Commutative algebra

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Contents

Part I

Chapter 1

Elementary Results

1 General Rings

(1.A) Let A be a ring and $\mathfrak{a}$ be an ideal of A. Then the set

$$\sqrt{\mathfrak{a}} = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n\}$$

is an ideal of A, called the radical of $\mathfrak{a}$.

An ideal $\mathfrak{p}$ is called a prime ideal of A if $A/\mathfrak{p}$ is an integral domain; in other words, if $\mathfrak{p} \neq A$ and if $A - \mathfrak{p}$ is closed under multiplication. If $\mathfrak{p}$ is a prime, and if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals not contained in $\mathfrak{p}$, then $\mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}$.

An ideal $\mathfrak{q}$ is called primary if $\mathfrak{q} \neq A$ and if the only zero divisors of $A/\mathfrak{q}$ are nilpotent elements, i.e. $xy \in \mathfrak{q}$, $x \notin \mathfrak{q}$ implies $y^n \in \mathfrak{q}$ for some n. If $\mathfrak{q}$ is primary then its radical $\mathfrak{p}$ is a prime, and $\mathfrak{p}$ and $\mathfrak{q}$ are said to belong to each other. If $\mathfrak{q}$ is an ideal contained some power $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$, then $\mathfrak{q}$ is a primary ideal belong to $\mathfrak{m}$.

The set of the prime ideals of A is called the spectrum of A and is denoted by $\mathrm{Spec}(A)$; the set of maximal ideals of A is called the maximal spectrum of A and we denote it by $\mathrm{Max}(A)$. The set $\mathrm{Spec}(A)$ is topologized as follows. For any subset S of A, put

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid S \subseteq \mathfrak{p}\},$$

and take as the closed sets in $\mathrm{Spec}(A)$ all subsets of the form $V(S)$. This topology is called the Zariski topology. If $f \in A$, we put $D(f) = \mathrm{Spec}(A) - V(f)$ and call it an elementary open set of $\mathrm{Spec}(A)$. The elementary open sets form a basis of open sets of the Zariski topology in $\mathrm{Spec}(A)$.

Let $\varphi : A \to B$ be a ring homomorphism. To each $\mathfrak{q} \in \mathrm{Spec}(B)$ we associate the ideal $\varphi^{-1}(\mathfrak{q}) = \mathfrak{q} \cap A$ of A. Since $\mathfrak{q} \cap A$ is prime in A, we then get a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, which is denoted by $\mathrm{Spec}(\varphi)$. The map $\mathrm{Spec}(\varphi)$ is continuous since $\mathrm{Spec}(\varphi)(D(f)) = D(\varphi(f))$. It does not necessary map $\mathrm{Max}(B)$ into $\mathrm{Max}(A)$. When $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$, we say that $\mathfrak{q}$ lies over $\mathfrak{p}$.

(1.B) Let A be a ring, and let I, $\mathfrak{p}_1$, …, $\mathfrak{p}_r$ be ideals in A. Suppose that all but possibly two of the $\mathfrak{p}_i$'s are prime ideals. Then, if $I \not\subseteq \mathfrak{p}_i$ for each i, the ideal $I \not\subseteq \bigcup\limits_i \mathfrak{p}_i$.

Proof. Omitting those $\mathfrak{p}_i$ which are contained in some other $\mathfrak{p}_j$, we may suppose that there are no inclusion relations between the $\mathfrak{p}_i$'s. We use induction on r. When $r = 2$, suppose $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$. Take $x \in I - \mathfrak{p}_1$ and $y \in I - \mathfrak{p}_2$. Then $x \in \mathfrak{p}_2$, hence $x + y \notin \mathfrak{p}_2$, therefore both y and $x + y$ must be in $\mathfrak{p}_1$. Then $x \in \mathfrak{p}_1$ and we get a contradiction.

When $r > 2$, assume that $\mathfrak{p}_r$ is prime. Then $I\mathfrak{p}_1 \dots \mathfrak{p}_{r-1} \not\subseteq \mathfrak{p}_r$ since there are no inclusion relations. Take an element $x \in I\mathfrak{p}_1 \dots \mathfrak{p}_{r-1} - \mathfrak{p}_r$. Put

$$S = I - (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_{r-1}).$$

By induction hypothesis S is not empty. Suppose $I \subseteq \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$. Then S is contained in $\mathfrak{p}_r$. But if $s \in S$ then $s + x \in S$ and therefore both s and $s + x$ are in $\mathfrak{p}_r$, hence $x \in \mathfrak{p}_r$, contradiction. ∎

Remark. When A contains an infinite field k, the condition that $\mathfrak{p}_3, \dots, \mathfrak{p}_r$ be prime is superfluous, because the ideals are k-vector spaces and $I = \bigcup\limits_i (I \cap \mathfrak{p}_i)$ cannot happen if $I \cap \mathfrak{p}_i$ are proper subspaces of I.

(1.C) (Chinese Remainder Theorem) Let A be a ring, and I_1, …, I_r be ideals of A

such that $I_i + I_j = A$ $(i \neq j)$. Then

$$I_1 I_2 \dots I_r = I_1 \cap \dots \cap I_r$$

and

$$A/\prod_i I_i = A/\bigcap_i I_i \cong (A/I_1) \times \dots \times (A/I_r).$$

(1.D) Any ring $A \neq 0$ has at least one maximal ideal. In fact, the set

$$\Sigma = \{\text{ideal } J \text{ of } A \mid 1 \notin J\}$$

is not empty since $(0) \in \Sigma$, and one can apply Zorn's lemma to find a maximal element of Σ. It follows that $\mathrm{Spec}(A)$ is empty iff $A = 0$.

If $A \neq 0$, $\mathrm{Spec}(A)$ has also minimal elements. In fact, any prime $\mathfrak{p} \in \mathrm{Spec}(A)$ contains at least one minimal prime. This is proves by reversing the inclusion-order of $\mathrm{Spec}(A)$ and applying Zorn's lemma.

If $J \neq A$ is an ideal, the map $\mathrm{Spec}(A/J) \to \mathrm{Spec}(A)$ obtained from the natural homomorphism $A \to A/J$ is an order preserving bijection from $\mathrm{Spec}(A/J)$ onto $V(J)$. Therefore $V(J)$ has maximal as well as minimal elements. We shall call a minimal element of $V(J)$ a minimal prime over J.

(1.E) A subset S of a ring A is called a multiplicative subset of A if $1 \in S$ and if the products of elements of S are again in S.

Let S be a multiplicative subset of A not containing 0, and let Σ be the set of ideals of A which do not meet S. Since $(0) \in \Sigma$ the set Σ is not empty, and it has a maximal element $\mathfrak{p}$ by Zorn's lemma. Such an ideal $\mathfrak{p}$ is prime; in fact, if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then both $(x) + \mathfrak{p}$ and $(y) + \mathfrak{p}$ meet S, hence there exist elements $a, b \in A$ and $s, t \in S$ such that $ax - s, by - t \in \mathfrak{p}$. Then $abxy - st \in \mathfrak{p}$, $st \in S$, therefore $st \notin \mathfrak{p}$ and hence $xy \notin \mathfrak{p}$. A maximal element of Σ is called a maximal ideal with respect to the multiplicative set S.

We list a few corollaries of the above result.

(i) If S is a multiplicative subset of a ring A and if $0 \notin S$, then there exists a prime $\mathfrak{p} \in \mathrm{Spec}(A)$ with $\mathfrak{p} \cap S = \varnothing$.

(ii) The set of nilpotent elements in A,

$$\mathfrak{N}(A) = \sqrt{(0)} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

(iii) Let A be a ring and J a proper ideal of A. Then

$$\sqrt{J} = \bigcap_{J \subseteq \mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

Proof. (i) is already proved. (ii): Clearly any prime ideal contains $\mathfrak{N}(A)$. Conversely, if $a \notin \mathfrak{N}(A)$, then $S = \{1, a, a^2, \ldots\}$ is multiplicative and $0 \notin S$, therefore there exists a prime $\mathfrak{p}$ with $a \notin \mathfrak{p}$. (iii) is nothing but (ii) applied to A/J. ■

We say a ring A is reduced if $\mathfrak{N}(A) = (0)$. This is equivalent to saying that (0) is an intersection of prime ideals. For any ring A, we put $A_{\mathrm{red}} = A/\mathfrak{N}(A)$. The ring A_{red} is of course reduced.

(1.F) Let S be a multiplicative subset of a ring A. Then the localization (or quotient ring of ring of fractions) of A with respect to S, denoted by $S^{-1}A$ or by A_S, is the ring

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A,\ s \in S \right\} / \sim$$

where the equivalence relation $\sim$ is defined by

$$\frac{a}{s} \sim \frac{b}{t} \iff u(at - bs) = 0 \text{ for some } u \in S$$

and the addition and the multiplication are defined by the usual formulas about fractions. We have $S^{-1}(A) = 0$ iff $0 \in S$.

The natural map $\varphi : A \to S^{-1}A$ given by $\varphi(a) = a/1$ is a homomorphism, and its kernel is

$$\{a \in A \mid \exists s \in S : sa = 0\}.$$

The A-algebra $S^{-1}A$ has the following universal mapping property: if $\psi : A \to B$ is a ring homomorphism such that the images of the element of S are invertible in B, then there exists a unique homomorphism $\psi_S : S^{-1}A \to B$ such that $\psi = \psi_S \circ \varphi$.

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d] & A \arrow[r, "\varphi"] \arrow[d, "\psi"] & S^{-1}A \arrow[dl, dashed, "\exists ! \psi_S"] \\
B^{\times} \arrow[r, hook] & B &
\end{tikzcd}$$

Of course one can use this property as a definition of $S^{-1}A$. It is the basis of all functorial properties of localization.

If $\mathfrak{p}$ is a prime (resp. primary) ideal of A such that $\mathfrak{p} \cap S = \varnothing$, then $\mathfrak{p}(S^{-1}A)$ is prime (resp. primary). Conversely, all the prime and the primary ideals of $S^{-1}A$ are obtained in this way. For any ideal of $S^{-1}A$ we have $I = (I \cap A)(S^{-1}A)$. If J is an ideal of A, then we have $J(S^{-1}A) = S^{-1}A$ iff $J \cap S \neq \varnothing$. The canonical map $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is an order preserving bijection and homeomorphism from $\mathrm{Spec}(S^{-1}A)$ onto the subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \varnothing\}$$

of $\mathrm{Spec}(A)$.

(1.G) Let S be a multiplicative subset of a ring A and let M be an A-module. One defines

$$S^{-1}M = \left\{ \frac{x}{s} \,\middle|\, x \in M,\ s \in S \right\}$$

in the same way as $S^{-1}A$. The set $S^{-1}M$ is an $S^{-1}A$-module, and there is a natural isomorphism of $S^{-1}A$-modules

$$S^{-1}M \cong S^{-1}A \otimes_A M$$

given by $x/s \mapsto (1/s) \otimes x$.

If M and N are A-modules, we have

$$\begin{aligned}
S^{-1}(M \otimes_A N) &= S^{-1}A \otimes_A (M \otimes_A N) \\
&= (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} (S^{-1}A \otimes_A N) \\
&= (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N).
\end{aligned}$$

When M is of finite presentation, i.e. when there is an exact sequence of the form $A^m \to A^n \to M \to 0$, we have also

$$S^{-1}(\mathrm{Hom}_A(M, N)) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

(1.H) When $S = A - \mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Spec}(A)$, we write $A_\mathfrak{p}$, $M_\mathfrak{p}$ for $S^{-1}A$, $S^{-1}M$.

Lemma 1. If an element x of M is mapped to 0 in $M_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$, then $x = 0$. In other words, the natural map

$$M \to \prod_{\mathfrak{m} \in \mathrm{Max}(A)} M_\mathfrak{m}$$

is injective.

Proof. $x = 0$ in $M_\mathfrak{m}$ iff $s \in A - \mathfrak{m}$ such that $sx = 0$ in M iff $\mathrm{Ann}(x) \not\subseteq \mathfrak{m}$. Therefore, if $x = 0$ in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$, the annihilator $\mathrm{Ann}(x)$ of x is not contained in any maximal ideal and hence $\mathrm{Ann}(x) = A$. This implies $x = 1 \cdot x = 0$. ■

Lemma 2. When A is an integral domain with quotient field K, all localizations of A can be viewed as subrings of K. In this sense, we have

$$A = \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} A_\mathfrak{m}.$$

Proof. Given $x \in K$, we put $D = \{a \in A \mid ax \in A\}$; we might call D the ideal of denominators of x. The element x is in A iff $D = A$, and x is in $A_\mathfrak{m}$ iff $D \not\subseteq \mathfrak{m}$. Therefore, if $x \notin A$, there exists a maximal ideal $\mathfrak{m}$ such that $D \subseteq \mathfrak{m}$, and $x \notin A_\mathfrak{m}$ for this $\mathfrak{m}$. ■

(1.I) Let $\varphi : A \to B$ be a homomorphism of rings and S a multiplicative subset of A; put $S' = \varphi(S)$. Then the localization $S^{-1}B$ of B as an A-module coincides with $S'^{-1}B$:

$$S'^{-1}B = S^{-1}B = S^{-1}A \otimes_A B.$$

In particular, if I is an ideal of A and if S' is the image of S in A/I, one obtains

$$S'^{-1}(A/I) = S^{-1}A/I(S^{-1}A).$$

In this sense, quotient commutes with localization.

(1.J) Let A be a ring and S a multiplicative subset of A; let $A \xrightarrow{\varphi} B \xrightarrow{\psi} S^{-1}A$ be homomorphisms such that (1) $\psi \circ \varphi$ is the natural map and (2) for any $b \in B$ there exists $s \in S$ with $\varphi(s)b \in \varphi(A)$. Then $S^{-1}B = \varphi(S)^{-1}B = S^{-1}A$, as one can easily check. In particular, let A be a domain, $\mathfrak{p} \in \operatorname{Spec}(A)$ and B a subring of $A_\mathfrak{p}$ such that $A \subseteq B \subseteq A_\mathfrak{p}$. Then $A_\mathfrak{p} = B_\mathfrak{q} = B_\mathfrak{p}$, where $\mathfrak{q} = \mathfrak{p}A_\mathfrak{p} \cap B$ and $B_\mathfrak{p} = B \otimes A_\mathfrak{p}$.

(1.K) A ring A which has only one maximal ideal $\mathfrak{m}$ is called a local ring, and $A/\mathfrak{m}$ is called the residue field of A. When we say that "$(A, \mathfrak{m})$ is a local ring" or "$(A, \mathfrak{m}, k)$ is a local ring", we mean that A is a local ring, that $\mathfrak{m}$ is the unique maximal ideal of A and that k is the residue field of A. When A is an arbitrary ring and $\mathfrak{p} \in \operatorname{Spec}(A)$, the ring $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. The residue field of $A_\mathfrak{p}$ is denoted by $\kappa(\mathfrak{p})$. Thus $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = (A/\mathfrak{p})_{(0)}$, which is the quotient field of the integral domain $A/\mathfrak{p}$ by (1.I).

If $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ are local rings, a homomorphism $\psi : A \to B$ is called a local homomorphism if $\psi(\mathfrak{m}) \subseteq \mathfrak{n}$. In this case ψ induces a homomorphism $k \to k'$.

Let A and B be rings and $\psi : A \to B$ a homomorphism. Consider the map $\operatorname{Spec}(\psi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. If $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\operatorname{Spec}(\psi)(\mathfrak{q}) = \mathfrak{q} \cap A = \mathfrak{p}$, we have $\psi(A - \mathfrak{p}) \subseteq B - \mathfrak{q}$, hence ψ induces a homomorphism $\psi_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q}$, which is a local homomorphism since

$$\psi_\mathfrak{q}(\mathfrak{p}A_\mathfrak{p}) \subseteq \psi(\mathfrak{p})B_\mathfrak{q} \subseteq \mathfrak{q}B_\mathfrak{q}.$$

Note that $\psi_\mathfrak{q}$ can be factored as

$$A_\mathfrak{p} \to B_\mathfrak{p} = A_\mathfrak{p} \otimes_A B \to B_\mathfrak{q}$$

and $B_\mathfrak{q}$ is the localization of $B_\mathfrak{q}$ by $\mathfrak{q}B_\mathfrak{q} \cap B_\mathfrak{p}$. In general $B_\mathfrak{p}$ is not a local ring, and the maximal ideals of $B_\mathfrak{p}$ which contain $\mathfrak{p}B_\mathfrak{p}$ correspond to the pre-images of $\mathfrak{p}$ in $\operatorname{Spec}(B)$. ($B_\mathfrak{p}$ can have maximal ideals other than these.) But if $B_\mathfrak{p}$ is a local ring, then $B_\mathfrak{p} = B_\mathfrak{q}$, because if $(R, \mathfrak{m})$ is a local ring then $R - \mathfrak{m}$ is the set of units of R and hence $R_\mathfrak{m} = R$.

(1.L) Let $A \neq 0$ be a ring. The Jacobson radical of A is defined by

$$\operatorname{rad}(A) = \bigcap_{\mathfrak{m} \in \operatorname{Max}(A)} \mathfrak{m}.$$

Thus, if $(A, \mathfrak{m})$ is a local ring then $\mathfrak{m} = \operatorname{rad}(A)$. We say that a ring $A \neq 0$ is a semi-local ring if it has only a finite number of maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. (We express this situation by saying "$(A, \mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ is a semi-local ring".) In this case $\operatorname{rad}(A) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r = \prod_i \mathfrak{m}_i$ by (1.C).

Any element of the form $1 + x$, $x \in \operatorname{rad}(A)$ is a unit in A, because $1 + x$ is not contained in any maximal ideal. Conversely, if I is an ideal and if $1 + x$ is a unit for each $x \in I$, we have $I \subseteq \operatorname{rad}(A)$.

(1.M)

Lemma (Nakayama). Let A be a ring, M a finite A-module and I an ideal of A. Suppose that $IM = M$, Then there exists an element $a \in A$ of the form $a = 1 + x$, $x \in I$, such that $aM = 0$. If moreover $I \subseteq \operatorname{rad}(A)$, then $M = 0$.

Proof. Let $M = Aw_1 + \ldots + Aw_s$. We use induction on s. Put $M' = M/Aw_s$. By induction hypothesis there exists $x \in I$ such that $(1+x)M' = 0$, i.e., $(1+x)M \subseteq Aw_s$ (when $s = 1$, take $x = 0$). Since $M = IM$, we have

$$(1 + x)M = I(1 + x)M \subseteq I(Aw_s) = Iw_s,$$

hence we can write $(1 + x)w_s = yw_s$ for some $y \in I$. Then

$$(1 + x - y)(1 + x)M \subseteq (1 + x - y)Aw_s = 0,$$

and $(1 + x - y)(1 + x) - 1 \in I$, proving the first assertion. The second assertion follows from this and from (1.L). ■

This lemma is often used in the following form.

Corollary. Let A be a ring, M an A-module, N and N' submodules of M, and I an ideal of A. Suppose that $M = N + IN'$, and that either (a) I is nilpotent, or (b) $I \subseteq \operatorname{rad}(A)$ and N' is finitely generated. Then $M = N$.

Proof. In case (a) we have

$$M/N = I(M/N) = I^2(M/N) = \cdots = 0.$$

In case (b), apply Nakayama's lemma to M/N. ■

(1.N) In particular, let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finite. Then a subset G of M generates M iff its image $\overline{G}$ in $M/\mathfrak{m}M = M \otimes k$ generates $M \otimes k$. In fact, if N is the submodule generated by G, and if $\overline{G}$ generates $M \otimes k$, then $M = N + \mathfrak{m}M$, whence $M = N$ by the corollary. Since $M \otimes k$ is a vector space over the field k, it has a basis, say $\overline{G}$, and if we lift $\overline{G}$ arbitrarily to a subset G of M, then G is a system of generator of M. Such a system of generators is called a minimal basis of M. Note that a minimal basis is not necessarily a basis of M (but it is so in an important case, cf. (3.G)).

(1.O) Let A be a ring and M an A-module. An element a of A is said M-regular if it is not a zero-divisor on M, i.e., if $M \xrightarrow{[a]} M$ is injective. The set of the M-regular elements is a multiplicative subset of A.

Let S_0 be the set of A-regular elements. Then $S_0^{-1}A$ is called the total quotient ring of A. In this book we shall denote it by ΦA. When A is an integral domain, ΦA is nothing but the quotient field of A.

(1.P) Let A be a ring and $\alpha : \mathbb{Z} \to A$ be the canonical homomorphism. Then $\ker \alpha = n\mathbb{Z}$ for some $n \geq 0$. We call n the characteristic of A and denote it by $\mathrm{char}(A)$. If A is local the characteristic $\mathrm{char}(A)$ is either 0 or a prime number.

2 Noetherian Rings and Artinian Rings

(2.A) A ring is called noetherian (resp. artinian) if the ascending chain condition (resp. descending chain condition) for ideals holds in it. A ring A is noetherian iff every ideal of A is a finite A-module.

An A-module M is a noetherian (resp. artinian) module if the ascending chain condition (resp. descending chain condition) for submodules holds in it. If there's an exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then M is noetherian (resp. artinian) iff M' and M'' are noetherian (resp. artinian).

A polynomial ring $A[X_1, \ldots, X_n]$ over a noetherian ring A is again noetherian. Similarly for a formal power series ring $A[[X_1, \ldots, X_n]]$ (Hilbert Basis Theorem). If B is an A-algebra of finite type and if A is noetherian, then B is noetherian since it is a homomorphic image of $A[X_1, \ldots, X_n]$ for some n.

If A is a noetherian ring and M a finite A-module, then M is noetherian and every submodule of M is a finite A-module. From this, it follows easily that a finite module M over a noetherian ring has a projective resolution

$$\cdots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is a finite free A-module. In particular, M is of finite presentation.

(2.B) Any proper ideal I of a noetherian ring has a primary decomposition, i.e. $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ with primary ideals $\mathfrak{q}_i$. (We shall discuss this topic again in Chap. 5.)

(2.C)

Proposition. A ring A is artinian iff the length of A as A-module is finite.

Proof. If $\operatorname{len}_A(A) < \infty$ then A is certainly artinian (and noetherian). Conversely, suppose A is artinian. Then A has only a finite number of maximal ideals. Indeed, if there were an infinite sequence of maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ then $\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supset \cdots$ would be a strictly descending infinite chain of ideals, contradicting the hypothesis. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be all maximal ideals of A (we may assume $A \neq 0$, so $r > 0$), and put $I = \mathfrak{m}_1 \ldots \mathfrak{m}_r$. The descending chain $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stops, so there exists $s > 0$ such that $I^s = I^{s+1}$. Put $((0) : I^s) = J$. Then

$$(J : I) = (((0) : I^s) : I) = ((0) : I^{s+1}) = J.$$

Claim. $J = A$, so $1 \cdot I^s \subseteq (0)$, i.e. $I^s = (0)$.

Suppose the contrary, and let J' be a minimal member of the set of ideals strictly containing J. Then $J' = Ax + J$ for any $x \in J' - J$. Since $I = \mathrm{rad}(A)$, the ideal $Ix + J$ is not equal to J' by Nakayama's lemma (Cor. of (1.K)). So we must have $Ix + J = J$ by the minimality of J', hence $Ix \subseteq J$ and $x \in (J : I) = J$, contradiction. Thus $J = A$.

Consider the descending chain

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq I \supseteq I\mathfrak{m}_1 \supseteq \cdots \supseteq I^2 \supseteq \cdots \supseteq I^s = (0).$$

Each factor module of this chain is a vector space over the field $A/\mathfrak{m}_i = k_i$ for some i, and its subspaces correspond bijectively to the intermediate ideals. Thus, the descending chain condition in A implies that this factor module is of finite dimension over k_i, therefore it is of finite length as A-module. Since $\mathrm{len}_A(A)$ is the sum of the length of the factor modules of the chain above, we see that $\mathrm{len}_A(A)$ is finite. ∎

A ring $A \neq 0$ is said to have dimension zero if all prime ideals are maximal (cf. 12.A).

Corollary. A ring $A \neq 0$ is artinian iff it is noetherian and of dimension zero.

Proof. If A is artinian, then it is noetherian since $\mathrm{len}_A(A) < \infty$.

Let $\mathfrak{p}$ be any prime ideal of A. In the notation of the above proof, we have $(\mathfrak{m}_1 \ldots \mathfrak{m}_r)^s = I^s = (0) \subseteq \mathfrak{p}$, hence $\mathfrak{p} = \mathfrak{m}_i$ for some i. Thus A is of dimension zero.

To prove the converse, let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a primary decomposition of the zero ideal in A, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Since $\mathfrak{p}_i$ is finitely generated over A, there is a positive integer n such that $\mathfrak{p}_i^n \subseteq \mathfrak{q}_i$ for $1 \leq i \leq r$. Then $(\mathfrak{p}_1 \ldots \mathfrak{p}_r)^n = (0)$. After this point we can imitate the last part of the proof of the proposition to conclude that $\mathrm{len}_A(A) < \infty$. ∎

(2.D) I.S.Cohen prove that a ring is noetherian iff every prime ideal is finitely generated (cf. Nagata, LOCAL RINGS, p.8). Recently(?) P.M.Eakin (Math. Annalen 177(1968), 278-282) proved that, if A is a ring and A' is a subring over which A is finite, then A' is noetherian if (and of course only if) A is so. (The theorem was independently obtained by Nagata, but the priority is Eakin's.)

Chapter 2

Flatness

3 Flatness

(3.A)

Definition. Let A be a ring and M an A-module; when

$$N_\bullet : \cdots \longrightarrow N_{i-1} \longrightarrow N_i \longrightarrow N_{i+1} \longrightarrow \cdots$$

is any sequence of A-modules (and of A-linear maps), let $N_\bullet \otimes M$ denote the sequence

$$\cdots \longrightarrow N_{i-1} \otimes M \longrightarrow N_i \otimes M \longrightarrow N_{i+1} \otimes M \longrightarrow \cdots$$

obtained by tensoring $N_\bullet$ with M.

We say M is flat over A, or A-flat, if $N_\bullet \otimes M$ is exact whenever $N_\bullet$ is exact. We say that M is faithfully flat (f.f.) over A, if $N_\bullet \otimes M$ is exact iff $N_\bullet$ is exact.

In general, the functor $- \otimes M$ is right exact, so we can define its left derived functor $\operatorname{Tor}_i^A(-, M)$. It can be proved that $\operatorname{Tor}_i^A(M, N) = \operatorname{Tor}_i^A(N, M)$. Since tensor product commutes with direct limits, we have

$$\operatorname{Tor}_j^A(\varinjlim N_i, M) \cong \varinjlim \operatorname{Tor}_j^A(N_i, M),\ \forall j \geq 0$$

for any direct system of A-modules $\{N_i\}_{i \in I}$.

Example. Projective modules are flat. Free modules are f.f.. If B and C are rings and $A = B \times C$, then B is a projective module (hence flat) over A but not f.f. over A.

Theorem 1. The following conditions are equivalent:

(i) M is A-flat;

(ii) if $0 \to N' \to N$ is an exact sequence of A-modules, then

$$0 \to N' \otimes M \to N \otimes M$$

is exact;

(iii) $\operatorname{Tor}_1^A(M, A/I) = 0$ for any finitely generated ideal I of A;

(iv) for any finitely generated ideal I of A, the sequence $0 \to I \otimes M \to M$ is exact, in other words we have $I \otimes M \cong IM$;

(v) $M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an injective A-module;

(vi) $\operatorname{Tor}_1^A(M, N) = 0$ for any finite A-module N;

(vii) if $a_i \in A$, $x_i \in M$ $(1 \leq i \leq r)$ and $\sum_i a_i x_i = 0$, then there exist an integer s and elements $b_{ij} \in A$ and $y_j \in M$ $(1 \leq j \leq s)$ such that $\sum_i a_i b_{ij} = 0$ for all j and $x_i = \sum_j b_{ij} y_j$ for all i.

Proof. (i) $\iff$ (ii) $\implies$ (iii) $\implies$ (iv) and (i) $\implies$ (vi) are trivial.

First we prove (ii) $\iff$ (v), let $N' \hookrightarrow N$. Then we have the induced map

$$\begin{aligned}(N \otimes M)^* = \operatorname{Hom}_{\mathbb{Z}}(N \otimes M, \mathbb{Q}/\mathbb{Z}) &\cong \operatorname{Hom}_A(N, M^*) \\ &\longrightarrow \operatorname{Hom}_A(N', M^*) \cong \operatorname{Hom}_{\mathbb{Z}}(N' \otimes M, \mathbb{Q}/\mathbb{Z}) = (N' \otimes M)^*.\end{aligned}$$

One can easily check that $B \hookrightarrow C$ iff $C^* \twoheadrightarrow B^*$ for any A-modules B and C. Therefore M^* is injective is equivalent to

$$(N \otimes M)^* \twoheadrightarrow (N' \otimes M)^* \ \forall N' \hookrightarrow N \iff N' \otimes M \hookrightarrow N \otimes M \ \forall N' \hookrightarrow N,$$

which is (ii).

For (v) $\Longrightarrow$ (iv), let I be an ideal of A. Similar to above, we have

$$M^* \text{ is injective } \implies (A \otimes M)^* \twoheadrightarrow (I \otimes M)^* \implies I \otimes M \hookrightarrow A \otimes M = M.$$

For (iv) $\Longrightarrow$ (v), let I be an ideal of A. Consider the set

$$\Sigma = \{J \subseteq I \mid J \text{ is a finitely generated ideal}\}.$$

It's a direct system, and $I = \varinjlim J$. So we have $I \otimes M = \varinjlim(J \otimes M)$ and

$$0 \longrightarrow I \otimes M \longrightarrow M$$

is exact by the exactness of $0 \longrightarrow J \otimes M \longrightarrow M$.

For (vi) $\Longrightarrow$ (ii), let $\varphi : N' \to N$ be an A-linear monomorphism. Consider the set

$$\Sigma = \{N'' \subseteq \operatorname{coker} \varphi \mid N'' \text{ is a finite } A\text{-module}\}.$$

It's a direct system, and $\operatorname{coker} \varphi = \varinjlim N''$. So we have

$$\operatorname{Tor}_1^A(\operatorname{coker} \varphi, M) = \operatorname{Tor}_1^A(\varinjlim N'', M) = \varinjlim \operatorname{Tor}_1^A(N'', M) = 0,$$

and hence $0 = \operatorname{Tor}_1^A(\operatorname{coker} \varphi, M) \to N' \otimes M \to N \otimes M$ is exact.

For (vii), first suppose that M is flat and $\sum_i a_i x_i = 0$. Consider the exact sequence

$$K \xrightarrow{i} A^r \xrightarrow{\varphi} A,$$

where φ is defined by $\varphi(b_1, \ldots, b_r) = \sum_i a_i b_i$, $K = \ker \varphi$ and i is the inclusion map. Then $K \otimes M \to M^r \xrightarrow{\varphi_M} M$ is exact, where $\varphi_M(t_1, \ldots, t_r) = \sum_i a_i t_i$; therefore $(x_1, \ldots, x_r) = \sum_j \beta_j \otimes y_j$ with $\beta_j \in K$, $y_j \in M$. Writing $\beta_j = (b_{1j}, \ldots, b_{rj})$, we get the wanted result. Next let us prove (vii) $\Longrightarrow$ (iv). Let $a_1, \ldots, a_r \in I$ and $x_1, \ldots, x_r \in M$ such that $\sum_i a_i x_i = 0$. Then by assumption $x_i = \sum_j b_{ij} y_j$, $\sum_i a_i b_{ij} = 0$, hence in $I \otimes M$ we have

$$\sum_i a_i \otimes x_i = \sum_i a_i \otimes \sum_j b_{ij} y_j = \sum_j \left(\sum_i a_i b_{ij} \otimes y_j \right) = 0. \qquad \blacksquare$$

(3.B) (Transitivity) Let $\varphi : A \to B$ be a homomorphism of rings and suppose that φ makes B a flat A-module. (In this case we shall say that φ is a flat homomorphism.) Then a flat B-module N is also flat over A.

Proof. Let $M_\bullet$ be a sequence of A-modules. Then

$$M_\bullet \otimes_A N = M_\bullet \otimes_A (B \otimes_B N) = (M_\bullet \otimes_A B) \otimes_B N.$$

Thus, $M_\bullet$ is exact $\implies M_\bullet \otimes_A B$ is exact $\implies M_\bullet \otimes_A N$ is exact. ∎

(3.C) (Change of base) Let $\varphi : A \to B$ be any homomorphism of rings and let M be a flat A-module. Then $M_{(B)} = M \otimes_A B$ is a flat B-module.

Proof. Let $N_\bullet$ be a sequence of B-modules. Then

$$N_\bullet \otimes_B B \otimes_A M = N_\bullet \otimes_A M,$$

which is exact if $N_\bullet$ is exact. ∎

(3.D) (Localization) Let A be a ring, and S a multiplicative subset of A. Then $S^{-1}A$ is flat over A.

Proof. Let M be an A-module and N a submodule. We have $M \otimes S^{-1}A = S^{-1}M$ and $N \otimes S^{-1}A = S^{-1}N$. A typical element of $S^{-1}N$ is of the form x/s, $x \in N$, $s \in S$; if $x/s = 0$ in $S^{-1}M$, this means that there exists $s' \in S$ with $s'x = 0$ in M, which is equivalent to saying that $s'x = 0$ in N, hence $x/s = 0$ in $S^{-1}N$. Thus $0 \to S^{-1}N \to S^{-1}M$ is exact. ∎

(3.E) Let $\varphi : A \to B$ be a flat homomorphism of rings, and let M and N be A-modules. Then

$$\operatorname{Tor}_i^A(M, N) \otimes_A B = \operatorname{Tor}_i^B(M_{(B)}, N_{(B)}).$$

If A is noetherian and M is finite over A, we also have

$$\operatorname{Ext}_A^i(M, N) \otimes_A B = \operatorname{Ext}_B^i(M_{(B)}, N_{(B)}).$$

Proof. Let $P_\bullet \to M \to 0$ be a projective resolution of the A-module M. Then, since B is flat, the sequence $P_{\bullet(B)} \to M_{(B)} \to 0$ is a projective resolution of $M_{(B)}$. We have therefore

$$\begin{aligned}\operatorname{Tor}_i^A(M,N) \otimes_A B = h_i(P_\bullet \otimes_A N) \otimes_A B &= h_i(P_\bullet \otimes_A N \otimes_A B)\\ &= h_i(P_{\bullet(B)} \otimes_B N_{(B)}) = \operatorname{Tor}_i^B(M_{(B)}, N_{(B)}),\end{aligned}$$

since the exact functor $- \otimes_A B$ commutes with taking homology. If A is noetherian and M is finite over A, we can assume that the P_i's are finite free A-modules. So

$$\begin{aligned}\operatorname{Ext}_A^i(M,N) \otimes_A B = h_i(\operatorname{Hom}_A(P_\bullet, N)) \otimes_A B &= h_i(\operatorname{Hom}_B(P_\bullet, N) \otimes_A B)\\ &= h_i(\operatorname{Hom}_B(P_\bullet \otimes_A B, N \otimes_A B)) = \operatorname{Ext}_B^i(M_{(B)}, N_{(B)}).\end{aligned}$$ ∎

In particular, for $\mathfrak{p} \in \operatorname{Spec} A$, we have

$$\operatorname{Tor}_i^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{Tor}_i^A(M,N)_\mathfrak{p}, \quad \operatorname{Ext}_{A_\mathfrak{p}}^i(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{Ext}_A^i(M,N)_\mathfrak{p},$$

the latter being valid for A noetherian and M finite.

(3.F)

Proposition. Let A be a ring and M an A-module. Then an A-regular element $x \in A$ is also M-regular iff $\operatorname{Tor}_1^A(A/(x), M) = 0$.

Proof. Consider the exact sequence

$$0 \longrightarrow A \xrightarrow{[x]} A \longrightarrow A/(x) \longrightarrow 0.$$

Then we get the exact sequence

$$0 = \operatorname{Tor}_1^A(A,M) \longrightarrow \operatorname{Tor}_1^A(A/(x), M) \longrightarrow M \xrightarrow{[x]} M,$$

so x is M-regular iff $M \xrightarrow{[x]} M$ is injective iff $\operatorname{Tor}_1^A(A/(x), M) = 0$. ∎

Corollary. Let A be a ring and M a flat A-module. Then an A regular element $x \in A$ is also M-regular.

(3.G)

Proposition. Let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finite over A. Then

$$M \text{ is free } \iff M \text{ is projective } \iff M \text{ is flat }.$$

Proof. We have only to prove that if M is flat then it is free. We prove that any minimal basis of M (cf.(1.N)) is a basis of M. For that purpose it suffices to prove that, if $x_1, \dots, x_n \in M$ are such that their images $\overline{x_1}, \dots, \overline{x_n}$ in $M/\mathfrak{m}M = M \otimes_A k$ are linearly independent over k, then they are linearly independent over A. We use induction on n. When $n = 1$, let $ax = 0$. Then by Theorem 1 there exist $y_1, \dots, y_r \in M$ and $b_1, \dots, b_r \in A$ such that $ab_i = 0$ for all i and such that $x = \sum_i b_i y_i$. Since $\overline{x} \neq 0$ in $M/\mathfrak{m}M$, not all b_i are in $\mathfrak{m}$. Suppose $b_1 \notin \mathfrak{m}$. Then b_1 is a unit in A and $ab_1 = 0$, hence $a = 0$.

Suppose $n > 1$ and $\sum_i a_i x_i = 0$. Then there exist $y_1, \dots, y_r \in M$ and $b_{ij} \in A$ such that $x_i = \sum_j b_{ij} y_j$ and $\sum_i a_i b_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we have $b_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum_i a_i b_{ij} = 0$ and b_{nj} is a unit, we have

$$a_n = \sum_{i=1}^{n-1} c_i a_i \quad (c_i = -b_{nj}^{-1} b_{ij}).$$

Then

$$0 = \sum_i a_i x_i = \sum_{i=1}^{n-1} a_i (x_i + x_i x_n).$$

Since the elements $\overline{x_i} + \overline{c_i x_n}$, $1 \leq i \leq n-1$, are linearly independent over k, by the induction hypothesis we get $a_1 = \cdots = a_{n-1} = 0$, and $a_n = \sum_{i=1}^{n-1} c_i a_i = 0$. ∎

(3.H) Let $A \to B$ be a flat homomorphism of rings, and let I_1 and I_2 be ideals of A. Then

(1) $(I_1 \cap I_2)B = I_1 B \cap I_2 B$,

(2) $(I_1 : I_2)B = I_1B : I_2B$ if I_2 is finitely generated.

Proof. (1) Consider the exact sequence of A-modules

$$0 \longrightarrow I_1 \cap I_2 \longrightarrow A \longrightarrow A/I_1 \oplus A/I_2.$$

Tensoring it with B, we get an exact sequence

$$0 \longrightarrow (I_1 \cap I_2) \otimes_A B = (I_1 \cap I_2)B \longrightarrow B \longrightarrow B/I_1B \oplus B/I_2B.$$

This means $(I_1 \cap I_2)B = I_1B \cap I_2B$.

(2) When I_2 is a principal ideal aA, we use the exact sequence

$$0 \longrightarrow (I_1 : aA) \xrightarrow{i} A \xrightarrow{q\circ[a]} A/I_1,$$

where q is the quotient map. Tensoring it with B we get

$$0 \longrightarrow (I_1 : aA)B \longrightarrow B \longrightarrow B/I_1B.$$

This means $(I_1 : aA)B = (I_1B : aB)$. In the general case, if $I_2 = a_1A + \cdots + a_nA$, we have $(I_1 : I_2) = \bigcap_i (I_1 : a_i)$ so that by (1)

$$(I_1 : I_2)B = \bigcap_i (I_1 : a_iA)B = \bigcap_i (I_1B : a_iB) = (I_1B : I_2B). \qquad \blacksquare$$

(3.I)

Example 1. Let $A = k[x, y]$ be a polynomial ring over a field k, and put $B = A/xA \cong k[y]$. Then B is not flat over A by (3.F). Let $I_1 = (x + y)A$ and $I_2 = yA$. Then $I_1 \cap I_2 = (xy + y^2)A$, $I_1B = I_2B = yB$, $(I_1 \cap I_2)B = y^2B \neq I_1B \cap I_2B$.

Example 2. Let k, x, y be as above and put $z = y/x$, $A = k[x, y]$, $B = k[x, y, z] = k[x, z]$. Let $I_1 = xA$, $I_2 = yA$. Then $I_1 \cap I_2 = xyA$, $(I_1 \cap I_2)B = x^2zB$, $I_1B \cap I_2B = xzB$. Thus B is not flat over A. The map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponds to the projection to (x, y)-plane of the surface $F : xz = y$ in the (x, y, z)-space. Note F contains the whole z-axis and hence does not look 'flat' over the (x, y)-plane.

Example 3. Let $A = k[x,y]$ be as above and $B = k[x,y,z]$ with $z^2 = f(x,y) \in A$. Then $B = A \oplus Az$ as an A-module, so that B is free, hence flat, over A. Geometrically, the surface $z^2 = f(x,y)$ appears indeed to lie rather flatly over the (x,y)-plane. A word of caution: such intuitive pictures are not enough to guarantee flatness.

(3.J) Let $A \to B$ be a homomorphism of rings. Then the following conditions are equivalent:

(i) B is flat over A;

(ii) $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ ($\mathfrak{p} = \mathfrak{q} \cap A$) for all $\mathfrak{q} \in \mathrm{Spec}(B)$;

(iii) $B_{\mathfrak{n}}$ is flat over $A_{\mathfrak{m}}$ ($\mathfrak{m} = \mathfrak{n} \cap A$) for all $\mathfrak{n} \in \mathrm{Max}(B)$.

Proof. (i) $\Longrightarrow$ (ii): the ring $B_{\mathfrak{p}} = B \otimes A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ (base change), and $B_{\mathfrak{q}}$ is a localization of $B_{\mathfrak{p}}$, so that $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ by transitivity. (ii) $\Longrightarrow$ (iii): trivial. (iii) $\Longrightarrow$ (i): it suffice to show that $\mathrm{Tor}_1^A(B,N) = 0$ for any A-module N. We use the following

Lemma. Let B be an A-algebra, $\mathfrak{q}$ a prime ideal of B, $\mathfrak{p} = \mathfrak{q} \cap A$ and N and A-module. Then

$$\mathrm{Tor}_i^A(B,N)_{\mathfrak{q}} = \mathrm{Tor}_i^{A_{\mathfrak{p}}}(B_{\mathfrak{q}}, N_{\mathfrak{p}}).$$

Proof. Let $P_\bullet \to N \to 0$ be a free resolution of the A-module N. We have $P_\bullet \otimes A_{\mathfrak{p}}$ is a free resolution of the $A_{\mathfrak{p}}$-module $N_{\mathfrak{p}}$ and

$$\begin{aligned} \mathrm{Tor}_i^A(B,N) \otimes_B B_{\mathfrak{q}} &= h_i(P_\bullet \otimes_A B \otimes_B B_{\mathfrak{q}}) = h_i(P_\bullet \otimes_A B_{\mathfrak{q}}) \\ &= h_i((P_\bullet \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}) = \mathrm{Tor}_i^{A_{\mathfrak{p}}}(B_{\mathfrak{q}}, N_{\mathfrak{p}}). \end{aligned}$$

Thus the lemma is proved.

Now, if $B_{\mathfrak{n}}$ is flat over $A_{\mathfrak{m}}$ for all $\mathfrak{n} \in \mathrm{Max}(B)$, then $\mathrm{Tor}_1^A(B,N)_{\mathfrak{n}} = 0$ for all $\mathfrak{n} \in \mathrm{Max}(B)$ by the lemma, therefore $\mathrm{Tor}_1^A(B,N) = 0$ by (1.H) as wanted. ■

4 Faithful Flatness

(4.A)

Theorem 2. Let A be a ring and M an A-module. The following are equivalent:

(i) M is faithfully flat over A;

(ii) M is flat over A, and for any A-module $N \neq 0$ we have $N \otimes_A M \neq 0$;

(iii) M is flat over A, and for any maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}M \neq M$.

Proof. (i) $\Longrightarrow$ (ii): suppose $N \otimes_A M = 0$. Let us consider the sequence $0 \to N \to 0$. As $0 \to N \otimes_A M \to 0$ is exact, so is $0 \to N \to 0$. Therefore $N = 0$.

(ii) $\Longrightarrow$ (iii): since $A/\mathfrak{m} \neq 0$, we have $(A/\mathfrak{m}) \otimes_A M = M/\mathfrak{m}M \neq 0$ by hypothesis.

(iii) $\Longrightarrow$ (ii): take an element $x \in N$, $x \neq 0$. The submodule Ax is a homomorphic image of A as A-module, hence $Ax \cong A/I$ for some ideal $I \neq A$. Let $\mathfrak{m}$ be a maximal ideal of A containing I. Then $M \supset \mathfrak{m}M \supseteq IM$, therefore $(A/I) \otimes_A M = M/IM \neq 0$. By flatness $0 \to (A/I) \otimes_A M \to N \otimes_A M$ is exact, hence $N \otimes_A M \neq 0$.

(ii) $\Longrightarrow$ (i): let $N' \xrightarrow{\varphi} N \xrightarrow{\psi} N''$ be a sequence of A-modules, and suppose that

$$N' \otimes_A M \xrightarrow{\varphi_M} N \otimes_A M \xrightarrow{\psi_M} N'' \otimes_A M$$

is exact. As M is flat, the exact functor $- \otimes_A M$ transforms kernel into kernel and image into image. Thus $\mathrm{Im}\,(\psi_M \circ \varphi_M) = 0$, and by assumption we get $\mathrm{Im}\,(\psi \circ \varphi) = 0$, i.e. $\psi \circ \varphi = 0$. Hence the sequence is a complex, and if H denotes its homology at N, we have $H \otimes_A M = \ker \varphi / \mathrm{Im}\,\psi = 0$. Using again assumption (ii) we obtain $H = 0$, which implies that the sequence is exact. ∎

Corollary. Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be local rings, and $\psi : A \to B$ a local homomorphism. Let $M(\neq 0)$ be a finite B-module. Then

$$M \text{ is flat over } A \iff M \text{ is f.f. over } A.$$

In particular, B is flat over A iff it is f.f over A.

Proof. Since ψ is local, $\mathfrak{m}M \subseteq \mathfrak{n}M$, and $\mathfrak{n}M \neq M$ by Nakayama's lemma, hence the assertion follows from the theorem. ■

(4.B) Just as flatness, faithful flatness is transitive (B is f.f. A-algebra and M is f.f B-module $\implies$ M is f.f over A) and is preserved by change of base (M is f.f. A-module and B is any A-algebra $\implies$ $M \otimes_A B$ is f.f. B-module).

Faithful flatness has moreover, the following descent property: if B is an A-algebra and if M is a f.f. B-module which is also f.f. over A, then B is f.f. over A.

Proof. Let $N' \to N \to N''$ be a sequence of A-modules, and suppose that

$$N' \otimes_A B \longrightarrow N \otimes_A B \longrightarrow N'' \otimes_A B$$

is exact. Then the flatness of M over B implies that

$$N' \otimes_A M \longrightarrow N \otimes_A M \longrightarrow N'' \otimes_A M$$

is also exact, then $N' \to N \to N''$ is exact by the faithful flatness of M over A. ■

(4.C) Faithful flatness is particularly important in the case of a ring extension. Let $\psi : A \to B$ be a f.f. homomorphism of rings. Then:

(1) For any A-module N, the map $N \to N \otimes_A B$ defined by $x \mapsto x \otimes 1$ is injective. In particular ψ is injective and A can be viewed as a subring of B.

