Étale cohomology

Spec $\mathbb{Z}$

March 29, 2021

Contents

1 Commutative algebra

1.1 Étale

Definition 1.1. A local homomorphism $f : A \to B$ of local rings is unramified if $\mathfrak{m}_B = f(\mathfrak{m}_A)B$ and $\kappa(B)$ is finite and separable over $\kappa(A)$.

Proposition 1.2. Let $f : A \to B$ be a local homomorphism of local rings. If

(a) f is injective,

(b) $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism,

(c) f is unramified, and

(d) B is a finite A-algebra,

then f is an isomorphism.

Definition 1.3. A homomorphism $f : A \to B$ of rings is étale if B is a finitely generated flat A-algebra and for all maximal ideals $\mathfrak{n}$ of B, $A_{(\operatorname{Spec} f)(\mathfrak{n})} \to B_{\mathfrak{n}}$ is unramified.

1.2 Henselian

Definition 1.4. A local ring A is said to be Henselian if it satisfies the following condition:

let $f \in A[T]$ be monic, and let $\bar{f}$ denote its image in $k[T]$, where $k = \kappa(A)$. If $\bar{f} = g_0 h_0$ with g_0 and h_0 monic and relatively prime in $k[T]$, then $f = gh$ with g and h monic, $\bar{g} = g_0$, and $\bar{h} = h_0$.

Remark. The g and h above are coprime and unique.

Definition 1.5. A local ring A is said to be strictly Henselian if it is Henselian and $\kappa(A)$ is saparably algebraically closed.

Theorem 1.6. Every complete local ring is Henselian.

Proposition 1.7. Let A be a local ring, and let $k = \kappa(A)$. The following statements are equivalent:

(i) A is Henselian;

(ii) let $f \in A[T]$; if $\bar{f} = g_0 h_0$ with g_0 monic and $\gcd(g_0, h_0) = 1$, then $f = gh$ with g monic and $\bar{g} = g_0$ and $\bar{h} = h_0$.

(iii) let $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$; every common zero v in k^n of the $\bar{f}_i$ for which $|\partial f / \partial T|(v)$ is nonzero lifts to a common zero of the f_i in A^n.

(iv) let B be an étale A-algebra; a decomposition $B/\mathfrak{m}_A B = k \times \bar{B}'$ lifts to a decomposition $B = A \times B'$.

Definition 1.8. Let A be a local ring. A local homomorphism $A \to A^h$ of local rings with A^h Henselian is called the Henselization of A if it satisfies the following universal property:

every other local homomorphism $A \to B$ with B Henselian factors uniquely into $A \to A^h \to B$.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A^h \arrow[d, dashed] \\ & B \end{tikzcd}$$

Proposition 1.9. For a local ring A, $A^h = \varinjlim B$, where the limit is over all pairs $(B, \mathfrak{q})$ consisting of an étale A-algebra B and $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap A = \mathfrak{m}_A$ and the induced map $\kappa(A) \to B/\mathfrak{q}$ is an isomorphism.

Definition 1.10. Let A be a local ring. A local homomorphism $A \to A^{sh}$ of local rings with A^{sh} strictly Henselian is called the strict Henselization of A if it satisfies the following universal property:

every other local homomorphism $A \to B$ with B strictly Henselian factors uniquely into $A \to A^{sh} \to B$.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A^{sh} \arrow[d, dashed] \\ & B \end{tikzcd}$$

2 Basic theory

Definition 2.1. An étale morphism $\varphi : Y \to X$ is said to be standard if it is isomorphic to

$$\operatorname{Spec}(A[T]/f)_b \to \operatorname{Spec} A,$$

where $f \in A[T]$ is monic and f' is invertible in $(A[T]/f)_b$.

Proposition 2.2. For any étale morphism $\varphi : Y \to X$ and $y \in Y$, there exist open affine neighborhoods V of y and U of $\varphi(y)$ such that $\varphi(V) \subseteq U$ and $\varphi|_V : V \to U$ is standard.

Proposition 2.3. (a) Every open immersion is étale.

(b) The composition of two étale morphisms is étale.

(c) Every base change of an étale morphism is étale.

(d) If $\varphi \circ \psi$ and φ are étale, then so is ψ.

Proposition 2.4. Let $\varphi : Y \to X$ be an étale morphism.

(a) For all $y \in Y$, $\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{X,\varphi(y)}$.

(b) φ is quasi-finite and open.

(c) If X is (reduced/normal/regular), then so is Y.

Proposition 2.5. Let $\varphi : Y \to X$ be a morphism of finite type. The set of points where φ is étale is open in Y.

Proposition 2.6. Let $\varphi : Y \to X$ be an étale morphism of varieties. If X is connected, then every section to φ is an isomorphism of X onto a connected component of Y.

Corollary 2.7. Let $p : X \to S$ and $q : Y \to S$ be morphisms of varieties over an algebraically closed field. Assume that p is étale and that Y is connected. Let φ, φ' be morphisms $Y \to X$ such that $p \circ \varphi = p \circ \varphi' = q$. If φ and φ' agree at a single point of Y, then they are equal on Y.

3 Local ring

Definition 3.1. A geometric point of a scheme X is a morphism $\bar{x} : \operatorname{Spec} K \to X$ with K a separably closed field, sometimes we write $\bar{x} = \operatorname{Spec} K$. An étale neighborhood of such a point $\bar{x}$ is an étale map $U \to X$ together with a geometric point $\bar{u} : \operatorname{Spec} K \to U$ lying over $\bar{x}$. The local ring at $\bar{x}$ for the étale topology is

$$\mathcal{O}_{X,\bar{x}} := \varinjlim_{(U,\bar{u})} \Gamma(U, \mathcal{O}_U)$$

where the limit is over the connected affine étale neighborhoods $(U, \bar{u})$ of $\bar{x}$.

When X is a variety and $x = \bar{x}$ is a closed point of X, then we have

Theorem 3.2. For any point x is X, a variety over an algebraically closed field k, $\mathcal{O}_{X,\bar{x}}$ is Henselian.

Proposition 3.3. If $\varphi : Y \to X$ is étale at y, then the map $\mathcal{O}_{X,\overline{\varphi(y)}} \to \mathcal{O}_{Y,\bar{y}}$ is an isomorphism.

Proposition 3.4. Let X be a variety over an algebraic closed field k. For every $x \in X$, $\mathcal{O}_{X,\bar{x}}$ is the Henselization of $\mathcal{O}_{X,x}$.

4 Sheaves

Definition 4.1. An étale covering of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}$ such that each φ_i is étale and $U = \bigcup_{i \in I} \varphi_i(U_i)$.

Definition 4.2. Let X be a scheme, a presheaf $\mathscr{F}$ on $X_{\text{ét}}$ is a contravariant functor

$$\mathsf{\acute{E}t}\,/X \to \mathsf{Sets}\,.$$

It is a sheaf if

$$(\spadesuit) \qquad 0 \to \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathscr{F}(U_i \times_U U_j)$$

is exact for every $U \to X$ étale and every étale covering $\{U_i \to U\}$.

Note that a sheaf $\mathscr{F}$ on $X_{\text{ét}}$ defines by restriction a sheaf on U for every $U \to X$ étale. In particular, if $U = \coprod U_i$, then $\mathscr{F}(U) \xrightarrow{\sim} \prod \mathscr{F}(U_i)$. On applying the sheaf condition in the case that I is the empty set, one finds that $\mathscr{F}(\varnothing)$ is a set with one element.

Proposition 4.3. A presheaf $\mathscr{F}$ on $X_{\text{ét}}$ is a sheaf if $\mathscr{F}$ satisfies the condition $(\spadesuit)$ for Zariski open coverings and for étale coverings $V \to U$ with V and U both affine.

Proof. If $\mathscr{F}$ satisfies the sheaf condition for Zariski open coverings, then

$$\mathscr{F}\left(\coprod U_i\right) = \prod \mathscr{F}(U_i).$$

From this it follows that $(\spadesuit)$ for a covering $\{U_i \to U\}$ in which the indexing set if finite is isomorphic to $(\spadesuit)$ arising from a single morphism $\coprod U_i \to U$ because

$$\left(\coprod U_i\right) \times_U \left(\coprod U_j\right) = \coprod U_i \times_U U_j.$$

Since a finite disjoint union of affine schemes is again affine, the second condition in the statement of the proposition implies that $(\spadesuit)$ is exact for coverings $\{U_i \to U\}$ in which the indexing set is finite and U and the U_i are affine.

Let $f : U' \to U$ be the morphism $\coprod U'_i \to U$, where $\{U'_i \to U\}$ is a covering, and write U as a union of open affines $U = \bigcup U_j$. Then $f^{-1}(U_j)$ is a union of open affines $f^{-1}(U_j) = \bigcup U'_{jk}$. Each $f(U'_{jk})$ is open in U_j, and U_j is quasi-compact, and therefore there is a finite set K_j of k's such that $\{U'_{jk} \to U_j\}$ is a covering.

Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}(U' \times_U U') \arrow[d] \\
0 \arrow[r] & \prod_j \mathscr{F}(U_j) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_j \prod_k \mathscr{F}(U'_{jk}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_j \prod_{k,\ell} \mathscr{F}(U'_{jk} \times_U U'_{j\ell}) \\
0 \arrow[r] & \prod_{i,j} \mathscr{F}(U_i \cap U_j) \arrow[r] & \prod_{i,j} \prod_{k,\ell} \mathscr{F}(U'_{ik} \cap U'_{j\ell}) &
\end{tikzcd}$$

From the assumption and the first part of the proof, the two columns and the middle row are exact. It follows that $\mathscr{F}(U) \to \mathscr{F}(U')$ is injective and hence the third row and the third column are exact. A simple diagram chasing shows that the first row is exact. ■

Proposition 4.4. For every faithfully flat homomorphism $A \to B$, the sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{b \mapsto 1 \otimes b - b \otimes 1} B \otimes_A B$$

is exact.

Proof. Since $A \to B$ is faithfully flat, it suffices to show that

$$0 \longrightarrow B \xrightarrow{b \mapsto 1 \otimes b} B \otimes_A B \xrightarrow{b \otimes b' \mapsto 1 \otimes b \otimes b' - b \otimes 1 \otimes b'} B \otimes_A B \otimes_A B$$

is exact. Consider the homotopies $h_1 : B \otimes_A B \to B$ defined by $b \otimes b' \mapsto bb'$ and $h_2 : B \otimes_A B \otimes_A B \to B \otimes_A B$ defined by $b \otimes b' \otimes b'' \mapsto -b \otimes b'b''$. Then the cohomology is 0 and hence exact. ■

Example 4.5 (Structure Sheaf). For any $U \to X$ étale, define $\mathcal{O}_{X_{\text{ét}}}(U) = \Gamma(U, \mathcal{O}_U)$. To show that $\mathcal{O}_{X_{\text{ét}}}$ is a sheaf, by (4.3), we only need to prove the condition (♠) for affine maps $V = \operatorname{Spec} B \to U = \operatorname{Spec} A$ over X. Since $A \to B$ is faithfully flat, the assertion follows from (4.4).

Example 4.6. An X-scheme Z defines a covariant functor

$$Z = \mathscr{F}_Z : \textsf{Ét}/X \to \textsf{Sets}, \quad Z(U) = \operatorname{Hom}_X(U, Z).$$

We need to check that

$$0 \to \mathrm{Hom}_X(U, Z) \to \prod \mathrm{Hom}_X(U_i, Z) \rightrightarrows \prod \mathrm{Hom}(U_i \times_U U_j, Z)$$

is exact. Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_X(U, Z) \arrow[r] \arrow[d, hook] & \prod \mathrm{Hom}_X(U_i, Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \prod \mathrm{Hom}_X(U_i \times_U U_j, Z) \arrow[d, hook] \\
0 \arrow[r] & \mathrm{Hom}(U, Z) \arrow[r] & \prod \mathrm{Hom}(U_i, Z) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathrm{Hom}(U_i \times_U U_j, Z)
\end{tikzcd}$$

If the bottom row is exact, then the first row is also exact, so we don't need to consider the base scheme X.

It is easy to check that $\mathscr{F}_Z$ satisfies (♠) for open Zariski coverings. Thus it suffices to show that

$$0 \to \mathrm{Hom}(\operatorname{Spec} A, Z) \to \mathrm{Hom}(\operatorname{Spec} B, Z) \rightrightarrows \mathrm{Hom}(\operatorname{Spec} B \otimes_A B, Z)$$

is exact for any faithfully flat map $A \to B$. If Z is affine, say $Z = \operatorname{Spec} C$, then the sequence is isomorphic to

$$0 \to \mathrm{Hom}_A(C, A) \to \mathrm{Hom}_A(C, B) \rightrightarrows \mathrm{Hom}_A(C, B \otimes_A B),$$

which is exact by (4.4). For general Z, write $Z = \bigcup Z_i$, where Z_i is open affine. Suppose that $f, g : \operatorname{Spec} A \to Z$ are morphisms such that the compositions with $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ are the same. Since φ is surjective, $f = g$ as maps between topological spaces. Let $U_i = f^{-1}(Z_i) = g^{-1}(Z_i)$, and $V_i = \varphi^{-1}(U_i)$, then writing U_i and V_i into union of affine open subschemes, we reduce to the affine case and hence $f = g$ as maps between schemes.

Suppose that $h : \operatorname{Spec} B \to Z$ such that the compositions with $\varphi_1, \varphi_2 : \operatorname{Spec} B \otimes_A B \to B$ are the same. If $a_1, a_2 \in \operatorname{Spec} B$ maps to the same element in $\operatorname{Spec} A$, then they maps to the same element in Z, so f factors through φ as maps between topological space. Write $g : U \to X$ such that $f = \varphi \circ g$. Again, let $U_i = g^{-1}(X_i)$ and $V_i = \varphi^{-1}(U_i)$, and write U_i and V_i into union of affine open subschemes, we reduce to the affine case and hence f factors through φ as maps between schemes.

Example 4.7. (a) Let μ_n be the scheme defined by the equation

$$T^n - 1 = 0$$

in affine space $\mathbb{A}^1_k$, then $\mu_n(U)$ is the group of n^{th} roots of unity in $\Gamma(U, \mathcal{O}_U)$.

(b) Let $\mathbb{G}_a$ be the affine line regarded as a group under addition. Then $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$ regarded as an abelian group.

(c) Let $\mathbb{G}_m$ be the affine line with origin omitted, regarded as a group under multiplication. Then $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$.

(d) Let GL_n be the scheme defined by the single equation

$$T \cdot \det(T_{ij}) = 1$$

in the $n^2 + 1$ variables T, T_{11}, ..., T_{nn}. Then $\mathrm{GL}_n(U) = \mathrm{GL}_n(\Gamma(U, \mathcal{O}_U))$.

Example 4.8 (Constant Sheaf). Let X be a quasi-compact scheme. For every set Λ, define

$$\mathscr{F}_\Lambda(U) = \Lambda^{\pi_0(U)}$$

—product of copies of Λ indexed by the set $\pi_0(U)$ of connected components of U. With the obvious restriction maps, this is a sheaf, called the constant sheaf on $X_{\text{ét}}$ defined by Λ. If Λ is finite, then it is the sheaf defined by scheme $X \times \Lambda$. When Λ is a group, then $\mathscr{F}_\Lambda$ is a sheaf of groups.

Example 4.9 (Coherent Sheaf). Let $\mathscr{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X. For every étale map $\varphi : U \to X$, we obtain a coherent $\mathcal{O}_U$-module $\varphi^*\mathscr{F}$ on U. Check this locally, if $U = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine, then $\mathscr{F} = \widetilde{M}$ for some M and hence $\varphi^*\mathscr{F} = \widetilde{M \otimes_A B}$ is coherent. There is a presheaf $U \mapsto \Gamma(U, \varphi^*\mathscr{F})$ on $X_{\text{ét}}$, which we denote $\mathscr{F}^{\text{ét}}$. For example, $(\mathcal{O}_X)^{\text{ét}} = \mathcal{O}_{X_{\text{ét}}}$. To verify that $\mathscr{F}^{\text{ét}}$ is a sheaf it suffices to show that the sequence

$$0 \to M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M$$

is exact whenever $A \to B$ is faithfully flat, which is similar to the proof of (4.4).

5 Stalks

Definition 5.1. Let $\mathscr{F}$ be sheaf on $X_{\text{ét}}$. The stalk of $\mathscr{F}$ at a geometric point $\bar{x} \to X$ is

$$\mathscr{F}_{\bar{x}} = \varinjlim_{(U,\bar{u})} \mathscr{F}(U),$$

where the limit is over the connected affine étale neighborhoods of $\bar{x}$.

Example 5.2. (a) The stalk of $\mathcal{O}_{X_{\text{ét}}}$ at $\bar{x}$ is $\mathcal{O}_{X,\bar{x}}$, the strictly local ring at $\bar{x}$.

(b) The stalk of $\mathscr{F}_Z$ at $\bar{x}$, Z a scheme of finite type over X is $Z(\mathcal{O}_{X,\bar{x}})$.

(c) The stalk of $\mathscr{F}^{\text{ét}}$, $\mathscr{F}$ a coherent $\mathcal{O}_X$-module, at $\bar{x}$ is $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\bar{x}}$.

Definition 5.3. A sheaf $\mathscr{F}$ is said to be a skyscraper sheaf if $\mathscr{F}_{\bar{x}} = 0$ except for a finite number of x.

Example 5.4. Let X be a scheme, Λ an abelian group, and $\bar{x}$ a geometric point in X. For any étale map $\varphi : U \to X$, we define

$$\Lambda^{\bar{x}}(U) = \Lambda^{\oplus \mathrm{Hom}_X(\bar{x}, U)}.$$

Then $\Lambda^{\bar{x}}$ is a sheaf on $X_{\text{ét}}$. Let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. To give a map $\mathscr{F}_{\bar{x}} \to \Lambda$ is equivalent to give a compatible family of maps $\mathscr{F}(U) \to \Lambda$ for every étale neighborhood $(U, \bar{u})$ of $\bar{x}$. This is equivalent to give maps $\mathscr{F}(U) \to \Lambda(U)$. Thus

$$\mathrm{Hom}(\mathscr{F}, \Lambda^{\bar{x}}) \cong \mathrm{Hom}(\mathscr{F}_{\bar{x}}, \Lambda).$$

When x in X is closed, then $(\Lambda^{\bar{x}})_{\bar{y}} = 0$ when $y \neq x$. So $\Lambda^{\bar{x}}$ is a skyscraper sheaf.

6 Category of sheaves

Let X be a scheme, the presheaves of abelian groups on $X_{\text{ét}}$, by definition, are contravariant functors $\mathsf{Ét}/X \to \mathsf{Ab}$. We can replace $\mathsf{Ét}/X$ by a small category $\mathsf{PreSh}(X_{\text{ét}})$. For example, by the set of étale maps $U \to X$ of the following form: U is obtained by patching the schemes attached to quotients of rings of the form $A[T_1, T_2, \ldots]$ where $A = \Gamma(V, \mathcal{O}_X)$ for some open affine $V \subseteq X$ amd $\{T_1, T_2, \ldots\}$ is a fixed set of symbols. It is an abelian category, in which $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact iff $\mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$ is exact for all $U \to X$ étale.

We define the category $\mathsf{Sh}(X_{\text{ét}})$ to be the full subcategory of $\mathsf{PreSh}(X_{\text{ét}})$ whose objects are the sheaves of abelian groups on $X_{\text{ét}}$. Clearly, $\mathsf{Sh}(X_{\text{ét}})$ is an additive category.

Definition 6.1. A morphism $\alpha : \mathscr{F} \to \mathscr{G}$ of sheaves is said to be locally surjective if for every U and $s \in \mathscr{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\mathscr{F}(U_i) \to \mathscr{G}(U_i)$ for each i.

Lemma 6.2. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves on $X_{\text{ét}}$. The following are equivalent:

(i) the sequence of sheaves $\mathscr{F} \to \mathscr{G} \to 0$ is exact;

(ii) the map α is locally surjective;

(iii) for each geometric point $\bar{x} \to X$, the map $\alpha_{\bar{x}} : \mathscr{F}_{\bar{x}} \to \mathscr{G}_{ox}$ is surjective.

Proof. (ii) $\Longrightarrow$ (i): Let $\beta : \mathscr{G} \to \mathscr{A}$ be a map of sheaves such that $\beta \circ \alpha = 0$; we have to prove that this implies $\beta = 0$. Let $t \in \mathscr{G}(U)$ for some étale $U \to X$. By assumption, there exists a covering $\{U_i \to U\}$ and $s_i \in \mathscr{F}(U_i)$ such that $\alpha(s_i) = t|_{U_i}$. Now

$$\beta(t)|_{U_i} = \beta(t|_{U_i}) = (\beta \circ \alpha)(s_i) = 0$$

for all i. Since $\mathscr{A}$ is a sheaf, this implies $\beta(t) = 0$.

(i) $\Longrightarrow$ (iii): Suppose $\alpha_{\bar{x}}$ is not surjective for some $\bar{x} \to X$, and let $A \neq 0$ be the cokernel of $\mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$. Let $A^{\bar{x}}$ be the sheaf defined in (5.4), then the map $\mathscr{G}_{\bar{x}} \to A$ defines a nonzero morphism $\mathscr{G} \to A^{\bar{x}}$, whose composite with $\mathscr{F} \to \mathscr{G}$ is zero. Therefore $\mathscr{F} \to \mathscr{G} \to 0$ is not exact, a contradiction.

(iii) $\Longrightarrow$ (ii): Note that for any étale map $U \to X$, a geometric point $\bar{u} \to U$ gives a geometric point $\bar{x} \to X$, and $\mathscr{A}_{\bar{u}} = \mathscr{A}_{\bar{x}}$ for any sheaf $\mathscr{A}$ on $X_{\text{ét}}$. Thus, the hypothesis implies that $\mathscr{F}_{\bar{u}} \to \mathscr{G}_{\bar{u}}$ is surjective for every geometric point $\bar{u} \to U$ of U. Let $t \in \mathscr{G}(U)$ and let $\bar{u} \to U$ be a geometric point of U. Because $\mathscr{F}_{\bar{u}} \to \mathscr{G}_{\bar{u}}$ is surjective, there exists an étale map $V \to U$ whose image contains u and which is such that $s|_V$ is in the image of $\mathscr{F}(V) \to \mathscr{G}(V)$. On applying this argument for sufficient many $u \in U$, we obtain a covering sought. ∎

Definition 6.3. Let $\mathscr{F}$ be a presheaf, a sheaf $\mathscr{F}^+$ is said to be the sheafification of $\mathscr{F}$ is all morphisms from $\mathscr{F}$ to a sheaf $\mathscr{G}$ factors through $\mathscr{F}^+$, i.e.,

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}(\mathscr{F}^+, \mathscr{G}).$$

Theorem 6.4. For every presheaf $\mathscr{F}$ on $X_{\text{ét}}$, there exists an sheafification $i : \mathscr{F} \to \mathscr{F}^+$. The map i induces isomorphisms $\mathscr{F}_{\bar{x}} \to \mathscr{F}^+_{\bar{x}}$ on stalks. The functor $(\cdot)^+ : \mathsf{PreSh}(X_{\text{ét}}) \to \mathsf{Sh}(X_{\text{ét}})$ is exact.

If we define $i : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{PreSh}(X_{\text{ét}})$ to be the inclusion functor, then for any presheaf $\mathscr{F}$ and any sheaf $\mathscr{G}$,

$$\operatorname{Hom}(\mathscr{F}, i(\mathscr{G})) = \operatorname{Hom}(\mathscr{F}^+, \mathscr{G}).$$

Proposition 6.5. Let

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

be a sequence of sheaves on $X_{\text{ét}}$. The following are equivalent:

(i) the sequence is exact in $\mathsf{Sh}(X_{\text{ét}})$;

(ii) the sequence

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$$

is exact for all étale map $U \to X$;

(iii) the sequence

$$0 \to \mathscr{F}'_{\bar{x}} \to \mathscr{F}_{\bar{x}} \to \mathscr{F}''_{\bar{x}}$$

is exact for every geometric point $\bar{x} \to X$.

Proof. (a) $\iff$ (b): Since i has a left adjoint, i is left exact.

(b) $\implies$ (c): Direct limits of exact sequence of abelian groups are exact.

(c) $\implies$ (b): $s \in \mathscr{G}(U)$ is zero iff $s_{\bar{u}}$ for all geometric points $\bar{u} \to U$. ■

Combining (6.2) and (6.5), we get:

Proposition 6.6. Let

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

be a sequence of sheaves on $X_{\text{ét}}$. The following are equivalent:

(i) the sequence is exact in $\mathsf{Sh}(X_{\text{ét}})$;

(ii) the map $\mathscr{F} \to \mathscr{F}''$ is locally surjective and the sequence

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$$

is exact for all étale map $U \to X$;

(iii) the sequence

$$0 \to \mathscr{F}'_{\bar{x}} \to \mathscr{F}_{\bar{x}} \to \mathscr{F}''_{\bar{x}} \to 0$$

is exact for every geometric point $\bar{x} \to X$.

Example 6.7 (Kummer).] Let n be an integer that is not divisible by the characteristic of any residue field of X. For example, if X is a variety over a field k of characteristic $p \neq 0$, then we require that $p \nmid n$. Consider the sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 0.$$

In order the prove that this is exact, we have to check that

$$0 \to \mu_n(A) \to A^\times \xrightarrow{t \mapsto t^n} A^\times \to 0$$

is exact for every strictly local ring $A = \mathcal{O}_{X,\bar{x}}$ of X. This follows from the fact that $d(T^n - a)/dT = nT^{n-1} \neq 0$ and so $T^n - a$ splits in $A[T]$.

Example 6.8 (Artin-Schreier). Let X be a variety over a field k of characteristic $p \neq 0$. Consider the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0.$$

In order to prove that this sequence is exact, we have to check that

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A \xrightarrow{t \mapsto t^p - t} A \to 0$$

is exact for every strictly local ring $A = \mathcal{O}_{X,\bar{x}}$ of X. This follows from the fact that $d(T^p - T - a)/dT = -1 \neq 0$ and so $T^p - T - a$ splits in $A[T]$.

7 Direct image of sheaves

Let $\pi : Y \to X$ be a morphism of sheaves, and let $\mathscr{F}$ be a presheaf on $Y_{\text{ét}}$. For $U \to X$ étale, define

$$\pi_* \mathscr{F}(U) = \mathscr{F}(U \times_X Y).$$

With the obvios restriction maps, $\pi_* \mathscr{F}$ becomes a presheaf on $X_{\text{ét}}$.

Lemma 7.1. If $\mathscr{F}$ is a sheaf, then so is $\pi_* \mathscr{F}$.

Proof. For a scheme V over X, let V_Y denote the scheme $V \times_X Y$ over Y. Then $V \mapsto V_Y$ is a functor. Let $\{U_i \to U\}$ be an étale covering in $\mathsf{Ét}/X$, then $\{U_{iY} \to U_Y\}$ is an étale covering in $\mathsf{Ét}/Y$, and so

$$0 \to \mathscr{F}(U_Y) \to \prod \mathscr{F}(U_{iY}) \rightrightarrows \prod \mathscr{F}(U_{iY} \times_Y U_{jY})$$

is exact. But this is equal to

$$0 \to \pi_*\mathscr{F}(U) \to \prod \pi_*\mathscr{F}(U_i) \rightrightarrows \prod \pi_*\mathscr{F}(U_i \times_X U_j),$$

which is therefore also exact. ∎

Obviously, the functor

$$\pi_* : \mathsf{PreSh}(Y_{\text{ét}}) \to \mathsf{PreSh}(X_{\text{ét}})$$

is exact. Therefore, its restriction

$$\pi_* : \mathsf{Sh}(Y_{\text{ét}}) \to \mathsf{Sh}(X_{\text{ét}})$$

is left exact.

