0 \to \mathcal{O}_Y(\lceil (j\mathbf{D}_i + \mathbf{A})_Y \rceil) \to \mathcal{O}_Y(\lceil (j\mathbf{D}_i + \mathbf{A}')_Y \rceil) \to \mathcal{O}_{S'}(\lceil (j(\mathbf{D}_i^0)_{S'} + \mathbf{A}(S, \mathrm{Diff}_S(B))_{S'} \rceil) \to 0.$$

This gives

$$\mathrm{H}^0(Y, \lceil (j\mathbf{D}_i + \mathbf{A}')_Y \rceil) \longrightarrow \mathrm{H}^0(S', \lceil (j(\mathbf{D}_i^0)_{S'} + \mathbf{A}(S, \mathrm{Diff}_S(B))_{S'} \rceil) \longrightarrow \mathrm{H}^1(Y, \lceil (j\mathbf{D}_i + \mathbf{A})_Y \rceil) = 0$$

by (5.5) since

$$(j\mathbf{D}_i + \mathbf{A})_Y = K_Y + \frac{j}{i}(\mathbf{M}_i)_Y - f^*(K_X + S + B).$$

Therefore, to prove the claim, we can change to compare $\mathrm{Mob}\lceil (j\mathbf{D}_i^0 + \mathbf{A}')_Y \rceil$ with $j(\mathbf{D}_j)_Y$. Then

$$\begin{aligned}
\mathrm{Mob}\lceil (j\mathbf{D}_i^0 + \mathbf{A}')_Y \rceil &= \mathrm{Mob}\lceil \tfrac{i}{j}\ \mathrm{Mob}(f^*(iD)) + \mathbf{A}'_Y \rceil \\
&\leq \mathrm{Mob}\lceil \tfrac{i}{j}\ \mathrm{Mob}(f^*(iD)) + F \rceil && (\lceil -f_*^{-1}B \rceil = 0) \\
&\leq \mathrm{Mob}\lceil \tfrac{i}{j}\, f^*(iD) + F \rceil = \mathrm{Mob}\lceil f^*(jD) \rceil = j(\mathbf{D}_j)_Y. && \square
\end{aligned}$$

Hence, R_S is a-saturated. ■

The finite generation conjecture implies R_S is finitely generated, and hence R is finitely generated.

23 A Shokurov algebra with mobile system on surface is finitely generated

Let $R(X, \mathbf{D}_\bullet)$ be a Shokurov algebra with $M_i = i\mathbf{D}_i$ is mobile for each i.

Proposition 23.1. Let (X, B) be a klt pair of dimension n. Assume that klt MMP holds in dimension n. Then there exists $(X^{\mathrm{ter}}, B^{\mathrm{ter}})$ a terminal pair and a projective birational morphism $\varphi\colon X^{\mathrm{ter}} \to X$ such that $K_{X^{\mathrm{ter}}} + B^{\mathrm{ter}} = \varphi^*(K_X + B)$.

Remark. We say that $(X^{\mathrm{ter}}, B^{\mathrm{ter}})$ is a terminal model of (X, B). It is unique when $n = 2$.

If (X, B) is a 2-dimensional klt pair and there is a birational weak (log) Fano contraction $f\colon X \to Z$, i.e., $-(K_X + B)$ is f-big and f-nef with $f_*\mathcal{O}_X = \mathcal{O}_Z$. Then $-(K_{X^{\mathrm{ter}}} + B^{\mathrm{ter}}) = -\varphi^*(K_X + B)$ is $f\varphi$-big and $f\varphi$-nef. Without loss of generality, we may assume that (X, B) is a 2-dimensional terminal pair since $R(X, \mathbf{D}_\bullet) = R(X^{\mathrm{ter}}, \mathbf{D}_\bullet)$.

Theorem 23.2. Let (X, B) be a 2-dimensional terminal pair, i.e., X is smooth and $\operatorname{mult}_x B < 1$ for each $x \in X$, with a birational weak (log) Fano contraction $f\colon X \to Z$. Let $\mathbf{M}$ be a mobile, canonically saturated b-divisor on X. Then

(1) $\mathbf{M}$ descends to X;

(2) $\mathbf{M}_X$ is nef (note that this is not true in higher dimensions).

Proof. Let $g\colon Y \to X$ be a high enough log resolution of (X, B) such that

(a) canonical saturation holds on Y;

(b) $\overline{\mathbf{M}}_Y = \mathbf{M}$ and $|\mathbf{M}_Y|$ is free.