(2) For any ideal I of A, we have $IB \cap A = I$.

(3) $\mathrm{Spec}(\psi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof. (1) Let $0 \neq x \in N$. Then $0 \neq Ax \subseteq N$, hence $Ax \otimes_A B \subseteq N \otimes_A B$ by flatness of B. Then $Ax \otimes_A B = (x \otimes 1)B$, therefore $x \otimes 1 \neq 0$ by Theorem 2.

(2) By change of base, $B \otimes_A (A/I) = B/IB$ is f.f. over A/I. Now the assertion follows from (1).

(3) Let $\mathfrak{p} \in \operatorname{Spec}(A)$. The ring $B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is f.f. over $A_\mathfrak{p}$, hence $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$ by Theorem 2. Take a maximal ideal $\mathfrak{m}$ of $B_\mathfrak{p}$ which contains $\mathfrak{p}B_\mathfrak{p}$. Then $\mathfrak{m} \cap A_\mathfrak{p} \supseteq \mathfrak{p}A_\mathfrak{p}$, therefore $\mathfrak{m} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ because $\mathfrak{p}A_\mathfrak{p}$ is maximal. Putting $\mathfrak{q} = \mathfrak{m} \cap B$, we get

$$\mathfrak{q} \cap A = (\mathfrak{m} \cap B) \cap A = \mathfrak{m} \cap A = (\mathfrak{m} \cap A_\mathfrak{p}) \cap A = \mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}. \qquad \blacksquare$$

(4.D)

Theorem 3. Let $\psi : A \to B$ be a homomorphism of rings. The following conditions are equivalent.

(i) ψ is f.f.;

(ii) ψ is flat, and $\operatorname{Spec}(\psi)$ is surjective;

(iii) ψ is flat, and for any $\mathfrak{m} \in \operatorname{Max}(A)$ there exists $\mathfrak{n} \in \operatorname{Max}(B)$ lying over $\mathfrak{m}$.

Proof. (i) $\Longrightarrow$ (ii) is already proved.

(ii) $\Longrightarrow$ (iii). By assumption there exists $\mathfrak{q} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{m}$. If $\mathfrak{n}$ is any maximal ideal of B containing $\mathfrak{q}$, we have $\mathfrak{n} \cap A = \mathfrak{m}$ as $\mathfrak{m}$ is maximal.

(iii) $\Longrightarrow$ (i). The existence of $\mathfrak{n}$ implies $\mathfrak{n}B \neq B$. Therefore B is f.f. over A by Theorem 2. $\blacksquare$

Remark. In algebraic geometry one says that a morphism $f : X \to Y$ of preschemes is faithfully flat if f is flat (i.e. for all $x \in X$ the associated homomorphisms $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are flat) and surjective.

(4.E) Let A be a ring and B a f.f A-algebra. Let M be an A-module. Then

(1) M is flat (resp. f.f.) over $A \iff M \otimes_A B$ is so over B,

(2) when A is local and M is finite over A we have M is A-free $\iff M \otimes_A B$ is B-free.

Proof. (i). The implication ($\Longrightarrow$) is nothing but a change of base ((3.C) and (4.B)), while ($\Longleftarrow$) follows from the fact that, any sequence $N_\bullet$ of A-modules, we have

$$(N_\bullet \otimes_A M) \otimes_A B = (N_\bullet \otimes_A B) \otimes_B (M \otimes_A B).$$

(ii). ($\Longrightarrow$) is trivial. ($\Longleftarrow$) follows from (i) because, under the hypothesis, freeness of M is equivalent to flatness as we saw in (3.G). ■

(4.F)

Remark. Let V be an algebraic variety over $\mathbb{C}$ and let $x \in V$ (or more generally, let V be an algebraic scheme over $\mathbb{C}$ and let x be a closed point on V). Let V^h denote the complex space obtained from V (for the precise definition see Serre's paper cited below), and let $\mathcal{O}$ and $\mathcal{O}^h$ be the local rings of x on V and on V^h respectively. Locally, one can assume that V is an algebraic subvariety of the affine n-space $\mathbb{A}^n$. Then V is defined by an ideal I of $R = \mathbb{C}[X_1, \ldots, X_n]$, and taking the coordinate system in such way that x is the origin we have $I \subseteq \mathfrak{m} = (X_1, \ldots, X_n)$ and $\mathcal{O} = R_\mathfrak{m}/IR_\mathfrak{m}$. Furthermore, denoting the ring of convergent power series in $X_1, \ldots, X_n$ by $S = \mathbb{C}\{\{X_1, \ldots, X_n\}\}$, we have $\mathcal{O}^h = S/IS$ by definition. Let F denote the formal power series ring $F = \mathbb{C}[[X_1, \ldots, X_n]]$. If has been known long since that $\mathcal{O}$ and $\mathcal{O}^h$ are noetherian local rings. J.-P. Serre observed that the completion $(\mathcal{O}^h)\widehat{\ }$(cf. Chap. 3) of $\mathcal{O}^h$ is the same as the completion $\widehat{\mathcal{O}} = F/IF$ of $\mathcal{O}$, and that $\widehat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$ as well as over $\mathcal{O}^h$. It follows by descent that $\mathcal{O}^h$ is faithfully flat over $\mathcal{O}$, and this fact was made the basis of Serre's famous paper GAGA (Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Vol.6, 1955/56). It was in the appendix to this paper that the notions of flatness and faithful flatness were defined and studied for the first time.

5 Going-up and Going-down

(5.A) Let $\varphi : A \to B$ be a homomorphism of rings. We say that the going-up theorem holds for φ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $\mathfrak{q}' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

Similarly, we say that the going-down theorem holds for φ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $\mathfrak{q}' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

$$\begin{tikzcd}
\mathfrak{q} \arrow[r, hook] \arrow[d, mapsto] & \mathfrak{q}' \arrow[r, hook] \arrow[d, mapsto] & B \\
\mathfrak{p} \arrow[r, hook] & \mathfrak{p}' \arrow[r, hook] & A \arrow[u, "\varphi"']
\end{tikzcd}$$

(5.B) The condition (GD) is equivalent to:

(GD') for any $\mathfrak{p} \in \mathrm{Spec}(A)$, and for any minimal prime over-ideal $\mathfrak{q}$ of $\mathfrak{p}B$, we have $\mathfrak{q} \cap A = \mathfrak{p}$.

Proof. (GD) $\Longrightarrow$ (GD'): let $\mathfrak{p}$ and $\mathfrak{q}$ as in (GD'). Then $\mathfrak{q} \cap A \supseteq \mathfrak{p}$ since $\mathfrak{q} \supseteq \mathfrak{p}B$. If $\mathfrak{q} \cap A \neq \mathfrak{p}$, by (GD) there exists $\mathfrak{q}_1 \in \mathrm{Spec}(B)$ such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$ and $\mathfrak{q} \supset \mathfrak{q}_1$. Then $\mathfrak{q} \supset \mathfrak{q}_1 \supseteq \mathfrak{p}B$, contradicting the minimality of $\mathfrak{q}$.

(GD') $\Longrightarrow$ (GD): let $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}'$ as in (GD). Then $\mathfrak{q}' \supseteq \mathfrak{p}B$ since $\mathfrak{q}' \cap A \supset \mathfrak{p}$. Take a minimal prime $\mathfrak{q} \subseteq \mathfrak{q}'$ over $\mathfrak{p}B$. Then $\mathfrak{q} \cap A = \mathfrak{p}$, and $\mathfrak{q} \neq \mathfrak{q}'$ since $\mathfrak{p} \neq \mathfrak{p}'$. ■

Remark. Put $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $f = \mathrm{Spec}(\varphi) : Y \to X$, and suppose B is noetherian. Then (GD') can be formulated geometrically as follow: let $\mathfrak{p} \in X$, put $X' = V(\mathfrak{p}) \subseteq X$ and let Y' be an arbitrary irreducible component of $f^{-1}(X')$. Then f maps Y' generically onto X' in the sense that the generic point of Y' is mapped to the generic point $\mathfrak{p}$ of X'. (See (6.A) and (6.D) for the definition of irreducible component and of generic point.)

(5.C)

Example. Let $k[x]$ be a polynomial ring over a field k, and put $x_1 = x(x-1)$, $x_2 = x^2(x-1)$. Then $k(x) = k(x_1, x_2)$, and the inclusion $k[x_1, x_2] \subseteq k[x]$ induces a birational morphism

$$f : C = \mathrm{Spec}(k[x]) \to C' = \mathrm{Spec}(k[x_1, x_2])$$

where C is the affine line and C' is the affine curve $x_1^3 - x_2^3 + x_1x_2 = 0$. The morphism f maps the points $Q_1 : x = 0$ and $Q_2 : x = 1$ of C to the same point $P = (0, 0)$ of C', which is an ordinary double point of C', and f maps $C - \{Q_1, Q_2\}$ bijectively onto $C - \{P\}$.

Let y be another indeterminate, and put $B = k[x, y]$, $A = k[x_1, x_2, y]$. Then $Y = \mathrm{Spec}(B)$ is a plane and $X = \mathrm{Spec}(A)$ is $C' \times$ line; X is obtained by identifying the line defined by $y = ax$, $a \neq 0$. Let $g : Y \to X$ be the natural morphism. Then $g(L_3) = X'$ is an irreducible curve on X, and

$$g^{-1}(X') = L_3 \cup \{(0, a), (1, 0)\}.$$

Therefore the going down theorem does not hold for $A \subset B$.

(5.D)

Theorem 4. Let $\varphi : A \to B$ be a flat homomorphism of rings. Then the going-down theorem holds for φ.

Proof. Let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ with $\mathfrak{p} \subset \mathfrak{p}'$, and let $\mathfrak{q}' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$. Then $B_{\mathfrak{q}'}$ is flat over $A_{\mathfrak{p}'}$ by (3.J), hence faithfully flat since $A_{\mathfrak{p}'} \to B_{\mathfrak{q}'}$ is local by (4.A). Therefore $\mathrm{Spec}(B_{\mathfrak{q}'}) \to \mathrm{Spec}(A_{\mathfrak{p}'})$ is surjective by (4.C). Let $\mathfrak{q}^*$ be a prime ideal of $B_{\mathfrak{q}'}$ lying over $\mathfrak{p}A_{\mathfrak{p}'}$. Then $\mathfrak{q} = \mathfrak{q}^* \cap B$ is a prime ideal of B lying over $\mathfrak{p}$ and contained in $\mathfrak{q}'$. ■

(5.E)

Theorem 5 (Cohen-Seidenberg)**.** Let B be a ring and A a subring over which B is integral. Then:

(1) The canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

(2) There is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.

(3) The going-up theorem holds for $A \subset B$.

(4) If A is a local ring and $\mathfrak{m}$ is its maximal ideal, then the prime ideals of B lying over $\mathfrak{m}$ are precisely the maximal ideals of B.

Suppose furthermore that A and B are integral domains and that A is integrally closed (in its quotient field ΦA). Then we also have the following.

(5) The going-down theorem holds for $A \subset B$.

(6) If B is the integral closure of A in a normal extension field L of $K = \Phi A$, then any two prime ideals of B lying over the same prime $\mathfrak{p} \in \mathrm{Spec}(A)$ are conjugate to each other by some automorphism of L over K.

Proof. (4) First let $\mathfrak{n} \in \mathrm{Max}(B)$ and put $\mathfrak{m}' = \mathfrak{n} \cap A$. Then $\overline{B} = B/\mathfrak{n}$ is a field which is integral over the subring $\overline{A} = A/\mathfrak{m}'$. Let $0 \neq A \in \overline{A}$. Then $1/x \in \overline{B}$, hen ce

$$\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in \overline{A}.$$

Multiplying by x^{n-1} we get $1/x = -(a_1 + a_2x + \cdots + a_nx^{n-1}) \in \overline{A}$. Therefore $\overline{A}$ is a field, i.e. $\mathfrak{m}' = \mathfrak{n} \cap A$ is the maximal ideal $\mathfrak{m}$ of A. Next, let $\mathfrak{q} \in \mathrm{Spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{m}$. Then $\overline{B} = B/\mathfrak{q}$ is a domain which is integral over the field $\overline{A} = A/\mathfrak{m}$. Let $0 \neq y \in \overline{B}$; let

$$y^n + a_1y^{n-1} + \cdots + a - n = 0 \ (a_i \in \overline{A})$$

be a relation of integral dependence for y, and assume that the degree n is the smallest possible. Then $a_n \neq 0$ (otherwise we could divide hte equation by y to get a relation of degree $n-1$). Then $y^{-1} = -a_n^{-1}(y^{n-1} + a_1y^{n-2} + \cdots + a_{n-1}) \in \overline{B}$, hence $\overline{B}$ is a field and $\mathfrak{q}$ is maximal.

(1) and (2). Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}B$ is integral over $A_{\mathfrak{p}}$ and contains it as a subring. The prime ideals of B lying over $\mathfrak{p}$ correspond to the prime ideals of $B_{\mathfrak{p}}$ lying over $\mathfrak{p}A_{\mathfrak{p}}$, which are the maximal ideals of $B_{\mathfrak{p}}$ by (4).

Since $A_{\mathfrak{p}} \neq 0$, $B_{\mathfrak{p}}$ is not zero and has maximal ideals. Of course there is no inclusion relation between maximal ideals. Thus (1) and (2) are proved.

(3) Let $\mathfrak{p} \subset \mathfrak{p}'$ be in $\operatorname{Spec}(A)$ and $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Then $B/\mathfrak{q}$ contains, and is integral over, $A/\mathfrak{p}$. By (1) there exists a prime $\mathfrak{q}'/\mathfrak{q} \in \operatorname{Spec}(B/\mathfrak{q})$ lying over $\mathfrak{p}'/\mathfrak{p} \in \operatorname{Spec}(A/\mathfrak{p})$. Then $\mathfrak{q}'$ is a prime ideal of B lying over $\mathfrak{p}'$.

(6) Put $G = \operatorname{Aut}(L/K)$. First assume L is finite over K. Then G is finite: $G = \{\sigma_1, \dots, \sigma_n\}$. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be prime ideals of B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$. Put $\sigma_i(\mathfrak{q}) = \mathfrak{q}_i$. (Note that $\sigma_i(B) = B$ so that $\mathfrak{q}_i \in \operatorname{Spec}(B)$.) If $\mathfrak{q}' \neq \mathfrak{q}_i$ for $i = 1, \dots, n$, then $\mathfrak{q}' \not\subseteq \mathfrak{q}_i$ by (2), and there exists an element $x \in \mathfrak{q}'$ which is not in any $\mathfrak{q}_i$ by (1.B). Put

$$y = \left(\prod_i \sigma_i(x)\right)^q, \quad \text{where } q = \begin{cases} 1, & \text{if } \operatorname{char}(K) = 0; \\ p^\nu \text{ with } \nu \text{ sufficiently large }, & \text{if } \operatorname{char}(K) = p. \end{cases}$$

Then $y \in K$, and since A is integrally closed and $y \in B$ we get $y \in A = K \cap B$. But $y \notin \mathfrak{q}$ (for, we have $x \notin \sigma_i^{-1}(\mathfrak{q})$ hence $\sigma_i(x) \notin \mathfrak{q}$) while $y \in \mathfrak{q}' \cap A = \mathfrak{q} \cap A$, contradiction.

When L is infinite over K, let K' be the invariant subfield of G; then L is Galois over K', and K' is purely inseparable over K. If $K' \neq K$, let $p = \operatorname{char}(K)$. It is easy to see that the integral closure B' of A in K' has one and only one prime $\mathfrak{p}'$ which lies over $\mathfrak{p}$, namely

$$\mathfrak{p}' = \{x \in B' \mid \exists\, q = p^\nu \text{ such that } x^q \in \mathfrak{p}\}.$$

Thus we can replace K by K' and $\mathfrak{p}$ by $\mathfrak{p}'$ in this case. Assume, therefore, that L is Galois over K. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be in $\operatorname{Spec}(B)$ and let $\mathfrak{q} \cap A = \mathfrak{q}' \cap A = \mathfrak{p}$. Let L' be any finite Galois extension of K contained in L, and put

$$F(L') = \{\sigma \in G = \operatorname{Aut}(L/K) \mid \sigma(\mathfrak{q} \cap L') = \mathfrak{q}' \cap L'\}.$$

This set is not empty by what we have proved, and is closed in G with respect to the Krull topology. Clearly $F(L') \supseteq F(L'')$ if $L' \subseteq L''$. For any finite number of finite Galois extension L'_i $(1 \leq i \leq n)$ there exists a finite Galois extension L'' containing all L'_i, therefore $\bigcap_i F(L'_i) \supseteq F(L'') \neq \varnothing$. As G is compact this means $\bigcap_{\text{all } L'} F(L') \neq \varnothing$. If σ belongs to this intersection we get $\sigma(\mathfrak{q}) = \mathfrak{q}'$.

$$
\begin{tikzcd}
\mathfrak{r} \arrow[r, hook] \arrow[dddr, mapsto] & \mathfrak{r}'_1 \arrow[drr, hook] \arrow[dddr, mapsto] & & & \\
& \sigma(\mathfrak{r}) \arrow[r, mapsto] \arrow[d, mapsto] & \mathfrak{r}' = \sigma(\mathfrak{r}'_1) \arrow[r, hook] \arrow[d, mapsto] & C \arrow[r, hook] & L \arrow[dd, dash, bend left=60, "\text{Galois}"] \\
& \mathfrak{q} \arrow[r, hook] \arrow[d, mapsto] & \mathfrak{q}' \arrow[r, hook] \arrow[d, mapsto] & B \arrow[r, hook] \arrow[u, hook] & L_1 = \Phi B \arrow[u, hook] \\
& \mathfrak{p} \arrow[r, hook] & \mathfrak{p}' \arrow[r, hook] & A \arrow[r, hook] \arrow[u, hook] & K = \Phi A \arrow[u, hook]
\end{tikzcd}
$$

Diagram in the proof of (5)

(5) Let $L_1 = \Phi B$, $K = \Phi A$, and let L be a normal extension of K containing L_1; let C denote the integral closure of A (hence also of B) in L. Let $\mathfrak{q}' \in \mathrm{Spec}(B)$, $\mathfrak{p}' = \mathfrak{q}' \cap A$, $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{p} \subset \mathfrak{p}'$. Take a prime ideal $\mathfrak{r} \in \mathrm{Spec}(C)$ lying over $\mathfrak{p}$, and using the going-up theorem for $A \subset C$, take $\mathfrak{r}'_1 \in \mathrm{Spec}(C)$ lying over $\mathfrak{p}'$ such that $\mathfrak{r} \subset \mathfrak{r}'_1$. Let $\mathfrak{r}'$ be a prime ideal of C lying over $\mathfrak{q}'$. Then by (6) there exists $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma(\mathfrak{r}'_1) = \mathfrak{r}'$. Put $\mathfrak{q} = \sigma(\mathfrak{r}) \cap B$. Then $\mathfrak{q} \subset \mathfrak{q}'$ and

$$\mathfrak{q} \cap A = \sigma(\mathfrak{r}) \cap A = \mathfrak{r} \cap A = \mathfrak{p}. \qquad \blacksquare$$

Remark. In the example of (5.C), the ring $B = k[x, y]$ is integral over $A = k[x_1, x_2, y]$ since $x^2 - x - x_1 = 0$. Therefore the going-up theorem holds for $A \subset B$ while the going-down does not.

6 Constructible Sets

(6.A) A topological space X is said to be noetherian if the descending chain condition holds for the closed sets in X. The spectrum $\mathrm{Spec}(A)$ of a noetherian ring is noetherian. If a space is covered by a finite number of noetherian subspaces then it is noetherian. Any subspace of a noetherian space is noetherian. A noetherian space is quasi-compact.

A closed set Z in a topological space is irreducible if it is not expressible as the sum of two proper closed subsets. In a noetherian space X any closed set Z is uniquely decomposed into a finite number of irreducible closed subsets:

$$Z = Z_1 \cup \cdots \cup Z_r \text{ such that } Z_i \not\subseteq Z_j \text{ for } i \neq j.$$

This follows easily from the definitions. The Z_i's are called the irreducible components of Z.

(6.B) Let X be a topological space and Z a subset of X. We say Z is locally closed in X if, for any point z of Z, there exists an open neighborhood U of z in X such that $U \cap Z$ is closed in U. It is easy to see that Z is locally closed in X iff it is expressible as the intersection of an open set in X and a closed set in X.

Let X be a noetherian space. We say a subset Z of X is a constructible set in X if Z is a finite union of locally closed sets in X:

$$Z = \bigcup_{i=1}^{m} (U_i \cap F_i), \ U_i \text{ open }, \ F_i \text{ closed.}$$

(When X is not noetherian, the definition of a constructible set is more complicated, cf. EGA 0_{III}.)

If Z and Z' are constructible in X, so are $Z \cup Z'$, $Z \cap Z'$ and $Z - Z'$. This is clear for $Z \cup Z'$. Repeated use of the formula

$$\begin{aligned}(U \cap F) - (U' \cap F') &= U \cap F \cap ((X - U') \cup (X - F')) \\ &= (U \cap (F \cap (X - U'))) \cup ((U \cap (X - F')) \cap F),\end{aligned}$$

shows that $Z - Z'$ is constructible. Taking $Z = X$ we see the complement of a constructible set is constructible. Finally, $Z \cap Z' = X - ((X - Z) \cup (X - Z'))$ is constructible.

We say a subset Z of a noetherian space X is pro-constructible (resp. ind-constructible) if it is the intersection (resp. union) of an arbitrary collection of constructible sets in X.

(6.C)

Proposition. Let X be a noetherian space and Z a subset of X. Then Z is constructible in X iff the following condition is satisfied.

($\spadesuit$) For each irreducible closed set X_0 in X, either $X_0 \cap Z$ is not dense in X_0, or $X_0 \cap Z$ contains a non-empty open set of X_0.

Proof. ($\Longrightarrow$) If Z is constructible we can write

$$X_0 \cap Z = \bigcup_{i=1}^{m} (U_i \cap F_i),$$

where U_i is open in X, F_i is closed and irreducible in X and $U_i \cap F_i$ is not empty for each i. Then $\overline{U_i \cap F_i} = F_i$ since F_i is irreducible, therefore $\overline{X_0 \cap Z} = \bigcup_i F_i$. If $X_0 \cap Z$ is dense in X_0, we have $X_0 = \bigcup_i F_i$ so that some F_i, say F_1, is equal to X_0 since X_0 is irreducible. Then $U_1 \cap X_0 = U_1 \cap F_1$ is a non-empty open set of X_0 contained in $X_0 \cap Z$.

($\Longleftarrow$) Suppose ($\spadesuit$) holds. We prove the constructibility of Z by induction on the smallness of X, using the fact that X is noetherian (noetherian induction). The empty set being constructible, we suppose that $Z \neq \varnothing$ and that any subset Z' of Z which satisfies ($\spadesuit$) and is such that $\overline{Z'} \subset \overline{Z}$ is constructible.

Let $\overline{Z} = F_1 \cup \cdots \cup F_r$ be the decomposition of $\overline{Z}$ into irreducible components. Then $F_1 \cap Z$ is dense in F_1 as one can easily check, whence there exists, by ($\spadesuit$), a non-empty open set of F_1, say U, is contained in $F_1 \cap Z$. Then $F' = F_1 - U$ is a proper closed subset of F_1 such that $F_1 - F' \subseteq Z$. Then, putting $F^* = F' \cup F_2 \cup \cdots \cup F_r$, we have $Z = (F_1 - F') \cup (Z \cap F^*)$. The set $F_1 - F^*$ is locally closed in X. On the other hand $Z \cap F^*$ satisfies the condition ($\spadesuit$) because, if X_0 is irreducible and if $\overline{Z \cap F^* \cap X_0} = X_0$, the closed set F^* must contain X_0 and so $Z \cap F^* \cap X_0 = Z \cap X_0$. Since $\overline{Z \cap F^*} \subseteq F^* \subset \overline{Z}$, the set $Z \cap F^*$ is constructible by the induction hypothesis, Therefore Z is constructible. ∎

(6.D)

Lemma 1. Let A be a ring and F a closed subset of $X = \mathrm{Spec}(A)$. Then F is irreducible iff $F = V(\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$. This $\mathfrak{p}$ is unique and is called the generic point of F.

Proof. Suppose that F is irreducible. Since it is closed it can be written $F = V(I)$ with $I = \bigcap_{\mathfrak{p}\in F} \mathfrak{p}$. If I is not a prime we would have elements a and b of $A - I$ such that $ab \in I$. Then $F \not\subseteq V(a)$, $F \not\subseteq V(b)$ and $H \subseteq V(a) \cup V(b) = V(ab)$, hence $F = (F \cap V(a)) \cup (F \cap V(b))$, which contradicts the irreducibility. The converse is proved by noting $\mathfrak{p} \in V(\mathfrak{p})$. The uniqueness comes from the fact that $\mathfrak{p}$ is the smallest element of $V(\mathfrak{p})$. ■

Lemma 2. Let $\varphi : A \to B$ be a homomorphism of rings. Put $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $f = \operatorname{Spec}(\varphi) : Y \to X$. Then $f(Y)$ is dense in X iff $\ker \varphi \subseteq \mathfrak{N}(A)$. If, in particular, A is reduced, $f(Y)$ is dense in X iff φ is injective.

Proof. The closure $\overline{f(Y)}$ in $\operatorname{Spec}(A)$ is the closed set $V(I)$ defined by the ideal

$$I = \bigcap_{\mathfrak{p}\in Y} \varphi^{-1}(\mathfrak{p}) = \varphi^{-1}\left(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\right),$$

which is equal to $\varphi^{-1}(\mathfrak{N}(B))$ by (1.E). Clearly $\ker \varphi \subseteq I$. Suppose that $f(Y)$ is dense in X. Then $V(I) = X$, whence $I = \mathfrak{N}(A)$ by (1.E). Therefore $\ker \varphi \subseteq \mathfrak{N}(A)$. Conversely, suppose $\ker \varphi \subseteq \mathfrak{N}(A)$. Then it is clear that $I = \varphi^{-1}(\mathfrak{N}(B)) = \mathfrak{N}(A)$, which means $\overline{f(Y)} = V(I) = X$. ■

(6.E)

Theorem 6 (Chevalley)**.** Let A be a noetherian ring and B an A-algebra of finite type. Let $\varphi : A \to B$ be the canonical homomorphism; put $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $f = \operatorname{Spec}(\varphi) : Y \to X$. Then the image $f(Y')$ of a constructible set Y' in Y is constructible in X.

Proof. First we show that (6.C) can be applied to the case when $Y' = Y$. Let X_0 be an irreducible closed set in X. Then $X_0 = V(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec}(A)$. Put $A' = A/\mathfrak{p}$, and $B' = B/\mathfrak{p}B$. Suppose that $X_0 \cap f(Y)$ is dense in X_0. The map $\varphi' : A' \to B'$ induced by φ is then injective by Lemma 2. We want to show $X_0 \cap f(Y)$ contains a non-empty open subset of X_0. By replacing A, B and φ by A', B' and φ' respectively, it is enough to prove the following assertion:

(♣) if A is a noetherian domain, and if B is a ring which contains A and which is finitely generated over A, there exists $0 \neq a \in A$ such that the elementary open set $D(a)$ of $X = \operatorname{Spec}(A)$ is contained in $f(Y)$, where $Y = \operatorname{Spec}(B)$ and $f : Y \to X$ is the canonical map.

Write $B = A[x_1, \ldots, x_n]$, and suppose that $x_1, \ldots, x_r$ are algebraically independent over A while each x_j $(r < j \leq n)$ satisfies algebraic relations over $A[x_1, \ldots, x_r]$. Put $A^* = A[x_1, \ldots, x_r]$, and choose for each $r < j \leq n$ a relation

$$g_{j0}(x)x_j^{d_j} + g_{j1}(x)x_j^{d_j-1} + \cdots = 0,$$

where $g_{jv}(x) \in A^*$, $g_{j0}(x) \neq 0$. Then $\prod_{j=r+1}^{n} g_{j0}(x_1, \ldots, x_r)$ is a non-zero polynomial in $x_1, \ldots, x_r$ with coefficients in A. Let $a \in A$ be any one of the non-zero coefficients of this polynomial.

Claim. The element a satisfies the requirement in (♣).

In fact, suppose $\mathfrak{p} \in \operatorname{Spec}(A)$, $a \notin \mathfrak{p}$, and put $\mathfrak{p}^* = \mathfrak{p}A^* = \mathfrak{p}[x_1, \ldots, x_r]$. Then $\prod g_{j0} \notin \mathfrak{p}^*$, so that $B_{\mathfrak{p}^*}$ is integral over $A^*_{\mathfrak{p}^*}$. Thus there exists a prime $\mathfrak{q}$ of $B_{\mathfrak{p}^*}$ lying over $\mathfrak{p}^* A^*_{\mathfrak{p}^*}$ by Theorem 5. We have

$$\mathfrak{q} \cap A = \mathfrak{q} \cap A^* \cap A = \mathfrak{p}[x_1, \ldots, x_r] \cap A = \mathfrak{p},$$

therefore $\mathfrak{p} = \mathfrak{q} \cap A = (\mathfrak{q} \cap B) \cap A \in f(\operatorname{Spec}(B))$. Thus (♣) is proved.

The general case follows from the special case treated above and (take $A \to B'$) from the following

Lemma. Let B be a noetherian ring and let Y' be a constructible set in $Y = \operatorname{Spec}(B)$. Then there exists a B-algebra of finite type B' such that the image of $\operatorname{Spec}(B')$ in $\operatorname{Spec}(B)$ is exactly Y'.

Proof. First suppose $Y' = U \cap F$, where U is an elementary open set $U = D(b)$, $b \in B$, and F is a closed subset $V(I)$ defined by an ideal I of B. Put $S = \{1, b, b^2, \ldots\}$ and $B' = S^{-1}(B/I)$. Then B' is a B-algebra of finite type generated by $1/\overline{b}$, where $\overline{b}$ is the image of b in B', and the image of $\operatorname{Spec}(B')$ in $\operatorname{Spec}(B)$ is clearly $U \cap F$.

When Y' is an arbitrary constructible set, we can write it as a finite union of locally closed sets $U_i \cap F_i$ $(1 \leq i \leq m)$ with U_i elementary open, because any open set in the noetherian space Y is a finite union of elementary open sets. Choose a B-algebra B'_i of finite type such that $U_i \cap F_i$ is the image of $\mathrm{Spec}(B'_i)$ for each i, and put $B' = B'_1 \times \cdots \times B'_m$. Then we can view $\mathrm{Spec}(B')$ as the disjoint union of $\mathrm{Spec}(B'_i)$'s, so the image of $\mathrm{Spec}(B')$ in Y is Y' as wanted. ■

(6.F)

Proposition 1. Let A be a noetherian ring, $\varphi : A \to B$ a homomorphism of rings, $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $f = \mathrm{Spec}(\varphi) : Y \to X$. Then $f(Y)$ is pro-constructible in X.

Proof. We have $B = \varinjlim B_\lambda$, where the B_λ's are the subalgebras of B which are finitely generated over A. Put $Y_\lambda = \mathrm{Spec}(B_\lambda)$ and let $g_\lambda : Y \to Y_\lambda$ and $f_\lambda Y_\lambda \to X$ denote the canonical maps. Clearly $f(Y) \subseteq \bigcap_\lambda f_\lambda(Y_\lambda)$. Actually the equality holds, for suppose that $\mathfrak{p} \in X - f(Y)$. Then $\mathfrak{p}B_\mathfrak{p} = B_\mathfrak{p} \ni 1$ (if not, then there exists $\mathfrak{m} \in \mathrm{Max}(B_\mathfrak{p})$ such that $\mathfrak{p}B_\mathfrak{p} \subset \mathfrak{m}$, then $\mathfrak{m}$ lies over $\mathfrak{p}$), so that there exist elements $\pi_\alpha \in \mathfrak{p}$, $b_\alpha \in B$ $(1 \leq \alpha \leq m)$ and $s \in A - \mathfrak{p}$ such that $\sum_\alpha \pi_\alpha(b_\alpha/s) = 1$ in $B_\mathfrak{p}$, i.e., $s'(\sum_\alpha \pi_\alpha b_\alpha - s) = 0$ in B for some $s' \in A - \mathfrak{p}$. If B_λ contains $b_1, \ldots, b_m$ we have $1 \in \mathfrak{p}(B_\lambda)_\mathfrak{p}$, therefore $\mathfrak{p} \notin f_\lambda(Y_\lambda)$ for such λ. Thus we have proved $f(Y) = \bigcap_\lambda f_\lambda(Y_\lambda)$. Since each $f_\lambda(Y_\lambda)$ is constructible by Theorem 6, $f(Y)$ is pro-constructible. ■

Remark. [EGA Ch.IV, §1] contains many other results on constructible sets, including generalization to non-noetherian case.

(6.G) Let A be a ring and let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$. We say that $\mathfrak{p}'$ is a specialization of $\mathfrak{p}$ and that $\mathfrak{p}$ is a generalization of $\mathfrak{p}'$ iff $\mathfrak{p} \subseteq \mathfrak{p}'$. If a subset Z of $\mathrm{Spec}(A)$ contains all specializations (resp. generalizations) of its points, we say Z is stable under specialization (resp. generalization). A closed (resp. open) set in $\mathrm{Spec}(A)$ is stable under specialization (resp. generalization).

Lemma. Let A be a noetherian ring and $X = \mathrm{Spec}(A)$. Let Z be a pro-constructible set in X under specialization. Then Z is closed in X.

Proof. Let $Z = \bigcap_{\lambda} E_\lambda$ with E_λ constructible in X. Let W be an irreducible component of $\overline{Z}$ and let x be its generic point. Then $W \cap Z$ is dense in W, hence a fortiori $W \cap E_\lambda$ is dense in W. Therefore $W \cap E_\lambda$ contains a non-empty open set of W by (6.C), so that $x \in E_\lambda$. Thus $x \in \bigcap_{\lambda} E_\lambda = Z$. This means $W \subseteq Z$ by our assumption, and so we obtain $Z = \overline{Z}$. ■

(6.H) Let $\varphi : A \to B$ be a homomorphism of rings, and put $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $f = \mathrm{Spec}(\varphi) : Y \to X$. We say that f is (or: φ is) submersive if f is surjective and if the topology of X is the quotient of that of Y (i.e. a subset X' of X is closed in X iff $f^{-1}(X')$ is closed in Y). We say f is (or: φ is) universally submersive if, for any A-algebra C, the homomorphism $\varphi_C : C \to B \otimes_A C$ is submersive. (Submersiveness and universal submersiveness for morphisms of preschemes are defined in the same way, cf. EGA IV (15.7.8).)

Theorem 7. Let A, B, φ, X, Y and f be as above. Suppose that (1) A is noetherian, (2) f is surjective and (3) the going-down theorem holds for $\varphi : A \to B$. Then φ is submersive.

Remark. The conditions (2) and (3) are satisfied, e.g., in the following cases:

(a) when φ is f.f., or

(b) when φ is injective, assume B is a integral domain over A and A is an integrally closed integral domain.

In the case (a), φ is even universally submersive since faithful flatness is preserved by change of base.

Proof. Let $X' \subseteq X$ be such that $f^{-1}(X')$ is closed. We have to prove X' is closed. Take an ideal J of B such that $f^{-1}(X') = V(J)$. As $X' = f(f^{-1}(X'))$ by (2), application of (6.F) to the composite map $A \to B \to B/J$ shows X' is pro-constructible. Therefore it suffices, by (1) and (6.G), to prove that X' is stable under specialization. For that purpose, let $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(A)$, $\mathfrak{p}_1 \supset \mathfrak{p}_2 \in X'$. Take $\mathfrak{q}_1 \in Y$ lying over $\mathfrak{p}_1$ (by (2)) and $\mathfrak{q}_2 \in Y$ lying over $\mathfrak{p}_2$ such that $\mathfrak{q}_1 \supset \mathfrak{q}_2$ (by (3)). Then $\mathfrak{q}_2$ is in the closed set $f^{-1}(X')$, so $\mathfrak{q}_1$ is also in $f^{-1}(X')$. Thus $\mathfrak{p}_1 = f(\mathfrak{q}_1) \in f(f^{-1}(X')) = X'$, as wanted. ■

(6.I)

Theorem 8. Let A be a noetherian ring and B an A-algebra of finite type. Suppose that the going down theorem holds between A and B. Then the canonical map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open map.

Proof. Let U be an open set in $\mathrm{Spec}(B)$. Then $f(U)$ is a constructible set (Theorem 6). On the other hand the going-down theorem shows that $f(U)$ is stable under generalization. Therefore, applying (6.G) to $\mathrm{Spec}(A) - f(U)$ we see that $f(U)$ is open. ■

(6.J) Let A and B be rings and $\varphi : A \to B$ a homomorphism. Suppose B is noetherian and that the going-up theorem holds for φ. Then $\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map.

Proof. Let Z be a closed set in $\mathrm{Spec}(B)$. Then Z can be decomposed into a finite number of irreducible closed sets: $Z = Z_1 \cup \cdots \cup Z_r$. Then $Z_i = V(\mathfrak{q}_i)$ for some $\mathfrak{q}_i \in \mathrm{Spec}(B)$.

Claim. $\mathrm{Spec}(\varphi)(Z) = \bigcup\limits_i V(\mathfrak{q}_i \cap A)$, which is closed.

Proof. ($\subseteq$) is trivial. For ($\supseteq$), let $\mathfrak{p} \in \bigcup\limits_i V(\mathfrak{q}_i \cap A)$, then $\mathfrak{p} \in V(\mathfrak{q}_i \cap A)$ for some i. By going-up, there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \supset \mathfrak{q}_i$ and $\mathfrak{q} \cap A = \mathfrak{p}$. So

$\mathfrak{p} \in \operatorname{Spec}(\varphi)(Z)$. ■

Chapter 3

Associated Primes

7 Ass(M)

(7.A) Throughout this section let A denote a noetherian ring and M an A-module. We say $\mathfrak{p} \in \operatorname{Spec}(A)$ is an associated prime of M, if one of the following equivalent conditions holds:

(i) there exists an element $x \in M$ with $\operatorname{Ann}(x) = \mathfrak{p}$;

(ii) M contains a submodule isomorphic to $A/\mathfrak{p}$.

The set of the associated primes of M is denoted by $\operatorname{Ass}_A(M)$ or by $\operatorname{Ass}(M)$. It's clear that if $N \subseteq M$ then $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$.

(7.B)

Proposition. Let $\mathfrak{p}$ be a maximal element of the set of ideals

$$\{\operatorname{Ann}(x) \mid x \in M,\ x \neq 0\}.$$

Then $\mathfrak{p} \in \operatorname{Ass}(M)$.

Proof. We have to show that $\mathfrak{p}$ is a prime. Let $\mathfrak{p} = \operatorname{Ann}(x)$, and suppose $ab \in \mathfrak{p}$, $b \notin \mathfrak{p}$. Then $bx \neq 0$ and $abx = 0$. Since $\operatorname{Ann}(bx) \supseteq \operatorname{Ann}(x) = \mathfrak{p}$, we have $\operatorname{Ann}(bx) = \mathfrak{p}$ by the maximality of $\mathfrak{p}$. Thus $a \in \mathfrak{p}$. ∎

Corollary 1. $\mathrm{Ass}(M) = \varnothing \iff M = 0$.

Corollary 2. The set of the zero-divisors for M is the union of the associated primes of M, i.e.,

$$\{a \in A \mid \exists\, x \in M - \{0\},\ ax = 0\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

Proof. ($\subseteq$): let $a \in A$ such that $ax = 0$ for some $x \in M \setminus \{0\}$. Then there exists a maximal element $\mathfrak{p} \supseteq \mathrm{Ann}(x)$, and $\mathfrak{p} \in \mathrm{Ass}(M)$. So $a \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.

($\supseteq$): let $a \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$, then $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Then $a \in \mathrm{Ann}(x) = \mathfrak{p}$ for some $x \in M$. So we see have $ax = 0$ and $x \neq 0$. ■

(7.C)

Lemma. Let S be a multiplicative subset of A, and put $A' = S^{-1}A$, $M' = S^{-1}M$. Then

$$\mathrm{Ass}_A(M') = f(\mathrm{Ass}_{A'}(M')) = \mathrm{Ass}_A(M) \cap \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \varnothing\},$$

where f is the natural map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$.

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_A(M')$, then $\mathfrak{p} = \mathrm{Ann}_A(x/s)$ for some $x \in M$, $s \in S$. Since A is noetherian, $\mathfrak{p}$ is finitely generated, say $\mathfrak{p} = \langle a_1, \ldots, a_r \rangle$. Then

$$\mathfrak{p} = \left\{ a \in A \,\middle|\, \frac{ax}{s} = 0 \text{ in } M' \right\} = \{a \in A \mid \exists\, t \in S,\ tax = 0 \text{ in } M\}.$$

Then there exist $t_1, \ldots, t_r \in S$ such that $t_i a_i x = 0$, then for any $a = \sum_i b_i a_i \in \mathfrak{p}$, $a(t_1 t_2 \ldots t_r x) = 0$. So $\mathfrak{p} = \mathrm{Ann}_A(t_1 t_2 \ldots t_n x) \in \mathrm{Ass}(M)$. If $\mathfrak{p} \cap S \neq \varnothing$, say a is in the intersection. Then $ax = 0$ implies $x = 0$ in M', so $\mathfrak{p} = \mathrm{Ann}(x) = A$, a contradiction. Hence

$$\mathrm{Ass}_A(M') \subseteq \mathrm{Ass}_A(M) \cap \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \varnothing\}.$$

Let $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and $\mathfrak{p} \cap S = \varnothing$, then $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M - \{0\}$. Since $\mathfrak{p} \cap S = \varnothing$, $S^{-1}\mathfrak{p} \in \mathrm{Spec}(A')$ and $f(S^{-1}\mathfrak{p}) = \mathfrak{p}$.

Claim. $S^{-1}\mathfrak{p} = \mathrm{Ann}_{A'}(x/1)$.

Proof. ($\subseteq$) is trivial. For ($\supseteq$), note that

$$\mathrm{Ann}_{A'}\left(\frac{x}{1}\right) = \left\{ \frac{a}{u} \in A' \,\middle|\, \exists\, t \in S,\ tax = 0 \text{ in } M \right\}.$$

If $a/u \in \mathrm{Ann}_{A'}(x/1) - S^{-1}\mathfrak{p}$, then $\exists\, t \in S$ such that $ta \in \mathfrak{p}$, but $t \notin \mathfrak{p}$, a contradiction.

So the claim is proved, and hence

$$\mathrm{Ass}_A(M) \cap \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \varnothing\} \subseteq f(\mathrm{Ass}_{A'}(M')).$$

Let $\mathfrak{p} \in f(\mathrm{Ass}_{A'}(M'))$, then $S^{-1}\mathfrak{p} = \mathrm{Ann}_{A'}(y)$ for some $y \in M'$. Then it's clear that $\mathfrak{p} = \mathrm{Ann}_A(y)$. So $f(\mathrm{Ass}_{A'}(M')) \subseteq \mathrm{Ass}_A(M')$. ■

(7.D)

Theorem 9. Let A be a noetherian ring and M an A-module. Then $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$, and any minimal element of $\mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$.

Proof. If $\mathfrak{p} \in \mathrm{Ass}(M)$ there exists an exact sequence $0 \to A/\mathfrak{p} \to M$, and since $A_\mathfrak{p}$ is flat over A the sequence $0 \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to M_\mathfrak{p}$ is also exact. As $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \neq 0$ we have $M_\mathfrak{p} \neq 0$, i.e. $\mathfrak{p} \in \mathrm{Supp}(M)$. Next let $\mathfrak{p}$ be a minimal element of $\mathrm{Supp}(M)$. By (7.C), $\mathfrak{p} \in \mathrm{Ass}(M)$ iff $\mathfrak{p}A_\mathfrak{p} \in \mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p})$, therefore replacing A and M by $A_\mathfrak{p}$ and $M_\mathfrak{p}$ we can assume that $(A, \mathfrak{p})$ is a local ring, that $M \neq 0$ and that $M_\mathfrak{q} = 0$ for any prime $\mathfrak{q} \subset \mathfrak{p}$. Thus $\mathrm{Supp}(M) = \{\mathfrak{p}\}$. Since $\mathrm{Ass}(M)$ is not empty and is contained in $\mathrm{Supp}(M)$, we must have $\mathfrak{p} \in \mathrm{Ass}(M)$. ■

Corollary. Let I be an ideal. Then the minimal associated primes of the A-module A/I are precisely the minimal prime over-ideals of I.

Proof. Just noting that $\mathfrak{p} \in \mathrm{Supp}(A/I)$ iff $0 \neq (A/I)_\mathfrak{p} = A_p/I_p$ iff $I \subset \mathfrak{p}$. ■

Remark. By the above theorem the minimal associated primes of M are minimal elements of $\mathrm{Supp}(M)$. Associated primes which are not minimal are called embedded primes.

(7.E)

Theorem 10. Let A be a noetherian ring and M a finite A-module, $M \neq 0$. Then there exists a chain of submodules

$$0 = M_0 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec}(A)$ $(1 \leq i \leq n)$.

Proof. Since $M \neq 0$ we can choose $M_1 \subseteq M$ such that $M_1 \cong A/\mathfrak{p}_1$ for some $\mathfrak{p}_1 \in \operatorname{Ass}(M)$. If $M_1 \neq M$ then we apply the same procedure to M/M_1 to find M_2, and so on. Since the ascending chain condition for submodules holds in M, the process must stop in finite steps. ∎

(7.F)

Lemma. If $0 \to M' \to M \to M''$ is an exact sequence of A-modules, then $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$.

Proof. Take $\mathfrak{p} \in \operatorname{Ass}(M)$ and choose a submodule N of M isomorphic to $A/\mathfrak{p}$. If $N \cap M' = 0$ then N is isomorphic to a submodule of $M/M' \subseteq M''$, so that $\mathfrak{p} \in \operatorname{Ass}(M'')$. If $N \cap M' \neq 0$, pick $0 \neq x \in N \cap M'$. Since $N \cong A/\mathfrak{p}$ and since $A/\mathfrak{p}$ is a domain we have $\operatorname{Ann}(x) = \mathfrak{p}$, therefore $\mathfrak{p} \in \operatorname{Ass}(M')$. ∎

(7.G)

Proposition. Let A be a noetherian ring and M a finite A-module. Then $\operatorname{Ass}(M)$ is a finite set.

Proof. Using the notation of Theorem 10, we have

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2/M_1) \cup \cdots \cup \operatorname{Ass}(M_n/M_{n-1})$$

by the lemma. On the other hand we have $\mathrm{Ass}(M_i/M_{i-1}) = \mathrm{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$, therefore $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. ■

8 Primary Decomposition

As in the preceding section, A denotes a noetherian ring and M an A-module.

(8.A)

Definition. An A-module is said to be co-primary if it has only one associated prime. A submodule N of M is said to be a primary submodule of M if M/N is co-primary. If $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, we say N is $\mathfrak{p}$-primary or that N belongs to $\mathfrak{p}$.