Example 7.2. Let $i : \bar{x} \to X$ be a geometric point of X. The functor $\mathscr{F} \mapsto \mathscr{F}(\bar{x})$ identifies $\mathsf{Sh}(\bar{x}_{\text{ét}})$ with Ab. Let Λ be an abelian group regarded as a sheaf on $\bar{x}$. Then $i_*\Lambda = \Lambda^{\bar{x}}$ defined in (5.4).

A geometric point $\bar{y} \to Y$ defines a geometric point $\bar{y} \to Y \to X$ of X, which we denote $\bar{x}$ (or $\pi(\bar{y})$). Since

$$(\pi_*\mathscr{F})_{\bar{x}} = \varinjlim_{(U,\bar{u})} \mathscr{F}(U \times_X Y) \ \varinjlim_{(U_Y,\bar{u})} \mathscr{F}(U_Y),$$

there is a canonical map $(\pi_*\mathscr{F})_{\bar{x}} \to \mathscr{F}_{\bar{y}}$. In general, this map will be neither injective nor surjective.

Proposition 7.3. Let $\pi : Y \to X$ be a morphism and $\mathscr{F}$ a sheaf on $Y_{\text{ét}}$.

(a) If π is an open immersion, then $(\pi_*\mathscr{F})_{\bar{x}} = \mathscr{F}_{\bar{x}}$ for $x \in Y$.

(b) If π is a closed immersion, then

$$(\pi_*\mathscr{F})_{\bar{x}} = \begin{cases} \mathscr{F}_{\bar{x}}, & x \in Y, \\ 0, & x \notin Y. \end{cases}$$

(c) If π is finite, then

$$(\pi_*\mathscr{F})_{\bar{x}} = \bigoplus_{y\mapsto x} \mathscr{F}_{\bar{y}}^{d(y)},$$

where $d(y)$ is the separable degree of $\kappa(y)$ over $\kappa(x)$.

Proof. (a): Trivial. (b): If $x \notin Y$, then for any sufficiently small étale neighborhood $U \to X$, $U \times_X Y = \varnothing$; thus $\mathscr{F}(U_Z) = 0$. When $x \in Y$, we have to show that every étale map $\bar{\varphi} : \bar{U} \to Y$ extends to an étale map $\varphi : U \to X$. In terms of rings, this amounts to showing that an étale homomorphism $\bar{A} \to \bar{B}$, $\bar{A} = A/\mathfrak{a}$, lifts to an étale homomorphism $A \to B$. We may assume that $\bar{B} = (\bar{A}[T]/\bar{f})_{\bar{b}}$ where $\bar{f} \in \bar{A}[T]$ is monic such that $\bar{f}'$ is invertible in $(\bar{A}[T]/\bar{f})_{\bar{b}}$. Choose $f \in A[T]$ lifting $\bar{f}$, and set $B = (A[T]/f)_b$ for an appropriate b.

(c): Localize at x, we may assume that $X = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$. Since $\mathcal{O}_{X,\bar{x}}$ is Henselian, the connected components of Y are 1-1 correspondence with the connected components of $\pi^{-1}(\bar{x})$. Thus,

$$Y = \coprod_{y\mapsto x} \operatorname{Spec} \mathcal{O}_{Y,\bar{y}}^{d(y)}. \qquad \blacksquare$$

Corollary 7.4. The functor π_* is exact if π is finite or a closed immersion.

8 Inverse image of sheaves

Let $\pi : Y \to X$ be a morphism of schemes. We shall define a left adjoint for the functor π_*. Let $\mathscr{F}$ be a presheaf on $X_{\text{ét}}$. For $V \to Y$ étale, define

$$\mathscr{F}'(V) = \varinjlim \mathscr{F}(U),$$

where the limit is over the commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with $U \to X$ étale. One sees easily that, for any presheaf $\mathscr{G}$ on Y, there are natural 1-1 correspondence between $\operatorname{Hom}(\mathscr{F}', \mathscr{G})$, $\operatorname{Hom}(\mathscr{F}, \pi_*\mathscr{G})$. Unfortunately, $\mathscr{F}'$ need not to be a sheaf even when $\mathscr{F}$ is. Thus, for $\mathscr{F}$ a sheaf on $X_{\text{ét}}$, we define $\pi^*\mathscr{F} = (\mathscr{F}')^+$. Then, for any sheaf $\mathscr{G}$ on $Y_{\text{ét}}$,

$$\operatorname{Hom}(\pi^*\mathscr{F}, \mathscr{G}) = \operatorname{Hom}(\mathscr{F}', \mathscr{G}) = \operatorname{Hom}(\mathscr{F}, \pi_*\mathscr{G}),$$

and so π^* is a left adjoint to $\pi_* : \mathsf{Sh}(Y_{\text{ét}}) \to \mathsf{Sh}(X_{\text{ét}})$.

Example 8.1. Let $\pi : U \to X$ be an étale morphism. For any sheaves $\mathscr{F}$ on $X_{\text{ét}}$ and $\mathscr{G}$ on $U_{\text{ét}}$,

$$\mathrm{Hom}(\mathscr{F}|_{U_{\text{ét}}}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \pi_*\mathscr{G}).$$

So $\pi^*\mathscr{F} = \mathscr{F}|_{U_{\text{ét}}}$.

Example 8.2. Let $i : \bar{x} \to X$ be a geometric point of X. For any sheaf on $X_{\text{ét}}$,

$$(i^*\mathscr{F})(\bar{x}) = \mathscr{F}_{\bar{x}}$$

from the definition of $\mathscr{I}^*$ and $\mathscr{F}_{\bar{x}}$. Therefore, for any morphism $\pi : Y \to X$ and geometric point $i : \bar{y} \to Y$ of Y,

$$(\pi^*\mathscr{F})_{\bar{y}} = (i^*\pi^*\mathscr{F})(\bar{y}) = \mathscr{F}_{\bar{x}}$$

where $\bar{x}$ is the geometric point $\bar{y} \to Y \to X$ of X.

Since this is true for all geometric points of Y, we see that π^* is exact and therefore π_* preserves injectives.

Let X be a scheme and let $j : U \to X$ be an open immersion. Let $\mathscr{F}$ be a presheaf on $U_{\text{ét}}$. For any $\varphi : V \to X$ étale, define

$$\mathscr{F}_!(V) = \begin{cases} \mathscr{F}(V) & , \text{ if } \varphi(V) \subseteq U, \\ 0 & , \text{ otherwise.} \end{cases}$$

Then $\mathscr{F}_!$ is a presheaf on $X_{\text{ét}}$, and for any other presheaf $\mathscr{G}$ on $X_{\text{ét}}$, a morphism $\mathscr{F} \to \mathscr{G}|_U$ extends uniquely to a morphism $\mathscr{F}_! \to \mathscr{G}$. Thus,

$$\mathrm{Hom}(\mathscr{F}_!, \mathscr{G}) \cong \mathrm{Hom}(f, \mathscr{G}|_U)$$

functorially. Unfortunately, $\mathscr{F}_!$ need not to be a sheaf even when $\mathscr{F}$ is. Thus, for $\mathscr{F}$ a sheaf on $U_{\text{ét}}$, we define $j_!\mathscr{F}$ to be $\mathscr{F}_!^+$. Then, for any sheaf $\mathscr{G}$ on $X_{\text{ét}}$,

$$\mathrm{Hom}(j_!\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}_!, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{G}|_U),$$

and so $j_!$ is a left adjoint to $j^* : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{Sh}(U_{\text{ét}})$.

Proposition 8.3. Let $j : U \to X$ be an open immersion. For any sheaf $\mathscr{F}$ on $U_{\text{ét}}$ and geometric point $\bar{x} \to X$,

$$(j_!\mathscr{F})_{\bar{x}} = \begin{cases} \mathscr{F}_{\bar{x}} & , \text{ if } x \in U, \\ 0 & , \text{ if } x \notin U. \end{cases}$$

Corollary 8.4. The functor $j_!$ is exact and j^* preserves injectives.

Let X be the complement of U in X, and denote the inclusion $Z \hookrightarrow X$ by i. Let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. There is a canonical morphism $j_!j^*\mathscr{F} \to \mathscr{F}$, corresponding by

$$\mathrm{id}_{j^*\mathscr{F}} \in \mathrm{Hom}(j^*\mathscr{F}, j^*\mathscr{F}) \cong \mathrm{Hom}(j_!j^*\mathscr{F}, \mathscr{F}),$$

and a canonical morphism $\mathscr{F} \to i_*i^*\mathscr{F}$, corresponding by

$$\mathrm{id}_{i^*\mathscr{F}} \in \mathrm{Hom}(i^*\mathscr{F}, i^*\mathscr{F}) \cong \mathrm{Hom}(\mathscr{F}, i_*i^*\mathscr{F}).$$

Proposition 8.5. For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$, the sequence

$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to i_*i^*\mathscr{F} \to 0$$

is exact.

Proof. Check on stalks. ■

Remark. It is possible to define $j_!$ for any étale map $j; U \to X$. Let $\mathscr{F}$ be sheaf on $U_{\text{ét}}$. For any $\varphi : V \to X$ étale, define

$$\mathscr{F}_!(V) = \mathscr{F}(V)^{\oplus \mathrm{Hom}_X(V,U)}.$$

Then $\mathscr{F}_!$ is a presheaf on $X_{\text{ét}}$, and we define $j_!\mathscr{F} = \mathscr{F}_!^+$. Again, $j_!$ is the left adjoint of j^* and is exact. Hence j^* preserves injectives.

9 Cohomology

We know that the functor

$$\Gamma(X, \cdot\,) : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{Ab}$$

is left exact. In order to define the right derived functor of $\Gamma(X, \cdot\,)$, we need to show that $\mathsf{Sh}(X_{\text{ét}})$ has enough injectives.

Proposition 9.1. Let X be a scheme. Every sheaf $\mathscr{F}$ on $X_{\text{ét}}$ can be embedded into an injective sheaf.

Proof. For each $x \in X$, choose a geometric point $i_x : \bar{x} \to X$ with image x and an embedding $\mathscr{F}_{\bar{x}} \hookrightarrow I(x)$ of the abelian group $\mathscr{F}_{\bar{x}}$ into an injective abelian group. Then $\mathscr{I}^x = i_{x*}(I(x))$ is injective. Since a product of injective objects is injective, $\mathscr{I} = \prod \mathscr{I}^x$ will be an injective sheaf. The composite

$$\mathscr{F} \hookrightarrow \prod (\mathscr{F}_{\bar{x}})^{\bar{x}} \hookrightarrow \prod \mathscr{I}$$

is the embedding sought. ■

Definition 9.2. We define $H^r(X_{\text{ét}}, \cdot\,)$ to be the r^{th} right derived functor of $\Gamma(X, \cdot\,)$.

Remark. Let $\varphi : U \to X$ be an étale morphism. $\varphi^* : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{Sh}(U_{\text{ét}})$ is exact and preserves injectives. Since the composite $\Gamma(U, \cdot\,) \circ \varphi^*$ is $\Gamma(U, \cdot\,)$, we see that

$$H^r(U_{\text{ét}}, \mathscr{F}|_U) = H^r(U_{\text{ét}}, \mathscr{F}).$$

We define $\mathscr{H}^r(\mathscr{F})$ to be the presheaf $U \mapsto H^r(U_{\text{ét}}, \mathscr{F})$.

Proposition 9.3. Let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$, then $\mathscr{H}^r(\mathscr{F})^+ = 0$ for $r > 0$.

Proof. Let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Then $\mathscr{H}^r(\mathscr{F})$ is the r^{th} cohomology presheaf of the complex $i(\mathscr{I}^\bullet)$, where i is the inclusion functor. Since $(\cdot)^+$ is exact, it commutes with taking cohomology, and so $\mathscr{H}^r(\mathscr{F})^+$ is the r^{th} cohomology sheaf of the complex $i(\mathscr{I}^\bullet)^+ = \mathscr{I}^\bullet$. Thus $\mathscr{H}^r(\mathscr{F})^+ = 0$. ■

Let Z be a closed subscheme of X, and let $U = X \setminus Z$. For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$, define

$$\Gamma_Z(X, \mathscr{F}) = \ker(\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F})),$$

i.e., the sections of $\mathscr{F}$ with support on Z. The functor $\Gamma_Z(X, \cdot\,)$ is obviously left exact, and we denote its r^{th} right derived functor by $H^r_Z(X, \cdot\,)$ (cohomology of $\mathscr{F}$ with support on Z).

For a fixed sheaf $\mathscr{F}_0$,

$$\mathrm{Hom}_X(\mathscr{F}_0, \cdot\,) : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{Ab}$$

is left exact, and we denote its r^{th} right derived functor by $\mathrm{Ext}^r_X(\mathscr{F}_0, \cdot\,)$.

Example 9.4. Let $\mathbb{Z}$ denote the constant sheaf on X. For any sheaf $\mathscr{F}$ on X, the map $\alpha \mapsto \alpha(1)$ is an isomorphism

$$\mathrm{Hom}_X(\mathbb{Z}, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(X).$$

Thus, $\mathrm{Hom}_X(\mathbb{Z}, \cdot\,) \cong \Gamma(X, \cdot\,)$, and so

$$\mathrm{Ext}^r_X(\mathbb{Z}, \cdot\,) \cong H^r(X_{\text{ét}}, \cdot\,).$$

Proposition 9.5. A short exact sequence

$$0 \to \mathscr{F}_0' \to f_0 \to \mathscr{F}_0'' \to 0$$

of sheaves on $X_{\text{ét}}$ gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^r_X(\mathscr{F}_0'', \mathscr{F}) \to \mathrm{Ext}^r_X(\mathscr{F}_0, \mathscr{F}) \to \mathrm{Ext}^r_X(\mathscr{F}_0', \mathscr{F}) \to \cdots$$

for any sheaf $\mathscr{F}$.

Proof. If $\mathscr{I}$ is injective, then

$$0 \to \mathrm{Hom}_X(\mathscr{F}_0'', \mathscr{I}) \to \mathrm{Hom}_X(\mathscr{F}_0, \mathscr{I}) \to \mathrm{Hom}_X(\mathscr{F}_0', \mathscr{I}) \to 0$$

is exact. For any injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$,

$$0 \to \mathrm{Hom}_X(\mathscr{F}_0'', \mathscr{I}^\bullet) \to \mathrm{Hom}_X(\mathscr{F}_0, \mathscr{I}^\bullet) \to \mathrm{Hom}_X(\mathscr{F}_0', \mathscr{I}^\bullet) \to 0$$

is an exact sequence of complexes, which give rise to a long exact sequence of cohomology groups. ■

Theorem 9.6. Let Z be a closed subscheme of X and let $U = X \setminus Z$. For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$, there is a long exact sequence

$$\cdots \to H^r_Z(X, \mathscr{F}) \to H^r(X, \mathscr{F}) \to H^r(U, \mathscr{F}) \to H^{r+1}_Z(X, \mathscr{F}) \to \cdots.$$

The sequence is functorial in the pairs (X, U) and $\mathscr{F}$.

Proof. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$. Let $\mathbb{Z}$ denote the constant sheaf on X defined by $\mathbb{Z}$, and consider the exact sequence

$$(\spadesuit) \qquad 0 \to j_! j^* \mathbb{Z} \to \mathbb{Z} \to i_* i^* \mathbb{Z} \to 0.$$

For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$,

$$\mathrm{Hom}_X(j_!j^*\mathbb{Z},\mathscr{F}) = \mathrm{Hom}_U(j^*\mathbb{Z},j^*\mathscr{F}) = \mathscr{F}(U).$$

and so $\mathrm{Ext}^r_X(j_!j^*\mathbb{Z},\mathscr{F}) = H^r(U_{\text{ét}},\mathscr{F})$. From the exact sequence

$$0 \to \mathrm{Hom}(i_*i^*\mathbb{Z},\mathscr{F}) \to \mathrm{Hom}(\mathbb{Z},\mathscr{F}) \to \mathrm{Hom}(j_!j^*\mathbb{Z},\mathscr{F})$$

we find that $\mathrm{Hom}_X(i_*i^*\mathbb{Z},\mathscr{F}) = \Gamma_Z(X,\mathscr{F})$, and so $\mathrm{Ext}^r_X(i_*i^*\mathbb{Z},\mathscr{F}) = H^r_Z(X,\mathscr{F})$. Then the sequence is just applying (♠) to (9.5). ■

Let $\pi : Y \to X$ be a morphism. Recall that, the functor $\pi_* : \mathsf{Sh}(Y_{\text{ét}}) \to \mathsf{Sh}(X_{\text{ét}})$ is left exact, and hence we can consider its right derived functors $R^r\pi_*$. We call the sheaves $R^r\pi_*\mathscr{F}$ the higher direct images of $\mathscr{F}$.

Proposition 9.7. For any $\pi : Y \to X$ and sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, $R^r\pi_*\mathscr{F}$ is a sheaf on $X_{\text{ét}}$ associated with the presheaf $U \mapsto H^r(U \times_X Y, \mathscr{F})$.

Proof. Let π_p be the functor $\mathsf{PreSh}(Y_{\text{ét}}) \to \mathsf{PreSh}(X_{\text{ét}})$ defined by $\mathscr{F} \mapsto [U \mapsto \Gamma(U_Y,\mathscr{F})]$, it is obviously exact. From the definition of π_*, $\pi_* = (\cdot)^+ \circ \pi_p \circ i$, where i is the inclusion functor. Let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Then, because $(\cdot)^+$ and π_p are exact,

$$R^r\pi_*\mathscr{F} = h^r(\pi_*\mathscr{I}) = G^r(((\pi_p \circ i)(T^\bullet))^+) = (\pi_p h^r(i\mathscr{I}^\bullet))^+.$$

Since $h^r(i(\mathscr{I}^\bullet))$ is the presheaf $\mathscr{H}^r(\mathscr{F})$ by the proof of (9.3), and so $\pi_p h^r(i(\mathscr{I}^\bullet))$ is the presheaf $U \mapsto H^r(U_Y,\mathscr{F})$. ■

Corollary 9.8. The stalk $(R^r\pi_*\mathscr{F})_{\bar{x}}$ at $\bar{x} \to X$ is

$$\varinjlim_{(U,\bar{u})} H^r(U_Y,\mathscr{F})$$

where the limit is over all étale neighborhood of $\bar{x}$..

Example 9.9. If $\pi : Y \to X$ is finite, then π_* is exact, so $R^r\pi_*\mathscr{F} = 0$ for $r > 0$.

10 Čech cohomology

It is hard to compute the cohomology groups in terms of derived functors directly. Under mild hypotheses on X, the derived functor groups agree with the Čech groups, which are sometimes more manageable.

Let $\mathcal{U} = \{U_i \to X\}$ be an étale covering of X, and let $\mathscr{F}$ be a presheaf of abelian groups on $X_{\text{ét}}$. Define

$$C^r(\mathcal{U}, \mathscr{F}) = \prod_{(i_0,\dots,i_r)\in I^{r+1}} \mathscr{F}(U_{i_0\dots i_r}), \qquad \text{where } U_{i_0\dots i_r} = U_{i_0} \times_X \cdots \times_X U_{i_r}.$$

For $s = (s_{i_0\dots i_r}) \in C^r(\mathcal{U}, \mathscr{F})$, define $d^r s \in C^{r+1}(\mathcal{U}, \mathscr{F})$ by

$$(d^r s)_{i_0\dots i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j s_{i_0\dots\widehat{i_j}\dots i_{r+1}}|_{U_{i_0\cdots i_{r+1}}}.$$

As in the classical case,

$$C^\bullet(\mathcal{U}, \mathscr{F}) = C^0(\mathcal{U}, \mathscr{F}) \to \cdots \to C^r(\mathcal{U}, \mathscr{F}) \to C^{r+1}(\mathcal{U}, \mathscr{F}) \to \cdots$$

is a complex. Define

$$\check{H}^r(\mathcal{U}, \mathscr{F}) = h^r(C^\bullet(\mathcal{U}, \mathscr{F})),$$

called the r^{th} Čech cohomology group of $\mathscr{F}$ relative to the covering $\mathcal{U}$. Note that

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$$

for a sheaf $\mathscr{F}$.

As in the classical case, if $\mathcal{V}$ is a refinement of U, then there's a map on cohomology groups

$$\rho_{\mathcal{V}\mathcal{U}} : \check{H}^r(\mathcal{U}, \mathscr{F}) \to \check{H}^r(\mathcal{V}, \mathscr{F}).$$

We may pass to the limit of all coverings, and so obtain the Čech cohomology groups

$$\check{H}^r(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} \check{H}^r(\mathcal{U}, \mathscr{F}).$$

If $U \to X$ is an étale map and $\mathscr{F}$ is a presheaf on $X_{\text{ét}}$, then, as above, we may define cohomology groups $\check{H}^r(U, \mathscr{F})$. Note that $\check{H}^r(U, \mathscr{F})$ is defined intrinsically in terms of $\mathscr{F}$. The mapping $U \mapsto \check{H}^r(U, \mathscr{F})$ extends to give a presheaf on $X_{\text{ét}}$, which we denoted by $\check{\mathscr{H}}^r(X_{\text{ét}}, \mathscr{F})$ or simply $\check{\mathscr{H}}^r(\mathscr{F})$.

Proposition 10.1. Let X be a scheme.

(a) For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$, $\check{H}^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

(b) $\check{H}^r(\mathcal{U}, \mathscr{I}) = 0$, $r > 0$, for all coverings $\mathcal{U}$ and injective sheaves $\mathscr{I}$.

Proof. (a) is obvious. (b): Let $\mathcal{U} = \{U_i \to X\}$. For $U_i \to X$, there's a presheaf $\mathbb{Z}_{i!}$ on $X_{\text{ét}}$ such that

$$\operatorname{Hom}(\mathbb{Z}_{i!}, \mathscr{I}) = \operatorname{Hom}(\mathbb{Z}, \mathscr{I}|_{U_i}) = \mathscr{I}(U).$$

So

$$C^r(\mathcal{U}, \mathscr{I}) = \prod \mathscr{I}(U_{i_0 \dots i_r}) = \prod \operatorname{Hom}(\mathbb{Z}_{i_0 \dots i_r!}, \mathscr{I}) = \operatorname{Hom}\left(\bigoplus \mathbb{Z}_{i_0 \dots i_r!}, \mathscr{I}\right).$$

Since $\mathscr{I}$ is injective as a sheaf, it is injective as a presheaf (because $(\cdot)^+$ is an exact left adjoint to the inclusion functor). So it suffices to show that

$$0 \leftarrow \bigoplus \mathbb{Z}_{i_0!} \leftarrow \cdots \leftarrow \bigoplus \mathbb{Z}_{i_0 \dots i_r!} \leftarrow \cdots$$

is exact in $\mathsf{PreSh}(X_{\text{ét}})$, i.e.,

$$0 \leftarrow \bigoplus \mathbb{Z}_{i_0!}(V) \leftarrow \cdots \leftarrow \bigoplus \mathbb{Z}_{i_0 \dots i_r!}(V) \leftarrow \cdots$$

is exact for all $V \to X$ étale. This follows from the fact that

$$\bigoplus \mathbb{Z}_{i_0 \dots i_r!}(V) = \mathbb{Z}^{\oplus \coprod_{\varphi \in \operatorname{Hom}_X(V,X)} \left(\coprod_i \operatorname{Hom}_\varphi(V, U_i)\right)^{r+1}}. \qquad \blacksquare$$

It follows that the isomorphism $\check{H}^0(X, \mathscr{F}) \cong H^0(X, \mathscr{F})$ extends to an isomorphism for all r and $\mathscr{F}$ iff every short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

of sheaves gives a long exact sequence

$$\cdots \to \check{H}^r(X, \mathscr{F}') \to \check{H}^r(X, \mathscr{F}) \to \check{H}^r(X, \mathscr{F}'') \to \cdots$$

of Čech cohomology groups, i.e., a covariant δ-functor since it's effaceble.

Proposition 10.2. Let $\mathcal{U}$ be a covering of U, and let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. There are spectral sequences

$$\check{H}^p(\mathcal{U}, \mathscr{H}^q(\mathscr{F})) \Rightarrow H^n(U, \mathscr{F}) \quad \text{and} \quad \check{H}^p(U, \mathscr{H}^q(\mathscr{F})) \Rightarrow H^n(U, \mathscr{F}).$$

Proof. It suffices to show that

$$\check{H}^0(\mathcal{U}, \mathscr{H}^0(\mathscr{F})) = \mathscr{H}^0(U, \mathscr{F}) = \check{H}^0(U, \mathscr{H}^0(\mathscr{F})),$$

for any sheaf $\mathscr{F}$, and that

$$\check{H}^r(\mathcal{U}, \mathscr{H}^0(\mathscr{I})) = 0 = \check{H}^r(U, \mathscr{H}^0(\mathscr{I}))$$

for $r > 0$ and $\mathscr{I}$ injective. The first statement follows from the definitions. Since $\mathscr{H}^0$ is simply the inclusion functor $\mathsf{Sh}(X_{\text{ét}}) \to \mathsf{PreSh}(X_{\text{ét}})$, it preserves injectives, so the second statement follows from (10.1). ■

Corollary 10.3. There is a spectral sequence

$$\check{\mathscr{H}}^p(\mathscr{H}^q(\mathscr{F})) \Rightarrow \mathscr{H}^n(X, \mathscr{F}).$$

Proposition 10.4. We have $\check{\mathscr{H}}^0(X, \mathscr{H}^q(\mathscr{F})) = 0$ for $q > 0$, i.e.,

$$\check{H}^0(U, \mathscr{H}^q(\mathscr{F})) = 0$$

for all $q > 0$ and all $U \to X$ étale.

Proof. By (9.3), $\mathscr{H}^q(\mathscr{F})^+ = 0$. Since $\check{\mathscr{H}}^0(\mathscr{H}^q(\mathscr{F}))$ is a subpresheaf of $\mathscr{H}^q(\mathscr{F})^+$, $\check{\mathscr{H}}^0(X, \mathscr{H}^q(\mathscr{F})) = 0$. ■

Corollary 10.5. For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$, there are isomorphisms

$$\check{H}^0(X, \mathscr{F}) \cong H^0(X, \mathscr{F}), \quad \check{H}^1(X, \mathscr{F}) \cong H^1(X, \mathscr{F})$$

and an exact sequence

$$0 \to \check{H}^2(X, \mathscr{F}) \to H^2(X, \mathscr{F}) \to \check{H}^1(X, \mathscr{H}^1(\mathscr{F})) \to \check{H}^3(X, \mathscr{F}) \to H^3(X, \mathscr{F}).$$

Proposition 10.6. Let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. The following are equivalent:

(i) $\mathscr{F}$ is flabby;

(ii) $\check{H}^q(\mathcal{U}/U, \mathscr{F}) = 0$, for any $U \to X$ étale, any covering $\mathcal{U}$ of U and any $q > 0$;

(iii) $\check{H}^q(U, \mathscr{F}) = 0$, for any $U \to X$ étale and any $q > 0$.

Proof. (a) $\implies$ (b): Since $\mathscr{F}$ is flabby, $\mathscr{H}^q(\mathscr{F}) = 0$ for $q > 0$. Hence,

$$\check{H}^q(\mathcal{U}/U, \mathscr{F}) \cong H^q(U, \mathscr{F}) = 0$$

for $q > 0$.

(b) $\implies$ (c): Pass to the direct limit.

