Derived Category

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1 Categories

1.1 Definitions

Definition 1.1. A locally small category $\mathcal{C}$ consists of

- A class $\operatorname{Ob}\mathcal{C}$ of objects.
- For all X, $Y \in \operatorname{Ob}\mathcal{C}$, a set of morphisms

$$\operatorname{Hom}(X, Y) = \{\varphi : X \to Y\}.$$

- A collection of maps: for all X, Y, $Z \in \operatorname{Ob}\mathcal{C}$,

$$\begin{array}{ccc} \operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) & \to & \operatorname{Hom}(X, Z) \\ (\varphi, \psi) & \mapsto & \psi \circ \varphi \end{array}$$

  subject to the following conditions.

- The sets $\operatorname{Hom}(X, Y)$ are pairwise disjoint.
- For each $X \in \operatorname{Ob}\mathcal{C}$, there exists $\mathrm{id}_X \in \operatorname{Hom}(X, X)$ such that $\mathrm{id}_X \circ \varphi = \varphi$ and $\psi \circ \mathrm{id}_X = \psi$.
- $(\varphi \circ \psi) \circ \chi = \varphi \circ (\psi \circ \chi)$

It follows from the definition that id_X is unique.

Definition 1.2. A morphism $\varphi : X \to Y$ in a category $\mathcal{C}$ is called isomorphism if there exists $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$. We say that X and Y are isomorphic.

Definition 1.3. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$, $\mathcal{D}$ consists of

- A map

$$\begin{array}{ccc} \operatorname{Ob}\mathcal{C} & \to & \operatorname{Ob}\mathcal{D} \\ X & \mapsto & F(X). \end{array}$$

- A map
$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{C}}(X,Y) & \to & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \\ \varphi & \mapsto & F(\varphi)\end{array}$$
for all X, $Y \in \mathrm{Ob}\,\mathcal{C}$ such that $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Proposition 1.4. A covariant functor sends isomorphisms to isomorphisms.

Proof. Let $\varphi : X \to Y$ be an isomorphism. Then there exists a morphism $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. Since

$$\begin{aligned}\mathrm{id}_{F(X)} &= F(\mathrm{id}_X) = F(\psi \circ \varphi) = F(\psi) \circ F(\varphi), \\ \mathrm{id}_{F(Y)} &= F(\mathrm{id}_Y) = F(\varphi \circ \psi) = F(\varphi) \circ F(\psi),\end{aligned}$$

$F(\varphi)$ is an isomorphism. ■

Definition 1.5. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is defined similarly, with

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{C}}(X,Y) & \to & \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X)) \\ \varphi & \mapsto & F(\varphi)\end{array}$$

We may view the contravariant functor F as a functor from the opposite category $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition 1.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is called

- full if $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ is surjective for all X, $Y \in \mathrm{Ob}\,\mathcal{C}$;
- faithful if $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ is injective for all X, $Y \in \mathrm{Ob}\,\mathcal{C}$.

Proposition 1.7. Given a fully faithful functor $F : \mathcal{C} \to \mathcal{D}$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Then f is an isomorphism if and only if $F(f)$ is an isomorphism.

Proof. The only if part follows from (1.4). Suppose that $F(f)$ is an isomorphism. Then there exists $\varphi : F(Y) \to F(X)$ such that $\varphi \circ F(f) = \mathrm{id}_{F(X)}$ and $F(f) \circ \varphi = \mathrm{id}_{F(Y)}$. Since $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ is surjective, there exists $g \in \mathrm{Hom}(X,Y)$ such that $F(g) = \varphi$. Then

$$F(g \circ f) = \varphi \circ F(f) = \mathrm{id}_{F(X)} = F(\mathrm{id}_X), \quad F(f \circ g) = F(f) \circ \varphi = \mathrm{id}_{F(Y)} = F(\mathrm{id}_Y).$$

It follows from the injectivity of $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, which shows that f is an isomorphism. ■

Definition 1.8. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is a category $\mathcal{D}$ such that

- $\mathrm{Ob}\,\mathcal{D} \subseteq \mathrm{Ob}\,\mathcal{C}$;
- $\mathrm{Hom}_{\mathcal{D}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all X, $Y \in \mathrm{Ob}\,\mathcal{D}$ and is compatible with compositions and the identity.

We call $\mathcal{D}$ a full subcategory if $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$

1.2 Equivalence of categories

Definition 1.9. Two categories $\mathcal{C}$ and $\mathcal{D}$ are called isomorphic if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.

Remark. Equality of objects is a very restrictive notion. Even objects defined by universal properties (e.g. $X \times Y$) are only unique up to unique isomorphism.

Definition 1.10. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if

- F is fully faithful;
- F is essentially surjective, i.e., for each $Y \in \mathcal{D}$, there exists $X \in \mathcal{C}$ such that $F(X) \cong Y$.

Definition 1.11. Given functors F, $G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\eta : F \to G$ is a collection of morphisms

$$\{\eta(X) : F(X) \to G(X)\}_{X \in \mathrm{Ob}\,\mathcal{C}}$$

such that for each $\varphi : X \to Y$, the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "\eta(X)"] & F(Y) \arrow[d, "\eta(X)"] \\ G(X) \arrow[r, "G(\varphi)"] & G(Y). \end{tikzcd}$$

We define $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ to be the category of functors $F : \mathcal{C} \to \mathcal{D}$, with natural transformations as morphisms. Then a natural isomorphism is an isomorphism in this category.

Theorem 1.12. Two category $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there exists $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \circ \mathrm{id}_{\mathcal{C}}$.

We say that G is a quasi-inverse of F.

Theorem 1.13 (Yoneda's lemma)**.** Let $\mathcal{C}$ be a category, and let $\mathcal{C}^{\mathrm{op}}$ be its opposite category. The functor

$$\begin{array}{rccl} h_\bullet : & \mathcal{C} & \to & \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Sets}) \\ & X & \mapsto & h_X = \mathrm{Hom}(-, X) \end{array}$$

is fully faithful.

A contravariant functor $F : \mathcal{C} \to \mathsf{Sets}$ is called representable if $F \cong h_X$ for some $X \in \mathrm{Ob}\,\mathcal{C}$. Such X is unique up to unique isomorphism by Yoneda's lemma.

Definition 1.14. For all $X, Y \in \mathrm{Ob}\,\mathcal{C}$. We define $X \times Y$ to be the object satisfying the universal property:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times Y \arrow[r] \arrow[d] & X \\ & Y & \end{tikzcd}$$

Proposition 1.15. The object $X \times Y$ is the unique object represents $Z \mapsto \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$.

2 Additive and abelian categories

2.1 Additive category

Let $\mathcal{C}$ be a category.

Definition 2.1. An object $* \in \mathcal{C}$ is called

- initial if $\#\operatorname{Hom}(*, X) = 1$ for each X;
- final if $\#\operatorname{Hom}(X, *) = 1$ for each X.

If $*$ is both initial and final, we call $*$ a zero object.

All these objects are unique up to unique isomorphism.

Definition 2.2. A category $\mathcal{C}$ is called additive if

1) $\operatorname{Hom}(X, Y)$ is an abelian group for all X, $Y \in \operatorname{Ob}\mathcal{C}$ and compositions are bi-additive;

2) the zero object 0 exists;

3) $X \times Y$ (or equivalently, $X \oplus Y$) exists for all X, $Y \in \operatorname{Ob}\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive category is called additive if $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$ is a group homomorphism.

Proposition 2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Then for all X, $Y \in \mathcal{C}$, $F(X \oplus Y) = F(X) \oplus F(Y)$.

Proof. Let $p_X : X \oplus Y \to X$, $p_Y : X \oplus Y \to Y$ be the projections, and let $i_X : X \to X \oplus Y$, $i_Y : Y \to X \oplus Y$ be the inclusions. Then $i_X \circ p_X + i_Y \circ p_Y = \mathrm{id}_{X \oplus Y}$. Given any $f : Z \to X$, $g : Z \to Y$, we see that $h = f \circ F(p_X) + g \circ F(p_Y) : Z \to F(X \oplus Y)$ satisfies

$$h \circ F(i_X) = f \circ F(\mathrm{id}_X) = f, \quad h \circ F(i_Y) = g \circ F(\mathrm{id}_Y) = g.$$

If $h' : Z \to F(X \oplus Y)$ is another morphism such that $h' \circ F(i_X) = f$ and $h' \circ F(i_Y) = g$, then

$$h' = h' \circ F(i_X \circ p_X + i_Y \circ p_Y) = f \circ F(p_X) + g \circ F(p_Y) = h.$$

Hence, $F(X \oplus Y) = F(X) \oplus F(Y)$. ■

2.2 Abelian category

Let $\mathcal{C}$ be an additive catrgory.

Definition 2.4. For each $f : X \to Y$. Define the kernel of f to be the fiber product (if exists)

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y. \end{tikzcd}$$

Define the cokernel of f to be the fiber product (if exists)

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ Y \arrow[r] & \operatorname{coker} f. \end{tikzcd}$$

Proposition 2.5. The kernel of $f : X \to Y$ exists if and only if

$$\begin{array}{rcc} \mathcal{C}^{\mathrm{op}} & \to & \mathsf{Sets} \\ Z & \mapsto & \ker(\operatorname{Hom}(Z, X) \xrightarrow{f\circ} \operatorname{Hom}(Z, Y)) \end{array}$$

is representable.

⚠ The naive analogous statement for $\operatorname{coker} f$ is wrong. The correct statement is $\operatorname{coker}(f)$ exists if and only if

$$\begin{array}{rcc} \mathcal{C} & \to & \mathsf{Sets} \\ Z & \mapsto & \ker(\operatorname{Hom}(Y, Z) \xrightarrow{\circ f} \operatorname{Hom}(X, Z)) \end{array}$$

is co-representable.

We define the image of f to be $\operatorname{Im} f = \ker(Y \to \operatorname{coker} f)$, and the coimage to be $\operatorname{coIm} f = \operatorname{coker}(\ker f \to X)$. The universal properties gives a unique factorization

$$X \to \operatorname{coIm} f \to \operatorname{Im} f \to Y.$$

Definition 2.6. An abelian category is an additive category $\mathcal{C}$ such that

4) for each morphism $f : X \to Y$, $\ker f$ and $\operatorname{coker} f$ exists;

5) the canonical map $\operatorname{coIm} f \to \operatorname{Im} f$ is an isomorphism.

2.3 Exact sequences

Let $\mathcal{C}$ be an abelian category. A sequence

$$\cdots \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \cdots$$

in $\mathcal{C}$ is called exact if $\ker f_i = \operatorname{Im} f_{i-1}$ for each i. A short exact sequence is an exact sequence of the form

$$0 \to X \to Y \to Z \to 0.$$

A covariant additive functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is called

- left exact if for every short exact sequence
$$0 \to X \to Y \to Z \to 0,$$
the sequence
$$0 \to F(X) \to F(Y) \to F(Z)$$
is exact;
- right exact if for every short exact sequence
$$0 \to X \to Y \to Z \to 0,$$
the sequence
$$F(X) \to F(Y) \to F(Z) \to 0$$
is exact;
- exact if F is both left and right exact.

Theorem 2.7 (Freyd-Mitchell Embedding Theorem)**.** For every small abelian category $\mathcal{C}$, there exists a fully faithful exact functor $F : \mathcal{C} \to R\text{-}\mathsf{Mod}$ for some ring R.

This allows us to manipulate small abelian category as if they were category of R-modules.

2.4 Adjoint functors

Let $\mathcal{C}$, $\mathcal{D}$ be arbitrary categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

Definition 2.8. We say that F is the left adjoint of G, and G is the right adjoint of F, if

$$\operatorname{Hom}_{\mathcal{D}}(F(-), -) \cong \operatorname{Hom}_{\mathcal{C}}(-, G(-))$$

in $\operatorname{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathsf{Sets})$. In this case, we write $F \dashv G$.

We may represent an adjoint pair as a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\dashv"] & \mathcal{D}. \arrow[l, bend left, "G"] \end{tikzcd}$$

Proposition 2.9. Left adjoint and right adjoint are unique.

Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories, and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are additive functors.

Proposition 2.10. If $F \dashv G$, then F is right exact and G is left exact.

3 Derived categories: first definition

3.1 Complexes

Let $\mathcal{A}$ be an abelian category. A complex $K^\bullet$ in $\mathcal{A}$ is a sequence

$$\cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$$

of morphisms in $\mathcal{A}$ such that $d^i \circ d^{i-1}$ for all $i \in \mathbb{Z}$. A morphism of complexes $f^\bullet : K^\bullet \to L^\bullet$ is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & K^{i-1} \arrow[r, "d"] \arrow[d, "f^{i-1}"] & K^i \arrow[r, "d"] \arrow[d, "f^i"] & K^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r] & L^{i-1} \arrow[r, "d"] & L^i \arrow[r, "d"] & L^{i+1} \arrow[r] & \cdots . \end{tikzcd}$$

A morphism of complexes $f^\bullet : K^\bullet \to L^\bullet$ is called a quasi-isomorphism if

$$H^n(f^\bullet) : H^n(K^\bullet) \xrightarrow{\sim} H^n(L^\bullet)$$

for all n. Define $\mathrm{Kom}(\mathcal{A})$ to be the complexes in $\mathcal{A}$. Quite often, we will also consider various full subset of bounded complexes:

$$\mathrm{Kom}^+(\mathcal{A}) := \{K^\bullet \mid K^i = 0,\ i \le i_0(K^\bullet)\},$$

$$\mathrm{Kom}^-(\mathcal{A}) := \{K^\bullet \mid K^i = 0,\ i \ge i_0(K^\bullet)\},$$

$$\mathrm{Kom}^{\mathrm{b}}(\mathcal{A}) := \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A}).$$

Throughout this section, let ? be $\varnothing$, $+$, $-$, or b.

Proposition 3.1. If $\mathcal{A}$ is abelian, then $\mathrm{Kom}^?(\mathcal{A})$ is also abelian.

We define the shift functor $-[n] : \mathrm{Kom}^?(\mathcal{A}) \to \mathrm{Kom}^?(\mathcal{A})$ as follows: for $K^\bullet \in \mathrm{Kom}^?(\mathcal{A})$, we define $K^\bullet[n]$ by $K[n]^i = K^{n+i}$ and $d_{K^\bullet[n]} = (-1)^n d_{K^\bullet}$. Given $f^\bullet : K^\bullet \to L^\bullet$, we define $f^\bullet[n] : K^\bullet[n] \to L^\bullet[n]$ by $f[n]^i = f^{n+i}$.

3.2 Localization of a category

Let $\mathcal{B}$ be a category and let S be a collection of morphisms (in $\mathcal{B}$).

Definition 3.2. A strict localization of $\mathcal{B}$ by S is

- a category $S^{-1}\mathcal{B}$;
- a functor $Q : \mathcal{B} \to S^{-1}\mathcal{B}$ that sends S to isomorphisms and satisfies the following universal property: for every functor $F : \mathcal{B} \to \mathcal{D}$ sending S to isomorphisms, we have

$$\begin{tikzcd} \mathcal{B} \arrow[r, "Q"] \arrow[dr, "F"'] & S^{-1}\mathcal{B} \arrow[d, "\exists!"] \\ & \mathcal{D} \end{tikzcd}$$

Definition 3.3. If $Q : \mathcal{B} \to S^{-1}\mathcal{B}.$ sends S to isomorphisms and the following weaker universal property:

> for every functor $F : \mathcal{B} \to \mathcal{D}$ sending S to isomorphisms, there exists $\Phi : S^{-1}\mathcal{B} \to \mathcal{D}$, unique up to (natural) isomorphisms, such that $\Phi \circ Q =\cong F$,

we call $S^{-1}\mathcal{B}$ a localization of $\mathcal{B}$ by S.

Theorem 3.4. Strict localization exists as a large category.

Definition 3.5. The derived category $D^?(\mathcal{A})$ of $\mathcal{A}$ is the localization of $\mathrm{Kom}^?(\mathcal{A})$ by the quasi-isomorphisms.

The proof of theorem is easy but useless in practice: Simply set $\mathrm{Ob}(S^{-1}\mathcal{B}) = \mathrm{Ob}\,\mathcal{B}$ and

$$\mathrm{Hom}_{S^{-1}\mathcal{B}}(X, Y) = \{X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_n = Y\}/\sim .$$

Here, each $X_i \leftrightarrow X_{i+1}$ is either $X_i \to X_{i+1} \in \mathrm{Hom}_{\mathcal{B}}(X_i, X_{i+1})$ or $X_i \leftarrow X_{i+1} \in S$, and the equivalence relation $\sim$ is defined by: $\varphi \sim \psi$ if we can transform φ to ψ through the following:

$$(W_1 \xrightarrow{f} W_2 \xrightarrow{g} W_3) \sim (W_1 \xrightarrow{g \circ f} W_3)$$
$$(W_1 \xrightarrow{s} W_2 \xleftarrow{s} W_1) \sim (W_1 \xrightarrow{\mathrm{id}_{W_1}} W_1)$$
$$(W_1 \xleftarrow{s} W_2 \xrightarrow{s} W_1) \sim (W_1 \xrightarrow{\mathrm{id}_{W_1}} W_1)$$

It is difficult to tell whether $\varphi = \psi$ in $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y)$ and not clear whether $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y)$ is a set. So we want good representative of φ, e.g., $\varphi = X \leftarrow X' \to Y$.

To deal with these problems, instead of working with $\mathrm{Kom}(\mathcal{A})$, we work with the homotopy category $K(\mathcal{A})$.

4 Derived categories: definition through homotopy categories

4.1 Homotopy category

Let $K^\bullet$, $L^\bullet \in \mathrm{Kom}(\mathcal{A})$. Let $k^i : K^i \to L^{i-1}$ be morphisms. Define $h^i = k^{i+1}d + dk^i$: $K^i \to L^i$.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{i-1} \arrow[r, "d"] \arrow[d, "h^{i-1}"] & K^i \arrow[r, "d"] \arrow[d, "h^i"] \arrow[dl, "k^i"] & K^{i+1} \arrow[r] \arrow[d, "h^{i+1}"] \arrow[dl, "k^{i+1}"] & \cdots \\
\cdots \arrow[r] & L^{i-1} \arrow[r, "d"] & L^i \arrow[r, "d"] & L^{i+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Then $h^\bullet : K^\bullet \to L^\bullet$ is a morphism of complexes.

Definition 4.1. The morphism $h^\bullet$ is said to be homotopic to 0, written as $h \sim 0$.

Proposition 4.2. The collection $\{h \sim 0\}$ forms an ideal, i.e., for all $h_1^\bullet$, $h_2^\bullet : K^\bullet \to L^\bullet$ homotopic to 0,

- $h_1^\bullet + h_2^\bullet \sim 0$;
- $f^\bullet \circ h_1^\bullet \sim 0$ for all $f^\bullet : L^\bullet \to M^\bullet$

- $h_1^\bullet \circ g^\bullet \sim 0$ for all $g : N^\bullet \to K^\bullet$.

We say that φ, $\psi : K^\bullet \to L^\bullet$ are homotopic ($\varphi \sim \psi$) if $\varphi - \psi \sim 0$.

Proposition 4.3. If $\varphi \sim \psi$, then the morphisms

$$H^\bullet(\varphi),\, H^\bullet(\psi) : H^\bullet(K^\bullet) \to H^\bullet(L^\bullet)$$

are equal. In particular, if φ is a quasi-isomorphism and $\varphi \sim \psi$, then ψ is also a quasi-isomorphism.

Definition 4.4. The homotopy category $K^?(\mathcal{A})$ is defined by

- $\operatorname{Ob} K^?(\mathcal{A}) = \operatorname{Ob} \operatorname{Kom}^?(\mathcal{A})$; and
- $\operatorname{Mor} K^?(\mathcal{A}) = \operatorname{Mor} \operatorname{Kom}^?(\mathcal{A})/\sim$.

Proposition 4.5. The localization of $K^?(\mathcal{A})$ by quasi-isomorphisms is canonically isomorphic to $D^?(\mathcal{A})$

Proof. Let S be the collection of quasi-isomorphisms and let $\tilde{D}(\mathcal{A}) = S^{-1}K(\mathcal{A})$. Then $\operatorname{Kom}(\mathcal{A}) \to K(\mathcal{A}) \to \tilde{D}(\mathcal{A})$ sends quasi-isomorphisms to isomorphisms, so that there exists a unique functor $G : D(\mathcal{A}) \to \tilde{D}(\mathcal{A})$ such that the following diagram commute:

$$\begin{tikzcd}
\operatorname{Kom}(\mathcal{A}) \arrow[r] \arrow[drr] & K(\mathcal{A}) \arrow[r] & \tilde{D}(\mathcal{A}) \\
& & D(\mathcal{A}). \arrow[u, "G"]
\end{tikzcd}$$

It is clear that G is a bijection on objects. Choose a section $K(\mathcal{A}) \to \operatorname{Kom}(\mathcal{A})$ of $\operatorname{Kom}(\mathcal{A}) \to K(\mathcal{A})$. The universal property gives

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r] \arrow[d, "\tilde{Q}"] \arrow[rr, bend left, "\mathrm{id}"] & \operatorname{Kom}(\mathcal{A}) \arrow[r] \arrow[d, "Q"] & K(\mathcal{A}) \arrow[d, "\tilde{Q}"] \\
\tilde{D}(\mathcal{A}) \arrow[r, "\exists"] \arrow[rr, bend right, "\mathrm{id}"'] & D(\mathcal{A}) \arrow[r, "G"] & \tilde{D}(A)
\end{tikzcd}$$

Thus G is surjective on morphisms. That G is injective on morphisms follows from

Lemma 4.6. If $f^\bullet \sim g^\bullet : K^\bullet \to L^\bullet$, then $Q(f^\bullet) = Q(g^\bullet)$.

■

4.2 Morphisms in $\boldsymbol{D}^?(\mathcal{A})$

Proposition 4.7.

1) $\operatorname{Hom}_{D^?(\mathcal{A})}(X, Y) = \{(s, f) \mid X \overset{s \in S}{\longleftarrow} X' \overset{f}{\to} Y\}/\sim$, where $(s, f) \sim (s', f')$ if we have the following commutative diagram in $K^?(\mathcal{A})$:

$$\begin{tikzcd}
 & & X''' \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "S\ni"'] & & \\
 & X' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X'' \arrow[dlll, "s'" near end] \arrow[dr, "f'"] & \\
X & & & & Y.
\end{tikzcd}$$

$\sim$ is an equivalence relation.

2) Given $X \overset{s}{\leftarrow} X' \overset{f}{\to} Y$ and $Y \overset{t}{\leftarrow} Y' \overset{g}{\to} Z$ with s, t quasi-isomorphism. There exists a commutative diagram in $K^?(\mathcal{A})$:

$$\begin{tikzcd}
 & & X'' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

such that u is a quasi-isomorphism.

Definition 4.8. Given a category $\mathcal{C}$ and a collection of morphisms S, we say that S is a localizing system if

(LS1) $\mathrm{id}_X \in S$ for all object S, $S \circ S \subseteq S$;

(LS2) (extension property) for all such diagrams

$$\begin{tikzcd}
 & Z \arrow[d, "s\in S"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "s'\in S"] & X' \\
Z', &
\end{tikzcd}$$

there exists $W \xrightarrow{t} X \in S$, $W \xrightarrow{g} Z$, $X' \xrightarrow{t'} W' \in S$ and $Z' \xrightarrow{g'} W'$ such that the following diagrams commute:

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "s'"] & X' \arrow[d, "t'"] \\ Z' \arrow[r, "g'"] & W'; \end{tikzcd}$$

(LS3) for all $f,\, g : X \to Y$, $f \circ s = g \circ s$ for some $s \in S$ if and only if $t \circ f = t \circ g$ for some $t \in S$.

Lemma 4.9. The collection of quasi-isomorphisms in $K^?(\mathcal{A})$ forms a localizing system.

We show that the lemma implies the proposition:

Step 1. By (LS1) and (LS2), the elements in $\mathrm{Hom}_{D^?(\mathcal{A})}(X,Y)$ can be represented by $(X \xrightarrow{s} X', X' \xrightarrow{f} Y)$ and (LS2) also gives the existence of composition (we still need to check that it is well-defined).

Step 2. We prove that $\sim$ is an equivalence relation. Transitivity is the least obvious. Assume $(X \xleftarrow{s} X' \xrightarrow{f} Y) \sim (X \xleftarrow{s'} X'' \xrightarrow{f'} Y)$ and $(X \xleftarrow{s'} X'' \xrightarrow{f'} Y) \sim (X \xleftarrow{s''} X''' \xrightarrow{f''} Y)$. Then, by definition, we have the following diagram:

$$\begin{tikzcd} & Z \arrow[dl, "t"'] \arrow[dr, "g"] & & Z' \arrow[dl, "t'"'] \arrow[dr, "g'"] & \\ X' \arrow[d, "s"'] \arrow[ddrrrr, "f" near end] & & X'' \arrow[ddll, "s'"' near start] \arrow[ddrr, "f'"] & & X''' \arrow[d, "f''"] \arrow[ddllll, "s''" near end] \\ X & & & & Y, \end{tikzcd}$$

with $s \circ t$ and $s' \circ t'$ being quasi-isomorphisms. It follows from (LS2) that we may complete the diagram

$$\begin{tikzcd} \color{blue}{W} \arrow[r, blue, "h"] \arrow[d, blue, "u \in S"] & Z' \arrow[d, "s' \circ t'"] \\ Z \arrow[r, "s \circ t"] & X. \end{tikzcd}$$

Since $s' \circ (g \circ u) = s \circ t \circ u = s' \circ (t \circ h)$, (LS3) gives a quasi-isomorphism $v : V \to W$ such that $(g \circ u) \circ v = (t \circ h) \circ v$.

$$\begin{tikzcd} V \arrow[r, "v"] & W \arrow[r, bend left, "g \circ u"] \arrow[r, bend right, "t \circ h"'] & X'' \arrow[r, "s'"] & X \end{tikzcd}$$

We verify that

$$\begin{tikzcd}
& & V \arrow[dl, "t\circ u\circ v"'] \arrow[dr, "g'\circ h\circ v"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "f"] & & X''' \arrow[dlll, "s''"] \arrow[dr, "f''"] & \\
X & & & & Y.
\end{tikzcd}$$

defines the desired equivalence.

Step 3. We check the composition is well-defined. First, given

$$\begin{tikzcd}
& X'' \arrow[d, "u"'] \arrow[drr, "h"] & & X''' \arrow[dll, "u'"] \arrow[d, "h'"] & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z.
\end{tikzcd}$$

We need to show that $(s\circ u, g\circ h) \sim (s\circ u', g\circ h')$. Take W so that we may complete the diagram

$$\begin{tikzcd}
W \arrow[r, "v\in S"] \arrow[d, "k"] & X'' \arrow[d, "u"] \\
X''' \arrow[r, "u'"] & X'.
\end{tikzcd}$$

Since $t\circ(h\circ v) = f\circ u\circ v = f\circ u'\circ k = t\circ(h'\circ k)$, (LS3) gives a quasi-isomorphism $w : V \to W$ such that $(h\circ v)\circ w = (h'\circ k)\circ w$.

$$\begin{tikzcd}
V \arrow[r, "w"] & W \arrow[r, bend left, "h\circ v"] \arrow[r, bend right, "h'\circ k"'] & Y' \arrow[r, "t"] & Y
\end{tikzcd}$$

We check that

$$\begin{tikzcd}
& & V \arrow[dl, "t\circ u\circ v"'] \arrow[dr, "g'\circ h\circ v"] & & \\
& X'' \arrow[dl, "s\circ u"'] \arrow[drrr, "g\circ h"] & & X''' \arrow[dlll, "s\circ u'"] \arrow[dr, "g\circ h'"] & \\
X & & & & Y.
\end{tikzcd}$$

defines the desired equivalence. This shows that given (s, f) and (t, g), the composition $(t, g)\circ(s, f)$ is well-defined.