(8.B)

Proposition. The following are equivalent:

(i) the module M is co-primary;

(ii) $M \neq 0$, and if $a \in A$ is a zero-divisor for M then a is locally nilpotent on M (by this we mean that, for each $x \in M$, there exists an integer $n > 0$ such that $a^n x = 0$),

Proof. (i) $\Longrightarrow$ (ii). Suppose $\mathrm{Ass}(M) = \{\mathfrak{p}\}$. If $0 \neq x \in M$, then $\mathrm{Ass}(Ax) = \{\mathfrak{p}\}$ and hence $\mathfrak{p}$ is the unique minimal element of $\mathrm{Supp}(Ax) = V(\mathrm{Ann}(x))$ by (7.D). Thus $\mathfrak{p}$ is the radical of $\mathrm{Ann}(x)$, therefore $a \in \mathfrak{p}$ implies $a^n x = 0$ for some $n > 0$.

(ii) $\Longrightarrow$ (i). Put $\mathfrak{p} = \{a \in A \mid a \text{ is locally nilpotent on } M\}$. Clearly this is an ideal. Let $q \in \mathrm{Ass}(M)$. Then there exists an element x of M with $\mathrm{Ann}(x) = \mathfrak{q}$, therefore $\mathfrak{p} \subseteq \mathfrak{q}$ by the definition of $\mathfrak{p}$. Conversely, since $\mathfrak{p}$ coincides with the union of the associated primes by assumption and (7.B), we get $\mathfrak{q} \subseteq \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{q}$ and $\mathrm{Ass}(M) = \{\mathfrak{p}\}$, so that M is co-primary. ■

Remark. When $M = A/\mathfrak{q}$, the condition (ii) reads as follows:

(ii') all zero-divisors of the ring $A/\mathfrak{q}$ are nilpotent.

This is precisely the classical definition of a primary ideal $\mathfrak{q}$, cf. (1.A).

(8.C) Let $\mathfrak{p}$ be a prime of A, and let Q_1 and Q_2 be $\mathfrak{p}$-primary submodules of M. Then the intersection $Q_1 \cap Q_2$ is also $\mathfrak{p}$-primary.

Proof. There is an obvious monomorphism $M/(Q_1 \cap Q_2) \to M/Q_1 \oplus M/Q_2$ and an exact sequence $0 \to M/Q_1 \to M/Q_1 \oplus M/Q_2 \to M/Q_2$. Hence by (7.F),

$$\varnothing \neq \mathrm{Ass}(M/(Q_1 \cap Q_2)) \subseteq \mathrm{Ass}(M/Q_1) \cup \mathrm{Ass}(M/Q_2) = \{\mathfrak{p}\}. \qquad \blacksquare$$

(8.D) Let N be a submodule of M. A primary decomposition of N is an equation $N = Q_1 \cap \cdots \cap Q_r$ with Q_i primary in M. Such a decomposition is said to be irredundant if no Q_i can be omitted and if the associated primes of M/Q_i $(1 \leq i \leq r)$ are all distinct. Clearly any primary decomposition simplified to an irredundant one.

(8.E)

Lemma. If $N = Q_1 \cap \cdots \cap Q_r$ is an irredundant primary decomposition and if Q_i belongs to $\mathfrak{p}_i$, then we have

$$\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\},$$

Proof. There is a natural monomorphism $M/N \to M/Q_1 \oplus \cdots \oplus M/Q_r$, whence

$$\mathrm{Ass}(M/N) \subseteq \bigcup_i \mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$$

by (7.F). Conversely, $(Q_2 \cap \cdots \cap Q_r)/N$ is isomorphic to a non-zero submodule of M/Q_1 so that $\mathrm{Ass}((Q_2 \cap \cdots \cap Q_r)/N) = \{\mathfrak{p}_1\}$, and since $(Q_2 \cap \cdots \cap Q_r)/N \subseteq M/N$ we have $\mathfrak{p}_1 \in \mathrm{Ass}(M/N)$. Similarly for other $\mathfrak{p}_i$'s. $\blacksquare$

(8.F)

Proposition. Let N be a $\mathfrak{p}$-primary submodule of an A-module M, and let $\mathfrak{q}$ be a prime ideal. Put $M' = M_\mathfrak{q}$ and $N' = N_\mathfrak{q}$ and let $\nu : M \to M'$ be the canonical map. Then

(1) $N' = M'$ if $\mathfrak{p} \not\subseteq \mathfrak{q}$.

(2) $N = \nu^{-1}(N')$ if $\mathfrak{p} \subseteq \mathfrak{q}$ (symbolically one may write $N = M \cap N'$).

Proof. (1) We have $M'/N' = (M/N)_\mathfrak{q}$ and

$$\mathrm{Ass}_A(M'/N') = \mathrm{Ass}_A(M/N) \cap \{ \text{ primes contained in } \mathfrak{q} \} = \varnothing$$

by (7.C). Hence $M'/N' = 0$.

(2) Since $\mathrm{Ass}(M/N) = \mathfrak{p}$ and since $\mathfrak{p} \subseteq \mathfrak{q}$, by (7.B), the multiplicative set $A - \mathfrak{q}$ does not contain zero-divisors for M/N. Therefore the natural map $M/N \to (M/N)_\mathfrak{q} = M'/N'$ is injective. ■

Corollary. Let $N = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary decomposition of a submodule N of M, let Q_1 be $\mathfrak{p}_1$-primary and suppose $\mathfrak{p}_1$ is minimal in $\mathrm{Ass}(M/N)$. Then $Q_1 = M \cap N_{\mathfrak{p}_1}$, hence the primary component Q_1 is uniquely determined by N and by $\mathfrak{p}_1$.

Proof. Localize $N = Q_1 \cap \cdots \cap Q_r$ at $\mathfrak{p}_1$. Since $\mathfrak{p}_1$ is minimal in $\mathrm{Ass}(M/N)$, by (1) of the proposition, we have $N_{\mathfrak{p}_1} = (Q_1)_{\mathfrak{p}_1}$. By (2) of the proposition, $Q_1 = M \cap N_{\mathfrak{p}_1}$. ■

Remark. If $\mathfrak{p}_i$ is an embedded prime of M/N then the corresponding primary component Q_i is not necessarily unique.

(8.G)

Theorem 11. Let A be a noetherian ring and M an A-module. Then one can choose a $\mathfrak{p}$-primary submodule $Q(\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{Ass}(M)$ in such way that

$$0 = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} Q(\mathfrak{p}).$$

Proof. Fix an associated prime $\mathfrak{p}$ of M, and consider the set of submodules $\Sigma = \{N \subseteq M \mid \mathfrak{p} \notin \operatorname{Ass}(N)\}$. This set is not empty since 0 is in it, and if $C = \{N_\lambda\}_\lambda$ is chain in Σ then $\bigcup_\lambda N_\lambda$ is an element of Σ (because $\operatorname{Ass} \bigcup_\lambda N_\lambda = \bigcup_\lambda \operatorname{Ass}(N_\lambda)$ by the definition of Ass). Therefore Σ has maximal elements by Zorn's lemma; choose one of them and call it $Q = Q(\mathfrak{p})$. Since $\mathfrak{p}$ is associated to M and not to Q we have $M \neq Q$.

On the other hand, if M/Q had an associated prime $\mathfrak{q}$ other than $\mathfrak{p}$, then M/Q would contain a submodule $Q'/Q \cong A/\mathfrak{q}$ and Q' would belong to Σ contradicting the maximality of Q. Thus $Q = Q(\mathfrak{p})$ is a $\mathfrak{p}$-primary submodule of M. As

$$\operatorname{Ass}\left(\bigcap_{\mathfrak{p}\in\operatorname{Ass}(M)} Q(\mathfrak{p})\right) = \bigcap_{\mathfrak{p}\in\operatorname{Ass}(M)} \operatorname{Ass}(Q(\mathfrak{p})) = \varnothing$$

we have $\bigcap_{\mathfrak{p}\in\operatorname{Ass}(M)} Q(\mathfrak{p}) = 0$. ■

Corollary. If M is finitely generated then any submodule N of M has a primary decomposition.

Proof. Apply the theorem to M/N and notice that $\operatorname{Ass}(M/N)$ is finite. ■

(8.H) Let $\mathfrak{p}$ be a prime ideal of a noetherian ring A, and let $n > 0$ be an integer. Then $\mathfrak{p}$ is the unique minimal prime over-ideal of $\mathfrak{p}^n$, therefore the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$ is uniquely determined; this is called the n-th symbolic power of $\mathfrak{p}$ and is denoted by $\mathfrak{p}^{(n)}$. Thus $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_\mathfrak{p} \cap A$. It can happen that $\mathfrak{p}^n \neq \mathfrak{p}^{(n)}$. Example: let k be a field and $B = k[x, y]$ the polynomial ring in the indeterminates x and y. Put $A = k[x, xy, y^2, y^3]$ and $\mathfrak{p} = yB \cap A = (xy, y^2, y^3)$. Then $\mathfrak{p}^2 = (x^2y^2, xy^3, y^4, y^5)$. Since $y = xy/x \in A_\mathfrak{p}$, we have $B = k[x, y] \subseteq A_\mathfrak{p}$ and hence $A_\mathfrak{p} = B_{yB}$. Thus

$$\mathfrak{p}^{(2)} = y^2 B_{yB} \cap A = y^2 B \cap A = (y^2, y^3) \neq \mathfrak{p}^2.$$

An irredundant primary decomposition of $\mathfrak{p}^2$ is given by

$$\mathfrak{p}^2 = (y^2, y^3) \cap (x^2, xy^3, y^4, y^5).$$

9 Homomorphisms and Ass

(9.A)

Proposition. Let $\varphi : A \to B$ be a homomorphism of noetherian rings and M a B-module. We can view M as an A-module by φ. Then

$$\mathrm{Ass}_A(M) = \mathrm{Spec}(\varphi)(\mathrm{Ass}_B(M)).$$

Proof. Let $\mathfrak{q} \in \mathrm{Ass}_B(M)$. Then there exists an element x of M such that $\mathrm{Ann}_B(x) = \mathfrak{q}$. Since $\mathrm{Ann}_A(x) = \mathrm{Ann}_B(x) \cap A = \mathfrak{q} \cap A$ we have $\mathfrak{q} \cap A \in \mathrm{Ass}_A(M)$. Conversely, let $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and take en element $x \in M$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$. Put $\mathrm{Ann}_B(x) = I$, let $I = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary decomposition of the ideal I and let Q_i be $\mathfrak{q}_i$-primary. Since $M \supseteq Bx \cong B/I$ the set $\mathrm{Ass}_B(M)$ contains $\mathrm{Ass}_B(B/I) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. We will prove $\mathfrak{q}_i \cap A = \mathfrak{p}$ for some i. Since $I \cap A = \mathfrak{p}$ we have $\mathfrak{q}_i \cap A \supseteq \mathfrak{p}$ for all i. Suppose $\mathfrak{q}_i \cap A \neq \mathfrak{p}$ for all i. Then there exists $a_i \in \mathfrak{q}_i \cap A$ such that $a_i \notin \mathfrak{p}$, for each i. Then $a_i^m \in Q_i$ for all i if m is sufficiently large, hence $a = \prod_i a_i^m \in I \cap A = \mathfrak{p}$, contradiction. Thus $\mathfrak{q}_i \cap A = \mathfrak{p}$ for some i and $\mathfrak{p} \in \mathrm{Spec}(\varphi)(\mathrm{Ass}_B(M))$. ■

(9.B)

Theorem 12 (Bourbaki). Let $\varphi : A \to B$ be a homomorphism of noetherian rings, E an A-module and F a B-module. Suppose F is flat as an A-module. Then:

(1) for any $\mathfrak{p} \in \mathrm{Spec}(A)$,

$$\mathrm{Spec}(\varphi)(\mathrm{Ass}_B(F/\mathfrak{p}F)) = \mathrm{Ass}_A(F/\mathfrak{p}F) = \begin{cases} \{\mathfrak{p}\}, & \text{if } F/\mathfrak{p}F \neq 0, \\ \varnothing, & \text{if } F/\mathfrak{p}F = 0. \end{cases}$$

(2) $\mathrm{Ass}_B(E \otimes_A F) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(E)} \mathrm{Ass}_B(F/\mathfrak{p}F)$.

Corollary. Let A and B be as above and suppose B is A-flat. Then

$$\mathrm{Ass}_B(B) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(A)} \mathrm{Ass}_B(B/\mathfrak{p}B),$$

and $\mathrm{Spec}(\varphi)(\mathrm{Ass}_B(B)) = \{\mathfrak{p} \in \mathrm{Ass}(A) \mid \mathfrak{p}B \neq B\}$. We have $\mathrm{Spec}(\varphi)(\mathrm{Ass}_B(B)) = \mathrm{Ass}(A)$ if B is faithfully flat over A by (4.A).

Proof. (1) The module $F/\mathfrak{p}F$ is flat over $A/\mathfrak{p}$ (base change), and $A/\mathfrak{p}$ is a domain, therefore $F/\mathfrak{p}F$ is torsion-free as an $A/\mathfrak{p}$-module by (3.F). The assertion follows from this.

(2) The inclusion ($\supseteq$) is immediate: if $\mathfrak{p} \in \mathrm{Ass}(E)$ then E contains a submodule isomorphic to $A/\mathfrak{p}$, whence $E\otimes F$ contains a submodule isomorphic to $(A/\mathfrak{p})\otimes_A F = F/\mathfrak{p}F$ by the flatness of F. Therefore $\mathrm{Ass}_B(F/\mathfrak{p}F) \subseteq \mathrm{Ass}_B(E \otimes F)$. To prove the other inclusion ($\subseteq$) is more difficult.

Step 1. Suppose E is finitely generated and coprimary with $\mathrm{Ass}(E) = \{\mathfrak{p}\}$. Then any associated prime $\mathfrak{q} \in \mathrm{Ass}_B(E \otimes F)$ lies over $\mathfrak{p}$. In fact, the elements of $\mathfrak{p}$ are locally nilpotent (on E by (8.B), hence) on $E \otimes F$, therefore $\mathfrak{p} \in \mathfrak{q} \cap A$. On the other hand the elements of $A - \mathfrak{p}$ are E-regular by (7.B), hence $E \otimes F$-regular by the flatness of F. Therefore $A-\mathfrak{p}$ does not meet $\mathfrak{q}$, so $\mathfrak{q}\cap A = \mathfrak{p}$. Now, take a chain of submodules

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

such that $E_i/E_{i-1} = A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}(A)$ (which exists by (7.E)). Then

$$0 = E_0 \otimes F \subseteq E_1 \otimes F \subseteq \cdots \subseteq E_r \otimes F = E \otimes F$$

and $E_i \otimes F/E_{i-1} \otimes F \cong F/\mathfrak{p}_iF$ by the flatness of F, so that

$$\mathrm{Ass}_B(E \otimes F) \subseteq \bigcup_i \mathrm{Ass}_B(F/\mathfrak{p}_iF).$$

But if $\mathfrak{q} \in \mathrm{Ass}_B(F/\mathfrak{p}_iF)$ and if $\mathfrak{p}_i \neq \mathfrak{p}$ then by (1), $\mathfrak{q} \cap A = \mathfrak{p}_i \neq \mathfrak{p}$, hence $\mathfrak{q} \notin \mathrm{Ass}_B(E \otimes F)$ by what we have just proved.

Step 2. Suppose E is finitely generated. Let $0 = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary decomposition of 0 in E and let Q_i be $\mathfrak{p}_i$-primary. Then E is isomorphic to a submodule of $E/Q_1 \oplus \cdots \oplus E/Q_r$, and so $E \otimes F$ is isomorphic to a submodule of

$$(E/Q_1) \otimes F \oplus \cdots (E/Q_r) \otimes F.$$

Then

$$\mathrm{Ass}_B(E \otimes F) \subseteq \bigcup_i \mathrm{Ass}_B((E/Q_i) \otimes F) = \bigcup_i \mathrm{Ass}_B(F/\mathfrak{p}_i F)$$

by *Step 1.*

Step 3. General case. Write $E = \bigcup_\lambda E_\lambda$ with finitely generated submodules E_λ. Then it follow from the definition of the associated primes that $\mathrm{Ass}(E) = \bigcup_\lambda \mathrm{Ass}(E_\lambda)$ and

$$\mathrm{Ass}(E \otimes F) = \mathrm{Ass}\left(\bigcup_\lambda E_\lambda \otimes F\right) = \bigcup_\lambda \mathrm{Ass}(E_\lambda \otimes F).$$

Therefore the proof is reduced to the case of finitely generated E. ■

(9.C)

Theorem 13. Let $A \to B$ be a flat morphism of noetherian rings; let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of A and assume that $\mathfrak{p}B \in \mathrm{Spec}(B)$. Then $\mathfrak{q}B$ is $\mathfrak{p}B$-primary.

Proof. Replacing A by $A/\mathfrak{q}$ and B by $B/\mathfrak{q}B$, then $A \to B$ is still flat by base change, so one may assume $\mathfrak{q} = 0$ since

$$\sqrt{(0)} = \mathfrak{p}(B/\mathfrak{q}B) = \mathfrak{p}B/\mathfrak{q}B \text{ in } B/\mathfrak{q}B \implies \sqrt{\mathfrak{q}B} = \mathfrak{p}B \text{ in } B.$$

Then $\mathrm{Ass}(A) = \{\mathfrak{p}\}$, whence

$$\mathrm{Ass}(B) = \mathrm{Ass}_B(B/\mathfrak{p}B) = \{\mathfrak{p}B\}$$

by the preceding corollary. ■

(9.D) We say a homomorphism $\varphi : A \to B$ of noetherian rings is non-degenerate if $\operatorname{Spec}(\varphi)$ maps $\operatorname{Ass}(B)$ into $\operatorname{Ass}(A)$. A flat homomorphism is non-degenerate by the Corollary of Theorem 12.

Proposition. Let $\varphi : A \to B$ and $\psi : B \to C$ be homomorphisms of noetherian rings. Suppose (i) $B \otimes_A C$ is noetherian, (2) φ is flat and (3) ψ is non-degenerate. Then $1_B \otimes \psi : B \to B \otimes_A C$ is also non-degenerate. (In short, the property of being non-degenerate is preserved by flat base change.)

Proof. Since B is flat over A, $B \otimes_A C$ is flat over C by base change. So $\varphi_C : C \to B \otimes_A C$ is non-degenerate. Hence

$$\begin{aligned}\operatorname{Spec}(1_B \otimes \psi)(\operatorname{Ass}(B \otimes_A C)) &= (\operatorname{Spec}(\psi) \circ \operatorname{Spec}(\varphi_C))(\operatorname{Ass}(B \otimes_A C)) \\ &\subseteq \operatorname{Spec}(\psi)(\operatorname{Ass}(C)) \subseteq \operatorname{Ass}(B).\end{aligned}$$ ■

Chapter 4

Graded Rings

10 Graded Rings and Modules

(10.A) A graded ring is a ring A equipped with a direct decomposition of the underlying additive group, $A = \bigoplus_{n \geq 0} A_n$, such that $A_n A_m \subseteq A_{n+m}$. A graded A-module is an A-module M, together with a direct decomposition as a group $M = \bigoplus_{n \in \mathbb{Z}} M_n$ such that $A_n M_m \subseteq M_{n+m}$. Elements of A_n (or M_n) are called homogeneous submodule if $N = \bigoplus_n (N \cap M_n)$. It is easy to see that this condition is equivalent to

($\spadesuit$) N is generated over A by homogeneous elements,

and also to

($\clubsuit$) if $x = x_r + x_{r+1} + \cdots + x_s \in N$, $x_i \in M_i$ (for all i), then each x_i is in N.

If N is a graded submodule of M, then M/N is also a graded A-module, in fact

$$M/N = \bigoplus_n (M_n/(N/\cap M_n)).$$

A graded ideal of A is a graded submodule of A as an A-module.

(10.B)

Proposition. Let A be a noetherian graded ring, and M a graded A-module. Then

(1) any associated prime $\mathfrak{p}$ of M is a graded ideal, and there exists a homogeneous element x of M such that $\mathfrak{p} = \mathrm{Ann}(x)$;

(2) one can choose a $\mathfrak{p}$-primary graded submodule $Q(\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{Ass}(M)$ in such way that

$$0 = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} Q(\mathfrak{p}).$$

Proof. (1) Let $\mathfrak{p} \in \mathrm{Ass}(M)$. Then $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M$. Write

$$x = \sum_{i=-r}^{r} x_i,\ x_i \in M_i.$$

Let $f = \sum_{j=0}^{s} f_j \in \mathfrak{p}$, $f_j \in A_j$. We shall prove that all f_j are in $\mathfrak{p}$. We have

$$0 = fx = \sum_{n=-r}^{r+s} \left(\sum_{i+j=n} f_j x_i \right).$$

Hence $\sum_{i+j=n} f_j x_i = 0$. In particular, $f_s x_r = 0$. So induction backward on k, we can prove that

$$f_s^{r+1-k} x_k = -f_s \sum_{\substack{i+j=k+s \\ i>k}} f_s^{r-k} f_j x_i = 0.$$

It follows that $f_s^{2r+1} x_k = 0$ for $-r \leq k \leq r$. Hence $f_s^{2r+1} x = 0$, $f_s^{2r+1} \in \mathfrak{p}$, therefore $f_r \in \mathfrak{p}$. By descending induction we see that all f_i are in $\mathfrak{p}$, so that $\mathfrak{p}$ is a graded ideal. Then $\mathfrak{p} \subset \mathrm{Ann}(x_i)$ for all i, and clearly $\mathfrak{p} = \bigcap_i \mathrm{Ann}(x_i)$. Since $\mathfrak{p}$ is prime this means $\mathfrak{p} = \mathrm{Ann}(x_i)$ for some i.

(2) A slight modification of the proof of (8.G) Theorem 11 proves the assertion. Alternatively, we can derive it from Theorem 11 and from the following

Lemma. Let $\mathfrak{p}$ be a graded ideal and let $Q \subset M$ be a $\mathfrak{p}$-primary submodule. Then the largest graded submodule Q' contained in Q (i.e. the submodule generated by the homogeneous elements in Q) is again $\mathfrak{p}$-primary.

Proof. Let $\mathfrak{p}'$ be an associated prime of M/Q'. Since both $\mathfrak{p}$ and $\mathfrak{p}'$ are graded, $\mathfrak{p}' = \mathfrak{p}$ iff $\mathfrak{p}' \cap H = \mathfrak{p} \cap H$ where H is the set of homogeneous elements of A.

If $a \in \mathfrak{p} \cap H$, then a is locally nilpotent on M/Q': for any $x_m \in M_m$, Q is $\mathfrak{p}$-primary and $a \in \mathfrak{p}$, by (7.B) there exists $k > 0$ such that $a^k x_m \in Q$, therefore $a^k x_m \in Q'$ since it's homogeneous. This means $\mathfrak{p} \cap H \subseteq \mathfrak{p}' \cap H$.

If $a \in H$, $a \notin \mathfrak{p}$, then for $x \in M$ satisfying $ax \in Q'$, $x = \sum_i x_i$, $x_i \in M_i$, we have $ax_i \in Q' \subseteq Q$ for each i, hence $x_i \in Q$ for each i, hence $x \in Q'$. Thus $a \notin \mathfrak{p}'$. This means $\mathfrak{p}' \cap H \subseteq \mathfrak{p} \cap H$. ■

(10.C) In this book we define a filtration of a ring A to be a descending sequence of ideals

$$(\heartsuit)\ A = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$$

satisfying $J_n J_m \subseteq J_{n+m}$. Given a filtration $(\heartsuit)$, we construct a graded ring A' as follows. The underlying additive group is

$$A' = \bigoplus_{n \geq 0} A'_n = \bigoplus_{n \geq 0} J_n / J_{n+1},$$

and if $\overline{x} \in A'_n = J_n/J_{n+1}$ and $\overline{y} \in A'_m = J_m/J_{m+1}$, where $\overline{x}$ and $\overline{y}$ denote the image of x and y in J_n/J_{n+1} and J_m/J_{m+1}, respectively. Then put

$$\overline{x}\,\overline{y} = \overline{xy} \in A'_{n+m} = J_{n+m}/J_{n+m+1}.$$

This multiplication is well-defined and makes A' a graded ring.

When I is an ideal A, its powers define a filtration

$$A = I^0 \supseteq I^1 \supseteq I^2 \supseteq \cdots .$$

This is called the I-adic filtration, and its associated graded ring is denoted by $\mathrm{gr}^I(A)$.

(10.D)

Proposition. If A is a noetherian ring and I an ideal, then $\mathrm{gr}^I(A)$ is noetherian.

Proof. Write $\mathrm{gr}^I(A) = \bigoplus_{n \geq 0} A'_n$, $A'_n = I^n/I^{n+1}$. Then $A'_0 = A/I$ is a noetherian ring. Let $I = a_1 A + \cdots + a_r A$ and let $\overline{a_i}$ denote the image of a_i in I/I^2. Then $\mathrm{gr}^I(A)$ is generated by $\overline{a_1}, \ldots, \overline{a_r}$ over A'_0, therefore is noetherian. ■

(10.E) Let A be an artinian ring, and $B = A[X_1, \ldots, X_m]$ the polynomial ring with its natural grading. Let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated, graded B-module. Put $F_M(n) = \ell(M_n)$ for $n \geq 0$, where $\ell(-) = \mathrm{len}_A(-)$ denotes the length of A-module. The numerical function F_M measures the largeness of M. Note that if the sequence of B-modules $0 \to M' \to M \to M'' \to 0$ is exact, then $F_M(n) = F_{M'}(n) + F_{M''}(n)$.

Proposition. The number $F_M(n)$ is finite for any n.

Proof. If M is generated over B by homogeneous elements $\xi_1, \ldots, \xi_p$ with $\deg(\xi_i) = d_i$ then the map

$$f : \bigoplus_i B(d_i) \to M, \quad f(b_1, \ldots, b_p) = \sum_i b_i \xi_i,$$

where $B(d) = B$ as a module but $B(d)_n = B_{n-d}$, is a degree-preserving epimorphism of B-modules. Note that, since the number of monomials of degree n in $X_1, \ldots, X_m$ is $\binom{n+m-1}{m-1}$, by (2.C) we have

$$\ell(M_n) \leq \sum_i \ell(B_{n-d_i}) = \sum_i \binom{n-d_i+m-1}{m-1} \ell(A) < \infty. \qquad \blacksquare$$

(10.F)

Theorem 14. Let A, B and M be as above. Then there is a polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $F_M(n) = P_M(n)$ for $n \gg 0$ (i.e. for all sufficient large n).

Proof. Let $\mathscr{P}(M)$ denote the assertion for M. We consider the graded submodules N of M and will prove $\mathscr{P}(M/N)$ by noetherian induction on the largeness of N (note that M is noetherian over B). For $N = M$ the assertion is obvious. Supposing $\mathscr{P}(M/N')$ is true for any graded submodule N' of M properly containing N, we prove $\mathscr{P}(M/N)$.

Case 1. If $N = N_1 \cap N_2$ with $N_i \supset N$ $(i = 1, 2)$, then using the exact sequences

$$0 \longrightarrow N_2/N \longrightarrow M/N \longrightarrow M/N_2 \longrightarrow 0$$
$$0 \longrightarrow (N_1 + N_2)/N_1 \longrightarrow M/N_1 \longrightarrow M/(N_1 + N_2) \longrightarrow 0$$

and the isomorphism $(N_1 + N_2)/N_1 \cong N_2/N$ we get

$$F_{M/N} = F_{M/N_2} + F_{N_2/N} = F_{M/N_2} + F_{(N_1+N_2)/N_1} = F_{M/N_1} + F_{M/N_2} - F_{M/(N_1+N_2)}$$

and the assertion $\mathscr{P}(M/N)$ follows from $\mathscr{P}(M/N_1)$, $\mathscr{P}(M/N_2)$ and $\mathscr{P}(M/(N_1 + N_2))$.

Case 2. If N is irreducible (in the sense that it is not the intersection of two larger submodules) then N is a primary submodule of M (otherwise it has a nontrivial primary decomposition); let $\mathrm{Ass}_B(M/N) = \{\mathfrak{p}\}$. Put $I = X_1B + \cdots + X_mB$ and $M' = M/N$. If $I \subseteq \mathfrak{p}$ then we claim that $M'_n = 0$ for large n. In fact, if $\{\xi_1, \ldots, \xi_p\}$ is a set of homogeneous generators of M' over B and if $d = \max(\deg \xi_i)$, then $M'_{d+n} = I^n M'_d$. On the other hand we have $\mathfrak{p}^e M' = 0$ for some $e > 0$, since M/N and $\mathfrak{p}$ are finitely generated and elements in $\mathfrak{p}$ are locally nilpotent by (8.B). Thus $M'_n = 0$ for $n > d + e$, and $\mathscr{P}(M')$ holds with $P'_M = 0$. It remains to show that the case $I \not\subseteq \mathfrak{p}$. We may suppose that $X_1 \notin \mathfrak{p}$. Then the sequence

$$0 \longrightarrow (M/N)_{n-1} \xrightarrow{[X_1]} (M/N)_n \longrightarrow (M/(N + X_1M))_n \longrightarrow 0$$

is exact for $n > 0$. Since $N + X_1M \supset N$ there is a polynomial $P(z) \in \mathbb{Q}[z]$ satisfying $P(M/N + X_1M)$, say

$$P(z) = c_0\binom{x}{0} + \ldots + c_r\binom{x}{r}, \; c_i \in \mathbb{Q}.$$

Thus there is an integer $n_0 > 0$ such that

$$F_{M/N}(n) - F_{M/N}(n-1) = P(n) \quad (n > n_0).$$

Then

$$F_{M/N}(n) = c_0\binom{n+1}{1} + \ldots + c_r\binom{n+1}{r+1} + \text{ const } \text{ for } n > n_0.$$

Which means $F_{M/N}(n)$ is a polynomial in n for $n > n_0$, as wanted. ∎

Remark. $P_M(z)$ of the theorem is called the Hilbert polynomial of the Hilbert characteristic function of M. Write

$$P_M(z) = c_0\binom{x}{0} + \ldots + c_r\binom{x}{r}, \ c_i \in \mathbb{Q}.$$

Since $P_M(n) \in \mathbb{Z}$ for $n \gg 0$, $c_i \in \mathbb{Z}$ for all i (and so $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$).

11 Artin-Rees Theorem

(11.A) Let A be a ring, I an ideal of A and M an A-module. We define a filtration of M to be a descending sequence of submodules

$$(\diamondsuit)\ M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

The filtration is said to be I-admissible if $IM_i \subseteq M_{i+1}$ for all i, I-adic if $M_i = I^iM$, and essentially I-adic if it is I-admissible and if there is an integer i_0 such that $IM_i = M_{i+1}$ for $i > i_0$.

Given a filtration $(\diamondsuit)$, we can define a topology on M by taking

$$\{x + M_n \mid n = 1, 2, \ldots\}$$

as a fundamental system of neighborhoods of x for each $x \in M$. This topology is Hausdorff iff $\bigcap_n M_n = 0$. The topology defined by the I-adic filtration is called the I-adic topology of M. An essentially I-adic filtration defines the I-adic topology on M, since

$$I^iM \subseteq M_i \subseteq I^{i-i_0}M_{i_0} \subseteq I^{i-i_0}M.$$

(11.B)

Lemma. Let A, I and M be as above. Let

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

be an I-admissible filtration such that all M_i are finite A-modules, let X be an indeterminate and put $A' = \sum_n I^nX^n$ and $M' = \sum_n M_nX^n$. Then the filtration is essentially I-adic iff M' is finitely generated over A'.

Proof. A' is a graded subring of $A[X]$ and M' is a subgroup of $M \otimes_A A[X]$ such that $A'M' \subset M'$, hence M' is a graded A'-module. If

$$M' = A'\xi_1 + \cdots + A'\xi_r,\ \xi_i \in M'_{d_i},$$

then $M'_n = (IX)M'_{n-1}$ (hence $M_n = IM_{n-1}$) for $n > \max_i d_i$. Conversely, if $M_n = IM_{n-1}$ for $n > d$, then M' is generated over A' by

$$M_{d-1}X^{d-1} + \cdots + M_1X + M_0,$$

which is, in turn, generated by a finite number of elements over A. ■

(11.C)

Theorem 15 (Artin-Rees)**.** Let A be a noetherian ring, I an ideal, M a finite A-module and N a submodule. Then there exists an integer $r > 0$ such that

$$I^nM \cap N = I^{n-r}(I^rM \cap N) \ \text{ for } n > r.$$

Proof. In other words, the theorem asserts that the filtration $\{I^nM \cap N\}_{n \geq 0}$ of N (induced on N by the I-adic filtration of M) is essentially I-adic. The filtration is I-admissible since

$$I(I^nM \cap N) \subseteq I^{n+1}M \cap IN \subseteq I^{n+1}M \cap N,$$

and $N' = \sum_n (I^nM \cap N)X^n$ is a submodule of the finite A'-module $M' = \sum_n I^nMX^n$, where $A' = \sum_n I^nX^n$. If $I = a_1A + \cdots + a_rA$ then $A' = A[a_1X, \ldots, a_rX]$, so that A' is noetherian. Therefore N' is finite over A'. Thus the assertion follows from the preceding lemma. ■

Remark. It follows that the I-adic topology on M induces the I-adic topology on N. This is not always true if M is infinite over A.

(11.D)

Theorem 16 (Intersection theorem)**.** Let A, I and M be as in the preceding theorem, and put $N = \bigcap_n I^n M$. Then we have $IN = N$.

Proof. For sufficiently large n we get

$$N = I^n M \cap N = I^{n-r}(I^r M \cap N) \subseteq IN \subseteq N. \qquad \blacksquare$$

Corollary 1. If $I \subseteq \operatorname{rad}(A)$ then $\bigcap_n I^n M = 0$ (by Nakayama's lemma). In other words, M is I-adically Hausdorff in that case.

Corollary 2 (Krull)**.** Let A be a noetherian ring and $I = \operatorname{rad}(A)$. Then $\bigcap_n I^n = (0)$ (since I is finitely generated).

Corollary 3 (Krull)**.** Let A be a noetherian domain and let I be any proper ideal then $\bigcap_n I^n = (0)$.

Proof. Putting $N = \bigcap_n I^n$ we have $IN = N$, whence there exists $x \in I$ such that $(1 + x)N = (0)$ by (1.M). Since A is an integral domain and since $1 + x \neq 0$, we have $N = (0)$. $\blacksquare$

(11.E)

Proposition. Let A be a noetherian ring, M a finite A-module, I an ideal, and J an ideal generated by M-regular elements. Then there exists $r > 0$ such that

$$(I^n M : J) = I^{n-r}(I^r M : J) \quad \text{for } n > r.$$

Proof. Let $J = a_1 A + \cdots + a_p A$ where the a_i are M-regular. Let S be the multiplicative subset of A generated by $a_1, \ldots, a_p$, and consider the A-submodules $a_j^{-1} M$ of $S^{-1}M$. Put

$$L = a_1^{-1} M \oplus \cdots \oplus a_p^{-1} M$$

and let Δ_M be the image of the diagonal map $x \mapsto (x, x, \ldots, x)$ from M to L. Then

$M \cong \Delta_A$, and

$$(I^n M : J) = \bigcap_j (I^n M : a_j) = \bigcap_j (I^n a_j^{-1} M \cap M) \cong I^n L \cap \Delta_M,$$

so that the assertion follows from the Artin-Rees theorem applied to L and Δ_M. $\blacksquare$

Chapter 5

Dimension

12 Dimension

(12.A) Let A be a ring, $A \neq 0$. A finite sequence of $n+1$ primes ideals

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

is called a prime chain of length n. If $\mathfrak{p} \in \operatorname{Spec}(A)$, the supremum of the length of the prime chains with $\mathfrak{p} = \mathfrak{p}_0$ is called the height of $\mathfrak{p}$ and denoted by $\operatorname{ht}(\mathfrak{p})$. Thus $\operatorname{ht}(\mathfrak{p}) = 0$ means that $\mathfrak{p}$ is a minimal prime ideal of A.

Let I be a proper ideal of A. We define the height of I to be the minimum of the heights of the prime ideals containing I:

$$\operatorname{ht}(I) = \inf \{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in V(I)\}.$$

The dimension of A is defined to be the supremum of the heights of the prime ideals in A:

$$\dim(A) = \sup \{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(A)\}.$$

It is also called the Krull dimension of A. If $\dim(A)$ is finite then it is equal to the length of the longest prime chains in A. For example, a principal ideal domain has dimension one.

It follows from the definition that

$$\operatorname{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) \quad (\mathfrak{p} \in \operatorname{Spec}(A)),$$

and that, for any ideal I of A,

$$\dim(A/I) + \mathrm{ht}(I) \leq \dim(A).$$

(12.B) Let $M \neq 0$ be an A-module. We define the dimension of M by

$$\dim(M) = \dim(A/\operatorname{Ann}(M)).$$

(When $M = 0$ we put $\dim(M) = -1$.) Under the assumption that A is noetherian and $M \neq 0$ is finite over A, the following conditions are equivalent:

(i) M is an A-module of finite length,

(ii) the ring $A/\operatorname{Ann}(M)$ is artinian,

(iii) $\dim(M) = 0$.

In fact, (ii) $\Longleftrightarrow$ (iii) is just (2.C). For (ii) $\Longrightarrow$ (i), since M is finitely generated, M is a homomorphic image of $(A/\operatorname{Ann}(M))^r$ for some r. Then M is of finite length since the latter one is of finite length by (2.C). Let us prove (i) $\Longrightarrow$ (iii). We suppose $\operatorname{len}_A(M)$ is finite, and replacing A by $A/\operatorname{Ann}(M)$ we assume that $\operatorname{Ann}(M) = (0)$. If $\dim(A) > 0$, take a minimal prime $\mathfrak{p}$ of A which is not maximal. If $M_\mathfrak{p} = 0$, let $M = Ax_1 + \cdots + Ax_r$, then

$$\frac{x_i}{1} = 0 \text{ in } M_\mathfrak{p} \implies \exists\, s_i \in A - \mathfrak{p},\ s_i x_i = 0 \text{ in } M.$$

So $(s_1 \ldots s_r)x_i = 0$ for all i, which means $s_1 \ldots s_r \in \operatorname{Ann}(M) = (0)$, a contradiction. Hence $M_\mathfrak{p} \neq 0$, then $\mathfrak{p}$ is a minimal member of $\operatorname{Supp}(M)$, so that $\mathfrak{p} \in \operatorname{Ass}(M)$. Then M contains a submodule isomorphic to $A/\mathfrak{p}$, and since $\dim(A/\mathfrak{p}) > 0$ we have $A/\mathfrak{p}$ is not Artinian, so $\operatorname{len}_A(A/\mathfrak{p}) = \infty$ by (2.C), contradiction. Therefore $\dim(M) = \dim(A) = 0$.

(12.C) Let A be a noetherian semi-local ring (i.e. $\#\operatorname{Max}(A) < \infty$), and $\mathfrak{J} = \operatorname{rad}(A)$. An ideal I is called an ideal of definition of A if $\mathfrak{J}^\nu \subseteq I \subseteq \mathfrak{J}$ for some $\nu > 0$. This is equivalent to saying that

$$I \subseteq \mathfrak{J}, \quad \text{and } A/I \text{ is artinian.}$$

($\Longrightarrow$): We have A/I is noetherian, then it suffices to show that A/I is of dimension zero by (2.C). Let $\mathfrak{p}/I \in \operatorname{Spec}(A/I)$, then

$$\mathfrak{p} \supseteq I \implies \mathfrak{p} = \sqrt{\mathfrak{p}} \supseteq \sqrt{I} = \mathfrak{J} = \bigcap_{\mathfrak{m}\in\operatorname{Max}(A)} \mathfrak{m}.$$

Which means $\mathfrak{p} = \mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{Max}(A)$, thus $\dim(A/I) = 0$.

($\Longleftarrow$): By the intersection theorem (Theorem 16), $\bigcap_n (\mathfrak{J}/I)^n = 0$. Since A/I is artinian $(\mathfrak{J}/I)^n = 0$ for some n, which means $\mathfrak{J}^n \subseteq I$.

Let I be an ideal of definition and M a finite A-module. Put

$$A^* = \operatorname{gr}^I(A) = \bigoplus_n I^n/I^{n+1} \text{ and } M^* = \operatorname{gr}^I(M) = \bigoplus_n I^n M/I^{m+1}M.$$

Let $I = Ax_1 + \cdots + Ax_r$. Then the graded ring A^* is a homomorphic image of $B = (A/I)[X_1, \ldots, X_r]$, and M^* is a finite, graded A^*-module. Therefore $F_{M^*}(n) = \operatorname{len}_A(I^n M/I^{n+1}M)$ is a polynomial in n, of degree $\leq r-1$ (since F_B is of degree $r-1$), for $n \gg 0$. It follows that the function

$$\chi(M, I; n) := \operatorname{len}_A(M/I^n M) = \sum_{j=0}^{n-1} F_{M^*}(j)$$

is also a polynomial in n, of degree $\leq r$, for $n \gg 0$. The polynomial which represents $\chi(M, I; n)$ for $n \gg 0$ is called the Hilbert polynomial of M with respect to I. If J is another ideal of definition of A, then $J^s \subseteq I$ for some $s > 0$, so that we have $\chi(M, I; n) \leq \chi(M, J; sn)$. Thus, if

$$\begin{aligned} \chi(M, I; n) &= a_d n^d + \cdots + a_0 \text{ and} \\ \chi(M, J; n) &= b_{d'} n^{d'} + \cdots + b_0, \end{aligned}$$

then $d \leq d'$. By symmetry we get $d = d'$. Thus the degree d of the Hilbert polynomial is independent of the choice of I. We denote it by $d(M)$. Remember that, if there exists an ideal of definition of A generated by r elements, then $d(M) \leq r$.

(12.D)

Proposition. Let A be a noetherian semi-local ring, I an ideal of definition of A and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

an exact sequence of finite A-modules. Then $d(M) = \max\{d(M'), d(M'')\}$. Moreover,

$$\chi(M, I; n) - \chi(M', I; n) - \chi(M'', I; n)$$

is a polynomial of degree $< d(M')$ for $n \gg 0$.

Proof. We have

$$M''/IM'' \cong \frac{M/M'}{I^n(M/M')} \cong \frac{M/M'}{(M' + I^n M)/M'} \cong M/(M' + I^n M).$$

So

$$\mathrm{len}_A(M''/I^n M'') = \mathrm{len}_A(M/(M' + I^n M)) \leq \mathrm{len}_A(M/I^n M),$$

we get $d(M'') \leq d(M)$. Furthermore,

$$\begin{aligned}\chi(M, I; n) - \chi(M'', I, n) &= \mathrm{len}_A(M/I^n M) - \mathrm{len}_A(M''/I^n M'') \\ &= \mathrm{len}_A((M' + I^n M)/I^n M) = \mathrm{len}_A(M'/(M' \cap I^n M)),\end{aligned}$$

and there exists $r > 0$ such that

$$M' \cap I^n M = I^{n-r}(M' \cap I^r M) \subseteq I^{n-r} M'$$

for $n > r$ by Artin-Rees. Thus

$$\mathrm{len}_A(M'/I^n M') \geq \mathrm{len}_A(M'/(M' \cap I^n M)) \geq \mathrm{len}_A(M'/I^{n-r} M').$$

This means that $\chi(M, I; n) - \chi(M'', I, n)$ and $\chi(M', I; n)$ have the same degree and the same leading term. ∎

(12.E)

Lemma 1. Let A be a noetherian semi-local ring. Then $d(A) \geq \dim(A)$.

Proof. Induction on $d(A)$. Let $\mathfrak{J} = \mathrm{rad}(A)$. If $d(A) = 0$ then $\mathfrak{J}^\nu = \mathfrak{J}^{\nu+1} = \cdots$ for some $\nu > 0$. By the intersection theorem (Theorem 16), this implies $\mathfrak{J}^\nu = (0)$. Hence $\mathrm{len}_A(A)$ is finite and $\dim(A) = 0$.

Suppose $d(A) > 0$. As the case $\dim(A) = 0$ is trivial, we assume $\dim(A) > 0$. Let

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_e = \mathfrak{p}$$

be a chain of length $e > 0$, and take an element $x \in \mathfrak{p}_{e-1}$ such that $x \notin \mathfrak{p}$. Then $\dim(A/(xA + \mathfrak{p})) \geq e - 1$ since we have the prime chain

$$\mathfrak{p}_0/(xA + \mathfrak{p}) \supset \mathfrak{p}_1/(xA + \mathfrak{p}) \supset \cdots \supset \mathfrak{p}_{e-1}/(xA + \mathfrak{p}).$$

Applying the preceding proposition to the exact sequences

$$0 \longrightarrow A/\mathfrak{p} \xrightarrow{[x]} A/\mathfrak{p} \longrightarrow A/(xA + \mathfrak{p}) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow A/\mathfrak{p} \longrightarrow 0,$$

we have $d(A/(xA + p)) < d(A/\mathfrak{p}) \leq d(A)$. Thus, by induction hypothesis we get

$$e - 1 \leq \dim(A/(xA + \mathfrak{p})) \leq d(A/(xA + \mathfrak{p})) < d(A).$$

Hence $e \leq d(A)$, therefore $\dim(A) \leq d(A)$. ■

Remark. The lemma shows that the dimension of A is finite. When A is an arbitrary noetherian ring and $\mathfrak{p}$ is a prime ideal, we have $\mathrm{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p})$ so that $\mathrm{ht}(\mathfrak{p})$ is finite. (This was first proved by Krull by a different method.) Thus the descending chain condition holds for prime ideals in a noetherian ring. On the other hand, there are noetherian rings with infinite dimension.