(c) $\implies$ (a): We have $\check{\mathscr{H}}^q(\mathscr{F}) = 0$ for $q > 0$. It follows from (10.5) that $\mathscr{H}^1(\mathscr{F}) = 0$. Induction on q. We know that $\check{\mathscr{H}}^2(\mathscr{F}) \cong \mathscr{H}^2(\mathscr{F}) = 0$, $\check{\mathscr{H}}^1(\mathscr{H}^1(\mathscr{F})) = 0$, and

$\check{\mathscr{H}}^0(\mathscr{H}^2(\mathscr{F})) = 0$. Using the spectral sequence $\check{\mathscr{H}}^p(\mathscr{H}^q(\mathscr{F})) \Rightarrow \mathscr{H}^n(\mathscr{F})$, we get $\mathscr{H}^2(\mathscr{F}) = 0$. Assume now that $\mathscr{H}^i(\mathscr{F}) = 0$ for all $i < q$, then the same argument shows that $\check{\mathscr{H}}^i(\mathscr{H}^j(\mathscr{F})) = 0$ for all $i + j \leq q$, and hence $\mathscr{H}^q(\mathscr{F}) = 0$. ■

Corollary 10.7. (a) If a sheaf $\mathscr{F}$ is flabby, then $\mathscr{F}|_U$ is flabby for all $U \to X$ étale.

(b) If $\pi : Y_{\text{ét}} \to X_{\text{ét}}$ is a morphism and $\mathscr{G}$ is flabby, then $\pi_*\mathscr{G}$ is flabby.

(c) If $\mathscr{I}$ is injective, then $\mathscr{H}om(\mathscr{F}, \mathscr{I})$ is flabby for any sheaf $\mathscr{F}$.

Proof. (a): $\mathscr{F}|_U$ satisfies (c) in (10.6).

(b): Let $\mathcal{U} = \{U_i \to U\}$ be a covering of U over X. Let $V_i = U_i \times_X Y$. Then $\mathcal{V} = \{V_i \to V\}$ is a covering of $V = U \times_X Y$ and $V_{i_0 \dots i_r} = U_{i_0 \dots i_r} \times_X Y$. It follows from the definition that $C^\bullet(\mathcal{V}, \mathscr{F}) \cong C^\bullet(\mathcal{U}, \pi_*\mathscr{F})$. Since the former is exact, so is the latter.

(c): Too long. ■

Theorem 10.8. Assume that every finite subset of X is contained in an open affine subset and that X is quasi-compact. Then, for any short exact sequence of sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

the direct limit of the complexes

$$0 \to C^\bullet(\mathcal{U}, \mathscr{F}') \to C^\bullet(\mathcal{U}, \mathscr{F}) \to C^\bullet(\mathcal{U}, \mathscr{F}'') \to 0$$

over the étale coverings $\mathcal{U}$ of X is exact, and so gives rise to a long exact sequence of Čech cohomology groups. Thus

$$\check{H}^r(X, \mathscr{F}) \cong H^r(X, \mathscr{F})$$

for all r and all sheaves $\mathscr{F}$.

First, we need a result from commutative algebra.

Lemma 10.9. Let A be a ring, $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ prime ideals of A, and $A^{\text{sh}}_{\mathfrak{p}_i}$ strict Henselizations of $A_{\mathfrak{p}_i}$. Then $A' = A^{\text{sh}}_{\mathfrak{p}_1} \otimes_A \cdots \otimes_A A^{\text{sh}}_{\mathfrak{p}_r}$ has the property that any faithfully flat étale map $A' \to B$ has a section $B \to A$.

Proof. ■

For any morphism $U \to X$ we write $U^0 = X$ and $U^n = U^{n-1} \times_X U$. Also, if $\bar{x} = (\bar{x}_1, \dots, \bar{x}_r)$ is an r-tuple of geometric points, then $X_{\bar{x}} = X$ if $r = 0$ and

$$X_{\bar{x}} = \operatorname{Spec} \mathcal{O}_{\bar{x}_1} \times_X \cdots \times_X \operatorname{Spec} \mathcal{O}_{\bar{x}_r}$$

if $r \neq 0$.

Lemma 10.10. Let X be as in the statement of the theorem, let $U \to X$ be étale of finite-type, and let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_r)$ be a family of geometric points of X. Let $W \to U^n \times_X X_{\bar{x}}$ be an étale surjective morphism. Then there is an étale surjective morphism $U' \to U$ such that the induced morphism $(U')^n \times_X X_{\bar{x}} \to U^n \times_X X_{\bar{x}}$ factors through W (provided either $n > 0$ or $r > 0$).

Proof. Induction on n.

First assume $n = 0$. Then $x_1, \dots, x_r$ are contained in some open affine subscheme $\operatorname{Spec} A$ of X, and it follows that

$$X_{\bar{x}} = \operatorname{Spec}(A^{\mathrm{sh}}_{\mathfrak{p}_1} \otimes \cdots \otimes A^{\mathrm{sh}}_{\mathfrak{p}_r})$$

for some primes ideals $\mathfrak{p}_i$ of A. Thus $W \to X_{\bar{x}}$ has a section by (10.9).

The case $n = 1$, $r = 0$ is obvious (take $U' = W$).

For general n, suppose that $V \to U$ is étale and of finite type but not surjective and that the map $V^n \times X_{\bar{x}} \to U^n \times X_{\bar{x}}$ factors through W (for example, take V to be empty). Let $\bar{u}$ be a geometric point of U not covered by V, and let $X_{\bar{u}} \cong U_{\bar{u}} = \operatorname{Spec} \mathcal{O}_{U,\bar{u}}$. Consider the morphism $V \sqcup X_{\bar{u}} \to U$ and the commutative diagram:

$$\begin{tikzcd}
W \times_{U^n} (V \sqcup X_{\bar{u}})^n \times_X X_{\bar{x}} \arrow[r] \arrow[d, two heads] & W \arrow[d, two heads] \\
(V \sqcup X_{\bar{u}})^n \times_X X_{\bar{x}} \arrow[r] & U^n \times_X X_{\bar{x}}
\end{tikzcd}$$

Note that the $(V \sqcup X_{\bar{u}})^n$ is a disjoint union of schemes of the form $V^i \times X^j_{\bar{u}}$ $(i + j = n)$. Let W_i be the pullback of W to $V^i \times X^j_{\bar{u}} \times X_{\bar{x}}$. For $i < n$, the induction hypothesis shows that there's an étale surjective map $V'_i \to V$ such that $(V'_i)^i \times (X^j_{\bar{u}} \times X_{\bar{x}})$ factors through $V^i \times (X^j_{\bar{u}} \times X_{\bar{x}})$. Replace V by $V' = V'_1 \times_V \cdots \times_V V'_{n-1}$. For $i = n$, the map factors by assumption. Thus, we may assume that

$$(V \sqcup X_{\bar{u}})^n \times X_{\bar{x}} \to U^n \times U^n \times X_{\bar{x}}$$

factors through W. Now, $X_{\bar{u}}$ is a limit of schemes Y étale over U, and so for some such Y, the map

$$(V \sqcup Y)^n \times X_{\bar{x}} \to U^n \times X_{\bar{x}}$$

also factors through W. Replace V by $V \sqcup Y$. By proceeding in this way, we obtain a sequence $V_i \to U$, $i = 1, 2, \dots$ whose image form a strictly increasing sequence of open subsets of U. As U is a Noetherian topological space, we arrive at a surjective map after only finitely many steps. ■

We now prove (10.8).

Proof. Let $\mathscr{F} \to \mathscr{F}''$ be a surjective map of sheaves, and let $V \to X$ étale. For a covering $\mathcal{U} = \{U_i \to V\}$ of V, since V is quasi-compact, we may assume that $\mathcal{U}$ is a finite cover $\{U_i \to V\}_{i=1}^r$. Let $U = \coprod U_i$, then

$$C^\bullet(\mathcal{U}, \mathscr{G}) = C^\bullet(\{U \to V\}, \mathscr{G})$$

for any sheaf $\mathscr{G}$. Hence, the coverings $U \to V$ consisting of a single morphism are cofinal in the set of all coverings of V. Let $s'' \in \mathscr{F}''(U^n)$. There exists an étale covering $W \to U^n$ and an $s \in \mathscr{F}(W)$ such that $s \mapsto s''|_W$. According to (10.10), there exists a covering $U' \to U$ such that $(U')^n \to U^n$ factors through W. Thus $s|_{(U')^n} \mapsto s''|_{(U')^n}$, which proves the theorem. ■

11 Principal homogeneous spaces and H^1

Let $\mathcal{U} = \{U_i \to X\}$ be an étale covering of X, and let $\mathscr{G}$ be a sheaf of groups on $X_{\text{ét}}$. A 1-cocycle for $\mathcal{U}$ with values in $\mathscr{G}$ is a family $g = \{g_{ij}\}_{(i,j)\in I^2}$ with $g_{ij} \in \mathscr{G}(U_{ij})$ such that

$$g_{jk}|_{U_{ijk}} \cdot g_{ki}|_{U_{ijk}} \cdot g_{ij}|_{U_{ijk}} = 1$$

for all i, j, k. Two cocycles g and g' are cohomologous, denotes $g \sim g'$, if there is a family $h = \{h_i\}_{i\in I}$ with $h_i \in \mathscr{G}(U_i)$ such that

$$g'_{ij} = h_i|_{U_{ij}} \cdot g_{ij} \cdot h_j|_{U_{ij}}^{-1}$$

for all i, j. The set of 1-cocycles module $\sim$ is denoted $\check{H}^1(X_{\text{ét}}, \mathscr{G})$. It is not in general a group, but it does have a distinguished element represented by the 1-cocycle $\{1\}$.

A sequence

$$1 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$$

of sheaves of groups is said to be exact if

$$1 \to \mathscr{G}'(U) \to \mathscr{G}(U) \to \mathscr{G}''(U)$$

is exact for all $U \to X$ étale and $\mathscr{G} \to \mathscr{G}''$ is locally surjective. Such a sequence gives rise to a sequence of sets

$$1v \to \mathscr{G}'(X) \to \mathscr{G}(X) \to \mathscr{G}''(X) \to \check{H}^1(X, \mathscr{G}') \to \check{H}^1(X, \mathscr{G}) \to \check{H}^1(X, \mathscr{G}'')$$

that is exact in the following sense: the image of each arrow is exactly the set mapped to the distinguished element by the following arrow.

Definition 11.1. Let $S \times G \to S$ be a right group action. S is said to be a principal homogeneous space (or torsor) for G if, for one $s \in S$, the map $g \mapsto sg : G \to S$ is a bijection.

Definition 11.2. Let $\mathscr{G}$ be a sheaf of groups on $X_{\text{ét}}$, and let $\mathscr{S}$ be a sheaf of sets on which $\mathscr{G}$ acts on the right. Then $\mathscr{S}$ is called a principal homogeneous space for $\mathscr{G}$ if

(a) there exists an étale covering $\{U_i \to X\}$ of X such that for all i, $\mathscr{S}(U_i) \neq \varnothing$; and

(b) for every $U \to X$ étale and $s \in \Gamma(U, \mathscr{S})$, the map $g \mapsto sg : \mathscr{G}|_U \to \mathscr{S}|_U$ is an isomorphism of sheaves.

Definition 11.3. A principal homogeneous space $\mathscr{S}$ is trivial if it is isomorphic to $\mathscr{G}$ acting on itself by right multiplication, or, equivalently, if $\mathscr{S}(X) \neq \varnothing$. We say that a covering $\{U_i \to X\}$ splits $\mathscr{S}$ if $\mathscr{S}(U_i) \neq \varnothing$ for each $i \in I$.

Let $\mathscr{S}$ be a principal homogeneous space for $\mathscr{G}$. Let $\mathcal{U} = \{U_i \to X\}$ be an étale covering of X that splits $\mathscr{S}$, and choose an $s_i \in \mathscr{S}(U_i)$ for each i. Because of condition (b), there exists a unique $g_{ij} \in \mathscr{G}(U_{ij})$ such that

$$s_i|_{U_{ij}} \cdot g_{ij} = s_j|_{U_{ij}}.$$

Then $\{g_{ij}\}$ is a cocycle, because

$$s_j \cdot g_{jk} \cdot g_{ki} \cdot g_{ij} = s_k \cdot g_{ki} \cdot g_{ij} = s_i \cdot g_{ij} = s_j.$$

Moreover, replacing s_i with $s'_i = s_i \cdot h_i$, $h_i \in \mathscr{G}(U_i)$, leads to a cohomologous cocycle. Thus, $\mathscr{S}$ defines a class $c(\mathscr{S})$ in $\check{H}^1(\mathcal{U}, \mathscr{G})$.

Proposition 11.4. The map $\mathscr{S} \mapsto c(\mathscr{S})$ defines a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of principal} \\ \text{homogeneous spaces for } \mathscr{G} \text{ split by } \mathcal{U} \end{array} \right\} \longleftrightarrow \check{H}^1(\mathcal{U}, \mathscr{G}).$$

Proof. Let $\alpha : \mathscr{S} \to \mathscr{S}'$ be an isomorphism of $\mathscr{G}$-sheaves, and choose $s_i \in \mathscr{S}(U_i)$. Then $\alpha(s_i) \in \mathscr{S}'(U_i)$, and

$$s_i \cdot g_{ij} = s_j \implies \alpha(s_i) \cdot g_{ij} = \alpha(s_j).$$

Therefore the 1-cocycle defined by the family $\{\alpha(s_i)\}$ equals that defined by $\{s_i\}$. This shows that $c(\mathscr{S})$ depends only on the isomorphism class of $\mathscr{S}$, and so $[\mathscr{S}] \mapsto \mathfrak{c}(\mathscr{S})$ is well-defined.

Suppose that $c(\mathscr{S}) = c(\mathscr{S}')$. Then we may choose $s_i \in \mathscr{S}(U_i)$ and $s'_i \in \mathscr{S}'(U_i)$ that define the same 1-cocycle $\{g_{ij}\}$. Suppose there exists a $t \in \mathscr{S}(V)$ for some $V \to X$ étale. Let $\{V_i \to V\}$ be the covering of V with $V_i = U_i \times_X V$. Then $t|_{V_i} = s_i|_{V_i} \cdot g_i$ for a unique $g_i \in \mathscr{G}(V_i)$; from the equality $(t|_{V_i})|_{V_{ij}} = (t|_{V_j})|_{V_{ij}}$, we find that

$$(\spadesuit) \qquad g_i = g_{ij} \cdot g_j$$

Because $\mathscr{S}$ is a sheaf, $t \mapsto \{g_i\}$ is a bijection from $\mathscr{S}(V)$ onto the set of families $\{g_i\}$, $g_i \in \mathscr{G}(U_i \times_X V)$, satisfying $(\spadesuit)$. A similar statement holds for $\mathscr{S}'$, and so there is a canonical bijection $\mathscr{S}(V) \to \mathscr{S}'(V)$. The family of these bijections is an isomorphism of $\mathscr{G}$-sheaves $\mathscr{S} \to \mathscr{S}'$.

Thus, the map is an injection into $\check{H}^1(\mathcal{U}, \mathscr{G})$, and it remains to prove that it is surjective. Let $\{g_{ij}\}$ be a 1-cocycle for the covering $\mathcal{U}$. For any $V \to X$ étale, define $\mathscr{S}(V)$ to be the set of families $\{g_i\}$, $g_i \in \mathscr{G}(V_i)$ such that

$$g_i|_{V_{ij}} = g_{ij} \cdot g_j|_{V_{ij}}.$$

This defines a sheaf of $\mathscr{G}$-sets, and that $c(\mathscr{S})$ is represented by $\{g_{ij}\}$. ∎

Let $L_n(X)$ be the set of isomorphism classes of locally free sheaves of $\mathcal{O}_X$-modules of rank n on X for the Zariski topology.

Theorem 11.5. There are natural bijections

$$L_n(X) \leftrightarrow \check{H}^1(X, \mathrm{GL}_n) \leftrightarrow \check{H}^1(X_{\text{ét}}, \mathrm{GL}_n).$$

Corollary 11.6. There is a canonical isomorphism

$$H^1(X_{\text{ét}}, \mathbb{G}_m) \cong \mathrm{Pic}(X).$$

Proof. By (11.5), we have $\mathrm{Pic}(X) \cong \check{H}^1(X_{\text{ét}}, \mathbb{G}_m)$. But, because $\mathbb{G}_m$ is commutative, this equals to $H^1(X_{\text{ét}}, \mathbb{G}_m)$. ■

12 Galois

For $X = \operatorname{Spec} k$, k a field, the étale morphisms $Y \to X$ are the spectra of étale k-algebras A, and each is finite. The choice of a geometric point for X amounts to the choice of a separably algebraically closed field K containing k. Define $F : \text{Ét}/k \to \mathsf{Sets}$ by

$$F(A) = \mathrm{Hom}_k(A, K).$$

Let $\tilde{k} = \{k_i\}$ be the projective system consisting of all finite Galois extensions of k contained in K. Then $\tilde{k}$ ind-represents F, i.e.,

$$F(A) \cong \mathrm{Hom}_k(A, \tilde{k}) = \varinjlim_{i \in I} \mathrm{Hom}_k(A, k_i)$$

functorially in A. Define

$$\mathrm{Aut}_k(\tilde{k}) = \varprojlim_{i \in I} \mathrm{Aut}_k(k_i).$$

Thus

$$\mathrm{Aut}_k(\tilde{k}) = \varprojlim_{i \in I} \mathrm{Gal}(k_i/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k).$$

Moreover, F defines an equivalence of categories from $\text{Ét}/k$ to the category of finite discrete $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-sets.

Let $\varphi : Y \to X$ be a morphism, and let G be a finite group. A right action G on Y over X is a map $\alpha : G \to \mathrm{Aut}_X(Y)$ such that $\alpha(gh) = \alpha(h) \circ \alpha(g)$. If X and Y are affine, so that φ corresponds to a map of rings $A \to B$, then to give a right action of G on Y over X is the same as to give a left action of G on the A-algebra B.

Definition 12.1. Let $Y \to X$ be a faithfully flat map, and let G be a finite group acting on Y over X on the right. Then $Y \to X$ is called a Galois covering of X with group G if the morphism $Y \times G \to Y \times_X Y$, $(y, g) \mapsto (y, yg)$ is an isomorphism.

Here $Y \times G$ denotes a disjoint union copies of Y indexed by the elements of G:

$$Y \times G = \coprod_g Y_g, \quad Y_g \cong Y.$$

The maps $\mathrm{id} : Y_g \to Y$ and $g : Y_g \to Y$ define a map $Y_g \to Y \times_X Y$ and the condition on the definition requires that these maps induce an isomorphism.

If $\varphi : Y \to X$ is a Galois covering, then φ is surjective, finite, and étale of degree equal to the order of G. Conversely, if $\varphi : Y \to X$ is surjective, finite, and étale of degree equal to the order of $\mathrm{Aut}_X(Y)$, then it is Galois with group $\mathrm{Aut}_X(Y)$.

An A-algebra B is said to be Galois if there is a group G acting on B (by A-algebra automorphisms) in such a way that $\operatorname{Spec} B \to \operatorname{Spec} A$ is Galois with group G. Explicitly, this means that $A \to B$ is faithfully flat and that the homomorphism

$$B \otimes_A B \to \prod_{g \in G} B, \quad b \otimes b' \mapsto (\ldots, b \cdot g(b'), \ldots)$$

is an isomorphism.

Let $\pi : Y \to X$ be Galois with group G. Then G acts on Y on the the right, and hence, for any presheaf $\mathscr{F}$, it acts on $\mathscr{F}(Y)$ on the left because $\mathscr{F}$ is a covariant functor.

Proposition 12.2. Let $Y \to X$ be Galois with group G, and let $\mathscr{F}$ be a presheaf on $X_{\text{ét}}$ that takes disjoint unions to products. Then $\mathscr{F}$ satisfies the sheaf condition for the covering $Y \to X$ if and only if the restriction map $\mathscr{F}(X) \to \mathscr{F}(Y)$ identifies $\mathscr{F}(X)$ with the $\mathscr{F}(Y)^G$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & Y & \Longleftarrow & Y \times_X Y \\ \| & & \| & & \uparrow \wr \\ X & \longleftarrow & Y & \Longleftarrow & Y \times G. \end{array}$$

in which the projection maps $(y, y') \mapsto y$ and $(y, y') \mapsto y'$ on the top row correspond respectively to the maps $(y, g) \mapsto y$ and $(y, g) \mapsto yg$ on the bottom row. On applying $\mathscr{F}$ to the diagram, we obtain a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r] \arrow[d, equal] & \mathscr{F}(Y) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathscr{F}(Y \times_X Y) \arrow[d, "\wr"] \\
\mathscr{F}(X) \arrow[r] & \mathscr{F}(Y) \arrow[r, shift left] \arrow[r, shift right] & \prod_{g \in G} \mathscr{F}(Y).
\end{tikzcd}$$

The maps $\mathscr{F}(Y) \to \prod_{g \in G} \mathscr{F}(Y)$ are

$$s \mapsto (s, \ldots, s), \quad s \mapsto (\ldots, gs, \ldots),$$

respectively. These maps agree on $s \in \mathscr{F}(Y)$ iff $gs = s$ for all $g \in G$. ■

Example 12.3. Let k be a field. A presheaf $\mathscr{F}$ of abelian groups on $\operatorname{Spec} k_{\text{ét}}$ can be regarded as a covariant functor $\mathsf{Ét}/k \to \mathsf{Ab}$ (recall $\mathsf{Ét}/k$ is the category of étale k-algebras). Such a functor will be a sheaf if and only if

$$f\left(\prod A_i\right) = \bigoplus \mathscr{F}(A_i)$$

for every finite familt $\{A_i\}$ of étale k-algebras and $\mathscr{F}(K) \cong \mathscr{F}(L)^{\mathrm{Gal}(L/K)}$ for every finite Galois extension L/K of fields with K finite over k.

Choose a separable closure k^{sep} of k, and let $G = \mathrm{Gal}(k^{\text{sep}}/k)$. For $\mathscr{F}$ a sheaf on $\operatorname{Spec} k_{\text{ét}}$, define

$$M_{\mathscr{F}} = \varinjlim_{K} \mathscr{F}(K),$$

where K runs through the subfields of k^{sep} that are finite and Galois over k. Then M is a discrete G-module.

Conversely, if M is a discrete G-module, we define

$$\mathscr{F}_M(A) = \mathrm{Hom}_G(\mathrm{Hom}_{k-\mathrm{alg}}(A, k^{\text{sep}}), M).$$

Then $\mathscr{F}_M$ is a sheaf on $\operatorname{Spec} k_{\text{ét}}$.

The functor $\mathscr{F} \mapsto M_{\mathscr{F}}$ and $M \mapsto \mathscr{F}_M$ define an equivalence between $\mathsf{Sh}(\operatorname{Spec} k_{\text{ét}})$ and the category of discrete G-modules.

On Čech cohomology, we have

Example 12.4. Let $\mathcal{U}$ be the covering of X consisting of a single Galois covering $Y \to X$ with Galois group G. If $\mathscr{F}$ is a presheaf on $X_{\text{ét}}$ carrying disjoint union into products, then we have the complex

$$\mathscr{F}(Y) \to \mathscr{F}(Y \times_X Y) \to \mathscr{F}(Y \times_X Y \times_X Y) \to \cdots.$$

Note that $Y \times_X \cdots \times_X Y \cong Y \times G \times \cdots \times G$, so we get the complex

$$\mathscr{F}(Y) \to \prod_{g_1 \in G} \mathscr{F}(Y) \to \prod_{g_1 \in G} \prod_{g_2 \in G} \mathscr{F}(Y) \to \cdots.$$

So

$$\check{H}^r(\mathcal{U}, \mathscr{F}) = H^r(G, \mathscr{F}(Y)).$$

Theorem 12.5. Let $\pi : Y \to X$ be a finite Galois covering with Galois group G, and let $\mathscr{F}$ be a sheaf for the étale topology on X. There is a spectral sequence

$$H^p(G, H^q(Y_{\text{ét}}, \mathscr{F})) \Rightarrow H^n(X_{\text{ét}}, \mathscr{F}).$$

Proof. Since G acts on Y on the right, it acts on $\mathscr{F}(Y)$ on the left. The composite of the functors

$$\mathscr{F} \mapsto \mathscr{F}(Y) : \mathsf{Sh}(X_{\text{ét}}) \to G\text{-}\mathsf{mod} \quad \text{and} \quad M \mapsto M^G : G\text{-}\mathsf{mod} \to \mathsf{Ab}$$

is the section functor $\Gamma(X, \cdot\,) : \mathsf{Sh}(X_{\text{ét}}) \to \mathsf{Ab}$. Then it suffices to show that

$$H^r(G, \mathscr{I}(Y)) = 0, \quad \forall r > 0$$

for any injective sheaf $\mathscr{I}$ on X. By (10.1),

$$H^r(G, \mathscr{I}(Y)) = \check{H}^r(Y/X, \mathscr{I}) = 0$$

because $\mathscr{I}$ is also injective as a presheaf. ∎

Example 12.6. Assume X is connected and normal, and let $g : \eta \to X$ be the inclusion of the generic point of X. Then

$$(R^r g_* \mathscr{F})_{\bar{\eta}} = H^r(\operatorname{Spec} K_{\bar{\eta}}, \mathscr{F}),$$

where $K_{\bar{\eta}}$ is the field of fractions of $\mathcal{O}_{X,\bar{\eta}}$. Moreover, in this case g_* takes a constant sheaf on η to a constant sheaf.

Let K^{sep} be a separable closure of K, and let $G = \operatorname{Gal}(K^{\text{sep}}/K)$. Let $M = M_{\mathscr{F}}$, the G-module corresponding to $\mathscr{F}$. Then $\mathscr{F}$ is constant if G acts trivially on M and locally constant if the action of G on M factors through a finite quotient.

13 The Fundamental group

Let X be a scheme and $s : \operatorname{Spec}\Omega \to X$ a geometric point. A scheme

$$f : Y \to X$$

is called a covering space if f is an étale and finite mapping. The covering spaces of X form a full subcategory of the category $\mathsf{Ét}\,(X)$. We consider the functor of geometric points over s:

$$Y(s) = \operatorname{Hom}_X(\operatorname{Spec}\Omega, Y).$$

A pointed covering space (over s) is a pair

$$(Y, \alpha)$$

consisting of a covering space Y of X and a geometric point $\alpha \in Y(s)$. A mapping of pointed covering spaces

$$f : (Y_1, \alpha_1) \to (Y_2, \alpha_2)$$

is an X-morphism $f : Y_1 \to Y_2$ with $f \circ \alpha_1 = \alpha_2$.

1. Let (Y_1, α_1) and (Y_2, α_2) be two pointed covering spaces with Y_1 connected. Then there is at most one mapping

$$(Y_1, \alpha_1) \to (Y_2, \alpha_2).$$

2. For any two nonempty connected covering spaces $X_i \to X$, $i = 1, 2$, there is a third nonempty connected covering space

$$X_3 \to X$$

that dominates the two given ones, i.e., there are X-morphisms

$$X_3 \to X_i, \quad i = 1, 2.$$

3. Let $Y \to X$ be a connected nonempty covering space of X and $G(X/Y)$ the group of automorphisms of Y over X and n the degree of the covering. Here $n = |Y(\beta)|$ for every geometric point of X for which $Y(\beta) \neq \varnothing$ (e.g. $\beta = s$, if X is connected). Because of (1), we have

$$|G(Y/X)| \leq n.$$

Y/X is called Galois if

$$|G(Y/X)| = n.$$

Then because of (1), $G(Y/X)$ acts transitively on every "geometric fiber" $Y(b)$. $G(Y/X)$ always acts on the right. For every nonempty covering space $Z \to X$ there is a Galois covering space $Y \to X$ that dominates it, i.e., such that there is an X-morphism

$$Y \to Z.$$

4. Let $X \to X$ and $Y \to X$ be two Galois covering spaces of X, and

$$f : Z \to Y$$

an X-morphism. For every "cover transformation" $\sigma \in G(Z/X)$ there is exactly one element $\bar{\sigma} \in G(Y/X)$ with $\bar{\sigma} \circ f = f \circ \sigma$. The mapping thus defined,

$$\begin{array}{ccc} G(Z/X) & \to & G(Y/X) \\ \sigma & \mapsto & \bar{\sigma}, \end{array}$$

is a surjective homomorphism. The kernel is precisely the "cover transformation group" $G(Z/Y)$ of the Galois covering space $Z \to Y$ of Y.