Next, we need to show that if $(X \xleftarrow{s} X' \xrightarrow{f} Y) \sim (X \xleftarrow{s'} X'' \xrightarrow{f'} Y)$, then for all $(Y \xleftarrow{t} Y' \xrightarrow{g} Z)$, $(t, g)\circ(s, f) \sim (t, g)\circ(s', f')$. Since $(s, f) \sim (s', f')$, there exists X''' and morphisms $X''' \xrightarrow{h} X'$, $X''' \xrightarrow{h'} X''$ such that $s \circ h = s' \circ h'$ is a quasi-isomorphism. So we may replace X'' by X''' so that there exists a morphism $X'' \xrightarrow{k} X'$ such that $s' = s \circ k$.

$$\begin{tikzcd}
 & & & W \arrow[dl, "u"'] \arrow[dr, "\ell"] & & \\
 & X' \arrow[dl, "s"'] \arrow[drr, "f"] & X'' \arrow[l, "k"'] \arrow[dll, "s'"] \arrow[dr, "f'"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
 X & & & Y & & Z
\end{tikzcd}$$

Take W that complete the above diagram with $u \in S$, then

$$(t \circ g) \circ (s, f) = (s \circ (k \circ u), g \circ \ell) = (s' \circ u, g \circ \ell) = (t \circ g) \circ (f', s').$$

Finally, we need to show that if $(Y \xleftarrow{t} Y' \xrightarrow{g} Z) \sim (Y \xleftarrow{t'} Y'' \xrightarrow{g'} Z)$, then for all $(X \xleftarrow{s} X' \xrightarrow{f} Y)$, $(t, g) \circ (s, f) \sim (t', g') \circ (s, f)$. Since $(t, g) \sim (t', g')$, there exists Y''' and morphisms and morphisms $Y''' \xrightarrow{h} Y'$, $Y''' \xrightarrow{h'} X''$ such that $t \circ h = t' \circ h'$ is a quasi-isomorphism. So we may replace Y'' by Y''' so that there exists a morphism $Y'' \xrightarrow{k} Y'$ such that $t' = t \circ k$.

$$\begin{tikzcd}
 & & W \arrow[dl, "u"'] \arrow[dr, "\ell"] & & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y'' \arrow[r, "k"] \arrow[dl, "t'"'] \arrow[drr, "g'"'] & Y' \arrow[dll, "t"] \arrow[dr, "g"] & \\
 X & & Y & & & Z
\end{tikzcd}$$

Take W that complete the above diagram with $u \in S$, then

$$(t \circ g) \circ (s, f) = (s \circ u, g \circ k \circ \ell) = (s \circ u, g' \circ \ell) = (t' \circ g') \circ (f, s).$$

Step 4. Define the category $\mathrm{Roof}(\mathcal{A})$ by taking $\mathrm{Ob}\,\mathrm{Roof}(\mathcal{A}) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ and

$$\mathrm{Hom}_{\mathrm{Roof}(\mathcal{A})}(X, Y) = \{X \xleftarrow{s \in S} Z \xrightarrow{f} Y\} / \sim .$$

We need to show that the composition is associative. Given

$$\begin{tikzcd}
 & W' \arrow[dl, "s"'] \arrow[dr, "f"] & & X' \arrow[dl, "t"'] \arrow[dr, "g"] & & Y' \arrow[dl, "u"'] \arrow[dr, "h"] & \\
 W & & X & & Y & & Z.
\end{tikzcd}$$

By (LS2) we can take W'', X'', and W''' that completes the diagram

$$\begin{tikzcd}
 & & & W''' \arrow[dl, "s''"'] \arrow[dr, "m"] & & & \\
 & & W'' \arrow[dl, "s'"'] \arrow[dr, "k"] & & X'' \arrow[dl, "t'"'] \arrow[dr, "\ell"] & & \\
 & W' \arrow[dl, "s"'] \arrow[dr, "f"] & & X' \arrow[dl, "t"'] \arrow[dr, "g"] & & Y' \arrow[dl, "u"'] \arrow[dr, "h"] & \\
W & & X & & Y & & Z
\end{tikzcd}$$

with s', t', and s'' being quasi-isomorphisms. Then

$$\begin{aligned}
(u,h)\circ((t,g)\circ(s,f)) = (u,h)\circ(s'\circ s, g\circ k) &= (s\circ s'\circ s'', h\circ\ell\circ m) \\
&= (t\circ t', h\circ\ell)\circ(s,f) = ((u,h)\circ(t,g))\circ(s,f).
\end{aligned}$$

Let $F : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Roof}(\mathcal{A})$ be a functor defined by $F(X) = X$ and $F(f) = (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y)$. It remains to show that $D(\mathcal{A}) = \mathrm{Roof}(\mathcal{A})$. We show that for each functor $H : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ that sends quasi-isomorphisms to isomorphisms, there exists a unique functor $G : \mathrm{Roof}(\mathcal{A}) \to \mathcal{D}$ such that $H = G \circ F$.

Step 5. (uniqueness) Assuming G exists. Since $F = \mathrm{id}$ on objects, $G \circ F = H$ gives $G(X) = H \circ F^{-1}(X)$. Let $\varphi = (s,f)$ be a morphism in $\mathrm{Roof}(\mathcal{A})$. Then $\varphi \circ F(s) = F(f)$. Apply G to both sides gives $G(\varphi) \circ H(s) = H(f)$. Since $H(s)$ is invertible, $G(\varphi) = H(f) \circ H(s)^{-1}$. So G is uniquely determined.

Step 6. (existence) The uniqueness of G suggest us to construct G as follows: for $X \in \mathrm{Roof}(\mathcal{A})$, define $G(X) = H \circ F^{-1}(X)$; for $\varphi = (s,f) \in \mathrm{Hom}_{\mathrm{Roof}(\mathcal{A})}$, define $G(\varphi) = H(f) \circ H(s)^{-1}$.

We check that G is a well-defined functor. If $(X' \xrightarrow{s} X, f) \sim (X'' \xrightarrow{s'} X, f') \in \mathrm{Hom}_{\mathrm{Roof}(\mathcal{A})}(X,Y)$, then there exists X''' and morphisms $X''' \xrightarrow{h} X'$, $X''' \xrightarrow{h'} X''$ such that $s'' = s \circ h = s' \circ h'$ is a quasi-isomorphism.

$$\begin{tikzcd}
 & & X''' \arrow[dl, "h"'] \arrow[dr, "h'"] \arrow[ddll, bend right, "s''"'] & & \\
 & X' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X'' \arrow[dlll, "s'" near end] \arrow[dr, "f'"] & \\
X & & & & Y.
\end{tikzcd}$$

Hence,

$$\begin{aligned} H(f) \circ H(s)^{-1} &= H(f) \circ H(h) \circ H(s'')^{-1} \\ &= H(f') \circ H(h') \circ H(s'')^{-1} = H(f') \circ H(s')^{-1}. \end{aligned}$$

So G is well-defined on the morphisms. It remains to check that $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$ and $G(\psi \circ \varphi) = G(\psi) \circ G(\varphi)$. Since $\mathrm{id}_X = (\mathrm{id}_X, \mathrm{id}_X)$,

$$G(\mathrm{id}_X) = H(\mathrm{id}_X) \circ H(\mathrm{id}_X)^{-1} = \mathrm{id}_{H(X)} \mathrm{id}_{H(X)}^{-1} = \mathrm{id}_{G(X)} .$$

Let $\varphi = (s, f)$ and let $\psi = (t, g)$. Then $\psi \circ \varphi = (s \circ u, g \circ h)$ for some u, h such that $f \circ u = t \circ h$ and u is a quasi-isomorphism.

$$\begin{tikzcd} & & X'' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

Hence,

$$\begin{aligned} H(\psi \circ \varphi) &= H(g \circ h) \circ H(s \circ u)^{-1} = H(g) \circ H(h) \circ H(u)^{-1} \circ H(s)^{-1} \\ &= H(g) \circ H(t)^{-1} \circ H(f) \circ H(s)^{-1} = H(\psi) \circ H(\varphi). \end{aligned}$$

This completes the proof of (4.7).

4.3 Mapping cones

Given $f^\bullet : K^\bullet \to L^\bullet$, the cone of f is the complex $C(f)^\bullet$ defined by $C(f)^i = K^{i+1} \oplus L^i$, i.e., $C(f)^\bullet = K^\bullet[1] \oplus L^\bullet$, and

$$d^i_{C(f)} \begin{pmatrix} k^{i+1} \\ \ell^i \end{pmatrix} = \begin{pmatrix} -d_K^{i+1} & 0 \\ f^{i+1} & d_L^i \end{pmatrix} \begin{pmatrix} k^{i+1} \\ \ell^i \end{pmatrix}.$$

We can easily check that

$$d^{i+1}_{C(f)} d^i_{C(f)} = \begin{pmatrix} -d_K^{i+2} & 0 \\ f^{i+2} & d_L^{i+1} \end{pmatrix} \begin{pmatrix} -d_K^{i+1} & 0 \\ f^{i+1} & d_L^i \end{pmatrix} = \begin{pmatrix} (-d_K^{i+2})(-d_K^{i+1}) & 0 \\ -f^{i+2} d_K^{i+1} + d_L^{i+1} f^{i+1} & d_L^{i+1} d_L^i \end{pmatrix} = 0.$$

We have natural maps

$$\begin{aligned} K^\bullet \xrightarrow{\ f\ } L^\bullet &\xrightarrow{\quad \tau \quad} C(f)^\bullet \xrightarrow{\qquad} K^\bullet[1] \\ \ell^i &\longmapsto \left(\begin{smallmatrix} 0 \\ \ell^i \end{smallmatrix}\right), \ \left(\begin{smallmatrix} k^{i+1} \\ \ell^i \end{smallmatrix}\right) \longmapsto k^{i+1}. \end{aligned}$$

The short exact sequence

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

induces to a long exact sequence

$$\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(C(f)^\bullet) \to H^{i+1}(K^\bullet) \to \cdots .$$

Given morphisms f_1, f_2, α, β with $\beta \circ f_1 = f_2 \circ \alpha$. We can find $\gamma : C(f_1)^\bullet \to C(f_2)^\bullet$ such that the following diagram commute.

$$\begin{tikzcd}
K_1^\bullet \arrow[r, "f_1"] \arrow[d, "\alpha"] & L_1^\bullet \arrow[r] \arrow[d, "\beta"] & C(f_1)^\bullet \arrow[r] \arrow[d, dashed, "\gamma"] & K_1^\bullet[1] \arrow[d, "\alpha[1]"] \\
K_2^\bullet \arrow[r, "f_2"] & L_2^\bullet \arrow[r] & C(f_2)^\bullet \arrow[r] & K_2^\bullet[1].
\end{tikzcd} \tag{TR3}$$

In fact, we can simply take $\gamma = \alpha[1] \oplus \beta$, so that

$$\begin{aligned}
\gamma^{i+1} d^i_{C(f_1)} - d^i_{C(f_2)} \gamma^i &= \begin{pmatrix} \alpha^{i+2} & 0 \\ 0 & \beta^{i+1} \end{pmatrix} \begin{pmatrix} -d^{i+1}_{K_1} & 0 \\ f_1^{i+1} & d^i_{L_1} \end{pmatrix} - \begin{pmatrix} -d^{i+1}_{K_2} & 0 \\ f_2^{i+1} & d^i_{L_2} \end{pmatrix} \begin{pmatrix} \alpha^{i+1} & 0 \\ 0 & \beta^i \end{pmatrix} \\
&= \begin{pmatrix} -\alpha^{i+2} d^{i+1}_{K_1} + d^{i+1}_{K_2} \alpha^{i+1} & 0 \\ \beta^{i+1} f_1^{i+1} - f_2^{i+1} \alpha^{i+1} & \beta^{i+1} d^i_{L_1} - d^i_{L_2} \beta^i \end{pmatrix} = 0.
\end{aligned}$$

Proposition 4.10 (TR2)**.** Given $f : K^\bullet \to L^\bullet$, there is a morphism $g : K^\bullet[1] \to C(\tau)$ such that in $K(\mathcal{A})$, g is an isomorphism and the following diagram commute.

$$\begin{tikzcd}
L^\bullet \arrow[r, "\tau"] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, equal] & K^\bullet[1] \arrow[r, "{f[1]}"] \arrow[d, dashed, "g"] & L^\bullet[1] \arrow[d, equal] \\
L^\bullet \arrow[r, "\tau"] & C(f)^\bullet \arrow[r] & C(\tau)^\bullet \arrow[r] & L^\bullet[1].
\end{tikzcd}$$

Proof. Note that $C(\tau)^i = L^{i+1} \oplus C(f)^i = L^{i+1} \oplus K^{i+1} \oplus L^i$. We define $g^i(k^{i+1}) = (-f^{i+1}(k^{i+1}), k^{i+1}, 0)$. It can be checked easily that g makes the above diagram commute. In $K(\mathcal{A})$, we can check that the projection $C(\tau)^i = L^{i+1} \oplus K^{i+1} \oplus L^i \to K^{i+1}$ is the inverse of g. ∎

Using the mapping cone, we can now prove (4.9).

Proof of (4.9). (LS1) is obvious. Given

$$\begin{tikzcd}
 & Z \arrow[d, "s \in S"] & & Y' \arrow[r, "f'"] \arrow[d, "s' \in S"] & X' \\
X \arrow[r, "f"] & Y & & Z', &
\end{tikzcd}$$

We have the morphisms

$$Z \xrightarrow{s} Y \xrightarrow{\tau} C(s) \longrightarrow Z[1],$$

which is isomorphic to

$$C(\tau)[-1] \longrightarrow Y \xrightarrow{\tau} C(s) \longrightarrow C(\tau),$$

in $K^?(a)$ by (4.10). (TR3) gives us a commutative diagram

$$\begin{tikzcd}
C(\tau\circ f)[-1] \arrow[r, "t"] \arrow[d] & X \arrow[r, "\tau\circ f"] \arrow[d, "f"] & C(s) \arrow[r] \arrow[d, equal] & C(\tau\circ f) \arrow[d] \\
C(\tau)[-1] \arrow[r] & Y \arrow[r, "\tau"] & C(s) \arrow[r] & C(\tau),
\end{tikzcd}$$

Define $W = C(\tau\circ f)[-1]$. Since s is a quasi-isomorphism, $H^\bullet(C(s)) = 0$, thus $t : W \to X$ is also a quasi-isomorphism. The other square can be complete similarly.

For (LS3), it is enough to show that for all $f \in \mathrm{Hom}_{K^?(\mathcal{A})}(K^\bullet, L^\bullet)$, $sf \sim 0$ for some quasi-isomorphism s if and only if $ft \sim 0$ for some quasi-isomorphism t. We only prove the only if part, the if part can be done similarly. Suppose $s : L^\bullet \to M^\bullet$ and $sf = hd + dh$ for some $h : K^\bullet \to M^\bullet[-1]$. Define $g^\bullet : K^\bullet \to C(s)^\bullet[-1] = L^\bullet \oplus M^\bullet[-1]$ by letting $g^i(k^i) = (f^i(k^i), -h^i(k^i))$. We check that g is indeed a morphism:

$$\begin{aligned}
g^{i+1}d_K^i - d^i_{C(s)[-1]}g^i &= \begin{pmatrix} f^{i+1} \\ -h^{i+1} \end{pmatrix} d_K^i - \begin{pmatrix} d_L^i & 0 \\ -s^i & -d_M^{i-1} \end{pmatrix}\begin{pmatrix} f^i \\ -h^i \end{pmatrix} \\
&= \begin{pmatrix} f^{i+1}d_K^i - d_L^i f^i \\ -h^{i+1}d_K^i + s^i f^i - d_M^{i-1}h^i \end{pmatrix} = 0.
\end{aligned}$$

It is clear that f is the composition of g and the projection map $C(s)^\bullet[-1] \to L^\bullet$.

$$\begin{tikzcd}
C(g)^\bullet[-1] \arrow[r, "t"] & K^\bullet \arrow[r, "g"] \arrow[drr, "f"] & C(s)^\bullet[-1] \arrow[d, equal] & & \\
 & & C(s)^\bullet[-1] \arrow[r] & L^\bullet \arrow[r, "s"] & M^\bullet
\end{tikzcd}$$

Since s is a quasi-isomorphism, $H^\bullet(C(s)^\bullet) = 0$, and thus the projection map $t : C(g)[-1] \to K^\bullet$ is also a quasi-isomorphism. Define $k^\bullet$ by

$$\begin{tikzcd}
C(g)^\bullet[-1] \arrow[r, "k^\bullet"] \arrow[d, equal] & C(s)^\bullet[-2] \arrow[d, equal] \\
K^\bullet \oplus L^\bullet[-1] \oplus M^\bullet[-2] \arrow[r, "\left(\begin{smallmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)"] & L^\bullet[-1] \oplus M^\bullet[-2].
\end{tikzcd}$$

Since

$$
\begin{aligned}
d^i_{C(s)[-1]} &= \begin{pmatrix} d^i_L & 0 \\ -s^i & -d^{i-1}_{M[-1]} \end{pmatrix}, \\
d^i_{C(g)[-1]} &= \begin{pmatrix} d^i_K & 0 \\ -g^i & -d^{i-1}_{C(s)[-1]} \end{pmatrix} = \begin{pmatrix} d^i_L & 0 & 0 \\ -f^i & -d^{i-2}_L & 0 \\ h^i & s^{i-1} & d^{i-2}_{M[-1]}, \end{pmatrix}
\end{aligned}
$$

we see that

$$
\begin{aligned}
k^{i+1}d^i_{C(g)[-1]} + d^{i-1}_{C(s)[-1]}k^i &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d^i_L & 0 & 0 \\ -f^i & -d^{i-2}_L & 0 \\ h^i & s^{i-1} & d^{i-2}_{M[-1]}, \end{pmatrix} \\
&\quad + \begin{pmatrix} d^i_L & 0 \\ -s^i & -d^{i-1}_{M[-1]} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} f^i & 0 & 0 \\ -h^i & 0 & 0 \end{pmatrix} = gt,
\end{aligned}
$$

i.e., $gt \sim 0$. Hence, $ft \sim 0$. ■

4.4 Derived categories are additive

Let $\mathcal{A}$ be an abelian category. Given φ, $\varphi' \in \mathrm{Hom}_{D^?(\mathcal{A})}(X, Y)$, which are represented by $(Z \xrightarrow{s} X, f)$ and $(Z' \xrightarrow{s'} X, f')$, respectively. We construct $\varphi + \varphi' \in \mathrm{Hom}_{D^?(\mathcal{A})}(X, Y)$ as follows. By extension property (LS2), there exists Z'' and morphisms $Z'' \xrightarrow{h} Z$, $Z'' \xrightarrow{h'} Z'$ such that $s \circ h = s' \circ h'$ is a quasi-isomorphism. Then we define $\varphi + \varphi' = (s \circ h = s' \circ h', f \circ h + f' \circ h')$.

$$
\begin{tikzcd}
 & Z'' \arrow[dl, "h"'] \arrow[dr, "h'"] & \\
Z \arrow[d, "s"'] \arrow[drr, "f"] & & Z' \arrow[dll, "s'"'] \arrow[d, "f'"] \\
X & & Y
\end{tikzcd}
$$

With this definition, we verify that

Proposition 4.11. $D^?(\mathcal{A})$ is additive.

4.5 Localization of subcategories

Proposition 4.12. We have $D^{\mathrm{b}}(X) \subseteq D^{\pm}(X) \subseteq D(X)$ as full subcategory.

Let $\mathcal{C}$ be a category, $S \subseteq \operatorname{Mor} \mathcal{C}$. For $\mathcal{B} \subseteq \mathcal{C}$ a full subcategory, define $S_\mathcal{B} = S \cap \operatorname{Mor} \mathcal{C}$.

Question. When does the natural functor $S_\mathcal{B}^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ realize S_B^{-1} as a full subcategory?

Lemma 4.13. Assume that S is a localizing system satisfying

a) $S_\mathcal{B}$ is a localizing system;

b) For each $X' \xrightarrow{s \in S} X$ with $X \in \mathcal{B}$, there exists $X'' \to X'$ such that the composition $(X'' \to X) \in S_\mathcal{B}$,

   or b') with all arrows reversed.

Then $S_\mathcal{B}^{-1}\mathcal{B} \hookrightarrow S^{-1}\mathcal{C}$ is fully faithful.

4.6 $\mathcal{A} \to \boldsymbol{D}^?(\mathcal{A})$ is a fully faithful embedding

Let $K^\bullet$ be an object of $\operatorname{Kom}^?(\mathcal{A})$. We call $K^\bullet$ an H^0-complex if $H^i(K^\bullet) = 0$ for each $i \neq 0$.

Proposition 4.14. The functor $\Phi : \mathcal{A} \to \operatorname{Kom}^?(\mathcal{A}) \to D^?(\mathcal{A})$ yields an equivalence of $\mathcal{A}$ with the full subcategory of $D^?(\mathcal{A})$ consisting of H^0-complexes.

Proof. Consider

$$\begin{array}{rccc} H^0: & D^?(\mathcal{A}) & \to & \mathcal{A} \\ & X^\bullet & \mapsto & H^0(X^\bullet). \end{array}$$

For all X, $Y \in \mathcal{A}$, we have

$$\begin{tikzcd} \operatorname{Hom}_\mathcal{A}(X,Y) \arrow[r, "\Phi"] \arrow[rr, bend right, "\mathrm{id}"'] & \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y) \arrow[r, "H^0"] & \operatorname{Hom}_\mathcal{A}(X,Y) \end{tikzcd}$$

by definition. Now we prove that

$$\begin{tikzcd} \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y) \arrow[r, "H^0"] \arrow[rr, bend right, "\mathrm{id}"'] & \operatorname{Hom}_\mathcal{A}(X,Y) \arrow[r, "\Phi"] & \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y). \end{tikzcd}$$

Given $\varphi = (Z \xrightarrow{s} X, f) \in \operatorname{Hom}_{D^?(\mathcal{A})}(X,Y)$. $\Phi(H^0(\varphi))$ is represented by $H^0(f) \circ H^0(s)^{-1}$.

Let

$$\begin{tikzcd}
V \arrow[d, "r"] \arrow[r, equal] & \cdots \arrow[r, "d_Z^{-3}"] & Z^{-2} \arrow[r, "d_Z^{-2}"] \arrow[d, "\mathrm{id}"] & Z^{-1} \arrow[r, "d_Z^{-1}"] \arrow[d, "\mathrm{id}"] & \ker d_Z^0 \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
Z \arrow[r, equal] & \cdots \arrow[r, "d_Z^{-3}"] & Z^{-2} \arrow[r, "d_Z^{-2}"] & Z^{-1} \arrow[r, "d_Z^{-1}"] & Z^0 \arrow[r, "d_Z^0"] & Z^1 \arrow[r, "d_Z^1"] & \cdots ,
\end{tikzcd}$$

and let $h : V \to X$ defined by $h^0 = H^0(s) \circ (\ker d_Z^0 \to H^0(Z))$, and $h^{i \neq 0} = 0$. We verify that

$$\begin{tikzcd}
& & V \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
& Z \arrow[dl, "s\mathrm{id}"'] \arrow[drrr, "f"] & & X \arrow[dlll] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

commutes in $\mathrm{Kom}^?(\mathcal{A})$ and r is a quasi-isomorphism.

Given an H^0-complex Z, both morphisms $r : V \to Z$ and $V \to H^0(Z)$ are quasi-isomorphism. Hence Z lies in the essential image of $\mathcal{A} \to D^?(\mathcal{A})$. ■

Let $\mathrm{Kom}_0^?(\mathcal{A}) \subseteq \mathrm{Kom}^?(\mathcal{A})$ be the full subcategory consisting of complexes with $d = 0$. We regard as the full subcategory of $\mathrm{Kom}^?(\mathcal{A})$, by sending $X \in \mathcal{A}$ to $[X]^\bullet$ with $[X]^0 = X$ and $[X]^{i \neq 0} = 0$.

Proposition 4.15. Assume that $\mathcal{A}$ is semi-simple, i.e., every short exact sequence splits in $\mathcal{A}$. Show that

$$\begin{tikzcd}
\mathrm{Kom}_0^?(\mathcal{A}) \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathrm{Kom}^?(\mathcal{A}) \arrow[r] & D^?(\mathcal{A}).
\end{tikzcd}$$

So $K^\bullet \cong \bigoplus_i H^i(K^\bullet)[-i]$.

Proposition 4.16. Show that the essential image of $D^+(\mathcal{A}) \to D(\mathcal{A})$ consists of complexes $X^\bullet$ with $H^i(X^\bullet) = 0$ for all $i \ll 0$. The similar statement hold for $D^-(\mathcal{A})$ and $D^{\mathrm{b}}(\mathcal{A})$.

5 Triangulated categories

We've seen that $D(\mathcal{A})$ is additive. Though $D(\mathcal{A})$ is not abelian for most $\mathcal{A}$, it is nevertheless a triangulated category.

5.1 The data of triangulated categories

A triangulated category $\mathcal{T}$ is an additive category together with:

- an additive automorphism $[1] : \mathcal{T} \to \mathcal{T}$, called the shift functor; from this, a triangle is a diagram of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

  where a morphism of triangles is a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]; \end{tikzcd}$$

- and a collection of triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

  called exact or distinguished triangles, denoted by $\triangle$ (we also use the notation

$$\left.\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}\right)$$

subject to the axioms (TR1) ~ (TR4) below.

Example 5.1. For $K^?(\mathcal{A})$ and $D^?(\mathcal{A})$, $[1]$ is the shift functor induced by $[1] : \mathrm{Kom}^?(\mathcal{A}) \to \mathrm{Kom}^?(\mathcal{A})$. The distinguished triangles are all triangles isomorphic to

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1]$$

for some morphism $f : X^\bullet \to Y^\bullet$ of complexes. Recall that if $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ is a distinguished triangle, then we have a long exact sequence

$$\cdots \to H^i(X^\bullet) \to H^i(Y^\bullet) \to H^i(Z^\bullet) \to H^{i+1}(X^\bullet) \to \cdots .$$

Proposition 5.2. Let

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

be a short exact sequence in $\mathrm{Kom}^?(\mathcal{A})$. Then

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{0} X[1].$$

is a distinguished triangle in $D^?(\mathcal{A})$.

Given $X^\bullet \in D^?(\mathcal{A})$, we define the canonical truncations

$$\begin{aligned}
\tau_{\leq i}X^\bullet &= (\cdots \to X^{i-2} \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} \ker d^i \to 0 \to \cdots) \\
\tau_{>i}X^\bullet &= (\cdots \to 0 \to \operatorname{Im} d^i \to X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots).
\end{aligned}$$

(5.2) implies that the triangle

$$\tau_{\leq i}X^\bullet \xrightarrow{f} X^\bullet \longrightarrow \tau_{>i} \longrightarrow \tau_{\leq i}X^\bullet[1]$$

is distinguished. We have

$$H^j(\tau_{\leq i}X^\bullet) = \begin{cases} H^j(X^\bullet), & \text{if } j \leq i \\ 0, & \text{else,} \end{cases} \qquad H^j(\tau_{>i}X^\bullet) = \begin{cases} H^j(X^\bullet), & \text{if } j > i \\ 0, & \text{else.} \end{cases}$$

Remark. Shouldn't confound $\tau_{\leq i}$, $\tau_{>i}$ with the naive truncation

$$\begin{aligned}
\sigma_{\leq i}X^\bullet &= (\cdots \to X^{i-1} \xrightarrow{d^{i-1}} X^i \to 0 \to \cdots) \\
\sigma_{>i}X^\bullet &= (\cdots \to 0 \to X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots).
\end{aligned}$$

The triangle

$$\sigma_{\leq i}X^\bullet \xrightarrow{f} X^\bullet \longrightarrow \sigma_{>i}X^\bullet \longrightarrow \sigma_{\leq i}X^\bullet[1]$$

is also distinguished.

5.2 Axioms and properties of triangulated categories

(TR1)

- $\triangle$: $X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$.

- Given an isomorphism of triangles

$$\begin{tikzcd}
\triangle' : & X' \arrow[r] \arrow[d, "\wr"] & Y' \arrow[r] \arrow[d, "\wr"] & Z' \arrow[r] \arrow[d, "\wr"] & X'[1] \arrow[d, "\wr"] \\
\triangle : & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1],
\end{tikzcd}$$

  $\triangle'$ is distinguished if and only if $\triangle$ is distinguished.

- Any morphism $f : X \to Y$ can be completed to a distinguished triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dl] \\
& Z. \arrow[ul, dashed] &
\end{tikzcd}$$

(TR2) A triangle

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$$

is distinguished if and only if

$$Y^\bullet \xrightarrow{v} Z \xrightarrow{w} X^\bullet[1] \xrightarrow{-u[1]} Y^\bullet[1]$$

is distinguished.

(TR3) Given two distinguished triangles

$$\triangle = \begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z, \arrow[ul, dashed] & \end{tikzcd} \qquad \triangle = \begin{tikzcd} X' \arrow[rr] & & Y' \arrow[dl] \\ & Z' \arrow[ul, dashed] & \end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[d, "g"] \\ X \arrow[r] & Y, \end{tikzcd}$$

there exists $h : Z' \to Z$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, "h"] & X'[1] \arrow[d, "f[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

commutes.

Remark. The map $h : Z' \to Z$ above is not required to be unique (in any sense). If $\mathcal{T}$ satisfies (TR1), (TR2), (TR3), we call $\mathcal{T}$ a pre-triangulated category.

Proposition 5.3. Given a distinguished triangle

$$\begin{tikzcd} X \arrow[rr, "u"] & & Y \arrow[dl, "v"] \\ & Z. \arrow[ul, dashed, "w"] & \end{tikzcd}$$

(i) $v \circ u = 0$.

(ii) For each $U \in \mathcal{T}$, the sequence

$$\cdots \to \mathrm{Hom}(U, X[i]) \xrightarrow{u[i]\circ} \mathrm{Hom}(U, Y[i]) \xrightarrow{v[i]\circ} \mathrm{Hom}(U, Z[i]) \to \mathrm{Hom}(U, X[i{+}1]) \to \cdots$$

is exact (similar for $\mathrm{Hom}(X[i], U)$ etc.)

Proof. (i) Consider the morphism of distinguished triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, "f"] & X[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1],
\end{tikzcd}$$

where f exists by (TR3). Hence $v \circ u = f \circ 0 = 0$.