(12.F)

Lemma 2. Let A be a noetherian semi-local ring, $M \neq 0$ a finite A-module, and $x \in \mathrm{rad}(A)$. Then

$$d(M) \geq d(M/xM) \geq d(M) - 1.$$

Proof. The first inequality follows from the proposition above. Let I be an ideal of definition containing x. Then

$$\begin{aligned}\chi(M/xM, I; n) &= \mathrm{len}_A(M/(xM + I^n M)) \\ &= \mathrm{len}_A(M/I^n M) - \mathrm{len}_A((xM + I^n M)/I^n M),\end{aligned}$$

$$(xM + I^nM)/I^nM \cong xM/(xM \cap I^nM) \cong M/(I^nM : x)$$

and $I^{n-1}M \subseteq (I^nM : x)$, therefore

$$\begin{aligned}\chi(M/xM, I; n) &\geq \mathrm{len}_A(M/I^nM) - \mathrm{len}_A(M/I^{n-1}M)\\ &= \chi(M, I; n) - \chi(M, I; n-1).\end{aligned}$$

It follows that $d(M/xM) \geq d(M) - 1$. ■

(12.G)

Lemma 3. Let A and M be as above, and let $\dim(M) = r$. Then there exist r elements $x_1, \ldots, x_r$ of $\mathrm{rad}(A)$ such that

$$\mathrm{len}_A(M/(x_1M + \cdots + x_rM)) < \infty.$$

Proof. Let I be an ideal of definition of A. When $r = 0$ we have $\mathrm{len}_A(M) < \infty$ and the assertion holds. Suppose $r > 0$, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be those minimal prime over-ideals of $\mathrm{Ann}(M)$ which satisfy $\dim(A/\mathfrak{p}_i) = r$. Then no maximal ideals are contained in any $\mathfrak{p}_i$, hence $\mathrm{rad}(A) \not\subseteq \mathfrak{p}_i$ $(1 \leq i \leq t)$ since A is a semi-local ring. Thus by (1.B) there exists $x_1 \in \mathrm{rad}(A)$ which is not contained in any $\mathfrak{p}_i$. Then $I := \mathrm{Ann}(M/x_1M) = \mathrm{Ann}(M) + x_1A \not\subseteq \mathfrak{p}_i$ for any i, hence for any prime chain

$$\mathfrak{q}_0/I \supset \mathfrak{q}_1/I \supset \cdots \supset \mathfrak{q}_n/I$$

in A/I, $\mathfrak{q}_n \neq \mathfrak{p}_i$ for any i, so $n \leq \dim(A/\mathfrak{q}_n) < r$. Then $\dim(M/x_iM) \leq r - 1$, and the assertion follows by induction on $\dim(M)$. ■

(12.H)

Theorem 17. Let A be a noetherian semi-local ring, $\mathfrak{J} = \mathrm{rad}(A)$ and $M \neq 0$ a finite A-module. Then $d(M) = \dim(M)$ and is the smallest integer r such that there exist elements $x_1, \ldots, x_r$ of $\mathfrak{J}$ satisfying

$$(\spadesuit) \quad \mathrm{len}_A(M/(x_1M + \cdots + x_rM)) < \infty.$$

Proof. If (♠) holds we have $d(M) \leq r$ by Lemma 2. When r is the smallest possible we have $r \leq \dim(M)$ by Lemma 3. It remains to prove $\dim(M) \leq d(M)$. Take a sequence of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$, $\mathfrak{p}_i \in \operatorname{Spec}(A)$ (which exists by (7.E)). Then $\mathfrak{p}_i \supseteq \operatorname{Ann}(M)$ and $\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ by (7.F). Since $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$ all the minimal over-ideals of $\operatorname{Ann}(M)$ are in $\operatorname{Ass}(M)$ (hence also in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$) by (7.D). Therefore

$$\begin{aligned} d(M) &= \max_i d(A/\mathfrak{p}_i) && \text{by (12.D)} \\ &\geq \max_i \dim(A/\mathfrak{p}_i) && \text{by Lemma 1} \\ &= \dim(A/\operatorname{Ann}(M)) = \dim(M), \end{aligned}$$

which completes the proof. ■

(12.I)

Theorem 18. Let A be a noetherian ring and $I = (a_1, \ldots, a_r)$ be an ideal generated by r elements. Then any minimal prime over-ideal $\mathfrak{p}$ of I has height $\leq r$. In particular, $\operatorname{ht}(I) \leq r$.

Proof. Since $\mathfrak{p}A_\mathfrak{p}$ is the only prime ideal of $A_\mathfrak{p}$ containing $IA_\mathfrak{p}$, the ring

$$A_\mathfrak{p}/IA_\mathfrak{p} = A_\mathfrak{p}/(a_1A_\mathfrak{p} + \cdots + a_rA_\mathfrak{p})$$

is artinian. Therefore $\operatorname{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p}) \leq r$ by Theorem 17. ■

(12.J) Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. In this case, an ideal of definition of A and a primary ideal belonging to $\mathfrak{m}$ are the same thing. We know from Theorem 17 that no ideals of definition are generated by less then d elements, and that there are ideals of definition generated by exactly d elements.

If $(x_1, \dots, x_d)$ is an ideal of definition then we say that $\{x_1, \dots, x_d\}$ is a system of parameters of A. If there exists a system of parameters generating the maximal ideal $\mathfrak{m}$, then we say that A is a regular local ring and we call such a system of parameters a regular system of parameters. Since the number of elements of a minimal basis of $\mathfrak{m}$ is equal to $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ by Nakayama's lemma, we have in general

$$\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2),$$

and the equality holds iff A is regular.

(12.K)

Proposition. Let $(A, \mathfrak{m})$ be a noetherian local ring and $x_1, \dots, x_d$ a system of parameters of A. Then

$$\dim(A/(x_1, \dots, x_k)) = d - k = \dim(A) - k$$

for each $1 \leq k \leq d$.

Proof. Put $\overline{A} = A/(x_1, \dots, x_k)$. Then $\dim(\overline{A}) \leq d - k$ since $\overline{x_{k+1}}, \dots, \overline{x_d}$ generate an ideal of definition of $\overline{A}$. On the other hand, if $\dim(\overline{A}) = e$ and if $\overline{y_1}, \dots, \overline{y_e}$ is a system of parameters of $\overline{A}$, then $x_1, \dots, x_k, y_1, \dots, y_e$ generate an ideal of definition of A so that $e + k \geq d$, that is, $e \geq d - k$. ■

13 Homomorphism and Dimension

(13.A) Let $\varphi : A \to B$ be a homomorphism of rings. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, and put $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Then $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ is called the fiber over $\mathfrak{p}$ (of the canonical map $\operatorname{Spec}(\varphi)$). There is a canonical homeomorphism between $\operatorname{Spec}(\varphi)^{-1}(\mathfrak{p})$ and $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$. If $\mathfrak{q}$ is a prime ideal of B lying over $\mathfrak{p}$, the corresponding prime of $B \otimes_A \kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is $\mathfrak{q}B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$; denote by $\mathfrak{q}^*$. Then the local ring $(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}^*}$ can be identified with $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = B_\mathfrak{q} \otimes_A \kappa(\mathfrak{p})$; in fact, we have $(B_\mathfrak{p})_{\mathfrak{q}B_\mathfrak{p}} = B_\mathfrak{q}$ and so

$$(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}^*} = (B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})_{\mathfrak{q}B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}} = B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$$

by (1.I). Now we have the following theorem.

(13.B)

Theorem 19. Let $\varphi : A \to B$ be a homomorphism of noetherian rings; let $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Then

(1) $\operatorname{ht}(\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}B)$, in other words

$$\dim(B_{\mathfrak{q}}) \leq \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p}));$$

(2) the inequality holds in (1) if the going-down theorem holds for φ (e.g. if φ is flat);

(3) if $\operatorname{Spec}(\varphi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective and if the going-down theorem holds, then we have (a) $\dim(B) \geq \dim(A)$, and (b) $\operatorname{ht}(I) = \operatorname{ht}(IB)$ for any ideal of A.

Proof. (1) Replacing A and B by $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$, we may suppose that $(A, \mathfrak{p})$ and $(B, \mathfrak{q})$ are local rings such that $\mathfrak{q} \cap A = \mathfrak{p}$. We have to prove $\dim(B) \leq \dim(A) + \dim(B/\mathfrak{p}B)$. Let $a_1, \ldots, a_r$ be a system of parameters of A and put $I = (a_1, \ldots, a_r)$. Then $\mathfrak{p}^n \subseteq I$ for some $n \geq 0$, so that $\mathfrak{p}^n B \subseteq IB \subseteq \mathfrak{p}B$. Thus $\sqrt{\mathfrak{p}B} = \sqrt{IB}$. Therefore it follows from the definition that $\dim(B/\mathfrak{p}B) = \dim(B/IB)$. If $\dim(B/IB) = s$ and if $\{\overline{b_1}, \ldots, \overline{b_s}\}$ is a system of parameters of B/IB, then $b_1, \ldots, b_s, a_1, \ldots, a_r$ generate an ideal of definition of B. Hence $\dim(B) \leq r + s$.

(2) We use the same notation as above. If $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}B) = s$ there exists a prime chain in B of length s,

$$\mathfrak{q} = \mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_s,$$

such that $\mathfrak{q}_s \supseteq \mathfrak{p}B$. As $\mathfrak{p} = \mathfrak{q} \cap A \supseteq \mathfrak{q}_i \cap A \supseteq \mathfrak{p}$, all the $\mathfrak{q}_i$ lie over $\mathfrak{p}$. If $\operatorname{ht}(\mathfrak{p}) = r$ then there exists a prime chain

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$$

in A, and by going-down there exists a prime chain

$$\mathfrak{q}_s = \mathfrak{r}_0 \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_r$$

of B such that $\mathfrak{r}_i \cap A = \mathfrak{p}_i$. Thus

$$\mathfrak{q} = \mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_s = \mathfrak{r}_0 \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_r$$

is a prime chain of length $r + s$, therefore $\mathrm{ht}(\mathfrak{q}) \geq r + s$.

(3) (a) follows from (2). (b) Take a minimal prime over-ideal $\mathfrak{q}$ of IB such that $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(IB)$, and put $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is minimal over $\mathfrak{p}B$, so $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = 0$, hence by (2) we get

$$\mathrm{ht}(IB) = \mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(I).$$

Conversely, let $\mathfrak{p}$ be a minimal prime over-ideal of I such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$, and take a prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Replacing $\mathfrak{q}$ if necessary we may suppose that $\mathfrak{q}$ is a minimal prime over-ideal of $\mathfrak{p}B$. Then $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = 0$, so by (2),

$$\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(IB). \qquad \blacksquare$$

(13.C)

Theorem 20. Let B be a noetherian ring, and let A be a noetherian subring which B is integral. Then

(1) $\dim(A) = \dim(B)$,

(2) for any $\mathfrak{q} \in \mathrm{Spec}(B)$ we have $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{q} \cap A)$,

(3) if, moreover, the going-down theorem holds between A and B, then for any ideal J of B we have $\mathrm{ht}(J) = \mathrm{ht}(J \cap A)$.

Proof. (1) Let $m = \dim(A)$, $n = \dim(B)$ and let

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_n$$

be a prime chain in B. Then by (5.E) (2) we have

$$\mathfrak{q}_0 \cap A \supset \mathfrak{q}_1 \cap A \supset \cdots \supset \mathfrak{q}_n \cap A,$$

so $n \leq m$. On the other hand, let

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_m$$

be a prime chain in A. Let $\mathfrak{q}_0 \in \mathrm{Spec}(B)$ such that $\mathfrak{q}_m \cap A = \mathfrak{p}_m$, then the going-up theorem by (5.E) (3) implies that there exists $\mathfrak{q}_{m-1}, \dots, \mathfrak{q}_0$ such that

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_m$$

and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$. So $n \geq m$.

(2) follows from Theorem 19 (1), since $\mathrm{ht}(\mathfrak{q}/(\mathfrak{q} \cap A)B) = 0$ by (5.E) (2).

(3) First take $\mathfrak{q} \in \mathrm{Spec}(B)$ minimal over J such that $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(J)$. Then $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}((\mathfrak{q} \cap A)B) = \mathrm{ht}(\mathfrak{q} \cap A)$ by Theorem 19 (3), hence $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{q} \cap A)$ by (2), so that

$$\mathrm{ht}(J) = \mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{q} \cap A) \geq \mathrm{ht}(J \cap A).$$

Next let $\mathfrak{p}$ be a prime ideal of A containing $J \cap A$ such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(J \cap A)$. Since B/J integral over the subring $A/J \cap A$, by (5.E) there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ containing J and lying over $\mathfrak{p}$. Then by (2),

$$\mathrm{ht}(J \cap A) = \mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(J). \qquad \blacksquare$$

(13.D)

Theorem 21. Let $\varphi : A \to B$ be a homomorphism of noetherian rings and suppose that the going-up theorem holds for φ. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be prime ideals of A such that $\mathfrak{p} \subset \mathfrak{p}'$. Then

$$\dim(B \otimes_A \kappa(\mathfrak{p})) \leq \dim(B \otimes_A \kappa(\mathfrak{p}')).$$

Proof. Put $r = \dim(B \otimes_A \kappa(\mathfrak{p}))$ and $s = \mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$. Take a prime chain

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_r$$

in B such that $\mathfrak{q}_i \cap A = \mathfrak{p}$ for all i, and a prime chain

$$\mathfrak{p}' = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_s = \mathfrak{p}$$

in A. By going-up we can find a prime chain.

$$\mathfrak{r}_0 \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_s = \mathfrak{q}_0$$

in B such that $\mathfrak{r}_i \cap A = \mathfrak{p}_i$. Then $\mathfrak{r}_0$ lies over $\mathfrak{p}'$ and $\mathrm{ht}(\mathfrak{r}_0/\mathfrak{q}_r) \geq r + s$. Applying Theorem 19 (1) to $A/\mathfrak{p} \to B/\mathfrak{q}_r$ we get

$$\begin{aligned}
\mathrm{ht}(\mathfrak{r}_0/\mathfrak{q}_r) &\leq \mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) + \mathrm{ht}((\mathfrak{r}_0/\mathfrak{q}_r)/\mathfrak{p}'(B/\mathfrak{q}_r)) \\
&= s + \mathrm{ht}(\mathfrak{r}_0/(\mathfrak{q}_r + \mathfrak{p}'B)) \\
&\leq s + \mathrm{ht}(\mathfrak{r}_0/\mathfrak{p}'B) \\
&\leq s + \dim(B \otimes_A \kappa(\mathfrak{p}'))
\end{aligned}$$

since $\mathfrak{r}_0/\mathfrak{p}'B \in \mathrm{Spec}(B_{\mathfrak{p}'}/\mathfrak{p}'B_{\mathfrak{p}'}) \hookrightarrow \mathrm{Spec}(B)$. Thus $r \leq \dim(B \otimes_A \kappa(\mathfrak{p}))$. ■

(13.E)

Remark. The local form of Theorem 21 is inconvenient for applications in algebraic geometry. The global counterpart of the going-up theorem is the closedness of a morphism. Thus, we have the following geometric theorem: Let $f : X \to Y$ be a closed morphism (e.g. a proper morphism) between noetherian schemes, and let y and y' be points of Y such that y' is a specialization of y. Then $\dim f^{-1}(y') \geq \dim f^{-1}(y)$. The proof is essentially the same as above.

14 Finitely Generated Extensions

(14.A)

Theorem 22. Let A be a noetherian ring and let $A[X_1, \ldots, X_n]$ be a polynomial ring in n variables. Then

$$\dim(A[X_1, \ldots, X_n]) = \dim(A) + n.$$

Proof. Enough to prove the case $n = 1$. Put $B = A[X]$. Let $\mathfrak{p}$ be a prime ideal of A and let $\mathfrak{q}$ be a prime ideal of B which is maximal among the prime ideals lying over $\mathfrak{p}$. We claim that $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = 1$. In fact, localizing A and B by the multiplicative set $A - \mathfrak{p}$ we can assume that $\mathfrak{p}$ is a maximal ideal, and then $B/\mathfrak{p}B = (A/\mathfrak{p})[X]$ is a polynomial ring in one variable over a field. Therefore $B/\mathfrak{p}B$ is a principal ideal domain and every maximal ideal has height one. Thus $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = 1$.

Since B is free over A, and hence flat, so the going-down theorem holds and we have

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = \mathrm{ht}(\mathfrak{p}) + 1$$

by Theorem 19 (2). As the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective (since B is flat over A), we obtain

$$\begin{aligned}\dim(B) &= \sup\{\mathrm{ht}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Spec}(B)\} \\ &= \sup\{\mathrm{ht}(\mathfrak{p}) + 1 \mid \mathfrak{p} \in \mathrm{Spec}(A)\} = \dim(A) + 1.\end{aligned}$$ ■

Corollary. Let k be a field. Then $\dim(k[X_1, \ldots, X_n]) = n$, and the ideal $(X_1, \ldots, X_i)$ is a prime ideal of height i, for $1 \le i \le n$.

Proof. Since

$$(X_1, \ldots, X_n) \supset (X_1, \ldots, X_{n-1}) \supset \cdots \supset (X_1) \supset (0)$$

is a prime chain of length n and since $\dim(k[X_1, \ldots, X_n]) = \dim(k) + n = n$, the assertion is obvious. ■

(14.B) A ring A is said to be catenary if, for each pair of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$, $\mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$ is finite and is equal to the length of any maximal prime chain between $\mathfrak{p}$ and $\mathfrak{p}'$. (When A is noetherian, the condition $\mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) < \infty$ is automatically satisfied.) Thus if A is noetherian domain the following conditions are equivalent:

(i) A is catenary,

(ii) for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ we have $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$,

(iii) for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ with $\mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$, we have $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathfrak{p}) + 1$.

The proof is trivial by noting that $(0) \in \mathrm{Spec}(A)$. If A is catenary, then clearly any localization $S^{-1}A$ and any homomorphic image A/I of A are also catenary.

A ring A is said to be universally catenary (u.c. for short) if A is noetherian and if every A-algebra of finite type is catenary. Since any A-algebra of finite type of a homomorphic image of $A[X_1, \ldots, X_n]$ for some n, a noetherian ring A is u.c. iff $A[X_1, \ldots, X_n]$ is catenary for every $n \geq 0$.

If A is u.c., so are the localizations of A, the homomorphic images of A and any A-algebras of finite type.

(14.C)

Theorem 23. Let A be a noetherian domain, and let B be a finitely generated over-domain of A. Let $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Then we have

$$(\clubsuit) \quad \operatorname{ht}(\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{trdeg}_A B - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}).$$

And the equality holds if A is universally catenary, or if B is a polynomial ring $A[X_1, \ldots, X_n]$. (Here, $\operatorname{trdeg}_A B$ means the transcendence degree of the quotient field of B over that of A, and $\kappa(\mathfrak{p})$ (resp. $\kappa(\mathfrak{q})$) is the quotient field of $A/\mathfrak{p}$ (resp. $B/\mathfrak{q}$).)

Proof. Let $B = A[x_1, \ldots, x_n]$. By induction on n it is enough to consider the case $n = 1$. So let $B = A[x]$. Replacing A by $A_\mathfrak{p}$, and B by $B_\mathfrak{p} = A_\mathfrak{p}[x]$, we may assume that $(A, \mathfrak{p})$ is a local domain. Put $k = \kappa(\mathfrak{p}) = A/\mathfrak{p}$ and

$$I = \{f(X) \in A[X] \mid f(x) = 0\}.$$

Thus $B = A[X]/I$ and $I \in \operatorname{Spec}(A[X])$ since B is a domain.

Case 1. $I = (0)$. Then $B = A[X]$, $\operatorname{trdeg}_A B = 1$ and $B/\mathfrak{p}B = k[X]$ is a PID. Therefore

$$\operatorname{ht}(\mathfrak{q}/\mathfrak{p}B) = \begin{cases} 1, & \text{if } \mathfrak{q} \supset \mathfrak{p}B, \\ 0, & \text{if } \mathfrak{q} = \mathfrak{p}B. \end{cases}$$

and

$$\operatorname{trdeg}_k \kappa(\mathfrak{q}) = \begin{cases} \operatorname{trdeg}_k k[x]/(\mathfrak{q}/\mathfrak{p}[X]) = 0, & \text{if } \mathfrak{q} \supset \mathfrak{p}B, \\ \operatorname{trdeg}_k k[x] = 1, & \text{if } \mathfrak{q} = \mathfrak{p}B. \end{cases}$$

In other words, $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = 1 - \mathrm{trdeg}_k\,\kappa(\mathfrak{q})$. On the other hand, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p}B)$ by Theorem 19 (2) since B is free over A, and hence flat. Thus the equality holds in (♣).

Case 2. $I \neq (0)$. Then $\mathrm{trdeg}_A B = 0$. Let $\mathfrak{q}^*$ be the inverse image of $\mathfrak{q}$ in $A[X]$, so that $\mathfrak{q} = \mathfrak{q}^*/I$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}^*)$. Since A is s subring of $B = A[X]/I$ we have $A \cap I = (0)$. If K denotes the quotient field of A and $S = A - \{0\}$, then $K[X] = S^{-1}A[X]$ and $I \cap S = \varnothing$, therefore

$$\mathrm{ht}(I) = \mathrm{ht}(IK[X]) \leq \dim(K[X]) = 1.$$

Since $I \neq (0)$ we have $\mathrm{ht}(I) = 1$. Hence

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{q}^*/I) \leq \mathrm{ht}(\mathfrak{q}^*) - \mathrm{ht}(I) = \mathrm{ht}(\mathfrak{q}^*) - 1,$$

where the equality holds if A is u.c.. On the other hand we have

$$\mathrm{ht}(\mathfrak{q}^*) = \mathrm{ht}(\mathfrak{p}) + 1 - \mathrm{trdeg}_k\,\kappa(\mathfrak{q}^*)$$

by *Case 1.*, and $\kappa(\mathfrak{q}^*) = \kappa(\mathfrak{q})$. Our assertions follow immediately from these. ■

Definition. We shall call the inequality (♣) the dimension inequality. If B is a finitely generated over-domain of A and if the equality in (♣) holds for any prime ideal of B, then we say that the dimension formula holds between A and B.

(14.D)

Corollary. A noetherian ring A is universally catenary iff the following is true:

A is catenary and for any prime $\mathfrak{p}$ of A and for any finitely generated over-domain B of $A/\mathfrak{p}$, the dimension formula holds between $A/\mathfrak{p}$ and B.

Proof. If A (hence $A/\mathfrak{p}$) is u.c., then the condition holds by the theorem. Conversely, suppose the condition holds. Let B be any A-algebra of finite type and let $\mathfrak{q} \subset \mathfrak{q}'$ be prime ideals of B. We have to show that all maximal prime chains between $\mathfrak{q}$ and $\mathfrak{q}'$ have same lengths.

Replacing B by $B/\mathfrak{q}$ and A by $A/(\mathfrak{q} \cap A)$ we can assume that B is a finitely generated over-domain of A. We are going to prove that for any prime ideals $\mathfrak{q} \subset \mathfrak{q}'$ in B we have $\mathrm{ht}(\mathfrak{q}') = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}(\mathfrak{q}'/\mathfrak{q})$.

Put $\mathfrak{p} = \mathfrak{q} \cap A$, $\mathfrak{p}' = \mathfrak{q}' \cap A$ and $n = \mathrm{trdeg}_A B$. Then

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + n - \mathrm{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}),$$
$$\mathrm{ht}(\mathfrak{q}') = \mathrm{ht}(\mathfrak{p}') + n - \mathrm{trdeg}_{\kappa(\mathfrak{p}')} \kappa(\mathfrak{q}')$$

and by the assumption applied to $\mathfrak{p}'/\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{p})$ and $\mathfrak{q}'/\mathfrak{q} \in \mathrm{Spec}(B/\mathfrak{q})$, we also have

$$\mathrm{ht}(\mathfrak{q}'/\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) + \mathrm{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) - \mathrm{trdeg}_{\kappa(\mathfrak{p}')} \kappa(\mathfrak{q}').$$

Since A is catenary we have $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$. It follows that

$$\mathrm{ht}(\mathfrak{q}') = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}(\mathfrak{q}'/\mathfrak{q}). \qquad \blacksquare$$

(14.E)

Example. All noetherian rings that appear in algebraic geometry are catenary. And many algebraists had in vain tried to know if all noetherian rings are catenary, until Nagata constructed counterexamples in 1956 (cf. Local Rings, p.203, Example 2). In particular, he produced a noetherian local domain which is catenary but not universally catenary. We will sketch here his construction in its simplest form.

Let k be a field and let $S = k[[x]]$ be the formal power series ring over k in one variable x. Take an element $z = \sum_i a_i x^i$ of S which is algebraically independent over $k(x)$. (It is well-known that the quotient field of S has an infinite transcendence degree over $k(x)$. Cf. e.g. Zariski-Samuel, Commutative Algebra, Vol.II, p.220.) Put

$$z_j = (z - \sum_{i<j} a_i x^i)/x^{j-1}$$

for $j = 1, 2, , \ldots$, (note that $z_1 = z$), and let R be the subring of S which is generated over k by x and by all the z_j's: $R = k[x, z_1, z_2, \ldots]$. Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x-1, z_1, z_2, \ldots)$ of R. Since $x(z_{j+1} + a_j) = z_j$ we have $z_j \in \mathfrak{m}$ for all j, and

$\mathfrak{m}$ is a maximal ideal of R with $R/\mathfrak{m} = k$. The local ring $R_\mathfrak{m}$ is a subring of S and $\mathfrak{m}R_\mathfrak{m} = xR_\mathfrak{m} \subset xS$. Hence

$$\bigcap_n x^n R \subseteq \bigcap_n x^n S = (0).$$

Then it is easy to see that any non-zero ideal of $R_\mathfrak{m}$ is of the form $x^i R_\mathfrak{m}$. Thus $R_\mathfrak{m}$ is noetherian, and is a regular local ring of dimension 1.

On the other hand, R is a subring of the rational function field in two variables $k(x, z)$, and so we have

$$R/(x-1) = k[x, z_1, z_2, \ldots]/(x-1) \cong k[z],$$

hence $\mathfrak{n} = (x-1, z)$ and $R/\mathfrak{n} \cong k$. The local ring $R_\mathfrak{n}$ contains x^{-1} and hence it is a localization of the ring $R[x^{-1}] = k[x, x^{-1}, z]$. This show that $R_\mathfrak{n}$ is noetherian. Clearly $R_\mathfrak{m}$ is a regular local ring of dimension 2. Let B be the localization of R with respect to the multiplicative closed subset $(R - \mathfrak{m}) \cap (R - \mathfrak{n})$. Then $\mathfrak{m}B$ and $\mathfrak{n}B$ are the only maximal ideals of B by (1.B), and local rings $B_{\mathfrak{m}B} = R_\mathfrak{m}$ and $B_{\mathfrak{n}B} = R_\mathfrak{n}$ are noetherian. It follows easily (using (1.H)) that any ideal of B is finitely generated. Thus B is a semi-local noetherian domain. Put $\mathfrak{J} = \mathrm{rad}(B)$ and $A = k + \mathfrak{J}$. Then A is a subring of B, and it is easy to see that $(A, \mathfrak{J})$ is a local ring. As

$$B/\mathfrak{J} \cong B/\mathfrak{m}B \oplus B/\mathfrak{n}B \cong k \oplus k$$

the ring B is a finite A-module. It follows (e.g. by Eakin's theorem cited in (2.D)) that A is also noetherian. We haveht$(\mathfrak{m}B) = 1$ and $\mathrm{ht}(\mathfrak{n}B) = 2$, hence $\dim(A) = \dim(B) = 2$ by (13.C) Theorem 20 (1). If A were u.c. then we would have

$$\mathrm{ht}(\mathfrak{m}B) = \mathrm{ht}(\mathfrak{m}B \cap A) = \mathrm{ht}_A(I) = \dim(A) = 2$$

by the dimension formula. Therefore A is not u.c.. But A is catenary because it is a local domain of dimension 2.

(14.F)

Lemma. Let A be a noetherian ring and let I be an ideal of A of height r. Then there exists an ideal J of A such that $J \subset I$ and $\mathrm{ht}(J) = r - 1$.

Proof. Take $f_1 \in I$ from outside of the minimal prime ideals over-ideals of $J_0 := (0)$, and $f_2 \in I$ from outside of the minimal prime over-ideals of $J_1 := (f_1)$, and $f_3 \in I$ from outside of the minimal prime over-ideals of $J_2 := (f_1, f_2)$, and so on, and put $J = J_{r-1}$. Then $\mathrm{ht}(J_{i+1}) > \mathrm{ht}(J_i)$ for any i, so

$$\mathrm{ht}(J_{r-1}) \geq \mathrm{ht}(J_{r-2}) + 1 \geq \cdots \geq \mathrm{ht}(J_0) + r - 1 = r - 1$$

and $\mathrm{ht}(J) \leq r - 1$ by Theorem 18. ■

Theorem 24. Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k, and let I be an ideal of A with $\mathrm{ht}(I) = r$. Then we can choose $Y_1, \ldots, Y_n \in A$ in such way that

(1) A is integral over $k[Y] = k[Y_1, \ldots, Y_n]$, and

(2) $I \cap k[Y] = (Y_1, \ldots, Y_r)$.

Proof. Induction on r. If $r = 0$ then $I = (0)$ and we can take $Y_i = X_i$. When $r = 1$, let $Y_1 = f(X)$ be any non-zero element of I. Write $f(X) = \sum_{j=1}^{s} a_j M_j(X)$, where $0 \neq a_j \in k$ and $M_j(X)$ are distinct monomials in $X_1, \ldots, X_n$, and take n positive integers $d_1 = 1$, $d_2, \ldots, d_n$. If $M(X) = \prod_i X_i^{a_i}$ then let us call the integer $\sum_i a_i d_i$ the weight of the monomial $M(X)$. By a suitable choice of $d_2, \ldots, d_n$ we can see to it that no two of the monomials $M_1, \ldots, M_s$ that appear in $f(X)$ have the same weight. (If p is a given prime number, we can take

$$d_2 = p^{\nu_2}, \ldots, d_n = p^{\nu_n},$$

where $\nu_i - \nu_{i-1}$ $(i = 2, \ldots, n;\ \nu_1 = 0)$ are large integers. This remark will be useful for some applications.)

Put $Y_i = X_i - X_1^{d_i}$ $(i = 2, \ldots, n)$. Then

$$Y_1 = f(X) = f(X_1, Y_2 + X_1^{d_2}, \ldots, Y_n + X_1^{d_n}) = aX_1^e + g(X_1, Y_2, \ldots, Y_n),$$

where g is a polynomial whose degree in X_1 is less than e and a is the coefficient of the term with highest weight in $f(X)$. Then X_1 is integral over $k[Y]$, and hence

$X_i = Y_i + X_1^{d_1}$ $(i = 2, \ldots, n)$ are also integral over $k[Y]$. The ideal (Y_1) of $k[Y]$ is prime of height 1, $(Y_1) \subseteq I \cap k[Y]$, and

$$\mathrm{ht}(I \cap k[Y]) = \mathrm{ht}(I) = 1$$

by Theorem 20 (3). (Note that $k[Y]$ is integrally closed and so the going-down theorem holds between $k[X]$ and $k[Y]$.) Therefore $(Y_1) = I \cap k[Y]$, as wanted.

When $r > 1$, let J be an ideal of $k[X]$ such that $J \subset I$, $\mathrm{ht}(J) = r - 1$ (which exists by the lemma above). By induction hypothesis there exist $Z_1, \ldots, Z_n \in k[X]$ such that $k[X]$ is integral over $k[Z]$ and that $k[Z] \cap J = (Z_1, \ldots, Z_{r-1})$. Put $I' = I \cap k[Z]$. Then $\mathrm{ht}(I') = \mathrm{ht}(I) = r$, again by Theorem 20 (3), and so $I' \supset (Z_1, \ldots, Z_{r-1})$. Thus we can choose an element $0 \neq f(Z_r, \ldots, Z_n)$ of I'. Following the method we used for the case $r = 1$, we put

$$Y_i = \begin{cases} Z_i, & \text{if } i < r, \\ f(Z_r, \ldots, Z_n), & \text{if } i = r, \\ Z_i - Z_r^{e_i}, & \text{if } i > r. \end{cases}$$

Then, for a suitable choice of $e_{r+1}, \ldots, e_n$, $k[Z]$ integral over $k[Y]$. Moreover, $I \cap k[Y]$ contains the prime ideal $(Y_1, \ldots, Y_r)$ of height r and so coincides with it. ■

Remark. The above proof shows that we can choose the Y_i's in such way that $Y_{r+1}, \ldots, Y_n$ have the form

$$Y_{r+j} = X_{r+j} + F_j(X_1, \ldots, X_r),$$

where F_j is a polynomial with coefficients in the prime subring k_0 of k (i.e. the canonical image of $\mathbb{Z}$ in k). If $\mathrm{char}(k) = p > 0$ then we can see to it that $F_j(X_1, \ldots, X_r) \in k_0[X_1^p, \ldots, X_r^p]$ for all j.

(14.G)

Corollary 1 (Noether Normalization Theorem)**.** Let $A = k[x_1, \ldots, x_n]$ be a finitely generated algebra over a field k. Then there exist $y_1, \ldots, y_r \in A$ which are algebraically independent over k such that A is integral over $k[y_1, \ldots, y_r]$. We have $r = \dim(A)$. If A is a domain we also have $r = \operatorname{trdeg}_k A$.

Proof. Write $A = k[X_1, \ldots, X_n]/I$, and put $\operatorname{ht}(I) = n-r$. According to the theorem there exist elements $Y_1, \ldots, Y_n$ of $k[X_1, \ldots, X_n]$ such that $k[X]$ is integral over $k[Y]$ and that $I \cap k[Y] = (Y_{r+1}, \ldots, Y_n)$. Putting $y_i \equiv Y_i \pmod{I}$ $(1 \le i \le r)$ we get the inclusion

$$k[y_1, \ldots, y_r] = k[Y]/(I \cap k[Y]) \hookrightarrow k[X]/I = A.$$

Then A is integral over $k[y_1, \ldots, y_r]$ since integrality is preserved by quotient. The equality $r = \dim(A)$ follows from Theorem 20 (1). The last assertion is obvious, as A is algebraic over $k(y_1, \ldots, y_r)$. ■

Corollary 2. Let k be an algebraically closed field. Then any maximal ideal $\mathfrak{m}$ of $k[X_1, \ldots, X_n]$ is of the form

$$\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n),\ a_i \in k.$$

Proof. Let $A = k[X_1, \ldots, X_n]$. Since $0 = \dim(A/\mathfrak{m}) = \operatorname{trdeg}_k(A/\mathfrak{m})$ by above corollary, we get $A/\mathfrak{m} \cong k$. Hence $X_i \equiv a_i \pmod{\mathfrak{m}}$ for some $a_i \in k$ for each i. Since $(X_1 - a_1, \ldots, X_n - a_n)$ is obviously a maximal ideal, it is $\mathfrak{m}$. ■

(14.H)

Corollary 3. Let A be a finitely generated algebra over a field k. Then

(1) if A is an integral domain, we have

$$\operatorname{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$$

for any $\mathfrak{p} \in \operatorname{Spec}(A)$, and

(2) A is universally catenary.

Proof. (1) Take $y_1, \ldots, y_r \in A$ as in Corollary 1, and put $\mathfrak{p}' = \mathfrak{p} \cap k[y]$. Then $\dim(A) = r$, $\dim(A/\mathfrak{p}) = \dim(k[y]/\mathfrak{p}')$ and $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}')$. As $k[y]$ is isomorphic to the polynomial ring in r variables, we have

$$\mathrm{ht}(\mathfrak{p}') + \dim(k[y]/\mathfrak{p}') = r$$

by Theorem 20 and Theorem 24.

(2) It suffices to prove that k is universally catenary. This is a consequence of (1) and (14.D), but we will give a direct proof. We are going to prove $k[X_1, \ldots, X_n]$ is catenary. Let $\mathfrak{p} \subset \mathfrak{p}'$ be prime ideals of $k[X] = k[X_1, \ldots, X_n]$. Then we have

$$\begin{aligned} \mathrm{ht}(\mathfrak{p}) &= n - \dim(k[X]/\mathfrak{p}), \\ \mathrm{ht}(\mathfrak{p}') &= n - \dim(k[X]/\mathfrak{q}), \text{ and} \\ \mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) &= \dim(k[X]/\mathfrak{p}) - \dim(k[X]/\mathfrak{p}') \qquad \text{by (1).} \end{aligned}$$

Therefore $\mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}') - \mathrm{ht}(\mathfrak{p})$. ■

(14.K)

Corollary 4 (Dimension of intersection in an affine space)**.** Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals in a polynomial ring $R = k[X_1, \ldots, X_n]$ over a field k, with $\dim(R/\mathfrak{p}_1) = r_1$, $\dim(R/\mathfrak{p}_2) = r_2$. Let $\mathfrak{q}$ be any minimal prime over-ideal of $\mathfrak{p}_1 + \mathfrak{p}_2$. Then

$$\dim(R/\mathfrak{q}) \geq r_1 + r_2 - n.$$

(In geometric terms this means that, if V_1 and V_2 are irreducible closed sets of dimension r_1 and r_2 respectively, in an affine n-space $\mathrm{Spec}(k[X_1, \ldots, X_n])$). Then any irreducible component of $V_1 \cap V_2$ has dimension not less than $r_1 + r_2 - n$.)

Proof. Let $Y_1, \ldots, Y_n$ be another set of n indeterminates and let $\mathfrak{p}_2'$ be the image of $\mathfrak{p}_2$ in $k[Y_1, \ldots, Y_n]$ by the isomorphism $k[X] \cong k[Y]$ over k which maps $X_i \to Y_i$ $(1 \leq i \leq n)$. Let I be the ideal of

$$k[X, Y] = k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$$

generated by $\mathfrak{p}_1$ and $\mathfrak{p}_2'$, and D the ideal $(X_1 - Y_1, \ldots, X_n - Y_n)$ of $k[X, Y]$. (Geometrically, $V(D)$ is the diagonal of the affine $2n$-space.)

Then $k[X,Y]/I \cong (R/\mathfrak{p}_1) \otimes_k (R/\mathfrak{p}_2)$, $k[X,Y]/D \cong k[X]$. Take $\xi_1, \ldots, \xi_{r_1} \in R/\mathfrak{p}_1$ and $\eta_1, \ldots, \eta_{r_2} \in R/\mathfrak{p}_2$ such that $R/\mathfrak{p}_1$ (resp. $R/\mathfrak{p}_2$) is integral over $k[\xi]$ (resp. over $k[\eta]$). Then $k[X,Y]/I$ is integral over $k[\xi,\eta]$ which is a polynomial ring in $r_1 + r_2$ variables. Thus

$$\dim(k[X,Y]/I) = \dim(k[\xi,\eta]) = r_1 + r_2.$$

Writing $k[X,Y]/I = k[x,y]$ we have

$$k[x,y]/(x_1 - y_1, \ldots, x_n - y_n) = k[X,Y]/(D + I) \cong k[X]/(\mathfrak{p}_1 + \mathfrak{p}_2).$$

So the prime $\mathfrak{q}$ of $k[X]$ corresponds to a minimal prime over-ideal $\mathfrak{Q}$ of $I + D$ in $k[X,Y]$ such that $k[X]/\mathfrak{q} \cong k[X,Y]/\mathfrak{Q}$. Then Q/I is a minimal prime over-ideal of $(x_1 - y_1, \ldots, x_n - y_n)$ of $k[x,y]$, hence $\mathrm{ht}(\mathfrak{Q}/I) \leq n$ by Theorem 18. Therefore

$$\dim(k[X]/\mathfrak{q}) = \dim(k[x,y]/(\mathfrak{Q}/I)) = \dim(k[x,y]) - \mathrm{ht}(\mathfrak{Q}/I) \geq r_1 + r_2 - n$$

by the preceding corollary. ∎

(14.L)

Theorem 25 (Hilbert's Nullstellensatz). Let k be a field, A be a finitely generated k-algebra and I be a proper ideal of A. Then the radical of I is the intersection of all maximal ideals containing I.

Proof. Let N denote the intersection of all maximal ideals containing I, and suppose that there is an element $a \in N$ which is not in the radical of I. Put $S = \{1, a, a^2, \ldots\}$ and $A' = S^{-1}A$. Then $I \cap S = \varnothing$, so $IA' \neq A'$ and there is a maximal ideal $\mathfrak{m}'$ of A' containing IA'. Since A' is also finitely generated over k, by Corollary 1 we have

$$0 = \dim(A'/\mathfrak{m}') = \mathrm{trdeg}_k\, A'/\mathfrak{m}'.$$

Putting $A \cap \mathfrak{m}' = \mathfrak{m}$ we have $k \subseteq A/\mathfrak{m} \subseteq A'/\mathfrak{m}'$, hence, again by Corollary 1,

$$0 = \mathrm{trdeg}_k\, A/\mathfrak{m} = \dim(A/\mathfrak{m}).$$

Thus $\mathfrak{m}$ is a maximal ideal of A containing I, and $a \notin \mathfrak{m}$, contradiction. ∎

Remark. The theorem can be stated as follows: if A is a k-algebra of finite type, then the correspondence which maps each closed set $V(I)$ of $\mathrm{Spec}(A)$ to $V(I) \cap \mathrm{Max}(A)$ is a bijection between the closed sets of $\mathrm{Spec}(A)$ and the closed sets of $\mathrm{Max}(A)$. When k is algebraically closed and $A \cong k[X_1, \ldots, X_n]/I$ one can identify $\mathrm{Max}(A)$ with the algebraic variety in k^n defined by the ideal I (i.e. the set of zero-points of I in k^n).

Chapter 6

Depth

15 M-regular Sequences

(15.A) Let A be a ring, M be an A-module and $a_1, \ldots, a_r$ be a sequence of elements of A. We write $(\underline{a})$ for the ideal $(a_1, \ldots, a_r)$, and $\underline{a}M$ for the submodule $\sum_i a_i M = (\underline{a})M$.

We say $a_1, \ldots, a_r$ is an M-regular sequence (or simply M-sequence) if the following conditions are satisfied:

(i) for each $1 \leq i \leq r$, a_i is not a zero-divisor on $M/(a_1, \ldots, a_{i-1})M$, and

(ii) $M \neq \underline{a}M$.

When all a_i belong to an ideal I we say $a_1, \ldots, a_r$ is an M-regular sequence in I. If, moreover, there is no $b \in I$ such that $a_1, \ldots, a_r, b$ is M-regular, then $a_1, \ldots, a_r$ is said to be a maximal M-regular in I. Notice that the notion of M-regular sequence depends on the order of the elements in the sequence.

Lemma 1. Suppose that $a_1, \ldots, a_r$ is M-regular and

$$a_1\xi_1 + \cdots + a_r\xi_r = 0, \ \xi_i \in M.$$

Then $\xi \in \underline{a}M$ for all i.

Proof. Induction on r. For $r = 1$, $a_1\xi_1 = 0$ implies $\xi_1 = 0$. Let $r > 1$. Since a_r is

$M/(a_1, \dots, a_{r-1})M$-regular we have

$$\xi_r = \sum_{i=1}^{r-1} a_i \eta_i,$$

hence

$$\sum_{i=1}^{r-1} a_i(\xi_i + a_r \eta_i) = 0.$$

By induction hypothesis, for $i < r$ we get $\xi_i + a_r + \eta_i \in (a_1, \dots, a_{r-1})M$, so $\xi_i \in (\underline{a})M$. ■

Theorem 26. Let A, M be as above and $a_1, \dots, a_r \in A$ be an M-regular sequence. Then for every sequence $\nu_1, \dots, \nu_r$ of positive integers, the sequence $a_1^{\nu_1}, \dots, a_r^{\nu_r}$ is M-regular.

Proof. It suffices to prove that $a_1^{\nu}, a_2, \dots, a_r$ is M-regular because then $a_2, \dots, a_r$ will be $M/a_1^{\nu}M$-regular and we can repeat the argument. We use induction on ν, the case $\nu = 1$ being true by assumption. Let $\nu > 1$ and assume that $a_1^{\nu-1}, a_2, \dots, a_r$ is M-regular. a_1^{ν} is certainly M-regular. Let $i > 1$ and assume that $a_1^{\nu}, a_2, \dots, a_{i-1}$ is an M-regular sequence. Let

$$a_i\omega = a_1^{\nu}\xi_1 + a_2\xi_2 + \cdots + a_{i-1}\xi_{i-1}.$$

Then

$$\omega = a_1^{\nu-1}\eta_1 + a_2\eta_2 + \cdots + a_{i-1}\eta_{i-1}$$

by the induction hypothesis. So

$$a_1^{\nu-1}(a_1\xi_1 - a_i\eta_1) + a_2(\xi_2 - a_i\eta_2) + \cdots + a_{i-1}(\xi_{i-1} - a_i\eta_{i-1}) = 0,$$

hence $a_1\xi_1 - a_i\eta_1 \in (a_1^{\nu-1}, a_2, \dots, a_{i-1})M$ by Lemma 1. It follows that $a_i\eta_1 \in (a_1, a_2, \dots, a_{i-1})M$, hence $\eta_1 \in (a_1, \dots, a_{i-1})M$ (since $a_1, \dots, a_i$ is M-regular) and so $\omega \in (a_1^{\nu}, a_2, \dots, a_{i-1})M$. ■

(15.B) Let A be a ring, $X_1, \dots, X_n$ be indeterminates over A and M be an A-module. An element of $M \otimes_A A[X_1, \dots, X_n]$ can be viewed as a polynomial $F(X) =$

$F(X_1, \dots, X_n)$ with coefficients in M. Therefore we write $M[X_1, \dots, X_n]$, or simply $M[X]$, for $M \otimes_A A[X_1, \dots, X_n]$. If $a_1, \dots, a_n \in A$ then $F(a) \in M$.

Let $a_1, \dots, a_n \in A$, $I = (\underline{a})$. We say that $a_1, \dots, a_n$ is an M-quasiregular sequence if the following condition is satisfied.

($\spadesuit$) For every $\nu > 0$ and for every homogeneous polynomial $F(X) \in M[X_1, \dots, X_n]$ of degree ν such that $F(a) \in I^{\nu+1}M$, we have $F \in IM[X]$.

Obviously this concept does not depend on the order of the elements. But $a_1, \dots, a_i$ ($i < n$) need not be M-quasiregular. Since $I = \underline{a}$, the condition ($\spadesuit$) can be stated in the following form.

($\spadesuit'$) If $F(X) \in M[X_1, \dots, X_n]$ is homogeneous and $F(a) = 0$, then the coefficients of F are in IM.

Define a map

$$\varphi : (M/IM)[X_1, \dots, X_n] \longrightarrow \mathrm{gr}^I(M) = \bigoplus_{\nu \geq 0} I^\nu M / I^{\nu+1} M$$

as follows. If $F[X] \in M[X]$ is homogeneous of degree ν, let $\psi(F)$ be the image of $F(a)$ in $I^\nu M / I^{\nu+1} M$. Then ψ is a degree-preserving additive map from $M[X]$ to $\mathrm{gr}^I(M)$, and since it maps $IM[X]$ to 0 it induces φ. This is clearly surjective, and ($\spadesuit$) is equivalent to

($\spadesuit''$) φ is an isomorphism: $(M/IM)[X_1, \dots, X_n] \cong \mathrm{gr}^I(M)$.

Theorem 27. Let A be a ring, M an A-module, $a_1, \dots, a_n \in A$ and $I = (\underline{a})$. Then:

(1) if $a_1, \dots, a_n$ is M-quasiregular and $x \in A$, $(IM : x) = IM$, then $(I^\nu M : x) = I^\nu M$ for all $\nu > 0$,

(2) if $a_1, \dots, a_n$ is M-regular then it is M-quasiregular;

(3) if M, $M/a_1 M$, $M/(a_1, a_2)M$, …, $M/(a_1, \dots, a_{n-1})M$ are Hausdorff in the I-adic topology, then the converse of (2) is also true.

Proof. (1) Induction on ν. Let $\nu > 1$, $\xi \in M$ and suppose $x\xi \in I^\nu M$. Then $\xi \in I^{\nu-1}M$ by induction hypothesis, hence there exists a homogeneous polynomial $F(X) \in M[X_1, \ldots, X_n]$ of degree $\nu - 1$ such that $\xi = F(a)$. Since

$$x\xi = xF(a) \in I^\nu M,$$

by (♠) the coefficients of F are in $(IM : x) = IM$. Therefore $\xi = F(a) \in I^\nu M$.

(2) Induction on n. For $n = 1$, $I = (a_1)$. If $F(X) \in M[X]$ is homogeneous, say $F(X) = \xi X^\nu$, and $F(a_1) = 0$. Then since a_1 is M-regular, $\xi = 0 \in IM$.