Now we consider the filtered category of all Galois pointed covering spaces (Y, α) of X. Because of 1., there is at most one morphism between two objects of this category. The set of isomorphism classes is thus a partially ordered test.

Definition 13.1. The fundamental group of X is

$$\pi_1(X, s) = \varprojlim_{(Y,\alpha)} G(Y/X).$$

Here (Y, α) runs over the category of pointed Galois covering spaces of X. It follows from the properties of this projective limit that the natural mapping

$$\pi_1(X, s) \to G(Y/X)$$

is surjective. The kernel is $\pi_1(Y, \alpha)$.

14 The Weil-divisor exact sequence and the cohomology of $\mathbb{G}_m$

We wish now to try to compute $H^r(X_{\text{ét}}, \mathbb{G}_m)$ for all r. We shall need to use some results from commutative algebra.

Proposition 14.1. Let A be an integrally closed integral domain. Then

$$A = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

So, for an integrally closed integral domain A,

$$A = \{a \in K \mid \text{ord}_{\mathfrak{p}}(a) \geq 0, \ \forall \, \text{ht}(\mathfrak{p}) = 1\},$$

which implies that

$$A^{\times} = \{a \in K \mid \text{ord}_{\mathfrak{p}}(a) = 0, \ \forall \, \text{ht}(\mathfrak{p}) = 1\}.$$

In other words, the sequence

$$0 \to A^{\times} \to K^{\times} \xrightarrow{a \mapsto (\text{ord}_{\mathfrak{p}}(a))} \bigoplus_{\text{ht}(\mathfrak{p})=1} \mathbb{Z}$$

is exact. The second map will not in general be surjective.

We shall need two further results from commutative algebra.

Proposition 14.2. (a) A noetherian integral domain A is a UFD iff every prime ideal $\mathfrak{p}$ of height 1 in A is principal.

(b) A regular local ring is a UFD.

Thus, when A is an integral domain,

$$0 \to A^{\times} \to K^{\times} \to \bigoplus_{\text{ht}(\mathfrak{p})=1} \mathbb{Z} \to 0$$

is exact iff A is a UFD.

Recall that a scheme is said to be normal if $\Gamma(U, \mathcal{O}_X)$ is an integrally closed integral domain for every connected open affine $U \subseteq X$, or, equivalently, if $\mathcal{O}_{X,x}$ is an integrally closed integral domain for all x in X.

We assume X to be connected and normal.

Proposition 14.3. There is a sequence of sheaves on X

$$0 \to \mathcal{O}_X^\times \to K^\times \to \mathrm{Div} \to 0$$

where $\Gamma(U, K^\times) = K^\times$ for all nonempty open U and $\mathrm{Div}(U)$ is the group of divisors on U. The sequence is always left exact, and it is exact when X is regular.

Proof. Follows from (14.1) and (14.2). ■

Let η be the generic point of X, then $\Gamma(U, g_* K^\times) = K^\times$. Let z be the generic point of a prime divisor $Z \subset X$, and let $i_Z : z \to X$ be the inclusion of z into X. For an open subset $U \subseteq X$, $z \in U$ iff $U \cap Z$ is nonempty. Therefore,

$$\Gamma(U, \oplus_{\mathrm{codim}(z)=1}) i_{z*}\mathbb{Z} = \mathrm{Div}(U).$$

On combining these remarks, we see that the sequence in can be rewritten

$$0 \to \mathcal{O}_X^\times \to g_* K^\times \to \bigoplus_{\mathrm{codim}(z)=1} i_{z*}\mathbb{Z} \to 0.$$

Proposition 14.4. For any connected normal scheme X, there is a sequence of sheaves on $X_{\text{ét}}$

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_{m,K} \to \bigoplus_{\mathrm{codim}(z)=1} i_{z*}\mathbb{Z} \to 0.$$

It is always left exact, and it is exact if X is regular.

Proof. For any étale $U \to X$ with U connected, the restriction of the sequence to U_{zar} is the sequence in (14.3). Since U is regular if X is, the statement follows. ■

Let X be a complete nonsingular algebraic curve over an algebraically closed field k. We shall use the Weil-divisor exact sequence to compute the cohomology of the sheaf $\mathbb{G}_m$ on X.

Definition 14.5. A field k is said to be quasi-algebraically closed if every nonconstant homogeneous polynomial $f(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$ of degree $d < n$ has a nontrivial zero in k^n.

Proposition 14.6. The following fields are quasi-algebraically closed:

(a) a finite field;

(b) a function field of dimension 1 over an algebraically closed field;

(c) the field of fraction K of a Henselian DVR R with algebraically closed residue field provided that the completion of K is separable over K.

The relevance of these results to Galois cohomology is shown by the following proposition.

Proposition 14.7. Let k be a quasi-algebraically closed field, and let $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then

(a) the Brauer group of k is zero, i.e., $H^2(G, (k^{\mathrm{sep}})^\times) = \mathrm{Br}(k^{\mathrm{sep}}/k) = 0$;

(b) $H^r(G, M) = 0$ for $r > 1$ and any torsion discrete G-module M;

(c) $H^r(G, M) = 0$ for $r > 2$ and any discrete G-module M.

Proof. (a) We must show that every central division algebra D over k has degree 1. Let $[D : k] = n^2$, and choose a basis for $e_1, \ldots, e_{n^2}$ for D as a k-vector space. Then there is a homogeneous polynomial $f(X_1, \ldots, X_{n^2})$ of degree n such that $f(a_1, \ldots, a_{n^2})$ is the reduced norm of the element $\alpha = \sum a_i e_i$ of D. The reduced norm of α in D/k is

$$\mathrm{Nr}_{D/k}(\alpha) = N_{\mathbb{Q}[\alpha]/\mathbb{Q}}(\alpha)^{n/[\mathbb{Q}[\alpha]:\mathbb{Q}]} \neq 0$$

if $\alpha \neq 0$. Thus $f(X_1, \ldots, X_{n^2})$ has no nontrivial zero, which implies that $n^2 \leq n$. So $n = 1$.

(b) Any finite extension of k is quasi-algebraically closed. Together with (a), this shows that, for any finite extension L/K of finite extensions of k,

$$H^1(\mathrm{Gal}(L/K), L^\times) = H^2(\mathrm{Gal}(L/K), L^\times) = 0.$$

Now Tate's theorem implies that $H^r(\mathrm{Gal}(L/k), L^\times) = 0$ for all $r > 0$ and any L/k finite and Galois. On passing to the inverse limit, one finds that $H^r(G, (k^{\mathrm{sep}})^\times) = 0$ for $r > 0$. From the cohomology sequence of the Kummer sequence

$$0 \to \mu_n \to (k^{\mathrm{sep}})^\times \to (k^{\mathrm{sep}})^\times \to 0$$

we find that $H^r(G, \mu_n) = 0$ for all $r > 1$ and n relatively prime to the characteristic of k. Let p be a prime $\neq \mathrm{char}\, k$. There exists a finite Galois extension K of k of degree prime to p such that K contains a primitive p^{th} root of 1. The composite

$$H^r(G, \mathbb{Z}/p\mathbb{Z}) \to H^r(G', \mathbb{Z}/p\mathbb{Z}) \to H^r(G, \mathbb{Z}/p\mathbb{Z}), \quad G' = \mathrm{Gal}(k^{\mathrm{sep}}/K),$$

is multiplication by $[K : k]$, and so is an isomorphism. Because

$$H^r(G', \mathbb{Z}/p\mathbb{Z}) = H^r(G', \mu_p) = 0,$$

$H^r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r > 1$.

From the Artin-Scheier exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to k \to k \to 0$$

and one finds that $H^r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r > 1$ fr any field k of characteristic p.

Now let M be a finite G-module. We want to show that $H^r(G, M) = 0$ for $r > 1$. We may suppose that M has order a power of a prime p. The Sylow theorems and the restriction-corestriction argument used above allow us to assume that G acts on M through a finite p-group $\bar{G}$. Now a standard result shows that the only simple $\bar{G}$-module of p-power is $\mathbb{Z}/p\mathbb{Z}$ with the trivial action, and so M has a composition series whose quotients are all $\mathbb{Z}/p\mathbb{Z}$. An induction argument now shows that $H^r(G, M) = 0$ for $r > 1$.

As any torsion G-module is a union of its finite submodules, and cohomology commutes with direct limits, this completes the proof.

(c) ■

Theorem 14.8. For a connected nonsingular curve X over an algebraically closed field,

$$H^r(X_{\text{ét}}, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^\times) & , \ r = 0, \\ \mathrm{Pic}(X) & , \ r = 1, \\ 0 & , \ r > 1. \end{cases}$$

Proof. Using the exact sequence

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_{m,K} \to \bigoplus_{\mathrm{codim}(z)=1} i_{z*}\mathbb{Z} \to 0,$$

it suffices to show that the cohomology groups

$$H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta}) = H^r(X_{\text{ét}}, \mathrm{Div}_X) = 0$$

for all $r > 0$.

For x a closed point of X, i_{x*} is exact, and so

$$H^r(X_{\text{ét}}, i_{x*}\mathscr{F}) = H^r(x_{\text{ét}}, \mathscr{F}) = 0$$

for any sheaf $\mathscr{F}$ on $X_{\text{ét}}$. Hence

$$H^r(X_{\text{ét}}, \mathrm{Div}_X) = \bigoplus_{\mathrm{codim}(x)=1} H^r(X_{\text{ét}}, i_{x*}\mathbb{Z}) = 0$$

for $r > 0$.

Now consider $R^r g_* \mathbb{G}_{m,\eta}$. According to (12.6),

$$(R^r g_* \mathbb{G}_{m,\eta})_{\bar{x}} = \begin{cases} 0 & \text{, if } x = \eta \text{ and } r > 0, \\ H^r(\operatorname{Spec} K_{\bar{x}}, \mathbb{G}_m) & \text{, if } x \neq \eta. \end{cases}$$

Here $K_{\bar{x}}$ os the field of fractions of the Henselian DVR $\mathcal{O}_{X,\bar{x}}$, and so (14.6(c)) shows that $H^r(\operatorname{Spec} K_{\bar{x}}, \mathbb{G}_m) = 0$ for $r > 0$. Therefore $R^r g_* \mathbb{G}_m = 0$ for $r > 0$, and so the Leray spectral sequence for g shows that

$$H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta}) = H^r(\bar{\eta}_{\text{ét}}, \mathbb{G}_{m,\eta}) = H^r(G, (K^{\mathrm{sep}})^\times)$$

for all r, where $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Now $H^1(G, (K^{\mathrm{sep}})^\times) = 0$ by Hilbert's Theorem 90, and $H^r(G, (K^{\mathrm{sep}})^\times) = 0$ for $r > 1$ by (14.6(b)) and (14.7(c)). ∎

Let X be a regular integral quasi-compact scheme and let $K = R(X)$. Similar arguments to the above show that there are exact sequences

$$0 \to H^0(X_{\text{ét}}, \mathbb{G}_m) \to K^\times \to \bigoplus_{\mathrm{codim}(z)=1} \mathbb{Z} \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0$$

and

$$0 \to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(K, \mathbb{G}_{m,\eta}).$$

Moreover, $H^r(X_{\text{ét}}, \mathbb{G}_m)$ is torsion for $r > 1$.

15 The cohomology of curves

Let X be a connected nonsingular curve over an algebraically closed field k. The Picard group of X can be defined by the exact sequence

$$K(X)^\times \to \bigoplus_{x \in X} \mathbb{Z} \to \mathrm{Pic}(X) \to 0.$$

Now assume that X is complete. The divisor (f) of a nonzero rational function f on X has degree zero. Let $\mathrm{Div}^0(X)$ denote the group of divisors of degree 0, and $\mathrm{Pic}^0(X)$ the quotient of $\mathrm{Div}^0(X)$ by the subgroup of principal divisors.

Proposition 15.1. The sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0$$

is exact. For any integer n relative prime to $\mathrm{char}(k)$,

$$D \mapsto nD : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$$

is surjective with kernel equal to $(\mathbb{Z}/n\mathbb{Z})^{2g}$, where g is the genus of X.

Let X be a complete connected nonsingular curve over an algebraically closed field k of genus g. For any n primes to $\mathrm{char}(k)$,

$$H^r(X_{\text{ét}}, \mu_n) = \begin{cases} \mu_n(k) & , \text{ if } r = 0, \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & , \text{ if } r = 1, \\ \mathbb{Z}/n\mathbb{Z} & , \text{ if } r = 2, \\ 0 & , \text{ if } r > 2. \end{cases}$$

Proof. ■

16 Constructible sheaf

For every sheaf of sets $\mathscr{F}$ on the scheme X, there is a family X_α of étale X-schemes and a surjective sheaf mapping

$$\coprod \mathrm{Hom}_X(\,\cdot\,, X_\alpha) \to \mathscr{F}.$$

Proposition 16.1. Let k be a separably closed field and X a finitely generated k-scheme, $\dim X \leq 1$. Let $\mathscr{F}$ be a constructible sheaf whose sections have order relatively prime to the characteristic of k. Then:

(a) The cohomology groups $H^q(X, \mathscr{F})$ are finite.

(b) $H^q(X, \mathscr{F}) = 0$ for $q \geq 3$.

(c) If X is affine, then actually

$$H^q(X, \mathscr{F}) = 0 \quad \text{for } q \geq 2.$$

Statements (b) and (c) can be extended from constructible sheaves to torsion sheaves by passage to limits.

Proof. We can assume at the start that k is algebraically closed and that X is reduced (by a property of purely inseparable morphism). Let $f : X' \to X$ be the normalization of X. It is an isomorphism outside finitely many points, say $U \subseteq X$. Hence the adjunction mapping

$$\mathscr{F} \to f_* f^* \mathscr{F}$$

is also an isomorphism on U. Then

$$H^q(X, \mathscr{F}) = H^q(X, f_* f^* \mathscr{F}) = H^q(X', f^* \mathscr{F})$$

since $X' \to X$ is finite. Hence, we may assume that X' is a smooth curve.

(a): Let

$$0 \to \mathscr{F} \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to$$

be an exact sequence of sheaves. Suppose the cohomology groups of the sheaves $\mathscr{F}_i$ are finite. Then obviously the cohomology groups of $\mathscr{F}$ are finite. It is enough therefore to show:

Every constructible sheaf $\mathscr{F}$ on X is embeddable in a constructible sheaf with finite cohomology groups.

Let $\mathscr{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X ($\operatorname{char} k \nmid n$). Then $\mathscr{F}$ is a subsheaf of a finite direct sum of sheaves

$$f_{j*}((\mathbb{Z}/n\mathbb{Z})_{Y_j})$$

for finite morphisms $f_j : Y_j \to X$. We can assume that the schemes Y_j are normal. Then

$$\dim Y_j \leq \dim X_j \leq 1.$$

We have already seen that the cohomology groups

$$H^q(Y_j, (\mathbb{Z}/n\mathbb{Z})_{Y_j}) = H^q(X, f_{j*}((\mathbb{Z}/n\mathbb{Z})_{Y_j}))$$

are finite groups.

(b): Let $\mathscr{F}$ be a torsion sheaf on an irreducible reduced curve X, and suppose $\mathscr{F}_{\bar{\eta}} = 0$, where $\bar{\eta}$ is the generic geometric point. If $\mathscr{F}$ is constructible, then the restriction to an open dense subscheme is 0, and thus

$$H^q(X, \mathscr{F}) = 0 \quad \text{for } q > 0.$$

Since an arbitrary torsion sheaf is a direct limit of constructible subsheaves, this extends to arbitrary torsion sheaves $\mathscr{F}$ with $\mathscr{F}_{\bar{\eta}} = 0$.

Let $\mathscr{F}$ now be an arbitrary torsion sheaf on an irreducible reduced curve X. Since the limit of the étale neighborhoods of $\bar{\eta}$ is

$$\operatorname{Spec} K^{\text{sep}} \times_X \operatorname{Spec} K = \operatorname{Spec} K^{\text{sep}},$$

we get

$$R^q\eta_*(\eta^*\mathscr{F})_{\bar{\eta}} = H^q(\operatorname{Spec} K^{\text{sep}}, \pi^*\eta^*\mathscr{F}) = \begin{cases} \mathscr{F}_{\bar{\eta}}, & \text{if } q = 0 \\ 0, & \text{if } q > 0, \end{cases}$$

where $\pi : \operatorname{Spec} K^{\text{sep}} \to \operatorname{Spec} K$ is the natural map. In particular, the map

$$\mathscr{F} \to \eta_*\eta^*\mathscr{F}$$

is an isomorphism at $\bar{\eta}$, and it is sufficient to prove

$$H^q(X, \eta_*\eta^*\mathscr{F}) = 0 \quad \text{for } q \geq 3.$$

It follows from Tsen's theorem that

(1) $H^n(\operatorname{Spec} K, \eta^* \mathscr{F}) = 0$ for $n \geq 2$,

(2) $R^q \eta_* (\eta^* \mathscr{F}) = 0$ for $q \geq 2$.

We know besides that

(3) $H^p(X, R^q \eta_*(\eta^* \mathscr{F})) = 0$ for $p, q > 0$.

Using the Leray spectral sequence

$$H^p(X, R^q \eta_* \eta^* \mathscr{F}) \Rightarrow H^{p+q}(\operatorname{Spec} K, \eta^* \mathscr{F}),$$

we get

$$H^p(X, \eta_* \eta^* \mathscr{F}) = 0 \quad \text{ for } p \geq 3.$$

(c): Let X be an affine irreducible smooth nonempty curve over k. We can assume that $\mathscr{F}$ is a constructible sheaf of $\Lambda = \mathbb{Z}/n\mathbb{Z}$-modules on X with $\operatorname{char} k \nmid n$. We have already shown that the functor $H^p(X, \cdot\,)$ vanishes for $p \geq 3$. In particular, $H^2(X, \cdot\,)$ is right exact.

Since $\mathscr{F}$ is constructible, there is a sheaf

$$\mathscr{G} = \bigoplus_i \widetilde{Y_i} = \operatorname{Hom}_X(\,\cdot\,, Y_i)$$

for some Y_i étale over X and a surjective mapping $\mathscr{G} \to \mathscr{F}$. Then there is a surjective Λ-homomorphism $\mathscr{G}^\Lambda \to \mathscr{F}$. We may assume that Y_i are smooth affine curves over k. It is enough to show

$$H^2(X, \widetilde{Y_i}^\Lambda) = 0$$

for each i. Let $Y = Y_i$ for some i.

Since $\pi : Y \to X$ is étale, Y can be represent by an open dense subscheme of a scheme $\overline{Y}$ finite and normal over X. Then $\overline{Y}$ is also a smooth affine curve over k. Consider the homomorphism

$$\widetilde{Y}^\Lambda \to \pi_* \Lambda_Y$$

which corresponds to

$$s \in \Gamma(Y \times_X Y, \Lambda_Y) = \Gamma(Y, \pi_* \Lambda_Y) = \operatorname{Hom}(\widetilde{Y}, \pi_* \Lambda_Y) = \operatorname{Hom}_\Lambda(\widetilde{Y}^\Lambda, \pi_* \Lambda_Y),$$

where $s|_\Delta = 1$ and $s|_{Y \times_X Y \setminus \Delta} = 0$ (note that Δ is clopen in $Y \times_X Y$).

Note that the homomorphism $\widetilde{Y}^{\Lambda} \to \pi_* \Lambda_Y$ is an isomorphism out of finite point on X, so

$$H^2(X, \widetilde{Y}^{\Lambda}) = H^2(X, \pi_* \Lambda_Y).$$

Also, $\bar{\pi}_* \Lambda_{\overline{Y}} \to \pi_* \Lambda_Y$ is an isomorphism out of finite point on X, so

$$H^2(X, \pi_* \Lambda_Y) = H^2(X, \bar{\pi}^* \Lambda_{\overline{Y}}) = H^2(\overline{Y}, \Lambda_{\overline{Y}}) = 0. \qquad \blacksquare$$

Theorem 16.2. Let X be a finitely generated affine scheme over an separably algebraically closed base field k. Let $\mathscr{F}$ be a torsion sheaf on X with torsion relatively prime to the characteristic of k. Let n be a given natural number, and suppose the stalk $\mathscr{F}_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X with

$$\dim \overline{\{x\}} > n.$$

Then

$$H^p(X, \mathscr{F}) = 0 \quad \text{ for } p > n.$$

Proof. Induction on n. Since $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$ is a universal homeomorphism, $\varphi : \overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \to X$ is a universal homeomorphism too, so $U \mapsto U \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ defines an equivalence of categories

$$\text{Ét}/X \to \text{Ét}/\overline{X}.$$

Hence,

$$H^p(\overline{X}, \varphi^* \mathscr{F}) = H^p(X, \mathscr{F})$$

and we may assume that k is algebraically closed. Since

$$\mathscr{F} = \varinjlim \mathscr{F}_0$$

where $\mathscr{F}_0$ runs over all constructible subsheaves of $\mathscr{F}$,

$$H^p(X, \mathscr{F}) = \varinjlim H^p(X, \mathscr{F}_0)$$

and we may assume that $\mathscr{F}$ is constructible. We then replace X by $\operatorname{Supp} \mathscr{F}$, we can even assume that

$$\dim X \leq n.$$

All sheaves we consider here will be sheaves of $\Lambda = \mathbb{Z}/m\mathbb{Z}$-modules, with $\operatorname{char} k \nmid m$.

Base case: $n = 1$.

Now, we assume $n \geq 2$. By our induction proof we assume WLOG that $\dim X = n \geq 2$. Let $A = k[X_1, \ldots, X_n]$ and let $B = k[X_1, \ldots, X_{n-1}]$. The Noether normalization theorem says that there is a finite mapping

$$\pi : X \to \operatorname{Spec} A.$$

Since π_* is exact,

$$H^p(X, \mathscr{F}) = H^p(\operatorname{Spec} A, \pi_* \mathscr{F}).$$

Hence, we may assume that $X = \operatorname{Spec} A$.

Lemma 16.3. Consider the natural mapping

$$f : X = \operatorname{Spec} B[X_n] \to \operatorname{Spec} B = Y.$$

Let $\bar{y}$ be a geometric point of Y with $\dim \overline{\{y\}} > n - q$. Then

$$(R^q f_* \mathscr{F})_{\bar{y}} = 0.$$

Proof. We distinguish two cases:

Case 1: $\dim \overline{\{y\}} > 0$. Let $\mathfrak{p}$ be the prime ideal of B associated with y. By hypothesis we have

$$\dim \overline{\{y\}} = s = \operatorname{trdeg}_k (B_\mathfrak{p} / \mathfrak{p} B_\mathfrak{p}).$$

We select elements $t_1, \ldots, t_s \in B$ whose images in $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ form a separating transcendence basis of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.

Let K be the separable algebraic closure of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. Then $\widetilde{B} = B \otimes_{k[t_1,\ldots,t_s]} K$ is an affine K-algebra. Since $t_1, \ldots, t_s$ are algebraically independent over k, we have

$$\dim \widetilde{B} = \dim B - s = n - 1 - s \leq n - 2.$$

There is a geometric point

$$\widetilde{y} : \operatorname{Spec} K \to \operatorname{Spec} \widetilde{B}$$

of $\operatorname{Spec} \widetilde{B}$ lying over $\bar{y}$. Since K is separable over $k(t_1, \ldots, t_s)$, $\widetilde{B}$ is a direct limit of étale algebras over B. Therefore the strict Henselization of $\widetilde{B}$ with respect to $\widetilde{y}$ coincides with that of B with respect to $\bar{y}$. Set $S = \operatorname{Spec} \widetilde{B}$, $T = \operatorname{Spec} \widetilde{B}[X_n]$, and $\widetilde{\mathscr{F}}$ the inverse image of $\mathscr{F}$ on T. Consider

$$\widetilde{f} = f \times_Y S : T \to S.$$

Then

$$(R^p f_* \mathscr{F})_{\tilde{y}} = (R^p \widetilde{f}_* \widetilde{\mathscr{F}})_{\tilde{y}} = \varinjlim H^q(U \times_S T, \widetilde{\mathscr{F}}_U).$$

Here $\widetilde{\mathscr{F}}_U$ is the inverse image of $\widetilde{\mathscr{F}}$ on $U \times_S T$, and U runs through the affine neighborhoods of $\bar{y}$.

$$\dim U \times_S T = \dim U + 1 = n - s < n, \quad n - s < n - (n - q) = q.$$

By induction hypothesis, $H^q(U \times_S T, \widetilde{\mathscr{F}}_U)$ vanishes for all U, and hence so does $(R^p f_* \mathscr{F})_{\bar{y}}$.
Case 2: $\dim \overline{\{y\}} = 0$, $q > n \geq 2$. We consider the natural open embedding

$$j : X \to \mathbb{P}^1 \times_k Y.$$

Let

$$\bar{f} : \mathbb{P}^1 \times_k Y \to Y$$

be the projection. Since $f = \bar{f} \circ j$, by Leray spectral sequence, it is enough to show

$$R^r \bar{f}_*(R^s j_* \mathscr{F}) = 0 \quad \text{ for } r + s > n.$$

For $s = 0$: We need to show that $R^r f_!(\mathscr{F}) = 0$ for all $r > 2$. After a base change, we only need to do the case when $X = \mathbb{A}^1$ and $Y = \operatorname{Spec} k$, which is true by (16.1).

For $s > 0$: $\operatorname{Supp} R^s j_*(\,\cdot\,) \subseteq \{\infty\} \times Y \cong Y$. So

$$R^r \bar{f}_*(R^s j_* \mathscr{F}) = 0 \quad \text{ for } r, s > 0.$$

Hence, it is enough then to assume that

$$r = 0, \quad s = q > n.$$

We must show that

$$R^q j_*(\mathscr{F})_{(\infty, a)} = 0$$

for every closed geometric point a of Y. We have

$$R^q j_*(\mathscr{F})_{(\infty, a)} = H^q(\operatorname{Spec} R_t, \mathscr{G})$$

where $t = 1/X_n$, R is the strict Henselization of the local ring $\mathbb{P}^1 \times_k Y$ at the point (∞, a), and $\mathscr{G}$ is the inverse image of $\mathscr{F}$ on $\operatorname{Spec} R_t$. The assertion follows from the following lemma.