(ii) By (TR2), it is enough to show that the sequence is exact at $\mathrm{Hom}(U, Y[i])$. By (i), $v[i] \circ u[i] \circ = (v \circ u)[i] \circ = 0$, so $\mathrm{Im}(u[i]\circ) \subseteq \ker(v[i]\circ)$. For each $f \in \ker(v[i]\circ)$, we have the morphism of distinguished triangles

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "g"] & V \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & V[1] \arrow[d, "{g[1]}"] \\
X[i] \arrow[r, "{u[i]}"] & Y[i] \arrow[r, "{v[i]}"] & Z[i] \arrow[r, "{w[i]}"] & X[i+1],
\end{tikzcd}$$

where g exists by (TR3). Hence $f = u[i] \circ g \in \mathrm{Im}(u[i]\circ)$. ■

Proposition 5.4. In (TR3), if f and g are isomorphisms, so is h. In particular, for each $f : X \to Y$, the isomorphism class of the object Z in the distinguished triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dl, "g"] \\
& Z \arrow[ul, dashed] &
\end{tikzcd}$$

completing f is unique.

We write $\mathrm{Cone}(f) := Z$ and call it the mapping cone of f.

Proof. For each $U \in \mathcal{T}$, apply $\mathrm{Hom}_{\mathcal{T}}(U, -)$ to the diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, "h"] & X'[1] \arrow[r] \arrow[d, "{f[1]}"] & Y'[1] \arrow[d, "{g[1]}"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \arrow[r] & Y[1].
\end{tikzcd}$$

It follows from (5.3) that the two rows of

$$\begin{tikzcd}
\mathrm{Hom}(U, X') \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}(U, Y') \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}(U, Z') \arrow[r] \arrow[d, "h_*"] & \mathrm{Hom}(U, X'[1]) \arrow[r] \arrow[d, "{f[1]_*}"] & \mathrm{Hom}(U, Y'[1]) \arrow[d, "{g[1]_*}"] \\
\mathrm{Hom}(U, X) \arrow[r] & \mathrm{Hom}(U, Y) \arrow[r] & \mathrm{Hom}(U, Z) \arrow[r] & \mathrm{Hom}(U, X[1]) \arrow[r] & \mathrm{Hom}(U, Y[1])
\end{tikzcd}$$

are both exact. Since f_*, g_*, $f[1]_*$, $g[1]_*$ are isomorphisms, the five lemma shows that h_* is also an isomorphism. Hence, by Yoneda's lemma, $h : Z' \to Z$ is an isomorphism. ■

Remark. Given $f : X \to Y$ in $\mathcal{T}$. Only the isomorphism class of $\operatorname{Cone}(f)$ is well-defined. In general there is no functorial construction of $\operatorname{Cone}(f)$.

(TR4) Assume we have the following diagram

$$\begin{tikzcd}
 & & Y \arrow[dl] \arrow[ddr] & & \\
 & Z \arrow[dl] & & & X \arrow[ull] \arrow[lll] \\
X' \arrow[uurr, dashed] \arrow[rrr, dashed] & & & Z' \arrow[ur, dashed] &
\end{tikzcd}$$

containing the distinguished triangles

$$\begin{tikzcd}
Y \arrow[rr] & & Z \arrow[dl] \\
 & X', \arrow[ul, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
 & Z' \arrow[ul, dashed] &
\end{tikzcd}$$

and the commutative diagrams

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
 & Z, &
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[rr, dashed] \arrow[dr, dashed] & & Y \arrow[dl] \\
 & Z'. &
\end{tikzcd}$$

Then we can complete the diagram by

$$\begin{tikzcd}
 & & Y \arrow[dl] \arrow[ddr] & & \\
 & Z \arrow[dl] \arrow[ddr] & & & X \arrow[ull] \arrow[lll] \\
X' \arrow[uurr, dashed] \arrow[rrr, dashed] & & & Z' \arrow[ur, dashed] \arrow[dl] & \\
 & & Y' \arrow[ull] \arrow[uurr, dashed] & &
\end{tikzcd}$$

with

$$\begin{tikzcd}
X \arrow[rr] & & Z \arrow[dl] \\
 & Y', \arrow[ul, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z' \arrow[rr] & & Y' \arrow[dl] \\
 & X' \arrow[ul, dashed] &
\end{tikzcd}$$

distinguished and

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\
 & X', &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z' \arrow[rr] \arrow[dr, dashed] & & Y' \arrow[dl, dashed] \\
 & X &
\end{tikzcd}$$

commute.

(TR4) implies that given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exists a distinguished triangle

$$\begin{tikzcd}
Z' \arrow[d, equal] & & Y' \arrow[d, equal] \\
\mathrm{Cone}(f) \arrow[rr] & & \mathrm{Cone}(g \circ f) \arrow[dl] \\
& \mathrm{Cone}(g) \arrow[ul, dashed] \arrow[d, equal] & \\
& X'. &
\end{tikzcd} \tag{TR4'}$$

Unfolding the octahedron, we have

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & \mathrm{Cone}(f) \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r, "g\circ f"] & Z \arrow[r] \arrow[d] & \mathrm{Cone}(g \circ f) \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
& \mathrm{Cone}(g) \arrow[r, equal] \arrow[d, "h"] & \mathrm{Cone}(g) \arrow[r, "h"] \arrow[d] & Y[1] \\
& Y[1] \arrow[r] & \mathrm{Cone}(f)[1] &
\end{tikzcd} \tag{Υ}$$

(TR4) implies the following

Lemma 5.5. Each commutative diagram

$$\begin{tikzcd}
E_1 \arrow[r, "e"] \arrow[d, "\varphi_1"] & E_2 \arrow[d, "\varphi_2"] \\
F_1 \arrow[r, "f"] & F_2
\end{tikzcd}$$

can be completed to a commutative diagram

$$\begin{tikzcd}
E_1 \arrow[r, "e"] \arrow[d, "\varphi_1"] & E_2 \arrow[r] \arrow[d, "\varphi_2"] & E_3 \arrow[r] \arrow[d] & E_1[1] \arrow[d] \\
F_1 \arrow[r, "f"] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & F_1[1] \arrow[d] \\
G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G_3 \arrow[r] \arrow[d] & G_1[1] \\
E_1[1] \arrow[r] & E_2[1] \arrow[r] & E_3[1]. &
\end{tikzcd}$$

Proof. We already know that G_1, G_2, E_3, F_3 exist (mapping cone). Let $H = \mathrm{Cone}(E_1 \to F_2)$. Applying (TR4') and ($\Upsilon$) to both $E_1 \xrightarrow{e} E_2 \xrightarrow{\varphi_2} F_2$ and $E_1 \xrightarrow{\varphi_1} F_1 \xrightarrow{f} F_2$ gives

$$E_3 \xrightarrow{u} H \xrightarrow{v'} G_2 \longrightarrow E_3[1]$$

$$G_1 \xrightarrow{u'} H \xrightarrow{v} F_3 \longrightarrow G_1[1]$$

such that

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d] \arrow[dr] & E_2 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d, "v \circ u"'] \arrow[dd, bend left, "u"] & E_1[1] \arrow[ddd, bend left, equal] \\
F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] \arrow[dr] & F_3 \arrow[r] & F_1[1] \\
G_1 \arrow[r, "v' \circ u'"] \arrow[d] \arrow[rr, bend right, "u'"'] & G_2 \arrow[d] & H \arrow[l, "v'"'] \arrow[u, "v"] \arrow[dr] & \\
E_1[1] \arrow[rrr, bend right, equal] & E_2[1] & & E_1[1] \arrow[ll] \arrow[uu]
\end{tikzcd}$$

is commutative. Define $G_3 = \mathrm{Cone}(E_3 \xrightarrow{u} H \xrightarrow{v} F_3)$. We finish the proof by applying (TR4') and (Υ) to $E_3 \xrightarrow{u} H \xrightarrow{v} F_3$. ∎

Theorem 5.6. The homotopy categories $K^?(\mathcal{A})$ are triangulated.

We've already proven that $K^?(\mathcal{A})$ satisfies (TR1) $\sim$ (TR3).

Remark. The homotopy category $K^?(\mathcal{A})$ of $\mathcal{A}$ makes sense as long as $\mathcal{A}$ is an additive category. $K^?(\mathcal{A})$ is again a triangulated category with [1] and distinguished triangles are defined the same way.

5.3 Exact functors

Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. A triangle functor (or exact functor) is an additive functor $\Phi : \mathcal{C} \to \mathcal{D}$ such that $\Phi \circ [1] = [1] \circ \Phi$ and Φ preserves distinguished triangles, i.e.,

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z \arrow[ul, dashed] \arrow[u, phantom, "\triangle"] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\Phi(X) \arrow[rr] & & \Phi(Y) \arrow[dl] \\
& \Phi(Z). \arrow[ul, dashed] \arrow[u, phantom, "\triangle"] &
\end{tikzcd}$$

Example 5.7. For every additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, the induced functor $K^?(\mathcal{A}) \to K^?(\mathcal{B})$ is exact.

Exactness is preserved under adjunction.

Proposition 5.8. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor. If $G \dashv F$ or $F \dashv G$, then $G : \mathcal{D} \to \mathcal{C}$ is exact. In particular, if F is an equivalence of category, then its quasi-inverse is also an exact functor.

Proof. We prove the proposition when $F \dashv G$.

$G \circ [1] = [1] \circ G$: Since $F \circ [n] = [n] \circ F$, we have

$$\begin{aligned} \mathrm{Hom}(A, G(B[1])) &\cong \mathrm{Hom}(F(A), B[1]) \cong \mathrm{Hom}(F(A)[-1], B) \\ &\cong \mathrm{Hom}(F(A[-1]), B) \cong \mathrm{Hom}(A[-1], G(B)) \cong \mathrm{Hom}(A, G(B)[1]), \end{aligned}$$

and all these isomorphisms are functorial in $A \in \mathcal{C}$ and $B \in \mathcal{D}$. By Yoneda's lemma, $G(B[1]) \cong G(B)[1]$, functorial in B. Hence $G \circ [1] = [1] \circ G$.

G preserves distinguished triangles: Given distinguished triangle

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

in D. We complete $G(X) \to G(Y)$ into a distinguished triangle

$$\begin{tikzcd} G(X) \arrow[rr] & & G(Y) \arrow[dl] \\ & Z_0. \arrow[ul, dashed] & \end{tikzcd}$$

Apply F to this distinguished triangle, we get

$$\begin{tikzcd} FG(X) \arrow[r] \arrow[d] & FG(Y) \arrow[r] \arrow[d] & F(Z_0) \arrow[r] \arrow[d, blue] & FG(X)[1] \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1], \end{tikzcd}$$

where $FG(X) \to X$ and $FG(Y) \to Y$ are adjunction morphisms, and the blue arrow

$F(Z_0) \to Z$ comes from (TR3). We have morphisms of triangles

$$\begin{tikzcd}
\triangle_1 : \arrow[d, "GF"] & G(X) \arrow[r] \arrow[d] & G(Y) \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & G(X)[1] \arrow[d] \\
\triangle_2 : & GFG(X) \arrow[r] \arrow[d] & GFG(Y) \arrow[r] \arrow[d] & GF(Z_0) \arrow[r] \arrow[d] & GFG(X)[1] \arrow[d] \\
\triangle_3 : & G(X) \arrow[r] & G(Y) \arrow[r] & G(Z) \arrow[r] & G(X)[1],
\end{tikzcd}$$

where $\triangle_1$ is distinguished and we want to show $\triangle_3$ is also distinguished. For each $A \in \mathcal{C}$, as

$$\begin{tikzcd}
\mathrm{Hom}(F(A), Y) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(F(A), Z) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(F(A), X[1]) \arrow[d, "\wr"] \\
\mathrm{Hom}(A, G(Y)) \arrow[r] & \mathrm{Hom}(A, G(Z)) \arrow[r] & \mathrm{Hom}(A, G(X)[1])
\end{tikzcd}$$

is exact, (*) and the five lemma implies

$$\mathrm{Hom}(A, G(Z)) \cong \mathrm{Hom}(A, Z_0),$$

functorial in A. So $\triangle_1 \to \triangle_3$ implies $G(Z) \cong Z_0$ by Yoneda's lemma. Hence $\triangle_3 \cong \triangle_1$ and therefore $\triangle_3$ is exact. ∎

Definition 5.9. An equivalence of triangulated categories is an exact functor $\Phi : \mathcal{C} \to \mathcal{D}$ which is an equivalence of categories. A triangulated subcategory of $\mathcal{D}$ is a subcategory $\mathcal{C} \subseteq \mathcal{D}$ carrying a structure of triangulated category such that $\mathcal{C} \hookrightarrow \mathcal{D}$ is exact.

Proposition 5.10. Let $\mathcal{C}$ be a full subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{C} \subseteq \mathcal{D}$ is a triangulated subcategory if and only if

(i) $[1]$ restricts to an automorphism on $\mathcal{C}$;

(ii) for each distinguished triangle

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z \arrow[ul, dashed] \arrow[u, phantom, "\triangle"] &
\end{tikzcd}$$

in $\mathcal{D}$, X, $Y \in \mathcal{C}$ implies Z is isomorphic to an object in $\mathcal{C}$.

Example 5.11. $K^{\mathrm{b}}(\mathcal{A}) \subseteq K^{\pm}(\mathcal{A}) \subseteq K(\mathcal{A})$ and $D^{\mathrm{b}}(\mathcal{A}) \subseteq D^{\pm}(\mathcal{A}) \subseteq D(\mathcal{A})$ are inclusions of full triangulated subcategory.

Remark. The canonical truncations $\tau_{\leq i}$, $\tau_{>i} : D^{?}(\mathcal{A}) \to D^{?}(\mathcal{A})$ are not exact functors.

5.4 Localizations of triangulated categories

When does a localization of a triangulated category $\mathcal{T}$ carry a natural structure of triangulated category?

Let S be a collection of morphisms in $\mathcal{T}$. We say that S is compatible with triangulation if

- $f \in S$ if and only if $f[1] \in S$
- for each diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f\in S"] & Y' \arrow[r] \arrow[d, "g\in S"] & Z' \arrow[r] & X'[1] \arrow[d, "f[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1], \end{tikzcd}$$

  there exists $h : Z' \to Z$ so that we can complete the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, "h"] & X'[1] \arrow[d, "f[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]. \end{tikzcd}$$

Theorem 5.12. Let $\mathcal{T}$ be a triangulated category, and let S be a localized system compatible with triangulation. Then $S^{-1}\mathcal{T}$ is a triangulated category, with respect to:

- The natural shift functor on $S^{-1}\mathcal{T}$.
- A triangle in $S^{-1}\mathcal{T}$ is called distinguished if it is isomorphic to the image of a distinguished triangle under $\mathcal{T} \to S^{-1}\mathcal{T}$.

Corollary 5.13. The derived category $D^?(\mathcal{A})$, together with the shift functor and distinguished triangles defined before, is a triangulated category.

Interlude: On diagram chasing

Let $\mathcal{A}$ be an abelian category. As we mentioned before, in many situations we can't regard objects of $\mathcal{A}$ as sets. Even if they are sets (e.g., $\mathcal{A} = \mathsf{Ab}$), in general we can't understand set-theoretically $\ker f$ and $\operatorname{coker} f$. For instance,

- monomorphism $\neq$ injection (e.g., $\mathcal{A}$ = the category of divisible abelian groups, $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$);
- epimorphism $\neq$ surjection (e.g., $\mathcal{A} = \mathsf{Ring}$, $\mathbb{Z} \to \mathbb{Q}$).

Fortunately, there exists a formalism allowing us to understand morphisms in $\mathcal{A}$, as if they were sets. Here we summarize the rules.

Elements

Let Y be an object of $\mathcal{A}$. An element y of Y (denoted $y \in Y$) is an equivalence class of pairs

$$(X \in \mathcal{A}, h : X \to Y)$$

by the equivalence relation $(X, h) \sim (X', h')$ if there exists

$$\begin{tikzcd} & X \arrow[dr, "h"] & \\ Z \arrow[ur, "g"] \arrow[dr, "g'"'] & & Y \\ & X' \arrow[ur, "h'"'] & \end{tikzcd}$$

with g, g' epic.

Maps induced by morphisms

Given a morphism $f : Y_1 \to Y_2$ in $\mathcal{A}$, we get

$$\begin{array}{rccc} f: & \{\text{elements of } Y_1\} & \to & \{\text{elements of } Y_2\} \\ & (X, h) & \mapsto & (X, f \circ h). \end{array}$$

Diagram chasing rules

The element $0 \in Y$ is defined by $0 \mapsto Y$.

Proposition. Let $f : Y_1 \to Y_2$ be a morphism in $\mathcal{A}$.

(i) f is a monomorphism if and only if for each $y \in Y_1$, $f(y) = 0$ implies $y = 0$, or equivalently, for all $y, y' \in Y_1$, $f(y) = f(y')$ implies $y = y'$.

(ii) f is an epimorphism if and only if for each $y_2 \in Y_2$, there exists $y_1 \in Y_1$ such that $f(y_1) = y_2$.

(iii) $f = 0$ if and only if for each $y \in Y$, $f(y) = 0$.

(iv) $Y_1 \xrightarrow{f} Y \xrightarrow{g} Y_2$ is exact if and only if $g \circ f = 0$ and for each $y \in Y$ with $g(y) = 0$, there exists $y_1 \in Y_1$ such that $f(y_1) = y$.

For each $y = (X, h) \in Y$, define $-y = (X, -h) \in Y$.

Proposition. Let $g : Y_1 \to Y_2$ be a morphism in $\mathcal{A}$. Let $y, y' \in Y_1$ such that $g(y) = g(y')$. Then exists $z \in Y_1$ such that $g(z) = 0$ and

- for each $f : Y_1 = Y$ such that $f(y) = 0$, we have $f(z) = -f(y')$;
- for each $f : Y_1 = Y$ such that $f(y') = 0$, we have $f(z) = -f(y)$.

6 Derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant additive functors between abelian categories. Assume F is left exact (e.g., $F = \Gamma(X, -) : \mathsf{QCoh}(X/k) \to \mathsf{Vect}_k$). We want to construct the right derived functor

$$\mathsf{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

which is an exact functor such that $R^iF = H^i(\mathsf{R}F)$. That $\mathsf{R}F$ is exact implies, e.g., for each short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$.

$$\mathsf{R}F(X) \to \mathsf{R}F(Y) \to \mathsf{R}F(Z) \to \mathsf{R}F(X)[1]$$

is an distinguished triangle in $D^+(\mathcal{B})$, and for each left exact sequence

$$0 \to X \to Y \to Z$$

in $\mathcal{A}$. We have the long exact sequence

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(X) \to \cdots$$

Rough plan of construction:

(1) $\mathsf{R}F$ maps complexes to complexes, but we're not able to give a general explicit construction of $\mathsf{R}F(X^\bullet)$ for every $X^\bullet \in \mathrm{Kom}(\mathcal{A})$.

(2) Instead, we only define $\mathsf{R}F(X^\bullet)$ for complexes $X^\bullet$ consisting of F-adapted objects $\mathcal{I}_F \subseteq \mathcal{A}$.

(3) The full subcategory $\mathcal{I}_F \subseteq \mathcal{A}$ satisfies

$$S^{-1}K^+(\mathcal{I}_F) \xrightarrow{\sim} D^+(\mathcal{A}),$$

so enough to define $\mathsf{R}F$ on $K^+(\mathcal{I}_F)$.

Sometimes we will only consider functors $F : \mathcal{A} \to \mathcal{B}$ which are left exact. For right exact functors $G : \mathcal{A} \to \mathcal{B}$, the construction of left derived functor $\mathsf{L}G : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and other statements are similar.

6.1 $\boldsymbol{F}$-adapted objects

A complex $X^\bullet \in \mathrm{Kom}(\mathcal{A})$ is called acyclic if $H^i(X^\bullet) = 0$ for all i.

Definition 6.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor. A class of objects $\mathcal{I}_F \subseteq \mathcal{A}$ is called F-adapted if the following conditions hold.

(i) $\mathcal{I}_F$ is stable under finite direct sums.

(ii) F sends acyclic complexes in $\mathrm{Kom}^+(\mathcal{I}_F)$ (resp. $\mathrm{Kom}^-(\mathcal{I}_F)$) to acyclic complexes.

(iii) Any object in $\mathcal{A}$ is a subobject (resp. quotient) of some object of $\mathcal{I}_F$.

If $\mathcal{I}_F$ satisfies (i) and (iii) we also say that $\mathcal{A}$ contains sufficiently many objects in $\mathcal{I}_F$, or $\mathcal{I}_F$ is sufficiently large.

Example 6.2. An object $I \in \mathcal{A}$ is called injective if for each monomorphism $f : A \to B$ and for each morphism $g : A \to I$, there exists a lifting

$$\begin{tikzcd} & I \\ A \arrow[ur, "g"] \arrow[r, "f"] & B. \arrow[u, "\tilde{g}"'] \end{tikzcd}$$

Let $\mathcal{I} \subseteq \mathcal{A}$ be the full subcategory of injective objects.

Proposition 6.3. If $\mathcal{A}$ contains enough injective objects, then $\mathcal{I}$ is F-adapted for each left-exact functor F.

Proof. Given $I,\ J \in \mathcal{I}$. For any monomorphism $f : A \to B$ and any morphism $(g,h) : A \to I \oplus J$, we have liftings

$$\begin{tikzcd} & I \\ A \arrow[ur, "g"] \arrow[r, "f"] & B, \arrow[u, "\tilde{g}"'] \end{tikzcd} \qquad \begin{tikzcd} & J \\ A \arrow[ur, "g"] \arrow[r, "h"] & B. \arrow[u, "\tilde{h}"'] \end{tikzcd}$$

We see that $(\tilde{g}, \tilde{h})$ gives a lifting

$$\begin{tikzcd} & I \oplus J \\ A \arrow[ur, "{(g,h)}"] \arrow[r, "f"] & B. \arrow[u, "{(\tilde{g},\tilde{h})}"'] \end{tikzcd}$$

So $\mathcal{I}$ is stable under finite direct sums.

Let

$$I^\bullet : \quad 0 \to I^\ell \xrightarrow{d^\ell} I^{\ell+1} \xrightarrow{d^{\ell+1}} I^{\ell+2} \to \cdots$$

be an acyclic complex in $\mathrm{Kom}^+(\mathcal{I}_F)$. We want to show that

$$F(I^\bullet) : \quad 0 \to F(I^\ell) \xrightarrow{F(d^\ell)} F(I^{\ell+1}) \xrightarrow{F(d^{\ell+1})} F(I^{\ell+2}) \to \cdots$$

is also acyclic. Since $H^i(I^\bullet) = 0$ for each i, we may decompose $I^\bullet$ into exact sequences

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & I^\ell \arrow[r] & I^{\ell+1} \arrow[rr] \arrow[dr] & & I^{\ell+2} \arrow[rr] \arrow[dr] & & I^{\ell+3} \arrow[rr] \arrow[dr] & & \cdots \\ & & & \ker d^{\ell+2} \arrow[ur] \arrow[dr] & & \ker d^{\ell+3} \arrow[ur] \arrow[dr] & & \cdots & \\ & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0. \arrow[ur] & & \end{tikzcd}$$

For each $i \geq \ell + 1$, since I^{i+1} is injective, the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker d^{i+1} \arrow[r, "d^i"] \arrow[d, equal] & \ker d^{i+1} \arrow[r] \arrow[dl, "s"] & \ker d^{i+2} \arrow[r] & 0 \\ & I^i & & & \end{tikzcd}$$

splits. Hence, the left-exactness of F gives a short exact sequence

$$0 \longrightarrow F(\ker d^{i+1}) \xrightarrow{F(d^i)} F(I^{i+1}) \longrightarrow F(\ker d^{i+2}) \longrightarrow 0$$

for each i. These exact sequences patches together into a long exact sequence $F(I^\bullet)$. ∎

Proposition 6.4. For each $X^\bullet \mathcal{I} K^+(\mathcal{A})$ and for each $I^\bullet \in K^+(\mathcal{I})$, the natural map

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet)$$

is an isomorphism.

Proof. ■

Dually, an object $P \in \mathcal{A}$ is called projective if for each epimorphism $f : B \to A$ and for each morphism $g : P \to A$, there exists a lifting

$$\begin{tikzcd} P \arrow[d, "g"] \arrow[dr, "\tilde{g}"] & \\ B \arrow[r, "f"] & A. \end{tikzcd}$$

Let $\mathcal{P} \subseteq \mathcal{A}$ be the full subcategory of projective objects. Again, if $\mathcal{A}$ contains enough projective object then $\mathcal{P}$ is F-adapted for each right exact functor G.

Let $\mathcal{I}_F \subseteq \mathcal{A}$ be a full subcategory of F-adapted objects for some left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then $\mathcal{I}_F$ is an additive subcategory of $\mathcal{A}$. Recall that $K^?(\mathcal{I}_F)$ is an triangulated category.

Lemma 6.5. Let S be the class of quasi-isomorphisms in $K^+(\mathcal{I}_F)$. Then S is a localizing system in $K^+(\mathcal{I}_F)$ compatible with triangulation.

Proof. Adapt the proof that quasi-isomorphisms in $K^?(\mathcal{A})$ form a localizing system compatible with triangulation. ■

Proposition 6.6. The natural functor $\Psi : S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$ is an equivalence of triangulated categories.

Proof. First, we show that $S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$ is essentially surjective. Namely, for each $C^\bullet \in \mathrm{Kom}^+(\mathcal{A})$, there exists $I^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ such that $C^\bullet$ is quasi-isomorphic to $I^\bullet$. We may assume that $C^i = 0$ for each $i < 0$. Construct $C^\bullet \to I^\bullet$ by induction: For the initial step, since $\mathcal{I}_F$ is F-adapted, we can find a monomorphism $C^0 \hookrightarrow I^0 \in \mathcal{I}_F$. Consider the fibered coproduct

$$\begin{tikzcd} 0 \arrow[r] & C^0 \arrow[r] \arrow[d, hook] & C^1 \arrow[d] \\ & I^0 \arrow[r] & C^1 \sqcup_{C_0} I^0, \end{tikzcd}$$

We then take a monomorphism $C^1 \sqcup_{C_0} I^0 \hookrightarrow I^1 \in \mathcal{I}_F$. Then, we complete the diagram

$$\begin{tikzcd} 0 \arrow[r] & C^0 \arrow[r] \arrow[d, hook] & C^1 \arrow[d] \\ 0 \arrow[r] & I^0 \arrow[r] & I^1. \end{tikzcd}$$

Now, assume that we have

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r] \arrow[d, "s^0"] & \cdots \arrow[r] & C^i \arrow[d, "s^i"] \\
0 \arrow[r] & I^0 \arrow[r] & \cdots \arrow[r, "d^{i-1}"] & I^i.
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
 & C^i \arrow[dl, "t^i"'] \arrow[rr] & & C^{i+1} \arrow[d] \\
I^i \arrow[r] & \operatorname{coker} d^{i-1} \arrow[rr] & & C^{i+1} \sqcup_{C_i} \operatorname{coker} d^{i-1}.
\end{tikzcd}$$

Then take a monomorphism $C^{i+1} \sqcup_{C_i} \operatorname{coker} d^{i-1} \hookrightarrow I^{i+1} \in \mathcal{I}_F$ so that we can complete the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & C^i \arrow[r] \arrow[d] & C^{i+1} \arrow[d] \\
0 \arrow[r] & I^0 \arrow[r] & \cdots \arrow[r] & I^i \arrow[r] & I^{i+1}.
\end{tikzcd}$$

This gives us a complex $I^\bullet \in \mathrm{Kom}^+(\mathcal{I}_F)$ and a morphism $s^\bullet : C^\bullet \to I^\bullet$. We still need to show that $H^i(C^\bullet) \to H^i(I^\bullet)$ is an isomorphism for each $i \geq 0$, which is a simple diagram chasing. (*)

We show that $S^{-1}K^+(\mathcal{I}_F) \to D^+(\mathcal{A})$ is fully faithful. By (4.13), it is enough show for each quasi-isomorphism $X \xrightarrow{s} X'$ with $X \in K^+(\mathcal{I}_F)$, there exists a morphism $X' \to X''$ such that the composition $X \to X'$ is a quasi-isomorphism in $K^+(\mathcal{I}_F)$. We just prove the existence of the quasi-isomorphism $X' \to X''$.

To show that the equivalence $S^{-1}K^+(\mathcal{I}_F) \xrightarrow{\sim} D^+(\mathcal{A})$ is triangulated, it suffices to verify that

- [1] restricts to an automorphism on $S^{-1}K^+(\mathcal{I}_F)$;
- for each distinguished triangle

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
 & Z \arrow[ul, dashed] &
\end{tikzcd}$$

  with X, $Y \in \mathrm{Kom}^+(\mathcal{I}_F)$, Z is quasi-isomorphic to some object in $\mathrm{Kom}^+(\mathcal{I}_F)$.