Let $n > 1$. By induction hypothesis $a_1, \ldots, a_{n-1}$ is M-quasiregular. Let $F(X) \in M[X_1, \ldots, X_n]$ be homogeneous of degree ν such that $F(a) = 0$. We will prove $F \in IM[X]$ by induction on ν. Write

$$F(X) = G(X_1, \ldots, X_{n-1}) + X_n H(X_1, \ldots, X_n).$$

Then G and H are homogeneous of degree ν and $\nu - 1$, respectively. Since a_n is $M/(a_1, \ldots, a_{n-1})$-regular, by (1) we have

$$H(a) \in ((a_1, \ldots, a_{n-1})^\nu M : a_n) = (a_1, \ldots, a_{n-1})^\nu M \subseteq I^\nu M,$$

therefore by induction hypothesis on ν we have $H \in IM[X]$. Since $H(a) \in (a_1, \ldots, a_{n-1})^\nu M$ there exists $h(X) \in M[X_1, \ldots, X_{n-1}]$ which is homogeneous of degree ν such that

$$H(a) = h(a_1, \ldots, a_{n-1}).$$

Putting $G(X_1, \ldots, X_{n-1}) + a_n h(X_1, \ldots, X_{n-1}) = g(X)$ we have $g(a_1, \ldots, a_{n-1}) = 0$, hence by induction hypothesis on n we have $g \in IM[X]$, hence $G = g - a_n h \in IM[X]$ so $F \in IM[X]$.

(3) If $a_1\xi = 0$ then ξ, the coefficient of the polynomial $F(X) = \xi X_1$, is in IM, hence $\xi = \sum_i a_i \eta_i$ and $\sum_i a_1 a_i \eta_i = 0$, hence similarly $\eta_i \in IM$ so that $\eta \in I^2 M$. In this way we see $\xi \in \bigcap_\nu I^\nu M = 0$. Thus a_1 is M-regular. Put $M_1 = M/a_1M$. If $a_2, \ldots, a_n$ is M-quasiregular then our assertion will be proved by induction on n. ($M \neq IM$ follows from the Hausdorff condition.) Let $F(X_2, \ldots, X_n) \in M[X_2, \ldots, X_n]$ be homogeneous of degree ν such that $F(a_2, \ldots, a_n) \in a_1 M$. Put

$F(a_2, \dots, a_n) = a_1\omega$, and assume $\omega \in I^i M$. Then $\omega = G(a_1, \dots, a_n)$ for some homogeneous polynomial of degree i, and

$$(\heartsuit) \quad a_1 G(a_1, \dots, a_n) = F(a_2, \dots, a_n) \in I^\nu M.$$

If $i < \nu - 1$, then $a_1 G(a) \in I^{i+1}M$, so $X_1 G \in IM[X]$ and hence $G \in IM[X]$. Therefore $\omega \in I^{i+1}M$. We thus conclude that $\omega \in I^{\nu-1}M$. If $i = \nu - 1$ in ($\heartsuit$), then $F(X_2, \dots, X_n) - X_1 G(X)$ is homogeneous and the vanishes at a, so it is in $IM[X]$, and since F does not contain X_1 we have $F \in IM[X]$. Therefore

$$F \bmod a_1 M[X] \in (a_2, \dots, a_n) M_1[X]. \qquad \blacksquare$$

Remark. The theorem shows that, under the assumptions of (3), any permutation of M-regular sequence is M-regular. The Hausdorff condition of (3) is satisfied in either of the following cases:

(a) A is noetherian, M is finitely generated and $I \subseteq \mathrm{rad}(A)$ by (11.D),

(b) A is a graded ring $A = \bigoplus_{\nu \geq 0} A_\nu$, M is a graded A-module $M = \bigoplus_{\nu \geq 0} M_\nu$ and each a_i is homogeneous of degree > 0. In this case, $M/(a_1, \dots, a_i)$ is a graded module and

$$I^\nu M \cap (S_0 + \cdots + S_{\nu-1}) = 0 \implies \bigcap_\nu I^\nu M = 0.$$

Example 1. Let k be a field and $A = k[X, Y, Z]$. Put $a_1 = X(Y-1)$, $a_2 = Y$ and $a_3 = Z(Y-1)$. Then a_1, a_2, a_3 is an A-regular sequence, while a_1, a_3, a_2 is not.

Example 2. There exists a non-noetherian local ring $(A, \mathfrak{m})$ such that $\mathfrak{m} = (x_1, x_2)$ where x_1, x_2 is an A-regular sequence but x_2 is a zero-divisor in A. (T. Dieudonne, Nagoya Math. J. 27-1 (1966), 355-356.)

(15.C) If $a_1, a_2, \ldots \in A$ is an M-regular sequence, then the sequence of submodules $a_1 M$, $(a_1, a_2)M$, $\ldots$ is strictly increasing. If A is noetherian such a sequence must stop. Therefore each M-regular sequence in I can be extended to a maximal M-regular sequence in I. The next theorem shows that any two maximal M-regular sequence in I have the same length if M is finitely generated.

Theorem 28. Let A be a noetherian ring, M a finite A-module and I an ideal of A with $IM \neq M$. Let $n > 0$ be an integer. Then the following are equivalent:

(i) $\mathrm{Ext}^i_A(N, M) = 0$ $(i < n)$ for every finite A-module N with $\mathrm{Supp}(N) \subseteq V(I)$;

(ii) $\mathrm{Ext}^i_A(A/I, M) = 0$ $(i < n)$;

(iii) there exists a finite A-module N with $\mathrm{Supp}(N) = V(I)$ such that

$$\mathrm{Ext}^i_A(M, N) = 0 \qquad (i < n);$$

(iv) there exists an M-regular sequence $a_1, \ldots, a_n$ of length n in I.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is trivial. (iii) $\Longrightarrow$ (iv): We have $\mathrm{Ext}^0_A(N, M) = \mathrm{Hom}_A(N, M) = 0$. If no elements of I are M-regular, then I is contained in the union of the associated primes of M by (7.B), hence in one of them by (1.B): $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Then there exists an injection $A/\mathfrak{p} \to M$. Localization at $\mathfrak{p}$ we get $\mathrm{Hom}_{A_\mathfrak{p}}(k, M_\mathfrak{p}) \neq 0$, where $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Since $\mathfrak{p} \in V(I) = \mathrm{Supp}(N)$, we have $N_\mathfrak{p} \neq 0$ and so $N \otimes_A k = N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \neq 0$ by Nakayama's lemma. Then $\mathrm{Hom}_k(N \otimes_A k, k) \neq 0$. Therefore $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. But the left hand side is a localization of $\mathrm{Hom}_A(N, M)$ by (1.G), which is 0. This is a contradiction, therefore there exists an M-regular element $a_1 \in I$.

If $n > 1$, put $M_1 = M/a_1M$. From the exact sequence

$$(\diamondsuit) \quad 0 \longrightarrow M \xrightarrow{[a_1]} M \longrightarrow M_1 \longrightarrow 0$$

we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_A(N, M) \longrightarrow \mathrm{Ext}^i_A(N, M_1) \longrightarrow \mathrm{Ext}^{i+1}_A(N, M) \longrightarrow \cdots$$

which shows that $\mathrm{Ext}^i_A(N, M_1) = 0$ $(i < n-1)$. So by induction on n there exists an M_1-regular sequence $a_2, \ldots, a_n$ in I.

(4) $\Longrightarrow$ (1): Put $M_1 = M/a_1M$. Then $\mathrm{Ext}^i_A(N, M_1) = 0$ $(i < n-1)$ by induction on n. From $(\diamondsuit)$ we get exact sequences

$$0 \longrightarrow \mathrm{Ext}^i_A(N, M) \xrightarrow{[a_1]} \mathrm{Ext}^i_A(N, M) \qquad (i < n).$$

But $\mathrm{Supp}(N) = V(\mathrm{Ann}(N)) \subseteq V(I)$, hence $I \subseteq \sqrt{\mathrm{Ann}(N)}$, and so $a_1^r N = 0$ for some $r > 0$. Therefore a_1^r annihilates $\mathrm{Ext}_A^i(N, M)$ as well. Thus we have $\mathrm{Ext}_A^i(N, M) = 0$ $(i < n)$. ■

Under the assumption of the theorem, we call the length of the maximal M-regular sequences in I the I-depth of M and denote it by $\mathrm{depth}_I(M)$. The theorem shows that

$$\mathrm{depth}_I(M) = \inf\{i \geq 0 \mid \mathrm{Ext}_A^i(A/I, M) \neq 0\}.$$

and

Corollary. If $a_1, \dots, a_n$ is an M-regular sequence of length in I, then

$$\mathrm{depth}_I(M/\underline{a}M) = \mathrm{depth}_I(M) - n.$$

Proof. It suffices to prove the case $n = 1$. $\mathrm{depth}_I(M/a_1M) \geq \mathrm{depth}_I(M) - 1$ follows from the proof of (iii) $\Longrightarrow$ (iv). If $\mathrm{depth}_I(M/a_1M) \geq \mathrm{depth}_I(M)$, then we can find a longer regular sequence in M, a contradiction. ■

When $(A, \mathfrak{m})$ is a local ring we write $\mathrm{depth}(M)$ or $\mathrm{depth}_A(M)$ for $\mathrm{depth}_{\mathfrak{m}}(M)$ and call it simply the depth of M. Thus $\mathrm{depth}(M) = 0$ iff there's a non-zero map $k = A/\mathfrak{m} \to M$, since k is a field, the map is injective so it is equivalent to $\mathfrak{m} \in \mathrm{Ass}(M)$. If A is an arbitrary noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(A)$, by (7.C) we have

$$\mathrm{depth}(M_{\mathfrak{p}}) = 0 \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \iff \mathfrak{p} \in \mathrm{Ass}_A(M) \implies \mathrm{depth}_{\mathfrak{p}}(M) = 0.$$

In general we have $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \mathrm{depth}_{\mathfrak{p}}(M)$, because by (3.E),

$$\mathrm{Ext}_A^i(A/\mathfrak{p}, M)_{\mathfrak{p}} = \mathrm{Ext}_{A_{\mathfrak{p}}}^i(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0 \implies \mathrm{Ext}_A^i(A/\mathfrak{p}, M) \neq 0.$$

When $IM = M$ we define $\mathrm{depth}_I(M) = \infty$. For instance $\mathrm{depth}_I(M) = 0$ if $M = 0$.

(15.D) D. Rees introduced the notion of grade, which is closely related to depth, in 1957. (The grade of an ideal or module, Proc. Camp. Phil. Soc. 53, 28-42.) Let A be a noetherian ring, $M \neq 0$ be a finite A-module and $I = \mathrm{Ann}(M)$. Then he puts

$$\mathrm{grade}\, M = \inf\{i \geq 0 \mid \mathrm{Ext}_A^i(M, A) \neq 0\}.$$

According to the above theorem, we have

$$\mathrm{grade}(M) = \mathrm{depth}_I(A).$$

Also, it follows from the definition that

$$\mathrm{grade}(M) \leq \mathrm{pd}(M).$$

When I is an ideal of A, grade(A/I) is called the grade of I. [Thus grade I can have two meanings according to whether I is viewed as an ideal or as a module. When confusion can arise, the depth notation should be used.] The grade of an ideal I is depth$_I(A)$, the length of maximal A-sequence in I. If $a_1, \dots, a_r$ is an A-regular sequence, since a_{i+1} is outside of the minimal prime ideals over-ideals of $(a_1, \dots, a_i)$ by (7.B) and (7.D),

$$\mathrm{ht}(a_1, \dots, a_{i+1}) = \mathrm{ht}(a_1, \dots, a_i) + 1$$

and hence $\mathrm{ht}(a_1, \dots, a_r) = r$ by Theorem 18. So we have

$$\mathrm{grade}\, I \leq \mathrm{ht}(I).$$

Proposition 2. Let A be a noetherian ring, $M \neq 0$ and N be finite A-modules, $\mathrm{grade}\, M = k$ and $\mathrm{pd}(N) = \ell < k$. Then

$$\mathrm{Ext}_A^i(M, N) = 0 \qquad (i < k - \ell).$$

Proof. Induction on ℓ. If $\ell = 0$ then N is projective and hence a direct summand of a free module. Since our assertion holds for A by definition, it holds for N also. If $\ell > 0$ take an exact sequence

$$0 \longrightarrow N' \longrightarrow L \longrightarrow N \longrightarrow 0$$

with L free. Then $\mathrm{pd}(N') = \ell - 1$ and we have the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_A^i(M, L) \longrightarrow \mathrm{Ext}_A^i(M, N) \longrightarrow \mathrm{Ext}_A^{i+1}(M, N') \longrightarrow \cdots,$$

so our assertion is proved by induction. ∎

(15.E)

Lemma 2 (Ischebeck)**.** Let $(A, \mathfrak{m})$ be a noetherian local ring and $M \neq 0$ and $N \neq 0$ be finite A-modules. Put $\operatorname{depth}(M) = k$, $\dim(N) = r$. Then

$$\operatorname{Ext}^i_A(M, N) = 0 \qquad (i < k - r).$$

Proof. Induction on r. If $r = 0$ then for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$, $\mathfrak{p} \not\supseteq \operatorname{Ann}(N)$, so $\operatorname{Supp}(N) = \{\mathfrak{m}\}$ and the assertion follows from Theorem 28.

Let $r > 0$. We do the case $N = A/\mathfrak{p}$ first. Since $r = \dim(A/\mathfrak{p}) > 0$ we can pick $x \in \mathfrak{m} - \mathfrak{p}$, and then

$$0 \longrightarrow N \xrightarrow{[x]} N \longrightarrow N' \longrightarrow 0$$

is exact, where $N' = A/(\mathfrak{p}+Ax)$ has dimension $< r$. Then using induction hypothesis we get exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_A(N, M) \xrightarrow{[x]} \operatorname{Ext}^i_A(N, M) \longrightarrow \operatorname{Ext}^{i+1}_A(N', M) = 0$$

for $i < k - r$, and these Ext's must vanish by Nakayama's lemma.

For general N, by Theorem 10 there exists a chain of submodules

$$(\clubsuit) \quad 0 = N_0 \subset \cdots \subset N_{n-1} \subset N_n = N$$

such that $N_i/N_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec}(A)$ $(1 \leq i \leq n)$. Then there's an exact sequence

$$0 \longrightarrow N_1 = A/\mathfrak{p}_1 \longrightarrow N \longrightarrow N/N_1 \longrightarrow 0.$$

So we get the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_A(A/\mathfrak{p}_1, M) \longrightarrow \operatorname{Ext}^i_A(N, M) \longrightarrow \operatorname{Ext}^i_A(N/N_1, M) \longrightarrow \cdots.$$

Then induction on length of $(\clubsuit)$ we get $\operatorname{Ext}^i_A(N, M) = 0$ when $i < k - r$. ■

Theorem 29. Let $(A, \mathfrak{m})$ be a noetherian local ring and let $M \neq 0$ be a finite A-module. Then we have

$$\operatorname{depth}(M) \leq \dim(A/\mathfrak{p}) \quad \text{for every } \mathfrak{p} \in \operatorname{Ass}(M).$$

Proof. If $\mathfrak{p} \in \mathrm{Ass}(M)$ then $\mathrm{Hom}_A(A/\mathfrak{p}, M) \neq 0$, hence $\mathrm{depth}(M) \leq \dim(A/\mathfrak{p})$ by Lemma 2. ■

(15.F)

Lemma 3. Let A be a ring, and let E and F be finite A-modules. Then

$$\mathrm{Supp}(E \otimes_A F) = \mathrm{Supp}(E) \cap \mathrm{Supp}(F).$$

Proof. For $\mathfrak{p} \in \mathrm{Spec}(A)$ we have

$$(E \otimes_A F)_{\mathfrak{p}} = (E \otimes_A F) \otimes_A A_{\mathfrak{p}} = E_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}.$$

Therefore the assertion is equivalent to the following:

Let $(A, \mathfrak{m}, k)$ be a local ring and E and F be finite A-modules. Then

$$E \otimes_A F = 0 \iff E \neq 0 \text{ and } F \neq 0.$$

Now ($\implies$) is trivial. Conversely, if $E \neq 0$ and $F \neq 0$ then $E \otimes_A k = E/\mathfrak{m}E \neq 0$ by Nakayama's lemma. Similarly $F \otimes_A k \neq 0$. Since k is a field we get

$$(E \otimes_A F) \otimes_A k = (E \otimes_A k) \otimes_k (F \otimes_A k) \neq 0,$$

so $E \otimes_A F \neq 0$. ■

Lemma 4. Let A be a noetherian local ring and M be a finite A-module. Let $a_1, \ldots, a_r$ be an M-regular sequence. Then

$$\dim(M/\underline{a}M) = \dim(M) - r.$$

Proof. We have $\dim(M/\underline{a}M) \geq \dim(M) - r$ by Theorem 17. On the other hand, suppose f is an M-regular element. We have

$$\mathrm{Supp}(M/fM) = \mathrm{Supp}(M) \cap \mathrm{Supp}(A/fA) = \mathrm{Supp}(M) \cap V(f)$$

by Lemma 3, and f is not in any minimal element of $\mathrm{Supp}(M)$ by Theorem 9, in other words $V(f)$ does not contain any irreducible component of $\mathrm{Supp}(M)$. Hence $\dim(M/fM) < \dim(M)$. This proves $\dim(M/\underline{a}M) \leq \dim(M) - r$. ■

Proposition 3. Let A be a noetherian ring, M a finite A-module and I an ideal. Then

$$\operatorname{depth}_I(M) = \inf\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}.$$

Proof. Let n denote the value of the right hand side. If $n = 0$ then $\operatorname{depth}(M_{\mathfrak{p}}) = 0$ for some $\mathfrak{p} \supseteq I$, and then $I \subseteq \mathfrak{p} \in \operatorname{Ass}(M)$. Thus $\operatorname{depth}_I(M) = 0$ since $\operatorname{Hom}_A(A/I, M) \neq 0$. If $0 < n < \infty$, then I is not contained in any associated prime of M (otherwise $n = 0$), and so there exists by (1.B) an M-regular element $a \in I$. Put $M' = M/aM$. Then for $\mathfrak{p} \in V(I)$, $a/1 \in A_{\mathfrak{p}}$ is an $M_{\mathfrak{p}}$-regular element since $A_{\mathfrak{p}}$ is flat over A. By the corollary in (15.C),

$$\operatorname{depth}(M'_{\mathfrak{p}}) = \operatorname{depth}(M_{\mathfrak{p}}/aM_{\mathfrak{p}}) = \operatorname{depth}(M_{\mathfrak{p}}) - 1,$$
$$\operatorname{depth}_I(M') = \operatorname{depth}_I(M/aM) = \operatorname{depth}_I(M) - 1.$$

Therefore our assertion is proved by induction on n.

If $n = \infty$ then $\mathfrak{p}M_{\mathfrak{p}} = M_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$. If $IM \neq M$ we would have $(M/IM)_{\mathfrak{p}} \neq 0$ for every

$$\mathfrak{p} \in \operatorname{Supp}(M/IM) = V(I) \cap \operatorname{Supp}(M).$$

If $\mathfrak{p}$ is a minimal element of $\operatorname{Supp}(M/IM)$ then $\operatorname{Supp}_{A_{\mathfrak{p}}}(M/IM)_{\mathfrak{p}} = \{\mathfrak{p}A_{\mathfrak{p}}\}$, hence the $A_{\mathfrak{p}}$-module $(M/IM)_{\mathfrak{p}} = M_{\mathfrak{p}}/IM_{\mathfrak{p}}$ is coprimary in $M_{\mathfrak{p}}$ and $\mathfrak{p}^s M_{\mathfrak{p}} \subseteq IM_{\mathfrak{p}}$ for some $s > 0$ by (8.B). Hence $\mathfrak{p}M_{\mathfrak{p}} \neq M_{\mathfrak{p}}$, contradiction. Therefore $IM = M$ and $\operatorname{depth}_I(M) = \infty$. ■

16 Cohen-Macaulay Rings

(16.A) Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. We know that $\operatorname{depth}(M) \leq \dim(M)$ since

$$0 \neq \operatorname{Hom}_A(M, M) = \operatorname{Ext}^0_A(M, M)$$

and by Lemma 2, provided that $M \neq 0$. We say that M is Cohen-Macaulay (briefly, C.M.) if $M = 0$ or if $\operatorname{depth}(M) = \dim(M)$. If the local ring A is C.M. as an A-module then we call A a Cohen-Macaulay ring.

Theorem 30. Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. Then

(1) if M is a C.M. module and $\mathfrak{p} \in \mathrm{Ass}(M)$, then we have $\mathrm{depth}(M) = \dim(A/\mathfrak{p})$. Consequently M has no embedded primes;

(2) if $a_1, \ldots, a_r$ is an M-regular sequence in $\mathfrak{m}$ and $M' = M/\underline{a}M$, then

$$M \text{ is C.M} \iff M' \text{ is C.M. ;}$$

(3) if M is C.M., then for every $\mathfrak{p} \in \mathrm{Spec}(A)$ the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is C.M., and if $M_\mathfrak{p} \neq 0$ we have

$$\mathrm{depth}_\mathfrak{p}(M) = \mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. (1) Since $\mathrm{Ass}(M) \neq \varnothing$, M is not 0 and so $\mathrm{depth}(M) = \dim(M)$. Since $\mathfrak{p} \in \mathrm{Supp}(M)$, there exists $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{q} \in \mathrm{Ass}(M)$ by Theorem 9, then we have

$$\dim(M) \geq \dim(A/\mathfrak{q}) \geq \dim(A/\mathfrak{p}),$$

and $\dim(A/\mathfrak{p}) \geq \mathrm{depth}(M)$ by Theorem 29.

(2) By Nakayama's lemma we have $M = 0$ iff $M' = 0$. Suppose $M \neq 0$. Then

$$\begin{aligned} \dim(M') &= \dim(M) - r && \text{by Lemma 4,} \\ \mathrm{depth}(M') &= \mathrm{depth}(M) - r && \text{by (15.C).} \end{aligned}$$

(3) We may assume that $M_\mathfrak{p} \neq 0$. Hence $\mathfrak{p} \supseteq \mathrm{Ann}(M)$. We know that

$$\dim(M_\mathfrak{p}) \geq \mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geq \mathrm{depth}_\mathfrak{p}(M).$$

So we will prove $\mathrm{depth}_\mathfrak{p}(M) = \dim(M_\mathfrak{p})$ by induction on $\mathrm{depth}_\mathfrak{p}(M)$. If $\mathrm{depth}_\mathfrak{p}(M) = 0$ then $\mathfrak{p}$ is contained in some $\mathfrak{p}' \in \mathrm{Ass}(M)$, but $\mathrm{Ann}(M) \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$ and the associated primes of M are the minimal prime over-ideals of $\mathrm{Ann}(M)$ by (1). Hence $\mathfrak{p} = \mathfrak{p}'$, and

$$\dim(M_\mathfrak{p}) = \dim_{A_\mathfrak{p}}(\mathrm{Ann}(M)_\mathfrak{p}) = 0.$$

Next suppose $\mathrm{depth}_\mathfrak{p}(M) > 0$; take an M-regular element $a \in \mathfrak{p}$ and put $M' = M/aM$. Since $A_\mathfrak{p}$ is flat over A, $a/1 \in A_\mathfrak{p}$ is $M_\mathfrak{p}$-regular. By Lemma 4 and the corollary in (15.C) we have

$$\dim(M'_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}}/aM_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}}) - 1,$$
$$\operatorname{depth}_{\mathfrak{p}}(M') = \operatorname{depth}_{\mathfrak{p}}(M/aM) = \operatorname{depth}_{\mathfrak{p}}(M) - 1.$$

Since M' is C.M. by (2), by induction hypothesis we have

$$\dim(M'_{\mathfrak{p}}) = \operatorname{depth}_{\mathfrak{p}}(M').$$ ■

(16.B)

Theorem 31. Let $(A, \mathfrak{m})$ be a C.M. local ring. Then

(1) for every proper ideal I of A, we have

$$\operatorname{ht}(I) = \operatorname{depth}_I(A) = \operatorname{grade} I,$$
$$\operatorname{ht}(I) + \dim(A/I) = \dim(A);$$

(2) A is catenary;

(3) for every sequence $a_1, \ldots, a_r$ in $\mathfrak{m}$, the following conditions are equivalent:

(i) the sequence $a_1, \ldots, a_r$ is A-regular,

(ii) $\operatorname{ht}(a_1, \ldots, a_i) = i$ $(1 \leq i \leq r)$,

(iii) $\operatorname{ht}(a_1, \ldots, a_r) = r$,

(iv) there exist $a_{r+1}, \ldots, a_n$ $(n = \dim(A))$ in $\mathfrak{m}$ such that $\{a_1, \ldots, a_n\}$ is a system of parameters of A.

Proof. (3) (i) $\Longrightarrow$ (ii) is in (15.D). (ii) $\Longrightarrow$ (iii) is trivial. (iii) $\Longrightarrow$ (iv): if $\dim(A) = r$, then

$$\dim(A/(a_1, \ldots, a_r)) \leq \dim(A) - \operatorname{ht}(a_1, \ldots, a_r) = 0,$$

so $A/(a_1, \ldots, a_r)$ is an artinian local ring, so $(a_1, \ldots, a_r)$ is an ideal of definition.

If $\dim(A) > r$ then $\mathfrak{m}$ is not a minimal prime over-ideal of $(a_1, \ldots, a_r)$, so we can take $a_{r+1} \in \mathfrak{m}$ which is not in any minimal prime over-ideal of $(a_1, \ldots, a_r)$ by

(1.B). Then $\mathrm{ht}(a_1, \ldots, a_{r+1}) = r + 1$, and we can continue. [Thus these implications are true for any noetherian local ring.]

(iv) $\Longrightarrow$ (i): It suffices to show that every system of parameters $x_1, \ldots, x_n$ of A is an A-regular sequence. If $\mathfrak{p} \in \mathrm{Ass}(A)$ then every element in $\mathfrak{p}$ is a zero-divisor by (7.B), hence $x_1 \notin \mathfrak{p}$. Put $A' = A/(x_1)$. Then A' is a local ring of dimension $n - 1$ by Theorem 30, and the images of $x_2, \ldots, x_n$ in A' form a system of parameters of A' since

$$(x_2, \ldots, x_n) \supseteq \mathfrak{m}^\nu/(x_1) \text{ whenever } (x_1, \ldots, x_n) \supseteq \mathfrak{m}^\nu.$$

Thus $x_2, \ldots, x_n$ is A'-regular by induction on n.

(1) Let $\mathrm{ht}(I) = r$. Then one can choose $a_1, \ldots, a_r \in I$ in such a way that $\mathrm{ht}(a_1, \ldots, a_i) = i$ holds for $1 \leq i \leq r$ (by the same way in (14.F)). Then the sequence $a_1, \ldots, a_r$ is A-regular by (3). Hence $r \leq \operatorname{grade} I$. Conversely if $b_1, \ldots, b_s$ is an A-regular sequence in I then

$$s = \mathrm{ht}(b_1, \ldots, b_s) \leq \mathrm{ht}(I).$$

Hence $\operatorname{grade} I = \mathrm{ht}(I)$.

Since

$$\mathrm{ht}(I) = \inf\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in V(I)\} \text{ and}$$
$$\dim(A/I) = \sup\{\dim(A/\mathfrak{p}) \mid \mathfrak{p} \in V(I)\},$$

so $\mathrm{ht}(I) = \dim(A) - \dim(A/I)$ holds in general if it holds for all prime ideal. Let $\mathfrak{p}$ be a prime ideal. Put

$$\dim(A) = \mathrm{ht}(\mathfrak{m}) = \mathrm{depth}(A) = n, \ \mathrm{ht}(\mathfrak{p}) = r.$$

By Theorem 30 (3) $A_\mathfrak{p}$ is a C.M. ring and

$$\dim A_\mathfrak{p} = \mathrm{ht}(\mathfrak{p}) = \mathrm{depth}_\mathfrak{p}(A).$$

So we can find an A-regular sequence $a_1, \ldots, a_r$ in $\mathfrak{p}$. Then $A/(a_1, \ldots, a_r)$ is C.M. of dimension $n - r$, and $\mathfrak{p}$ is a minimal prime over-ideal of $(\underline{a})$. Therefore $\dim A/\mathfrak{p} = n - r$ by Theorem 30 (1).

(2) If $\mathfrak{p} \subset \mathfrak{p}'$ are two prime ideals of A, since $A_\mathfrak{p}$ is C.M. we have

$$\dim(A_{\mathfrak{p}'}) = \mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}'}) + \dim(A_{\mathfrak{p}'}/\mathfrak{p}A_{\mathfrak{p}'})$$

by (1), i.e. $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}'/\mathfrak{p})$. Therefore A is catenary. ■

(16.C) We say a noetherian ring A is Cohen-Macaulay if $A_\mathfrak{p}$ is a C.M. local ring for every $\mathfrak{p} \in \mathrm{Spec}(A)$. By Theorem 30 this is equivalent to saying that $A_\mathfrak{m}$ is a C.M. local ring for every $\mathfrak{m} \in \mathrm{Max}(A)$.

Let A be a noetherian ring and I an ideal; let

$$\mathrm{Ass}_A(A/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}.$$

We say that I is unmixed if $\mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(I)$ for all i. We say that the unmixedness holds in A if the following is true: if $I = (a_1, \ldots, a_r)$ is an ideal of height r generated by r elements, then I is unmixed. (Note that such an ideal is unmixed iff A/I has no embedded primes by (7.D) and Theorem 18.) The condition implies in particular (for $r = 0$) that A has no embedded primes. If I is as above and if it possesses an embedded prime $\mathfrak{p}$, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then in $A_\mathfrak{m}$ the ideal $IA_\mathfrak{m}$ has $\mathfrak{p}A_\mathfrak{m}$ as embedded prime by (7.C). Therefore, the unmixedness theorem holds in A if it holds in $A_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.

Theorem 32. Let A be a noetherian ring. Then A is C.M. iff the unmixedness theorem holds in A.

Proof. Suppose the unmixedness theorem holds in A. Let $\mathfrak{p}$ be a prime ideal of height r. Then we can find $a_1, \ldots, a_r \in \mathfrak{p}$ such that $\mathrm{ht}(a_1, \ldots, a_i) = i$ for $1 \le i \le r$ (by the same way in (14.F)). The ideal $(a_1, \ldots, a_i)$ is unmixed by assumption, so a_{i+1} lies in no associated primes of $A/(a_1, \ldots, a_i)$. Thus $a_1, \ldots, a_r$ is an A-regular sequence in $\mathfrak{p}$, hence

$$r \le \mathrm{depth}_\mathfrak{p}(A) \le \mathrm{depth}(A_\mathfrak{p}) \le \dim(A_\mathfrak{p}) = r,$$

so that $A_\mathfrak{p}$ is a C.M. local ring.

Conversely, suppose A is C.M.. To prove the unmixedness theorem we may localize, so we assume that A is a C.M. local ring. We know that the ideal $I = (0)$ is unmixed by Theorem 30 (1). Let $I = (a_1, \dots, a_r)$ be an ideal of height $r > 0$. Then $a_1, \dots, a_r$ is an A-regular sequence by Theorem 31 (3), hence $A/(a_1, \dots, a_r)$ is C.M. by Theorem 30 and so $(a_1, \dots, a_r)$ is unmixed by the case $I = (0)$. ■

(16.D)

Theorem 33. Let A be a Cohen-Macaulay ring. Then $A[X_1, \dots, X_n]$ is also Cohen-Macaulay. As a consequence, any homomorphic image of a C.M. ring is universally catenary.

Proof. Enough to consider the case of $n = 1$. Let $\mathfrak{q}$ be a prime ideal of $B = A[X]$, and put $\mathfrak{p} = \mathfrak{q} \cap A$. We want to prove that the local ring $B_\mathfrak{q}$ is C.M.. Since $B_\mathfrak{q}$ is a localization of $A_\mathfrak{p}[X]$ and since $A_\mathfrak{p}$ is C.M., we may assume that A is a C.M. local ring and $\mathfrak{p}$ is the maximal ideal.

Then $B/\mathfrak{p}B = k[X]$ is a PID, where $k = A/\mathfrak{p}A$ is a field. Therefore we have either $\mathfrak{q} = \mathfrak{p}B$, or $\mathfrak{q} = \mathfrak{p}B + fB$ where $f \in B$ is a monic polynomial of positive degree. As B is flat over A, so is $B_\mathfrak{q}$ (transitivity). It follows that any A-regular sequence $a_1, \dots, a_r$ ($r = \dim(A)$) in $\mathfrak{p}$ is also $B_\mathfrak{q}$-regular by (3.F).

If $\mathfrak{q} = \mathfrak{p}B$ we have $\dim(B_\mathfrak{q}) = \dim(A)$ by Theorem 19 since $B_\mathfrak{q}$ is flat over A, and as $\operatorname{depth}(B_\mathfrak{q}) \geq \dim(A)$ by above we see that $B_\mathfrak{q}$ is C.M.. If $\mathfrak{q} = \mathfrak{p}B + fB$ then $\dim(B_\mathfrak{q}) = \dim(A) + 1$ by Theorem 19, and since any monic polynomial is a non-zero-divisor in $(A/(a_1, \dots, a_r))[X]$ we have

$$\operatorname{depth}(B_\mathfrak{q}) \geq r + 1 = \dim(B_\mathfrak{q}).$$

Thus $B_\mathfrak{q}$ is C.M. in this case also. The last assertion follows from the fact that a ring is catenary if its localization at every prime ideal is catenary. ■

(16.E)

Example 1. A polynomial ring $k[X_1, \dots, X_n]$ over a field k is C.M. by Theorem 33. (Macaulay proved the unmixedness theorem for polynomial rings before 1916. Kaplansky says "In many aspects Macaulay was far ahead of his time, and some aspects of his work won full appreciation only recently".)

Example 2. Let $A = k[x, y]$ be a polynomial ring in two variables x, y over a field k, and put $B = k[x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$. Then A and B have the same quotient field and A is integral over B. Put $\mathfrak{m} = (xA + yA) \cap B$. Then we have $x^4 \notin x^3B$ and $x^4\mathfrak{m} \subseteq x^3B$, so that $\mathfrak{m} \in \mathrm{Ass}_B(B/x^3B)$. It follows that the local ring $B_{\mathfrak{m}}$ is not Cohen-Macaulay.

(16.F)

Proposition. Let A be a C.M. ring, and $J = (a_1, \dots, a_r)$ be an ideal of height r. Then A/J^ν is C.M., and hence J^ν is unmixed, for every $\nu > 0$.

Proof. Localize at every prime ideal that contains J, we may assume that A is local. Let k be its residue field and put $d = \dim(A/J)$. Since $a_1, \dots, a_r$ is an A-regular sequence, by Theorem 27 it is A-quasiregular and $J^\nu/J^{\nu+1}$ is isomorphic to a free A/J-module (by ($\spadesuit''$) in(15.B)). Since A/J is C.M. (by Theorem 30) with

$$\mathrm{depth}(A/J) = \dim(A/J) = d,$$

and since $\mathrm{depth}_A(A/J) = \mathrm{depth}_{A/J}\, A/J$, we have $\mathrm{Ext}^i_A(k, A/J) = 0$ $(i < d)$ by Theorem 28. Then $\mathrm{Ext}^i_A(k, J^\nu/J^{\nu+1}) = 0$ $(i < d)$. By the exact sequence

$$0 \longrightarrow J^\nu/J^{\nu+1} \longrightarrow A/J^{\nu+1} \longrightarrow A/J^\nu \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_A(k, J^\nu/J^{\nu+1}) \longrightarrow \mathrm{Ext}^i_A(k, A/J^{\nu+1}) \longrightarrow \mathrm{Ext}^i_A(k, A/J^\nu) \longrightarrow \cdots.$$

By induction on ν we have $\mathrm{Ext}^i_A(k, A/J^\nu) = 0$ $(i < d)$ for all $\nu > 0$. Therefore by Theorem 28,

$$\mathrm{depth}(A/J^\nu) \geq d = \dim(A/J^\nu)$$

since $\sqrt{J^\nu} = \sqrt{J}$, so that A/J^ν is C.M.. ■

Chapter 7

Normal Rings and Regular Rings

17 Classical Theory

(17.A) Let A be an integral domain, and K be its quotient field. We say that A is normal if it is integrally closed in K. If A is normal, so is the localization $S^{-1}A$ for every multiplicative closed subset S of A not containing 0. Since $A = \bigcap_{\mathfrak{m}\in\mathrm{Max}(A)} A_{\mathfrak{m}}$ by (1.H), the domain A is normal iff $A_{\mathfrak{m}}$ is normal for every maximal ideal $\mathfrak{m}$.

An element u of K is said to be almost integral over A if there exists an element a of $A - \{0\}$ such that $au^n \in A$ for all $n > 0$. If u and v are almost integral over A, so are $u + v$ and uv. If $u \in K$ is integral over A then it is almost integral over A. The converse is also true when A is noetherian. In fact, if $a \neq 0$ and $au^n \in A$ $(n = 1, 2, \ldots)$, then $A[u]$ is a submodule of the finite A-module $a^{-1}A$, whence $A[u]$ itself is finite over A and u is integral over A.

We say that A is completely normal if every element u of K which is almost integral over A belongs to A. For a noetherian domain normality and complete normality coincide. Valuation rings of rank (= Krull dimension) greater than one (cf. Nagata: LOCAL RINGS or Zariski-Samuel: COMM. ALG. vol.II) are normal but not completely normal.

We say (in accordance with the usage of EGA) that a ring B is normal if $B_{\mathfrak{p}}$ is a normal domain for every prime ideal $\mathfrak{p}$ of B. A noetherian normal ring is a direct product of normal domains.

(17.B)

Proposition. (1) Let A be a complete normal domain. Then a polynomial ring $A[X_1, \ldots, X_n]$ over A is also completely normal. Similarly for a formal power series ring $A[[X_1, \ldots, X_n]]$.

(2) Let A be a normal ring. Then $A[X_1, \ldots, X_n]$ is normal.

Proof. Enough to treat the case of $n = 1$.

(1) Let K denote the quotient field of A. Then the quotient field of $A[X]$ is $K(X)$. Let $u \in K(X)$ be almost integral over $A[X]$. Since $A[X] \subseteq K[X]$ and since $K[X]$ is completely normal (because of unique factorization), the element u must belong to $K[X]$. Write

$$u = a_r x^r + a_{r+1} x^{r+1} + \cdots, \; a_r \neq 0.$$

Let $f(x) = b_s x^s + b_{s+1} x^{s+1} + \cdots \in A[X]$ be such that $f u^n \in A[X]$ for all n. Then $b_s a_r^n \in A$ for all n so that $a_r \in A$. Then $u - a_r x^r = a_{r+1} x^{r+1} + \cdots$ is almost integral over $A[X]$, so we get $a_{r+1} \in A$ as before, and so on. Therefore $u \in A[X]$. The case of $A[[X]]$ is proved similarly.

(2) Let $\mathfrak{q}$ be a prime ideal of $A[X]$ and let $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A[X]_{\mathfrak{q}}$ is a localization of $A_{\mathfrak{p}}[X]$ and $A_{\mathfrak{p}}$ is a normal domain. So we may assume that A is a normal domain with quotient field K. Let $u = P/Q$ ($P, Q \in A[X]$) be such that

$$u^d + f_1 u^{d-1} + \cdots + f_g = 0, \; f_i \in A[X].$$

In order to prove that $u \in A[X]$, we consider the subring A_0 of A generated by 1 and by the coefficients of P, Q and all the f_i's. Then u is in the quotient field of $A_0[X]$ and is integral over $A_0[X]$. The proof of (1) shows that u is a polynomial in X: $u = a_r x^r + \ldots + a_0$, and that each coefficient a_i is almost integral over A_0. As A_0 is noetherian (it is a finitely generated $\mathbb{Z}$-algebra), a_i is integral over A_0 and a fortiori over A. Therefore $a_i \in A$, as wanted. ∎

Remark. There exists a normal ring A such that $A[[X]]$ is not normal (A.Seidenberg).

(17.C) Let A be a ring and I an ideal with $\bigcap_n I^n = (0)$. Then for each non-zero element a of A there is an integer $n \geq 0$ such that $a \in I^n$ and $a \notin I^{n+1}$. We then write $n = \operatorname{ord}(a)$ (or $\operatorname{ord}_I(a)$) and call it the order of a (with respect to I). We have

$$\operatorname{ord}(a+b) \geq \min\{\operatorname{ord}(a), \operatorname{ord}(b)\} \text{ and } \operatorname{ord}(ab) \geq \operatorname{ord}(a) + \operatorname{ord}(b).$$

Put $A' = \operatorname{gr}^I(A) = \bigoplus_n I^n/I^{n+1}$. For an element a of A with $\operatorname{ord}(a) = n$, we call the image of a in $I^n/I^{n+1} = A'_n$ the leading form of a and denote it by a^*. We define $0^* = 0\,(\in A')$. The map $a \mapsto a^*$ is in general neither additive nor multiplicative, but if

(a) $a^*b^* \neq 0$ (i.e. if $\operatorname{ord}(ab) = \operatorname{ord}(a) + \operatorname{ord}(b)$) then we have $(ab)^* = a^*b^*$;

(b) $\operatorname{ord}(a) = \operatorname{ord}(b)$ and $a^* + b^* \neq 0$ then we have $(a+b)^* = a^* + b^*$.

It follows that, for any ideal J of A, the set

$$J^* := \{a^* \in A' \mid a \in J\}$$

is a graded ideal of A'.

⚠ Warning: if $J = \sum_i a_i A$ it does not necessarily follow that $J^* = \sum_i a_i^* A'$. But if J is a principal ideal aA and if A' is a domain, then we have $J^* = a^*A'$.

Put $\overline{A} = A/J$ and $\overline{I} = (I+J)/J$. Then it holds that $\operatorname{gr}^{\overline{I}}(\overline{A}) \cong \operatorname{gr}^I(A)/J^*$. In fact, we have

$$\begin{aligned}\overline{I}^n/\overline{I}^{n+1} &= (I^n + J)/(I^{n+1} + J) \cong I^n/(I^n \cap (I^{n+1} + J)) \\ &= I^n/((I^n \cap J) + I^{n+1}) = A'_n/J^*_n.\end{aligned}$$

(17.D)

Theorem 34 (Krull)**.** Let A, I and A' be as above. Then

(1) if A' is a domain, so is A;

(2) suppose that A is noetherian and that $I \subseteq \mathrm{rad}(A)$, then, if A' is a normal domain, so is A.

Proof. (1) Let a and b be elements of $A - \{0\}$. Then $a^* \neq 0$ and $b^* \neq 0$, hence $(ab)^* = a^*b^* \neq 0$ and so $ab \neq 0$.

(2) The ring A is a domain by (1). Let $a, b \in A$, $b \neq 0$, and suppose that a/b is integral over A. We have to prove $a \in bA$. The A-module A/bA is separated in the I-adic topology by (11.D) Corollary 1, in other words $bA = \bigcap_n (bA + I^n)$. Therefore it suffices to prove that $a \in bA + I^n$ for all n.

Suppose that $a \in bA + I^{n-1}$ is already proved. Then $a = br + a'$ with $r \in A$ and $a' \in I^{n-1}$, and $a'/b = a/b - r$ is integral over A. So we can replace a by a' and assume that $a \in I^{n-1}$. Since a/b is almost integral over A there exists $0 \neq c \in A$ such that $ca^m \in b^m$ for all m. As A' is a domain the map $[a \mapsto a^*]$ is multiplicative, hence we have $c^*a^{*m} \in b^{*m}A'$ for all m, and since A' is noetherian (by (10.D)) and normal we have $a^* \in b^*A'$. Let $c \in A$ be such that $a^* = b^*c^*$. Then

$$n - 1 \leq \mathrm{ord}(a) < \mathrm{ord}(a - bc),$$

whence $a - bc \in I^n$ so that $a \in bA + I^n$. ■

Remark. Even when A is a normal domain it can happen that A' is not a domain. Example: $A = k[x, y, z] = k[X, Y, Z]/(Z^2 - X^2 - Y^3)$, where k is a field of characteristic $\neq 2$, and $I = (x, y, z)$. We have $A' = \mathrm{gr}^I(A) \cong k[X, Y, Z]/(Z^2 - X^2)$, so $(x^* - z^*)(x^* + z^*) = 0$. On the other hand A is normal. In general, a ring of the form $k[X_1, \ldots, x_n, Z]/(X^2 - f(X))$ is normal provided that $f(X)$ is square-free.

(17.E) Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. Recall that the ring A is said to be regular if $\mathfrak{m}$ is generated by d elements, or what amounts to the same, if $d = \mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ (cf. (12.J)). A regular local ring of dimension 0 nothing but a field. The formal power series ring $k[[X_1, \ldots, X_d]]$ over a field k is a typical example of regular local ring.

Theorem 35. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then A is regular iff the graded ring $\operatorname{gr}(A) = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$ associated to the $\mathfrak{m}$-adic filtration is isomorphic (as a graded k-algebra) to a polynomial ring $k[X_1, \dots, X_d]$.

Proof. Suppose A is regular, and let $\{x_1, \dots, x_d\}$ be a regular system of parameters. Then $\operatorname{gr}(A) = k[x_1^*, \dots, x_d^*]$, hence $\operatorname{gr}(A)$ is of the form $k[X_1, \dots, X_d]/I$ where I is a graded ideal. If I contains a homogeneous polynomial $F(X) \neq 0$ of degree n_0 then we would have, for $n > n_0$,

$$\begin{aligned} \chi(A, \mathfrak{m}; n+1) &= \operatorname{len}_A(A/\mathfrak{m}^{n+1}) \\ &\leq \sum_{i=0}^{n} \dim_k((k[X]/(F))_i) \\ &= \binom{n+d}{d} - \binom{n-n_0+d}{d}, \end{aligned}$$

which is a polynomial of degree $d-1$ in n. But $\chi(A, \mathfrak{m}, n)$ is a polynomial in n (for $n \gg 0$) of degree $d = \dim(A)$ by (12.H). Therefore the ideal I must be (0).

Conversely, suppose $\operatorname{gr}(A) \cong k[X_1, \dots, X_d]$. Then we get

$$\chi(A, \mathfrak{m}; n+1) = \operatorname{len}_A(A/\mathfrak{m}^{n+1}) = \sum_{i=0}^{n} \dim_k(k[X]_i) = \binom{n+d}{d}.$$

So $\dim(A) = \deg \chi(A, \mathfrak{m}, z) = d$, while

$$\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = \operatorname{rank}_k(kX_1 + \cdots + kX_d) = d.$$

Thus A is regular. ∎

(17.F)

Theorem 36. Let A be a regular local ring and $\{x_1, \dots, x_d\}$ a regular system of parameters. Then:

(1) A is a normal domain;

(2) $x_1, \dots, x_d$ is an A-regular sequence, and hence A is a Cohen-Macaulay local ring;

(3) $(x_1, \ldots, x_i) = \mathfrak{p}_i$ is a prime ideal of height i for each $1 \leq i \leq d$, and $A/\mathfrak{p}_i$ is a regular local ring of dimension $d - i$;

(4) conversely, if $\mathfrak{p}$ is an ideal of A and if $A/\mathfrak{p}$ is regular and has dimension $d - i$, then there exists a regular system of parameters $\{y_1, \ldots, y_d\}$ such that $\mathfrak{p} = (y_1, \ldots, y_i)$.