Lemma 16.4. Let A be a regular affine k-algebra, $\mathfrak{m}$ a maximal ideal, R the strict Henselization of the local ring $A_{\mathfrak{m}}$, t a regular parameter of R, and $\mathscr{G}$ an étale Λ-sheaf on $S = \operatorname{Spec} R_t$. Suppose

$$\dim R \leq n.$$

Then

$$H^r(S, \mathscr{G}) = 0 \qquad \text{for } r > n.$$

Proof. We can assume that t is contained in $\mathfrak{m}$, that $\mathscr{G}$ is the inverse image of an étale Λ-sheaf $\mathscr{G}_0$ on $\operatorname{Spec} A$. We can further assume that

$$\dim A = \dim A_{\mathfrak{m}} = \dim R \leq n.$$

We consider the system of connected étale neighborhood $\operatorname{Spec} B$ of $\mathfrak{m}$ (more precisely, of an associated geometric point), setting

$$R = \varinjlim B.$$

Let

$$B(t) = B \otimes_{k[t]} k(t).$$

Then $B(t)$ is an affine $k(t)$-algebra with dimension

$$\dim B(t) \leq \dim B - 1 = \dim A - 1 \leq n - 1.$$

We have

$$R_t = R \otimes_{k[t]} k(t) = \varinjlim B(t).$$

Let $\mathscr{G}_B$ be the inverse image of $\mathscr{G}_0$ on

$$Y_B = \operatorname{Spec} B(t).$$

Then

$$H^r(\operatorname{Spec} R_t, \mathscr{G}) = \varinjlim H^r(Y_B, \mathscr{G}_B).$$

It is enough to show that for $r > n$,

$$H^r(Y_B, \mathscr{G}_B)$$

vanishes. We consider the Leray spectral sequence

$$E_2^{pq} = H^p(\operatorname{Spec} k(t), R^q h_*(\mathscr{G}_B)) \Rightarrow H^{p+q}(Y_B, \mathscr{G}_B)$$

associated with the mapping

$$h : Y_B \to \operatorname{Spec} k(t).$$

By Tsen's theorem,

$$H^p(\operatorname{Spec} k(t), \,\cdot\,) = 0$$

for $p > 1$. Let $k(t)^{\mathrm{sep}}$ be the separable algebraic closure of $k(t)$. We consider the geometric point

$$y : \operatorname{Spec} k(t)^{\mathrm{sep}} \to \operatorname{Spec} k(t).$$

It follows from the induction hypothesis that

$$R^q f_*(\mathscr{G}_B)_y = H^q(\operatorname{Spec}(Y_B \otimes_{k(t)} k(t)^{\mathrm{sep}}), \,\cdot\,) = 0$$

for $q > n-1$. This yields $E^{pq} = 0$ for $p+q > n$, and $H^r(Y_B, \mathscr{G}_B) = 0$ for $r > n$, and thus the lemma is proved. ■

17 Base change theorems

Theorem 17.1. Let $f : X \to S$ be a proper morphism of schemes, and let

$$\begin{array}{ccc} X \times_S T & \xrightarrow{f'} & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

be a cartesian diagram. For every torsion sheaf $\mathscr{F}$ on X, the base change homomorphism

$$g^* \mathsf{R} f_* \mathscr{F} \to \mathsf{R} f'_* (g')^* \mathscr{F}$$

is a quasi-isomorphism.

Theorem 17.2. Let $g : T \to S$ be a smooth morphism of schemes, and let

$$\begin{array}{ccc} X \times_S T & \xrightarrow{f'} & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

be a cartesian diagram. For every torsion sheaf $\mathscr{F}$ with orders of sections are relatively prime to the residue characteristics of S on X, the base change homomorphism

$$g^* \mathsf{R} f_* \mathscr{F} \to \mathsf{R} f'_* (g')^* \mathscr{F}$$

is a quasi-isomorphism.

Proposition 17.3. Let $f : X \to S$ be a smooth proper morphism, and let $\mathscr{F}$ be a locally constant torsion sheaf on X with torsion orders invertible on S. Then the specialization homomorphisms

$$(R^p f_* \mathscr{F})_s \to (R^p f_* \mathscr{F})_a$$

are isomorphisms for all p.

Proof. WLOG we may assume that $S = \operatorname{Spec} R$ is the spectrum of a strictly Henselian local ring with residue field k,

$$s : \operatorname{Spec} k \to S$$

is the special point of S, and

$$a : \operatorname{Spec} K \to S$$

is some geometric point.

We consider the diagram

$$\begin{tikzcd} X_a \arrow[r, "j"] \arrow[d, "f_a"] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\ \operatorname{Spec} K \arrow[r, "a"] & S & \operatorname{Spec} k. \arrow[l, "s"'] \end{tikzcd}$$

The restriction mapping

$$(\mathsf{R}f_* \mathscr{F})_s = \mathsf{R}\Gamma(X, \mathscr{F}) \to \mathsf{R}\Gamma(X_a, j^* \mathscr{F}) = (\mathsf{R}f_* \mathscr{F})_a$$

is the specialization homomorphism. To show that this is a quasi-isomorphism, since, by the proper base change theorem,

$$\mathsf{R}\Gamma(X_a, j^* \mathscr{F}) = \mathsf{R}\Gamma(X, \mathsf{R}j_* j^* \mathscr{F}) = \mathsf{R}\Gamma(X_s, i^*(\mathsf{R}j_* j^* \mathscr{F}))$$

and

$$\mathsf{R}\Gamma(X, \mathscr{F}) = \mathsf{R}\Gamma(X_s, i^* \mathscr{F}),$$

it suffices to show that

$$i^* \mathscr{F} \to i^*(\mathsf{R}j_* j^* \mathscr{F})$$

is a quasi-isomorphism. The question is local, so we may assume that $\mathscr{F} = M_X$ is constant. By the smooth base change theorem,

$$i^* \mathsf{R}j_* j^* M = i^* \mathsf{R}j_* f_a^* M = i^* f^* \mathsf{R}a_* M = f_s^* s^* \mathsf{R}a_* M.$$

The assertion follows from

$$R^p a_* M = 0$$

for $p > 0$ and

$$(a_* M)_s = (a_* M)(S) = M(\operatorname{Spec} K) = M. \qquad \blacksquare$$

18 Purity theorems

Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$. We consider only sheaves of Λ-modules and only schemes on which m is invertible.

Let X be a scheme and Y a closed subscheme of X,

$$i : Y \to X$$

the closed immersion and

$$j : U = X \setminus Y \to X$$

the open immersion. Let $\mathscr{F}$ be an étale sheaf on X. We then have the functors $\Gamma_Y(X, \cdot\,)$ and Γ_Y:

$$\begin{aligned} \Gamma_Y(X, \mathscr{F}) &= \{s \in \Gamma(X, \mathscr{F}) \mid s|_U = 0\} = \ker(\mathscr{F}(X) \to \mathscr{F}(U)) = \operatorname{Hom}_\Lambda(i_* i^* \Lambda, \mathscr{F}), \\ \Gamma_Y \mathscr{F} &= \ker(\mathscr{F} \to j_* j^* \mathscr{F}) = \mathscr{H}\!om_\Lambda(i_* i^* \Lambda, \mathscr{F}). \end{aligned}$$

For the derived functors of $\Gamma_Y(X, \cdot\,)$ we also write

$$H^i_Y(X, \mathscr{F}) = R^i \Gamma_Y(X, \mathscr{F}).$$

Note that for an injective sheaf $\mathscr{I}$,

$$\mathscr{H}\!om_\Lambda(\Lambda, \mathscr{I}) = \mathscr{I} \to j_* j^* \mathscr{I} = \mathscr{H}\!om_\Lambda(j_! j^* \Lambda, \mathscr{I})$$

is surjective, so we have the usual long exact sequences

$$\begin{gathered} \cdots \to H^n_Y(X, \mathscr{F}) \to H^n(X, \mathscr{F}) \to H^n(U, \mathscr{F}) \to H^{n+1}_Y(X, \mathscr{F}) \cdots \\ 0 \to \Gamma_Y \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \to R^1 \Gamma_Y \mathscr{F} \to 0 \\ R^n j_* j^* \mathscr{F} \cong R^{n+1} \Gamma_Y \mathscr{F} \quad \text{for } n \geq 1. \end{gathered}$$

Theorem 18.1. Let $X \to S$ and $Y \to S$ be smooth schemes over a base scheme S, with $j : Y \to X$ a closed immersion over S. Suppose Y has constant codimension d in all geometric fibers of $X \to S$. Let $\mathscr{F}$ be a locally constant constructible sheaf on X. Then

$$R^n\Gamma_Y\mathscr{F} = \begin{cases} 0, & \text{if } n \neq 2d \\ (R^{2d}\Gamma_Y\Lambda_X) \otimes \mathscr{F}, & \text{if } n = 2d. \end{cases}$$

The sheaf $R^{2d}\Gamma_Y\Lambda_X$ is concentrated on Y, locally constant there, and locally isomorphic to Λ_Y. This $R^{2d}\Gamma_Y\Lambda_X$ is invariant under base change of S.

19 Comparison theorems between étale cohomology and singular cohomology

For finitely generated schemes over $\mathbb{C}$. It is known that there is a naturally constructed functor from the category of these schemes to the category of complex spaces. There is a natural morphism of local ringed spaces

$$X_{\text{an}} \to X$$

compatible with the functor, that induces an isomorphism

$$\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X_{\text{an}}, Y)$$

for every complex space Y (the morphisms here are in the category of local ringed spaces). In particular, we identify the set

$$X(\mathbb{C}) = \operatorname{Hom}_{\operatorname{Spec}\mathbb{C}}(\operatorname{Spec}\mathbb{C}, X)$$

of geometric points of X having values in $\mathbb{C}$ (the set of closed points of X) with the points of X_{an}.

Let $p : U \to X$ be an étale scheme over the scheme X. Then $p_{\text{an}} : U_{\text{an}} \to X_{\text{an}}$ is a local isomorphism. Now a scheme is connected in the Zariski topology if and only if the associated analytic space is connected. Hence the connected components of U that are mapped isomorphically onto X by p correspond bijectively to the clopen subsets of U_{an} that are mapped isomorphically onto X_{an}. Thus

$$\operatorname{Hom}_X(X, U) \cong \operatorname{Hom}_{X_{\text{an}}}(X_{\text{an}}, U_{\text{an}}).$$

Since $X \to X_{\text{an}}$ is compatible with cartesian products, we can thus conclude:

Lemma 19.1. Let $U \to X$ and $V \to X$ be étale morphisms. Then the mapping

$$\mathrm{Hom}_X(U,V) \to \mathrm{Hom}_{X_{\mathrm{an}}}(U_{\mathrm{an}}, V_{\mathrm{an}})$$

is an isomorphism.

For a complex space Y, denote by $\textsf{Ét}\,(Y)$ the category of complex spaces

$$q : U \to Y$$

over Y for which q is a local isomorphism. Then the functor

$$\begin{array}{ccc} \textsf{Ét}\,(X) & \to & \textsf{Ét}\,(X_{\mathrm{an}}) \\ (U \to X) & \mapsto & (U_{\mathrm{an}} \to X_{\mathrm{an}}) \end{array}$$

is a fully faithful embedding of categories.

Let Y be an analytic space. We shall embed the category of sheaves on Y in the category of contravariant functors

$$\textsf{Ét}\,(Y) \to \textsf{Sets}\,.$$

Let $p : U \to Y$ be a local isomorphism. We set

$$\mathscr{F}(U) := q^*\mathscr{F}(U).$$

For a mapping

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & V \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

in $\textsf{Ét}\,(Y)$, the map $\mathscr{F}(\varphi) : \mathscr{F}(V) \to \mathscr{F}(U)$ is the map induced by φ,

$$\Gamma(V, q^*\mathscr{F}) \to \Gamma(U, p^*\mathscr{F}).$$

To a space $p : U \to Y$ in $\textsf{Ét}\,(Y)$ we can assign a sheaf $\widetilde{U}$ on Y,

$$\widetilde{U}(V) = \mathrm{Hom}_Y(V, U)$$

for V open in Y. There is then a natural isomorphism

$$\mathscr{F}(U) = p^*\mathscr{F}(U) \xrightarrow{\sim} \mathrm{Hom}(\widetilde{U}, \mathscr{F})$$

for every sheaf $\mathscr{F}$ on Y.

Lemma 19.2. Let $\mathscr{F}$ be a sheaf on the analyitc space Y. Then the following "sheaf properties" hold for the functor extension of $\mathscr{G}$ to $\textsf{Ét}\,(Z)$:

(1) Let $U_i \to Y$ be in $\textsf{Ét}\,(Y)$. Then

$$\mathscr{F}\left(\bigsqcup_{i\in I} U_i\right) = \prod_{i\in I} \mathscr{F}(U_i)$$

(2) Let $U \to V$ be a surjective mapping in $\textsf{Ét}\,(Y)$. Then the induced sequence

$$\mathscr{F}(V) \to \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_V U)$$

is exact.

Proof. (1) is trivial. (2) follows from $\widetilde{U} \to \widetilde{V}$ is a surjective mapping of sheaves, and we have $\widetilde{U \times_V U} = \widetilde{U} \times_{\widetilde{V}} \widetilde{U}$. Hence

$$\mathrm{Hom}(\widetilde{V}, \mathscr{F}) \to \mathrm{Hom}(\widetilde{U}, \mathscr{F}) \rightrightarrows \mathrm{Hom}(\widetilde{U} \times_{\widetilde{V}} \widetilde{U}, \mathscr{F})$$

is exact. ■

Now let X be a scheme over $\mathbb{C}$. There is then a "direct image functor" from the category of sheaves on X_{an} to the category of étale sheaves,

$$\mathscr{F} \mapsto \mathscr{F}_{\mathrm{al}} :$$

for $U \to X$ étale,

$$\mathscr{F}_{\mathrm{al}}(U) = \mathscr{F}(U_{\mathrm{an}}),$$

and for mapping $\varphi : U \to V$ in $\textsf{Ét}\,(X)$,

$$\mathscr{F}(\varphi) = \mathscr{F}(\varphi_{\mathrm{an}}) : \mathscr{F}_{\mathrm{al}}(V) \to \mathscr{F}_{\mathrm{al}}(U).$$

The functor is an étale sheaf by (19.2). Similarly, every mapping of sheaves on X_{an} defines an natural mapping of the associatd étale sheaves on $\textsf{Ét}\,(X)$.

Proposition 19.3. The functor $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{al}}$ has a left adjoint functor, the "inverse image functor"

$$\mathscr{G} \mapsto \mathscr{G}_{\mathrm{an}}$$

from the $\textsf{Ét}\,(X)$ to $\textsf{Sh}(X_{\mathrm{an}})$, i.e.,

$$\mathrm{Hom}(\mathscr{G}_{\mathrm{an}}, \mathscr{F}) = \mathrm{Hom}(\mathscr{G}, \mathscr{F}_{\mathrm{al}})$$

for sheaves $\mathscr{F}$ on X_{an}.

Proof. Let $\mathscr{G} \in \mathsf{Sh}(X_{\text{ét}})$. It is enough to prove that the covariant functor

$$\mathscr{F} \mapsto \operatorname{Hom}(\mathscr{G}, \mathscr{F}_{\text{al}})$$

from $\mathsf{Sh}(X_{\text{an}})$ to Sets is representable by a sheaf $\mathscr{G}_{\text{an}}$ on X_{an}.

Suppose $\mathscr{G}$ is representable. There is thus an étale scheme $Y \to X$ with $\widetilde{Y} = \mathscr{G}$. Then

$$\operatorname{Hom}(\widetilde{Y}, \mathscr{F}_{\text{al}}) = \mathscr{F}_{\text{al}}(Y) = \mathscr{F}(Y_{\text{an}}) = \operatorname{Hom}(\widetilde{Y_{\text{an}}}, \mathscr{F}).$$

Thus $\widetilde{Y_{\text{an}}}$ is the desired sheaf.

For an arbitrary étale sheaf $\mathscr{G}$, it is a direct (unfiltered) limit of representable sheaves $\mathscr{G}_\alpha = \widetilde{Y_\alpha}$,

$$\mathscr{G} = \varinjlim_\alpha \widetilde{Y_\alpha}.$$

Then the sheaf

$$\mathscr{G}_{\text{an}} = \varinjlim_\alpha \widetilde{(Y_\alpha)_{\text{an}}}$$

has the desired property. ■

Proposition 19.4. The functor $\mathscr{G} \mapsto \mathscr{G}_{\text{an}}$ has the following properties:

(1) For U étale over X, there is a natural mapping

$$\begin{array}{ccc} \mathscr{G}(U) & \to & \mathscr{G}_{\text{an}}(U_{\text{an}}) \\ s & \mapsto & s_{\text{an}}. \end{array}$$

(2) The functor is compatible with arbitrary direct limits.

(3) It is compatible with finite inverse limits.

(4) It is compatible with formation of inverse images, i.e., for a morphism $f : X \to Y$ of schemes over $\mathbb{C}$ and $\mathscr{F}$ an étale sheaf on Y,

$$f_{\text{an}}^* \mathscr{G}_{\text{an}} = (f^* \mathscr{G})_{\text{an}}.$$

(5) If $x \in X(\mathbb{C})$ is a geometric point of X, then the stalks $\mathscr{G}_x$ and $(\mathscr{G}_{\text{an}})_x$ are canonically isomorphic.

Proof. (2):

$$\begin{aligned}\operatorname{Hom}((\varinjlim \mathscr{G}_\alpha)_{\text{an}}, \mathscr{F}) &= \operatorname{Hom}(\varinjlim \mathscr{G}_\alpha, \mathscr{F}_{\text{al}}) = \varprojlim \operatorname{Hom}(\mathscr{G}_\alpha, \mathscr{F}_{\text{al}}) \\ &= \varprojlim \operatorname{Hom}(\mathscr{G}_{\alpha\text{an}}, \mathscr{F}) = \operatorname{Hom}(\varinjlim(\mathscr{G}_{\alpha\text{an}}), \mathscr{F})\end{aligned}$$

(4):

$$\begin{aligned}\mathrm{Hom}((f^*\mathscr{G})_{\mathrm{an}},\mathscr{F}) &= \mathrm{Hom}(f^*\mathscr{G},\mathscr{F}_{\mathrm{al}}) = \mathrm{Hom}(\mathscr{G},f_*\mathscr{F}_{\mathrm{al}})\\ &= \mathrm{Hom}(\mathscr{G},((f_{\mathrm{an}})_*\mathscr{F})_{\mathrm{al}}) = \mathrm{Hom}(\mathscr{G}_{\mathrm{an}},(f_{\mathrm{an}})_*\mathscr{F}) = \mathrm{Hom}(f_{\mathrm{an}}^*\mathscr{G}_{\mathrm{an}},\mathscr{F}).\end{aligned}$$

(5): Consider the map $f : x \to X$, by (4) we have

$$(\mathscr{G}_{\mathrm{an}})_x = f_{\mathrm{an}}^*(\mathscr{G}_{\mathrm{an}})(x_{\mathrm{an}}) = (f^*\mathscr{G})_{\mathrm{an}}(x_{\mathrm{an}}) = f^*\mathscr{G}(x) = \mathscr{G}_x.$$

(3) follows from (5). ■

Proposition 19.5. Let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves on a scheme X over $\mathbb{C}$. Then the mapping

$$\mathrm{Hom}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}(\mathscr{F}_{\mathrm{an}},\mathscr{G}_{\mathrm{an}})$$

is injective. It is actually bijective if $\mathscr{G}$ is constructible. In particular, for an étale scheme $U \to X$ over X and a constructible sheaf $\mathscr{G}$,

$$\mathscr{G}(U) \cong \mathscr{G}_{\mathrm{an}}(U_{\mathrm{an}}).$$

Proof. The injectivity follows from (19.4(5)). The bijectivity for a constructible sheaf $\mathscr{G}$: It suffices to assume that $\mathscr{F}$ is representible, and thus we have to show

$$\mathscr{G}(U) \cong \mathscr{G}_{\mathrm{an}}(U_{\mathrm{an}}).$$

Say for instance $U = X$. The bijectivity is clear in any case if $\mathscr{G}$ is also representable. In the general case, $\mathscr{G}$ is at least representible on a Zariski-open dense set. It follows by noetherian induction that for two sections $s, t \in \mathscr{G}_{\mathrm{an}}(X_{\mathrm{an}})$ the set of points $x \in X_{\mathrm{an}}$ at which the germs s_x and t_x agree is constructible in the Zariski topology and hence actually Zariski-open. By (19.4(5)), then, for every $s \in \mathscr{G}_{\mathrm{an}}(X_{\mathrm{an}})$ and every geometric point $x \in X(\mathbb{C})$, there is a Zariski-open neighborhood U and an element $t \in \mathscr{G}(U)$ with $t_{\mathrm{an}} = s|_{U_{\mathrm{an}}}$. From this the assertion follows easily. ■

Consider the derived category $D(X)$ for the category of all étale sheaves of abelian groups on the scheme X and the derived category $D(X_{\mathrm{an}})$ for the category of all sheaves of abelian groups on X_{an}. Since the functor $\mathscr{G} \mapsto \mathscr{G}_{\mathrm{an}}$ is exact, it automatically extends to a cohomological functor

$$\begin{array}{ccc} D(X) & \to & D(X_{\mathrm{an}}) \\ \mathscr{F}^\bullet & \mapsto & \mathscr{F}^\bullet_{\mathrm{an}}. \end{array}$$

We now consider a morphism $f : X \to S$ of schemes over $\mathbb{C}$ and a sheaf $\mathscr{F}$ on X. There is a natural mapping

$$f_*\mathscr{F} \to ((f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}})_{\mathrm{al}}$$

given as follows:

Let $U \to S$ be étale. We have a natural mapping

$$f_*\mathscr{F}(U) = \mathscr{F}(U \times_S X) \to \mathscr{F}_{\mathrm{an}}(U_{\mathrm{an}} \times_{S_{\mathrm{an}}} X_{\mathrm{an}}) = (f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}}(U_{\mathrm{an}}) = ((f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}})_{\mathrm{al}}(U).$$

The adjoint mapping is a mapping

$$(f_*\mathscr{F})_{\mathrm{an}} \to (f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}}.$$

It extends immediately to complexes, and by injective resolutions it yields a natural transformation of functors $D_+(X) \to D(Y_{\mathrm{an}})$.

Let $\mathscr{F}^\bullet$ be a complex in $D_+(X)$, $\mathscr{F}^\bullet \to \mathscr{I}^\bullet$ an injective resolution of $\mathscr{F}^\bullet$, and $\mathscr{F}^\bullet_{\mathrm{an}} \to \mathscr{J}^\bullet$ an injective resolution of $\mathscr{F}^\bullet_{\mathrm{an}}$. Since $\mathscr{F}^\bullet_{\mathrm{an}} \to (\mathscr{I}^\bullet)_{\mathrm{an}}$ is also a quasi-isomorphism, there is a homotopy-commutative diagram

$$\begin{tikzcd} \mathscr{F}^\bullet_{\mathrm{an}} \arrow[r] \arrow[dr] & \mathscr{J}^\bullet \\ & (\mathscr{I}^\bullet)_{\mathrm{an}}. \arrow[u, dashed] \end{tikzcd}$$

Using the mapping constructed before, we get a homomorphism

$$(\mathsf{R}f_*\mathscr{F}^\bullet)_{\mathrm{an}} = (f_*\mathscr{I}^\bullet)_{\mathrm{an}} \to (\mathscr{F}_{\mathrm{an}})_*\mathscr{J}^\bullet = \mathsf{R}(f_{\mathrm{an}})_*\mathscr{F}^\bullet_{\mathrm{an}}.$$

Passage to the cohomology sheaves yeilds

$$(R^n f_*\mathscr{F})_{\mathrm{an}} \to R^n(f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}}.$$

In the case when $S = \operatorname{Spec}\mathbb{C}$, the mappings here constructed are equivalent to mappings

$$\mathsf{R}\Gamma(X, \mathscr{F}^\bullet) \to \mathsf{R}\Gamma(X_{\mathrm{an}}, \mathscr{F}^\bullet_{\mathrm{an}}),$$
$$H^n(X, \mathscr{F}) \to H^n(X_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}}).$$

Let

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

be a compactification of f and $\mathscr{F}$ a sheaf on X. It is easy to see that $(j_!\mathscr{F})_{\mathrm{an}}$ is the "extension by zero" of $\mathscr{F}_{\mathrm{an}}$ to all of $\bar{X}_{\mathrm{an}}$,

$$(j_!\mathscr{F})_{\mathrm{an}} = (j_{\mathrm{an}})_!\mathscr{F}_{\mathrm{an}}.$$

It is known that the higher direct images $R^n(\bar{f}_{\mathrm{an}})_*((j_{\mathrm{an}})_!\mathscr{F}_{\mathrm{an}})$ are canonically isomorphic to the derived functors $R^n(f_{\mathrm{an}})_!\mathscr{F}_{\mathrm{an}}$ of direct image functor $(f_{\mathrm{an}})_!\mathscr{F}_{\mathrm{an}}$ with compact support. Combining this with the transformations previously constructed, we thus get a natural transformation of functors

$$D_+(X, \mathrm{tor}) \to D(S_{\mathrm{an}})$$

namely

$$(\mathsf{R}f_!\mathscr{F}^\bullet)_{\mathrm{an}} \to \mathsf{R}(f_{\mathrm{an}})_!\mathscr{F}^\bullet_{\mathrm{an}},$$

from this again homomorphisms

$$(R^n f_!\mathscr{F})_{\mathrm{an}} \to R^n(f_{\mathrm{an}})_!\mathscr{F}_{\mathrm{an}}$$

and, in the case $S = \operatorname{Spec}\mathbb{C}$,

$$\mathsf{R}\Gamma_c(X, \mathscr{F}^\bullet) \to \mathsf{R}\Gamma_c(X_{\mathrm{an}}, \mathscr{F}^\bullet_{\mathrm{an}}),$$
$$H^n_c(X, \mathscr{F}) \to H^n_c(X_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}}).$$

Theorem 19.6 (Comparison Theorem)**.** Let $f : X \to S$ be a morphism of schemes over $\mathbb{C}$ and $\mathscr{F}^\bullet$ a complex in $D_+(X, \mathrm{tor})$.

(1) The mapping

$$(\mathsf{R}f_!\mathscr{F}^\bullet)_{\mathrm{an}} \to \mathsf{R}(f_{\mathrm{an}})_!\mathscr{F}^\bullet$$

is an isomorphism in $D(X_{\mathrm{an}})$. In particular,

$$\mathsf{R}\Gamma_c(X, \mathscr{F}^\bullet) \to \mathsf{R}\Gamma_c(X_{\mathrm{an}}, \mathscr{F}^\bullet_{\mathrm{an}}),$$
$$H^n_c(X, \mathscr{F}) \to H^n_c(X_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}}).$$

are isomorphisms.

(2) Suppose in addition that the complex $\mathscr{F}^\bullet$ has constructible cohomology sheaves. Then the mapping

$$(\mathsf{R}f_*\mathscr{F}^\bullet)_{\mathrm{an}} \to \mathsf{R}(f_{\mathrm{an}})_*\mathscr{F}^\bullet$$

is an isomorphism.