Both are clear. ■

6.2 Construction of the derived functor

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, and let $\mathcal{I}_F \subseteq \mathcal{A}$ be a full subcategory of F-adapted objects. We have seen that

$$\Psi : S^{-1}K^{+}(\mathcal{I}_F) \to D^{+}(\mathcal{A})$$

defines an equivalence of categories.

Lemma 6.7. We have a factorization

$$\begin{tikzcd}
K^{+}(\mathcal{I}_F) \arrow[r, "I^\bullet \mapsto F(I^\bullet)"] \arrow[d] & D^{+}(\mathcal{B}) \\
S^{-1}K^{+}\mathcal{I}_F. \arrow[ur, "\exists \overline{F}"] &
\end{tikzcd}$$

Proof. By the universal property of localization, enough to show that F maps a quasi-isomorphism $f : X^\bullet \to Y^\bullet$ in $\mathrm{Kom}^{+}(\mathcal{I}_F)$ to a quasi-isomorphism in $\mathrm{Kom}^{+}(\mathcal{B})$.

Since f is a quasi-isomorphism, $C(f)$ is acyclic, so $C(F(f)) = F(C(f))$ is also acyclic. Hence, $F(f)$ is a quasi-isomorphism. ■

Choose a quasi-inverse

$$\Phi : D^{+}(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^{+}(\mathcal{I}_F)$$

of Ψ.

Definition 6.8. The right derived functor of F is

$$\begin{array}{rcccc}
\mathsf{R}F : & D^{+}(\mathcal{A}) & \xrightarrow{\Phi} & S^{-1}K^{+}(\mathcal{I}_F) & \xrightarrow{\overline{F}} & D^{+}(\mathcal{B}) \\
 & & & I^\bullet & \mapsto & F(I^\bullet).
\end{array}$$

Define $R^iF(X^\bullet) = H^i(\mathsf{R}F(X^\bullet))$.

Proposition 6.9. The functor $\mathsf{R}F$ is an exact functor.

Proof. It is clear that both Φ and $\overline{F}$ commute with $[1]$. Since $\overline{F}$ sends distinguished triangles to distinguished triangles (because $F(C(f)) = C(F(f))$). So $\overline{F}$ is exact. It remains to show that Φ sends distinguished triangles to distinguished triangles; namely, for each triangle

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] & & Y \arrow[dl] \\
& Z \arrow[ul, dashed] &
\end{tikzcd}$$

in $S^{-1}K^+(\mathcal{I}_F)$, which is distinguished in $D^+(\mathcal{A})$, it is isomorphic in $S^{-1}K^+(\mathcal{I}_F)$ to some distinguished triangle.

We represent $\varphi : X \to Y$ by $(T \xrightarrow{s} X, f)$, where $T \in \mathrm{Kom}^+(\mathcal{I}_F)$. Then there is a commutative diagram

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "s"] & Y \arrow[r] \arrow[d, equal] & C(s) \arrow[r] \arrow[d] & T[1] \arrow[d, "{s[1]}"] & \triangle \text{ in } S^{-1}K^+(\mathcal{I}_F) \\
X \arrow[r, "\varphi"] & Y \arrow[r] & Z \arrow[r] & X[1] & \triangle \text{ in } D^+(\mathcal{A}),
\end{tikzcd}$$

where the blue arrow (a morphism in $D^+(\mathcal{A})$) exists by (TR3). Since s is a quasi-isomorphism, $C(s)$ is quasi-isomorphic to Z. As Φ is fully faithful, it gives an isomorphism

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "\wr"] & Y \arrow[r] \arrow[d, equal] & C(s) \arrow[r] \arrow[d, "\wr"] & T[1] \arrow[d, "\wr"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1],
\end{tikzcd}$$

of triangles in $S^{-1}K^+(\mathcal{I}_F)$. ■

Remark. More generally, given any exact functor $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$. Assume that there exists a full triangulated subcategory $\mathcal{I}_F \subseteq K^+(\mathcal{A})$ such that

- for each $X^\bullet \in K^+(\mathcal{A})$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet \in \mathcal{I}_F$;
- $I^\bullet \in \mathcal{I}_F$ is acyclic implies $F(I^\bullet)$ is acyclic.

Then $\mathsf{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists.

6.3 How unique is $\mathsf{R}F$?

A priori, $\mathsf{R}F$ depends on $\mathcal{I}_F$ and $\Phi : D^+(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^+(\mathcal{I}_F)$. Given another $\mathcal{I}'_F$ and $\Phi' : D^+(\mathcal{A}) \xrightarrow{\sim} S^{-1}K^+(\mathcal{I}'_F)$, the two constructions of $\mathsf{R}F$ are related by a natural isomorphism

$$\begin{tikzcd}
 & S^{-1}K^+(\mathcal{I}_F) \arrow[dr, "\overline{F}"] \arrow[dd, Leftrightarrow, "\eta"] & \\
D^+(\mathcal{A}) \arrow[ur, "\Phi"] \arrow[dr, "\Phi'"'] & & D^+(\mathcal{B}) \\
 & S^{-1}K^+(\mathcal{I}'_F) \arrow[ur, "\overline{F}'"'] &
\end{tikzcd}$$

defined as follows:

Recall that Φ and Ψ are quasi-inverse. So there exists $\beta : \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi \circ \Phi$. Similarly, there exists $\beta' : \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi' \circ \Phi'$. Given $X^\bullet \in D^+(\mathcal{A})$, and let

$$I^\bullet = \Phi(X^\bullet), \quad J^\bullet = \Phi'(X^\bullet).$$

β and β' yields quasi-isomorphisms $X^\bullet \to I^\bullet$ and $X^\bullet \to J^\bullet$, which are functorial in $X^\bullet$. Since $\overline{F}$ maps quasi-isomorphisms to quasi-isomorphisms, we get quasi-isomorphisms $F(X^\bullet) \to F(I^\bullet)$ and $F(X^\bullet) \to F(J^\bullet)$, which gives a quasi-isomorphism

$$\overline{F}(\Phi(X^\bullet)) = F(I^\bullet) \to F(J^\bullet) = \overline{F}'(\Phi'(X^\bullet))$$

in $D^+(\mathcal{B})$, functorial in $X^\bullet$. Hence, $\overline{F}\Phi$ is natural isomorphic to $\overline{F}'\Phi'$.

Proposition 6.10 (Universal property of $\mathsf{R}F$)**.** We have a morphism of functors

$$\begin{tikzcd}
 & D^+(\mathcal{A}) \arrow[dr, "\mathsf{R}F"] \arrow[dd, Leftrightarrow, "\varepsilon"] & \\
K^+(\mathcal{A}) \arrow[ur, "Q_\mathcal{A}"] \arrow[dr, "K(F)"'] & & D^+(\mathcal{B}) \\
 & K^+(\mathcal{B}). \arrow[ur, "Q'_\mathcal{B}"'] &
\end{tikzcd}$$

Moreover, for each exact functor $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, and for each morphism $Q_\mathcal{B} \circ K(F) \to G \circ Q_\mathcal{A}$, there exists a $\nu : \mathsf{R}F \to G$ such that the following diagram commute.

$$\begin{tikzcd}
Q_\mathcal{B} \circ K(F) \arrow[r] \arrow[d, "\varepsilon"] & G \circ Q_\mathcal{A} \\
\mathsf{R}F \circ Q_\mathcal{A}. \arrow[ur, "\nu \circ Q_\mathcal{A}"'] &
\end{tikzcd}$$

Here we only define $\varepsilon : Q_\mathcal{B} \circ K(F) \to \mathsf{R}F \circ Q_\mathcal{A}$. As before, choose $\beta : \mathrm{id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi \circ \Phi$. Then for each $X \in K^+(\mathcal{A})$, there is a functorial resolution $Q_\mathcal{A}(X) \to I^\bullet = \Phi(Q_\mathcal{A}(X)) \in \mathrm{Kom}^+(\mathcal{I}_F)$ in $D^+(\mathcal{A})$, represented by $(X \xrightarrow{s} C, I^\bullet \xrightarrow{t} C)$ in $K^+(\mathcal{A})$. Apply $K(F)$, we get $(K(F)(X) \xrightarrow{K(F)(s)} K(F)(C), K(F)(I^\bullet) \xrightarrow{K(F)(t)} K(F)(C))$.

We can assume that $C \in \mathrm{Kom}^+(\mathcal{I}_F)$. As C, $I^\bullet$, t is a quasi-isomorphism implies $F(C(t)) = C(F(t))$ is acyclic, which means $K(F)(t)$ is a quasi-isomorphism. We get $Q_\mathcal{B}K(F)(X) \to Q_\mathcal{B}K(F)(I^\bullet)$ functorial in X. As the diagram

$$\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r, "K(F)"] & K^+(\mathcal{B}) \arrow[r, "Q_\mathcal{B}"] & D^+(\mathcal{B}) \\
K^+(\mathcal{I}_F) \arrow[u, hook] \arrow[r] & S^{-1}K^+(\mathcal{I}_F) \arrow[ur, "\overline{F}"'] &
\end{tikzcd}$$

commutes, we get

$$Q_{\mathcal{B}}K(F)(I^\bullet) = \overline{F}(I^\bullet) = \overline{F}(\Phi(Q_{\mathcal{A}}(X))) = \mathsf{R}F(Q_{\mathcal{A}}(X)).$$

This defines $\varepsilon : Q_{\mathcal{B}} \circ K(F) \to \mathsf{R}F \circ Q_{\mathcal{A}}$.

6.4 The largest F-adapted class

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Given an F-adapted class of objects $\mathcal{I}_F$, sometimes there aren't so many objects in $\mathcal{I}_F$ to compute $\mathsf{R}F$ in practice.

Example 6.11. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{I}_F$ be the class of injective $\mathcal{O}_X$-modules. $\mathcal{O}_X$-mod has enough injective objects, but essentially the only injective resolution we know is the "Godement" resolution ($\mathscr{F} \hookrightarrow \mathscr{I}$ with $\mathscr{I}(U) := \prod_{x \in U} I_x$, where I_x is an injective abelian group that contains $\mathscr{F}_X$), which is useless in practive, to compute e.g., $\mathsf{R}\Gamma$, where

$$\begin{array}{rccc} \Gamma : & \mathcal{O}_X - \mathsf{mod} & \to & \mathsf{Ab} \\ & \mathscr{F} & \mapsto & \Gamma(X, \mathscr{F}). \end{array}$$

We want to find $\mathcal{I}_F$ containing as many objects as possible. Assume that $\mathsf{R}F$ exists. An object $X \in \mathcal{A}$ is called F-acyclic if $R^iF(X) = 0$ for each $i \neq 0$.

Theorem 6.12. Let $\mathcal{Z} \subseteq \mathcal{A}$ be the full subcategory of F-acyclic objects.

(i) $\mathcal{I}_F \subseteq \mathcal{Z}$ for any F-adapted class $\mathcal{I}_F$.

(ii) $\mathcal{I}_F$ exists if and only if $\mathcal{Z}$ is sufficiently large.

In this case, $\mathcal{Z}$, and also all sufficiently large subclasses of $\mathcal{Z}$, are F-adapted.

Proof. (i) For each $X \in \mathcal{I}_F$, we have $R^iF(X) = H^i(F(X)[0]) = 0$ for each $i \neq 0$.

(ii) The only if part follows from (i). It remains to show that every sufficiently large subclass $R \subseteq \mathcal{Z}$ is F-adapted; namely, F sends acyclic complexes $K^\bullet \in \mathrm{Kom}^+(R)$ to acyclic complexes. Note that if

$$0 \to K_1 \to K_2 \to K_3 \to 0$$

is exact with K_1, K_2, $K_3 \in R$,

$$0 \to F(K_1) \to F(K_2) \to F(K_3) \to 0$$

is also exact (because $R^iF(K_j) = 0$ for $i \neq 0$ and $j = 1, 2, 3$). Now given an acyclic complex

$$(\cdots \to 0 \to K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots) \in \mathrm{Kom}^+(R),$$

we decompose it into short exact sequences

$$\begin{aligned} 0 \to K^0 &\to K^1 \to \operatorname{Im} d^1 \to 0, \\ 0 \to \operatorname{Im} d^1 &\to K^2 \to \operatorname{Im} d^2 \to 0, \\ &\vdots \\ 0 \to \operatorname{Im} d^i &\to K^{i+1} \to \operatorname{Im} d^{i+1} \to 0, \\ &\vdots \end{aligned}$$

Apply $\mathsf{R}F$ and induction on i, we get $\operatorname{Im} d^i \in R$. So

$$F(\operatorname{Im}(K^i \xrightarrow{d^i} K^{i+1})) = \operatorname{Im}(F(K^i) \xrightarrow{F(d^i)} F(K^{i+1}))$$

and

$$0 \to \operatorname{Im} F(d^i) \to F(K^{i+1}) \to \operatorname{Im} F(d^{i+1})$$

is exact. Hence $\operatorname{Im} F(d^i) = \ker F(d^{i+1})$, i.e., $F(K^\bullet)$ is acyclic. ■

6.5 Composition of derived functors

Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories such that $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$. Given $X \in D^+(\mathcal{A})$, it is easy to see that $\mathsf{R}(G \circ F)(X) \cong \mathsf{R}G(\mathsf{R}F(X))$: choose $I^\bullet \in \mathrm{Kom}^+(\mathcal{I}_F)$ that is quasi-isomorphic to X, then

$$\mathsf{R}(G \circ F)(X) \cong G(F(I^\bullet)) \cong \mathsf{R}G(F(I^\bullet)) \cong \mathsf{R}G(\mathsf{R}F(I^\bullet)).$$

Next proposition says that $\mathsf{R}(G \circ F)(X) \cong \mathsf{R}G(\mathsf{R}F(X))$ is functorial in $X \in D^+(\mathcal{A})$.

Proposition 6.13. Assume that $\mathcal{I}_F \subseteq\subseteq \mathcal{A}$ (resp. $\mathcal{I}_G \subseteq \mathcal{B}$) is an F-adapted (resp. G-adapted) class such that $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$. Then $\mathsf{R}F$, $\mathsf{R}G$, $\mathsf{R}(G \circ F)$ exist and

$$\mathsf{R}(G \circ F) \cong \mathsf{R}G \circ \mathsf{R}F.$$

Interlude 2: Spectral sequences

Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Assume that

- there exists F-adapted class $\mathcal{I}_F \subseteq \mathcal{A}$;
- there exists G-adapted class $\mathcal{I}_G \subseteq \mathcal{B}$;
- $F(\mathcal{I}_F) \subseteq \mathcal{I}_G$,

which implies that $\mathsf{R}F$, $\mathsf{R}G$, $\mathsf{R}(G \circ F)$ exist and $\mathsf{R}(G \circ F) \cong \mathsf{R}G \circ \mathsf{R}F$. We can use spectral sequence to relate $R^pG(R^qF(X))$ and $R^{p+q}(G \circ F)(X)$.

Proposition 6.14. For each $X \in D^+(\mathcal{A})$, there exists a spectral sequence

$$E_2^{p,q} = R^pG(R^qF(X)) \Rightarrow R^{p+q}(G \circ F)(X).$$

Definition

Let $\mathcal{A}$ be an abelian category. A spectral sequence in $\mathcal{A}$ consists of

- $E^n \in \mathcal{A}$ for each $n \in \mathbb{Z}$;
- for each $r \in \mathbb{Z}_{\geq 0}$, $\{E_r^{p,q} \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$, called the r^{th} page;
- $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that $d_r \circ d_r = 0$;
- isomorphisms
$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(d_r : E_r^{p-r,q+r_1} \to E_r^{p,q})},$$

subject to the following conditions:

- For each (p, q), there exists r_0 such that for all $r \geq r_0$,
$$E_r^{p-r,q+r-1} \xrightarrow{d_r=0} E_r^{p,q} \xrightarrow{d_r=0} E_r^{p+r,q-r+1}.$$

  This implies $E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong \cdots =: E_\infty^{p,q}$.

- Each E^n admits a decreasing filtration

$$\cdots \subseteq F^{p+1}E^n \subseteq F^pE^n \subseteq F^{p-1}E^n \subseteq \cdots,$$

with $\bigcap\limits_p F^pE^n = 0$ and $\bigcup\limits_p F^pE^n = E^n$ such that

$$E_\infty^{p,q} \cong {F^pE^{p+q}}\big/{F^{p+1}E^{p+q}} =: \operatorname{Gr}_F^p E^{p+q}.$$

We say that the spectral sequence $\{E_r^{p,q}\}$ converges to $\{E^n\}$, and write $E_r^{p,q} \Rightarrow E^{p+q}$.

We say that $\{E_r^{p,q}\}$ degenerate at page r_0 if $d_r^{p,q} = 0$ for each $r \geq r_0$ and for each p, q. In this case, $E_\infty^{p,q} = E_{r_0}^{p,q}$. If $\mathcal{A}$ is semisimple, then $E^n \cong \bigoplus\limits_{p+q=n} E_{r_0}^{p,q}$ (non-canonical).

An example

Let $G : \mathcal{A} \to \mathcal{B}$ be left exact. Assume there exists a G-adapted class $\mathcal{I}_G$. Then we get the right derived functor $\mathsf{R}G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.

Corollary 6.15. If $\mathsf{R}G(A) \in D^{\mathrm{b}}(\mathbb{B})$ for each $A \in \mathcal{A}$, then $\mathsf{R}G$ induces

$$\mathsf{R}G : D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{B}).$$

Proof. Let $X^\bullet \in D^{\mathrm{b}}(\mathcal{A})$. Then for each $k \in \mathbb{Z}$,

$$\begin{tikzcd}
\tau_{\leq k}X^\bullet \arrow[rr] \arrow[dr, phantom, "\triangle"] & & X^\bullet \arrow[dl] \\
& \tau_{>k}X^\bullet \arrow[ul, dashed] &
\end{tikzcd}
\quad\Longrightarrow\quad
\begin{tikzcd}
\mathsf{R}G(\tau_{\leq k}X^\bullet) \arrow[rr] \arrow[dr, phantom, "\triangle"] & & \mathsf{R}G(X^\bullet) \arrow[dl] \\
& \mathsf{R}G(\tau_{>k}X^\bullet). \arrow[ul, dashed] &
\end{tikzcd}$$

Then induction on the number of integers j such that $H^j(X^\bullet) \neq 0$ completes the proof. ∎

We present a proof using spectral sequences.

Proof. Let $X^\bullet \in D^{\mathrm{b}}(\mathcal{A})$. Apply (6.14) to G and $F = \mathrm{id}_A$, we get

$$E_2^{p,q} = R^pG(R^qF(X^\bullet)) = R^pG(H^q(X^\bullet)) \Rightarrow R^{p+q}(G)(X^\bullet).$$

Since $X^\bullet \in D^{\mathrm{b}}(\mathcal{A})$ and $\mathsf{R}G(H^q(X^\bullet)) \in D^{\mathrm{b}}(X)$, there exists $C > 0$ such that $E_2^{p,q} = R^pG(H^q(X^\bullet)) = 0$ for all $|p|$, $|q| > C$. As $E_\infty^{p,q}$ is a sub-quotient of $E_2^{p,q}$, we have $E_\infty^{p,q} = 0$ for all $|p|$, $|q| > C$. Hence $R^{p+q}G(X^\bullet) = E^{p+q} = 0$ whenever $|p+q| > 2C$, so $\mathsf{R}G(X^\bullet) \in D^{\mathrm{b}}(\mathcal{B})$. ∎

7 Examples of derived functors

7.1 Ext

Let $\mathcal{A}$ be an abelian category.

Definition 7.1. For each X, $Y \in \mathcal{A}$, we define

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{D(\mathcal{A})}(X[0], Y[i]).$$

Some immediate properties of Ext:

- $\mathrm{Ext}^0(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$.
- $\mathrm{Ext}^i(X, Y) = \mathrm{Hom}_{D(\mathcal{A})}(X[k], Y[k+i])$ for each $k \in \mathbb{Z}$.
- For each X, Y, $Z \in \mathcal{A}$, we have a bilinear map

  $$\mathrm{Ext}^i(X, Y) \times \mathrm{Ext}^j(Y, Z) \to \mathrm{Ext}^{i+j}(X, Z)$$

  defined by composition. It is called the Yoneda product.
- We have seen that every short exact sequence

  $$0 \to X' \to X \to X'' \to 0$$

  induces a long exact sequence

  $$\cdots \to \mathrm{Ext}^i(X'', Y) \to \mathrm{Ext}^i(X, Y) \to \mathrm{Ext}^i(X', Y) \to \mathrm{Ext}^{i+1}(X'', Y) \to \cdots$$

  and similar for $\mathrm{Ext}^i(Y, -)$.

Proposition 7.2. For each $i < 0$, $\mathrm{Ext}^i(X, Y) = 0$.

Proof. Given $X[0] \to Y[i]$, represented by $(K^\bullet \xrightarrow{s} X[0], f)$. Since $i < 0$, we have

$$\begin{tikzcd}
 & \tau_{\leq -i-1} K^\bullet \arrow[dl] \arrow[d, "g"] \arrow[dr, "0"] & \\
X[0] & K^\bullet \arrow[l, "s"] \arrow[r, "f"'] & Y[i],
\end{tikzcd}$$

where the composition $s \circ g$ is a quasi-isomorphism. Hence $\operatorname{Ext}^i(X, Y) = \operatorname{Hom}(X, Y[i]) = 0$. ■

Proposition 7.3. Assume that $\mathcal{A}$ has enough injectives (resp. projectives). Then $\operatorname{Ext}^i(X, -) \cong R^i \operatorname{Hom}(X, -)$ (resp. $\operatorname{Ext}^i(-, X) \cong R^i \operatorname{Hom}(-, X)$).

Proof. For each $A^\bullet$, $B^\bullet \in \operatorname{Kom}(\mathcal{A})$, define

$$\operatorname{Hom}^n(A^\bullet, B^\bullet) = \prod_{i \in \mathbb{Z}} \operatorname{Hom}(A^i, B^{n+i})$$

and

$$\begin{aligned} d^n : \quad \operatorname{Hom}^n(A^\bullet, B^\bullet) &\to \operatorname{Hom}^{n+1}(A^\bullet, B^\bullet) \\ f &\mapsto d_B \circ f - (-1)^n f \circ d_A. \end{aligned}$$

Then $(\operatorname{Hom}^n(A^\bullet, B^\bullet), d^n)$ is a complex. We have

$$\ker d^i = \operatorname{Hom}_{\operatorname{Kom}(\mathcal{A})}(A^\bullet, B^\bullet[i]), \quad H^i(\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)) = \operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet[i]).$$

Now given $Y \in \mathcal{A}$, let $Y \to I^\bullet$ be an injective resolution. Then by (6.4)

$$\begin{aligned} \operatorname{Ext}^i(X, Y) &= \operatorname{Hom}_{D(\mathcal{A})}(X, I^\bullet[i]) \cong \operatorname{Hom}_{K(\mathcal{A})}(X, I^\bullet[i]) \\ &= H^i(\operatorname{Hom}^\bullet(X, I^\bullet)) = H^i(\operatorname{Hom}(X, I^\bullet[i])) = R^i \operatorname{Hom}(X, Y). \end{aligned}$$

The second statement can be proved similarly. ■

7.2 Tensor product

7.2.1 R-modules

Let R be a ring with 1, and let N be a left R-module. Then $F = - \otimes_R N : \textsf{mod-}R \to \textsf{Ab}$ is a right exact functor. Flat modules form a class of F-adapted objects. Then we get

$$\mathsf{L}F = - \otimes^{\mathsf{L}}_R N : D^-(R\textsf{-mod}) \to D^-(\textsf{Ab})$$

We define $\operatorname{Tor}_i^R(M, N) = H^{-i}(M \otimes^{\mathsf{L}}_R N)$.

7.2.2 Coherent sheaves

Let X be a variety over a field k. Let $\mathscr{F} \in \textsf{Coh}\, X$. Then

$$- \otimes \mathscr{F} : \textsf{Coh}\, X \to \textsf{Coh}\, X$$

is right exact.

Assume X is quasi-projective. Then for each $\mathscr{E} \in \mathsf{Coh}\, X$, $\mathscr{E}$ has a resolution

$$\cdots \to \mathscr{L}^1 \to \mathscr{L}^0 \to \mathscr{E}$$

by locally free sheaves of finite rank. Moreover if $\mathscr{E}^\bullet \in \mathrm{Kom}^-(\mathsf{Coh}\, X)$ is an acyclic complex of locally free sheaves, then $\mathscr{E}^\bullet \otimes \mathscr{F}$ is also acyclic. So locally free sheaves on X form a class of objects adapted for $- \otimes \mathscr{F}$. Hence, we get

$$- \otimes^{\mathsf{L}} \mathscr{F} : D^-(X) \to D^-(X) =: D^-(\mathsf{Coh}\, X),$$

and $\mathcal{T}or_1(\mathscr{E}, \mathscr{F}) := H^{-i}(\mathscr{E} \otimes^{\mathsf{L}} \mathscr{F})$.

Assume X is smooth quasi-projective. Then every $\mathscr{E} \in \mathsf{Coh}\, X$ has a finite locally free resolution. Hence,

$$- \otimes^{\mathsf{L}} \mathscr{F} : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X) := D^{\mathrm{b}}(\mathsf{Coh}\, X).$$

7.2.3 Complexes of coherent sheaves

Let X be a quasi-projective variety over a field k. Given $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$. Define

$$(\mathscr{E}^\bullet \otimes \mathscr{F}^\bullet)^i = \bigoplus_{p+q=i} \mathscr{E}^p \oplus \mathscr{F}^q,$$

with $d^i = d_{\mathscr{E}} \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes d_{\mathscr{F}}$. Then

$$- \otimes \mathscr{F}^\bullet : K^-(\mathsf{Coh}\, X) \to K^-(\mathsf{Coh}\, X)$$

is right exact, and complexes of locally free sheaves in $K^-(\mathsf{Coh}\, X)$ are again adapted for $- \otimes \mathscr{F}^\bullet$. This gives the derived functor $- \otimes^{\mathsf{L}} \mathscr{F}^\bullet : D^-(X) \to D^-(X)$. Furthermore, we get

$$- \otimes^{\mathsf{L}} - : D^-(X) \times D^-(X) \to D^-(X)$$

with $D^{\mathrm{b}}(X) \times D^{\mathrm{b}}(X)$ if X is smooth.

7.3 Pullback and pushforward

Let $f : X \to Y$ be a morphism of ringed spaces. Then f^* is the left adjoint of f_*. Hence, f^* is right exact and f_* is left exact.

Now, let X, Y be quasi-projective varieties over a field k. Then $f^* : \mathsf{Coh}\, Y \to \mathsf{Coh}\, X$ is right exact. Since locally free sheaves are f^*-adapted, we get the left derived functor $\mathsf{L}f^* : D^-(Y) \to D^-(X)$, and $\mathsf{L}f^*$ maps $D^{\mathrm{b}}(Y)$ to $D^{\mathrm{b}}(X)$ if Y is smooth.

If X and Y are noetherian schemes, we have $f_* : \mathsf{QCoh} \to \mathsf{QCoh}$, which is left exact. Since QCoh has enough injectives, we have the right derived functor $\mathsf{R}f_* : D^+(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, X)$. We define the higher direct image $R^i f_*(\mathscr{F}^\bullet)$ to be $H^i(\mathsf{R}f_*\mathscr{F}^\bullet)$.

In particular, when X is defined over a field k,

$$\Gamma : \mathsf{QCoh}\, X \to \mathsf{Vect}_k$$

is the pushforward of $p : X \to \operatorname{Spec} k$. So we define

$$\mathsf{R}\Gamma = \mathsf{R}p_* : D^+(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, X),$$

and $H^i(X, \mathscr{F}^\bullet) = R^i\Gamma(\mathscr{F}^\bullet)$ the hypercohomology.

For each $\mathscr{F} \in \mathsf{QCoh}\, X$, as $R^i f_*\mathscr{F} = 0$ for $|i| \gg 1$, $\mathsf{R}f_*$ induces

$$Rf_* : D^{\mathrm{b}}(\mathsf{QCoh}\, X) \to D^{\mathrm{b}}(\mathsf{QCoh}\, Y).$$

Theorem 7.4. Assume $f : X \to Y$ is a proper morphism. Then $f_* : \mathsf{Coh}\, X \to \mathsf{Coh}\, Y$.

Assume $f : X \to Y$ is proper. In general, $\mathsf{Coh}\, X$ does not have enough injectives so we can't define $\mathsf{R}f_* : D^?(\mathsf{Coh}\, X) \to D^?(\mathsf{Coh}\, Y)$ as a derived functor of $f_* : \mathsf{Coh}\, X \to \mathsf{Coh}\, Y$. To define $\mathsf{R}f_* : D^{\mathrm{b}}(\mathsf{Coh}\, X) \to D^{\mathrm{b}}(\mathsf{Coh}\, Y)$, we need to proceed differently.

7.4 Derived category of coherent sheaves

Definition 7.5. A thick subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$ is a full abelian subcategory such that for each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with M', $M'' \in \mathcal{A}$, we have $M \in \mathcal{A}$.