Proof. (1) follows from Theorem 34, Theorem 35 and the fact that $k[X_1, \ldots, X_d]$ is a UFD.

(2) follows from (3) below. Alternatively, $x_1, \ldots, x_n$ is A-quasiregular since it satisfies ($\spadesuit''$) in (15.B). Therefore $x_1, \ldots, x_n$ is A-regular by Theorem 27. A is a C.M. local ring since

$$\dim(A) \geq \operatorname{depth}(A) = \operatorname{depth}(A/(x_1, \ldots, x_d)) + d \geq \dim(A).$$

(3) We have $\dim(A/\mathfrak{p}_i) = d - i$ by (12.K), while the maximal ideal $\mathfrak{m}/\mathfrak{p}_i$ of $A/\mathfrak{p}_i$ is generated by $d - i$ elements $\overline{x_{i+1}}, \ldots, \overline{x_d}$. Therefore $A/\mathfrak{p}_i$ is regular, and hence $\mathfrak{p}_i$ is a prime by (1).

(4) Put $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$. Then

$$d - i = \operatorname{rank}_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = \operatorname{rank}_k(\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{p})) = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) - \operatorname{rank}_k((\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2),$$

hence $i = \operatorname{rank}_k((\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2)$. Thus we can choose i elements $y_1, \ldots, y_i$ of $\mathfrak{p}$ which span $\mathfrak{p} + \mathfrak{m}^2$ mod $\mathfrak{m}^2$ over k, and $d - i$ elements $y_{i+1}, \ldots, y_d$ of $\mathfrak{m}$ which, together with $y_1, \ldots, y_i$, span $\mathfrak{m}$ mod $\mathfrak{m}^2$ over k. Then $\{y_1, \ldots, y_d\}$ is a regular system of parameters of A, so that $(y_1, \ldots, y_i) = \mathfrak{p}'$ is a prime ideal of height i by (3). As $\mathfrak{p} \supseteq \mathfrak{p}'$ and

$$\operatorname{ht}(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p}) = i,$$

we must have $\mathfrak{p} = \mathfrak{p}'$. ■

(17.G) Let A be a regular local ring of dimension 1, and let $\mathfrak{m} = (\pi)$ be the maximal ideal of A. Then the non-zero ideals of A are the powers $\mathfrak{m}^n = (\pi^n)$ $(n \geq 0)$ of $\mathfrak{m}$.

Proof. If I is an ideal and $I \neq 0$, then there exists $n \geq 0$ such that $I \subseteq \mathfrak{m}^n = (\pi^n)$ and $I \not\subseteq \mathfrak{m}^{n+1}$. Then $\pi^{-n}I$ is an ideal of A not contained in the maximal ideal $\mathfrak{m}$, therefore $\pi^{-n}I = A$, i.e. $I = (\pi^n)$, as claimed. ∎

Thus A is a PID. Furthermore, any fractional ideal (that is, finitely generated non-zero A-submodule of the quotient field A) is equal to some (π^n) $(n \in \mathbb{Z})$. If $0 \neq x \in K$ and $(x) = (\pi^n)$, then we write $n = \operatorname{ord}(x)$. Then $x \mapsto \operatorname{ord}(x)$ is a valuation of K with $\mathbb{Z}$ as the valuation group, and A is the ring of the valuation.

Conversely, let v be a valuation of K whose value group is discrete and of rank 1 (i.e. isomorphic to $\mathbb{Z}$); then the valuation ring R_v of v is called a principal valuation ring or a discrete valuation ring of rank 1, and is regular local ring of dimension 1. Thus a principal valuation ring and a one-dimensional regular local ring are the same thing. On the contrary, no other kinds of valuation rings are noetherian.

In the next paragraph we shall learn another characterization (Theorem 37) of the one-dimensional regular local rings.

(17.H) Let A be a noetherian domain with quotient field K. For any non-zero ideal I of A we put

$$I^{-1} = \{x \in K \mid xI \subseteq A\}.$$

We have $A \subseteq I^{-1}$ and $II^{-1} \subseteq A$.

Lemma 1. Let $0 \neq a \in A$ and $\mathfrak{p} \in \operatorname{Ass}_A(A/aA)$. Then $\mathfrak{p}^{-1} \neq A$.

Proof. Let $b \in A$ such that $\operatorname{Ann}_A\left(\bar{b}\right) = \mathfrak{p}$. Then $(aA : b) = \mathfrak{p}$. So $(b/a)\mathfrak{p} \subseteq A$ and $b/a \notin A$. ∎

Lemma 2. Let $(A, \mathfrak{m})$ be a noetherian domain such that $\mathfrak{m} \neq 0$ and $\mathfrak{m}\mathfrak{m}^{-1} = A$. Then $\mathfrak{m}$ is a principal ideal, and so A is regular of dimension 1.

Proof. Since $\bigcap_n \mathfrak{m}^n = (0)$ by (11.D) Corollary 3, we have $\mathfrak{m} \neq \mathfrak{m}^2$. Take $\pi \in \mathfrak{m} - \mathfrak{m}^2$. Then $\pi\mathfrak{m}^{-1} \subseteq A$, and if $\pi\mathfrak{m}^{-1} \subseteq \mathfrak{m}$ then $\pi A = \pi\mathfrak{m}^{-1}\mathfrak{m} \subseteq \mathfrak{m}^2$, contradicting the choice of π. Therefore we must have $\pi\mathfrak{m} = A$, that is, $\pi A = \pi\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$. ∎

Theorem 37. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is regular iff it is normal.

Proof. Suppose A is normal (hence a domain). By Lemma 2 it suffices to show $\mathfrak{m}\mathfrak{m}^{-1} = A$. Assume the contrary. Then $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$, and hence

$$a^n\mathfrak{m} \subseteq \mathfrak{m}(\mathfrak{m}^{-1})^n = \mathfrak{m} \subseteq A$$

for any $n > 0$ and $a \in \mathfrak{m}^{-1}$. Therefore all the elements of $\mathfrak{m}^{-1}$ are integral over A, whence $\mathfrak{m}^{-1} = A$ by the normality. But, as $\dim(A) = 1$, we have $\mathfrak{m} \in \mathrm{Ass}(A/\pi A)$ for any non-zero element π of $\mathfrak{m}$ so that $\mathfrak{m}^{-1} \neq A$ by Lemma 1. Thus $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$ cannot occur. ■

Theorem 38. Let A be a noetherian normal domain. Then any non-zero principal ideal is unmixed, and it holds that

$$A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

If $\dim(A) \leq 2$, then A is C.M..

Proof. Let $a \neq 0$ be a non-unit of A and let $\mathfrak{p} \in \mathrm{Ass}(A/aA)$. Replacing A by $A_{\mathfrak{p}}$ we may suppose that $(A, \mathfrak{p})$ is local. Then we have $\mathfrak{p}^{-1} \neq A$ by Lemma 1, and if $\mathrm{ht}(\mathfrak{p}) = \dim(A) > 1$ we would have a $\mathfrak{p}\mathfrak{p}^{-1} \neq A$ by Lemma 2. Similar to the preceding proof, this leads to a contradiction. Thus $\mathrm{ht}(\mathfrak{p}) = 1$. This implies that aA is unmixed. The second assertion of the theorem follows immediately from that.

If $\dim(A) \leq 2$, then by Theorem 32, we need to show that the unmixedness theorem holds in A, which follows from the first assertion and the definition of unmixedness. ■

(17.I) Let A be a noetherian ring. Consider the following conditions about A for $k = 0, 1, \ldots$;

(S_k) it holds $\mathrm{depth}(A_{\mathfrak{p}}) \geq \min\{k, \mathrm{ht}(\mathfrak{p})\}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$,

and

(R_k) if $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathrm{ht}(\mathfrak{p}) \leq k$, then $A_\mathfrak{p}$ is regular.

The condition (S_0) is trivial. The condition (S_1) holds iff $\mathrm{Ass}(A)$ has no embedded primes by

$$\mathrm{depth}(A_\mathfrak{p}) = 0 \iff \mathfrak{p} \in \mathrm{Ass}(A)$$

in (15.C). The condition (S_2), which is probably the most important, is equivalent to that not only $\mathrm{Ass}(A)$ but also $\mathrm{Ass}(A/fA)$ for every non-zero-divisor f of A have no embedded primes since

$$\mathrm{depth}(A_\mathfrak{p}) = 1 \iff \mathrm{depth}(A_\mathfrak{p}/fA_\mathfrak{p}) = 0 \iff \mathfrak{p} \in \mathrm{Ass}(A/fA)$$

for any non-zero-divisor $f \in \mathfrak{p}$ by the corollary in (15.C). The ring A is C.M. iff it satisfies all (S_k) since they are both equivalent to $\mathrm{depth}(A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

If (S_1) and (R_0) are satisfied, let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a primary decomposition. Then $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ is a minimal prime by (S_1) and $\mathfrak{p}_i A_{\mathfrak{p}_i} = (0)$ by (R_0). Let $a \in \mathfrak{p}_i$, then $ra = 0 \in \mathfrak{q}_i$ for some $r \notin \mathfrak{p}_i$. If $a \notin \mathfrak{q}_i$, then $r^n \in \mathfrak{q}_i \subseteq \mathfrak{p}_i$, a contradiction. So $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ is the nil-radical of A, this means A is reduced.

Conversely, if A is reduced, let $\mathfrak{p} = \mathrm{Ann}(a) \in \mathrm{Ass}(A)$. If $a \in \mathfrak{p}$, then $a^2 = 0$, which implies $a = 0$, a contradiction. Let $f/s \in \mathfrak{p}A_\mathfrak{p}$, then $f \in \mathfrak{p}$ and $fa = 0$, hence

$$\frac{f}{a} = \frac{f}{s}\frac{a}{a} = \frac{0}{sa} = 0.$$

So $\mathfrak{p}A_\mathfrak{p} = 0$, so $A_\mathfrak{p}$ is a field and $\mathrm{ht}(\mathfrak{p}) = 1$.

The following theorem is due to Krull(1931) in the case A is a domain, and to Serre in the general case.

Theorem 39 (Criterion of normality). A noetherian ring is normal iff it satisfies (S_2) and (R_1).

Proof. (After EGA IV_2 p.108). Let A be a noetherian ring. Suppose first that A is normal, and let $\mathfrak{p}$ be a prime ideal. Then $A_\mathfrak{p}$ is a field for $\mathrm{ht}(\mathfrak{p}) = 0$, and regular for $\mathrm{ht}(\mathfrak{p}) = 1$ by Theorem 37, hence the condition (R_1). Since a local normal ring is a domain, Theorem 38 implies that A satisfies (S_2).

Next suppose that A satisfies (S_2) and (R_1). Then A is reduced since it's (S_1) and (R_0). Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of A. Thus we have $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. The total quotient ring ΦA (cf. (1.N)) of A is isomorphic to the direct product $K_1 \times \cdots \times K_r$, where K_i is the quotient field of $A/\mathfrak{p}_i$; this follows from Chinese Remainder Theorem (1.C) applied to ΦA.

We shall prove that A is integrally closed in ΦA. Suppose this is done; then the unit element e_i of K_i belongs to A since $e_i^2 - e_i = 0$, and we have $1 = \sum_i e_i$ and $e_i e_j = 0$ $(i \neq j)$. Therefore $A = \prod_i Ae_i$, and Ae_i is a normal domain as it is integrally closed in K_i; thus A is a normal ring.

So suppose

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \ \text{ in } \Phi A,$$

where a, b and the c_i's are elements of A and b is A-regular. This is equivalent to $a^n + \sum_i c_i a^{n-i} b^i = 0$. We want to prove $a \in bA$. Let

$$bA = \mathfrak{q}_1 \cap \cdots \mathfrak{q}_r$$

be a primary decomposition and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then $\dim(A_{\mathfrak{p}_i}) = 1$ by (S_2), so it is normal by (R_1) and Theorem 37. We have

$$\left(\frac{a}{1}\right)^n + \sum_i \left(\frac{c_i}{1}\right) \left(\frac{a}{1}\right)^{n-i} \left(\frac{b}{1}\right)^i = 0,$$

in $A_{\mathfrak{p}_i}$, therefore $a/1 \in bA_{\mathfrak{p}_i}$. Then there exists $s \in A - \mathfrak{p}_i$ such that $as \in bA$, so $a \in \mathfrak{q}_i$ since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Hence $a \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = bA$. ∎

(17.J)

Theorem 40. Let A be a ring such that for every prime ideal $\mathfrak{p}$ the localization $A_{\mathfrak{p}}$ is regular. Then the polynomial ring $A[X_1, \dots, X_n]$ over A has the same property.

Proof. Enough to consider the case $n = 1$. Let $\mathfrak{q}$ be a prime ideal of $B = A[X]$, and put $\mathfrak{p} = \mathfrak{q} \cap A$. We want to prove that the local ring $B_{\mathfrak{q}}$ is regular. Since $B_{\mathfrak{q}}$ is a localization of $A_{\mathfrak{p}}[X]$ and since $A_{\mathfrak{p}}$ has the property, we may assume that A is a local ring and $\mathfrak{p}$ is the maximal ideal.

We have $\mathfrak{q} \supseteq \mathfrak{p}B$ and $B/\mathfrak{p}B = k[X]$ is a PID, where $k = A/\mathfrak{p}$ is a field. Therefore either $\mathfrak{q} = \mathfrak{p}B$, or $\mathfrak{q} = \mathfrak{p}B + fB$ with a monic polynomial $f \in A[X]$. Put $d = \dim(A)$. Then $\mathfrak{p}$ is generated by d elements, so $\mathfrak{q}$ is generated by d elements over B if $\mathfrak{q} = \mathfrak{p}B$, and by $d+1$ elements if $\mathfrak{q} = \mathfrak{p}B + fB$. On the other hand, since B is flat over A, by Theorem 19 we have

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = \begin{cases} d, & \text{if } \mathfrak{q} = \mathfrak{p}B; \\ d+1, & \text{if } \mathfrak{q} = \mathfrak{p}B + fB. \end{cases}$$

Therefore $B_{\mathfrak{q}}$ is regular. ■

In particular, all local rings of a polynomial ring $k[X_1, \ldots, X_n]$ over a field are regular.

18 Homological Theory

(18.A) Let A be a ring. The projective (resp. injective) dimension of an A-module M, denoted by $\mathrm{pd}(M)$ (resp. $\mathrm{id}(M)$), is the length of a shortest projective (resp. injective) resolution of M.

Lemma 1. Let M be an a module, then

(1) M is projective iff $\mathrm{Ext}^1_A(M, N) = 0$ for all A-modules N.

(2) M is injective iff $\mathrm{Ext}^1_A(A/I, M) = 0$ for all ideals I of A.

Proof. Immediate from the definitions. In (2) we use the fact (which is proved by Zorn's lemma) that if any homomorphism $f : N \to M$ can be extend to any A-module N' containing N such that $N' = N + A\xi$ for some $\xi \in N'$, then M is injective. ■

Lemma 2. Let A be a ring and $n \geq 0$ be an integer. Then the following conditions are equivalent:

(i) $\mathrm{pd}(M) \leq n$ for all A-modules M,

(ii) $\mathrm{pd}(M) \leq n$ for all finite A-modules M,

(iii) $\mathrm{id}(M) \leq n$ for all A-modules M,

(iv) $\mathrm{Ext}_A^{n+1}(M, N) = 0$ for all A-modules M and N.

Proof. (1) $\Longrightarrow$ (2): trivial. (2) $\Longrightarrow$ (3): take an exact sequence

$$0 \longrightarrow M \longrightarrow Q^0 \longrightarrow \cdots \longrightarrow Q^{n-1} \longrightarrow N \longrightarrow 0$$

with Q^j injective for all j. Let I be any ideal. Then we have $\mathrm{Ext}_A^1(A/I, N) \cong \mathrm{Ext}_A^{n+1}(A/I, M)$, which is zero by (2) since A/I is a finite A-module. So N is injective by Lemma 1, hence $\mathrm{id}(M) \leq n$.

(3) $\Longrightarrow$ (4): let

$$0 \longrightarrow N \longrightarrow Q^0 \longrightarrow \cdots \longrightarrow Q^{n-1} \longrightarrow Q^n \longrightarrow 0$$

be an injective resolution of N, then

$$\mathrm{Ext}_A^{n+1}(M, N) = H^{n+1}(\mathrm{Hom}(M, Q^\bullet)) = 0.$$

(4) $\Longrightarrow$ (1): similar to (2) $\Longrightarrow$ (3). Take an exact sequence

$$0 \longrightarrow N \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_j projective for all j. Let L be any A-module. Then we have $\mathrm{Ext}_A^1(N, L) \cong \mathrm{Ext}_A^{n+1}(M, L) = 0$. So N is projective by Lemma 1, hence $\mathrm{pd}(M) \leq n$. ∎

By virtue of Lemma 2 we have

$$\sup_M \mathrm{pd}(M) = \sup_M \mathrm{id}(M).$$

We call this common value (which may be ∞) the global dimension of A and denote it by $\mathrm{gldim}(A)$. (In EGA it is denoted by $\dim.\mathrm{coh}(A)$.)

(18.B)

Lemma 3. Let A be a noetherian ring and M a finite A-module. Then $\mathrm{pd}(M) \leq n$ iff $\mathrm{Ext}_A^{n+1}(M, N) = 0$ for all finite A-modules N.

Proof. The only if part is trivial. If $n = 0$, take a resolution

$$(\spadesuit) \quad 0 \longrightarrow R \xrightarrow{i} F \xrightarrow{p} M \longrightarrow 0$$

with F finite and free. Then R is also finite, hence we have $\mathrm{Ext}_A^1(M, R) = 0$. Thus $\mathrm{Hom}(F, R) \xrightarrow{i^*} \mathrm{Hom}(R, R) \longrightarrow 0$ is exact, and so there exists $s : F \to R$ with $s \circ i = \mathrm{id}_R$, i.e. the sequence $(\spadesuit)$ splits. Then M is a direct summand of a free module.

For general n, take an exact sequence

$$0 \longrightarrow N \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with F_j finite and free for all j. Then N is also finite. Let L be any finite A-module, then we have $\mathrm{Ext}_A^1(N, L) = \mathrm{Ext}_A^{n+1}(M, L) = 0$. So N is projective and hence $\mathrm{pd}(M) \leq n$. ∎

Lemma 4. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and M be a finite A-module. Then

$$\mathrm{pd}(M) \leq n \iff \mathrm{Tor}_{n+1}^A(M, k) = 0.$$

Proof. ($\implies$) is trivial. ($\impliedby$): If $n = 0$, let $x_1, \ldots, x_r \in M$ such that $\{\overline{x_1}, \ldots, \overline{x_r}\}$ form a basis of $M/\mathfrak{m}M$. Let $F = \bigoplus_{i=1}^{r} Ae_i$ and let $u : F \to M$ defined by $e_i \mapsto x_i$, then u is minimal, i.e. $u \otimes 1_k : F \otimes k \to M \otimes k$ is an isomorphism. We have the exact sequence

$$0 \longrightarrow \ker u \longrightarrow F \xrightarrow{u} M \longrightarrow 0.$$

Then

$$0 = \mathrm{Tor}_1^A(M, k) \longrightarrow \ker u \otimes k \longrightarrow F \otimes k \xrightarrow{u \otimes 1_k} M \otimes k \longrightarrow 0$$

is exact, hence $R \otimes k = 0$ and so $R = 0$ by Nakayama's lemma. Therefore M is free, as wanted.

For general n, let

$$0 \longrightarrow N \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be exact with P_i projective for all i. Then $\mathrm{Tor}_1^A(N, k) \cong \mathrm{Tor}_{n+1}^A(M, k) = 0$, hence N is free and so $\mathrm{pd}(M) \leq n$. ∎

Lemma 5. Let A be a noetherian ring.

(1) Let M be a finite A-module. Then

(a) $\mathrm{pd}_A(M) = \sup_{\mathfrak{m}\in\mathrm{Max}(A)} \mathrm{pd}_{A_\mathfrak{m}}(M_\mathfrak{m})$;

(b) we have $\mathrm{pd}_A(M) \leq n$ iff $\mathrm{Tor}^A_{n+1}(M, A/\mathfrak{m}) = 0$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.

(2) The following conditions are equivalent:

(i) $\mathrm{gldim}(A) \leq n$,

(ii) $\mathrm{pd}(M) \leq n$ for all finite A-modules M,

(iii) $\mathrm{id}(M) \leq n$ for all finite A-modules M,

(iv) $\mathrm{Ext}^{n+1}_A(M, N) = 0$ for all finite A-modules M and N,

(v) $\mathrm{Tor}^A_{n+1}(M, N) = 0$ for all finite A-modules M and N.

(3) Then

$$\mathrm{gldim}(A) = \sup_{\mathfrak{m}\in\mathrm{Max}(A)} \mathrm{gldim}(A_\mathfrak{m}).$$

Proof. (1) (a) Let N be a finite A-module, then by (3.E) we have $\mathrm{Ext}^{n+1}_A(M, N)_\mathfrak{m} = \mathrm{Ext}^{n+1}_{A_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$. So

$$\mathrm{Ext}^{n+1}_A(M, N) = 0 \iff \mathrm{Ext}^{n+1}_{A_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = 0 \ \forall\, \mathfrak{m} \in \mathrm{Max}(A).$$

By Lemma 3,

$$\begin{aligned}
\mathrm{pd}_A(M) &= \inf \left\{ n \geq 0 \mid \mathrm{Ext}^{n+1}_A(M, N) = 0,\ \forall\, N \text{ finite} \right\} \\
&= \inf \left\{ n \geq 0 \mid \mathrm{Ext}^{n+1}_{A_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = 0,\ \forall\, N \text{ finite},\ \mathfrak{m} \in \mathrm{Max}(A) \right\} \\
&= \sup_{\mathfrak{m}\in\mathrm{Max}(A)} \inf \left\{ n \geq 0 \mid \mathrm{Ext}^{n+1}_{A_\mathfrak{m}}(M_\mathfrak{m}, N) = 0,\ \forall\, N \text{ finite} \right\} \\
&= \sup_{\mathfrak{m}\in\mathrm{Max}(A)} \mathrm{pd}_{A_\mathfrak{m}}(M_\mathfrak{m}).
\end{aligned}$$

(b) By (a) and Lemma 4 since $(A/\mathfrak{m})_\mathfrak{m} = A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$.

(2) We already saw (ii) $\iff$ (i) $\implies$ (iii) in Lemma 2, and (iii) $\implies$ (iv) and (ii) $\implies$ (v) are trivial. Moreover, (v) $\implies$ (ii) by (1) above, and (iv) $\implies$ (ii) is easy to see by Lemma 3.

(3) By (1) and (2), we have

$$\begin{aligned}\operatorname{gldim}(A) &= \sup_{M \text{ finite}} \operatorname{pd}_A(M) = \sup_{M \text{ finite}} \left(\sup_{\mathfrak{m} \in \operatorname{Max}(A)} \operatorname{pd}_{A_\mathfrak{m}}(M_\mathfrak{m}) \right) \\ &= \sup_{\mathfrak{m} \in \operatorname{Max}(A)} \left(\sup_{M \text{ finite}} \operatorname{pd}_{A_\mathfrak{m}}(M_\mathfrak{m}) \right) \leq \sup_{\mathfrak{m} \in \operatorname{Max}(A)} \operatorname{gldim}(A_\mathfrak{m}).\end{aligned}$$

For each $\mathfrak{m} \in \operatorname{Max}(A)$, let M be a finite $A_\mathfrak{m}$-module. Then there is a finite A-module N such that $M = N_\mathfrak{m}$. So by (1),

$$\operatorname{pd}_A(N) \geq \operatorname{pd}_{A_\mathfrak{m}}(N_\mathfrak{m}) = \operatorname{pd}_{A_\mathfrak{m}}(M).$$

Hence $\operatorname{gldim}(A) \geq \operatorname{gldim}(A_\mathfrak{m})$. ■

Theorem 41. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then

$$\operatorname{gldim}(A) \leq n \iff \operatorname{Tor}_{n+1}^A(k, k) = 0.$$

Consequently, we have $\operatorname{gldim}(A) = \operatorname{pd}(k)$ (as A-module).

Proof. The only if part is trivial. For the if part, by Lemma 4 we have

$$\begin{aligned}\operatorname{Tor}_{n+1}^A(k, k) = 0 &\implies \operatorname{pd}(k) \leq n \\ &\implies \operatorname{Tor}_{n+1}^A(M, k) = 0, \ \forall\, M \text{ finite} \\ &\implies \operatorname{pd}(M) \leq n, \ \forall\, M \text{ finite} \\ &\implies \operatorname{gldim}(A) \leq n.\end{aligned}$$

■

(18.C)

Lemma 6. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finite A-module. If $\operatorname{pd}(M) = r < \infty$ and if x is an M-regular element in $\mathfrak{m}$, then $\operatorname{pd}(M/xM) = r + 1$.

Proof. The sequence

$$0 \longrightarrow M \xrightarrow{[x]} M \longrightarrow M/xM \longrightarrow 0$$

is exact by assumption, therefore the sequences

$$0 = \operatorname{Tor}_i^A(M, k) \longrightarrow \operatorname{Tor}_i^A(M/xM, k) \longrightarrow \operatorname{Tor}_{i-1}^A(M, k) = 0 \quad (i > r + 1)$$

and

$$\mathrm{Tor}^A_{r+1}(M,k)=0 \longrightarrow \mathrm{Tor}^A_{r+1}(M/xM,k) \longrightarrow \mathrm{Tor}^A_r(M,k) \xrightarrow{[x]} \mathrm{Tor}^A_r(M,k)$$

are also exact. Since $k = A/\mathfrak{m}$ is annihilated by x, the A-module $\mathrm{Tor}^A_r(M,k)$ is also annihilated by x. Therefore $\mathrm{Tor}^A_{r+1}(M/xM,k) \cong \mathrm{Tor}^A_r(M,k) \neq 0$ by Lemma 4 and $\mathrm{Tor}^A_i(M/xM,k) = 0$ for $i > r+1$. We then have $\mathrm{pd}(M/xM) = r+1$ by Lemma 4. ■

Theorem 42. Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension n. Then

$$\mathrm{gldim}(A) = n.$$

Proof. Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters. Then the sequence $x_1, \ldots, x_n$ is A-regular and $k = A/(x_1, \ldots, x_n)$, hence we have $\mathrm{pd}(k) = n$ by Lemma 6. So the theorem follows from Theorem 41. ■

Corollary (Hilbert Syzygy Theorem)**.** Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then $\mathrm{gldim}(A) = n$.

Proof. This follows from Theorem 22, Theorem 40, Theorem 42 and Lemma 5. ■

We are going to prove a converse (due to Serre) of Theorem 42, namely that a noetherian local ring of finite global dimension is regular (Theorem 45). This is more important than Theorem 42, and its proof is also more difficult. Roughly speaking there are two different proofs: one is due to Nagata (simplified by Grothendieck) and uses induction on $\dim(A)$. This proof is shorter and does not require big tools (cf. EGA IV_1 pp.46-48). The other is due to Serre and uses Koszul complex and minimal resolution; it has the merit of giving more information about the homology groups $\mathrm{Tor}_i(k,k)$. Here we shall follows Serre's proof. We begin with explaining the necessary homological techniques, which are useful in other situations also.

(18.D) Koszul Complex. Let A be a ring. A (chain) complex $M_\bullet$ is a sequence

$$M_\bullet : \quad \cdots \longrightarrow M_n \xrightarrow{\mathrm{d}} M_{n-1} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} M_0 \xrightarrow{\mathrm{d}} 0$$

of A-modules and A-linear maps such that $\mathrm{d}^2 = 0$. The module M_i is called the i-dimensional part of the complex and the map d is called the differentiation. If $L_\bullet$ and $M_\bullet$ are two complexes, their tensor product $L_\bullet \otimes M_\bullet$ is, by definition, the complex such that

$$(L_\bullet \otimes M_\bullet)_n = \bigoplus_{p+q=n} L_p \otimes_A M_q$$

and such that $\mathrm{d} : (L_\bullet \otimes M_\bullet)_n \to (L_\bullet \otimes M_\bullet)_{n-1}$ is defined on $L_p \otimes_A M_q$ by the formula $\mathrm{d}(x \otimes y) = d_L(x) \otimes y + (-1)^p x \otimes d_M(y)$.

Let $x_1, \ldots, x_n \in A$, and let Ae_i be a free A-module of rank one with a specified basis e_i for $1 \leq i \leq n$. Let

$$K_\bullet(x_i) : \quad 0 \longrightarrow Ae_i \xrightarrow{x_i} A \longrightarrow 0$$

denote the complex defined by

$$K_p(x_i) = \begin{cases} A, & \text{if } p = 0 \\ Ae_i, & \text{if } p = 1 \\ 0, & \text{else} \end{cases}$$

and by $\mathrm{d}(e_i) = x_i$. Then $h_0(K_\bullet(x_i)) = A/x_iA$ and $h_1(K_\bullet(x_i)) \cong \mathrm{Ann}(x_i)$. For any complex $C_\bullet$ of A-modules, we put

$$C_\bullet(x_1, \ldots, x_n) = C_\bullet \otimes K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n).$$

If M is an A-module we view it as a complex $M_\bullet$ with $M_n = 0$ $(n \neq 0)$ and $M_0 = M$, and we put $K_\bullet(x_1, \ldots, x_n; M) = M_\bullet(x_1, \ldots, x_n)$. If there is no danger of confusion we denote them by $C_\bullet(\underline{x})$ and by $K_\bullet(\underline{x}; M)$ respectively. These complexes are called Koszul complexes. We have $K_p(x_1, \ldots, x_n; M) = 0$ for $n < p$, while

$$K_p(x_1, \ldots, x_n; M) = \bigoplus_{i_1 < \cdots < i_p} M \otimes_A (Ae_{i_1} \otimes_A \cdots \otimes Ae_{i_p})$$

for $0 \leq p \leq n$. Put $e_{i_1 \ldots i_p} = e_{i_1} \otimes e_{i_p}$. Then

$$K_0(x_1, \ldots, x_n; M) = M,$$
$$K_p(x_1, \ldots, x_n; M) = \bigoplus_{i_1 < \cdots < i_p} Me_{i_1 \ldots i_p} \cong M^{\binom{n}{p}} \quad (1 \leq p \leq n).$$

and

$$(\heartsuit) \quad \mathrm{d}(me_{i_1\dots i_p}) = \sum_{r=1}^{p}(-1)^r x_{i_r} m e_{i_1\dots\widehat{i_r}\dots i_p}$$

(where $m \in M$, and $\widehat{i_r}$ indicates that i_r is omitted there). The formula ($\heartsuit$) for the operator d can be put into another form: let $\displaystyle\sum_{i_1<\dots<i_p} m_{i_1\dots i_p} e_{i_1\dots i_p}$ be an arbitrary element of $K_p(\underline{x}, M)$, and extend the $m_{i_1\dots i_p}$'s to an alternating function of the indices (i.e. such that $m_{\dots i\dots i\dots} = 0$ and $m_{\dots i\dots j\dots} = -m_{\dots j\dots i\dots}$). Then we have

$$(\diamondsuit) \quad \mathrm{d}\left(\sum_{i_1<\dots<i_p} m_{i_1\dots i_p} e_{i_1\dots i_p}\right) = \sum_{j=1}^{n} x_j \left(\sum_{i_1<\dots<i_{p-1}} m_{j i_1\dots i_{p-1}} e_{i_1\dots i_{p-1}}\right).$$

There is another interpretation of the Koszul complex. Let $F = AX_1 \oplus \dots \oplus AX_n$ be a free A-module of rank n with a basis $\{X_1, \dots, X_n\}$. Then the exterior product $\bigwedge^p F$ is a free module of rank $\binom{n}{p}$ with

$$\left\{X_{i_1} \wedge \dots \wedge X_{i_p} \mid 1 \le i_1 < \dots < i_p \le n\right\}$$

as a basis, so that there is an isomorphism of A-modules $M \otimes_A \bigwedge^p F \to K_p(\underline{x}, M)$ which maps $X_{i_1} \wedge \dots \wedge X_{i_p}$ to $e_{i_1\dots i_p}$. Thus we can define $K_\bullet(\underline{x}, M)$ to be the complex $M \otimes L_\bullet$ with $L_p = \bigwedge^p F$ and with

$$\mathrm{d}\left(X_{i_1} \wedge \dots \wedge X_{i_p}\right) = \sum_{r=1}^{p}(-1)^{r-1} x_{i_r} X_{i_1} \wedge \dots \widehat{X_{i_r}} \wedge \dots \wedge X_{i_p}.$$

If we adopt this definition then we have to check $\mathrm{d}^2 = 0$ on $L_\bullet$, which is straightforward anyway.

For any $x \in A$, we have an exact sequence of complexes

$$0 \longrightarrow A \longrightarrow K_\bullet(x) \longrightarrow A' \longrightarrow 0$$

where A' is the factor complex $K_\bullet(x)/A$, therefore $(A')_1 \cong A$ and $(A')_n = 0$ for $n \neq 1$. Let $C_\bullet$ be any complex. Then tensoring the exact sequence with $C_\bullet$ we get

$$0 \longrightarrow C_\bullet \longrightarrow C_\bullet(x) \longrightarrow C'_\bullet \longrightarrow 0 \qquad (C'_\bullet = C_\bullet \otimes A'),$$

which is again exact. The complex $C'_\bullet$ is obtained from $C_\bullet$ increasing the dimension by one: $C'_p = C_{p-1}$ and $d'_p = d_{p-1}$. In fact, we get the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{p+1} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "\mathrm{d}"] & C_{p+1}\oplus C_p \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}\mathrm{d} & (-1)^p x\\ 0 & \mathrm{d}\end{smallmatrix}\right)"] & C_p \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d] & C_p\oplus C_{p-1} \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d] & C_{p-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Thus $H_p(C'_\bullet) \cong H_{p-1}(C_\bullet)$, and we get a long exact sequence

$$\cdots \longrightarrow H_p(C_\bullet) \longrightarrow H_p(C_\bullet(x)) \longrightarrow H_{p-1}(C_\bullet) \xrightarrow{\delta_p} H_{p-1}(C_\bullet) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_1(C_\bullet) \longrightarrow H_1(C_\bullet(x)) \longrightarrow H_0(C_\bullet) \xrightarrow{\delta_1} H_0(C_\bullet) \longrightarrow H_0(C_\bullet(x)) \longrightarrow 0.$$

The connecting homomorphism δ_p comes from the following diagram:

$$\begin{tikzcd}
 & & & H_{p-1}(C_\bullet) \arrow[d] & \\
 & C_p/\,\mathrm{d}(C_{p+1}) \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "\mathrm{d}"] & \dfrac{C_p\oplus C_{p-1}}{\mathrm{d}(C_{p-1}\oplus C_{p-2})} \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}\mathrm{d} & (-1)^{p-1} x\\ 0 & \mathrm{d}\end{smallmatrix}\right)"] & C_{p-1}/\,\mathrm{d}(C_p) \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\
0 \arrow[r] & \ker \mathrm{d} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d] & \ker \begin{pmatrix}\mathrm{d} & (-1)^{p-2}x\\ 0 & \mathrm{d}\end{pmatrix} \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] & \ker \mathrm{d} & \\
 & H_{p-1}(C_\bullet) & & &
\end{tikzcd}$$

One immediately checks that δ_p is the multiplication by $(-1)^{p-1}x$. Therefore we get

Lemma 7. If $C_\bullet$ is a complex with $H_p(C_\bullet) = 0$ for all $p > 0$, then $H_p(C_\bullet(x)) = 0$ for all $p > 1$ and

$$0 \longrightarrow H_1(C_\bullet(x)) \longrightarrow H_0(C_\bullet) \xrightarrow{[x]} H_0(C_\bullet) \longrightarrow H_0(C_\bullet(x)) \longrightarrow 0$$

is exact. If, in particular, x is $H_0(C_\bullet)$-regular, then we have $H_p(C_\bullet(x)) = 0$ for all $p > 0$ and $H_0(C_\bullet(x)) = H_0(C_\bullet)/xH_0(C_\bullet)$.

Theorem 43. Let A be a ring, M an A-module and $x_1, \ldots, x_n$ an M-regular sequence in A. Then we have

$$H_p(K_\bullet(\underline{x}; M)) = \begin{cases} 0, & \text{if } p > 0; \\ M/(\underline{x})M, & \text{if } p = 0. \end{cases}$$

Proof. Induction on n. Let $C_\bullet = K_\bullet(x_1, \ldots, x_{n-1}; M)$ and let $x = x_n$, then by Lemma 7. ∎

Corollary. Let A be a ring and $x_1, \ldots, x_n$ be an A-regular sequence in A. Then $K_\bullet(x_1, \ldots, x_n; A)$ is a free resolution of the A-module $A/(x_1, \ldots, x_n)$.

(18.E) Minimal Resolution. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. We say that a homomorphism $u : L \to M$ is minimal if $u \otimes 1_k : L \otimes_A k \to M \otimes_A k$ is an isomorphism, or equivalently, if u is surjective with $\ker u \subseteq \mathfrak{m}L$. Let M be a finite A-module. A free resolution of M,

$$\cdots \longrightarrow L_i \xrightarrow{\mathrm{d}_i} L_{i-1} \longrightarrow \cdots \xrightarrow{\mathrm{d}_1} L_0 \xrightarrow{\epsilon} M \longrightarrow 0,$$

is called a minimal resolution if $\epsilon : L_0 \to M$ and $\mathrm{d}_i : L_i \to \ker(d_{i-1})$ are minimal for any i. In this case the complex

$$L_\bullet \otimes k : \cdots \longrightarrow \overline{L_i} \xrightarrow{\overline{\mathrm{d}_i}} \overline{L_{i-1}} \longrightarrow \cdots \xrightarrow{\overline{\mathrm{d}_1}} \overline{L_0},$$

where $\overline{L_i} = L_i \otimes_A k = L_i/\mathfrak{m}L_i$, has trivial differentiation (i.e. all $\overline{d_i} = 0$). Therefore we have $\mathrm{Tor}_i^A(M, k) \cong \overline{L_i}$ for all i, and so $\mathrm{rank}_A L_i = \dim_k \mathrm{Tor}_i^A(M, k)$. In particular, all L_i are finite over A.

Lemma 8. Let M be a finite module over a noetherian local ring A. Then a minimal resolution of M exists, and is unique up to (non-canonical) isomorphisms.

Proof. The existence is easy to see: one constructs a minimal resolution step by step, using minimal basis (as in the proof of Lemma 4). To prove the uniqueness, let $L_\bullet \to M$ and $L'_\bullet \to M$ be two minimal resolutions of M. Since $L_\bullet$ is a projective resolution there exists a homomorphism $f : L_\bullet \to L'_\bullet$ of complexes over M. Since

$$\begin{tikzcd} L_1 \arrow[r, "\mathrm{d}_1"] \arrow[d, "f_1"] & L_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & M \arrow[d, "1_M"] \\ L'_1 \arrow[r, "\mathrm{d}'_1"] & L'_0 \arrow[r, "\epsilon'"] & M \end{tikzcd}$$

is commutative and since ϵ and ϵ' are minimal, the map $\overline{f_0}$ is an isomorphism. Since both L_0 and L'_0 are free, the map f_0 is then defined by a square matrix T with $\det T \notin \mathfrak{m}$. Then f_0 itself is an isomorphism. Repeating the same reasoning we prove inductively that all f_i are isomorphisms. ■

Proposition (Auslander-Buchsbaum)**.** Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and $M \neq 0$ be a finite A-module. If $\mathrm{pd}(M)$ is finite, then

$$\mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(A).$$

Proof. Say $\mathrm{pd}(M) = n$, then M has a minimal free resolution $L_\bullet \to M$ with $L_p = 0$ for $n < p$.

Suppose $\mathrm{depth}(A) = 0$. Then $\mathfrak{m} \in \mathrm{Ass}(A)$, so there exists a short exact sequence:

$$0 \longrightarrow k \longrightarrow A \longrightarrow N \longrightarrow 0$$

From this we get a long exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}_p^A(A, M) \longrightarrow \mathrm{Tor}_p^A(N, M) \longrightarrow \mathrm{Tor}_{p-1}^A(k, M) \longrightarrow \mathrm{Tor}_{p-1}^A(A, M) \longrightarrow \cdots$$

So we have $\mathrm{Tor}_p^A(N, M) \cong \mathrm{Tor}_{p-1}^A(k, M)$ for all p and in particular for $p = n + 1$, $\mathrm{Tor}_{n+1}^A(N, M) \cong \mathrm{Tor}_n^A(k, M)$. If $n > 0$, then by Lemma 4, $\mathrm{Tor}_{n+1}^A(N, M) \cong \mathrm{Tor}_n^A(k, M) \neq 0$, a contradiction. So $n = 0$ and then M is projective, hence free. Thus,

$$\mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(M) = \mathrm{depth}(A).$$

Next, let $\mathrm{depth}(A) > 0$. If $\mathrm{depth}(M) > 0$, then $\mathfrak{m} \notin \mathrm{Ass}(M)$ and $\mathfrak{m} \notin \mathrm{Ass}(A)$. Therefore we can find an $x \notin \mathfrak{m}$ such that x is A-regular and M-regular. Then by (3.F), $\mathrm{Tor}_1^A(A/(x), M) = 0$ and $\mathrm{Tor}_1^A(A/(x), L_p) = 0$ for each p since L_p is free.

Hence, $L_\bullet \otimes A/(x)$ is exact. Therefore, it is a free resolution of M/xM over $A/(x)$. So we have

$$\mathrm{Tor}_i^{A/(x)}(k, M/xM) = H_i(L_\bullet \otimes_A A/(x) \otimes_{A/(x)} k) \cong H_i(L_\bullet \otimes_A k) = \mathrm{Tor}_i^A(k, M)$$

for all $i \geq 0$. Therefore

$$\begin{aligned}\mathrm{pd}_{A/(x)}(M/xM) &= \inf \left\{ i \geq 0 \,\middle|\, \mathrm{Tor}_{i+1}^{A/(x)}(k, M/xM) = 0 \right\} \\ &= \inf \left\{ i \geq 0 \,\middle|\, \mathrm{Tor}_{i+1}^{A}(k, M) = 0 \right\} \\ &= \mathrm{pd}_A(M).\end{aligned}$$

By the corollary in (15.C), we have

$$\begin{aligned}\mathrm{depth}_{A/(x)}(A/(x)) &= \mathrm{depth}_A(A) - 1, \\ \mathrm{depth}_{A/(x)}(M/xM) &= \mathrm{depth}_A(M) - 1.\end{aligned}$$

Therefore, by induction on $\mathrm{depth}(A)$,

$$\mathrm{pd}_{A/(x)}(M/xM) + \mathrm{depth}_{A/(x)}(M) = \mathrm{depth}_{A/(x)}(A/(x)),$$

and consequently,

$$\mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(A).$$

Therefore we only need to consider the case $\mathrm{depth}(M) = 0$. Take the exact sequence

$$0 \longrightarrow K \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

Then $\mathrm{pd}(K) = \mathrm{pd}(M) - 1$. From the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_A^i(k, M) \longrightarrow \mathrm{Ext}_A^{i+1}(k, K) \longrightarrow \mathrm{Ext}_A^{i+1}(k, L_0) \longrightarrow \mathrm{Ext}_A^{i+1}(k, M) \longrightarrow \cdots$$

and the fact that

$$\mathrm{depth}(N) = \inf \{ i \geq 0 \mid \mathrm{Ext}_A^i(k, N) \neq 0 \},$$

we get $\mathrm{depth}(K) = 1$ since $\mathrm{Ext}_A^0(k, L_0) \cong \mathrm{Hom}_A^0(k, A)^s = 0$ by $\mathrm{depth}(A) > 0$, where $s = \mathrm{rank}_A L_0$. Thus,

$$\mathrm{pd}(M) + \mathrm{depth}(M) = (\mathrm{pd}(K) + 1) + (\mathrm{depth}(K) - 1) = \mathrm{depth}(A). \qquad \blacksquare$$

Let $L_\bullet \to M$ be a minimal resolution and $F_\bullet \to M$ be an arbitrary free resolution. Then since $L_\bullet$ is a projective resolution there exists a homomorphism $f : L_\bullet \to F_\bullet$ of complexes over M. So we have the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & L_1 & \xrightarrow{\mathrm{d}_1} & L_0 & \xrightarrow{\epsilon} & M \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_M} \\
\cdots & \longrightarrow & F_1 & \xrightarrow{\mathrm{d}_1'} & F_0 & \xrightarrow{\epsilon'} & M
\end{array}$$

and it induces a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \overline{L_1} & \xrightarrow{\overline{\mathrm{d}_1}} & \overline{L_0} & \xrightarrow{\overline{\epsilon}} & \overline{M} \\
 & & \downarrow{\scriptstyle \overline{f_1}} & & \downarrow{\scriptstyle \overline{f_0}} & & \downarrow{\scriptstyle 1_{\overline{M}}} \\
\cdots & \longrightarrow & \overline{F_1} & \xrightarrow{\overline{\mathrm{d}_1'}} & \overline{F_0} & \xrightarrow{\overline{\epsilon'}} & \overline{M}
\end{array}$$

Since $\overline{\epsilon}$ and $1_{\overline{M}}$ are isomorphisms, $\overline{f_0}$ is injective. Then there exists $g_0 : F_0 \to L_0$ such that $g_0 \circ f_0 = 1_{L_0}$ since both L_0 and F_0 are free. So $F_0 \cong L_0 \oplus W_0$ for some W_0 and there's a map $\epsilon'' : W_0 \to M$ such that $\epsilon' = \epsilon \oplus \epsilon''$. Then we get the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \epsilon & \longrightarrow & \ker(\epsilon \oplus \epsilon'') & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0 & \longrightarrow & L_0 \oplus W_0 & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon \oplus \epsilon''} & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The first row is split exact by snake lemma and the fact that the second row and the third row are also split exact. So $\ker(\epsilon \oplus \epsilon'') \cong \ker \epsilon \oplus W_0$. Repeating the same reasoning we prove inductively that $F_\bullet \cong L_\bullet \oplus W_\bullet$ with some acyclic complex $W_\bullet$.