Proof. (1) Consider a compatification of f,

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & S. \end{tikzcd}$$

Let $S = \bigcup S_i$ with S_i open affine, for each i, take X_i open affine in X such that $X_i \subseteq f^{-1}(S_i)$. Then $U = \bigcap_i X_i$ is affine since X is separated. Let $j : U \to X$ and $i : Z \to X$ such that $X = U \sqcup Z$, and let $k : B = \bar{X} \setminus U \to \bar{X}$. Then we have two compatifications,

$$\begin{tikzcd} U \arrow[r, "\varphi\circ j"] \arrow[dr, "f\circ j"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "\varphi\circ i"] \arrow[dr, "f\circ i"'] & B \arrow[d, "\bar{f}|_B"] \\ & S. \end{tikzcd}$$

Apply $\mathsf{R}\bar{f}_*$ to the exact sequence

$$0 \to (\varphi \circ j)_! j^* \mathscr{F}^\bullet \to \varphi_! \mathscr{F}^\bullet \to k_*(\varphi \circ i)_! i^* \mathscr{F}^\bullet \to 0,$$

we get the triangle

$$\begin{tikzcd} & \mathsf{R}(f\circ i)_! i^*\mathscr{F}^\bullet \arrow[dl] & \\ \mathsf{R}(f \circ j)_! j^* \mathscr{F}^\bullet \arrow[rr] & & \mathsf{R}f_!\mathscr{F}^\bullet. \arrow[ul] \end{tikzcd}$$

Using this and noetherian induction, we may assume that $X \to S$ is affine. By base change theorem, it is enough to prove the case when $X \to S = \operatorname{Spec} \mathbb{C}$ is an affine scheme over $\mathbb{C}$. There is a factorization of f into affine mappings

$$f_i : X_i \to X_{i+1}, \quad f = f_n \circ \cdots \circ f_1$$

with $\dim X_n / X_{n+1} \leq 1$. Then it suffices to prove (1) for the individual mappings f_n. Again, by base change theorem, it is enough to prove that

$$\mathsf{R}\Gamma_c(X, \mathscr{F}^\bullet) \to \mathsf{R}\Gamma_c(X_{\mathrm{an}}, \mathscr{F}^\bullet_{\mathrm{an}})$$

is an isomorphism when X is an affine scheme of dimension $\dim X \leq 1$, or, equivalently,

$$H^n_c(X, \mathscr{F}) \cong H^n_c(X_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}})$$

for all torsion sheaf $\mathscr{F}$ on X. Note that X can be embedded as an open subscheme of a projective scheme $\bar{X}$ of dimension $\dim \bar{X} \leq 1$, and we need to prove that

$$H^n(\bar{X}, \mathscr{F}) \to H^n(\bar{X}_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}})$$

for all torsion sheaf $\mathscr{F}$ on $\bar{X}$. Let $\pi : \widetilde{X} \to \bar{X}$ be a normalization, then using the adjunction mapping

$$\mathscr{F} \to \pi^* \pi_* \mathscr{F}$$

we may assume that X is a projective smooth curve. Since $\mathscr{F}$ is direct limit of its constructible subsheaves, we may assume that $\mathscr{F}$ is constructible. Since $\mathscr{F}$ is a factor sheaf of a finite direct sum of sheaves of the type $\widetilde{Y}^{\Lambda}$, where $\Lambda = \mathbb{Z}/m\mathbb{Z}$. Using short four lemma argument, we may assume that $\mathscr{F} = \Lambda_X$.

Since $\mathbb{C}$ contains the m-th roots of unity, $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to the sheaf μ_m of m-th roots of unity, and thus to the kernel of

$$\begin{array}{ccc} \mathcal{O}_X^* & \to & \mathcal{O}_X^* \\ a & \mapsto & a^m. \end{array}$$

For a locally isomorphic mapping $q : Z \to X_{\mathrm{an}}$, we have $q^*\mathcal{O}_{X_{\mathrm{an}}} = \mathcal{O}_Z$. Thus

$$\mathcal{O}_{X_{\mathrm{an}}}(Z) = \mathcal{O}_Z(Z).$$

Let $U \to X$ be étale. Then there is a natural mapping

$$\mathcal{O}_X^*(U) = \mathcal{O}_U^*(U) \to \mathcal{O}_{U_{\mathrm{an}}}^*(U_{\mathrm{an}}) = \mathcal{O}_{X_{\mathrm{an}}}^*(U_{\mathrm{an}}) = (\mathcal{O}_{X_{\mathrm{an}}}^*)_{\mathrm{al}}(U).$$

Thus we have constructed a homomorphism

$$\mathcal{O}_X^* \to (\mathcal{O}_{X_{\mathrm{an}}}^*)_{\mathrm{al}}.$$

This homomorphism induces homomorphisms

$$H^n(X, \mathcal{O}_X^*) \to H^n(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}^*).$$

By the Kummer sequence

$$0 \to \mu_m \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0,$$

it suffices to prove that these homomorphisms are isomorphisms.

$n \geq 2$: $H^n(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}^*) = H^n(X, \mathcal{O}_X^*) = 0$

$n = 0$: $H^0(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}) = H^0(X, \mathcal{O}^*_X) = \mathbb{C}^\times$

$n = 1$: Using Čech cohomology, one shows that the homomorphism

$$H^1(X, \mathcal{O}^*_X) \to H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}})$$

is equivalent to the natural homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathrm{an}}).$$

This is an isomorphism from the GAGA theorems.

(2) Let

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & S. \end{tikzcd}$$

be a compatification of f. By (1),

$$(\mathsf{R}\bar{f}_*\mathscr{F}^\bullet)_{\mathrm{an}} \to \mathsf{R}(\bar{f}_{\mathrm{an}})_*\mathscr{F}^\bullet$$

is an isomorphism. Since $\mathsf{R}f_* = \mathsf{R}\bar{f}_* \circ \mathsf{R}\varphi$, so we may assume that $X \to S$ is an open immersion. Let $\pi : \widetilde{S} \to S$ be a resolution of singularity such that $\widetilde{S} \setminus \widetilde{X}$ is a smooth divisor. Since the kernel and the cokernel of

$$\pi^*\mathsf{R}f_*\mathscr{F}^\bullet \to \mathsf{R}f'_*\pi|^*_{\widetilde{X}}\mathscr{F}^\bullet$$

are supported in $\widetilde{S} \setminus \widetilde{X}$, so by induction we may assume that S is smooth and $S \setminus X$ is a smooth divisor, and we need to prove that

$$(R^n f_*\mathscr{F})_{\mathrm{an}} \to R^n(f_{\mathrm{an}})_*\mathscr{F}_{\mathrm{an}}$$

is an isomorphism. Again, using short four lemma argument, we may assume that $\mathscr{F} = \Lambda_X = (\mathbb{Z}/m\mathbb{Z})_X$. Then the result follows from the following theorem ■

Theorem 19.7. Let X be a smooth variety with $i : Z \to X$ be a closed smooth subvariety of codimension c in X at each point. Let $j : U = X \setminus Z \to X$ be the immersion. Then

$$R^r j_*\Lambda = \begin{cases} \Lambda, & \text{if } r = 0 \\ i_*\Lambda(-c), & \text{if } r = 2c-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The exact sequence of the pair (X, U) is

$$\cdots \to H^r_Z(X, \Lambda) \to H^r(X, \Lambda) \to H^r(U, \Lambda) \to \cdots .$$

For any $V \to X$ étale we get a similar sequence with U and Z replaced with $U_V = U \times_X V$ and $Z_V = Z \times_X V$. When we vary V, this becomes an exact sequence of presheaves, which remains exact after we apply the sheafification functor. Now the sheaf associated with $V \mapsto H^r(V, \Lambda)$ is zero for $r > 0$ and Λ for $r = 0$, and the sheaf associated with $V \mapsto H^r(U_V, \Lambda)$ is $R^r j_* \Lambda$. Finally, the sheaf associated with $V \mapsto H^r_{Z_V}(V, \Lambda)$ is $R^r \mathbf{\Gamma}_Z(X, \Lambda)$. So it suffices to show that

$$R^r \mathbf{\Gamma}_Z(X, \Lambda) = \begin{cases} 0, & \text{if } r \neq 2c \\ \Lambda(-c), & \text{if } r = 2c. \end{cases}$$

For $P \in Z$, we have $\operatorname{codim}_{T_P X} T_P Z = c$. Let $f_1, \ldots, f_n$ defined on V near P such that $\{df_i\}_{i=1}^{n-c}$ form a basis of $T^*_p Z$ and $\{df_i\}_{i=1}^{n}$ form a basis of $T^*_p X$. Consider the map

$$\begin{array}{rccc} \alpha : & V & \to & \mathbb{A}^n \\ & Q & \mapsto & (f_1(Q), \ldots, f_n(Q)). \end{array}$$

Then α is étale at P and $\alpha|_Z : Z \cap V \to \mathbb{A}^{n-c}$ is étale at P. So we may assume that $X = \mathbb{A}^n$ and $Z = \mathbb{A}^{n-c}$. ■

20 Poincaré duality

For every smooth compactifiable morphism $f : X \to Y$ with constant fiber dimension d, we will construct a homomorphism

$$R^{2d} f_!(\mu_n^{\otimes d}) \to \Lambda,$$

the relative trace mapping.

We first construct the case of an etalé mapping $f : X \to Y$. For every sheaf $\mathscr{F}$ on $Y_{\text{ét}}$ there is an adjunction mapping

$$S_{X/Y} : f_! f^* \mathscr{F} \to \mathscr{F},$$

which for every $\mathscr{G}$ on X induces an isomorphism

$$\begin{tikzcd} \mathrm{Hom}(\mathscr{G}, f^*\mathscr{F}) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Hom}(f_!\mathscr{G}, \mathscr{F}) \\ & \mathrm{Hom}(f_!\mathscr{F}, f_!f^*\mathscr{F}) \arrow[ur] & \end{tikzcd}$$

This trace mapping induces a natural mapping

$$H^p_c(X, f^*\mathscr{F}) = H^p_c(Y, f_!f^*\mathscr{F}) \to H^p_c(Y, \mathscr{F}).$$

Let $f : X \to Y$ be quasi-finite and flat, then $\mathcal{O}_{X,\bar{x}}$ is a finitely generated free $\mathcal{O}_{Y,\overline{f(x)}}$-module. We denote by $n(x)$ the rank of $\mathcal{O}_{X,\bar{x}}$ over $\mathcal{O}_{Y,\overline{f(x)}}$.

Lemma 20.1. Let $f : X \to Y$ be a quasi-finite flat morphism, $\mathscr{F}$ a sheaf on Y. There is a uniquely determined trace mapping

$$S_{X/Y} : f_!f^*\mathscr{F} \to \mathscr{F}$$

defined as follows on the stalk of a geometric point $\bar{y}$ of Y:

$$\begin{array}{rcccc} (f_!f^*\mathscr{F})_{\bar{x}} = & \bigoplus\limits_{x \in f^{-1}(y)} \mathscr{F}_{\bar{x}} & \to & \mathscr{F}_{\bar{x}} \\ & \oplus s_x & \mapsto & \sum n(x) s_x \end{array}$$

If f is étale, then this mapping agrees with the adjunction mapping.

Proof. Let

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

be a compactification of f. Since $\bar{f}$ is quasi-finite and proper, $\bar{f}$ is finite. ■

Let $f : \widetilde{X} \to X$ be a finite flat morphism. We denote the norm mapping by

$$N : f_*\mathcal{O}^*_{\widetilde{X}} \to \mathcal{O}^*_X.$$

Then the following diagram is exact and commutative:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & f_*\mu_n \arrow[r] \arrow[d] & f_*\mathcal{O}^*_{\widetilde{X}} \arrow[r] \arrow[d] & f_*\mathcal{O}^*_{\widetilde{X}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mu_n \arrow[r] & \mathcal{O}^*_X \arrow[r] & \mathcal{O}^*_X \arrow[r] & 0. \end{tikzcd}$$

From this we get a ommutative diagram of connecting homomorphisms

$$\begin{array}{ccc}
\operatorname{Pic}(\widetilde{X}) & \xrightarrow{\text{cl}} & H^2(\widetilde{X}, \mu_n) \\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle S} \\
\operatorname{Pic}(X) & \xrightarrow{\text{cl}} & H^2(X, \mu_n).
\end{array}$$

Lemma 20.2. Let $f : \widetilde{X} \to X$ be a finite flat morphism and $\mathscr{L}$ a line bundle on X. If again

$$S_{\widetilde{X}/X} : H^2(\widetilde{X}, \mu_n) \to H^2(X, \mu_n)$$

denotes the mapping induced by the trace mapping, then the norm $N(\mathscr{L})$ satisfies:

$$\text{cl}(N(\mathscr{L})) = S_{\widetilde{X}/X}(\text{cl}(\mathscr{L})).$$

We first construct a trace mapping for smooth curves $f : X \to \operatorname{Spec} k$ over an algebraically closed base field. Using the Kummer sequence for a smooth, irreducible, projective curve, we get an isomorphism

$$\operatorname{Pic}(X)/n \operatorname{Pic}(X) \cong H^2(X, \mu_n).$$

But $\operatorname{Pic}(X)/n \operatorname{Pic}(X)$ is canonically isomorphic to $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Thus we have obtained a natural isomorphism

$$\begin{array}{rccc}
S_{X/k}: & H^2(X, \mu_n) & \xrightarrow{\sim} & \Lambda \\
& \text{cl}(\mathscr{L}) & \mapsto & \deg \mathscr{L}.
\end{array}$$

If $X \neq \varnothing$ is irreducible but not complete, then X can be embedded in a unique smooth irreducible projective curve

$$j : X \to \bar{X}.$$

Since the complement of X in $\bar{X}$ is finite, j induces an isomorphism

$$H_c^2(X, \mu_n) \xrightarrow{\sim} H^2(\bar{X}, \mu_n).$$

Composing with $S_{\bar{X}/k}$, we get a trace mapping

$$S_{X/k} : H_c^2(X, \mu_n) \xrightarrow{\sim} \Lambda.$$

If X is not connected, then we get

$$S_{X/k} : H_c^2(X, \mu_n) \to \Lambda$$

by adding the trace mappings on the connected components.

In this case of curves, the mapping $H^2_c(X,\mu_n) \to \Lambda$ that we have constructed is equivalent to a mapping

$$S_{X/k} : R^2 f_! \mu_n \to \Lambda.$$

Definition 20.3. A finitely generated flat morphism $f : X \to Y$ of schemes will be called of type S if there is a natural number d such that all the nonempty geometric fibers are nonsingular and pure d-dimensional.

In particular, morphisms of type S are smooth. The fibers do not have to be connected. Étale morphisms are of type S.

The number $d = d(X/S)$ is the relative dimension of X over S.

Notation: If $f : X \to S$ is a morphism of schemes and d the largest dimension of a geometric fiber, we set

$$\mathscr{T}_{X/S} = \mu_n^{\otimes d}.$$

Let $g : X \to T$, $h : T \to S$, $f = h \circ g : X \to S$ be morphisms of type S. The vanishing theorem says that

$$R^p g_! \mathscr{F} = 0 \quad \text{if } p > 2d(X/T),$$
$$R^p h_! \mathscr{F} = 0 \quad \text{if } p > 2d(T/S).$$

The Leray spectral sequence therefore induces natural isomorphisms

$$R^{2(d(X/T)+d(T/S))} f_! \mathscr{F} \cong R^{2d(T/S)} h_! (R^{2d(X/T)} g_! \mathscr{F})$$

for every torsion sheaf $\mathscr{F}$ on X.

We also note that for a sheaf $\mathscr{F}$ on X and a locally constant locally free sheaf $\mathscr{G}$ on T, there is a natural isomorphism

$$R^p g_! (\mathscr{F} \otimes g^* \mathscr{G}) \cong R^p g_! \mathscr{F} \otimes \mathscr{G}.$$

Now suppose we have homomorphisms

$$\alpha : R^{2d(X/T)} g_! \mathscr{T}_{X/T} \to \Lambda$$

and

$$\beta : R^{2d(T/S)} h_! \mathscr{T}_{T/S} \to \Lambda.$$

We have

$$\mathscr{T}_{X/S} = \mathscr{T}_{X/T} \otimes g^* \mathscr{T}_{T/S}.$$

Then we have mappings

$$\begin{aligned}
R^{2d(X/S)} f_! \mathscr{T}_{X/S} &\cong R^{2d(T/S)} h_! (R^{2d(X/T)}) g_! (\mathscr{T}_{X/T} \otimes g^* \mathscr{T}_{T/S}) \\
&\cong R^{2d(T/S)} h_! (R^{2d(X/T)}) g_! \mathscr{T}_{X/T} \otimes \mathscr{T}_{T/S}) \\
&\to R^{2d(T/S)} h_! \mathscr{T}_{T/S} \to \Lambda.
\end{aligned}$$

We denote the composite mapping by $\beta \,\square\, \alpha$.

If α and β are isomorphisms, so is $\beta \,\square\, \alpha$.

Let

$$\begin{tikzcd}
X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

be a base change of schemes. Let the morphism f and hence also f' be of type S. Set $d = d(X/S) = d(X_T/T)$.

$$(g')^* \mathscr{T}_{X/S} = \mathscr{T}_{X_T/T}.$$

We say that two trace mappings α for X/S and β for X_T/T are compatible with base change if the following diagram is commutative:

$$\begin{tikzcd}
g^* R^{2d} f_! \mathscr{T}_{X/S} \arrow[r, "\sim"] \arrow[d, "g^*\alpha"] & R^{2d} f_! \mathscr{T}_{X_T/T} \arrow[d, "\beta"] \\
g^*\Lambda \arrow[r, equal] & \Lambda.
\end{tikzcd}$$

Theorem 20.4. One can assign to every compactifiable morphism $f : X \to S$ of type S a trace mapping

$$S_{X/S} : R^{2d} f_! \mathscr{T}_{X/S} \to \Lambda$$

which has the follwing properties and is uniquely determined by them:

(1) $S_{X/S}$ is compatible with base change.

(2) $S_{X/S}$ is transitive, i.e., $S_{X/S} = S_{T/S} \,\square\, S_{X/T}$.

(3) If f is étale, then

$$S_{X/S} : f_! f^* \Lambda \to \Lambda$$

is the canonical mapping.

(4) If $f : X \to S$ is a smooth curve over the spectrum S of an algebraically closed field k, then

$$S_{X/S} : R^2 f_! \mathscr{T}_{X/S} \to \Lambda$$

is the mapping $S_{X/k}$ that we considered before, the one comming from the Kummer sequence.

Now, we recast our work on the trace map in terms of derived category.

In general, $f_!$ does not have a sheaf theoretic right adjoint. However, $Rf_!$ has a right adjoint $f^!$ in the derived category. For a smooth morphism f of relative dimension d, we give an ad-hoc definition:

$$f^! \mathscr{F} = f^* \mathscr{F}[2d](d).$$

We'll need a version of the trace map in the derived category, which will take the form

$$Rf_! f^! \mathscr{F} \to \mathscr{F}.$$

To define this map, it will suffice to show that there is a morphism

$$Rf_! f^! \mathscr{F} \to R^{2d} f_! f^! \mathscr{F}[-2d]$$

with the latter sheaf regarded as a complex in the derived catogory concentrated in degree 0, since we can then compose this with the trace map that we have already constructed.

Now, $R^{2d} f_! f^! \mathscr{F}$ is just a cohomology sheaf of $Rf_! f^! \mathscr{F}$, so it suffices to prove that $Rf_! f^! \mathscr{F}$ is represented by a complex with support in degrees $[-2d, 0]$. This is a consequence of the fact that $f_!$ has cohomological dimension $2d$.

Let $F \in D_-(X)$ and $G \in D_+(X)$. Then $\mathsf{R}\mathscr{H}om(F, G) \in D(X)$. Now consider a smooth compactifiable morphism $f : X \to S$. We claim that there is a map

$$\mathsf{R}f_* \mathsf{R}\mathscr{H}om(F, G) \to \mathsf{R}\mathscr{H}om(\mathsf{R}f_! F, \mathsf{R}f_! G).$$

This map is essentially functoriality of the construction of $\mathsf{R}f_!$. It can be described as follows. Take a compactification $j : X \to \overline{X}$ and injective resolutions

$$j_! F \to I, \quad j_! G \to J.$$

To compute the left side, we replace G by the injective resolution $j^* J$ on X. Therefore,

$$j_* \mathsf{R}\mathscr{H}om(F, G) = j_* \mathscr{H}om(F, j^* J) = \mathscr{H}om(j_! F, J) = \mathscr{H}om(I, J).$$

Now apply $\mathsf{R}\bar{f}_*$, to obtain a map

$$\mathsf{R}\bar{f}_*(j_*\mathsf{R}\mathscr{H}om(F,G)) = \mathsf{R}\bar{f}_*\mathscr{H}om(I,J) \to \mathscr{H}om(\mathsf{R}\bar{f}_*I, \mathsf{R}\bar{f}_*J) = \mathsf{R}\mathscr{H}om(\mathsf{R}f_!F, \mathsf{R}f_!G),$$

as desired.

Applying $\mathsf{R}\Gamma$, we get

$$\mathsf{R}\operatorname{Hom}(F,G) \to \mathsf{R}\operatorname{Hom}(Rf_!F, Rf_!G)$$

. Taking cohomology, we get a map

$$\operatorname{Ext}^p(F,G) \to \operatorname{Ext}^p(Rf_!F, Rf_!G).$$

Finally, let's bring in the trace map. If $F = \mathscr{F}$ is a sheaf on X, and $G = f^!\mathscr{G}$ for a sheaf $\mathscr{G}$ on S, then we get maps

$$\begin{aligned}
\Delta^1_{X/S} &: \mathsf{R}f_*\mathsf{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \to \mathsf{R}\mathscr{H}om(\mathsf{R}f_!\mathscr{F}, \mathscr{G})\\
\Delta^2_{X/S} &: \mathsf{R}\operatorname{Hom}(\mathscr{F}, f^!\mathscr{G}) \to \mathsf{R}\operatorname{Hom}(Rf_!\mathscr{F}, \mathscr{G})\\
\Delta^3_{X/S} &: \operatorname{Ext}^p(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^p(Rf_!\mathscr{F}, \mathscr{G}).
\end{aligned}$$

Theorem 20.5 (Grothendieck)**.** The maps $\Delta^i_{X/S}$ are all isomorphisms.

This shows that the pairing induced by cup product and the trace mapping,

$$\operatorname{Ext}^p(\mathscr{F}, \mu^d) \times H_c^{2d-p}(X, \mathscr{F}) \to H_c^{2d}(X, \mu^d) \to \Lambda,$$

is nondegenerate.

21 Cohomology classes of algebraic cycles

Let $S = \operatorname{Spec} k$, where k is an algebraically closed field, and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ such that $\operatorname{char} k \nmid n$.

Let X be a reduced algebraic bariety of constant dimension d over S, $X_0 = X \setminus \operatorname{Sing} X$, the regular points on X. We have $\operatorname{Sing} X < d$. The natural mapping

$$H_c^{2d}(X_0, \mu_n^d) \to H_c^{2d}(X, \mu_n^d)$$

is an isomorphism. Composing its inverse with the trace mapping

$$H_c^{2d}(X_0, \mu_n^t) \to \Lambda,$$

we get a trace mapping

$$S_X : H_c^{2d}(X, \mu_n^d) \to \Lambda.$$

More generally, one can define a trace mapping for a non-reduced algebraic variety of constant dimension d,

$$S_X = \sum_i e_i S_{X^{(i)}} : H_c^{2d}(X, \mu_n^d) = H_c^{2d}(X_{\text{red}}, \mu_n^d) = \bigoplus_i H_c^{2d}(X^{(i)}, \mu_n^d) \to \Lambda.$$

Here $X^{(i)}$ are the irreducible components of X_{red} and the number e_i are the multiplicities of Y at the generic points of the components $Y^{(i)}$.

Let X be smooth now and let

$$i : Y \to X$$

an irreducible reduced closed subvariety of codimension s.

We compose the restriction mapping

$$i^* : H_c^{2(d-s)}(X, \mu_n^{d-s}) \to H_c^{2(d-s)}(Y, \mu_n^{(d-s)})$$

with the trace mapping

$$S_Y : H_c^{2(d-s)}(Y, \mu_n^{(d-s)}) \to \Lambda$$

and get an element α in

$$\mathrm{Hom}(H_c^{2(d-s)}(X, \mu_n^{(d-s)}), \Lambda) \cong H^{2s}(X, \mu_n^s).$$

We obtain a cohomology class $\mathrm{cl}_X(Y) \in H^{2s}(X, \mu_n^s)$ as the image of α under the duality isomorphism.

Extending linearly, we get a homomorphism

$$\mathrm{cl}_X : Z^s(X) \to H^{2s}(X, \mu_n^s).$$

The sheaf of ideals $\mathscr{I}$ of an irreducible reduced subvariety Y of X of codimension 1 is a line bundle. We set $\mathscr{L}(Y) = \mathscr{I}^{-1}$. Extending linearly, we get a homomorphism

$$\begin{array}{ccc} Z^1(X) & \to & \mathrm{Pic}(X) \\ D & \mapsto & \mathscr{L}(D). \end{array}$$

Proposition 21.1. Let $D \in Z^1(X)$ be a divisor on X, and $\mathscr{L}(D)$ the associated line bundle. Let $\mathrm{cl}(\mathscr{L}(D))$ be again the first Chern class of $\mathscr{L}(D)$, defined using the Kummer sequence. Then

$$\mathrm{cl}_X(D) = \mathrm{cl}(\mathscr{L}(D)).$$

Corollary 21.2. The cohomology class $\mathrm{cl}_X(D)$ of a divisor $D \in Z^1(X)$ depends only on the rational equivalence class. The mapping $\mathrm{cl}_X(D)$ is compatible with construction of inverse images in the sense of intersection theory:

For $f : \widetilde{X} \to X$ we have

$$f^*(\mathrm{cl}_X(D)) = \mathrm{cl}_{\widetilde{X}}(f^*D).$$

Let $\mathscr{R}_X$ be the sheaf of rational functions on the "étale topos" of X, $\mathscr{R}_X^*$ the sheaf of invertible rational functions, and $\mathcal{O}_X^*$ the sheaf of invertible holomorphic functions.

The line bundle $\mathscr{L}(Y)$ associated with Y is in a natural way contained in $\mathscr{R}_X$,

$$\mathcal{O}_X \subset \mathscr{L}(Y) \subset \mathscr{R}_X.$$

There is a covering $\{U_i\}$ by Zariski-open subsets U_i of X and rational functions $f_i \in \mathscr{R}_X^*(U_i)$ with

$$\mathcal{O}_{U_i} \cdot f_i = \mathscr{L}(D)|_{U_i}.$$

The family $\{f_i\}$ defines a section $\alpha \in \Gamma(X, \mathscr{R}_X^*/\mathcal{O}_X^*)$, whose support is contained in Y because $\mathscr{L}(D)|_{X\setminus Y} = \mathcal{O}_X(X \setminus Y)$. The section α is independent of the choice of the covering and the functions f_i.

For the short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathscr{R}_X^* \to \mathscr{R}_X^*/\mathcal{O}_X^* \to 0$$

and the Kummer sequence

$$0 \to \mu_n \to \mathcal{O}_X^* \xrightarrow{n} \mathcal{O}_X^* \to 0,$$

we consider the corresponding long exact cohomology sequences with support in Y:

$$\cdots \to \Gamma_Y(X, \mathscr{R}_X^*/\mathcal{O}_X^*) \to H_Y^1(X, \mathcal{O}_X^*) \to H_Y^2(X, \mu_n) \to \cdots .$$

Definition 21.3. We denote the image of α under the composite mapping

$$\Gamma_Y(X, \mathscr{R}_X^*/\mathcal{O}_X^*) \to H_Y^2(X, \mu_n)$$

by $\mathrm{cl}^0(Y) \in H_Y^2(X, \mu_n)$.

22 Fixed point formula

Let P be a finitely generated projective module over a commutative ring A, and $f : P \to P$ an endomorphism. We choose any projective module Q for which $P \oplus Q$ is free. Consider the mapping

$$\begin{array}{rccc} \widetilde{f}: & P \oplus Q & \to & P \oplus Q \\ & (p, q) & \mapsto & (f(p), 0). \end{array}$$

We then define the trace and the characteristic polynomial of f by

$$\operatorname{tr}(f) = \operatorname{tr}(\widetilde{f}) \in A, \ \det(1 - tf) = \det(\mathrm{id}_{P \oplus Q} - t\widetilde{f}) \in A[t].$$

The constructions are independent of the choice of Q. (Using the fact that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ and $\det(1 - tab) = \det(1 - tba)$.)