Assuming (1), $\mathbf{M}_Y = g^*\mathbf{M}_X$, so for each irreducible curve $C \subseteq X$,

$$\mathbf{M}_X \cdot [C] = g^*\mathbf{M}_X \cdot g^*[C] \geq 0$$

since $\mathbf{M}_Y$ is free, and hence nef. For (1), write

$$K_Y = g^*(K_X + B) - g_*^{-1}B + \sum a^i E_i, \quad a^i > 0.$$

Let $E = \lceil \mathbf{A}(X, B)_Y \rceil = \lceil \sum a^i E_i \rceil$.

Claim. $E \cap D = \varnothing$ for a general member $D \in |\mathbf{M}_Y|$. (If so, $|\mathbf{M}_Y|$ avoids $\operatorname{Supp} E = \operatorname{Exc}(g)$ altogether, and thus $\mathbf{M}_Y = g^*\mathbf{M}_X$.)

It follows from (a) that $\operatorname{Mob}\lceil \mathbf{M}_Y + E \rceil \leq \mathbf{M}_Y$. So

$$\mathbf{M}_Y + E - \operatorname{Mob}\lceil \mathbf{M}_Y + E \rceil \geq E.$$

This shows that $\operatorname{Fix}|\mathbf{M}_Y + E| \geq E$. Since $|\mathbf{M}_Y|$ is free, $E = \operatorname{Fix}|\mathbf{M}_Y + E|$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(\mathbf{M}_Y - D + E) \longrightarrow \mathcal{O}_Y(\mathbf{M}_Y + E) \longrightarrow \mathcal{O}_D(\mathbf{M}_Y + E) \longrightarrow 0.$$

We get (let $h = fg$)

$$h_*\mathcal{O}_Y(\mathbf{M}_Y + E) \longrightarrow h_*\mathcal{O}_D(\mathbf{M}_Y + E) \longrightarrow R^1h_*\mathcal{O}_Y(E)$$

by (5.5) since

$$E = K_Y + \lceil -g^*(K_X + B) \rceil$$

and $-g^*(K_X+B)$ is h-big and h-nef. Since Z is affine,

$$\mathrm{H}^0(Y, \mathbf{M}_Y + E) \longrightarrow \mathrm{H}^0(D, (\mathbf{M}_Y + E)|_D)$$

is surjective. Therefore, $E \cap D = \mathrm{Bs}\,|(\mathbf{M}_Y + E)|_D| = \varnothing$ (since the question is local, we may assume that X, Y, and hence D, are affine.) ■

Now, since $R(X^{\mathrm{ter}}, \mathbf{D}_\bullet)$ is bounded, there exists a divisor $G = \sum G_j$ on X^{ter} such that $\mathrm{Supp}(\mathbf{M}_i)_{X^{\mathrm{ter}}} \subseteq G$ for each i. Then all $\mathbf{M}_i$ descend to X^{ter} and if $X'' \to X^{\mathrm{ter}}$ is a log resolution of $(X^{\mathrm{ter}}, B^{\mathrm{ter}} + G)$, then canonical a-saturation holds uniformly on models $Y \to X''$ higher than X'', i.e.,

$$\mathrm{Mob}\lceil j(\mathbf{D}_i)_Y + \mathbf{A}_Y \rceil \le j(\mathbf{D}_j)_Y.$$

So we may assume that (X, B) is terminal and there exists a divisor G on X such that $\mathrm{Supp}(\mathbf{D}_i)_X \subseteq G$. Then

$$\mathbf{D} = \lim_{i\to\infty} \mathbf{D}_i \in \mathbf{WDiv}(X)_{\mathbb{R}}$$

and $\mathrm{Supp}\,\mathbf{D}_X \subseteq G$.

Lemma 23.3. The divisor $\mathbf{D}_X$ is semiample.