Lemma 9. Let

$$\cdots \longrightarrow L_i \xrightarrow{\mathrm{d}_i} L_{i-1} \longrightarrow \cdots \xrightarrow{\mathrm{d}_1} L_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

be a minimal resolution of M, and

$$\cdots \longrightarrow F_i \xrightarrow{\mathrm{d}_i'} F_{i-1} \longrightarrow \cdots \xrightarrow{\mathrm{d}_1'} F_0 \xrightarrow{\epsilon'} M \longrightarrow 0$$

be a complex, such that

(1) each F_i is finite and free over A,

(2) $\overline{\epsilon'} : \overline{F_0} \to \overline{M}$ is injective, and

(3) $\mathrm{d}'_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for each $i > 0$, and d'_i induces an injection

$$\overline{F_i} \longrightarrow \overline{\mathfrak{m}F_{i-1}} = \left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_A F_{i-1}.$$

Then there exists a homomorphism $f : F_\bullet \to L_\bullet$ of complexes over M such that each f_i maps F_i isomorphically onto a direct summand of L_i. Consequently, we have

$$\operatorname{rank}_A F_i \leq \operatorname{rank}_A L_i = \dim_k \operatorname{Tor}_i^A(M, k).$$

Proof. Since $L_\bullet$ is a resolution and since each F_i is free, there exists a homomorphism $f : F_\bullet \to L_\bullet$ over M. We have to prove that, for each i, there exists an A-linear map $g_i : L_i \to F_i$ with $g_i \circ f_i = 1_{F_i}$. Since both F_i and L_i are free, we can easily see that such g_i exists iff $\overline{f_i} : \overline{F_i} \to \overline{L_i}$ is injective. Similar to the discussion above, by (2) we have f_0 is also injective. So $L_0 \cong F_0 \oplus W_0$ for some W_0 and $\ker \epsilon \cong \ker \epsilon' \oplus W_0$. Since $\overline{\epsilon'}$ is injective, $\ker \epsilon' \subseteq \mathfrak{m}F_0$, then by (3) we can prove inductively that all $\overline{f_i}$ are injective. ■

(18.F)

Theorem 44. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and let $s = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right)$. Then we have

$$\dim_k \operatorname{Tor}_i^A(k, k) \geq \binom{s}{i} \qquad \text{for } 0 \leq i \leq s.$$

Proof. Take a minimal basis $\{x_1, \ldots, x_s\}$ of $\mathfrak{m}$, and consider the Koszul complex $F_\bullet = K_\bullet(x_1, \ldots, x_s; A)$. There is an obvious augmentation $F_0 = A \to k = A/\mathfrak{m}$, which satisfies the condition of Lemma 9. By the definition of $\mathrm{d}_p : F_p \to F_{p-1}$, or $(\diamondsuit)$ of (18.D), it is clear that $\mathrm{d}_p(F_p) \subseteq \mathfrak{m}F_{p-1}$. Moreover, we have

$$\overline{F_\bullet} = k \otimes_A F_\bullet = K_\bullet(\underline{x}; k)$$

and

$$\left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_A F_{p-1} = \left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_k k \otimes_A F_{p-1} = \left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_k K_{p-1}(\underline{x}, k).$$

Since the residue classes of the x_i's modulo $\mathfrak{m}^2$ form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, the formula ($\diamondsuit$) of (18.D) implies that d_p induces an injection $\overline{F_p} \to \left(\mathfrak{m}/\mathfrak{m}^2\right) \otimes_A F_{p-1}$. In fact,

$$\mathrm{d}\left(\sum_{i_1<\cdots<i_p} a_{i_1\ldots i_p} e_{i_1\ldots i_p}\right) = \sum_{j=1}^{n} x_j \left(\sum_{i_1<\cdots<i_{p-1}} a_{ji_1\ldots i_{p-1}} e_{i_1\ldots i_{p-1}}\right) \in \mathfrak{m}^2 F_{p-1}$$

implies that

$$\sum_{i_1<\cdots<i_{p-1}} a_{ji_1\ldots i_{p-1}} e_{i_1\ldots i_{p-1}} \in \mathfrak{m} F_{p-1}$$

for all j. So $a_{i_1\ldots i_p} \in \mathfrak{m}$ for all $i_1 < \cdots < i_p$ and hence

$$\sum_{i_1<\cdots<i_p} a_{i_1\ldots i_p} e_{i_1\ldots i_p} \in \mathfrak{m} F_p.$$

Thus the conditions of Lemma 9 are all satisfied. Therefore we have

$$\binom{s}{p} = \mathrm{rank}_A F_i \leq \dim_k \mathrm{Tor}_p^A(k,k). \qquad \blacksquare$$

Theorem 45 (Serre)**.** A noetherian local ring A is regular iff the global dimension of A is finite.

Proof. We have already prove the 'only if' part in Theorem 42. So suppose that $(A, \mathfrak{m}, k)$ is a noetherian local ring with $\mathrm{gldim}(A) = n < \infty$. Put $\dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right) = s$. Then $\mathrm{Tor}_s^A(k,k) \neq 0$ by Theorem 44, hence $\mathrm{gldim} \geq s$. On the other hand, it follows from the formula

$$\mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(A)$$

of Auslander-Buchsbaum in (18.E) and from Theorem 41 that

$$\mathrm{gldim}(A) = \mathrm{pd}(k) = \mathrm{depth}(A) - \mathrm{depth}(k) = \mathrm{depth}(A).$$

Therefore we get

$$\dim(A) \leq \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right) \leq \mathrm{gldim}(A) = \mathrm{depth}(A) \leq \dim(A).$$

Whence $\dim(A) = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right)$, which means A is regular. $\blacksquare$

Corollary. If A is a regular local ring then $A_\mathfrak{p}$ is a regular local ring for any $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proof. Let M be an $A_{\mathfrak{p}}$-module. As an A-module it has a projective resolution of finite length:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

$n \leq \operatorname{gldim}(A)$. By flatness of $A_{\mathfrak{p}}$ the sequence

$$0 \longrightarrow (P_n)_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (P_0)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} = M \longrightarrow 0$$

is exact, and gives a projective resolution of M as $A_{\mathfrak{p}}$-module. Hence

$$\operatorname{gldim}(A_{\mathfrak{p}}) \leq \operatorname{gldim}(A) < \infty \qquad \blacksquare$$

Definition. A ring A is called a regular ring if $A_{\mathfrak{m}}$ is a regular local ring for every $\mathfrak{m} \in \operatorname{Max}(A)$. In view of the above Corollary, this is equivalent to saying that $A_{\mathfrak{p}}$ is a regular local ring for every $\mathfrak{p} \in \operatorname{Spec}(A)$.

(18.H)

Theorem 46. Let A be a regular local ring, and B a domain containing A which is a finite A-module. Then B is flat (hence free) over A iff B is Cohen-Macaulay. In particular, if B is regular then it is a free A-module.

Proof. Suppose B if flat over A. If $\mathfrak{m} \in \operatorname{Max}(B)$ then $\dim(B_{\mathfrak{m}}) \leq \dim(A)$ by (13.C), while any A-regular sequence is also $B_{\mathfrak{m}}$-regular by the flatness and (3.E) and hence $\operatorname{depth}(B_{\mathfrak{m}}) \geq \operatorname{depth}(A)$. So B is C.M. as A is so.

Conversely, suppose B is C.M.. Since A is normal by (17.F), the going-down theorem holds between A and B by (5.E), so if $\mathfrak{m}$ is the maximal ideal of A we have $\operatorname{ht}(\mathfrak{m}B) = \operatorname{ht}(\mathfrak{m})$ by (13.B) Theorem 19 (3). By the unmixedness theorem in B (Theorem 32), if $\{x_1, \ldots, x_r\}$ is a regular system of parameters of A, then

$$\operatorname{ht}((\underline{x})B) = \operatorname{ht}(\mathfrak{m}B) = \operatorname{ht}(\mathfrak{m}) = r,$$

where $r = \dim(A)$. So $\{x_1, \ldots, x_r\}$ is a regular sequence in B. Therefore

$$\operatorname{depth}_A(B) \geq \dim(A) = \operatorname{depth}_A(A),$$

and by the formula of Auslander-Buchsbaum in (18.E) we have

$$\mathrm{pd}_A(B) \leq \mathrm{depth}_A(A) - \mathrm{depth}_A(B) = 0$$

so B is projective, i.e. B is A-free. ∎

19 Unique Factorization

(19.A) Let A be an integral domain. An element $a \neq 0$ of A is said to be irreducible if it is a non-unit of A and if it is not a product of two non-units of A. The ring A is called a unique factorization domain (UFD) if every non-zero element is a product of a unit and of a finite number of irreducible elements and if such a representation is unique up to order and units. A noetherian domain in which every irreducible element generates a prime ideal is UFD.

Theorem 47. Let A be a noetherian domain. Then the following are equivalent:

(i) A is UFD,

(ii) every prime ideal of height 1 is principal,

(iii) every irreducible element generates a prime ideal.

Proof. (i) $\Longrightarrow$ (ii): Let $\mathfrak{p}$ be a height 1 prime ideal. Let $a = up_1 \dots p_r \in \mathfrak{p} - \{0\}$, where $u \in A^\times$ and p_i are irreducible, then $p_i \in \mathfrak{p}$ for some i. Since A is UFD, $(p_i) \neq (0)$ is a prime ideal and it is contained in $\mathfrak{p}$, hence $\mathfrak{p} = (p_i)$ is principal.

(ii) $\Longrightarrow$ (iii): Let π be an irreducible element and let $\mathfrak{p}$ be a minimal prime over-ideal of (π). Then $\mathrm{ht}(\mathfrak{p}) = 1$ by Theorem 18, so that $\mathfrak{p}$ is principal: $\mathfrak{p} = (a)$. Then $\pi = au$ with some u, which must be a unit by the irreducibility of π. Thus $(\pi) = \mathfrak{p}$. As we remarked above, this means that A is UFD.

(iii) $\Longrightarrow$ (i): Consider the set Σ of ideals of A of the form (a) where $a \neq 0$ is not a product of a unit and of a finite number of irreducible elements. If $\Sigma \neq \varnothing$, then there's a maximal element, say (a). Since a is not irreducible, $a = bc$ for some $b, c \in A - A^\times$. Then $(a) \subset (b)$ and $(a) \subset (c)$. So we can write

$$b = up_1 \dots p_r, \quad c = vq_1 \dots q_s$$

with $u, v \in A^\times$ and p_i, q_j are irreducible. Then $a = (uv)p_1 \dots p_r q_1 \dots q_s$, a contradiction. So $\Sigma = \varnothing$ and hence every non-zero element of A a product of a unit and of a finite number of irreducible elements. If $a = up_1 \dots p_r = vq_1 \dots q_s$ with $u, v \in A^\times$ and p_i, q_j are irreducible, then $\prod_j (q_j) = (a) \subseteq (p_1)$ implies that $(q_j) \subseteq (p_1)$ for some j, since (p) is a prime ideal, say $j = 1$. So $(q_1) = (p_1)$ since they are both of height 1 prime ideals. So $q_1 = p_1 w$ for some $w \in A^\times$. Then

$$0 = (up_1 \dots p_r) - (vq_1 \dots q_s) = p_1(up_2 \dots p_r - (vw)q_2 \dots q_s).$$

So $up_2 \dots p_r = (vw)q_2 \dots q_s$ since A is a domain, and we get the uniqueness by induction on $\max\{r, s\}$. ■

Remark. The proof also shows that every noetherian domain is a factorization domain, i.e. every non-zero element can written as a finite product of irreducible elements.

(19.B)

Lemma. Let A be a noetherian domain and let $f \neq 0$ be an element such that $(f) \in \operatorname{Spec}(A)$. Put $A_f = S^{-1}A$, where $S = \{1, f, f^2, \dots\}$. Then A is UFD iff A_f is so.

Proof. Suppose A is UFD. Let $\mathfrak{p}A_f$ be a height 1 prime ideal of A_f, then $\mathfrak{p} \in \operatorname{Spec}(A)$ is also height 1. So we can write $\mathfrak{p} = (p)$ for some $p \in A$, hence $\mathfrak{p}A_f = pA_f$ is principal so by Theorem 47 we have A_f is UFD.

Conversely, suppose A_f is UFD. Let $\mathfrak{p}$ be a height 1 prime ideal of A, if $f \notin \mathfrak{p}$ then $\mathfrak{p}A_f \in \operatorname{Spec}(A_f)$ is also height 1. So $\mathfrak{p}A_f = (p/f^n)A_f$ for some $p \in A$, $n \geq 0$. Since A is a noetherian domain, by (11.D) Corollary 3. we can always find an $m \geq 0$ such that $p = f^m q$ where $q \notin (f)$. Then

$$\mathfrak{p} = (\mathfrak{p}A_f) \cap A = (qA_f) \cap A = (q),$$

which is principal. If $f \in \mathfrak{p}$. Then $(f) \neq (0)$ is a prime ideal contained in $\mathfrak{p}$, so $\mathfrak{p} = (f)$, which is principal. By Theorem 47 we have A is UFD. ■

Theorem 48. [Auslander-Buchsbaum, 1959] A regular local ring $(A, \mathfrak{m})$ is UFD.

Proof. (due to Kaplansky) We use induction on $\dim(A)$. If $\dim(A) = 0$ then A is a field, and if $\dim(A) = 1$ then A is a PID by (17.G). Suppose $\dim(A) > 1$. Let $x \in \mathfrak{m} - \mathfrak{m}^2$. Then (x) is prime by Theorem 36 (3), hence we have only to prove that A_x is UFD. Let $\mathfrak{q}$ be a prime ideal of height 1 in A_x and put $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q} = \mathfrak{p}A_x$. Since A is a regular local ring, the A-module $\mathfrak{p}$ has a resolution of finite length

$$(\spadesuit) \quad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathfrak{p} \longrightarrow 0$$

with F_i finite and free by (18.E). If $\mathfrak{r}$ is a prime ideal of A_x, the local ring $(A_x)_\mathfrak{r} = A_{A \cap \mathfrak{r}}$ is a UFD by induction assumption. Therefore $\mathfrak{q}(A_x)_\mathfrak{r}$ is principal. So we have

$$\mathrm{pd}(\mathfrak{q}) = \sup_{\mathfrak{m} \in \mathrm{Max}(A_x)} \mathrm{pd}(\mathfrak{q}(A_x)_\mathfrak{m}) = 0$$

by (18.B) Lemma 5, i.e. $\mathfrak{q}$ is projective. Localizing $(\spadesuit)$ with respect to $S = \{1, x, x^2, \ldots\}$ we see

$$(\spadesuit_x) \quad 0 \longrightarrow (F_n)_x \longrightarrow \cdots \longrightarrow (F_0)_x \longrightarrow \mathfrak{q} \longrightarrow 0$$

is exact, where $(F_i)_x = F_i \otimes A_x$ are finite and free over A_x. If we decompose $(\spadesuit_x)$ into short exact sequences

$$0 \longrightarrow K_0 \longrightarrow (F_0)_x \longrightarrow \mathfrak{q} \longrightarrow 0$$
$$0 \longrightarrow K_i \longrightarrow (F_i)_x \longrightarrow K_{i-1} \longrightarrow 0$$

then each K_i must be projective since $\mathfrak{q}$ is. Hence the short exact sequences split. It follows that

$$\bigoplus_{2\,|\,i} (F_i)_x \cong \bigoplus_{2\,\nmid\,i} (F_i)_x \oplus \mathfrak{q}.$$

Thus, we have finite free A_x-modules F and G such that $F \cong G \oplus \mathfrak{q}$. Put $\mathrm{rank}_{A_x} G = r$. Since $\mathfrak{q}$ is a non-zero ideal of the integral domain A_x we have

$$\mathrm{rank}_{A_x} \mathfrak{q} := \dim_{\Phi A_x}(\mathfrak{q} \otimes_{A_x} \Phi A_x) = 1$$

and $\mathrm{rank}_{A_x} F = r + 1$. Then

$$A_x \cong \bigwedge^{r+1} F \cong \bigwedge^{r+1} (G \oplus \mathfrak{q}) = \bigoplus_{p+q=r+1} \left(\bigwedge^p G \otimes_{A_x} \bigwedge^q \mathfrak{q} \right).$$

Since $\mathfrak{q}$ is projective, it is locally free of rank 1. Therefore

$$\left(\bigwedge^q \mathfrak{q}\right)_{\mathfrak{r}} = \bigwedge^q \mathfrak{q}_{\mathfrak{r}} \cong \bigwedge^q (A_x)_{\mathfrak{r}} = 0$$

for $q > 1$ and for all $\mathfrak{r} \in \operatorname{Spec}(A_x)$, so $\bigwedge^q \mathfrak{q} = 0$. Then

$$A_x \cong \left(\bigwedge^r G \otimes_{A_x} \bigwedge^1 \mathfrak{q}\right) \oplus \left(\bigwedge^{r+1} G \otimes_{A_x} \bigwedge^0 \mathfrak{q}\right) \cong \mathfrak{q},$$

so $\mathfrak{q}$ is free and hence principal. ■

Remark. 1. As Theorem 35 suggests, regular local rings are similar to polynomial rings or power series rings in many aspects. In particular, the inequality on the dimension (14.K) can be extended to an arbitrary regular local ring. Namely, in the non-local form one has the following theorem (due to Serre):

Let A be a regular ring, $\mathfrak{p}_1$, $\mathfrak{p}_2$ are prime ideals of A and $\mathfrak{q}$ a minimal prime over-ideal of $\mathfrak{p}_1 + \mathfrak{p}_2$. Then

$$\operatorname{ht}(\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{p}_1) + \operatorname{ht}(\mathfrak{p}_2).$$

For the proof see J.-P. Serre: Algèbre Locale, Multiplicités (2nd ed.) Ch.V, p.18.

2. A normal domain A is called a Krull ring if

(1) for any non-zero element x of A, the number of the prime ideals of A of height one containing x is finite, and

(2) $A = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$.

Noetherian normal rings are Krull, but not conversely. If A is a noetherian domain, then the integral closure of A in the quotient field of A is a Krull ring (Theorem of Y. Mori, cf. Nagata: Local Rings). On Krull rings, cf. Bourbaki: Alg. Comm. Ch.7.

3. P. Samuel has made an extensive study on the subject of unique factorization. Cf. his Tata lecture note.

4. We did not discuss valuation theory. On this topic the following paper contains important results in connection with algebraic geometry. Abhyankar: On the valuations centered in a local domain, Amer. J. Math. 78(1956), 321-348.

Chapter 8

Flatness II

20 Local Criteria of Flatness

(20.A) In (18.B) Lemma 4 we proved the following.

Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. Then M is flat iff $\mathrm{Tor}_1(M, A/\mathfrak{m}) = 0$.

The condition that M is finite over A is too strong; in geometric application it is often necessary to prove flatness of infinite modules. In this section we shall learn several criteria of flatness, due to Bourbaki, which are very useful.

Let A be a ring, I an ideal of A and M an A-module. We say that M is idealwise separated (i.s. for short) for I if, for each finitely generated ideal J of A, the A-module $J \otimes_A M$ is Hausdorff in the I-adic topology.

Example 1. Let B be a noetherian A-algebra such that $IB \subseteq \mathrm{rad}(B)$, and let M be a finite B-module. Then M is i.s. for I as an A-module: since $J \otimes_A M$ is a finite B-module and since the I-adic topology on $J \otimes_A M$ is nothing but the IB-adic topology, we can apply (11.D) Corollary 1.

Example 2. When A is a PID, any I-adically Hausdorff A-module M is i.s. for I since any ideal J of A is isomorphic to A as A-module.

Example 3. Let M be an I-adically Hausdorff flat A-module. Then M is i.s. for I. In fact, we have $J \otimes M \cong JM \subseteq M$.

(20.B) Put $\mathrm{gr}(A) = \mathrm{gr}^I(A) = \bigoplus_n I^n/I^{n+1}$, $\mathrm{gr}(M) = \mathrm{gr}^I(M) = \bigoplus_n I^n M/I^{n+1}M$, $A_0 = \mathrm{gr}_0(A) = A/I$ and $M_0 = \mathrm{gr}_0(M) = M/IM$. Then $\mathrm{gr}(M)$ is a graded $\mathrm{gr}(A)$-module. There are canonical epimorphisms

$$\gamma_n : (I^n/I^{n+1}) \otimes_{A_0} M_0 \longrightarrow I^n M/I^{n+1}M$$

for $n \geq 0$. In other words, there is a graded epimorphism $\gamma : \mathrm{gr}(A) \otimes_{A_0} M_0 \to \mathrm{gr}(M)$.

(20.C)

Theorem 49 (Local criteria of flatness)**.** Let A be a ring, I an ideal of A and M an A-module. Assume that either

(1) I is nilpotent, or

(2) A is noetherian and M is idealwise separated for I.

Then the following are equivalent:

(i) M is A-flat;

(ii) $\mathrm{Tor}_1^A(N, M) = 0$ for all A_0-module N;

(iii) M_0 is A_0-flat, and $I \otimes_A M \cong IM$ by the natural map, (note that, if I is a maximal ideal, the flatness over A_0 is trivial), or equivalently, $\mathrm{Tor}_1^A(A_0, M) = 0$;

(iv) M_0 is A_0-flat, and the canonical maps

$$\gamma_n : (I^n/I^{n+1}) \otimes_{A_0} M_0 \longrightarrow I^n M/I^{n+1}M$$

are isomorphisms;

(v) $M_n = M/I^{n+1}M$ is flat over $A_n = A/I^{n+1}$, for each $N \geq 0$,

(The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are true without any assumption on I and M.)

Proof. We first prove the equivalence of (i) and (v) under the assumption (1) or (2). The implication (i) $\Longrightarrow$ (v) is just a change of base (cf.(3.C)).

(v) $\Longrightarrow$ (i): The nilpotent case (1) is trivial since $A = A_n$ for some n). In the case (2), we prove the flatness of M by showing that, for every ideal J of A, the canonical map $j : J \otimes_A M \to M$ is injective. Since $J \otimes_A M$ is I-adically Hausdorff it suffices to prove that ker $j \subseteq I^n(J \otimes M)$ for all $n > 0$. Fix an n. Then there exists, by Artin-Rees (11.C), an integer $k > n$ such that $J \cap I^k \subseteq I^n J$. Consider the natural maps

$$J \otimes_A M \xrightarrow{\alpha} \big(J/(I^k \cap J)\big) \otimes_A M \xrightarrow{\beta} (J/I^n J) \otimes_A M = J \otimes_A M/I^n(J \otimes_A M).$$

Since M_{k-1} is A_{k-1}-flat, the natural map

$$\big(J/(I^k \cap J)\big) \otimes_A M = \big(J/(I^k \cap J)\big) \otimes_{A_{k-1}} M_{k-1} \longrightarrow M_{k-1} = M/I^k M$$

is injective. Therefore ker $j \subseteq$ ker α, and a fortiori

$$\ker j \subseteq \ker(\beta\alpha) = I^n(J \otimes_A M).$$

Thus our assertion is proved.

Next we prove (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) for arbitrary M. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii): Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be an exact sequence of A_0-modules. Then

$$0 = \mathrm{Tor}_1^A(N'', M) \longrightarrow N' \otimes_A M = N' \otimes_{A_0} M_0 \longrightarrow N \otimes_A M = N \otimes_{A_0} M_0$$

is exact, so M_0 is A_0-flat. From the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A_0 \longrightarrow 0$$

we get

$$0 = \mathrm{Tor}_1^A(A_0, M) \longrightarrow I \otimes M \longrightarrow M$$

is exact, which proves $I \otimes_A M \cong IM$.

(iii) $\Longrightarrow$ (ii): Let N be an A_0-module and take an exact sequence of A_0-modules

$$0 \longrightarrow R \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

where F_0 is A_0-free. Then

$$0 = \mathrm{Tor}_1^A(F_0, M) \longrightarrow \mathrm{Tor}_1^A(N, M) \longrightarrow R \otimes_{A_0} M_0 \longrightarrow F_0 \otimes_{A_0} M_0$$

is exact and M_0 is A_0-flat, hence $\mathrm{Tor}_1^A(N, M) = 0$.

(ii) $\Longrightarrow$ (iv): consider the exact sequences

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow I^n/I^{n+1} \longrightarrow 0$$

and the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1} \otimes_A M \arrow[r] \arrow[d, "\alpha_{n+1}"] & I^n \otimes_A M \arrow[r] \arrow[d, "\alpha_n"] & (I^n/I^{n+1}) \otimes_A M \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & I^{n+1}M \arrow[r] & I^nM \arrow[r] & I^nM/I^{n+1}M \arrow[r] & 0
\end{tikzcd}$$

The first row is exact by (ii) and the second row is of course exact. Since α_1 is injective by (iii), we see inductively that all α_n are injective by snake lemma. Thus they are isomorphisms, and consequently the γ_n are also isomorphisms (by snake lemma).

Before proving (iv) $\Longrightarrow$ (v) we remark the following fact: if (ii) holds, then, for any $n \geq 0$ and for any A_n-module N, we have $\mathrm{Tor}_1^A(N, M) = 0$. In fact, if N is an A_n-module and $n > 0$, then IN and N/IN are A_{n-1}-modules, so that the assertion is proved by induction on n.

(iv) $\Longrightarrow$ (v): We fix an integer $n \geq 0$ and we are going to prove that M_n is A_n-flat. For $n = 0$ this is included in the assumptions, so we suppose $n > 0$. Put $I_n = I/I^{n+1}$.

Consider the commutative diagrams with exact rows:

$$\begin{tikzcd}
 & (I^{i+1}/I^{n+1}) \otimes_A M \arrow[r] \arrow[d, "\overline{\alpha_{i+1}}"] & (I^i/I^{n+1}) \otimes_A M \arrow[r] \arrow[d, "\overline{\alpha_i}"] & (I^i/I^{i+1}) \otimes_A M \arrow[r] \arrow[d, "\gamma_i"] & 0 \\
0 \arrow[r] & I^{i+1}M/I^{n+1}M \arrow[r] & I^iM/I^{n+1}M \arrow[r] & I^iM/I^{i+1}M \arrow[r] & 0
\end{tikzcd}$$

for $i = 1, 2, \ldots, n$. Since the γ_i are isomorphisms by assumption, and since $\overline{\alpha_{n+1}} = 0$, we see by descending induction on i and snake lemma that all $\overline{\alpha_i}$ are isomorphisms. In particular,

$$\overline{\alpha_1} : (I/I^{n+1}) \otimes_A M = (IA_n) \otimes_{A_n} M_n \longrightarrow IM/I^{n+1}M = IM_n$$

is an isomorphism. Therefore the condition (iii) (hence also (ii)) holds for A_n, IA_n and M_n ($(A_n)_0 = A_0$, $(M_n)_0 = M_0$). From this and from what we have just remarked it follows that $\mathrm{Tor}_1^{A_n}(N, M_n) = 0$ for all A_n-modules N since $A_n = (A_n)_n$, hence M_n is A_n-flat. ■

(20.D)

Application 1 (Hartshorne). Let $(B, \mathfrak{n})$ be a noetherian local ring containing a field k and let $x_1, \ldots, x_n$ be a B-regular sequence in $\mathfrak{n}$. Then the subring $k[x_1, \ldots, x_n]$ of B is isomorphic to the polynomial ring $A = k[X_1, \ldots, x_n]$, and B is flat over it.

Proof. Considering the k-algebra homomorphism $\varphi : A \to B$ such that $\phi(X_i) = x_i$, we view B as an A-algebra. It suffices to prove B is flat over A. In fact, any non-zero element f of A is A-regular, so under the assumption of flatness it is also B-regular, hence $\varphi(f) \neq 0$.

We apply the criterion (iii) of Theorem 49 to A, $I = (X_1, \ldots, X_n)$ and $M = B$. The A-module B is idealwise separated for I as $IB \subseteq \mathrm{rad}(B)$. Since $A/I = k$ is a field we have only to prove $\mathrm{Tor}_1^A(k, B) = 0$. Now the Koszul complex $K_\bullet(X_1, \ldots, X_n; A)$ is a free resolution of the A-module $k = A/I$ by Corollary to Theorem 43. So we have

$$\mathrm{Tor}_i^A(k, B) = h_i(K_\bullet(X_1, \ldots, X_n; A) \otimes_A B) = h_i(K_\bullet(x_1, \ldots, x_n; B)),$$

which is zero for $i > 0$ by Theorem 43 as $x_1, \ldots, x_n$ is a B-regular sequence. ■

(20.E)

Application 2 (EGA 0_{III} (10.2.4))**.** Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings and $A \to B$ a local homomorphism. Let $u : M \to N$ be a homomorphism of finite B-modules, and assume that N is A-flat. Then the following are equivalent:

(i) u is injective, and $\operatorname{coker} u$ is A-flat;

(ii) $\overline{u} : M \otimes_A k \to N \otimes_A k$ is injective.

Proof. (i) $\Longrightarrow$ (ii). Tensoring the exact sequence

$$0 \longrightarrow M \xrightarrow{u} N \longrightarrow \operatorname{coker} u \longrightarrow 0$$

with k, we get

$$0 = \operatorname{Tor}_1^A(\operatorname{coker} u, k) \longrightarrow M \otimes_A k \xrightarrow{\overline{u}} N \otimes_A k.$$

(ii) $\Longrightarrow$ (i). Let $x \in \ker u$. Then $x \otimes 1 = 0$ in $M \otimes_A k = M/\mathfrak{m}M$, therefore $x \in \mathfrak{m}M$. We will show $x \in \bigcap\limits_n \mathfrak{m}^n M = (0)$ by induction. Suppose $x \in \mathfrak{m}^n M$, let $\{a_1, \ldots, a_p\}$ be a minimal basis of the ideal $\mathfrak{m}^n \pmod{\mathfrak{m}^{n+1}}$ and write $x = \sum a_i x_i$, $x_i \in M$. Then $u(x) = \sum a_i u(x_i) = 0$ in N. By flatness of N and (3.A) there exists $c_{ij} \in A$ and $x'_j \in N$ such that $\sum\limits_i a_i c_{ij} = 0$ (for all j) and such that $u(x_i) = \sum\limits_j c_{ij} x'_j$ (for all i). By the choice of $a_1, \ldots, a_p$ all the c_{ij} must belong to $\mathfrak{m}$. Thus $u(x_i) \in \mathfrak{m}N$, in other words $\overline{u}(x_i \otimes 1) = 0$. Since $\overline{u}$ is injective we get $x_i \in \mathfrak{m}M$, hence $x \in \mathfrak{m}^{n+1}M$. Thus u is injective and we get an exact sequence

$$0 \longrightarrow M \xrightarrow{u} N \longrightarrow \operatorname{coker} u \longrightarrow 0.$$

From this and from the hypotheses it follows that $\operatorname{Tor}_1^A(k, \operatorname{coker} u) = 0$, which shows the flatness of $\operatorname{coker} u$ by Theorem 49 ((iii) $\Longrightarrow$ (i)). ■

(20.F)

Corollary 1. Let A be a noetherian ring, B a noetherian A-algebra, M a finite B-module and $f \in B$. Suppose that

(1) M is A-flat, and

(2) for each maximal ideal $\mathfrak{n}$ of B, the element f is $M/(\mathfrak{n} \cap A)M$-regular.

Then f is M-regular and M/fM is A-flat.

Proof. If K denotes the kernel of $M \xrightarrow{[f]} M$, then $K = 0$ iff $K_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \mathrm{Max}(B)$. Similarly, by an obvious extension of (3.J), M/fM is A-flat if $M_{\mathfrak{m}}/fM_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}\cap A}$ for all $\mathfrak{m} \in \mathrm{Max}(B)$. The assumptions are also stable under localization. So we may assume that $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ are noetherian local rings and $A \to B$ is a local homomorphism. Take $u : M \xrightarrow{[f]} M$, then the assertion follows from (20.E). ■

Corollary 2. Let A be a noetherian ring and $B = A[X_1, \dots, X_n]$ a polynomial ring over A. Let $f(X) \in B$ be such that its coefficients generate over A the unit ideal A. Then f is not a zero-divisor of B, and B/fB is A-flat.

Proof. Take $M = B$ in Corollary 1, then we only need to prove that for each $\mathfrak{m} \in \mathrm{Max}(B)$, f is $B/(\mathfrak{m}\cap A)B$-regular. This is equivalent to show that $f \notin (\mathfrak{m}\cap A)B$ since $(\mathfrak{m} \cap A)B \in \mathrm{Spec}(B)$, which is trivial. ■

(20.G)

Application 3. Let $A \to B \to C$ be local homomorphisms of noetherian local rings and M be a finite C-module. Suppose B is A-flat. Let k denote the residue field of A. Then

$$M \text{ is } B\text{-flat} \iff M \text{ is } A\text{-flat and } M \otimes_A k \text{ is } B \otimes_A k\text{-flat}$$

Proof. ($\Longrightarrow$) Trivial. ($\Longleftarrow$) Use criterion (iv) in Theorem 49 (let $I = \mathfrak{m}B$), we need to show that $M/\mathfrak{m}M$ is $B/\mathfrak{m}B$ flat, which is by assumption, and the canonical maps

$$\gamma_n : (\mathfrak{m}^n B/\mathfrak{m}^{n+1}B) \otimes_{B/\mathfrak{m}B} M/\mathfrak{m}M \longrightarrow \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$$

are isomorphisms. They are already surjective. To show that they are injective, we use (20.E) (i) $\Longrightarrow$ (ii). Fix $n \geq 0$, we need to show that

$$u : (\mathfrak{m}^n B) \otimes_B M \to \mathfrak{m}^n M$$

is injective, and $\operatorname{coker} u$ is A-flat. Since B is A-flat and by assumption M is A-flat,

$$(\mathfrak{m}^n B) \otimes_B M = \mathfrak{m}^n \otimes_A B \otimes_B M = \mathfrak{m}^n \otimes_A M = \mathfrak{m}^n M. \qquad \blacksquare$$

21 Fibers of Flat Morphisms

(21.A) Let $\varphi : A \to B$ be a homomorphism of noetherian rings; let $\mathfrak{q} \in \operatorname{Spec}(B)$, $\mathfrak{p} = \mathfrak{q} \cap A$ and $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the residue field. Then the 'fiber over $\mathfrak{p}$' is $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, and 'the local ring of $\mathfrak{q}$ on the fiber' is $B_\mathfrak{q}/\mathfrak{p}B_q = B_\mathfrak{q} \otimes_A \kappa(\mathfrak{p})$ (cf. (13.A)). Suppose B is flat over A. Then we have

$$(\heartsuit) \quad \dim(B_\mathfrak{q}) = \dim(A_\mathfrak{p}) + \dim(B_\mathfrak{q} \otimes_A \kappa(\mathfrak{p}))$$

by (13.B) Theorem 19.

(21.B)

Theorem 50. Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings, and let $A \to B$ a local homomorphism. Let M be a finite A-module and N be a finite B-module which is A-flat. Then we have

$$\operatorname{depth}_B(M \otimes_A N) = \operatorname{depth}_A(M) + \operatorname{depth}_{B \otimes_A k}(N \otimes_A k).$$

Proof. Induction on $n = \operatorname{depth}_A(M) + \operatorname{depth}_{B \otimes_A k}(N \otimes_A k)$.

Case 1. $n = 0$. Then $\mathfrak{m} \in \operatorname{Ass}_A(M)$ and $\mathfrak{n} \in \operatorname{Ass}_B(N \otimes_A k)$, and we know from (9.B) that

$$\operatorname{Ass}_B(M \otimes_A N) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(M)} \operatorname{Ass}_B(N/\mathfrak{p}N).$$

Hence $\mathfrak{n} \in \operatorname{Ass}_B(M \otimes_A N)$, i.e. $\operatorname{depth}_B(M \otimes_A N) = 0$.

Case 2. $\mathrm{depth}(M) > 0$. Let $a \in \mathfrak{m}$ be an M-regular element. Then

$$\mathrm{depth}_A(M/aM) = \mathrm{depth}_A(M) - 1.$$

By the exactness of

$$0 \longrightarrow M \xrightarrow{[a]} M$$

and the A-flatness of N, we have

$$0 \longrightarrow M \otimes_A N \xrightarrow{[a]} M \otimes_A N$$

is also exact, so a is $M \otimes_A N$-regular. Hence

$$\mathrm{depth}_B((M/aM) \otimes_A B) = \mathrm{depth}_B(M \otimes_A B/a(M \otimes_A B)) = \mathrm{depth}_B(M \otimes_A B) - 1.$$

Case 3. $\mathrm{depth}_{B\otimes_A k}(N \otimes_A k) > 0$. Take $y \in \mathfrak{n}$ which is $N \otimes_A k$-regular. By (20.E) y is N-regular and N/yN is A-flat. From the exact sequence

$$0 \longrightarrow N \xrightarrow{[y]} N \longrightarrow N/yN \longrightarrow 0$$

it then follows that

$$0 = \mathrm{Tor}_1^A(M, N/yN) \longrightarrow M \otimes_A N \xrightarrow{[y]} M \otimes_A N \longrightarrow M \otimes_A (N/yN) \longrightarrow 0$$

is exact. Putting $\overline{N} = N/yN$ we get

$$\mathrm{depth}_B(M \otimes \overline{N}) = \mathrm{depth}_B(M \otimes N) - 1, \qquad \text{and}$$
$$\mathrm{depth}_{B\otimes_A k}(\overline{N} \otimes_A k) = \mathrm{depth}_{B\otimes_A k}(N \otimes_A k) - 1.$$

From these and from the induction hypothesis on $\overline{N}$ we get the desired formula. ■

(21.C)

Corollary 1. Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings, and let $A \to B$ a local homomorphism. Suppose that B is flat. Then we have

$$\mathrm{depth}(B) = \mathrm{depth}(A) + \mathrm{depth}(B \otimes_A k),$$

and B is C.M. $\iff$ A and $B \otimes_A k$ are C.M.

Corollary 2. Let A and B be noetherian rings and $\varphi : A \to B$ be a faithfully flat homomorphism. Let i be a positive integer. Then

(1) if B satisfies the condition (S_i) of (17.I), so does A;

(2) if A satisfies (S_i) and if all fibers satisfy (S_i) (i.e. $B \otimes_A \kappa(\mathfrak{p})$ satisfies (S_i) for every $\mathfrak{p} \in \operatorname{Spec}(A)$) then B satisfies (S_i).

Proof. (1) Given $\mathfrak{p} \in \operatorname{Spec}(A)$, since φ is a f.f. homomorphism, $\operatorname{Spec}(\varphi)$ is surjective. Take $\mathfrak{q} \in \operatorname{Spec}(B)$ which is minimal among prime ideals of B lying over $\mathfrak{p}$, and put $k = \kappa(\mathfrak{p})$. Then

$$\dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k) = \operatorname{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k) = 0,$$

whence $\operatorname{depth}(B_{\mathfrak{q}}) = \operatorname{depth}(A_{\mathfrak{p}})$ by above corollary and $\dim(B_{\mathfrak{q}}) = \dim(A_{\mathfrak{p}})$ by $(\heartsuit)$. Therefore

$$\operatorname{depth}(A_{\mathfrak{p}}) = \operatorname{depth}(B_{\mathfrak{q}}) \geq \inf\{i, \dim(B_{\mathfrak{q}})\} = \inf\{i, \dim(A_{\mathfrak{p}})\}.$$

(2) Given $\mathfrak{q} \in \operatorname{Spec}(B)$, put $\mathfrak{p} = \mathfrak{q} \cap A$ and $k = \kappa(\mathfrak{p})$. Then by above corollary and $(\heartsuit)$,

$$\begin{aligned}
\operatorname{depth}(B_{\mathfrak{q}}) &= \operatorname{depth}(A_{\mathfrak{p}}) + \operatorname{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k) \\
&\geq \inf\{i, \dim(A_{\mathfrak{p}})\} + \inf\{i, \dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k)\} \\
&\geq \inf\{i, \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k)\} \\
&= \inf\{i, \dim(B_{\mathfrak{q}})\}.
\end{aligned}$$

■

(21.D)

Theorem 51. Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings and $\varphi : A \to B$ a local homomorphism. Then:

(1) if B is flat over A and regular, then A is regular;

(2) if $\dim(B) = \dim(A) + \dim(B \otimes_A k)$ holds, and if A and $B \otimes_A k$ are regular, then B is flat over A and regular.

Proof. (1) Since a flat base change commutes with homology, we have

$$\mathrm{Tor}_q^A(k,k) \otimes_A B = \mathrm{Tor}_q^B(k \otimes_A B, k \otimes_A B) = 0$$

for $q > \dim(B)$ by Theorem 41 and Theorem 42. Since B is faithfully flat over A this implies $\mathrm{Tor}_q^A(k,k) = 0$ by Theorem 2, hence $\mathrm{gldim}(A)$ is finite by Theorem 41, so A is regular by Theorem 45.

(2) If $\{x_1, \ldots, x_r\}$ is a regular system of parameters of A and if $y_1, \ldots, y_s \in \mathfrak{n}$ are such that their images form a regular system of parameters of $B/\mathfrak{m}B = B \otimes_A k$, then $\{\varphi(x_1), \ldots, \varphi(x_r), y_1, \ldots, y_s\}$ generates $\mathfrak{n}$, and $r + s = \dim(B)$ by hypothesis. Thus B is regular. To prove the flatness it suffices, by the criterion (iii) of Theorem 49, to prove $\mathrm{Tor}_1^A(k, B) = 0$. The Koszul complex $K_\bullet(x_1, \ldots, x_r; A)$ is a free resolution of the A-module k, hence we have

$$\mathrm{Tor}_1^A(k, B) = H_1(K_\bullet(\underline{x}; A) \otimes_A B) = H_1(K_\bullet(\underline{x}; B)).$$

Since the sequence $\varphi(x_1), \ldots, \varphi(x_r)$ is a part of a regular system of parameters of B it is a B-regular sequence. Hence we have $H_i(K_\bullet(\underline{x}; B)) = 0$ for all $i > 0$ by Theorem 43, and we are done. ∎

Remark. Even if B is regular and A-flat, the local ring $B \otimes_A k$ on the fiber is not necessarily regular. Example: let k be a field, $k[x,y] = k[X,Y]/((X-1)^2 + Y^2 - 1)$, $B = k[x,y]_{(x,y)}$, $A = k[x]_{(x)}$ and $\mathfrak{m} = xA$. Then $B \otimes_A (A/\mathfrak{m}) \cong k[Y]/(Y^2)$ has nilpotent elements.

(21.E)

Corollary. Let A and B be noetherian rings and $\varphi : A \to B$ a faithfully flat homomorphism. Let i be a non-negative integer. Then

(1) if B satisfies the condition (R_i) of (17.I), so does A;

(2) if A and all fibers $B \otimes_A \kappa(\mathfrak{p})$ ($\mathfrak{p} \in \mathrm{Spec}(A)$) satisfy (R_i), then B satisfies (R_i);

(3) if B is normal (resp. C.M., resp. reduced), so is A. Conversely, if A and all fibers are normal (resp. C.M., resp. reduced) then B is normal (resp. C.M., resp. reduced).

Proof. (1) Given $\mathfrak{p} \in \mathrm{Spec}(A)$ with $\mathrm{ht}(\mathfrak{p}) \leq i$, since φ is a f.f. homomorphism, $\mathrm{Spec}(\varphi)$ is surjective. Take $\mathfrak{q} \in \mathrm{Spec}(B)$ which is minimal among prime ideals of B lying over $\mathfrak{p}$, then by ($\heartsuit$),

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = \mathrm{ht}(\mathfrak{p}) \leq i.$$

Therefore $B_\mathfrak{q}$ is regular by assumption and hence $A_\mathfrak{p}$ is regular by Theorem 51.

(2) Given $\mathfrak{q} \in \mathrm{Spec}(B)$ with $\mathrm{ht}(\mathfrak{q}) \leq i$, put $\mathfrak{p} = \mathfrak{q} \cap A$, then ($\heartsuit$) holds, so

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) - \mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) \leq \mathrm{ht}(\mathfrak{q}) \leq i, \quad \text{and}$$
$$\mathrm{ht}(\mathfrak{q}/\mathfrak{p}B) = \mathrm{ht}(\mathfrak{q}) - \mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{q}) \leq i$$

Therefore $A_\mathfrak{p}$ and $B_\mathfrak{q} \otimes \kappa(\mathfrak{p}) = (B \otimes_A \kappa(\mathfrak{p}))_{(\mathfrak{q}/\mathfrak{p}B)}$ are regular by assumption and hence $B_\mathfrak{q}$ is regular by Theorem 51.

(3) It is enough to recall (17.I) that normal $\iff$ (R_1) + (S_2), C.M. $\iff$ all (S_i), and reduced $\iff$ (R_0) + (S_1). ∎

22 Theorem of Generic Flatness

(22.A)

Lemma 1. Let A be a noetherian domain, B an A-algebra of finite type and M a finite B-module. Then there exists $f \in A - \{0\}$ such that $M_f = M \otimes_A A_f$ is A_f-free (where A_f is the localization of A with respect to $\{1, f, f^2, \ldots\}$).

Proof. We may suppose that $M \neq 0$. Then, by (7.E) Theorem 10 there exists a chain of submodules

$$0 = M_0 \subset \cdots \subset M_{n-1} \subset M_n = M$$

with $M_i/M_{i-1} \cong B/\mathfrak{q}_i$ for some $\mathfrak{q}_i \in \mathrm{Spec}(B)$ $(1 \leq i \leq n)$. Since an extension of free modules is again free, it suffices to prove the lemma for the case that B is a domain and $M = B$. If the canonical map $A \to B$ has a non-trivial kernel then $B_f = 0$ for any non-zero element f of the kernel, and our assertion is trivial. So we may assume that A is a subring of the domain B.

Let K be the quotient field of A. Then $B \otimes_A K = BK$ is a domain (contained in the quotient field of B) and is finitely generated as an algebra over K. Hence $\dim(BK) = \mathrm{trdeg}_K(BK) < \infty$ by (14.G).

Put $n = \dim(BK)$. We use induction on n. By the normalization theorem (14.G), the ring BK contains n algebraically independent elements $y_1, \ldots, y_n$ such that BK is integral over $K[y]$. We may assume that $y_i \in B$. Since B is finitely generated over A, we can write $B = A[x_1, \ldots, x_r]$. For each i, there's a polynomial $f_i \in (K[y])[X]$ such that $f_i(x_i) = 0$. Then there exists $g_i \in A - \{0\}$ such that $g_i f_i \in (A[y])[X]$. Let $g = \prod_i g_i \neq 0$, then $B_g = BA_g$ is integral over $A_g[y]$. Replacing A and B by A_g and B_g respectively, and put $C = A[y]$, we have that B is a finite module over the polynomial ring C.

Let $b_1, \ldots, b_m$ be a maximal set of linearly independent elements over C in B. Then we have an exact sequence

$$0 \longrightarrow C^m \longrightarrow B \longrightarrow B' \longrightarrow 0,$$

where B' is a finitely generated torsion C-module. By (7.E) Theorem 10 there exists a chain of submodules

$$0 = B'_0 \subset \cdots \subset B'_{m-1} \subset B'_m = B$$

with $B'_i/B'_{i-1} \cong C/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}(C)$ $(1 \leq i \leq m)$, then B' is torsion implies that $\mathfrak{p}_i \neq 0$ for each i. Since $(C/\mathfrak{p}) \otimes_C K = CK/\mathfrak{p}K$ has a smaller dimension than $n = \dim(CK)$ for any non-zero prime ideal $\mathfrak{p}$ of C, there exists by the induction assumption a non-zero element f of A such that B'_f is A_f-free. Since $C_f^m = (A_f[y])^m$ is also A_f-free, the localization B_f is also A_f-free. ■

An important special case of the Lemma is the following

Theorem 52. Let A be a noetherian domain and B an A-algebra of finite type. Suppose that the canonical map $\varphi : A \to B$ is injective. Then there exists $f \in$

$A - \{0\}$ such that B_f is A_f-free and non-zero. Thus, the map $\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is faithfully flat over the non-empty open set $D(f) = \mathrm{Spec}(A) - V(f)$ of $\mathrm{Spec}(A)$, that is, $\mathrm{Spec}(\varphi)^{-1}(D(f)) \to D(f)$ is faithfully flat.

(22.B)

Lemma 2. Let B be a noetherian domain and let U be a subset of $\mathrm{Spec}(B)$. Then U is open iff the following conditions are satisfied.

(1) U is stable under generalization (6.G),

(2) if $\mathfrak{q} \in U$ then U contains a non-empty open set of the irreducible closed set $V(\mathfrak{q})$.