Now let $P^\bullet$ be a bounded complex of finitely generated projective A-modules, and

$$f^\bullet : P^\bullet \to P^\bullet$$

a homomorphism. We set

$$\begin{aligned} \operatorname{tr}(f^\bullet) &= \sum_n (-1)^n \operatorname{tr}(f^n) \in A, \\ \det(1 - tf^\bullet) &= \prod_n \det(1 - tf^n)^{(-1)^n} \in A[[t]]. \end{aligned}$$

We have:

(a) If $h^n(P^\bullet) = 0$ for all n, then

$$\operatorname{tr}(f^\bullet) = 0, \ \det(1 - tf^\bullet) = 1.$$

(Only need to do the case when $P^\bullet$ is a short exact sequence.)

(b) Let $Q^\bullet$ be another complex of finitely generated projective A-modules. Suppose that there is a diagram that is commutative up to homotopy:

$$\begin{array}{ccc} P^\bullet & \xrightarrow{f^\bullet} & P^\bullet \\ \downarrow{\scriptstyle \varphi^\bullet} & & \downarrow{\scriptstyle \varphi^\bullet} \\ Q^\bullet & \xrightarrow{g^\bullet} & Q^\bullet. \end{array}$$

Let $C(\varphi^\bullet)$ be the mapping cone of $\varphi^\bullet$. We may define

$$h^\bullet : C(\varphi^\bullet) \to C(\varphi^\bullet)$$

in a natural way with respect to $f^\bullet$ and $g^\bullet$. Then we have

$$\operatorname{tr}(f^\bullet) + \operatorname{tr}(h^\bullet) = \operatorname{tr}(g^\bullet)$$
$$\det(1 - tf^\bullet)\det(1 - th^\bullet) = \det(1 - tg^\bullet).$$

Proof. (b) We omit the $^\bullet$. Let $C(\varphi) = Q \oplus P[1]$ and let $\varphi f - g\varphi = dk + kd$. Then

$$\begin{pmatrix} g & -k \\ 0 & f[1] \end{pmatrix}$$

defines the morphism $h : C(\varphi) \to C(\varphi)$ in the commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{\varphi} & Q & \xrightarrow{i} & C(\varphi) & \xrightarrow{\pi} & P[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ P & \xrightarrow{\varphi} & Q & \xrightarrow{i} & C(\varphi) & \xrightarrow{\pi} & P[1]. \end{array}$$

since

$$d = \begin{pmatrix} d & \varphi \\ 0 & -d \end{pmatrix},$$

$ig = hi$ and $\pi h = f\pi$. ■

So in particular the trace and characteristic polynomial depend only on the homotopy class.

A complex $K^\bullet \in D(A)$ is called perfect if there is a quasi-isomorphism

$$P^\bullet \to K^\bullet,$$

where $P^\bullet$ is a bounded complex of finitely generated projective A-modules.

Let $K^\bullet$ be a perfect complex and a morphism

$$f^\bullet : K^\bullet \to K^\bullet$$

a morphism in $D(A)$. We choose a bounded complex $P^\bullet$ of finitely generated projective modules and a quasi-isomorphism

$$P^\bullet \to K^\bullet$$

Then there is a homomorphism of complexes

$$\widetilde{f}^\bullet : P^\bullet \to P^\bullet$$

such that the diagram

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\tilde{f}^\bullet} & P^\bullet \\ \downarrow & & \downarrow \\ K^\bullet & \xrightarrow{f^\bullet} & K^\bullet \end{array}$$

is commutative in $D(A)$. We set

$$\mathrm{tr}(f^\bullet) = \mathrm{tr}(\widetilde{f}^\bullet),\ \det(1 - tf^\bullet) = \det(1 - t\widetilde{f}^\bullet).$$

Properties (a) and (b) imply that this definition is independent of the choice $P^\bullet \to K^\bullet$ and of $\widetilde{f}^\bullet$. Moreover, we get:

Lemma 22.1. The trace and characteristic polynomial are additive in the following sence:

Let

$$K^\bullet_{(1)} \to K^\bullet_{(2)} \to K^\bullet_{(3)} \to K^\bullet_{(1)}[1]$$

be a distinguished triangle in $D(A)$, and let $f^\bullet_{(i)} : K^\bullet_{(i)} \to K^\bullet_{(i)}$ be mappings in $D(A)$. Suppose two of the three complexes are perfect. Then the third is also perfect.

(1) The following additivity formula

$$\mathrm{tr}(f^\bullet_{(1)}) + \mathrm{tr}(f^\bullet_{(3)}) = \mathrm{tr}(f^\bullet_{(2)}),$$
$$\det(1 - tf^\bullet_{(1)}) \det(1 - tf^\bullet_{(3)}) = \det(1 - tf^\bullet_{(2)}).$$

(2) are valid, if the mappings $f^\bullet_{(i)}$ define a true endomorphism of the triangle in the following sense:

Let $E(A)$ be the category of endomorphisms $M \to M$ of A-modules and $D(E(A))$ its derived category. Each distinguished triangle of $D(E(A))$ defines in an obviuous way an distinguished triangle

$$K^\bullet_{(1)} \to K^\bullet_{(2)} \to K^\bullet_{(3)} \to K^\bullet_{(1)}[1]$$

in $D(A)$ and a system $f^\bullet_{(i)} : K^\bullet_{(i)} \to K^\bullet_{(i)}$ of mappings in $D(A)$. A system of mapping which arises in this way is called a true endomorphism of this triangle in $D(A)$.

For a (possibly commutative) ring Λ. Let $\Lambda^\natural$ be the additive abelian group defined by

$$\Lambda^\natural = \Lambda / \langle ab - ba \mid a, b \in \Lambda \rangle.$$

For an Λ-endomorphism $f : P \to P$ of a free Λ-module P, represent f as a matrix over Λ, we may define $\mathrm{tr}_\Lambda(f)$ to be the image of the sum of the diagonal of the matrix in $\Lambda \to \Lambda^\natural$. As before, we may extend this to endomorphisms of finite projective modules. Let A be a commutative ring with a ring map $A \to \Lambda$ which image lies in the center of Λ.

Let $\Lambda[G]$ be the group ring over Λ. Note that the map

$$\begin{array}{ccc} \Lambda[G] & \to & \Lambda^\natural \\ \sum_\sigma \lambda_\sigma \cdot \sigma & \mapsto & [\lambda_e] \end{array}$$

factors through $\Lambda[G]^\natural$. We denote $\varepsilon : \Lambda[G]^\natural \to \Lambda^\natural$ the induced map.

Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module P. We define

$$\mathrm{tr}_\Lambda^G(f; P) = \varepsilon(\mathrm{tr}_{\Lambda[G]}(f; P))$$

to be the G-trace of f on P.

Lemma 22.2. Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module P. Then

$$\mathrm{tr}_\Lambda(f; P) = |G| \cdot \mathrm{tr}_\Lambda^G(f; P).$$

Proof. By the additivity of trace, it suffices to prove the case when $P = \Lambda[G]$, which is trivial. ∎

Lemma 22.3. Let P be a finite projective $A[G]$-module and M a $\Lambda[G]$-module, finite projective as a Λ-module. Let $f : P \to P$ be an $A[G]$-endomorphism and $g : M \to M$ be a $\Lambda[G]$-endomorphism. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module for the structure induced by the diagonal action and

$$\mathrm{tr}_\Lambda^G(f \otimes g; P \otimes_A M) = \mathrm{tr}_A^G(f; P) \cdot \mathrm{tr}_\Lambda(g; M).$$

Proof. For any $\Lambda[G]$-module N, one has

$$\mathrm{Hom}_{\Lambda[G]}(A[G] \otimes_A M, N) = \mathrm{Hom}_{A[G]}(P, \mathrm{Hom}_\Lambda(M, N))$$

where the G-action on $\mathrm{Hom}_\Lambda(M, N)$ is given by $(g(\varphi))(m) = g\varphi(g^{-1}m)$.

For the formula, we reduce to the case $P = A[G]$. Then f is an element $\sum a_\sigma \sigma$ in $A[G]$. There is an isomorphism of $\Lambda[G]$-modules

$$\begin{array}{cccc} \varphi : & A[G] \otimes_A M & \xrightarrow{\sim} & (A[G] \otimes_A M)' \\ & g \otimes m & \mapsto & g \otimes g^{-1}m \end{array}$$

where $(A[G] \otimes_\Lambda M)'$ has the module structure given by the left G-action, together with the Λ linearity on M. This transport of structure changes $f \otimes g$ into $\sum a_\sigma \sigma \otimes \sigma^{-1} g$. We have to show

$$\operatorname{tr}_\Lambda^G \left(\sum a_\sigma \sigma \otimes \sigma^{-1} g; (A[G] \otimes_A M)' \right) = a_e \cdot \operatorname{tr}_\Lambda(g; M).$$

This is done by showing that

$$\operatorname{tr}_\Lambda^G(a_\sigma \sigma \otimes \sigma^{-1} g) = \begin{cases} 0, & \text{if } \sigma \neq e \\ a_e \operatorname{tr}_\Lambda(g; M), & \text{if } \sigma = e \end{cases}$$

by reducing to $M = \Lambda$. ■

Consider the monoid extension

$$1 \to G \to E \xrightarrow{\pi} \mathbb{N}_0 \to 0.$$

and let $\gamma \in E$. Then we write

$$Z_\gamma = \{\sigma \in G \mid \sigma\gamma = \gamma\sigma\} \leq G.$$

Lemma 22.4. Let P be a $\Lambda[E]$-module, finite projective as a $\Lambda[G]$-module, and $\gamma \in E$. Then

$$\operatorname{tr}_\Lambda(\gamma; P) = |Z_\gamma| \cdot \operatorname{tr}_\Lambda^{Z_\gamma}(\gamma; P).$$

Lemma 22.5. Let P be an $A[E]$-module, finite projective as an $A[G]$-module. Let M be a $\Lambda[E]$-module, finite projective as a Λ-module. Then

$$\operatorname{tr}_\Lambda^{Z_\gamma}(\gamma; P \otimes_\Lambda M) = \operatorname{tr}_A^{Z_\gamma}(\gamma; P) \cdot \operatorname{tr}_\Lambda(\gamma, M).$$

Lemma 22.6. Let P be a $\Lambda[E]$-module, finite and projective as a $\Lambda[G]$-module. Then $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective Λ-module, endowed with an action $E/G = \mathbb{Z}$. Moreover, we have

$$\operatorname{tr}_\Lambda(1; P_G) = \sum_{\gamma \mapsto 1}{}' \operatorname{tr}_\Lambda^{Z_\gamma}(\gamma; P)$$

where the sum means taking the sum over the G-conjugacy classes in E.

Proof. We first prove this after multiplying by $|G|$.

We have

$$|G| \operatorname{tr}_\Lambda(1; P_G) = \operatorname{tr}_\Lambda \left(\sum_{\gamma \mapsto 1} \gamma; P_G \right) = \operatorname{tr}_\Lambda \left(\sum_{\gamma \mapsto 1} \gamma; P \right)$$

where the second equality follows by considering the commutative diagram

$$
\begin{tikzcd}
P^G \arrow[dr, "i"'] & & P_G \arrow[ll, "\Sigma"'] \\
& P \arrow[ur, "q"'] &
\end{tikzcd}
$$

where i is the canonical inclusion, q the canonical quotient and $\Sigma = \sum_{\gamma \mapsto 1} \gamma$. Then we have

$$\sum_{\gamma \mapsto 1} \gamma\,|_P = i \circ \Sigma \circ q \quad \text{and} \quad \sum_{\gamma \mapsto 1} \gamma\,|_{P_G} = q \circ i \circ \Sigma$$

hence they have the same trace. We then have

$$|G| \operatorname{tr}_\Lambda(1; P_G) = \sum_{\gamma \mapsto 1}{}' \frac{|G|}{|Z_\gamma|} \operatorname{tr}_\Lambda(\gamma; P) = |G| \sum_{\gamma \mapsto 1}{}' \operatorname{tr}_\Lambda^{Z_\gamma}(\gamma, P).$$

For general case, choose an element $g \in E$ such that $\pi(g) = 1$ and let $\widetilde{g} = \operatorname{ad} g \in \operatorname{Aut}(\Lambda[G])$ be the induced automorphism. Then we may restate the lemma as follows:

> Let P be a finite projective $\Lambda[G]$-module with an $\widetilde{g}$-linear endomorphism $\gamma : P \to P$, i.e., $\gamma(\sigma p) = \widetilde{g}(\sigma)\gamma(p)$. We have
>
> $$\operatorname{tr}_\Lambda(\gamma; P) = \sum_h{}' \operatorname{tr}_\Lambda^{Z_g}(h\gamma; P)$$
>
> where the sum is over the $\widetilde{g}$-conjugacy classes of G.

After doing this, we may assume that $P = \Lambda[G]$. Then γ is of the form

$$x \mapsto \widetilde{g}(x) \sum \lambda_\sigma \cdot \sigma$$

with $\lambda_\sigma \in \Lambda$. It suffices to deal with the universal case where the λ_σ are indeterminate, i.e., may assume $\Lambda = \mathbb{Z}\langle \lambda_\sigma \rangle$ and $\Lambda^\natural$ is torsion free, so that we may divide $|G|$ on the both side of the equation above. ■

We say $\mathscr{K}^\bullet \in D(X)$ is perfect if it is isomorphic to a finite complex of finitely generated projective étale sheaves of Λ-modules.

Proposition 22.7. A bounded complex $\mathscr{K}^\bullet \in D(X)$ is perfect iff the following two conditions are satisfied:

(i) the cohomology sheaves $h^i(\mathscr{K}^\bullet)$ are constructible,

(ii) $\mathscr{K}^\bullet$ has finite Tor-dimension.

Proof. ($\Longrightarrow$) is trivial. For the converse, suppose that $h^i(\mathscr{K}^\bullet \otimes^{\mathsf{L}} \mathscr{F}) = 0$ for all étale sheaf $\mathscr{F}$ on X and $i \notin [a, b]$. Represent $\mathscr{K}^\bullet$ by a bounded above projective sheaf. We induction on $b - a$. If $a = b$, then we represent $\mathscr{K}^\bullet$ as $h^a(\mathscr{K}^\bullet)[-a]$, which is clearly true. Assume $b > a$. Consider the surjection

$$\ker(\mathscr{K}^b \to \mathscr{K}^{b+1}) \to h^b(\mathscr{K}^\bullet).$$

Since $\mathscr{H}^b(\mathscr{K}^\bullet)$ is constructible, there is a surjection

$$\bigoplus_\alpha (j_\alpha)_! \Lambda_{U_\alpha} \to h^b(\mathscr{K}^\bullet).$$

After replace U_α by standard étale coverings $\{U_{\alpha\beta} \to U_\alpha\}$ we may assume this surjection lifts to a map

$$\mathscr{G} = \bigoplus_\alpha (j_\alpha)_! \Lambda_{U_\alpha} \to \ker(\mathscr{K}^b \to \mathscr{K}^{b+1}).$$

This map determines a distinguished triangle

$$\mathscr{G}[-b] \to \mathscr{K}^\bullet \to L \to \mathscr{G}[-b+1].$$

Since $\mathscr{G}[-b]$ and $\mathscr{K}^\bullet$ satisfy (i) and (ii), L satisfies them too. In fact L has Tor amplitude in $[a, b-1]$ as $\mathscr{G}$ surjects onto $h^b(\mathscr{K}^\bullet)$. By induction hypothesis we can find a finite complex

$$\mathscr{F}^a \to \cdots \to \mathscr{F}^{b-1}$$

of flat constructible sheaves of Λ-modules representing L. The map $L \to \mathscr{G}[-b+1]$ corresponds to a map $\mathscr{F}^{b-1} \to \mathscr{G}$ annihilating the image of $\mathscr{F}^{b-2} \to \mathscr{F}^{b-1}$. Then $\mathscr{K}^\bullet$ is represented by the complex

$$\mathscr{F}^a \to \cdots \to \mathscr{F}^{b-1} \to \mathscr{G}$$

which finishes the proof. ■

Theorem 22.8. Let $f : X \to S$ be a map of schemes, If $\mathscr{K}^\bullet \in D(X)$ is perfect, then $\mathsf{R}f_!\mathscr{K}^\bullet \in D(S)$ is also perfect.

Proof. First we handle the constructibility. We have a spectral sequence

$$R^q f_!(h^p(\mathscr{K}^\bullet)) \implies R^{p+q} f_!(\mathscr{K}^\bullet)$$

and by assumption $h^p(\mathscr{K}^\bullet)$ is constructible on X. Since $f_!$ preserve constructibility, $R^q f_!(h^p(\mathscr{K}^\bullet))$ is also constructible, hence so is $R^{p+q} f_!(\mathscr{K}^\bullet)$.

Next let's examine the Tor-dimension. Let $\mathscr{F}$ be a sheaf on S. Then

$$(Rf_!\mathscr{K}^\bullet) \otimes^{\mathsf{L}} \mathscr{F} = Rf_!(\mathscr{K}^\bullet \otimes^{\mathsf{L}} f^*\mathscr{F}).$$

Since $\mathscr{K}^\bullet$ has finite Tor-dimension, $\mathscr{K}^\bullet \otimes^{\mathsf{L}} f^*\mathscr{F}$ has bounded cohomology, so $\mathsf{R}f_!(K \otimes^{\mathsf{L}} f^*\mathscr{F})$ also has bounded homology. ■

In the following, κ is the finite field $\mathbb{F}_q$ with $q = p^s$, and k is its algebraic closure. For every algebraic variety X over κ the Frobenius morphism with respect to κ,

$$\mathrm{Fr}_X : X \to X,$$

is defined over κ. Let $U \to X$ be a morphism. Since the diagram

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{Fr}_U} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{Fr}_X} & X \end{array}$$

is commutative, there is an induced mapping

$$\mathrm{Fr}_{U/X} : U \to X \times_X U = \mathrm{Fr}_X^{-1}(U).$$

Lemma 22.9. If $U \to X$ is étale, then $\mathrm{Fr}_{U/X} : U \to X \times_X U$ is an isomorphism, i.e., the diagram above is Cartesian.

Let $\mathscr{G}$ be a sheaf of sets on X. For every étale mapping $U \to X$, there is a mapping

$$\mathscr{G}(U) \xrightarrow{\mathscr{G}(\mathrm{Fr}_{U/X})} \mathscr{G}(\mathrm{Fr}_X^{-1}(U)) = (\mathrm{Fr}_X)_*\mathscr{G}(U).$$

The collection of these mapping defines an isomorphism

$$\mathscr{G} \to (\mathrm{Fr}_X)_*\mathscr{G}.$$

Because

$$\mathrm{Hom}(\mathscr{G}, (\mathrm{Fr}_X)_*\mathscr{G}) = \mathrm{Hom}(\mathrm{Fr}_X^*\mathscr{G}, \mathscr{G}),$$

this isomorphism induces a mapping

$$\mathrm{Fr}_X^*\mathscr{G} \to \mathscr{G}.$$

If $\mathscr{G}$ is representable by a scheme $V \to X$ étale over X, this mapping corresponds precisely to the isomorphism

$$(\mathrm{Fr}_{V/X})^{-1} : \mathrm{Fr}_X^{-1}(V) \to V.$$

Definition 22.10. The map thus constructed is called the Frobenius homomorphism of $\mathscr{G}$

$$\mathrm{Fr}_{\mathscr{G}} : \mathrm{Fr}_X^* \mathscr{G} \to \mathscr{G}.$$

$\mathrm{Fr}_{\mathscr{G}}$ is an isomorphism and compatible with all morphisms of sheaves.

Remark. $\mathrm{Fr}_{\mathscr{G}}$ has a series of compatibility properties: For a κ-morphism $\mathrm{Fr} : Y \to X$. $\mathrm{Fr}_{\mathscr{G}}$ is compatible with formation of inverse images, direct images, higher direct images, and with the "higher direct images with compact support". By d-fold repeated application, one gets a homomorphism

$$\mathrm{Fr}^d_{\mathscr{G}} : (\mathrm{Fr}^d_X)^* \mathscr{G} \to \mathscr{G}.$$

If κ' is a field extension of degree d of κ, then

$$\mathrm{Fr}^d_X \otimes \kappa' : X \times_\kappa \kappa' \to X \times_\kappa \kappa'$$

is precisely the Frobenius morphism $\mathrm{Fr}_{X \times_\kappa \kappa'}$ of $X \times_\kappa \kappa'$ with respect to κ'. The corresponding fact holds for the inverse image $\mathscr{G}_{\kappa'}$ of $\mathscr{G}$ on $X \times_\kappa \kappa'$ and the induced mapping

$$(\mathrm{Fr}^d_{\mathscr{G}})_{\kappa'} : \mathrm{Fr}^*_{X \times_\kappa \kappa'} \mathscr{G}_{\kappa'} \to \mathscr{G}_{\kappa'}$$

equals to $\mathrm{Fr}_{\mathscr{G}_{\kappa'}}$.

Let U_0 be an algebraic scheme over κ and $\mathscr{G}_0$ a sheaf on U_0. We denote by $U = U_0 \times_\kappa k$ the extension of U_0 to the algebraic closure k of κ, and by $\mathscr{G}$ the inverse image of $\mathscr{G}_0$ on U. After field extension, Fr_{U_0} and $\mathrm{Fr}_{\mathscr{G}_0}$ induce mappings

$$\mathrm{Fr}_U = \mathrm{Fr}_{U_0} \otimes k : U \to U$$

and

$$\mathrm{Fr}_{\mathscr{G}} : \mathrm{Fr}_U^* \mathscr{G} \to \mathscr{G}.$$

We consider fixed powers of Fr_U and $\mathrm{Fr}_{\mathscr{G}}$:

$$F_U = \mathrm{Fr}^d_U : U \to U$$
$$F_{\mathscr{G}} = \mathrm{Fr}^d_{\mathscr{G}} : (\mathrm{Fr}^d_U)^* \mathscr{G} \to \mathscr{G}.$$

The pairing $(F_U, F_{\mathscr{G}})$ induces a mapping

$$\mathsf{R}\Gamma_c(F_U, F_{\mathscr{G}}) : \mathsf{R}\Gamma_c(U, \mathscr{G}) \to \mathsf{R}\Gamma_c(U, \mathscr{G}).$$

Theorem 22.11. Let X_0 be a variety over κ and $\mathscr{F}_0$ a constructible $\mathbb{Q}_\ell$-sheaf on X_0. Let $X = X_0 \times_\kappa k$ and $\mathscr{F}$ be the pullback of $\mathscr{F}_0$ to X. Then we have

$$\sum_{x \in X(\kappa)} \operatorname{tr}(F_x : \mathscr{F}_x \to \mathscr{F}_x) = \sum_i (-1)^i \operatorname{tr}(F : H^i_c(X, \mathscr{F}) \to H^i_c(X, \mathscr{F})).$$

It suffices to prove that

$$\sum_{x \in X(\kappa)} \operatorname{tr}(F^\bullet_x : (\mathscr{F}_n)_x \to (\mathscr{F}_n)_x) = \sum_i (-1)^i \operatorname{tr}(F : H^i_c(X, \mathscr{F}_n) \to H^i_c(X, \mathscr{F}_n))$$

for each n, where $\mathscr{F}_n$ is a constructible $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$-module.

Theorem 22.12. Let X_0 be a variety over κ and $\mathscr{K}^\bullet_0 \in D(X_0)$ be a perfect complex. Let $X = X_0 \times_\kappa k$ and $\mathscr{K}^\bullet$ the pullback of $\mathscr{K}^\bullet_0$ to X. Then we have

$$\sum_{x \in X(\kappa)} \operatorname{tr}(F^\bullet_x : \mathscr{K}^\bullet_x \to \mathscr{K}^\bullet_x) = \operatorname{tr}(\mathsf{R}\Gamma_c(F_X, F_{\mathscr{K}^\bullet}))$$

Proof. Denote LHS by $T_\ell(X_0, \mathscr{K}^\bullet)$ and RHS by $T_g(X_0, \mathscr{K}^\bullet)$.

Claim 1. If $X_0 = U_0 \sqcup Y_0$ is a partition of X_0 into a closed subset Y_0 and its closed complement U_0, and the assertion holds for U_0 and Y_0, then it holds for X_0.

Proof of Claim 1. Let $j : U_0 \to X_0$ and $i : Y_0 \to X_0$ be the two inclusion maps. It is easy to see that if $\mathscr{K}^\bullet_0 \in D(X_0)$ is perfect, then $j^*\mathscr{K}^\bullet_0$ and $i^*\mathscr{K}^\bullet_0$ are perfect. Therefore, the assumption that the assertion holds for U_0 and Y_0 implies that

$$T_\ell(U_0, j^*\mathscr{K}^\bullet_0) = T_g(U_0, j^*\mathscr{K}^\bullet_0),\ T_\ell(Y_0, j^*\mathscr{K}^\bullet_0) = T_g(Y_0, j^*\mathscr{K}^\bullet_0)$$

Since T_ℓ is purely local, we manifestly have

$$T_\ell(U_0, j^*\mathscr{K}^\bullet_0) + T_\ell(Y_0, i^*\mathscr{K}^\bullet_0) = T_\ell(X_0, \mathscr{K}^\bullet_0).$$

By the additive of the trace in long exact sequences, applied to the cohomology long exact sequence induced by the exact triangle

$$j_!j^*\mathscr{K}^\bullet_0 \to \mathscr{K}^\bullet_0 \to i_*i^*\mathscr{K}^\bullet_0 \to j_!j^*\mathscr{K}^\bullet_0[1]$$

and the fact that

$$\mathsf{R}\Gamma_c(U_0, j^*\mathscr{K}_0^\bullet) = \mathsf{R}\Gamma_c(X_0, j_!j^*\mathscr{K}_0^\bullet)$$
$$\mathsf{R}\Gamma_c(Y_0, i^*\mathscr{K}_0^\bullet) = \mathsf{R}\Gamma_c(X_0, i_*i^*\mathscr{K}_0^\bullet),$$

we have also

$$T_g(U_0, j^*\mathscr{K}_0^\bullet) + T_g(Y_0, i^*\mathscr{K}_0^\bullet) = T_g(X_0, j^*\mathscr{K}_0^\bullet),$$

thus established the claim in light of the preceding equations. □

Claim 2. If $f : X_0 \to S_0$ is a map and the assertion holds for S_0, and all fibers of f, then it holds for X_0.

Proof of Claim 2. We know that $\mathsf{R}f_!\mathscr{K}^\bullet$ is perfect. Then the theorem applied to $\mathsf{R}f_!\mathscr{K}_0^\bullet$ tells us that

$$T_\ell(S_0, \mathsf{R}f_!\mathscr{K}_0^\bullet) = T_g(S_0, \mathsf{R}f_!\mathscr{K}_0^\bullet).$$

Since $\mathsf{R}\Gamma_c(S, \mathsf{R}f_!\mathscr{K}^\bullet) \cong \mathsf{R}\Gamma_c(X, \mathscr{K}^\bullet)$ we have

$$T_g(S_0, \mathsf{R}f_!\mathscr{K}_0^\bullet) = T_g(X_0, \mathscr{K}_0^\bullet).$$

On the other hand,

$$T_\ell(S_0, \mathsf{R}f_!\mathscr{K}_0^\bullet) = \sum_{s \in S(\kappa)} \mathrm{tr}(F_s : (\mathsf{R}f_!\mathscr{K}^\bullet)_s \to (\mathsf{R}f_!\mathscr{K}^\bullet)_s)$$

and the assertion applied to $f^{-1}(s)$ plus proper base change applied to

$$\begin{tikzcd} f^{-1}(s) \arrow[r, "i'"] \arrow[d, "f"] & X \arrow[d, "f"] \\ s \arrow[r, "i"] & S \end{tikzcd}$$

says that

$$\mathrm{tr}(F_s : (\mathsf{R}f_!\mathscr{K}^\bullet)_s \to (\mathsf{R}f_!\mathscr{K}^\bullet)_s) = T_g(f^{-1}(s), \mathscr{K}^\bullet_{f^{-1}(s)}) = T_\ell(f^{-1}(s), \mathscr{K}^\bullet_{f^{-1}(s)}).$$

Thus we find that

$$T_\ell(S_0, \mathsf{R}f_!\mathscr{K}^\bullet) = \sum_{s \in S_0(\kappa)} T_\ell(f^{-1}(s), \mathscr{K}^\bullet_{f^{-1}(s)}) = T_\ell(X_0, \mathscr{K}_0^\bullet),$$

as desired. □

Finally we are reduced to showing the assertion for a curve. We begin with some simplifications. We can easily prove directly the 0-dimensional case of the assertion, which by excision lets us cut out a finite set of closed points any time we want.