Let $\mathcal{A} \subseteq \mathcal{B}$ be a thick subcategory. Define $D^?_{\mathcal{A}}(\mathcal{B}) \subseteq D^?(\mathcal{B})$ as the full subcategory of complexes $X^\bullet$ with $H^i(X^\bullet) \in \mathcal{A}$ for each i.

Lemma 7.6. Let X be a noetherian scheme and let $\varphi : \mathscr{E} \to \mathscr{F}$ be a surjective morphism of quasi-coherent sheaves. Assume that $\mathscr{F}$ is coherent. Then there exists a coherent subsheaf $\mathscr{G} \subseteq \mathscr{E}$ such that $\varphi|_{\mathscr{G}}$ is surjective.

Proposition 7.7. For any noetherian scheme X, and for $? = -$ or b,

$$D^?(X) := D^?(\mathsf{Coh}\, X) \xrightarrow{\sim} D^?_{\mathsf{Coh}\, X}(\mathsf{QCoh}\, X).$$

Proof. Let $? = \mathrm{b}$, the other case is similar. Let $\mathscr{F}^\bullet \in D^{\mathrm{b}}_{\mathsf{Coh}\, X}(\mathsf{QCoh}\, X)$, and assume that $\mathscr{F}^i$ is coherent for all $i \geq i_0$ for some i_0. Since $\mathscr{F}^{i_0} \to \operatorname{Im} d^{i_0}$ and $\ker d^{i_0} \to \dfrac{\ker d^{i_0}}{\operatorname{Im} d^{i_0-1}}$ are surjections and the codomains are coherent, by (7.6) there exists coherent subsheaves $\mathscr{F}' \subseteq \mathscr{F}^{i_0}$ and $\mathscr{F}'' \subseteq \ker d^{i_0}$ such that $\mathscr{F}' \to \operatorname{Im} d^{i_0}$ and $\mathscr{F}'' \to \dfrac{\ker d^{i_0}}{\operatorname{Im} d^{i_0-1}}$ are also surjections. Replace $\mathscr{F}^{i_0}$ by the coherent subsheaf $\mathscr{F}^{i_0}_{\mathrm{new}}$ generated by $\mathscr{F}'$, $\mathscr{F}'' \subseteq \mathscr{F}^{i_0}$ and $\mathscr{F}^{i_0-1}$ by $(d^{i_0-1})^{-1}(\mathscr{F}^{i_0}_{\mathrm{new}})$. Do this inductively, we finally get a complex of coherent sheaves that is quasi-isomorphic to the original complex. ∎

Proposition 7.8. Let $f : X \to Y$ be a proper morphism between noetherian schemes. Then

$$\mathsf{R}f_* : D^+(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, Y)$$

induces $\mathsf{R}f_* : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$.

Proof. Enough to show that for each $\mathscr{F}^\bullet \in D^{\mathrm{b}}_{\mathsf{Coh}\, X}(\mathsf{QCoh}\, X)$, we have

$$\mathsf{R}f_*\mathscr{F}^\bullet \in D^{\mathrm{b}}_{\mathsf{Coh}\, Y}(\mathsf{QCoh}\, Y),$$

namely $R^i f_*\mathscr{F}^\bullet \in \mathsf{Coh}\, Y$. We have a spectral sequence

$$E_2^{p,q} = R^p f_*(H^q(\mathscr{F}^\bullet)) \Rightarrow R^{p+q} f_*\mathscr{F}^\bullet.$$

Since $R^p f_*(H^q(\mathscr{F}^\bullet))$ are coherent, $E_\infty^{p,q}$ are also coherent. Since $E_2^{p,q} = 0$ for all $|p|$, $|q| \gg 1$, $E_\infty^{p,q} = 0$ for all $|p|$, $|q| \gg 1$. Thus $R^{p+q} f_*\mathscr{F}^\bullet$ is an extension of finitely many coherent sheaves $E_\infty^{p,q}$, showing that $R^{p,q} f_*\mathscr{F}^\bullet$ is coherent. ∎

7.5 Local Hom

Let X be a quasi-projective variety over a field k. For $\mathscr{E}^\bullet \in \mathrm{Kom}^-(\mathsf{QCoh}\, X)$ and $\mathscr{F}^\bullet \in \mathrm{Kom}^+(\mathsf{QCoh}\, X)$, we define

$$\mathscr{H}om^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \prod_{j \in \mathbb{Z}} \mathscr{H}om(\mathscr{E}^j, \mathscr{F}^{j+i})$$

with $d^i(\{\mathscr{E}^j \xrightarrow{f^j} \mathscr{F}^{j+i}\}_j) = \{f^{j+1 \circ d - (-1)^i d \circ f^j}\}_j$. We get a left exact functor

$$\mathscr{H}om^\bullet(\mathscr{E}^\bullet, -) : K^+(\mathsf{QCoh}\, X) \to K^+(\mathsf{QCoh}\, X).$$

If $\mathscr{E}^\bullet \in \mathrm{Kom}^{\mathrm{b}}(\mathsf{Coh}\, X)$, it maps $K^{\mathrm{b}}(\mathsf{Coh}\, X)$ to itself.

Using the same argument constructing $\mathsf{R}f_* : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$, we get the right derived functor

$$\mathsf{R}\mathscr{H}om(\mathscr{E}^\bullet, -) : D^+(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, X).$$

Again, it maps $D^{\mathrm{b}}(X)$ to itself if $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$.

Likewise, using the same argument constructing $\mathsf{L}f^*$, we get the right derived functor

$$\mathsf{R}\mathscr{H}om(-, \mathscr{F}^\bullet) : D^-(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, X),$$

which maps $D^{\mathrm{b}}(X)$ to itself if $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$ and X is smooth. Using these derived functors, we get

$$\mathsf{R}\mathscr{H}om(-, -) : D^-(\mathsf{QCoh}\, X) \times D^+(\mathsf{QCoh}\, X) \to D^+(\mathsf{QCoh}\, X),$$

which maps $D^{\mathrm{b}}(X) \times D^{\mathrm{b}}(X)$ to $D^{\mathrm{b}}(X)$.

8 Some properties of derived functors in algebraic geometry

8.1 Composition

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of smooth quasi-projective varieties.

Proposition 8.1. We have

(1) $\mathsf{R}(g \circ f)_* \xrightarrow{\sim} \mathsf{R}g_* \circ \mathsf{R}f_*$;

(2) $\mathsf{L}(g \circ f)^* \xrightarrow{\sim} \mathsf{R}f^* \circ \mathsf{R}g^*$.

Proof. (1) Given an injective object $\mathcal{I} \in \mathsf{QCoh}\, X$. Then $\mathcal{I}$ is flasque, so $f_*\mathcal{I}$ is also flasque. Thus $R^j g_*(f_*\mathcal{I}) = 0$ for each $j \neq 0$, i.e., f_* sends injective objects to g_*-acyclic objects. This proves (1).

(2) If $\mathscr{F}$ is a locally free sheaf on Z, then $g^*\mathscr{F}$ is also locally free. This proves (2). ■

8.2 Projection formula

Proposition 8.2. Let $f : X \to Y$ be a proper morphism between smooth quasi-projective varieties. For all $\mathscr{E} \in D^{\mathrm{b}}(X)$ and $\mathscr{F} \in D^{\mathrm{b}}(Y)$, we have functorial isomorphisms

$$\mathsf{R}f_*\mathscr{E} \otimes^{\mathsf{L}} \mathscr{F} \xrightarrow{\sim} \mathsf{R}f_*(\mathscr{E} \otimes^{\mathsf{L}} \mathsf{L}f^*\mathscr{F}).$$

Proof. Let $\mathcal{I}^\bullet$ be a complex of injective objects that is quasi-isomorphic to $\mathscr{E}$, and let $\mathscr{L}^\bullet$ be a complex of locally free sheaves of finite length that is quasi-isomorphic to $\mathscr{F}$. Then it follows from the classical projection formula that

$$\mathsf{R}f_*\mathscr{E} \otimes^{\mathsf{L}} \mathscr{F} \cong (f_*\mathcal{I}^\bullet) \otimes \mathscr{L}^\bullet \cong f_*(\mathcal{I}^\bullet \otimes f^*\mathscr{L}^\bullet).$$

Since

$$\mathsf{R}f_*(\mathscr{E} \otimes^{\mathsf{L}} \mathsf{L}f^*\mathscr{F}) \cong \mathsf{R}f_*(\mathcal{I}^\bullet \otimes f^*\mathscr{L}^\bullet),$$

it suffices to show that the morphism

$$\alpha : f_*(\mathcal{I}^\bullet \otimes f^*\mathscr{L}^\bullet) \to \mathsf{R}f_*(\mathcal{I}^\bullet \otimes f^*\mathscr{L}^\bullet),$$

which comes from the natural transformation (see (6.10))

$$\begin{tikzcd}
 & D^+(\mathsf{QCoh}\, X) \arrow[dr, "\mathsf{R}f_*"] & \\
K^+(\mathsf{QCoh}\, X) \arrow[ur] \arrow[dr, "f_*"'] & \Uparrow & D^+(\mathsf{QCoh}\, Y) \\
 & K^+(\mathsf{QCoh}\, Y), \arrow[ur] &
\end{tikzcd}$$

is an isomorphism. The statement is local, so up to shrinking Y, we can assume that $\mathscr{L}^i = \bigoplus_{J_i} \mathcal{O}_Y$. Then $f^*\mathscr{L}^i = \bigoplus_{J_i} \mathcal{O}_X$ and thus $\mathcal{I}^j \otimes f^*\mathscr{L}^i \cong \bigoplus_{J_i} \mathcal{I}^j$ is injective. It follows from $\mathcal{I}^\bullet \otimes f^*\mathscr{L}^\bullet$ that α is an isomorphism. ■

Based on similar arguments, we can also show the following.

Proposition 8.3. Let $f : X \to Y$ be a morphism of smooth projective varieties. Given $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet \in D^{\mathrm{b}}(Y)$, we have

$$\mathsf{L}f^*\mathscr{E}^\bullet \otimes^{\mathsf{L}} \mathsf{L}f^*\mathscr{F}^\bullet \xrightarrow{\sim} \mathsf{L}f^*(\mathscr{E}^\bullet \otimes^{\mathsf{L}} f^\bullet)$$

and the isomorphism is functorial.

8.3 Adjunction

Proposition 8.4. Let $f : X \to Y$ be a morphism of smooth projective varieties. For all $\mathscr{E} \in D^{\mathrm{b}}(X)$ and $\mathscr{F} \in D^{\mathrm{b}}(Y)$, we have functorial isomorphisms

$$\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathsf{R}f_*\mathscr{E}) \xrightarrow{\sim} \mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathsf{L}f^*\mathscr{F}, \mathscr{E}).$$

In particular,

$$\mathsf{R}\operatorname{Hom}(\mathscr{F}, \mathsf{R}f_*\mathscr{E}) \xrightarrow{\sim} \mathsf{R}\operatorname{Hom}(\mathsf{L}f^*\mathscr{F}, \mathscr{E}),$$
$$\operatorname{Hom}(\mathscr{F}, \mathsf{R}f_*\mathscr{E}) \xrightarrow{\sim} \operatorname{Hom}(\mathsf{L}f^*\mathscr{F}, \mathscr{E}).$$

Proof. The proof is similar to the projection formula. With the same notation in that proof, we have

$$\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathsf{R}f_*\mathscr{E}) \cong \mathscr{H}om(\mathscr{L}^\bullet, f_*\mathcal{I}^\bullet) \cong f_*\mathscr{H}om(f^*\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

and

$$\mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathsf{L}f^*\mathscr{F}, \mathscr{E}) \cong \mathsf{R}f_*\mathscr{H}om)(f^*\mathscr{L}^\bullet, \mathcal{I}^\bullet).$$

Similar argument shows that $f_*\mathscr{H}om(f^*\mathcal{L}^\bullet, \mathcal{I}^\bullet) \to \mathsf{R}f_*\mathscr{H}om)(f^*\mathscr{L}^\bullet, \mathcal{I}^\bullet)$ is an isomorphism. ■

Let X be a smooth projective variety. We can also show the following in $D^{\mathrm{b}}(X)$: for all $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$, $\mathscr{G}^\bullet \in D^{\mathrm{b}}(X)$, we have

$$\mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet) \otimes^{\mathsf{L}} \mathscr{G}^\bullet \cong \mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes^{\mathsf{L}} \mathscr{G}^\bullet) \cong \mathsf{R}\mathscr{H}om(\mathsf{R}\mathscr{H}om(\mathscr{E}^\bullet, \mathscr{F}^\bullet), \mathscr{G}^\bullet),$$
$$\mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet, \mathsf{R}\mathscr{H}om(\mathscr{E}^\bullet, \mathscr{G}^\bullet) \cong \mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet \otimes^{\mathsf{L}} \mathscr{E}^\bullet, \mathscr{G}^\bullet).$$

In particular, if $\mathscr{G}^\bullet = \mathcal{O}_X$, then

$$\mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet) \cong (\mathscr{F}^\bullet)^\vee \otimes^{\mathsf{L}} \mathscr{E}^\bullet,$$

where $(\mathscr{F}^\bullet)^\vee = \mathsf{R}\mathscr{H}om(\mathscr{F}^\bullet, \mathcal{O}_X)$.

8.4 Grothendieck-Verdier duality

Here is a particular case of the GV duality

Theorem 8.5. Let $f : X \to Y$ be a proper morphism between smooth quasi-projective varieties over some field k. Then for all $\mathscr{E} \in D^{\mathrm{b}}(\mathscr{E})$ and $\mathscr{F} \in D^{\mathrm{b}}(\mathscr{F})$, there exists a functorial isomorphism

$$\mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathscr{E}, f^!\mathscr{F}) \cong \mathsf{R}\mathscr{H}om(\mathsf{R}f_*\mathscr{E}, \mathscr{F}),$$

where $f^!\mathscr{F} = \mathsf{L}f^*\mathscr{F} \otimes \omega_X \otimes f^*\omega_Y^\vee[\dim X - \dim Y]$ and ω_X, ω_Y are the canonical line bundles on X, Y, respectively.

In particular,

$$\mathrm{Hom}(\mathscr{E}, f^!\mathscr{F}) \cong \mathrm{Hom}(\mathsf{R}f_*\mathscr{E}, \mathscr{F}),$$

namely, $f^!$ is a right adjoint of f_*.

When $f : X \to \operatorname{Spec} k$, we obtain Serre duality

$$\mathrm{Hom}(\mathscr{E}, \omega_X[n]) \cong \mathrm{Hom}(\mathsf{R}\Gamma(\mathscr{E}^\bullet), k).$$

8.5 Serre functor

Let $\mathcal{D}$ be a k-linear triangulated category. Assume

$$\sum_i \dim \mathrm{Hom}^i(\mathscr{E}, \mathscr{F}) < \infty$$

for all $\mathscr{E}$, $\mathscr{F} \in \mathcal{D}$.

Definition 8.6. A Serre functor is an autoequivalence

$$S : \mathcal{D} \to \mathcal{D}$$

such that for all $\mathscr{E}$, $\mathscr{F} \in \mathcal{D}$, there exists an isomorphism

$$\mathrm{Hom}(\mathscr{E}, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{F}, S(\mathscr{E}))^\vee,$$

which is functorial in $\mathscr{E}$ and $\mathscr{F}$.

Example 8.7. Let X be a smooth projective variety over k. Then $S = -\otimes(\omega_X[\dim X]) : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$ is a Serre functor. In particular, when $\omega_X \cong \mathcal{O}_X$, then $[\dim X] : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$ is a Serre functor.

For a triangulated category $\mathcal{D}$, if $[n] : \mathcal{D} \to \mathcal{D}$ is a Serre functor, we call $\mathcal{D}$ a Calabi-Yau category of dimension n.

Proposition 8.8. Given a k-linear exact functor

$$F : \mathcal{C} \to \mathcal{D}$$

between k-linear triangulated category, admitting Serre functors $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $S_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$. Then

- F has a left adjoint if and only if F has a right adjoint;
- F is an equivalence implies that $F \circ S_{\mathcal{C}} \cong S_{\mathcal{D}} \circ F$.

8.6 Fourier-Mukai transforms

Let X, Y be smooth projective varieties. Let

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

be the projections.

Definition 8.9. A Fourier-Mukai transform is a functor of the form

$$\begin{array}{rccc} \Phi^{\mathscr{P}}_{X\to Y} : & D^{\mathrm{b}}(X) & \to & D^{\mathrm{b}}(Y) \\ & \mathscr{F}^{\bullet} & \mapsto & \mathsf{R}q_*(p^*\mathscr{F}^{\bullet} \otimes^{\mathsf{L}} \mathscr{P}) \end{array}$$

for some $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$. We call $\mathscr{P}$ the Fourier-Mukai kernel of $\Phi^{\mathscr{P}}$.

We see that $\Phi^{\mathscr{P}}_{X\to Y}$ is an exact functor. Given smooth projective varieties X, Y, Z and

$$\mathscr{P} \in D^{\mathrm{b}}(X \times Y), \quad \mathscr{Q} \in D^{\mathrm{b}}(Y \times Z),$$

the composition of the Fourier-Mukai transforms $\Phi^{\mathscr{Q}} \circ \Phi^{\mathscr{P}}$ is also a Fourier-Mukai transform with kernel

$$\mathscr{R} = \mathsf{R}(p_{ZX})_*(p^*_{XY}\mathscr{P} \otimes^{\mathsf{L}} p^*_{YZ}\mathscr{Q}),$$

where p_{YZ}, p_{ZX}, p_{XY} are the projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{YZ}"'] \arrow[d, "p_{ZX}"] \arrow[dr, "p_{XY}"] & \\ Y \times Z & Z \times X & X \times Y. \end{tikzcd}$$

We call $\mathscr{R}$ the convolution of $\mathscr{P}$ and $\mathscr{Q}$ and write $\mathscr{R} = \mathscr{P} * \mathscr{Q}$.

Proposition 8.10. Let $f : X \to Y$ be a morphism of smooth projective varieties and let $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$. Then $\mathsf{R}f_*$, $\mathsf{L}f^*$, $- \otimes \mathscr{F}^\bullet$, $[1]$ are Fourier-Mukai transforms.

Theorem 8.11 (Orlov)**.** Given an exact functor

$$F : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y).$$

If F is fully faithful, then $F = \Phi^{\mathscr{P}}_{X \to Y}$ for some $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$. Moreover, $\mathscr{P}$ is unique up to isomorphisms.

Remark. There exist examples of exact functor $F : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ which are not Fourier-Mukai transforms.

Proposition 8.12. Let $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$. We have

$$\Phi^{\mathscr{P}^\vee}_{Y \to X} \circ S_Y \dashv \Phi^{\mathscr{P}}_{X \to Y}, \quad \Phi^{\mathscr{P}}_{X \to Y} \dashv S_X \circ \Phi^{\mathscr{P}^\vee},$$

where S_X, S_Y are the Serre functors and

$$\mathscr{P}^\vee = \mathsf{R}\mathscr{H}om(\mathscr{P}, \mathcal{O}_{Y \times X}) \in D^{\mathrm{b}}(Y \times X).$$

Proof. We only prove $\Phi^{\mathscr{P}^\vee}_{Y \to X} \circ S_Y \dashv \Phi^{\mathscr{P}}_{X \to Y}$. Given $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$ and $\mathscr{F}^\bullet \in D^{\mathrm{b}}(Y)$, we have

$$\begin{aligned} \operatorname{Hom}(\Phi^{\mathscr{P}^\vee} \circ S_Y(\mathscr{F}^\bullet), \mathscr{E}^\bullet) &= \operatorname{Hom}(\mathsf{R}p_*(\Phi^{\mathscr{P}^\vee} \otimes^{\mathsf{L}} q^*\mathscr{F}^\bullet \otimes q^\bullet\omega_Y[\dim Y], \mathscr{E}^\bullet) \\ &\cong \operatorname{Hom}(\mathsf{R}p * (\mathscr{P}^\vee \otimes^{\mathsf{L}} q^*\mathscr{F}^\bullet \otimes q^*\omega_Y), \mathscr{E}^\bullet[-\dim Y]) \\ &\cong \operatorname{Hom}(\mathscr{P}^\vee \otimes^{\mathsf{L}} q^*\mathscr{F}^\bullet \otimes q^*\omega_Y, p^!\mathscr{E}^\bullet[-\dim Y]) \\ &= \operatorname{Hom}(\mathscr{P}^\vee \otimes^{\mathsf{L}} q^*\mathscr{F}^\bullet \otimes q^*\omega_Y, p^*\mathscr{E}^\bullet \otimes p^*\omega_X \otimes q^*\omega_Y \otimes p^*\omega_X^\vee) \\ &\cong \operatorname{Hom}(\mathscr{P}^\vee \otimes^{\mathsf{L}} q^*\mathscr{F}^\bullet, p^*\mathscr{E}^\bullet) \cong \operatorname{Hom}(q^*\mathscr{F}^\bullet, \mathscr{P}^{\otimes}p^*\mathscr{E}^\bullet) \\ &\cong \operatorname{Hom}(\mathscr{F}^\bullet, q_*(\mathscr{P} \otimes p^*\mathscr{E}^\bullet)) = \operatorname{Hom}(\mathscr{F}^\bullet, \Phi^{\mathscr{P}}(\mathscr{E}^\bullet)). \end{aligned}$$

∎

9 Semi-orthogonal decomposition

9.1 X is connected if and only if $D^{\mathrm{b}}(X)$ is indecomposable

9.1.1 Decomposition of triangulated category

Let $\mathcal{T}$ be a triangulated category.

Definition 9.1. We say that $\mathcal{T}$ is decomposed into triangulated subcategories $\mathcal{C}_1$, $\mathcal{C}_2 \subseteq \mathcal{T}$ if

i) for each $A \in \mathcal{T}$, there exists $B_i \in \mathcal{C}_i$ such that

$$\begin{tikzcd} B_1 \arrow[rr] & & A \arrow[dl] \\ & B_2; \arrow[ul, dashed] & \end{tikzcd}$$

ii) $\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0$ for all $B_i \in \mathcal{C}_i$.

Proposition 9.2. Given a distinguished triangle

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C. \arrow[ul, dashed, "u"] & \end{tikzcd}$$

- If $u = 0$, then $B = A \oplus C$.
- If $\mathcal{T}$ decomposed into $\mathcal{C}_1$, $\mathcal{C}_2$, then for each $A \in \mathcal{T}$, there exists $B_1 \in \mathcal{C}_1$, $B_2 \in \mathcal{C}_2$ such that $A = B_1 \oplus B_2$.

We say that $\mathcal{T}$ is indecomposable if for each decomposition of $\mathcal{T}$ into $\mathcal{C}_1$, $\mathcal{C}_2$, either $\mathcal{C}_1 = 0$ or $\mathcal{C}_2 = 0$.

Proposition 9.3. Let X be a noetherian scheme. Then $D^{\mathrm{b}}(X)$ is indecomposable if and only if X is connected.

9.1.2 Support

Definition 9.4. Let $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$. The support of $\mathscr{F}^\bullet$ is defined as

$$\operatorname{Supp} \mathscr{F}^\bullet = \bigcup_i \operatorname{Supp}(H^i(\mathscr{F}^\bullet)).$$

Lemma 9.5. Suppose $\mathscr{F}^\bullet$ is an object in $D^{\mathrm{b}}(X)$ such that $\operatorname{Supp}\mathscr{F}^\bullet = Z_1 \sqcup Z_2$ with Z_1, Z_2 closed. Then $\mathscr{F}^\bullet = \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ with $\operatorname{Supp}\mathscr{F}_i^\bullet \subseteq Z_i$.

Proof. We can assume that $H^k(\mathscr{F}^\bullet) = 0$ for all $k < 0$ and $H^0(\mathscr{F}^\bullet) \neq 0$. We induction on the length of $\mathscr{F}^\bullet$, i.e., the maximal number ℓ such that $H^\ell(\mathscr{F}^\bullet) \neq 0$.

If $\ell = 0$, then $\mathscr{F}^\bullet \cong \mathscr{H}[0]$ with $\mathscr{H} = H^0(\mathscr{F}^\bullet) \in \mathsf{Coh}\, X$. We have $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ with $\operatorname{Supp}\mathscr{H}_i \subseteq Z_i$.

For general case, consider the distinguished triangle

$$\begin{tikzcd}
\tau_{\leq 0}\mathscr{F}^\bullet \arrow[rr] & & \mathscr{F}^\bullet \arrow[dl] \\
& \tau_{>0}\mathscr{F}^\bullet. \arrow[ul, dashed, "u"] &
\end{tikzcd}$$

We have $\ell(\tau_{>0}\mathscr{F}^\bullet[1]) = \ell(\mathscr{F}^\bullet) - 1$, so $\tau_{>0}\mathscr{F}^\bullet \cong \mathscr{G}_1^\bullet \oplus \mathscr{G}_2^\bullet$ with $\operatorname{Supp}\mathscr{G}_i^\bullet \subseteq Z_i$. We have $\tau_{\leq 0}\mathscr{F}^\bullet \cong \mathscr{H}[0]$ for $\mathscr{H} = H^0(\mathscr{F}^\bullet)$. Write $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ with $\operatorname{Supp}\mathscr{H}_i \subseteq Z_i$.
Claim. $\operatorname{Hom}(\mathscr{G}_1^\bullet, \mathscr{H}_2[1]) = 0$.

Proof of Claim. Consider the spectral sequence (by taking the left exact functors $\operatorname{Hom}(-, \mathscr{H}_2)$ and id)

$$E_2^{p,q} = \operatorname{Ext}^p(H^{-q}(\mathscr{G}_1^\bullet), \mathscr{H}_2) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{G}_1^\bullet, \mathscr{H}_2).$$

For each q, we have the spectral sequence (by taking the left exact functors id and $\mathscr{H}om(-, \mathscr{H}_2)$)

$$(E^q)_2^{s,t} = H^s(X, \mathscr{E}xt^t(\mathscr{H}^{-q}(\mathscr{G}_1^\bullet), \mathscr{H}_2)) \Rightarrow \operatorname{Ext}^{s+t}(\mathscr{H}^{-q}(\mathscr{G}_1^\bullet), \mathscr{H}_2).$$

Since $\mathscr{E}xt^t(\mathscr{H}^{-q}(\mathscr{G}_1^\bullet), \mathscr{H}_2) = 0$ as $\operatorname{Supp}(\mathscr{H}^{-q}(\mathscr{G}_1^\bullet)) \cap \operatorname{Supp}\mathscr{H}_2 = \varnothing$,

$$\operatorname{Ext}^p(\mathscr{H}^{-q}(\mathscr{G}_1^\bullet), \mathscr{H}_2) = 0$$

for each p. Hence,

$$\operatorname{Hom}(\mathscr{G}_1^\bullet, \mathscr{H}_2[1]) = \operatorname{Ext}^1(\mathscr{G}_1^\bullet, \mathscr{H}_2) = 0. \qquad \square$$

Similarly, $\operatorname{Hom}(\mathscr{G}_2, \mathscr{H}_1[1]) = 0$. So $u : \tau_{>0}\mathscr{F}^\bullet \to \tau_{\leq 0}\mathscr{F}^\bullet[1]$ can be decomposed into $\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$, where $u_1 : \mathscr{G}_1^\bullet \to \mathscr{H}_1[1]$ and $u_2 : \mathscr{G}_2^\bullet \to \mathscr{H}_2[1]$. Take $\mathscr{F}_i^\bullet$ such that complete the distinguished triangle

$$\begin{tikzcd}
\tau_{\leq 0}\mathscr{F}_i^\bullet \arrow[rr] & & \mathscr{G}_i^\bullet \arrow[dl, "u_i"] \\
& \mathscr{H}_i^\bullet[1], \arrow[ul, dashed] &
\end{tikzcd}$$

then $\operatorname{Supp} \mathscr{F}_i^\bullet \subseteq Z_i$. By (TR3) and (*),

$$\begin{tikzcd}
\tau_{\leq 0}\mathscr{F}^\bullet \arrow[r] \arrow[d, equal] & \mathscr{F}^\bullet \arrow[r] \arrow[d, "\wr"] & \tau_{>0}\mathscr{F}^\bullet \arrow[r] \arrow[d, equal] & \tau_{\leq 0}\mathscr{F}^\bullet[1] \arrow[d, equal] \\
\mathscr{H}_1 \oplus \mathscr{H}_2 \arrow[r] & \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet \arrow[r] & \mathscr{G}_1^\bullet \oplus \mathscr{G}_2^\bullet \arrow[r] & \mathscr{H}_1[1] \oplus \mathscr{H}_2[1].
\end{tikzcd}$$

■

9.1.3 Proof of (9.3)

Assume that $X = X_1 \sqcup X_2$. Let $j_i : X_i \hookrightarrow X$ be the inclusion.

Lemma 9.6. The functor

$$\mathsf{R}(j_i)_* : D^{\mathrm{b}}(X_i) \to D^{\mathrm{b}}(X)$$

is fully faithful, whose essential image consists of $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$ with $\operatorname{Supp} \mathscr{F}^\bullet \subseteq X_i$.