Proof. The 'only if' part is trivial. Assume the conditions, and let F be the complement of U and $\mathfrak{q}_i$ $(1 \leq i \leq s)$ be the generic points of the irreducible components of the closure $\overline{F}$ of F. Then (2) implies that $\mathfrak{q}_i$ cannot lie in U. Hence $\mathfrak{q}_i \in F$, and so $F = \overline{F}$ by (1). ■

Theorem 53. Let A be a noetherian ring, B an A-algebra of finite type and M a finite B-module. Put

$$U = \{\mathfrak{q} \in \mathrm{Spec}(B) \mid M_{\mathfrak{q}} \text{ is flat over } A\}.$$

Then U is open in $\mathrm{Spec}(B)$.

Proof. Let $\mathfrak{q}' \supset \mathfrak{q}$ be prime ideals of B with $M_{\mathfrak{q}}$ flat over A. For any A-module N we have $N \otimes_A M_{\mathfrak{q}'} = (N \otimes_A M_{\mathfrak{q}}) \otimes_B B_{\mathfrak{q}'}$, therefore $M_{\mathfrak{q}'}$ is flat over A and the condition (1) of Lemma 2 is verified for U. As for the condition (2), let $\mathfrak{q} \in U$ and put $\mathfrak{p} = \mathfrak{q} \cap A$ and $\overline{A} = A/\mathfrak{p}$. Let $\mathfrak{q}' \in V(\mathfrak{q})$. Then $\mathfrak{p}B_{\mathfrak{q}'} \subseteq \mathrm{rad}(B_{\mathfrak{q}'})$, so we can apply the local criterion of flatness that $M_{\mathfrak{q}'}$ is flat over A iff $M_{\mathfrak{q}'}/\mathfrak{p}M_{\mathfrak{q}'}$ is flat over $\overline{A}$ and $\mathrm{Tor}_1^A(M_{\mathfrak{q}'}, \overline{A}) = 0$. Applying Lemma 1 to $(\overline{A},\ B/\mathfrak{p}B,\ M/\mathfrak{p}M)$ we see that there exists a neighborhood of $\mathfrak{q}$ in $V(\mathfrak{p}B)$ such that $M_{\mathfrak{q}'}/\mathfrak{p}M_{\mathfrak{q}'}$ is flat over $\overline{A}$ for each point $\mathfrak{q}'$ in it. On the other hand, since

$$0 = \mathrm{Tor}_1^A(M_{\mathfrak{q}}, \overline{A}) = \mathrm{Tor}_1^A(M, \overline{A}) \otimes_B B_{\mathfrak{q}}$$

and since $\mathrm{Tor}_1^A(M, \overline{A})$ is a finite B-module, its support in $\mathrm{Spec}(B)$ is closed, so there exists a neighborhood of $\mathfrak{q}$ in $\mathrm{Spec}(B)$ in which $\mathrm{Tor}_1^A(M_{\mathfrak{q}'}, \overline{A}) = 0$ for each $\mathfrak{q}'$ in it. Therefore there exists a non-empty open set of $V(\mathfrak{q})$ in which $M_{\mathfrak{q}'}$ is A-flat for all points $\mathfrak{q}'$ in it, in other words the set U in question contains a non-empty open set of $V(\mathfrak{q})$. Thus the theorem is proved. ■

Remark. (1) The set U may be empty. (2) It follows from (6.I) Theorem 8 that a flat morphism of finite type between noetherian preschemes is an open map. Therefore the image of U in $\mathrm{Spec}(A)$ is open in $\mathrm{Spec}(A)$.

(22.C) Let $\mathscr{P}$ be a property on noetherian local rings and let $\mathscr{P}(A)$ denote the set

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ has the property } \mathscr{P}\}.$$

Consider the following statement.

(NC) If A is a noetherian ring and if, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathscr{P}(A/\mathfrak{p})$ contains a non-empty open set of $\mathrm{Spec}(A/\mathfrak{p})$, then $\mathscr{P}(A)$ is open in $\mathrm{Spec}(A)$.

While Lemma 2 of (22.B) was topological, (NC) is ring-theoretical and its validity of course depends on $\mathscr{P}$. Both are invention of Nagata (NC means Nagata criterion), who proved (NC) for $\mathscr{P}$ is regular (cf. (32.A)).

Proposition. (NC) is valid for $\mathscr{P}$ is C.M..

Proof. C.M.(A) is stable under generalization by Theorem 30. We will prove (2) of Lemma 2. If $\mathfrak{p} \in$ C.M.(A) and $\mathrm{ht}(\mathfrak{p}) = n$, we can take an $A_{\mathfrak{p}}$-regular sequence $y_1, \ldots, y_n$ from $\mathfrak{p}$. Let K_i denote the kernel of the map

$$A/(y_1, \ldots, y_{i-1}) \xrightarrow{[y_i]} A/(y_1, \ldots, y_{i-1}),$$

then $(K_i)_{\mathfrak{p}} = 0$. Since $\operatorname{Supp}(K_i)$ is closed, there's an open neighborhood of $\mathfrak{p}$, say $D(a_i)$, such that $(K_i)_{\mathfrak{q}} = 0$ for any $\mathfrak{q} \in D(a)$. Then $(K_i)_{a_i} = 0$. So the map

$$A_{a_i}/(y_1, \ldots, y_{i-1}) \xrightarrow{[y_i]} A_{a_i}/(y_1, \ldots, y_{i-1})$$

is injective. Let $a' = \prod_i a_i$, then $y_1, \ldots, y_n$ is an $A_{a'}$-regular sequence. On the other hand, let $I = \sum_i y_i A = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a primary decomposition, then $I_{\mathfrak{p}}$ is $\mathfrak{p}$-primary implies that one of $\sqrt{\mathfrak{q}_i}$ is $\mathfrak{p}$, say $\sqrt{\mathfrak{q}_1}$. Take $a'' \in \sqrt{\mathfrak{q}_2} \cap \cdots \cap \sqrt{\mathfrak{q}_r}$ but not in $\mathfrak{p}$, then $I_{a''}$ is $\mathfrak{p}$-primary. Let $a = a'a''$, replacing A by $A_{\mathfrak{a}}$, we may assume that $y_1, \ldots, y_n$ is an A-regular sequence and I is a $\mathfrak{p}$-primary ideal. For $\mathfrak{q} \in V(\mathfrak{p})$, $A_{\mathfrak{q}}$ is C.M. iff $A_{\mathfrak{q}}/IA_{\mathfrak{q}}$ is so by Theorem 30. Hence we can replace A by A/I and assume that (0) is $\mathfrak{p}$-primary. So we have $\mathfrak{p}^r = 0$ for some $r > 0$. Since $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ is a finite $A/\mathfrak{p}$-module for each $0 \leq i < r$, we may assume (replacing A by some A_a) that the $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ are free $A/\mathfrak{p}$-modules by Lemma 1. Suppose a sequence $x_1, \ldots, x_n \in A$ is $A/\mathfrak{p}$-regular. Let $b \in \operatorname{Ann}(x_1)$, then $b \in \mathfrak{p}$. If $b \neq 0$, then $b \in \mathfrak{p}^k - \mathfrak{p}^{k+1}$ for some $1 \leq k < r$, but $\mathfrak{p}^k/\mathfrak{p}^{k+1}$ is $A/\mathfrak{p}$-free, a contradiction. So x_1 is A-regular and inductively we can prove that $x_1, \ldots, x_n$ is an A-regular sequence (♠). By the hypothesis of (NC) we may assume further that $A/\mathfrak{p}$ is C.M. (localize at a neighborhood of the generic point $\mathfrak{p}/\mathfrak{p} \in \operatorname{Spec}(A/\mathfrak{p})$). Then for any $\mathfrak{q} \in V(\mathfrak{p})$,

$$\begin{aligned}
\operatorname{depth}(A_{\mathfrak{q}}) &= \operatorname{depth}(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) && \text{by } (\spadesuit) \\
&= \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) && \text{by } A/\mathfrak{p} \text{ is C.M. and Theorem 30 (3)} \\
&= \dim(A_{\mathfrak{q}}) && \text{by } \sqrt{(0)} = \mathfrak{p},
\end{aligned}$$

hence $\mathfrak{q} \in \text{C.M.}(A)$. ■

Chapter 9

Completion

23 Completion

(23.A) Let A be a ring, and let $\mathcal{F}$ be a set of ideals of A such that for any two ideals I_1, $I_2 \in \mathcal{F}$ there exists $I_3 \in \mathcal{F}$ contained in $I_1 \cap I_2$. Then one can define a topology on A by taking

$$\{x + I \mid I \in \mathcal{F}\}$$

as a fundamental system of neighborhood of x for each $x \in A$. One sees immediately that in this topology the addition, the multiplication and the map $x \mapsto -x$ are continuous; in other words A is a topological ring. A topology on a ring obtained in this manner is called a linear topology. When M is an A-module one defied a linear topology on M in the same way, the only difference being that 'ideals' are replaced by 'submodules'. Let $\mathcal{M} = \{M_\lambda\}$ be a set of submodules which defines the topology. Then M is Hausdorff iff

$$\bigcap_\lambda M_\lambda = 0.$$

A submodule N of M is closed in M iff

$$\bigcap_\lambda (M_\lambda + N) = N,$$

the left hand side being the closure of N.

(23.B) Let A be a ring, M an A-module linearly topologized by a set of submodules $\{M_\lambda\}$ and N a submodule of M. Let $\overline{M_\lambda}$ be the image of M_λ in M/N. Then the linear topology on M/N defined by $\{M_\lambda\}$ is nothing but the quotient topology of the topology on M, as one can easily check. When we say "the quotient module M/N", we shall always mean the module with the quotient topology. It is Hausdorff iff N is closed.

(23.C) For simplicity, we shall consider in the following only such linear topologies that are defined by a countable set of submodules. This is equivalent to saying that the topology is first countable. If a linear topology on M is defined by $\{M_1, M_2, \ldots\}$, then the set

$$\{M_1, M_1 \cap M_2, M_2 \cap M_2 \cap M_3, \ldots\}$$

defines the same topology. Therefore we can assume without loss of generality that $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ (in other words, the topology is defined by a filtration of M, cf. (11.A)). A sequence $\{x_n\}$ of elements of M is a Cauchy sequence if, for every open submodule N of M, there exists an integer n_0 such that

$$(\clubsuit) \quad x_n - x_m \in N \text{ for all } n, m > n_0.$$

Since N is a submodule, the condition ($\clubsuit$) can also be written as $x_{n+1} - x_n \in N$ for all $n > n_0$. Therefore a sequence $\{x_n\}$ is Cauchy iff $x_{n+1} - x_n$ converges to zero when n tends to infinity. A continuous homomorphism of linearly topologized modules maps Cauchy sequences into Cauchy sequences. A topological A-module M is said to be complete if every Cauchy sequence in M has a limit in M. Note that the limit of a Cauchy sequence is not uniquely determined if M is not Hausdorff.

(23.D)

Proposition. Let A be a ring and let M be an A-module with a linear topology defined by a filtration

$$M_1 \supseteq M_2 \supseteq \cdots ;$$

let N be a submodule of N. If M is complete, then the quotient module M/N is also complete.

Proof. Let $\{\overline{x_n}\}$ be a Cauchy sequence in M/N. For each $\overline{x_n}$ choose a pre-image x_n in M. We have $\overline{x_{n+1}} - \overline{x_n} \in \overline{M_{i(n)}}$ with $i(n) \to \infty$, therefore we can write

$$x_{n+1} - x_n = y_n + z_n,\ y_n \in M_{i(n)},\ z_n \in N,$$

and the sequence $\{y_n\}$ converges to zero in M. Let $s \in M$ be a limit of the sequence

$$x_1, x_1 + y_1, x_1 + y_1 + y_2, \ldots;$$

then its image $\overline{s}$ in M/N is a limit of the sequence $\{\overline{x_n}\}$. Thus M/N is complete. ■

(23.E) Let A be a ring, I an ideal and M an A-module. The set of submodules $\{I^n M \mid n \in \mathbb{N}\}$ defines the I-adic topology of M. We also say that the topology is adic and that I is an ideal of definition for the topology. Clearly, any ideal J such that $I^n \subseteq J$ and $J^m \subseteq I$ for some $n, m > 0$ is an ideal of definition for the same topology. When both A and M are I-adically topologized, the map $(a, x) \mapsto ax$ ($a \in A$, $x \in M$) is a continuous map from $A \times M$ to M. When A is a semi-local ring with $\mathrm{rad}(A) = \mathfrak{J}$ then it is viewed as an $\mathfrak{J}$-adic topological ring, unless the contrary is explicitly stated.

(23.F) Let R be a ring, and let A and B be R-algebras with linear topology defined by $\mathcal{M} = \{I_m\}$ and $\mathcal{N} = \{J_n\}$, respectively. Put $C = A \otimes_R B$. Then a linear topology can be defined on C by the means of the set of ideals $\{I_m C + J_n C\}$. This is called the topology of tensor product. If A has the I-adic topology and B the J-adic topology, where I (resp. J) is an ideal of A (resp. B), then the topology of tensor product on C is the $(IC + JC)$-adic topology, for we have

$$(IC + JC)^{m+n-1} \subseteq I^m C + J^n C \quad \text{and} \quad I^n C + J^n C \subseteq (IC + JC)^n.$$

(23.G)

Proposition. Let A be a ring and I an ideal of A. Suppose that A is complete and Hausdorff for the I-adic topology. Then

$$\{u + x \in A \mid u \in A^\times, x \in I\} \subseteq A^\times$$

and $I \subseteq \mathrm{rad}(A)$.

Proof. We have $u + x = u(1 - y)$, where $y = -u^{-1}x \in I$. The infinite series

$$1 + y + y^2 + \cdots$$

converges in A since $y^n \in I^n$, and we have

$$(1 - y)(1 + y + y^2 + \cdots) = 1$$

since A is Hausdorff. Thus $1-y$ (hence also $u+x$) is a unit. For the second assertion, if $x \in I$, then $1 + ax \in A^\times$ for any $a \in A$ by the first assertion, so $x \in \mathrm{rad}(A)$. ■

(23.H) Let A be a ring and M a linearly topologized A-module. The completion of M is, by definition, an A-module $\widehat{M}$ with a complete Hausdorff linear topology, together with a continuous homomorphism $\iota : M \to \widehat{M}$, having the following universal mapping property: for any A-module M' with a complete Hausdorff linear topology and for any continuous homomorphism $\varphi : M \to M'$, there exists a unique continuous homomorphism $\widehat{\varphi} : \widehat{M} \to M'$ satisfying $\widehat{\varphi} \circ \iota = \varphi$.

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\iota"] & M' \\
\widehat{M} \arrow[ur, dashed, "\widehat{\varphi}"] &
\end{tikzcd}$$

The completion of M exists, and is unique up to isomorphisms. In fact the uniqueness is clear from definition, while the existence can be proved by several methods. First of all, note that, if K is the intersection of all open submodules of M, the canonical map $\iota : M \to \widehat{M}$ must factor through $M^h = M/K$ (which is called the Hausdorffization of M) and hence M and M^h have the same completion.

1. Take the completion of the uniform space M^h and call it $\widehat{M}$. The topological space $\widehat{M}$ becomes a linearly topologized A-module by extending the A-module structure of M^h to $\widehat{M}$ by uniform continuity. The universal mapping property of $\widehat{M}$ follows immediately, continuous homomorphisms $\varphi : M \to M'$ being uniformly continuous.

2. Let W be the set of Cauchy sequences in M, and make it an A-module by defining the addition and the scalar multiplication termwise. Then the set W_0 of the null sequences (i.e. the sequence which have zero as a limit) is a submodule of W. Put $\widehat{M} = W/W_0$, and define the canonical map $\iota : M \to \widehat{M}$ in the obvious way. For any open submodule N of M, let $\widehat{N}$ denote the image in $\widehat{M}$ of the set of Cauchy sequences in N. Then $\widehat{N}$ is a submodule of $\widehat{M}$, The set of all such $\widehat{N}$ defines a linear topology in $\widehat{M}$, and $\widehat{N}$ is the closure of $\iota(N)$ in this topology. It is easy to see that $\widehat{M}$ is complete and Hausdorff and has the universal property.

3. Denote by $\widehat{M}$ the inverse limit of the discrete A-modules M/M_n, where $\{M_n\}$ is a filtration of M defining the topology, and put the inverse limit topology (i.e. the topology as a subspace of the product space $\prod_n (M/M_n))$ on it. Let $\iota : M \to \widehat{M}$ be defined in the obvious way, and let $\widehat{M_n}$ denote that closure of $\iota(M_n)$ in $\widehat{M}$. Then $\widehat{M_n}$ consists of those vectors of $\widehat{M}$ of which the first n coordinates are zero, and the set of submodules $\{\widehat{M_n}\}$ defines a complete Hausdorff linear topology on $\widehat{M}$. Let M' be an A-module with a complete Hausdorff linear topology and $\varphi : M \to M'$ a continuous homomorphism. For any element $\widehat{x} = (\overline{x_1}, \overline{x_2}, \ldots)$ of $\widehat{M}$ ($\overline{x_n} \in M/M_n$), choose a pre-image x_n of $\overline{x_n}$ in M for each n. Then the sequence $x_1, x_2, \ldots$ is a Cauchy sequence in M, hence the image sequence $\varphi(x_1), \varphi(x_2), \ldots$ is a Cauchy sequence in M'. Therefore $\lim_n \varphi(x_n)$ exists in M', and this limit is easily seen to be independent of the choice of the pre-image x_n. Putting $\widehat{\varphi}(\widehat{x}) = \lim_n \varphi(x_n)$ we obtain $\widehat{\varphi} : \widehat{M} \to M'$ as wanted.

These constructions show that $\iota : M \to \widehat{M}$ is injective if M is Hausdorff.

(23.I) If $\varphi : M \to N$ is a continuous homomorphism of linearly topologized A-modules M and N, and if $\iota_M : M \to \widehat{M}$ and $\iota_N : N \to \widehat{N}$ are the canonical homomorphisms into the completions, then there exists a unique continuous homomorphism $\widehat{\varphi} : \widehat{M} \to \widehat{N}$ with $\iota_N \circ \varphi = \widehat{\varphi} \circ \iota_M$; this is a formal consequence of the definition. The map is called the completion of φ. Taking completions is, therefore, an additive covariant functor.

Proposition. Let M be a linearly topologized A-module, N a submodule and $\iota : M \to \widehat{M}$ the canonical map to the completion. Then

(1) the completion of N (for the topology induced from M) is the closure $\overline{\iota(N)}$ of $\iota(N)$ in $\widehat{M}$, and

(2) the quotient module $\widehat{M}/\overline{\iota(N)}$ is the completion of the quotient module M/N.

Proof. (1) This follows from the second construction of completion in (23.H). In fact, a sequence in N is Cauchy iff it is Cauchy in M.

(2) The quotient module $\widehat{M}/\overline{\iota(N)}$ is Hausdorff by (23.B), and complete by (23.D). The canonical map ι induces a map $\bar{\iota} : M/N \to \widehat{M}/\overline{\iota(N)}$. Let M' be any A-module with a complete Hausdorff linear topology and let $\varphi : M/N \to M'$ be any continuous homomorphism. Then there exists a unique continuous homomorphism $\widehat{f} : \widehat{M} \to M'$ such that the following diagram commute:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\iota"] & M/N \arrow[r, "\varphi"] \arrow[d, "\widehat{\varphi}"'] & M' \\
\widehat{M} \arrow[r] \arrow[urr, dashed] & \widehat{M}/\overline{\iota(N)} \arrow[ur, dashed] &
\end{tikzcd}$$

Since $\ker \widehat{f}$ is closed and

$$\iota(N) \subseteq \iota(\ker f) \subseteq \ker \widehat{f},$$

we have $\overline{\iota(N)} \subseteq \ker \widehat{f}$, so $\widehat{f}$ factors through $\widehat{M}/\overline{\iota(N)}$ and hence $\widehat{M}/\overline{\iota(N)}$ satisfies the universal property. ∎

Remark. (1) Taking $N = M$ we see that $\iota(M)$ is dense in $\widehat{M}$. (2) If N is an open submodule of M then M/N is discrete, hence complete and Hausdorff. Thus $M/N = \widehat{M}/\overline{\iota(N)}$.

Theorem 54. Let A be a noetherian ring and I an ideal. Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of finite A-modules, and let $\widehat{}$ denote that I-adic completion. Then the sequence

$$0 \longrightarrow \widehat{L} \longrightarrow \widehat{M} \longrightarrow \widehat{N} \longrightarrow 0$$

is also exact.

Proof. By Artin-Rees Theorem (11.C),

$$I^n L \subseteq I^n M \cap L \subseteq I^{n-r} L, \quad \forall n > r$$

for some $r > 0$. So the I-adic topology of L coincides with the topology induced by the I-adic topology of M. Therefore the assertion follows from the preceding proposition. ■

(23.J) Let A be a linear topologized ring. Then the completion $\widehat{A}$ of A is not only an A-module but also a ring, the multiplication in A being extended to $\widehat{A}$ by continuity. If $\iota : A \to \widehat{A}$ is the canonical map and I is an ideal of A, then the closure $\overline{\iota(I)}$ of $\iota(I)$ in $\widehat{A}$ is an ideal of $\widehat{A}$ by (23.I). Thus $\widehat{A}$ is a linearly topologized ring. Example: let k be a ring. Put $A = k[X_1, \ldots, X_n]$ and $I = \sum_i AX_i$. Then the ring of formal power series $k[[X_1, \ldots, X_n]]$ is the I-adic completion of A.

(23.K) Let A be a ring, I a finitely generated ideal of A, $\widehat{A}$ the I-adic completion of A and $\iota : A \to \widehat{A}$ the canonical map. Then, for any element x of $\widehat{A}$ there exists a Cauchy sequence $\{x_n\}$ in A such that $x = \lim \iota(x_n)$. Replacing $\{x_n\}$ by a suitable sub-sequence we may assume that $x_{n+1} - x_n \in I^n$ $(n \geq 0)$. Let $a_1, \ldots, a_m$ generate I, and put $a'_i = \iota(a_i)$. Then $x_{n+1} - x_n$ is a homogeneous polynomial of degree n in $a_1, \ldots, a_m$. Thus

$$x = \iota(x_0) + \sum_{n=0}^{\infty} \iota(x_{n+1} - x_n)$$

has a power series expansion in $a'_1, \ldots, a'_m$ with coefficients in $\iota(A)$. Consider the formal power series ring $A[[X]] = A[[X_1, \ldots, X_m]]$; let $u(X) \in A[[X]]$, and let $\widehat{u}(X)$ denote the power series obtained by applying ι to the coefficients of $u(X)$. Since $\widehat{A}$ is complete and Hausdorff, the series $\widehat{u}(a') = \widehat{u}(a'_1, \ldots, a'_m)$ converges in $\widehat{A}$. The map $[u(X) \mapsto \widehat{u}(a')]$ defines a surjective homomorphism $A[[X]] \to \widehat{A}$. Thus $\widehat{A} \cong A[[X]]/J$ with some ideal J in $A[[X]]$. As a consequence, $\widehat{A}$ is noetherian if A is so.

(23.L) Let A be a ring, I an ideal and M an A-module. Let $\widehat{\ }$ denote the I-adic completion. Then $\widehat{M}$ is an $\widehat{A}$-module in a natural way, therefore there exists a canonical map $M \otimes_A \widehat{A} \to \widehat{M}$.

Theorem 55. When A is noetherian and M is finite over A, the canonical map

$$M \otimes_A \widehat{A} \to \widehat{M}$$

is an isomorphism.

Proof. Take an exact sequence of A-modules

$$A^p \xrightarrow{\varphi} A^q \xrightarrow{\psi} M \longrightarrow 0.$$

Since the completion commutes with direct sum, we get a commutative diagram

$$\begin{array}{ccccccc}
A^p \otimes_A \widehat{A} & \longrightarrow & A^q \otimes_A \widehat{A} & \longrightarrow & M \otimes_A \widehat{A} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\widehat{A}^p & \xrightarrow{\widehat{\varphi}} & \widehat{A}^q & \xrightarrow{\widehat{\psi}} & \widehat{M} & \longrightarrow & 0
\end{array}$$

where the vertical arrows are the canonical maps and the horizontal sequences are exact by the right-exactness of tensor product and by Theorem 54. Since α and β are isomorphisms, γ is also an isomorphism by snake lemma. ■

Corollary 1. Let A be a noetherian ring and I an ideal of A. Then the I-adic completion $\widehat{A}$ of A is flat over A.

Corollary 2. Let A and I be as above and assume that A is I-adically complete and Hausdorff. Let M be a finite A-module. Then M is complete and Hausdorff, and any submodule N of M is closed in M, for the I-adic topology.

Proof. Since $A = \widehat{A}$ we have $\widehat{M} = M \otimes_A \widehat{A} = M$, i.e. M is its own completion. Similarly, a submodule N is complete in the I-adic topology, which coincides with the induced topology by Artin-Rees theorem (cf. Theorem 54). Since a complete subspace of M is necessarily closed, we are done. ■

Corollary 3. Let A be a noetherian ring, M a finite A-module, N a submodule of M and I an ideal of A. Let $\iota : M \to \widehat{M}$ be the canonical map to the I-adic completion $\widehat{M}$. Then we have

$$\widehat{N} \cong \overline{\iota(N)} = \widehat{A}\,\iota(N),$$

where $\overline{\iota(N)}$ is the closure of $\iota(N)$ in $\widehat{M}$.

Proof. By Theorem 54, $\widehat{N} \cong \overline{\iota(N)}$ is a submodule of $\widehat{M}$. Note that $\widehat{A}\,\iota(N)$ is the smallest $\widehat{A}$-submodule of $\widehat{M}$ containing $\iota(N)$, by Corollary 2 we have $\widehat{A}\,\iota(N)$ is the closure of $\iota(N)$. ■

Corollary 4. Let A and I be as in Corollary 3. Then the topology of the I-adic completion $\widehat{A}$ of A is the $I\widehat{A}$-adic topology.

Proof. By construction, the topology of $\widehat{A}$ is defined by the ideals $\left\{\overline{\varphi(I^n)}\right\}$, which is

$$I^n\widehat{A} = (I\widehat{A})^n$$

for each n by Corollary 3. ■

Corollary 5. Let A, I and $\widehat{A}$ be as above and suppose that $I = \sum_{i=1}^{m} a_i A$. Then

$$\widehat{A} \cong A[[X_1, \ldots, X_m]]/(X_1 - a_1, \ldots, X_m - a_m)$$

Proof. Put $B = A[X_1, \ldots, X_m]$, $I' = \sum_i X_i B$ and $J = \sum_i (X_i - a_i)B$. Then $B/J \cong A$, and the I'-adic topology on the B-algebra B/J corresponds to the I-adic topology on A. Denoting the I'-adic completion by $\widehat{\ }$, we thus obtain

$$\begin{aligned}
\widehat{A} \cong \widehat{B/J} &= \widehat{B}/\widehat{J} && \text{by Theorem 54}\\
&= \widehat{B}/J\widehat{B} && \text{by Theorem 55}\\
&= A[[X_1, \ldots, X_m]]/(X_1 - a_1, \ldots, X_m - a_m).
\end{aligned}$$ ■

24 Zariski Rings

(24.A)

Definition. A Zariski ring is a noetherian ring equipped with an adic topology, such that every ideal is closed in it.

Theorem 56. Let A be a noetherian ring with an adic topology, and let I be an ideal of definition. Then the following are equivalent.

(i) A is a Zariski ring;

(ii) $I \subseteq \operatorname{rad}(A)$;

(iii) every finite A-module M is Hausdorff in the I-adic topology;

(iv) in every finite A-module M, every submodule is closed in the I-adic topology;

(v) the completion $\widehat{A}$ of A is faithfully flat over A.

Proof. (i) $\Longrightarrow$ (ii): Suppose that a maximal ideal $\mathfrak{m}$ does not contain I. Then $I^n \nsubseteq \mathfrak{m}$ for all $n > 0$, so that $\mathfrak{m} + I^n = A$ and $\bigcap_n (\mathfrak{m} + I^n) = A \neq \mathfrak{m}$. Therefore $\mathfrak{m}$ is not closed, contradiction.

(ii) $\Longrightarrow$ (iii): By the intersection theorem (11.D).

(iii) $\Longrightarrow$ (iv): If N is a submodule of M, then M/N is Hausdorff by assumption so that N is closed in M.

(iv) $\Longrightarrow$ (i): Trivial.

(ii) $\Longrightarrow$ (v): Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathfrak{m} \supseteq I$, hence

$$\mathfrak{m} = \bigcup_{x \in \mathfrak{m}} (x + I)$$

is open in A and so $\widehat{A}/\mathfrak{m}\widehat{A} \cong A/\mathfrak{m}$ by the remark in (23.I). Thus $\mathfrak{m}\widehat{A} \neq \widehat{A}$. Since $\widehat{A}$ is flat over A by (23.L) Corollary 1, this implies by (4.A) Theorem 2 that $\widehat{A}$ is f.f. over A.

(v) $\Longrightarrow$ (ii): If $\mathfrak{m}$ is maximal ideal of A then there exists, by assumption and (4.D), a maximal ideal $\mathfrak{m}'$ of $\widehat{A}$ lying over $\mathfrak{m}$. Since $I\widehat{A} \subseteq \mathfrak{m}'$ by (23.G), we have

$$I \subseteq I\widehat{A} \cap A \subseteq \mathfrak{m}' \cap A = \mathfrak{m}. \qquad \blacksquare$$

Corollary. Let A be a Zariski ring and $\widehat{A}$ its completion. Then

(1) A is a subring of $\widehat{A}$, and

(2) the map $[\mathfrak{m} \mapsto \mathfrak{m}\widehat{A}]$ is a bijection from the set $\mathrm{Max}(A)$ to $\mathrm{Max}(\widehat{A})$, and we have

$$A/\mathfrak{m} \cong \widehat{A}/\mathfrak{m}\widehat{A} \quad \text{and} \quad \mathfrak{m}\widehat{A} \cap A = \mathfrak{m}.$$

Proof. (1) follows from (iii). (2): Since $A/I \cong \widehat{A}/I\widehat{A}$, $I \subseteq \mathrm{rad}(A)$ by (ii) and $I\widehat{A} \subseteq \mathrm{rad}(\widehat{A})$ by (23.G),

$$\mathrm{Max}(A) \cong \mathrm{Max}(A/I) \cong \mathrm{Max}(\widehat{A}/I\widehat{A}) \cong \mathrm{Max}(\widehat{A}). \qquad \blacksquare$$

(24.B) A noetherian semi-local ring is a Zariski ring. A noetherian ring with an adic topology which is complete and Hausdorff is also a Zariski ring by Theorem 56.

Let A be an arbitrary noetherian ring and I a proper ideal of A. Put

$$S = 1 + I = \{1 + x \mid x \in I\}, \quad A' = S^{-1}A \quad \text{and} \quad I' = S^{-1}I.$$

Then all elements of $1 + I'$ are invertible in A', and so $I' \in \mathrm{rad}(A')$. We equip A with the I-adic topology and A' with the I'-adic (or what is the same, the I-adic) topology. Then the canonical map $\psi : A \to A'$ is continuous, and has the universal mapping property for continuous homomorphisms from A to Zariski rings.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"] & B \\ A' \arrow[ur, dashed, "\varphi'"'] & \end{tikzcd}$$

In fact, if $\varphi : A \to B$ is such a homomorphism and if J is an ideal of definition for B, then $\varphi(I^n) \subseteq J \subseteq \mathrm{rad}(B)$ for some n, hence $\varphi(I) \subseteq \mathrm{rad}(B)$ and the elements of $\varphi(S)$ are invertible in B. Therefore φ factors through A'. In particular, the canonical map $\iota : A \to \widehat{A}$ of A into the completion $\widehat{A}$ of A factors through A', and it follows immediately that $\widehat{A}$ is also the completion of A'.

For a prime ideal $\mathfrak{p}$ of A, we have

$$\mathfrak{p} \cap S = \varnothing \iff \mathfrak{p} + I \neq A \iff V(\mathfrak{p}) \cap V(I) \neq \varnothing.$$

The localization $A \to A'$ has, geometrically, the effect of considering only the "subvarieties" of $\mathrm{Spec}(A)$ with intersect the closed set $V(I)$. Since $\widehat{A}$ is faithfully flat over A' by Theorem 56, the map $\mathrm{Spec}(\widehat{A}) \to \mathrm{Spec}(A')$ is surjective, so the set

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} + I \neq A\} \cong \mathrm{Spec}(A')$$

is also the image of $\mathrm{Spec}(\widehat{A})$ in $\mathrm{Spec}(A)$. The set of the maximal ideals of $\widehat{A}$ (resp. the prime ideals of $\widehat{A}$ containing $I\widehat{A}$) is in a natural 1-1 correspondence with the set of the maximal ideals (resp. prime ideals) of A containing I by the corollary in (24.A) (resp. by $A/I \cong \widehat{A}/I\widehat{A}$).

(24.C) Let A be a semi-local ring and $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be its maximal ideals. Put

$$A_i = A_{\mathfrak{m}_i}, \quad \mathfrak{m}'_i = \mathfrak{m}_i A_i \ (1 \leq i \leq r), \quad \text{and} \quad \mathfrak{J} = \mathrm{rad}(A) = \prod_i \mathfrak{m}_i.$$

Then $\mathfrak{m}^n = \prod_i \mathfrak{m}_i^n = \bigcap_i \mathfrak{m}_i^n$, hence

$$A/\mathfrak{m}^n = \prod_i A/\mathfrak{m}_i^n$$

by (1.C). Moreover, $A/\mathfrak{m}_i^n \cong A_i/(\mathfrak{m}'_i)^n$ as $A/\mathfrak{m}_i^n$ is a local ring. Therefore

$$\widehat{A} = \varprojlim A/\mathfrak{m}^n = \prod_i \widehat{A_i}.$$

(24.D) Let $(A, \mathfrak{m})$ be a noetherian local ring and $\widehat{A}$ its completion. Then $A/\mathfrak{m}^n \cong \widehat{A}/\mathfrak{m}^n\widehat{A}$ for all $n > 0$, hence $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}^n\widehat{A}/\mathfrak{m}^{n+1}\widehat{A}$ and $\mathrm{gr}(A) \cong \mathrm{gr}(\widehat{A})$. It follows that

(1) $\dim(A) = \dim(\widehat{A})$, and

(2) A is regular iff $\widehat{A}$ is so.

Next let A be an arbitrary noetherian ring, I an ideal of A and $\widehat{A}$ the I-adic completion of A. Let $\mathfrak{p}$ be a prime ideal of A containing I. Since $\mathfrak{p}$ is open in A, the ideal $\mathfrak{p}\widehat{A} = \widehat{\mathfrak{p}}$ is open and prime in $\widehat{A}$ and $A/\mathfrak{p}^n \cong \widehat{A}/\widehat{\mathfrak{p}}^n$ for all $n > 0$. Localizing both sides with respect to $\mathfrak{p}/\mathfrak{p}^n$ and $\widehat{\mathfrak{p}}/\widehat{\mathfrak{p}}^n$ respectively, we get

$$A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \cong \widehat{A}_{\widehat{\mathfrak{p}}}/\widehat{\mathfrak{p}}^n \widehat{A}_{\widehat{\mathfrak{p}}}.$$

Therefore

$$\widehat{A_{\mathfrak{p}}} = \varprojlim A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \cong (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }.$$

Two local rings are said to be analytically isomorphic if their completions are isomorphic. Thus, if $\mathfrak{p}$ and $\widehat{\mathfrak{p}}$ are corresponding open prime ideals of A and $\widehat{A}$, then the local rings $A_{\mathfrak{p}}$ and $\widehat{A}_{\widehat{\mathfrak{p}}}$ are analytically isomorphic. Since all maximal ideals of $\widehat{A}$ are open, it follows that

(1') $\dim(\widehat{A}) = \sup\limits_{\mathfrak{p}\supseteq I} \dim(A_{\mathfrak{p}})$, and

(2') if $A_{\mathfrak{p}}$ is regular for every prime ideal $\mathfrak{p}$ containing I, then $\widehat{A}$ is regular.

As a corollary of (2') we have the following

Proposition. Let A be a regular noetherian ring,. Then the ring of formal power series $A[[X_1, \ldots, X_m]]$ is also regular.

Proof. $A[X] = A[X_1, \ldots, X_m]$ is a regular ring by (17.J), and $A[[X]]$ is a completion of $A[X]$. ■

(24.E)

Proposition. Let A be a Zariski ring and $\widehat{A}$ its completion. Then:

(1) If $\mathfrak{a}$ is an ideal of A and if $\mathfrak{a}\widehat{A}$ is principal, then $\mathfrak{a}$ itself is principal.

(2) If $\widehat{A}$ is normal, then A is also normal.

Proof. (1) Suppose $\mathfrak{a}\widehat{A} = \alpha\widehat{A}$, $\alpha \in \widehat{A}$. Then $\alpha = \sum\limits_i a_i\xi_i$ with $a_i \in \mathfrak{a}$, $\xi_i \in \widehat{A}$. Put $\widehat{I} = I\widehat{A}$, where I is an ideal of definition of A. By Artin-Rees (11.C) we have

$$\alpha\widehat{A} \cap \widehat{I}^n \subseteq \widehat{I}\alpha\widehat{A}$$

for n sufficient large. Take $x_i \in A$ such that $x_i - \xi_i \in \widehat{I}^n$ and put $a = \sum\limits_i a_i x_i$. Then $\beta = a - \alpha \in \widehat{I}^n$, and $a \in \mathfrak{a} \subseteq \alpha\widehat{A}$. Therefore

$$\beta \in \alpha\widehat{A} \cap \widehat{I}^n \subseteq \widehat{I}\alpha\widehat{A},$$

hence $\alpha\widehat{A} \subseteq a\widehat{A} + \widehat{I}\alpha\widehat{A}$, and by Nakayama's lemma we get $\alpha\widehat{A} = a\widehat{A}$. Then

$$\mathfrak{a} = \alpha\widehat{A} \cap A = a\widehat{A} \cap A = aA.$$

(2) is a consequence of faithful flatness and was already proved in (21.E). ■

We shall set in Part II that noetherian local (or semi-local) rings have many good properies.

Part II

Chapter 10

Derivation

25 Extension of a Ring by a Module

(25.A) Let C be a ring and N an ideal of C with $N^2 = 0$; put $C' = C/N$. Then the C-module N can be viewed as a C'-module. Conversely, suppose that we are given a ring C' and a C'-module N. By an extension of C' by M we mean a triple (C, ε, i) of a ring C, a surjective homomorphism of rings $\varepsilon : C \to C'$ and a map $i : N \to C$, such that:

(a) $\ker(\varepsilon)$ is an ideal whose square is zero (hence a structure of C'-module on $\ker(\varepsilon)$), and

(b) the map i is an isomorphism from N onto $\ker(\varepsilon)$ as C'-modules.

Therefore, identifying N with $i(N)$ we get $C' \cong C/N$, $N^2 = 0$. An extension is often represented by the exact sequence

$$0 \longrightarrow N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \longrightarrow 0.$$

Two extensions $(C_1, \varepsilon_1, i_1)$ and $(C_2, \varepsilon_2, i_2)$ are said to be isomorphic if there exists a ring homomorphism $\varphi : C_1 \to C_2$ such that $\varepsilon_2 \circ \varphi = \varepsilon_1$ and $\varphi \circ i_1 = i_2$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{i_1} & C_1 & \xrightarrow{\varepsilon_1} & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{i_2} & C_2 & \xrightarrow{\varepsilon_2} & C' & \longrightarrow & 0
\end{array}$$

Such φ is necessarily unique.

(25.B) Given C' and N we can always construct an extension as follows: take an additive group $C' \oplus N$, and define a multiplication in this set by the formula

$$(a, x)(b, y) = (ab, ay + bx) \quad a, b \in C', \; x, y \in N.$$

This is bilinear and associative, and has $(1, 0)$ as the unit element. Hence we get a ring structure on $C' \oplus N$. We denote this ring by $C' \star N$. By the obvious definitions $\varepsilon(a, x) = a$ and $i(x) = (0, x)$ the ring $C' \star N$ becomes an extension of C' by N, which is called the trivial extension.

An extension (C, ε, i) of C' by N is isomorphic to $C' \star N$ iff there exists a section, i.e. a ring homomorphism $s : C' \to C$ satisfying $\varepsilon \circ s = \mathrm{id}_{C'}$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & C' \star N \arrow[r] \arrow[d, "\varphi"] & C' \arrow[r] \arrow[d, equal] \arrow[l, bend right, "s'"'] \arrow[dl, dashed, "s"] & 0 \\
0 \arrow[r] & N \arrow[r, "\varepsilon"] & C \arrow[r, "i"] & C' \arrow[r] & 0
\end{tikzcd}$$

In this case the extension (C, ε, i) is also said to be trivial, or to be split.

(25.C) Let us briefly mention the Hochschild extensions. An extension (C, ε, i) is called a Hochschild extension if the exact sequence of additive groups

$$0 \longrightarrow N \xrightarrow{i} C \xrightarrow{\varepsilon} 0$$

splits, i.e. if there exists an additive map $s : C' \to C$ such that $\varepsilon \circ s = \mathrm{id}_{C'}$. Then C is isomorphic to $C' \oplus N$ as additive group, while the multiplication is given by

$$(a, x)(b, y) = (ab, ay + by + f(a, b)) \quad a, b \in C', \; x, y \in N$$

where the map $f : C' \times C' \to N$ is symmetric and bilinear and satisfies the cocycle condition (corresponding to the associativity in C)

$$af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c = 0.$$

Conversely, any such function $f(a, b)$ gives rise to a Hochschild extension. Moreover, the extension is trivial iff there exists a function $g : C' \to N$ satisfying

$$f(a, b) = ag(b) - g(ab) + g(a)b.$$

(25.D) Let A be a ring, and let

$$0 \longrightarrow N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \longrightarrow 0$$

be an extension of a ring C' by a C'-module N such that C and C' are A-algebras and ε is a homomorphism of A-algebras. Then C is called an extension of the A-algebra C' of N. The extension is said to be A-trivial, or to split over A, if there exists a homomorphism of A-algebras $s : C' \to C$ with $\varepsilon \circ s = \mathrm{id}_{C'}$.

(25.E) Let

$$(\spadesuit) \quad 0 \longrightarrow M \xrightarrow{i} C \xrightarrow{\varepsilon} C' \longrightarrow 0$$

be an extension and let $\varphi : M \to N$ be a homomorphism of C'-modules. Then there exists an extension

$$(\psi_*(\spadesuit)) \quad 0 \longrightarrow N \xrightarrow{i} D \xrightarrow{\varepsilon} C' \longrightarrow 0$$

of C' by N and a ring homomorphism $\psi : C \to D$ such that

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "\psi"] & C' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & D \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is commutative. Such an extension $\psi_*(\spadesuit)$ is unique up to isomorphisms. The ring D is obtained as follows: we view the C'-module and form the trivial extension $C \star N$. Then $M' = \{(x, -g(x)) \mid x \in M\}$ is an ideal of $C \star N$, and we put $D = (C \star N)/M'$. Thus, as an additive group D is the pushout of C and N with respect to M. The uniqueness of $\psi_*(\spadesuit)$ follows from this construction.

Similarly, if $\mu : C'' \to C$ is a ring homomorphism then there exists an extension

$$(\mu^*(\spadesuit)) \quad 0 \longrightarrow M \longrightarrow E \longrightarrow C'' \longrightarrow 0$$

of C'' by M and a ring homomorphism $\psi : E \to C$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\psi"] & C'' \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & M \arrow[r] & C \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is commutative. Moreover, such $\mu^*(\spadesuit)$ is unique up to isomorphisms.

26 Derivations and Differentials

(26.A) Let A be a ring, B an A-algebra and M an A-module. A A-derivation, or a derivation over A, is an A-linear map $\mathrm{d} : B \to M$ satisfying

$$\mathrm{d}(b_1 b_2) = \mathrm{d}(b_1) b_2 + b_1 \,\mathrm{d}(b_2),\ \forall b_1, b_2 \in B.$$

The set of all A-derivations of B into M is denoted by $\mathrm{Der}_A(B, M)$; it is an A-module in the natural way. The element in $\mathrm{Der}_{\mathbb{Z}}(B, M)$ is simply called a derivation. We write $\mathrm{Der}_A(B)$ for $\mathrm{Der}_A(B, B)$.

For any A-derivation d, $\mathrm{d}(A) = 0$. When $A = \mathbb{Z}$, $\mathrm{d}^{-1}(0)$ is a subring of B. If furthermore B is a field, $\mathrm{d}^{-1}(0)$ is a subfield.

Suppose that A is a ring whose characteristic is a prime number p, and let A^p denote the subring $\{a^p \mid a \in A\}$. Then any derivation $\mathrm{d} : A \to M$ vanishes on A^p, for

$$\mathrm{d}(a^p) = p a^{p-1} \,\mathrm{d}(a) = 0.$$

(26.B) Let A and C be rings and N an ideal of C with $N^2 = 0$. Let $q : C \to C/N$ be the natural map. Let $u, u' : A \to C$ be two homomorphisms (of rings) satisfying $qu = qu'$, and put $\mathrm{d} = u' - u$. Then u and u' induce the same A-module structure on N, and $\mathrm{d} : A \to N$ is an derivation. In fact, we have

$$u'(ab) = u'(a)u'(b) = (u(a) + \mathrm{d}(a))(u(b) + \mathrm{d}(b)) = u(ab) + \mathrm{d}(a)b + a\,\mathrm{d}(b).$$

Conversely, if $u : A \to C$ is a homomorphism and $\mathrm{d} : A \to N$ is a derivation (with respect to the A-module structure on N induced by u), then $u' = u + \mathrm{d}$ is a homomorphism.

(26.C) Let A be a ring, B an A-algebra and $C = B \otimes_A B$. Consider the homomorphisms of A-algebras

$$\varepsilon : C \to B \quad \text{and} \quad \lambda_1, \lambda_2 : B \to C$$

defined by $\varepsilon(b_1 \otimes b_2) = b_1 b_2$, $\lambda_1(b) = b \otimes 1$ and $\lambda_2(b) = 1 \otimes b$. Once and for all, we make $C = B \otimes_A B$ an B-algebra via λ_1. We denote the kernel of ε by $I_{B/A}$ or simply by I, and we put $I/I^2 = \Omega_{B/A}$. The C-modules I, I^2 and $\Omega_{B/A}$ are also viewed as B-modules via $\lambda_1 : B \to C$. Then the B-module $\Omega_{B/A}$ is called the module of differentials (or of Kähler differentials) of B over A.

We have $\varepsilon \circ \lambda_1 = \varepsilon \circ \lambda_2 = \mathrm{id}_B$. Therefore, if we denote the natural homomorphism $C \to C/I^2$ by q and if we put $\mathrm{d}' = \lambda_2 - \lambda_1$ and $\mathrm{d} = q \circ \mathrm{d}'$, then we get a derivation $\mathrm{d} : B \to \Omega_{B/A}$ (as A-module). Identifying $\varepsilon(\lambda_1(B))$ with B, we get

$$C/I^2 = B \oplus \Omega_{B/A}.$$

In other words, C/I^2 is a trivial extension of B by $\Omega_{B/A}$.

Chapter 11

Formal Smoothness

Chapter 12

Nagata Rings

Chapter 13

Excellent Rings

(32.A)