Proof. Since $\mathbf{M}_i = i\mathbf{D}_i$ is mobile, $(\mathbf{M}_i)_X$ is nef, and hence $\mathbf{D}_X$ is also nef. Since Z is affine, it contains no projective curves. So every projective curve in X is in the fiber of f. This shows that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{(K_X+B+\varepsilon(\text{ample}))<0},$$

which is a finite rational polyhedral. The dual cone $\mathrm{Nef}(X)$ is generated by the semiample divisors supporting the contractions of its extremal faces. So all nef divisors on X are semiample. ■

Recall that $\mathrm{Supp}\,\mathbf{D}_X \subseteq G = \sum G_j$. Let $N'_{\mathbb{Z}} = \bigoplus \mathbb{Z} \cdot G_j \subseteq \mathrm{WDiv}(X)$. Since $\mathbf{D}_X$ is semiample, we can choose effective bpd divisors $P_k \in N'_{\mathbb{Z}}$ such that

$$\mathbf{D}_X \in \sum \mathbb{R}^+ P_k \subseteq \sum R^+ G_j \subseteq N'_{\mathbb{Z}} \otimes \mathbb{R}.$$

Assume that $\mathbf{D}_X$ is not rational.

Proposition 23.4. If $\mathbf{D}_X$ is not rational, then for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ and $M \subseteq N'_{\mathbb{Z}}$ such that

(1) $|M|$ is free;

(2) $\|m\mathbf{D}_X - M\| < \varepsilon$;

(3) $m\mathbf{D}_X - M$ is not effective.

Lemma 23.5. Choose $\gamma \in \mathbb{Q}^+$ small enough such that $(X, B + \gamma G)$ is klt. Set $\mathbf{A} = \mathbf{A}(X, B)$. We have $\lceil \mathbf{A} - \gamma\overline{G} \rceil \geq 0$. Assume $\mathbf{D}_X$ is not rational, and let $M \in N'_{\mathbb{Z}}$ as in the previous proposition. If $0 < \varepsilon < \gamma$, then on every model $f : Y \to X$ of X,

$$\operatorname{Mob}\lceil m\mathbf{D}_Y + \mathbf{A}_Y \rceil \geq f^*M.$$

Proof. Since $\mathbf{D}_i = \overline{(\mathbf{D}_i)_X}$ for each i, $\mathbf{D} = \overline{\mathbf{D}}_X$ and thus $\mathbf{D}_Y = f^*\mathbf{D}_X$. Set $F = m\mathbf{D}_X - M$. Then $f^*F > -\varepsilon f^*G$ by (2). So

$$m\mathbf{D}_Y + \mathbf{A}_Y = f^*M + f^*F + \mathbf{A}_Y > f^*M - \gamma f^*G + \mathbf{A}_Y.$$

Taking round up and consider their mobile part, we have

$$\operatorname{Mob}\lceil m\mathbf{D}_Y + \mathbf{A}_Y \rceil \geq \operatorname{Mob}(f^*M + \lceil \mathbf{A}_Y - \gamma f^*G \rceil) \geq \operatorname{Mob} f^*M = f^*M. \qquad \blacksquare$$

Now, we have

$$\operatorname{Mob}\lceil m(\mathbf{D}_i)_Y + \mathbf{A}_Y \rceil \leq m(\mathbf{D}_m)_Y \leq m\mathbf{D}_Y.$$

Taking limit in i, we get

$$f^*M \leq \operatorname{Mob}\lceil m\mathbf{D}_Y + \mathbf{A}_Y \rceil \leq m\mathbf{D}_Y,$$

i.e., $(m\mathbf{D} - \overline{M})_Y \geq 0$. But then, $(m\mathbf{D} - \overline{M})_X \geq 0$, a contradiction. Hence, $\mathbf{D}$ is rational.

Finally, the characteristic sequence $\mathbf{D}_\bullet$ is eventually constant and hence $R(X, \mathbf{D}_\bullet)$ is finitely generated by limiting criterion. Indeed, $\mathbf{D}_X$ is now rational and semiample, so there exists $m \in \mathbb{N}$ such that $M = m\mathbf{D}_X$ is integral and free. As before, $(X, B + \gamma G)$ klt, $\lceil \mathbf{A} - \gamma\overline{G} \rceil \geq 0$, and

$$\overline{M}_Y \geq m(\mathbf{D}_m)_Y \geq \operatorname{Mob}\lceil (\overline{M} + \mathbf{A})_Y \rceil \geq \operatorname{Mob}\lceil (\overline{M} + \mathbf{A} - \gamma\overline{G})_Y \rceil \geq \operatorname{Mob}\lceil \overline{M}_Y \rceil = \overline{M}_Y.$$

Hence $(\mathbf{D}_m)_Y = \mathbf{D}_Y$, as desired.