Regarding $D^{\mathrm{b}}(X_1)$, $D^{\mathrm{b}}(X_2)$ as full subcategories of $D^{\mathrm{b}}(X)$, we show that $D^{\mathrm{b}}(X)$ decomposed into $D^{\mathrm{b}}(X_1)$, $D^{\mathrm{b}}(X_2)$:

- For each $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$, (9.5) gives a decomposition $\mathscr{F}^\bullet \cong \mathscr{G}_1^\bullet \oplus \mathscr{G}_2^\bullet$, where $\operatorname{Supp} \mathscr{G}_i^\bullet \subseteq X_i$, So we have the distinguished triangle

$$\begin{tikzcd}
\mathscr{G}_1^\bullet \arrow[rr] & & \mathscr{F}^\bullet \arrow[dl] \\
& \mathscr{G}_2^\bullet. \arrow[ul, dashed] &
\end{tikzcd}$$

- For each $\mathscr{G}_1^\bullet \in D^{\mathrm{b}}(X_1)$ and each $\mathscr{G}_2^\bullet \in D^{\mathrm{b}}(X_2)$, we claim that

$$\operatorname{Hom}_{D^{\mathrm{b}}(X)}(\mathscr{G}_1^\bullet, \mathscr{G}_2^\bullet) = 0.$$

  Indeed, let $f : \mathscr{G}_1^\bullet \to \mathscr{G}_2^\bullet$ in $D^{\mathrm{b}}(X)$ represented by $(\mathscr{G}_1^\bullet \to \mathcal{I}^\bullet, \mathscr{G}_2^\bullet \to \mathcal{I}^\bullet)$ with $\mathscr{G}_2^\bullet \to \mathcal{I}^\bullet$ quasi-isomorphism. We can assume that $\operatorname{Supp} \mathscr{G}_1^j \subseteq X_1$ and $\operatorname{Supp} \mathcal{I}^j \subseteq X_2$ for each j, so $\mathscr{G}_1^\bullet \to \mathcal{I}^\bullet$ is zero.

  Same argument shows that $\operatorname{Hom}_{D^{\mathrm{b}}(X)}(\mathscr{G}_2^\bullet, \mathscr{G}_1^\bullet) = 0$.

Now assume that X is connected. Suppose that $D^{\mathrm{b}}(X)$ decomposes into D_1, $D_2 \subseteq D^{\mathrm{b}}(X)$. We have $\mathcal{O}_X \cong \mathscr{G}_1^\bullet \oplus \mathscr{G}_2^\bullet$ with $\mathscr{G}_i^\bullet \in D_i$. As $H^j(\mathscr{G}_1^\bullet) = H^j(\mathscr{G}_2^\bullet) = 0$ for each $j \neq 0$, we can assume that $\mathscr{G}_i^\bullet = \mathscr{G}_i \in \mathsf{Coh}\, X$. Hence, both $\mathscr{G}_1$ and $\mathscr{G}_2$ are ideal sheaves $\mathcal{I}_{X_1}$,

$\mathcal{I}_{X_2}$. Since X is connected and $\mathcal{O}_X \cong \mathcal{I}_{X_1} \oplus \mathcal{I}_{X_2}$, either $X_1 = \varnothing$ or $X_2 = \varnothing$. With loss of generality, we assume that $X_1 = \varnothing$, then $X_2 = X$, so $\mathscr{G}_2 = 0$ and thus $\mathcal{O}_X \in D_1$.

Let $x \in X$ be a closed point. As $\mathcal{O}_{X,x}$ is simple in $\mathsf{Coh}\, X$, either $\mathcal{O}_{X,x} \in D_1$ or $\mathcal{O}_{X,x} \in D_2$. As $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_{X,x}) \neq 0$, necessarily $\mathcal{O}_{X,x} \in D_1$.

Assume that $\mathscr{F}^\bullet$ is a nonzero object in D_2. We can assume that $H^0(\mathscr{F}^\bullet \neq 0)$ and $H^i(\mathscr{F}^\bullet) = 0$ for all $i > 0$. Then we have a distinguished triangle

$$\begin{tikzcd}
\tau_{\leq 0}\mathscr{F}^\bullet \arrow[rr] & & \mathscr{F}^\bullet \arrow[dl] \\
& \tau_{>0}\mathscr{F}^\bullet . \arrow[ul, dashed] &
\end{tikzcd}$$

Choose $x \in \operatorname{Supp}(H^0(\mathscr{F}^\bullet))$ and a surjection $H^0(\mathscr{F}^\bullet) \to \mathcal{O}_{X,x}$. We see that

$$\tau_{\leq 0}\mathscr{F}^\bullet \to H^0(\mathscr{F}^\bullet) \to \mathcal{O}_{X,x}$$

is nonzero, contradicting $\tau_{\leq 0}\mathscr{F}^\bullet \in D_2$ and $\mathcal{O}_{X,x} \in D_1$. Hence $D^{\mathrm{b}}(X)$ is indecomposable.

9.2 Semi-orthogonal decomposition

Analogy: Decomposition of triangulated category corresponds to $A = B \oplus C$ in an abelian category $\mathcal{A}$, while semi-orthogonal decomposition corresponds to a short exact sequence

$$0 \to B \to A \to C \to 0$$

in $\mathcal{A}$.

9.2.1 Definition

Let $\mathcal{T}$ be a triangulated category.

Definition 9.7. A semi-orthogonal decomposition of $\mathcal{T}$ is a sequence of strictly full triangulated subcategory $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that

i) $\operatorname{Hom}_{\mathcal{T}}(C_i, C_j) = 0$ for all $C_i \in \mathcal{C}_i$, $C_j \in \mathcal{C}_j$ whenever $i > j$;

ii) for each $X \in \mathcal{T}$, there exists a decomposition

$$0 \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 = X$$

such that $\operatorname{Cone}(f_i) \in \mathcal{C}_i$.

We write $\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$.

Here, a strictly full subcategory $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory such that for each $X \in \mathcal{B}$, we have

$$Y \cong X \text{ in } \mathcal{A} \implies Y \in \mathcal{B}.$$

An semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$ is called **maximal** if $\mathcal{C}_i$ does not admit any nontrivial semi-orthogonal decomposition for each i.

Given some additive full subcategories D_1, ... , $D_m \subseteq \mathcal{T}$. The **thick closure** of D_1, ... , D_m is the smallest strictly full triangulated subcategory $\mathcal{D} \subseteq \mathcal{T}$ such that $\mathcal{D} \supseteq \mathcal{D}_i$ for each i and $\mathcal{D}$ is thick, i.e., closed under taking direct summands.

If $\mathcal{D} = \mathcal{T}$, we say that $\mathcal{D}_1$, ... , $\mathcal{D}_m$ **classically generate** $\mathcal{T}$.

Proposition 9.8. Given strictly full thick triangulated subcategories $\mathcal{C}_1$, ... , $\mathcal{C}_m \subseteq \mathcal{T}$ which satisfying $\mathrm{Hom}(\mathcal{C}_i, \mathcal{C}_j) = 0$ for all $i > j$. Then the followings are equivalent:

- $\mathcal{C}_1$, ... , $\mathcal{C}_m$ classically generate $\mathcal{T}$;
- $\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$ is an semi-orthogonal decomposition.

Proposition 9.9. Given an semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$. For each $X \in \mathcal{T}$, the X_i's and the C_i's in the decomposition

$$0 \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 = X$$

are unique up to isomorphisms.

So we get projection functors

$$\begin{array}{rccc} P_i : & \mathcal{T} & \to & \mathcal{C}_i \\ & X & \mapsto & C_i, \end{array} \quad \begin{array}{rccc} P_{i,m} : & \mathcal{T} & \to & \langle \mathcal{C}_i, \ldots, \mathcal{C}_m \rangle \\ & X & \mapsto & X_i, \end{array}$$

One way of obtaining semi-orthogonal decomposition is by taking the orthogonal complement of an admissible subcategory.

9.2.2 Admissible subcategory

Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{D}' \subseteq \mathcal{D}$ be a full triangulated subcategory. The right orthogonal complement of $\mathcal{D}'$ (with respect to $\mathcal{D}$) is the full subcategory $(\mathcal{D}')^{\perp} \subseteq \mathcal{D}$

with

$$\mathrm{Ob}(\mathcal{D}')^{\perp} = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \ \forall Y \in \mathcal{D}'\}.$$

The left orthogonal complement ${}^{\perp}D'$ is defined similarly.

Proposition 9.10. If $\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$, then

$$\mathcal{T} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \langle \mathcal{C}_k, \ldots, \mathcal{C}_\ell \rangle, \mathcal{C}_{\ell+1}, \ldots, \mathcal{C}_m \rangle$$

and

$$\langle \mathcal{C}_k, \ldots, \mathcal{C}_\ell \rangle = {}^{\perp}\langle \mathcal{C}_1, \ldots, \mathcal{C}_{k-1} \rangle \cap \langle \mathcal{C}_{\ell+1}, \ldots, \mathcal{C}_m \rangle^{\perp}.$$

Definition 9.11. A subcategory $\mathcal{D}'$ of $\mathcal{D}$ is called right (resp. left) admissible if for each $X \in D$, there exists a distinguished triangle

$$\begin{tikzcd} Y \arrow[rr] & & X \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

such that $Y \in \mathcal{D}'$ and $Z \in (\mathcal{D}')^{\perp}$ (resp. $Y \in {}^{\perp}\mathcal{D}'$ and $Z \in \mathcal{D}'$).

We say that $\mathcal{D}'$ is admissible if $\mathcal{D}'$ is left and right admissible. By definition, if $\mathcal{D}'$ is right (resp. left) admissible, then $\mathcal{D} = \langle (\mathcal{D}')^{\perp}, \mathcal{D}' \rangle$ (resp. $\mathcal{D} = \langle \mathcal{D}', {}^{\perp}\mathcal{D}' \rangle$). The following proposition is useful.

Proposition 9.12. Let $\mathcal{D}' \subseteq \mathcal{D}$ be a full triangulated subcategory. Then $\mathcal{D}' \subseteq \mathcal{D}$ is right (resp. left) admissible if and only if the inclusion $\iota : \mathcal{D}' \to \mathcal{D}$ has a right (resp. left) adjoint $\pi : \mathcal{D} \to \mathcal{D}'$.

Proof. Suppose $\mathcal{D}' \to \mathcal{D}$ has a right adjoint. Then the element $g : \iota\pi(X) \to X$ corresponding to $\mathrm{id} : \pi(X) \to \pi(X)$ gives a distinguished triangle

$$\begin{tikzcd} \iota\pi(X) \arrow[rr, "g"] & & X \arrow[dl] \\ & Z. \arrow[ul, dashed] & \end{tikzcd}$$

For each $Y' \in \mathcal{D}'$, by the naturality of adjunction, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(\iota(Y'), \iota\pi(X)[j]) \arrow[r, "g\circ"] & \mathrm{Hom}_{\mathcal{D}}(\iota(Y'), X[j]) \\ & \mathrm{Hom}_{\mathcal{D}'}(Y', \pi(X)[j]). \arrow[ul] \arrow[u, "\wr"'] \end{tikzcd}$$

As $\iota : \mathcal{D}' \to \mathcal{D}$ is fully faithful, $\operatorname{Hom}_{\mathcal{D}'}(Y', \pi(X)[j]) \to \operatorname{Hom}_{\mathcal{D}}(\iota(Y'), \iota\pi(X)[j])$ is an isomorphism. If follows from the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}_D(\iota(Y'), \iota\pi(X)) \arrow[r, "\sim"] & \operatorname{Hom}_D(\iota(Y'), X) \arrow[r] & \operatorname{Hom}_D(Y', Z) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \operatorname{Hom}_D(\iota(Y'), \iota\pi(X)[1]) \arrow[r, "\sim"] & \operatorname{Hom}_D(\iota(Y'), X[1])
\end{tikzcd}$$

gives $\operatorname{Hom}_D(Y', Z) = 0$.

Suppose $\mathcal{D}' \subseteq \mathcal{D}$ is right admissible. For each $X \in \mathcal{D}$, choose $Y \in \mathcal{D}'$, $Z \in (\mathcal{D}')^{\perp}$ such that the triangle

$$\begin{tikzcd}
Y \arrow[rr, "g"] & & X \arrow[dl] \\
& Z \arrow[ul, dashed] \arrow[u, phantom, "\triangle"] &
\end{tikzcd}$$

is distinguished. Define $\pi(X) = Y$. Then for each $A \in \mathcal{D}'$,

$$g\circ : \operatorname{Hom}_{\mathcal{D}'}(A, \pi(X)) \to \operatorname{Hom}_D(\iota(A), X) \tag{♉}$$

is an isomorphism. The isomorphism is clearly functorial in A. It remains to show that it is functorial in X.

Given a morphism $X' \xrightarrow{f} X$. We get two distinguished triangles

$$\begin{tikzcd}
Y' \arrow[r] & X' \arrow[r] \arrow[d, "f"] & Z' \arrow[r] & Y'[1] \\
Y \arrow[r] & X \arrow[r] & Z \arrow[r] & Y[1].
\end{tikzcd}$$

Since $\operatorname{Hom}(Y', Z) = \operatorname{Hom}(Y', Z[-1]) = 0$, we have unique morphisms

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "\exists!"] & X' \arrow[r] \arrow[d, "f"] & Z' \arrow[r] \arrow[d, "\exists!"] & Y'[1] \arrow[d] \\
Y \arrow[r] & X \arrow[r] & Z \arrow[r] & Y[1],
\end{tikzcd}$$

which shows that (♉) is functorial in X. ■

Remark. In many references, when we define semi-orthogonal decomposition

$$\mathcal{T} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle,$$

we require that each $\mathcal{C}_i$ is admissible. When $\mathcal{T} = D^{\mathrm{b}}(X)$ where X is smooth and projective:

Theorem 9.13. If $D^{\mathrm{b}}(X) = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$, then each $\mathcal{C}_i$ is admissible.

9.2.3 Examples of admissible subcategories

Let X, Y be smooth projective varieties. Assume that we have $F : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$, which is fully faithful and exact. Orlov's theorem shows that F is Fourier-Mukai. So F has left and right adjoints. Thus $F : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ embeds $D^{\mathrm{b}}(X)$ as an admissible subcategory of $D^{\mathrm{b}}(Y)$.

Proposition 9.14. Let $f : X \to B$ be a projective morphism between smooth projective varieties. Assume that $\mathsf{R}f_*\mathcal{O}_X = \mathcal{O}_B$, e.g., f has connected fiber. Then $\mathsf{L}f^* : D^{\mathrm{b}}(B) \to D^{\mathrm{b}}(X)$ is fully faithful, thus realizing $D^{\mathrm{b}}(B)$ as an admissible subcategory of $D^{\mathrm{b}}(X)$.

Proof. By adjunction and projection formula, we have morphisms

$$\mathscr{F}^\bullet \to \mathsf{R}f_*\mathsf{L}f^*\mathscr{F}^\bullet \cong \mathscr{F}^\bullet \otimes^{\mathsf{L}} \mathsf{R}f_*\mathcal{O}_X \cong \mathscr{F}^\bullet.$$

The composition is an isomorphism by checking on a bounded locally free sheaf resolution $\mathscr{L}^\bullet \to \mathscr{F}^\bullet$ locally. So $\mathrm{id} \cong \mathsf{R}f_*\mathsf{L}f^*$. It follows from the diagram

$$\begin{tikzcd}
\operatorname{Hom}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \arrow[r, "\sim"] \arrow[dr, "\mathsf{L}f^*"'] & \operatorname{Hom}(\mathscr{E}^\bullet, \mathsf{R}f_*\mathsf{L}f^*\mathscr{F}^\bullet) \arrow[d, "\wr"] \\
& \operatorname{Hom}(\mathsf{L}f^*\mathscr{E}^\bullet, \mathsf{L}f^*\mathscr{F}^\bullet)
\end{tikzcd}$$

that $\mathsf{L}f^*$ is fully faithful. ■

Let $\mathcal{D}$ be a k-linear triangulated category.

Definition 9.15. An object $E \in \mathcal{D}$ is called exceptional if

$$\operatorname{Hom}(E, E[\ell]) = \begin{cases} k, & \text{if } \ell = 0 \\ 0, & \text{else.} \end{cases}$$

Proposition 9.16. The thick closure of an exceptional object $E \in \mathcal{D}$ consists of all object isomorphic to $\bigoplus_{i\in I} E[i]^{j_i}$, where $I \subseteq \mathbb{Z}$ is a finite set.

Proposition 9.17. Assume that $\sum_i \dim \operatorname{Hom}^i(A, B) < \infty$ for all A, $B \in \mathcal{D}$. Then the thick closure $\langle E \rangle \subseteq \mathcal{D}$ of an exceptional object E is an admissible subcategory.

Proof. Given $A \in \mathcal{D}$. We have the distinguished triangle

$$\triangle : \qquad B[-1] \longrightarrow \bigoplus_i \operatorname{Hom}(E, A[i]) \otimes E[-i] \xrightarrow{f} A \longrightarrow B,$$

where $\operatorname{Hom}(E, A[i]) \otimes E[-i]$ is in fact $E[-i]^{\oplus \dim \operatorname{Hom}(E, A[i])}$. As E exceptional, applying $\operatorname{Hom}(E[-i], -)$ to f gives

$$\operatorname{Hom}(E, A[i]) \underset{[i]}{\xrightarrow{\sim}} \operatorname{Hom}(E[-i], A).$$

Thus $\operatorname{Hom}(E[-i], B) = 0$ for each i, so $B \in \langle E \rangle^{\perp}$ and $\langle E \rangle$ is right admissible.

The proof of left-admissibility is similar. ■

9.2.4 Exceptional collection

Definition 9.18. Let $E_1, \ldots, E_m \in \mathcal{D}$ be exceptional objects.

- If $\operatorname{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and for all ℓ, we can $(E_1, \ldots, E_m)$ an exceptional collection.
- An exceptional collection is full if $E_1, \ldots, E_m$ classically generate $\mathcal{D}$.

If $E_1, \ldots, E_m$ is a full exceptional collection of $\mathcal{D}$, then $\langle E_1, \ldots, E_m \rangle := \big\langle \langle E_1 \rangle, \ldots, \langle E_1 \rangle \big\rangle$ is an semi-orthogonal decomposition of D. More generally, if $E_1, \ldots, E_m$ is an exceptional collection, then $\langle \mathcal{C}^{\perp}, E_1, \ldots, E_m \rangle$ is an semi-orthogonal decomposition of $\mathcal{D}$, where $\mathcal{C}^{\perp} := \langle E_1, \ldots, E_m \rangle$.

Interlude 3: Yoneda extensions

Let $\mathcal{A}$ be an abelian category.

Definition 9.19. Let A, $B \in \mathcal{A}$. A degree i **Yoneda extension** of B is an exact sequence of the form

$$E : \quad 0 \to A \to Z^{-(i-1)} \to Z^{-(i-2)} \to \cdots \to Z^0 \to B \to 0.$$

Two Yoneda extension E, E' are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
E: & 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Z^{-(i-1)} \arrow[r] & \cdots \arrow[r] & Z^0 \arrow[r] & B \arrow[r] \arrow[d, equal] & 0 \\
E'': & 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & (Z^{-(i-1)})'' \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & (Z^0)'' \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
E': & 0 \arrow[r] & A \arrow[r] & (Z^{i-1})' \arrow[r] & \cdots \arrow[r] & (Z^0)' \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

where E'' is also a Yoneda extension.

Proposition 9.20. The above definition defines an equivalence relation.

Let

$$\mathrm{Ex}^i(A,B) = {\{\text{ Yoneda extension of } B \text{ by } A\,\}}\big/{\{\text{ the equivalence }\}}.$$

Consider the map

$$\delta : \mathrm{Ex}^i(B,A) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(B, A[i]) = \mathrm{Ext}^i(B,A)$$

by sending $E : A \to Z^\bullet \to B$ to the roof

$$\begin{tikzcd}
 & & & & & B & & \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & Z^{i-1} \arrow[r] & \cdots \arrow[r] & Z^0 \arrow[r] \arrow[u] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots & & &
\end{tikzcd}$$

Lemma 9.21. The map $\delta : \mathrm{Ex}^i(B,A) \to \mathrm{Ext}^i(B,A)$ is a bijection.

In particular if $\delta(A \to Z^\bullet \to B) = 0$, then $Z^\bullet \cong A[i] \oplus B$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$.

10 Full exceptional collection

10.1 $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$

Theorem 10.1 (Beilinson)**.** The line bundles

$$\mathcal{O}(a),\ \mathcal{O}(a+1),\ \ldots, \mathcal{O}(a+n)$$

form a full exceptional collection in $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$.

Proof. Since $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) \cong H^\ell(\mathbb{P}^n, \mathcal{O}(j-i))$, which is 0 if $-n < j - i < 0$. These line bundle form an exceptional collection.

It remains to show that $\mathcal{O}(a), \ldots, \mathcal{O}(a+n)$ classically generate $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$. As $- \otimes \mathcal{O}(j)$ defines an equivalence of category from $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ to itself, it is enough to prove this for some a.

For each full additive subcategory of $\mathcal{C} \subseteq \mathcal{D}$, we say that $\mathcal{C}$ generates $\mathcal{D}$ if $\langle \mathcal{C} \rangle^\perp = 0$.

Proposition 10.2. The subcategory $\langle \mathcal{C} \rangle^\perp = 0$ if and only if for all $C \in \mathcal{C}$, $X \in \mathcal{D}$ and for all i, $\mathrm{Hom}_{\mathcal{D}}(C, X[i]) = 0$.

It is enough to prove the following

Lemma 10.3. Let X be a projective variety over a field k of dimension n. If $\mathscr{L}$ is an globally generated (base-point-free) ample line bundle, then $\bigoplus_{i=0}^{n} \mathscr{L}^{-i}$ generates $\mathcal{D}^{\mathrm{b}}(X)$.

Proof of Lemma. Let $\varphi = |\mathcal{L}| : X \to \mathbb{P}^N$, which is a finite morphism. Consider the Koszul resolution

$$\cdots \longrightarrow \bigwedge^{\ell+1} H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}(-\ell-1)$$

$$\xrightarrow{\ f_\ell\ } \bigwedge^{\ell} H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}(-\ell) \longrightarrow \cdots \longrightarrow \mathcal{O} \longrightarrow 0,$$

where f_ℓ is the composition ($H^0 := H^0(\mathbb{P}^N, \mathcal{O}(1))$)

$$\bigwedge^{\ell+1} H^0 \otimes \mathcal{O}(-\ell-1) \to \bigwedge^{\ell} H^0 \otimes H^0 \otimes \mathcal{O}(-\ell-1) \to \bigwedge^{\ell} H^0 \otimes \mathcal{O}(-\ell)$$

Pulling back the Koszul complex by the finite morphism φ, we obtain an exact sequence

$$0 \to \mathscr{L}^{-N-1} \to \cdots \to (\mathscr{L}^{-k})^{\oplus \binom{N+1}{k}} \to \cdots \to \mathcal{O}_X \to 0.$$

Let $K := \ker((\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to (\mathscr{L}^{-n})^{\oplus \binom{N+1}{n}})$. We get an exact sequence (i.e., a Yoneda extension of $\mathcal{O}_X$ by K)

$$0 \to K \to (\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathscr{L}^{-1})^{\oplus N+1} \to \cdots \to \mathcal{O}_X \to 0.$$

Since $\dim X = n$, we have $\mathrm{Ext}^{n+1}(\mathcal{O}_X, K) = 0$, so $\mathcal{O}_X$ is a direct summand of

$$0 \to (\mathscr{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathscr{L}^{-1})^{\oplus N+1} \to \cdots \to 0$$

in $\mathcal{D}^{\mathrm{b}}(X)$. Applying the exact functor $(-\otimes \mathscr{L}^{-n-j-1})\circ \mathsf{R}\operatorname{Hom}(-,\mathcal{O}_X)$ to this result shows that $\mathscr{L}^{-n-j-1}$ is a direct summand of

$$0 \to (\mathscr{L}^{-n-j})^{\oplus N+1} \to \cdots \to (\mathscr{L}^{-j})^{\oplus\binom{N+1}{n+1}} \to \cdots \to 0.$$

This for each $j \geq 0$, $\mathscr{L}^{-j} \in \langle \mathscr{L}^{-i} \mid 0 \leq i \leq n\rangle$.

Given $E^\bullet \in \mathcal{D}^{\mathrm{b}}(X)$ such that

$$\mathsf{R}\operatorname{Hom}\Big(\bigoplus_{i=0}^{n} \mathscr{L}^{-i}, E^\bullet\Big) = 0.$$

Then $\mathsf{R}\operatorname{Hom}(\mathscr{L}^{-j}, E^\bullet = 0$ for each $j \geq 0$. Namely, $\mathsf{R}\Gamma(X, E^\bullet \otimes \mathscr{L}^j) = 0$ for each $j \geq 0$. Up to shifting, we can assume that $H^i(E^\bullet) = 0$ for each $i > 0$. We show that $H^0(E^\bullet) = 0$; this proves the lemma by induction.

Recall that $\dim X - n$. We have

$$\mathsf{R}\Gamma(X, \tau_{\leq -n-1}(E^\bullet \otimes \mathscr{L}^j) \in \mathcal{D}^{<0}(\mathsf{Vect}_k).$$

This is because

$$E_2^{p,q} = R^p\Gamma(X, H^q(\tau_{\leq -n-1}(E^\bullet \otimes \mathscr{L}^j)) \Rightarrow R^{p+q}\Gamma(X, \tau_{\leq -n-1}(E^\bullet \otimes \mathscr{L}^j)),$$

and $R^p\Gamma(X, H^q(\tau_{\leq -n-1}(E^\bullet \otimes \mathscr{L}^j)) = 0$ if $p \geq n+1$ or $q \geq -n$.

Consider the distinguished triangle

$$\mathsf{R}\Gamma(X, \tau_{\leq -n-1}(E^\bullet \otimes \mathscr{L}^j)) \to \mathsf{R}\Gamma(X, E^\bullet \otimes \mathscr{L}^j) \to \mathsf{R}\Gamma(X, \tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j)) \xrightarrow{[1]} .$$

Since $\mathsf{R}\Gamma(X, E^\bullet \otimes \mathscr{L}^j) = 0$, we have

$$R^0\Gamma(X, \tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j)) = 0.$$

Since $H^i(E^\bullet) = 0$ for $i > 0$ and $\mathscr{L}$ is ample, we can choose $j_0 \in \mathbb{Z}$ such that for each $j \geq j_0$, $R^p\Gamma(X, H^q(\tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j))) = 0$ if $p \geq 1$. Consider

$$E_2^{p,q} = R^p\Gamma(X, H^q(\tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j))) \Rightarrow R^{p+q}\Gamma(X, \tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j)).$$

Since $E_2^{p,q} \neq 0$ only if $p \leq 0$ and $q \leq 0$,

$$0 = R^0\Gamma(X, \Gamma(X, \tau_{>-n-1}(E^\bullet \otimes \mathscr{L}^j)) = E_2^{0,0} = H^0(X, H^0(E^\bullet) \otimes \mathscr{L}^j)$$

for each $j \geq j_0$. As $\mathscr{L}$ is ample, necessarily $H^0(E^\bullet) = 0$. ■

Remark. Let $\mathcal{C} \subseteq \mathcal{D}$ be a full additive subcategory.

- If $\mathcal{C}$ classically generates $\mathcal{D}$, then $\mathcal{C}$ generates $\mathcal{D}$.
- If $\mathcal{C}$ is a right admissible full triangulated subcategory, then the converse also holds.

10.2 Strong exceptional collection

Let $\mathcal{D}$ be a triangulated category.

Definition 10.4. An exceptional collection $E_1, \ldots, E_m$ of $\mathcal{D}$ is called strong if $\operatorname{Hom}(E_i, E_j[k])$ for all i, j and for all $k \neq 0$.

The full exceptional collection

$$\mathcal{O}(a),\ \mathcal{O}(a+1),\ \ldots,\ \mathcal{O}(a+n)$$

of $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ is strong.

Theorem 10.5 (Bondal)**.** Let X be a smooth projective variety. If $(E_1, \ldots, E_m)$ is a strong full exceptional collection of $D^{\mathrm{b}}(X)$, then

$$\mathsf{R}\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(X)}\Big(\bigoplus_{i=1}^{m} E_i, -\Big) : \mathcal{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\textsf{mod-}A_{\mathrm{fin}}),$$

where A is the endomorphism ring $\operatorname{End}\Big(\bigoplus_{i=1}^{m} E_i\Big)$ and $\textsf{mod-}A_{\mathrm{fin}}$ is the category of right A-modules of finite type.

Given a triangulated category $\mathcal{D}$. If $\mathcal{D} \cong \mathcal{D}^{?}(\mathcal{A})$ for some abelian category $\mathcal{A}$, we call $\mathcal{A}$ the heart (of a triangulated structure) of $\mathcal{D}$. In bondal's theorem, the statement

$$\mathcal{D}^{\mathrm{b}}(\mathsf{Coh}\, X) \cong \mathcal{D}^{\mathrm{b}}(\textsf{mod-}A_{\mathrm{fin}}$$

provides two hearts of $\mathcal{D}^{\mathrm{b}}(X)$ of different nature.

Remark. It is rare that $\mathcal{D}^{\mathrm{b}}(X)$ admits of full exceptional collection

$$(E_1, \ldots, E_m).$$

For instance, we will see that this implies

- $H^{p,q}(X) = 0$ for all $p \neq q$;
- $K(\mathcal{D}^{\mathrm{b}}(X)) = \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_m]$.

Conjecture 10.6. If $\mathcal{D}^{\mathrm{b}}(X)$ has a full exceptional collection, then X is rational.

10.3 Quiver representations

10.3.1 Quiver

A quiver Q is an oriented graph. Formally, $Q = (Q_0, Q_1, s, t)$, where

- Q_0 is a set of vertices;
- Q_1 is a set of edges;
- s, $t : Q_1 \to Q_0$ source and target map, i.e., if $a \xrightarrow{\alpha} b \in Q_1$, $s(\alpha) = a$ and $t(\alpha) = b$.

10.3.2 Path algebra

Given a, $b \in Q_0$. A **path** from a to b is a sequence $\alpha_1, \dots, \alpha_n \in Q_1$ such that $s(\alpha_1) = a$, $t(\alpha_i) = s(\alpha_{i+1})$, and $t(\alpha_n) = b$. We write $p = (a \mid \alpha_1, \dots, \alpha_n \mid b)$, and define $s(p) = a$, $t(p) = b$. The length of the path $\ell(p) := n$. $n = 0$ is allowed: $e_a := (a \mid\mid a)$.

The **path algebra** of Q over a field k is the graded associative k-algebra kQ defined as

- the paths in Q form the basis of kQ;
- grading $(kQ)_n$ is defined by the length of the path;
- given two paths p_1, p_2, define

$$p_1 p_2 = \begin{cases} \text{concatenation of } p_1 \text{ and } p_2, & \text{if } t(p_1) = s(p_2) \\ 0, & \text{else.} \end{cases}$$

A **cycle** is a path p with $\ell(p) \geq 1$ such that $s(p) = t(p)$

Proposition 10.7. The dimension of the algebra kQ is finite if and only if Q is acyclic, i.e., without any cycle.

10.3.3 Quiver with relations

A **relation** ρ in a quiver Q is an element

$$\rho = \sum_i a_i p_i \in kQ$$

such that $\ell(p_i) \geq 2$ and $s(p_i) = s(p_j)$, $t(p_i) = t(p_j)$ for all i, j.

A **quiver with relations** (Q, ρ) is a quiver Q endowed with a set of relations $\rho = \{\rho_j\}$.

The **path algebra** of (Q, ρ) is $A_Q := {}^{kQ}\!\diagup_{I}$, where I is the two-sided ideal generated by ρ_i.

10.3.4 Quiver representations

A representation of the quiver Q is the data

$$W = ((W_i)_{i \in Q_0}, (w_\alpha)_{\alpha \in Q_1})$$

where each W_i is a k-vector space and each $w_\alpha : W_{s(\alpha)} \to W_{t(\alpha)}$ is a k-linear map.

Assume that (Q, ρ) is a quiver with relations. A representation of (Q, ρ) is a representation W of Q such that for each $\rho_i \in \rho$, the corresponding linear map

$$\rho_i : W_{s(rho_i)} \to W_{t(\rho_i)}$$

is zero. Morphisms of quiver representations are defined in the obvious way. We set $\dim W = \sum_{i \in Q_0} \dim W_i$, the dimension of the quiver representation.

Proposition 10.8. Let (Q, ρ) be a quiver with relations. Then $\textsf{mod-}\left({}^{kQ}\!\diagup_{I}\right)_{\text{fin}}$ is equivalent to the category of finite dimensional representation of (Q, ρ).

10.4 Full exceptional collection and quiver with relations

Let X be a smooth projective variety. Assume that $\mathcal{D}^{\mathrm{b}}(X)$ admits a strongly full exceptional collection

$$(E_1, \ldots, E_m).$$

Recall that they satisfy

$$\operatorname{Hom}(E_i, E_j) = \begin{cases} k, & \text{if } i = j \\ 0, & \text{if } i > j. \end{cases}$$

So

$$A = \operatorname{End}\Big(\bigoplus_{i=1}^{m} E_i\Big) = \Big(\bigoplus_{i=1}^{m} k e_i\Big) \oplus \Big(\bigoplus_{i<j} \operatorname{Hom}(E_i, E_j)\Big),$$

where e_i is the generator of $\operatorname{Hom}(E_i, E_i)$. We now construct acyclic (Q, ρ) such that $A \cong {}^{kQ}\!\diagup_{I}$. Let

- $Q_0 = \{1, \ldots, m\}$;
- $e_i \in A$ is the path of length 0 in Q at the vertex i;
- for all $i < j$, consider the linear map

$$\varphi_{i,j} : \prod_{i<k<j} \mathrm{Hom}(E_i, E_k) \times \mathrm{Hom}(E_k, E_j) \to \mathrm{Hom}(E_i, E_j)$$

  defined by composition. Choose a basis $\alpha_1, \ldots, \alpha_{n_{ij}}$ of ${\mathrm{Hom}(E_i, E_j)}\big/{\mathrm{Im}\,\varphi_{ij}}$. These $\alpha_1, \ldots, \alpha_{n_{ij}}$ define the edges $i \to j$ in Q_1.
- An element $p = \sum a_i p_i \in kQ$ with $s(p_i) = s(p_j)$, $t(p_i) = t(p)j)$ is in I if and only if the corresponding map $E_{s(p)} \to E_{t(p)}$ is zero.

As an example, for the exceptional collection

$$\mathcal{O},\ \mathcal{O}(1),\ \ldots,\ \mathcal{O}(n)$$

of $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$, the associated quiver is

$$\begin{tikzcd} 0 \arrow[r, bend left=30] \arrow[r, bend left=15] \arrow[r, bend right=30] \arrow[r, phantom, "\vdots"] & 1 & \cdots & (n-1) \arrow[r, bend left=30] \arrow[r, bend left=15] \arrow[r, bend right=30] \arrow[r, phantom, "\vdots"] & n \end{tikzcd}$$

$(n+1)$
arrows

Proposition 10.9. The ideal I is generated by the relations

$$x_k x'_\ell - x_\ell x'_k$$

for all $k \neq \ell \in \{1, \ldots, n+1\}$, where $\{x_1, \ldots, x_{n+1}\}$ (resp. $\{x'_1, \ldots, x'_{n+1}\}$) is the set of arrows $i \to (i+1)$ (resp. $(i+1) \to (i+2)$).

11 Grothendieck-Riemann-Roch

In this section, we work over $\mathbb{C}$. Let X be a smooth quasi-projective variety.

11.1 Chern classes of a vector bundle

For each vector bundle $\mathscr{E}$ over X, we have the **Chern classes** of $\mathscr{E}$,

$$\mathrm{c}_i(\mathscr{E}) \in H^{2i}(X, \mathbb{Z}).$$

They satisfy the following properties

1) $\mathrm{c}_0(\mathscr{E}) = 1$;

2) $\mathrm{c}_1(\mathcal{O}_X(D)) = [D]$ for any divisor D on X;

3) given short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0,$$

we have $\mathrm{c}(\mathscr{E}) = \mathrm{c}(\mathscr{E}')\,\mathrm{c}(\mathscr{E}'')$, where $\mathrm{c}(\mathscr{E}) = \sum_{i \geq 0} \mathrm{c}_i(\mathscr{E})$;

4) $\mathrm{c}_i(\mathscr{E}) = 9$ for each $i > \operatorname{rk}\mathscr{E}$.

11.2 Chern classes of a coherent sheaf

Let $\mathscr{F}$ be a coherent sheaf on X. As X is smooth we can choose a locally free resolution

$$0 \to \mathscr{L}_\ell \to \cdots \to \mathscr{L}_0 \to \mathscr{F} \to 0.$$

Define $\mathrm{c}(\mathscr{F}) = \prod_i \mathrm{c}(\mathscr{L}_i)^{(-1)^i}$. Here,

$$\mathrm{c}(\mathscr{L})^{-1} = 1 + (1 - \mathrm{c}(\mathscr{L})) + (1 - \mathrm{c}(\mathscr{L}))^2 + \cdots.$$

Lemma 11.1. The Chern class $\mathrm{c}(\mathscr{F})$ is independent of the choice of resolution and still satisfy 1) $\sim$ 4).

11.3 Chern character

Let X be a smooth quasi-projective variety. $\mathscr{E}$ a vector bundle on X.

11.3.1 A particular example

Assume that $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$. Then we define

$$\mathrm{ch}(\mathscr{E}) = \sum_i e^{\mathrm{c}_1(\mathscr{L}_i)}.$$

11.3.2 General definition

In general, $\mathscr{E}$ does not split into line bundles. Let $\mathrm{c}_t(\mathscr{E}) = \sum t^i\,\mathrm{c}_i(\mathscr{E})$, called the **Chern polynomial**. Write formally

$$\mathrm{c}_t(\mathscr{E}) = \prod_{i=1}^{\operatorname{rk}\mathscr{E}} (1 + \alpha_i t).$$

The formal variables α_i are called **Chern roots**. (When $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$, we can take $\alpha_i = \mathrm{c}_1(\mathscr{L}_i)$.)

We define the **Chern character** $\mathrm{ch}(\mathscr{E})$ to be $\sum e^{\alpha_i}$. $\mathrm{ch}(\mathscr{E})$ is actually a $\mathbb{Q}$-linear combination of product of Chern classes. Explicitly, the first few terms are

$$\begin{aligned}\mathrm{Ch}(\mathscr{E}) = \mathrm{rk}(\mathscr{E}) + \mathrm{c}_1 + \frac{1}{2}(\mathrm{c}_1^2 - 2\,\mathrm{c}_2) + \frac{1}{6}(\mathrm{c}_1^3 - 3\,\mathrm{c}_1\mathrm{c}_2 + 3\,\mathrm{c}_3) \\ + \frac{1}{24}(\mathrm{c}_1^4 - 4\,\mathrm{c}_1^2\mathrm{c}_2 + 4\,\mathrm{c}_1\mathrm{c}_3 + 2\,\mathrm{c}_2^2 - 4\,\mathrm{c}_4) + \cdots \in H^*(X, \mathbb{Q}).\end{aligned}$$

Lemma 11.2. Let $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{E}''$ be vector bundles.

1) If there is a short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0,$$

then $\mathrm{Ch}(\mathscr{E}) = \mathrm{Ch}(\mathscr{E}') + \mathrm{Ch}(\mathscr{E}'')$.

2) $\mathrm{Ch}(\mathscr{E} \otimes \mathscr{E}') = \mathrm{Ch}(\mathscr{E})\,\mathrm{Ch}(\mathscr{E}')$.

3) The definition of Chern character can be extended to coherent sheaves in a unique way, subject to 1).

11.4 Grothendieck group

11.4.1 Abelian category

Let $\mathcal{A}$ be an abelian category and let $\mathcal{B} \subseteq \mathcal{A}$ a additive subcategory. We define

$$\mathrm{K}(\mathcal{B}) = \frac{\bigoplus_{E \in \mathcal{B}} \mathbb{Z} \cdot [E]}{\langle [E] - [E'] - [E''] \mid 0 \to E' \to E \to E'' \to 0 \text{ exact in } \mathcal{A} \rangle}.$$

For example, $\mathrm{K}(\mathsf{Vect}_{k,\mathrm{fin}} \cong \mathbb{Z}$.

11.4.2 $\mathrm{K}(\mathsf{Coh}\, X)$

Let X be a smooth quasi-projective variety. Define $\mathrm{K}(X) = \mathrm{K}(\mathsf{Coh}\, X)$. Since X is smooth, we actually have

$$\mathrm{K}(\mathsf{Vect}\, X) \cong \mathrm{K}(X),$$

where $\mathsf{Vect}\, X$ is the category of vector bundles over X of finite rank. As tensoring with a vector bundle preserves exact sequence, given $\mathscr{E}$, $\mathscr{E}' \in \mathsf{Vect}\, X$, we can define

$$[\mathscr{E}] \cdot [\mathscr{E}'] = [\mathscr{E} \otimes \mathscr{E}'].$$

This defines $\mathrm{K}(X) \cong \mathrm{K}(\mathsf{Vect}\, X)$ as a ring. Chern character extends to a ring homomorphism

$$\begin{array}{rrcl} \mathrm{Ch}: & \mathrm{K}(X) & \to & H^*(X,\mathbb{Q}) \\ & [\mathscr{F}] & \mapsto & \mathrm{Ch}(\mathscr{F}). \end{array}$$

11.4.3 $\mathrm{K}(D^{\mathbf{b}}(X))$

Let $\mathcal{D}$ be a triangulated category. We have a similar definition:

$$\mathrm{K}(\mathcal{D}) := \frac{\bigoplus_{E\in\mathcal{D}} \mathbb{Z}\cdot[E]}{\langle [E]-[E']-[E''] \mid F\to E\to G\to F[1] \text{ distinguished} \rangle}$$

We call $\mathrm{K}(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$.

Proposition 11.3. Let $\mathcal{A}$ be an abelian category.

- For each $E^\bullet \in D^{\mathrm{b}}(\mathcal{A})$, we have

  $$[E^\bullet] = \sum_i (-1)^i [H^i(E^\bullet)] = \sum_i (-1)^i [E^i]$$

  in $\mathrm{K}(D^{\mathrm{b}}(\mathcal{A}))$.

- We have an equivalence of category

  $$\mathrm{K}(\mathcal{A}) \xrightarrow{\sim} \mathrm{K}(D^{\mathrm{b}}(\mathcal{A})).$$

In particular, if X is a smooth quasi-projective variety,

$$\mathrm{K}(D^{\mathrm{b}}(X)) \cong \mathrm{K}(X).$$

Proposition 11.4. Given $\mathscr{F}_1^\bullet, \mathscr{F}_2^\bullet \in D^{\mathrm{b}}(X)$, we have

$$[\mathscr{F}_1^\bullet]\cdot[\mathscr{F}_2^\bullet] = [\mathscr{F}_1^\bullet \otimes^{\mathsf{L}} \mathscr{F}_2^\bullet].$$

By construction, all exact functor between triangulated category $F : \mathcal{D}_1 \to \mathcal{D}_2$ induces $F : \mathrm{K}(\mathcal{D}_1) \to \mathrm{K}(\mathcal{D}_2)$. For example, let $f : X \to Y$ be a proper morphism between smooth quasi-projective varieties. Then the functor $\mathsf{R}f_* : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ induces $\mathsf{R}f_* : \mathrm{K}(X) \to \mathrm{K}(Y)$.

11.5 Grothendieck-Riemann-Roch

11.5.1 Todd class

Let X be a smooth quasi-projective variety, $\mathscr{E}$ a vector bundle of rank r on X, $\alpha_1, \dots,$ α_r the Chern roots of $\mathscr{E}$. We define the Todd class $\mathrm{Td}(\mathscr{E})$ to be

$$\prod_{i=1}^{r} Q(\alpha_i),$$

where $Q(x) = \dfrac{x}{1-e^{-x}}$. $\mathrm{Td}(\mathscr{E})$ is again a $\mathbb{Q}$-linear combination of products of Chern classes:

$$\begin{aligned}\mathrm{Td}(\mathscr{E}) = 1 &+ \frac{1}{2}\,\mathrm{c}_1 + \frac{1}{2}(\mathrm{c}_1^2 + \mathrm{c}_2) + \frac{1}{24}\,\mathrm{c}_1\mathrm{c}_2 \\ &+ \frac{1}{720}(-\mathrm{c}_1^4 + 4\,\mathrm{c}_1^2\mathrm{c}_2 + \mathrm{c}_1\mathrm{c}_3 + 3\,\mathrm{c}_2^2 - \mathrm{c}_4) + \cdots \in H^*(X,\mathbb{Q}).\end{aligned}$$

11.5.2 GRR

Theorem 11.5. Let $f : X \to Y$ be a proper morphism of smooth quasi-projective varieties. Then we have the following commutative diagram:

$$\begin{tikzcd} D^{\mathrm{b}}(X) \arrow[r, "\mathsf{R}f_*"] \arrow[d] & D^{\mathrm{b}}(Y) \arrow[d] \\ \mathrm{K}(X) \arrow[r, "\mathsf{R}f_*"] \arrow[d, "\mathrm{ch}(-)\,\mathrm{Td}_X"] & \mathrm{K}(Y) \arrow[d, "\mathrm{ch}(-)\,\mathrm{Td}_Y"] \\ H^\bullet(X,\mathbb{Q}) \arrow[r, "f_*"] & H^\bullet(Y,\mathbb{Q}), \end{tikzcd}$$

where $\mathrm{Td}_X = \mathrm{Td}(T_X)$, $\mathrm{Td}_Y = \mathrm{Td}(T_Y)$.

11.5.3 Hirzebruch-Riemann-Roch

When $f : X \to \{\mathrm{pt}\}$, for each $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$ we have

$$\mathsf{R}f_*[\mathscr{E}^\bullet] = [\mathsf{R}\Gamma(\mathscr{E}^\bullet)] = \sum (-1)^i \dim H^i(X,\mathscr{E}^\bullet) =: \chi(\mathscr{E}^\bullet).$$

Corollary 11.6 (HRR). For each $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$,

$$\chi(\mathscr{E}^\bullet) = \int_X \mathrm{ch}(\mathscr{E}^\bullet)\,\mathrm{Td}_X .$$

11.5.4 Fourier-Mukai transforms and GRR

Let X, Y be smooth projective varieties. For each $\alpha \in D^{\mathrm{b}}(X)$ (or $\alpha \in K(X)$), define

$$v(\alpha) = \mathrm{ch}(\alpha)\sqrt{\mathrm{Td}_X} \in H^{\bullet}(X, \mathbb{Q}).$$

We call $v(\alpha)$ the Mukai vector. Let $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$, and let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections.

Proposition 11.7. We have the commutative diagram:

$$\begin{tikzcd} D^{\mathrm{b}}(X) \arrow[r, "\Phi^{\mathscr{P}}"] \arrow[d, "v"] & D^{\mathrm{b}}(Y) \arrow[d, "v"] \\ H^{\bullet}(X, \mathbb{Q}) \arrow[r] & H^{\bullet}(Y, \mathbb{Q}) \\ \beta \arrow[r, mapsto] & q_*(v(\mathscr{P}) \cup p^*\beta). \end{tikzcd}$$

Remark. This induced map $H^{\bullet}(X, \mathbb{Q}) \to H^{\bullet}(Y, \mathbb{Q})$ is just $\mathbb{Q}$-linear. In general it does not preserve the grading, nor the cup-product.

11.6 Grothendieck group and semi-orthogonal decomposition

Let $\mathcal{D}$ be a triangulated category. Assume that $\mathcal{D}$ admits an semi-orthogonal decomposition $\langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$. Recall that we have the projection functors $p_i : \mathcal{D} \to \mathcal{C}_i$.

Proposition 11.8. The morphism

$$\begin{array}{ccc} \mathrm{K}(\mathcal{D}) & \xrightarrow{\sim} & \mathrm{K}(\mathcal{C}_1) \oplus \cdots \oplus \mathrm{K}(\mathcal{C}_m) \\ [F] & \mapsto & ([p_1(F)], \ldots, [p_m(F)]) \end{array}$$

is well-defined and s a group isomorphism.

In particular, if $E_1, \ldots, E_m$ is a full exceptional collection, then $K(\mathcal{D})$ is isomorphic to $\mathbb{Z}^m$. When $\mathcal{D} = D^{\mathrm{b}}(X)$, this is very rare, because usually $\mathrm{K}(\mathcal{D}) = \mathrm{K}(X)$ is usually infinitely dimensional.

12 Invariants under D-equivalence

Let X, Y be smooth projective varieties. Assume that $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$ as triangulated categories. Then X and Y share same common invariants. We will see some examples of such invariants.

12.1 Dimension

Proposition 12.1. Suppose there is an equivalence $\Phi : D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$. Then $\dim X = \dim Y$.

Proof. Let $\mathscr{P} \in D^{\mathrm{b}}(X \times Y)$ be the Fourier-Mukai kernel such that $\Phi = \Phi^{\mathscr{P}}$. As Φ is an equivalence, we have

$$\Phi^{-1} \dashv \Phi \dashv \Phi^{-1}.$$

Since the Fourier-Mukai kernel of left adjoint of Φ is $\mathscr{P}^\vee \otimes p_Y^* \omega_Y[\dim Y]$, and the Fourier-Mukai kernel of right adjoint of Φ is $\mathscr{P}^\vee \otimes p_X^* \omega_X[\dim X]$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections, the uniqueness of Fourier-Mukai kernel gives an isomorphism

$$\mathscr{P}^\vee \otimes p_Y^* \omega_Y[\dim Y] \cong \mathscr{P}^\vee \otimes \pi_X^* \omega_X[\dim X].$$

As they are nonzero in $D^{\mathrm{b}}(X \times Y)$ and $H^i(\mathscr{P}^\vee \otimes p_Y^* \omega_Y) = H^i(\mathscr{P}^\vee) \otimes p_Y^* \omega_Y$ is non-zero if and only if $H^i(\mathscr{P}^\vee \otimes p_X^* \omega_X) = H^i(\mathscr{P}^\vee) \otimes p_X^* \omega_X$ is non-zero (note that $p_Y^* \omega_Y$ and $p_X^* \omega_X$ are locally free), necessarily $\dim X = \dim Y$. ■

Question. Let X, Y be smooth projective varieties. Assume that there is an embedding $D^{\mathrm{b}}(X) \hookrightarrow D^{\mathrm{b}}(Y)$, do we have $\dim X \leq \dim Y$?

12.2 Grothendieck group

Let $\Phi : D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$. We have

$$K(X) \cong K(D^{\mathrm{b}}(X)) \cong K(D^{\mathrm{b}}(Y)) \cong K(Y),$$

where all $\cong$ are group isomorphisms. It may happen that $\Phi(\mathcal{O}_X) \neq \mathcal{O}_Y$. In this case, $K(X) \cong K(Y)$ is not a ring isomorphism.

12.3 Cohomology, Euler characteristic

Let X, Y be smooth projective varieties over $\mathbb{C}$.

Proposition 12.2. The equivalence $\Phi : D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$ implies

- $H^\bullet(X, \mathbb{Q}) \cong H^\bullet(Y, \mathbb{Q})$ as $\mathbb{Q}$-vector spaces;
- $e(X) = e(Y)$, where $e(-) = \sum(-1)^i \dim H^i(-, \mathbb{Q})$ is the Euler characteristic.

Proof. Let $\mathscr{P}$, $\mathscr{Q} \in D^{\mathrm{b}}(X \times Y)$ such that $\Phi = \Phi^{\mathscr{P}}$ and $\Phi^{-1} = \Phi^{\mathscr{Q}}$. We have

$$\begin{tikzcd} D^{\mathrm{b}}(X) \arrow[r, "\Phi^{\mathscr{P}}"] \arrow[d, "v"] & D^{\mathrm{b}}(Y) \arrow[r, "\Phi^{\mathscr{Q}}"] \arrow[d, "v"] & D^{\mathrm{b}}(X) \arrow[d, "v"] \\ H^\bullet(X, \mathbb{Q}) \arrow[r, "\varphi_{\mathscr{P}}"] & H^\bullet(Y, \mathbb{Q}) \arrow[r, "\varphi_{\mathscr{Q}}"] & H^\bullet(X, \mathbb{Q}), \end{tikzcd}$$

where $\varphi_{\mathscr{P}}(\beta) = q_*(v(\mathscr{P}) \cup p^*\beta)$ and $\varphi_{\mathscr{Q}}$ is defined similarly.

As $\varphi_{\mathscr{Q}} \circ \varphi_{\mathscr{P}}(\beta) = p_*(v(\mathscr{P} * \mathscr{Q}) \cup p^*\beta)$ and $\mathscr{P} * \mathscr{Q} \cong \mathcal{O}_{\Delta_X} \in D^{\mathrm{b}}(X \times X)$, we have $\varphi_{\mathscr{Q}} \circ \varphi_{\mathscr{P}} = \mathrm{id}$, and similarly $\varphi_{\mathscr{P}} \circ \varphi_{\mathscr{Q}} = \mathrm{id}$. Hence $H^\bullet(X, \mathbb{Q}) \cong H^\bullet(Y, \mathbb{Q})$.

Since $v(\mathscr{P}) = \mathrm{ch}(\mathscr{P}) \cup \sqrt{\mathrm{Td}_{X \times Y}} \in H^{\mathrm{even}}(X \times Y, \mathbb{Q})$. The morphism

$$\varphi_{\mathscr{P}} : H^{\mathrm{even}}(X, \mathbb{Q}) \oplus H^{\mathrm{odd}}(X, \mathbb{Q}) \xrightarrow{\sim} H^{\mathrm{even}}(Y, \mathbb{Q}) \oplus H^{\mathrm{odd}}(Y, \mathbb{Q})$$

preserves the $\mathbb{Z}/2\mathbb{Z}$-grading. Hence $e(X) = \dim H^{\mathrm{even}} - \dim H^{\mathrm{odd}} = e(Y)$. ∎

Conjecture 12.3. If $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$, then $h^{p,q}(X) = h^{p,q}(Y)$ for all p, q.

The conjecture is known to hold when $\dim X \leq 3$ (Popa-Schnell).

Proposition 12.4. Under the assumpstion $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$, the sums of vertical Hodge number are preserved, i.e., for each k,

$$\sum_{p-q=k} h^{p,q}(X) = \sum_{p-q=k} h^{p,q}(Y).$$

12.4 Canonical rings

Let X, Y, be smooth projective varieties over a field k. The **canonical ring** of X is defined to be

$$R(X) = \bigoplus_{i \geq 0} H^0(X, \omega_X^{\otimes i}).$$

Proposition 12.5 (Orlov)**.** If $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$, then $R(X) \cong R(Y)$ as graded k-algebra.

Proof. As before, let $\mathscr{P}, \mathscr{Q} \in D^{\mathrm{b}}(X \times Y)$ such that $\Phi = \Phi^{\mathscr{P}}_{X\to Y} : D^{\mathrm{b}}(X) \xrightarrow{\sim} D^{\mathrm{b}}(Y)$ and $\Phi^{-1} = \Phi^{\mathscr{Q}}_{Y\to X}$.

First, we prove that $\Phi^{\mathscr{Q}}_{X\to Y} : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ is also an equivalence. Indeed, $\mathscr{P} * \mathscr{Q} \cong \mathcal{O}_{\Delta_X}$, $\mathscr{Q} * \mathscr{P} \cong \mathcal{O}_{\Delta_Y}$. Let $\tau : X \times Y \to Y \times X$ be the isomorphism that sends (x, y) to (y, x). Then $(\tau^*\mathscr{Q}) * (\tau^*\mathscr{P}) \cong \mathcal{O}_{\Delta_X}$, $(\tau^*\mathscr{P}) * (\tau^*\mathscr{Q}) \cong \mathcal{O}_{\Delta_Y}$. Thus $\Phi^{\mathscr{Q}}_{X\to Y} = \Phi^{\tau^*\mathscr{Q}}_{X\to Y}$ is an equivalence.

Let $\iota : X \to X \times X$, $\iota : Y \to Y \times Y$ be the diagonal maps. We have

$$H^0(X, \omega_X^{\otimes k}) \cong \mathrm{Hom}_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X^{\otimes k}),$$
$$H^0(Y, \omega_Y^{\otimes k}) \cong \mathrm{Hom}_{Y\times Y}(\iota_*\mathcal{O}_Y, \iota_*\omega_Y^{\otimes k}).$$

Consider $(\tau^*\mathscr{Q}) \boxtimes \mathscr{P} \in D^{\mathrm{b}}((X\times X)\times(Y\times Y))$. We will show that $\Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}(\iota_*\omega_X^{\otimes k}) \cong \iota_*\omega_Y^{\otimes k}$, so that $R(X) \cong R(Y)$ as graded k-vector spaces.

Let $\mathscr{S} = \Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}(\iota_*\omega_X^k)$. Then $\Phi^{\mathscr{S}}$ can be factorized as

$$D^{\mathrm{b}}(Y) \xrightarrow{\Phi^{\mathscr{Q}}} D^{\mathrm{b}}(X) \xrightarrow{\Phi^{\iota_*\omega_X^{\otimes k}}} D^{\mathrm{b}}(X) \xrightarrow{\Phi^{\mathscr{P}}} D^{\mathrm{b}}(Y).$$

Since $\Phi^{\iota_*\omega_X^{\otimes k}} = (- \otimes \omega_X)^k = S_X^k[-kn]$, where $n = \dim X = \dim Y$, (8.8) gives

$$\Phi^{\mathscr{S}} = \Phi^{\mathscr{P}} \circ \Phi^{\iota_*\omega_X^{\otimes k}} \circ \Phi^{\mathscr{Q}} \cong \Phi^{\mathscr{P}} \circ S_X^k[-kn] \circ \Phi^{\mathscr{Q}} \cong S_Y^k[-kn] = \Phi^{\iota_*\omega_Y^{\otimes k}}.$$

Hence, $\iota_*\omega_Y^{\otimes k} \cong \mathscr{S} = \Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}(\iota_*\omega_X^k)$.

Finally, we show that $R(X) \xrightarrow{\sim} R(Y)$ is a ring homomorphism. In fact, this follows from the following commutative diagram:

$$\begin{tikzcd}
H^0(X, \omega_X^{\otimes k}) \otimes H^0(X, \omega_X^{\otimes \ell}) \arrow[r, "\cdot"] \arrow[d, "\wr"] & H^0(X, \omega_X^{\otimes (k+\ell)}) \arrow[d, "\wr"] \\
\mathrm{Hom}(\mathcal{O}_X, \omega_X^{\otimes k}) \otimes \mathrm{Hom}(\omega_X^{\otimes \ell}, \omega_X^{\otimes (k+\ell)}) \arrow[r, "\circ"] \arrow[d, "\wr"] & \mathrm{Hom}(\mathcal{O}_X, \omega_X^{\otimes (k+\ell)}) \arrow[d, "\wr"] \\
\mathrm{Hom}_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X^{\otimes k}) \otimes \mathrm{Hom}_{X\times X}(\iota_*\omega_X^{\otimes \ell}, \iota_*\omega_X^{\otimes (k+\ell)}) \arrow[r, "\circ"] \arrow[d, "\Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}"] & \mathrm{Hom}_{X\times X}(\iota_*\mathcal{O}_X, \omega_X^{\otimes (k+\ell)}) \arrow[d, "\Phi^{\tau^*\mathscr{Q}\boxtimes\mathscr{P}}"] \\
\mathrm{Hom}_{Y\times Y}(\iota_*\mathcal{O}_Y, \iota_*\omega_Y^{\otimes k}) \otimes \mathrm{Hom}_{Y\times Y}(\iota_*\omega_Y^{\otimes \ell}, \iota_*\omega_Y^{\otimes (k+\ell)}) \arrow[r, "\circ"] & \mathrm{Hom}_{Y\times Y}(\iota_*\mathcal{O}_Y, \omega_Y^{\otimes (k+\ell)}).
\end{tikzcd}$$

■

Remark. One can show that ω_X (resp. $\omega_X^\vee$) is ample if and only if ω_Y (resp. $\omega_Y^\vee$) is ample. Hence if ω_X or $\omega_X^\vee$ is ample,

$$D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y) \iff X \cong Y.$$

Corollary 12.6. We have $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$ if and only if $\mathrm{K}(X) \cong \mathrm{K}(Y)$.

12.5 Hochschild (co)homology

Let X be a smooth projective variety over a field k. Define the bigraded k-algebra

$$HH(X) = \bigoplus_{i,\ell \in \mathbb{Z}} HA_{i,\ell}(X),$$

where $HA_{i,\ell}(X) = \mathrm{Ext}^i_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X^\ell)$ with product defined by the Yoneda product.

The canonical ring $R(X) = \bigoplus_\ell HA_{0,\ell}(X) \subseteq HH(X)$ as subalgebra. We could have proven directly that $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(Y)$ implies $HH(X) \cong HH(Y)$ as bigraded k-algebra.

We have another sub-k-algebra

$$HH^\bullet(X) := \bigoplus_i HA_{i,0}(X) \cong HH(X),$$

called the **Hochschild cohomology** of X. The **Hochschild homology** of X is defined as

$$HH_\bullet(X) = \bigoplus_i HA_{i+\dim X,1}(X) = \bigoplus_i \mathrm{Ext}^{i+\dim X}(\iota_*\mathcal{O}_X, \iota_*\omega_X).$$

$HH_\bullet(X)$ is a graded $HH^\bullet(X)$-module. So $HH(X) \cong HH(Y)$ implies $HH^\bullet(X) \cong HH^\bullet(Y)$ as k-algebra and $HH_\bullet(X) \cong HH_\bullet(Y)$ as $HH^\bullet$-modules.

12.6 Hochschild-Kostant-Rosenberg isomorphism

Theorem 12.7. Let X be a smooth quasi-projective variety over a field k. Then

$$HH^i(X) \cong \mathrm{Ext}^i_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) \cong \bigoplus_{p+q=i} H^q(X, \textstyle\bigwedge^p T_X),$$

$$HH_i(X) \cong \mathrm{Ext}^i_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X) \cong \bigoplus_{q-p=i} H^q(X, \Omega^p_X).$$

When X is smooth over $\mathbb{C}$, this implies again that the sums of vertical Hodge numbers are invariant under D-equivalence.

Remark. In general, these isomorphism don't preserve product.

Proof. (sketch) Consider the local-to-global spectral sequences

$$H^p(X\times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) = HH^{p+q}(X),$$

$$H^p(X\times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\omega_X)) \Rightarrow \mathrm{Ext}^{p+q}_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X) = HH_{p+q-\dim X}(X).$$

One way to prove this theorem is to show that these spectral sequence degenerate at E_2, so that

$$HH^\ell(X) = \bigoplus_{p+q=\ell} H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X))$$
$$HH_{\ell-\dim X}(X) = \bigoplus_{p+q=\ell} H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\omega_X)).$$

Assuming this, we finish the proof as follows: Choose an embedding $\jmath : X \hookrightarrow \mathbb{P}^N = \mathbb{P}(V)$. We have the Euler sequence

$$0 \to \Omega_{\mathbb{P}^N}(1) \to V^\vee \to \mathcal{O}(1) \to 0.$$

Let $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$ be the diagonal. Then

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^N}(-1) \boxtimes \Omega_{\mathbb{P}^N}(1) \arrow[r] \arrow[d, hook] & \mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^N} \arrow[r] & \mathcal{O}_\Delta \arrow[r] & 0 \\ V \boxtimes V^\vee \arrow[ur] & & & \end{tikzcd}$$

is exact (because over $(\ell, \ell') \in \mathbb{P}^N \times \mathbb{P}^N$, $\mathcal{O}_{\mathbb{P}^N}(-1) \boxtimes \Omega_{\mathbb{P}^N}(1)|_{(\ell,\ell')} = \ell \otimes \ell'^\perp$). Apply $(\jmath \times \jmath)^*$, we get the exact sequence

$$\mathscr{E} = \mathcal{O}_X(-1) \boxtimes \Omega_X(1) \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0,$$

where $\mathscr{E}$ is a locally free sheaf of rank $d = \dim X$ over $X \times X$. Consider the Koszul resolution

$$0 \to \bigwedge^d \mathscr{E} \to \bigwedge^{d-1} \mathscr{E} \to \cdots \to \mathscr{E} \to \mathcal{O}_{X \times X} \to \iota_*\mathcal{O}_X \to 0.$$

We see that

$$\begin{aligned} \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) &\cong \mathscr{H}om_{X \times X}(\iota_*\mathcal{O}_X[-q], \iota_*\mathcal{O}_X) \\ &\cong \mathscr{H}om_{X \times X}\Big(\bigwedge^q \mathscr{E}, \iota_*\mathcal{O}_X\Big) \\ &\cong \iota_*\mathscr{H}om_X\Big(\iota^*\bigwedge^q \mathscr{E}, \mathcal{O}_X\Big) \\ &\cong \iota_*\mathscr{H}om_X\Big(\bigwedge^q(\iota^*\mathscr{E}), \mathcal{O}_X\Big). \end{aligned}$$

Since X is the zero locus of some section of $\mathscr{E}^\vee$, $\iota^* e \cong N^\vee_{X/X \times X} \cong \Omega_X$. So $\mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) \cong \iota_*\bigwedge^q T_X$. Hence

$$H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X)) \cong H^p\Big(X \times X, \iota_*\bigwedge^q T_X\Big) \cong H^p\Big(X, \bigwedge^q T_X\Big).$$

Similarly, we have $\mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\omega_X) \cong \iota_*\Omega_X^{d-q}$, thus

$$H^p(X \times X, \mathscr{E}xt^q(\iota_*\mathcal{O}_X, \iota_*\omega_X)) \cong H^p(X, \Omega_X^{d-q}). \qquad \blacksquare$$

13 Spanning class

13.1 Definition

Let $\mathcal{D}$ be a triangulated category.

Definition 13.1. A **spanning class** of $\mathcal{D}$ is a collection Ω of objects of $\mathcal{D}$ such that $\langle\Omega\rangle^{\perp} = {}^{\perp}\langle\Omega\rangle = 0$. Explicitly, this means that the following are all equivalent:

- $E \cong 0$;
- $\mathrm{Hom}(F, E[i]) = 0$ for each $F \in \Omega$ and for each $i \in \mathbb{Z}$;
- $\mathrm{Hom}(E[i], F) = 0$ for each $F \in \Omega$ and for each $i \in \mathbb{Z}$.

Spanning classes are like generators of groups, rings, etc., which are useful for practical reason.

Proposition 13.2. Assume that $\mathcal{D}$ has a Serre functor. Then for any collection Ω, $\langle\Omega\rangle^{\perp} = 0$ if and only if ${}^{\perp}\langle\Omega\rangle = 0$. Thus, Ω generates $\mathcal{D}$ if and only if Ω is a spanning class.

13.2 Examples

Let X be a smooth projective variety over a field k.

13.2.1 Closed points

Lemma 13.3. The collection

$$\Omega = \{\mathcal{O}_x \mid x \in X \text{ is a closed point,}\}$$

is a spanning class of $D^{\mathrm{b}}(X)$.

Proof. Let $E^\bullet \in D^{\mathrm{b}}(X)$ be a nonzero object. We may assume that $H^0(E^\bullet) \neq 0$ and $H^i(E^\bullet) = 0$ for each $i > 0$. Choose $x \in \mathrm{Supp}(H^0(E^\bullet))$. Consider

$$E_2^{p,q} = \mathrm{Ext}^p(H^{-q}(E^\bullet), \mathcal{O}_x) \Rightarrow \mathrm{Ext}^{p+q}(E^\bullet, \mathcal{O}_x).$$

We have $E_2^{p,q} = 0$ if $p < 0$ or $q < 0$. So

$$E_2^{0,0} \cong E_\infty^{0,0} \cong \mathrm{Hom}(E^\bullet, \mathcal{O}_x) \cong \mathrm{Hom}(H^0(E^\bullet), \mathcal{O}_x) \neq 0$$

as $x \in \mathrm{Supp}(H^0(E^\bullet))$. ■

13.2.2 Ample line bundles

Let $\mathscr{L}$ be an ample line bundle on X with $\dim X = n$. Assume that $\mathscr{L}$ is globally generated. We have seen in the proof of Beilinson's theorem (10.1) that

$$\bigoplus_{i=0}^{n} \mathscr{L}^{-i}$$

spans $D^{\mathrm{b}}(X)$. In particular, for every ample line bundle $\mathscr{L}$ on X,

$$\Omega = \{\mathcal{O}_X, \mathscr{L}^k, \dots, \mathscr{L}^{nk}\}$$

spans $D^{\mathrm{b}}(X)$ whenever $k \gg 0$ (such that $\mathscr{L}^k$ is globally generated).

13.3 Some applications

Theorem 13.4. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Assume that $G \dashv F \dashv H$. Let Ω be a spanning class of $\mathcal{C}$. Assume that for any A, $B \in \Omega$ and for any $i \in \mathbb{Z}$, the map

$$F : \mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B)[i])$$

is an isomorphism. Then $F : \mathcal{C} \to \mathcal{D}$ is fully faithful.

Proof. As $G \dashv F \dashv H$ and F is exact, both G and H are exact. For any A, $B \in \mathcal{C}$ and for any $i \in \mathbb{Z}$ we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, B[i]) \arrow[r] \arrow[d] \arrow[dr] & \mathrm{Hom}(A, HF(B)[i]) \arrow[d, "\wr"] \\
\mathrm{Hom}(GF(A), B[i]) \arrow[r, "\sim"] & \mathrm{Hom}(F(A), F(B)[i]).
\end{tikzcd}$$

Complete $GF(A) \to A$ to a distinguished triangle

$$\begin{tikzcd}
GF(A) \arrow[rr] & & A \arrow[dl] \\
& C. \arrow[ul, dashed] &
\end{tikzcd}$$

Apply $\mathrm{Hom}(-, B[i])$ to the triangle, we get

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(C, B[i]) \arrow[r] & \mathrm{Hom}(A, B[i]) \arrow[r] \arrow[dr, "F"'] & \mathrm{Hom}(GF(A), B[i]) \arrow[r] \arrow[d, "\wr"] & \cdots \\
& & & \mathrm{Hom}(F(A), F(B)[i]). &
\end{tikzcd}$$

Now assume A, $B \in \Omega$, then the F in the above diagram becomes an isomorphism. So $\mathrm{Hom}(C, B[i]) = 0$, and thus $C = 0$. Hence $GF(A) \xrightarrow{\sim} A$ if $A \in \Omega$.

Still assuming $A \in \Omega$. For each $B \in \mathcal{C}$, we now have

$$\begin{tikzcd}
\mathrm{Hom}(A, B[i]) \arrow[r, "\sim"] \arrow[d, "\wr"] \arrow[dr] & \mathrm{Hom}(A, HF(B)[i]) \arrow[d, "\wr"] \\
\mathrm{Hom}(GF(A), B[i]) \arrow[r, "\sim"] & \mathrm{Hom}(F(A), F(B)[i]).
\end{tikzcd}$$

Consider the distinguished triangle

$$\begin{tikzcd}
B \arrow[rr] & & HF(B) \arrow[dl] \\
& C'. \arrow[ul, dashed] &
\end{tikzcd}$$

We get

$$\cdots \longrightarrow \mathrm{Hom}(A, B[i]) \xrightarrow{\sim} \mathrm{Hom}(A, HF(B)[i]) \longrightarrow \mathrm{Hom}(A, C'[i]) \longrightarrow \cdots .$$

So $\mathrm{Hom}(A, C') = 0$, and hence $C' = 0$. This gives $B \xrightarrow{\sim} HF(B)$.

Hence, for any A, $B \in \mathcal{C}$,

$$F : \mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(A, HF(B)) \xrightarrow{\sim} \mathrm{Hom}(F(A), F(B))$$

is an isomorphism. ■

Recall that an equivalence of triangulated categories $F : \mathcal{C} \to \mathcal{D}$ commutes with Serre functors whenever they exist (8.8). We show that the converse is also true, and that verifying the commutativity for a spanning class is enough.

Theorem 13.5. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories with Serre functors $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$, respectively. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor and assume that $G \dashv F$. Let Ω be a spanning class of $\mathcal{C}$. Assume that $\mathcal{C} \ncong 0$ and $\mathcal{D}$ is indecomposable. If $F \circ S_{\mathcal{C}}(A) \cong S_{\mathcal{D}} \circ F(A)$ for all $A \in \Omega$, then $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of category.

We first prove another criterion:

Theorem 13.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful exact functor of triangulated categories. Assume that $\mathcal{D}$ is indecomposable and $\mathcal{C} \ncong 0$. Then F defines an equivalence of categories if and only if $G \dashv F \dashv H$ and $H(B) \cong 0$ implies $G(B) \cong 0$ for each $B \in \mathcal{D}$.

Proof. Assume that F defines an equivalence of categories. Then $F^{-1} \dashv F \dashv F^{-1}$ and the only if part is now clear.

Lemma 13.7. If $F \dashv H$ and F is fully faithful, then $HF \cong \mathrm{id}_{\mathcal{C}}$.

Proof of Lemma. For all A, $B \in \mathcal{C}$, we have

$$\mathrm{Hom}(B, HF(A)) \cong \mathrm{Hom}(F(B), F(B)) \cong \mathrm{Hom}(B, A)$$

and these isomorphisms are functorial in A, B. Thus $HF \cong \mathrm{id}_{\mathcal{C}}$ by Yoneda's lemma. ■

Assume that $G \dashv F \dashv H$ and $H(B) \cong 0$ implies $G(B) \cong 0$ for each $B \in \mathcal{D}$. Let $B \in \mathcal{D}$. Let $\mathcal{C} \in D$ such that

$$FH(B) \to B \to C \to FH(B)[1]$$

is a distinguished triangle. Apply H to this triangle, we get the distinguished triangle

$$HFH(B) \to H(B) \to H(C) \to HFH(B)[1],$$

where $HFH(B) \to H(B)$ is an isomorphism by the lemma. Hence $H(C) \cong 0$. We also have $FHFH(B) \cong FH(B)$.

Consider the full subcategories

$$\mathcal{D}_1 = \{B \in \mathcal{D} \mid FH(B) \cong B\} \subseteq \mathcal{D}$$

$$\mathcal{D}_2 = \{B \in \mathcal{D} \mid H(B) \cong 0\} \subseteq \mathcal{D}$$

We just showed that for each $B \in \mathcal{D}$. there exists a distinguished triangle

$$B_1 \to B \to B_2 \to B_1[1]$$

with $B_1 \in \mathcal{D}_1$ and $B_2 \in \mathcal{D}_2$. But for any $B_1 \in \mathcal{D}_1$ and any $B_2 \in \mathcal{D}_2$, we have

$$\mathrm{Hom}(B_1, B_2) \cong \mathrm{Hom}(FH(B_1), B_2) \cong \mathrm{Hom}(H(B_1), H(B_2)) \cong 0.$$

$$\mathrm{Hom}(B_2, B_1) \cong \mathrm{Hom}(B_2, FH(B_1)) \cong \mathrm{Hom}(G(B_2), H(B_1)) \cong 0$$

since $H(B_2) = 0$ and $G(B_2) = 0$ by assumption. Hence $\mathcal{D}$ decompses into $\mathcal{D}_1$ and $\mathcal{D}_2$. As $\mathcal{C} \neq 0$, $HF \cong \mathrm{id}_{\mathcal{C}}$ gives $\mathcal{D}_2 \ncong \mathcal{D}$. Since $\mathcal{D}$ is indecomposable, $\mathcal{D}_1 \cong \mathcal{D}$. So $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective. ■

We continue the proof of (13.5).

Proof. We want to show that: If $FS_{\mathcal{C}}(A) \cong S_{\mathcal{D}}F(A)$ for all $A \in \Omega$, then $F : \mathcal{C} \to \mathcal{D}$. We have $G \dashv F \dashv H := S_{\mathcal{C}} \circ G \circ S_{\mathcal{D}}^{-1}$. By the previous theorem, it is enough to show that for each $B \in \mathcal{D}$, $H(B) \cong 0$ implies $G(B) \cong 0$. Suppose $H(B) \cong 0$, then for each $A \in \Omega$,

$$\begin{aligned}
\mathrm{Hom}(A, G(B)[i]) &\cong \mathrm{Hom}(G(B)[i], S_{\mathcal{C}}(A))^{\vee} \cong \mathrm{Hom}(B[i], FS_{\mathcal{C}}(A))^{\vee} \\
&\cong \mathrm{Hom}(B[i], S_{\mathcal{D}}F(A))^{\vee} \cong \mathrm{Hom}(F(A), B[i]) \\
&\cong \mathrm{Hom}(A, H(B)[i]) \cong 0.
\end{aligned}$$

Hence $G(B) \cong 0$. ■

14 Autoequivalence

14.1 Definition, first examples

Let $\mathcal{D}$ be a triangulated category.

Definition 14.1. The group of autoequivalence is defined by

$$\mathrm{Aut}(\mathcal{D}) := \{\, \text{isomorphism classes of equivalences } \mathcal{D} \to \mathcal{D} \,\}.$$

Examples of autoequivalences:

- $[1] : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$
- $f^* : \mathcal{D}^?(X) \xrightarrow{\sim} \mathcal{D}^?(X)$, where X is a noetherian scheme and $f : X \xrightarrow{\sim} X$ is a morphism.
- $- \otimes L : \mathcal{D}^?(X) \xrightarrow{\sim} \mathcal{D}^?(X)$, where X is a noetherian scheme and L is a line bundle over X.

Lemma 14.2. Let X be a variety over a field k. We have an injective group homomorphism

$$\begin{aligned} \mathbb{Z} \times (\operatorname{Aut}(X) \ltimes \operatorname{Pic}(X)) &\hookrightarrow \operatorname{Aut}(D^?(X)) \\ (i, f, \mathscr{L}) &\longmapsto \left[\mathscr{F}^\bullet \mapsto f^*(\mathscr{F}^\bullet \otimes \mathscr{L})[i]\right]. \end{aligned}$$

Proof. The group homomorphism part is clear. Assume that $\Phi : D^?(X) \to D^?(X)$ defined by $\mathscr{F}^\bullet \mapsto f^*(\mathscr{F}^\bullet \otimes \mathscr{L})[i]$ satisfies $\Phi \cong \mathrm{id}$. Then we have

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & f^*\mathscr{L}[i] \arrow[r] & \mathcal{O}_X. \end{tikzcd}$$

This implies $\operatorname{Ext}^i(\mathcal{O}_X, f^*L) \neq 0$, $\operatorname{Ext}^{-i}(f^*\mathscr{L}, \mathcal{O}_X) \neq 0$. This implies $i = 0$. So we get $H^0(X, f^*\mathscr{L}) \neq 0$ and $H^0(X, f^*\mathscr{L}^\vee) \neq 0$, which gives $f^*L \cong \mathcal{O}_X$, and thus $L \cong \mathcal{O}_X$. Finally, for each closed point $x \in X$, we have

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathcal{O}_{X,f(x)}[i] \arrow[r] & \mathcal{O}_{X,x}. \end{tikzcd}$$

So $x = f(x)$. Hence $f = \mathrm{id}_X$. ■

Proposition 14.3 (Bondal-Orlov)**.** Let X be a smooth projective variety with ω_X or $\omega_X^\vee$. Then

$$\operatorname{Aut}(D^{\mathrm{b}}(X)) \cong \mathbb{Z} \times (\operatorname{Aut}(X) \ltimes \operatorname{Pic}(X)).$$

14.2 Spherical twists

14.2.1 Historical origin

Let X and $X^\vee$ be mirror pairs of Calabi-Yau manifolds. The homological mirror symmetry conjecture asserts that there should be an equivalence

$$D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(\operatorname{Fuk}(X^\vee)).$$

The Dehn twists along a Lagrangian sphere S on $D^{\mathrm{b}}(\operatorname{Fuk}(X^\vee))$ should give us spherical twists on $D^{\mathrm{b}}(X)$ associated to spherical objects $\mathscr{E}^\bullet$, and $\mathscr{E}^\bullet$ corresponds to S.

14.2.2 Spherical objects

Let X be a smooth projective variety over a field k.

Definition 14.4. An object $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$ is called spherical if

(i) $\mathscr{E}^\bullet \otimes \omega_X \cong \mathscr{E}^\bullet$,

(ii) $\mathrm{Hom}(\mathscr{E}^\bullet, \mathscr{E}^\bullet[i]) = k$ if $i = 0$ or $\dim X$ and equals to 0 otherwise.

Let $\mathscr{E}^\bullet \in D^{\mathrm{b}}(X)$ be a spherical object. For each $\mathscr{F}^\bullet \in D^{\mathrm{b}}(X)$, choose $T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet)$ such that

$$\mathsf{R}\mathscr{H}om(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet \to \mathscr{F}^\bullet \to T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet) \to \mathsf{R}\mathscr{H}om(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet[1]$$

is a distinguished triangle.

Theorem 14.5. The functor

$$T_{\mathscr{E}^\bullet} : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)$$

defines an equivalence.

We call $T_{\mathscr{E}^\bullet}$ the spherical twist associated to $\mathscr{E}^\bullet$. Let's first look at some example before proving the theorem.

14.2.3 Examples

(a) Let X be Calabi-Yau: $\omega_X \cong \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. Every line bundle $\mathscr{L}$ on X is spherical.

Interlude 4: *T*-structures and torsion pairs

14.3 *T*-structures

Let $\mathcal{D}$ be a triangulated category. We want to find some abelian category $\mathcal{A}$ in $\mathcal{D}$.

Definition 14.6. A t-structure on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full additive subcategory such that we have:

- Define $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$. $D^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$.

- $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.

- For each $E \in \mathcal{D}$, there exists a distinguished triangle

$$t_{\leq 0}E \to E \to t_{\geq 1}E \to t_{\leq 0}E[1]$$

with $t_{\leq 0}E \in \mathcal{D}^{\leq 0}$ and $t_{\geq 1}E \in \mathcal{D}^{\geq 1}$.

Example 14.7 (standard t-structure)**.** Assume that $\mathcal{D} = D^{?}(\mathcal{A})$ for some abelian category $\mathcal{A}$. Define

$$\begin{aligned}
\mathcal{D}^{\leq 0} &= \{E \in \mathcal{D} \mid H^i(E) = 0 \ \forall i > 0\}, \\
\mathcal{D}^{\geq 0} &= \{E \in \mathcal{D} \mid H^i(E) = 0 \ \forall i < 0\}.
\end{aligned}$$

Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure of $\mathcal{D}$ and $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \cong \mathcal{A}$.

Theorem 14.8. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure of $\mathcal{D}$. Then $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category. We call $\mathcal{A}$ the **heart** of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure of $\mathcal{D}$ and let $\mathcal{A}$ be its heart. Define

$$\begin{array}{rccc}
H^i_{\mathcal{A}}: & \mathcal{D} & \to & \mathcal{A} \\
& E & \mapsto & t_{\geq i}t_{\leq i}E,
\end{array}$$

where $t_{\leq i}E = (t_{\leq 0}(E[i]))[-i]$ and $t_{\geq i}E = (t_{\geq 1}(E[1-i]))[i-1]$. Then $H^i_{\mathcal{A}}$ is a cohomological functor, namely for each distinguished triangle

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z, \arrow[ul, dashed] &
\end{tikzcd}$$

the induced sequence

$$\cdots \longrightarrow H^i_{\mathcal{A}}(X) \longrightarrow H^i_{\mathcal{A}}(Y) \longrightarrow H^i_{\mathcal{A}}(Z) \longrightarrow H^{i+1}_{\mathcal{A}}(X) \longrightarrow \cdots$$

is exact. Assume X, $Y \in \mathcal{A}$, then $H^i_{\mathcal{A}}(X) = H^i_{\mathcal{A}}(Y) = 0$ for each $i \neq 0$, so

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}_{\mathcal{A}}(Z) \arrow[r] & H^0_{\mathcal{A}}(X) \arrow[r] \arrow[d, phantom, "\wr"] & H^0_{\mathcal{A}}(Y) \arrow[r] \arrow[d, phantom, "\wr"] & H^0_{\mathcal{A}}(Z) \arrow[r] & 0 \\
0 \arrow[r] & \ker f \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & \operatorname{coker} f \arrow[r] & 0.
\end{tikzcd}$$

In particular, for any X, Y, $Z \in \mathcal{A}$,

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is exact in $\mathcal{A}$ if and only if

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z, \arrow[ul, dashed] \arrow[u, phantom, "\triangle"] &
\end{tikzcd}$$

in $\mathcal{D}$.

Remark. We have $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{D}}(X,Y)$, but in general $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \ncong \mathrm{Ext}^i_{\mathcal{D}}(X,Y)$. So $D^?(\mathcal{A}) \ncong \mathcal{D}$ in general.

Definition 14.9. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ is called bounded if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j}).$$

Proposition 14.10. Let $\mathcal{A} \subseteq \mathcal{D}$ be a full additive subcategory. Then $\mathcal{A}$ is the heart of a bounded t-structure if and only if the following conditions are satisfied:

- for each $k \in \mathbb{Z}_{<0}$ and for any A, $B \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{D}}(A, B[k]) = 0$;
- for each $E \in \mathcal{D}$ with $E \neq 0$, there exists integers $k_1 > \cdots > k_n$ and

$$0 = E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow E$$

  such that $\mathrm{Cone}(E_{i-1} \to E_i) \in \mathcal{A}[k_i]$.

In this case, we have $\mathrm{K}(\mathcal{D}) \cong \mathrm{K}(\mathcal{A})$.

14.4 Torsion pairs

Let $\mathcal{A}$ be an abelian category.

Definition 14.11. A **torsion pair** is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ such that:

- $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$;
- for each $E \in \mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \to 0$$

  with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Proposition 14.12. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of $\mathcal{A}$. Then $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. For each $E \in \mathcal{A}$, the objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

is a short exact sequence are unique up to isomorphisms.

Example 14.13. Let X be a variety. In $\mathsf{Coh}(X)$,

$$\mathcal{T} = \{\text{ torsion sheaf }\}, \quad \mathcal{F} = \{\text{ torsion free sheaf }\},$$

form a torsion pair.

14.5 Tilting

Let $\mathcal{D}$ be a triangulated category, $\mathcal{A}$ the heart of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Suppose that $\mathcal{A}$ admits a torsion pair $(\mathcal{T}, \mathcal{F})$. Define

$$\begin{aligned} {}^{\dagger}\mathcal{D}^{\leq 0} &= \{E \in \mathcal{D}^{\leq 0} \mid H^0(E) \in \mathcal{T}\} \\ {}^{\dagger}\mathcal{D}^{\geq 0} &= \{E \in \mathcal{D}^{\geq -1} \mid H^{-1}(E) \in \mathcal{F}\} \end{aligned}$$

Theorem 14.14 (Happel-Reiten-Smalø)**.** The pair $({}^{\dagger}\mathcal{D}^{\leq 0}, {}^{\dagger}\mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$. We call it the **tilted *t*-structure** with respect to $(\mathcal{T}, \mathcal{F})$.