- We may assume that $\mathscr{K}^\bullet$ is a finite complex of finitely generated projective modules. By filtering $\mathscr{K}^\bullet$, we reduce to the case where $\mathscr{K}^\bullet = \mathscr{F}$ is a constructible Λ-sheaf with finitely generated projective stalks.

- By restricting to a stratum, we may assume that $\mathscr{F}$ is a locally constant.

- By replacing X by an open subset U, we may assume that X is smooth and irreducible.

- By partitioning X into the union of its rational points and their complement, we may assume that $X(\kappa) = \varnothing$. The goal is then to show that

$$\operatorname{tr}(F : \mathsf{R}\Gamma_c(X, \mathscr{F}) \to \mathsf{R}\Gamma_c(X, \mathscr{F})) = 0.$$

Since $\mathscr{F}$ is locally constant, we can find a finite étale Galois cover $f : Y_0 \to X_0$ such that $f^*\mathscr{F}$ is constant.

There is a counit map

$$f_* f^* \mathscr{F} \to \mathscr{F}.$$

Let $G = \mathrm{Gal}(Y_0/X_0)$. The sheaf $f_* f^* \mathscr{F}$ has an action of G coming from the fact that it is pushing forward from Y_0, while $\mathscr{F}$ doesn't, so this map factors through

$$(f_* f^* \mathscr{F})_G \to \mathscr{F}.$$

We can check on stalks that this is an isomorphism. Indeed, for $\bar{x} \to X$,

$$(f_* f^* \mathscr{F})_{\bar{x}} = \bigoplus_{f(\bar{y}) = \bar{x}} \mathscr{F}_{\bar{x}}$$

with the G-action induced by its permutation action on the G-torsor $f^{-1}(\bar{x})$.

Say $f^*\mathscr{F}$ has value group $M = H^0(Y, f^*\mathscr{F})$. Then M is a $\Lambda[G]$-module, and

$$f_* f^* \mathscr{F} = f_*(A \otimes_A f^*(M_X)) = f_* A \otimes_A M_X$$

for $A = \mathbb{Z}/\ell^n\mathbb{Z}$ with $n \gg 0$ with the G-action being the diagonal one. Therefore,

$$\mathscr{F} = \Lambda \otimes_{\Lambda[G]} ((f_* A) \otimes_A M).$$

We claim that these tensor products really coincide with the derived products. Note that $f_* A_{\bar{x}} = A[G]$ is a projective $A[G]$-module and $M_{\bar{x}}$ is a projective Λ-module, so $((f_* A) \otimes_A M)_{\bar{x}}$ is projective over $\Lambda[G]$.

Hence,

$$\begin{aligned}
\mathsf{R}\Gamma_c(X,\mathscr{F}) &= \mathsf{R}\Gamma_c(X,\Lambda\otimes_{\Lambda[G]}(f_*A\otimes_A M))\\
&= \mathsf{R}\Gamma_c(X,\Lambda\otimes^{\mathsf{L}}_{\Lambda[G]}(f_*A\otimes^{\mathsf{L}}_A M))\\
&= \Lambda\otimes_{\Lambda[G]}(\mathsf{R}\Gamma_c(X,f_*A)\otimes_A M)\\
&= \Lambda\otimes_{\Lambda[G]}(\mathsf{R}\Gamma_c(Y,A)\otimes_A M)\\
&= (\mathsf{R}\Gamma_c(Y,A)\otimes_A M)_G.
\end{aligned}$$

We want to understand the action of Frobenius on the LHS in terms of the action of Frobenius on $\Gamma_c(X,f_*A)$. We begin by understanding the effect of the G-coinvariants.

We have

$$\begin{aligned}
\mathrm{tr}_\Lambda(F;\mathsf{R}\Gamma_c(X,\mathscr{F})) &= {\sum_{g\in G}}' \mathrm{tr}_\Lambda^{Z_g}(g\circ F;\mathsf{R}\Gamma_c(Y,A)\otimes_A M)\\
&= {\sum_{g\in G}}' \mathrm{tr}_A^{Z_g}(g\circ F;\mathsf{R}\Gamma_c(Y,A))\cdot \mathrm{tr}_\Lambda(g;M).
\end{aligned}$$

Recall that we want to show that

$$\mathrm{tr}_\Lambda(F;\mathsf{R}\Gamma_c(X,\mathscr{F}))=0.$$

Thus, it suffices to show that

$$\mathrm{tr}_A^{Z_g}(g\circ F;\mathsf{R}\Gamma_c(Y,A))\mapsto 0$$

in $A\to\Lambda$. Recall that

$$|Z_g|\,\mathrm{tr}_A^{Z_g}(g\circ F;\mathsf{R}\Gamma_c(Y,A))=\mathrm{tr}_A(g\circ F;\mathsf{R}\Gamma_c(Y,A)).$$

Since $|Z_g|$ is fixed and $A=\mathbb{Z}/\ell^n\mathbb{Z}$ with $n\gg 0$, it suffices to show that

$$\mathrm{tr}(g\circ F;\mathsf{R}\Gamma_c(Y,A))=0$$

for every g. In this case the cohomology endomorphism $g\circ F$ is induced by a map of schemes $\varphi=g^{-1}\circ F_{Y_0}:Y\to Y$,

$$\mathrm{tr}(g\circ F;\mathsf{R}\Gamma_c(Y,A))=\mathrm{tr}(\varphi^*;\mathsf{R}\Gamma_c(Y,A)).$$

Consider the compactified covering map

$$\begin{array}{ccc}
Y & \hookrightarrow & \bar{Y}\\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}}\\
X & \hookrightarrow & \bar{X}.
\end{array}$$

By Weil's Theorem, we know that

$$\operatorname{tr}(\varphi^*; \mathsf{R}\Gamma_c(\bar{Y}, A)) = \sum_{y \in \operatorname{Fix}(\varphi)} \operatorname{tr}(g \circ F; A_y).$$

Now by exicion,

$$\operatorname{tr}(\varphi^*, \mathsf{R}\Gamma_c(\bar{Y}, A)) = \operatorname{tr}(\varphi^*, \mathsf{R}\Gamma_c(Y, A)) + \operatorname{tr}(\varphi^*, \mathsf{R}\Gamma_c(\bar{Y} \setminus Y, A)).$$

By the 0-dimensional case of Weil's theorem, we have

$$\operatorname{tr}(\varphi^*, \mathsf{R}\Gamma_c(\bar{Y} \setminus Y, A)) = \sum_{y \in \operatorname{Fix}(\varphi) \setminus Y} \operatorname{tr}(g \circ F, A_y).$$

Therefore, it suffices to see that there are no points $y \in Y$ fixed by φ. Indeed, if there were then $f(y) \in X$ would be fixed by F_{X_0}, but $X(\kappa) = \varnothing$. We are done. ■

Theorem 22.13 (Weil)**.** Let X be a smooth projective curve over k and $\varphi : X \to X$ be a morphism over k. Then

$$\Delta \cdot \Gamma_\varphi = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\varphi^* : H^i(X, \mathbb{Q}_\ell) \to H^i(X, \mathbb{Q}_\ell)).$$

In fact, we can take $\mathbb{Q}_\ell$ to be Λ.

Proof. First, we establish $\operatorname{cl}(\Gamma_\varphi)$. Let $p, q : X \times X \to X$ be the projections and $i : C \to X \times X$ be the graph morphism. Then $p \circ i = \operatorname{id}_X$ and $q \circ i = \varphi$.
Claim. For any $\alpha \in H^*(X, \mathbb{Q}_\ell)$,

$$p_*(\operatorname{cl}(\Gamma_\varphi) \cup q^*(\alpha)) = \varphi^*(\alpha).$$

Proof of Claim. Note that $\operatorname{cl}(\Gamma_\varphi) = i_*(\operatorname{cl}(X))$, so

$$\begin{aligned} p_*(\operatorname{cl}(\Gamma_\varphi) \cup q^*(\alpha)) &= p_*(i_*(\operatorname{cl}(X)) \cup q^*(\alpha)) \\ &= p_*(i_*(\operatorname{cl}(X) \cup i^* q^*(\alpha))) \\ &= (\operatorname{id}_X)_*(\operatorname{cl}(X) \cup \varphi^*(\alpha)) = \varphi^*(\alpha), \end{aligned}$$

as desired. □

Let $\{e_i^r\}$ be a basis of $H^r(X, \mathbb{Q}_\ell)$ and $\{f_i^{2-r}\}$ the dual basis with respect to $S_X(\,\cdot \cup \cdot\,)$, i.e., $S_X(f_j^{2-r} \cup e_i^r) = \delta_{ij}$. By Künneth formula, we can write

$$\operatorname{cl}(\Gamma_\varphi) = \sum_{i,r} p^*(\alpha_i^r) \cup q^*(f_i^{2-r})$$

for some $\alpha_i^r \in H^r(X, \mathbb{Q}_\ell)$. By **Claim**, we have

$$\begin{aligned}
\varphi^*(e_j^s) &= p_*(\mathrm{cl}(\Gamma_\varphi) \cup q^*(e_j^s)) \\
&= p_*(\sum_i p^*(\alpha_i^r) \cup q^*(f_i^{2-r}) \cup q^*(e_j^s)) \\
&= \sum_i \alpha_i \cup p_* q^*(f_i^{2-r} \cup e_j^s) = \alpha_j.
\end{aligned}$$

So

$$\mathrm{cl}(\Gamma_\varphi) = \sum_{i,r} p^*(\varphi^*(e_i^r)) \cup q^*(f_i^{2-r}).$$

Applying the same calculation to the morphism id_X for basis $\{f_i^r\}$ and its dual basis $\{(-1)^r e_i^{2-r}\}$, we get

$$\mathrm{cl}(\Delta) = \sum_{i,r} (-1)^r p^*(f_i^r) \cup q^*(e_i^{2-r}).$$

Hence it now follows that

$$\begin{aligned}
\Delta \cdot \Gamma_\varphi &= S_{X\times X}(\mathrm{cl}(\Delta) \cup \mathrm{cl}(\Gamma_\varphi)) \\
&= S_{X\times X}\left(\sum_{i,j,r,s} (-1)^r p^*(f_i^r) \cup q^*(e_i^{2-r}) \cup p^*(\varphi^*(e_j^s)) \cup q^*(f_j^{2-s})\right) \\
&= S_{X\times X}\left(\sum_{i,j,r,s} (-1)^{r+s(2-r)} p^*(f_i^r \cup \varphi^*(e_j^s)) \cup q^*(e_i^{2-r} \cup f_j^{2-s})\right) \\
&= \sum_{i,r} S_{X\times X}(p^*(f_i^r \cup \varphi^*(e_i^{2-r})) \cup q^*(e_i^{2-r} \cup f_i^r)) \\
&= \sum_{i,r} S_X(f_i^r \cup \varphi^*(e_i^{2-r})) = \mathrm{tr}(\varphi^*; H^*(X, \mathbb{Q}_\ell)).
\end{aligned}$$

■

23 Lefschetz pencils

Let X be a variety. A point x on X is an ordinary double point if the tangent cone at x is isomorphic to the subvariety of $\mathbb{A}^{\dim X+1}$ defined by a non-degenerate quadratic form $Q(T_1, \ldots, T_{d+1})$, or, equivalently, if

$$\widehat{\mathcal{O}}_{X,x} \cong k[[T_1, \ldots, T_{d+1}]]/(Q(T_1, \ldots, T_{d+1})).$$

Now, let X be a nonsingular projective variety of dimension $d \geq 2$, and embed X is some projective space $\mathbb{P}^m$.

Definition 23.1. An embedding

$$i : X \hookrightarrow \mathbb{P}^r$$

is called a Lefschetz embedding if there is a closed subset $S \subseteq \check{\mathbb{P}}^r$ of codimension ≥ 2 such that the following conditions are satisfied by every hyperplane H with coordinates in $\bar{k}$ that does not lie in S:

(a) H does not contain any connected components of X.

(b) The (scheme-theoretic) intersection $H \cap X$ contains at most one singular point, and the singularity is an ordinary double point if it exists.

Proposition 23.2. Let X be an irreducible smooth projective variety, $\dim X \geq 1$, over an infinite base field k. Let $i : X \hookrightarrow \mathbb{P}^r$ be a projective embedding. Denote by

$$i(d) : X \hookrightarrow \mathbb{P}^N$$

the composition of i with the Segre embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^N$ of degree d. Then $i(d)$ for $d \geq 3$ is a Lefschetz embedding.

Remark. In fact, we have:

(a) In the case of characteristic 0, $i = i(1)$ is a Lefschetz embedding.

(b) $i(d)$ is always a Lefschetz embedding for $d \geq 2$.

Proof.

Lemma 23.3. The set

$$Y_i = \{(x, H) \in X \times \check{\mathbb{P}}^r \mid \ H \text{ touches } X \text{ at } x \ \}$$

is an irreducible smooth closed subvariety of dimension $r - 1$ of $X \times \check{\mathbb{P}}^r$ (resp. $= \varnothing$, when $X = \mathbb{P}^r$). More precisely:

Let $\mathscr{I}$ be the sheaf of ideals in $\mathcal{O}_{\mathbb{P}^r}$ associated with (X, i). Then Y_i is isomorphic to the projective bundle space of the conormal bundle $(\mathscr{I}/\mathscr{I}^2)|_X$.

Lemma 23.4. With notation as in (23.3), the set

$$Y_i' = \{(x, H) \in Y_i \mid \ x \text{ is not an ordinary double point of } X \cap H \ \}$$

is Zariski-closed in Y_i.

Proof. Let $(x_0, H) \in Y_i \setminus Y_i'$. We may assume that $x_0 = [1 : 0 : \cdots : 0]$, $x_i = X_i/X_0$ and T_xX is defined by

$$x_{n+1} = \cdots = x_r = 0.$$

Then $\widehat{\mathcal{O}}_{\mathbb{P}^r,x_0} = k[[x_1, \ldots, x_r]]$ and the defining equations of X in $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathbb{P}^r,x_0}$ are of the form

$$g_i = x_{n+i} - h_i(x_1, \ldots, x_n)$$

where $h_i \in (x_1, \ldots, x_n)^2$. We may assume that H is defined by g_{r-n}. Let $V = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_0}$, then there's a trivialization

$$\begin{array}{ccc} \mathcal{O}_V^{r-n} & \xrightarrow{\sim} & \mathscr{I}/\mathscr{I}^2|_V \\ (\lambda_1, \ldots, \lambda_{r-n}) & \mapsto & \sum \lambda_i g_i. \end{array}$$

So there's a neighborhood W of (x_0, H) in $\mathbb{P}(\mathscr{I}/\mathscr{I}^2|_V)$ such that

$$\begin{array}{ccc} V \times \mathbb{A}^{r-n-1} & \xrightarrow{\sim} & W \\ (v, \lambda_1, \ldots, \lambda_{r-n-1}) & \mapsto & \left(v, g_{r-n} + \displaystyle\sum_{i=1}^{r-n-1} \lambda_i g_i \right) \end{array}$$

and hence $\widehat{\mathcal{O}}_{\mathbb{P}(\mathscr{I}/\mathscr{I}^2),(x_0,H)} \cong k[[x, \lambda]]$. Let $\varphi : \mathbb{P}(\mathscr{I}/\mathscr{I}^2) \to \check{\mathbb{P}}^r$ be the projection, write

$$\varphi(x, \lambda) = \left(\sum_{i=0}^{r} \varphi_i(x, \lambda) X_i = 0 \right).$$

Since $V(\sum \varphi_i X_i)$ contains $(x) = (1, x_1, \ldots, x_n, h_1(x), \ldots, h_{r-n}(x))$, we get, in $k[[x, \lambda]]$,

$$\varphi_0(x, \lambda) = -\sum_{i=1}^{n} x_i \varphi_i(x, \lambda) - \sum_{i=1}^{r-n} h_i(x) \varphi_{n+i}(x, \lambda)$$

and

$$\varphi_j(x, \lambda) = \partial_j \left(g_{r-n} + \sum_{i=1}^{r-n-1} \lambda_i g_i \right)(x).$$

Using $g_i = x_{n+i} - h_i$, we get

$$\varphi_j(x, \lambda) = \begin{cases} -\partial_j \left(h_{r-n} + \sum \lambda_i h_i \right), & \text{if } 1 \le j \le n \\ \lambda_{j-n}, & \text{if } n < j < r \\ 1, & \text{if } j = r. \end{cases}$$

Then

$$\frac{\partial\varphi}{\partial(x,\lambda)}(0) = \begin{pmatrix} 0 & 0 \\ -\dfrac{\partial^2 h_{r-n}}{\partial x_i \partial x_j} & 0 \\ 0 & I_{r-n-1} \\ 0 & 0 \end{pmatrix}.$$

So φ is unramified at (x_0, H) if and only if

$$\left(\frac{\partial^2 h_{r-n}}{\partial x_i \partial x_j}(0)\right)$$

is invertible. On the other hand,

$$\widehat{\mathcal{O}}_{X\cap H, x_0} \cong k[[x_1, \ldots, x_n]]/(h_{r-n}),$$

so Y_i' is the intersection of Y_i and the ramified points of φ, which is closed. □

We now consider the Segre embedding

$$i(d) : X \to \mathbb{P}^N, \quad N = \binom{r+d}{d} - 1.$$

There is a 1-1 correspondence between the hyperplanes in $\mathbb{P}^N$ and the hypersurfaces of degree d in $\mathbb{P}^r$.

Lemma 23.5. Assume $d \geq 2$. For every closed point $a \in X$, there is a hypersurface F of degree d which touches X at a and is such that a is an ordinary double point in $X \cap F$. In particular,

$$\dim Y'_{i(d)} \leq N - 2 \quad (= \dim Y_{i(d)} - 1).$$

We can choose a homogeneous coordinate system $y_0, \ldots, y_r$ in such a way that $[1 : 0 : \cdots : 0]$ are the homogeneous coordinates of $a \in X$ and the functions (y_1/y_0), …, (y_n/y_0) form a regular system of parameters for X at a. There exists a nonsingular quadratic form Q in the variables (y_1/y_0), …, (y_n/y_0). We can take our hypersurface the set of zeros of the homogeneous polynomial $y_0^{d-2}Q(y_1, \ldots, y_n)$. □

Lemma 23.6. The set

$$Z(d) = \{(x, y, H) \in X \times X \times \check{\mathbb{P}}^N \mid x \neq y,\ H \text{ touches } X \text{ at } x \text{ and } y\ \}$$

is Zariski closed in $(X \times X \setminus \Delta) \times \check{\mathbb{P}}^N$, and

$$\dim Z(d) \leq N - 2$$

for $d \geq 3$.

The Zariski closure follows from (23.3). For the bound on the dimension, it suffices to show that every fiber of the projection mapping

$$Z(d) \to X \times X - \Delta$$

has dimension at most $N - 2 - 2 \cdot \dim X$.

The fiber of this projection over (a, b) is the collection of all hypersurfaces of degree d that touches the projective tangent space of X at a and of X at b. The required bound then follows from the following assertion:

> Suppose we have two linear subspaces $V, W \subseteq \mathbb{P}^r$ and points $a \in V$, $b \in W$, $a \neq b$. Consider the vector space
>
> $$\bar{U} = \{f \in k[T_0, \ldots, T_r] \mid f \text{ is a homogeneous polynomial of degree } d\} \cup \{0\}.$$
>
> This space has dimension $N + 1$. The subspace of all equations f whose sets of zeros $V(f) \subseteq \mathbb{P}^r$ touch the space V at a and the space W at b is a linear subspace U of $\bar{U}$, and
>
> $$\operatorname{codim}_{\bar{U}} U \geq \dim V + \dim W + 2.$$

Let $a_0 = a$, $a_1, \ldots, a_{\dim V}$ be independent points in V, and $b_0 = b$, $b_1, \ldots, b_{\dim W}$ independent points in W. The condition on $V(f)$ is

(i) $a_0, b_0 \in V(f)$.

(ii) $a_i \in T_a(V(f))$ and $b_i \in T_b(V(f))$ for $i \geq 1$.

These conditions are linear conditions and there are $\dim V + \dim W + 2$ of them. We must show that they are linearly independent. Obviously it suffices to handle the case where $V = W = \mathbb{P}^r$. Then $V(f)$ is singular at a and b.

After suitable change of coordinates, we can assume that

$$a = [1 : 0 : \cdots : 0], \quad b = [0 : 1 : 0 : \cdots : 0].$$

Let $f = \sum_{|I|=d} a_I T^I$. Then the conditions say:

$$a, b \in V(f) \implies a_{(d,0,\ldots,0)} = a_{(0,d,0,\ldots,0)} = 0.$$

$V(f)$ is singular at a and b means

$$\frac{\partial f}{\partial T_j}[1:0:\cdots:0] = \frac{\partial f}{\partial T_j}[0:1:0:\cdots:0] = 0$$

for $0 \leq j \leq r$. The coefficients of $T_0^{d-1}T_j$ and $T_1^{d-1}T_j$ must therefore vanish. The vector space of equations f that satisfy these condition has codimension $2r+2$. Thus the $2r+2$ linear conditions are linear independent. $\square$

Back to the main theorem, consider the union S of the Zariski closures of the projections of $Y'_{i(d)}$ and $Z(d)$ in $\check{P}^N$. (23.5) and (23.6) show that $\dim S \leq N-2$. By the definition of $Y'_{i(d)}$ and $Z(d)$, every hyperplane not contained in S meets X in just one ordinary point. $\blacksquare$

A line D in $\check{\mathbb{P}}^m$ is called a pencil of hyperplanes in $\mathbb{P}^m$. The axis of D is

$$A = H_0 \cap H_\infty = \bigcap_{H \in D} H.$$

Definition 23.7. A line D in $\check{\mathbb{P}}^m$ is said to be a Lefschetz pencil for $X \subseteq \mathbb{P}^m$ if

(a) the axis A of the pencil cuts X transversally;

(b) $D \cap S = \varnothing$, i.e., the hyperplane sections $X_t = X \cap H_t$ of X are nonsingular for all t in some open dense subset U of D; and for $t \notin U$, X_t has only a single singularity, and the singularity is an ordinary double point.

The conditions (a) and (b) are satisfied by almost all lines. We now consider the following (reduced) projective subvariety

$$\widetilde{X} = \{(x, H) \in X \times D \mid x \in H\}$$

of $X \times D$. We denote the two projections by $f : \widetilde{X} \to D$ and $\pi : \widetilde{X} \to X$. The fibers of f are precisely the intersections of X with the hyperplanes in D. One can show that $\widetilde{X}$ arises from X by blowing up the subvariety $A \cap X$. In particular, $\widetilde{X}$ is nonsingular.

Proposition 23.8. The canonical mapping

$$H^p(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^p(X, \mathbb{Z}/\ell\mathbb{Z}),$$

where $\operatorname{char} k \nmid \ell$, is injective.

Proof. For $p = 2n = 2\dim X$. The assertion follows from the existence of the trace mapping. For $p < 2n$. This case is reduced to the first case by using the cup-product pairing

$$H^p(X, \mathbb{Z}/\ell\mathbb{Z}) \times H_c^{2n-p}(X, \mathbb{Z}/\ell Z) \to H_c^{2n}(X, \mathbb{Z}/\ell\mathbb{Z}).$$

We need the fact that the cup product is nondegenerate and compatible with inverse images. ∎

24 Classification of nondegenerate double points

Definition 24.1. Let A be a local ring, $V = \bigoplus_{i=1}^{n} Ae_i$ a finitely generated free A-module. A mapping $Q : V \to A$ is called a quadratic form if it can be expressed in coordinates as

$$Q(x_1e_1 + \cdots x_ne_n) = \sum a_{ij}x_ix_j.$$

Q is called nondegenerate if the associated bilinear form

$$B(v, w) = Q(v + w) - Q(v) - Q(w)$$

is nondegenerate.

A homogeneous polynomial $P \in A[x_1, \ldots, x_n]$ of degree 2 defines a quadratic form on A^n in a natural way. We identify P with this quadratic form.

Theorem 24.2 (Classification Theorem)**.** Let $Q : V \to A$ be a nondegenerate quadratic form over a strictly Henselian local ring A. Let $n = \dim V$. Then

Case 1. If $n = 2m$, V has a basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ such that

$$Q(x_1e_1 + \cdots + x_me_m + y_1f_1 + \cdots y_mf_m) = \sum_i x_iy_i.$$

Case 2. If $n = 2m + 1$, V has a basis $e_1, \ldots, e_m, f_1, \ldots, f_m, g$ such that

$$Q(x_1e_1 + \cdots + x_me_m + y_1f_1 + \cdots y_mf_m + zg) = \sum_i x_iy_i + z^2.$$

The second case can only occur when the residue field characteristic of A is different from 2.

The isomorphism class of the quadratic module (V, Q) thus depends only on $\dim V$.

Proposition 24.3. Let A be a local ring, $F \in A[[x_1, \dots, x_n]]$, and $\mathfrak{a} = \langle x_1, \dots, x_n \rangle \trianglelefteq A[[x_1, \dots, x_n]]$. Assume that there is a nondegenerate quadratic form $Q(x)$ over A and an element a in A with

$$F(x) = Q(x) + a \pmod{\mathfrak{a}^{m+1}}$$

for some integer $m \geq 2$.

Then there are elements $h_1, \dots, h_n$ in $\mathfrak{a}^m$ with

$$F(x_1 + h_1, \dots, x_n + h_n) = Q(x) + a,$$

i.e., $A[[x]]/(F)$ is A-isomorphic to $A[[x]]/(Q(x) + a)$.

Proof. Since $A[[x]]$ is complete in the $\mathfrak{a}$-adic topology, it is enough to show that for every m there are elements in $h_1, \dots, h_n$ in $\mathfrak{a}^m$ with

$$F(x + h) = F(x_1 + h_1, \dots, x_n + h_n) = Q(x) + a \pmod{\mathfrak{a}^{m+2}}.$$

Let $B(x, y) = \sum b_i(x_i) y_i$ be the bilinear form associated with $Q(x)$. Since it is nondegenerate, the linear polynomials $b_i(x)$ generate the ideal $\mathfrak{a}$. By hypothesis we have

$$F(x) = Q(x) + a - q(x), \quad q(x) \in \mathfrak{a}^{m+1}.$$

For an arbitrary n-tuple $h = (h_1, \dots, h_n)$ in $\mathfrak{a}^m$, we have $Q(h) \in \mathfrak{a}^{2m} \subseteq \mathfrak{a}^{m+2}$,

$$\begin{aligned} q(x+h) &= q(x) \pmod{a^{m+2}}, \\ F(x+h) &= Q(x+h) + a - q(x+h) \\ &= Q(x) + B(x,h) + Q(h) + a - q(x+h) \\ &= Q(x) + a + \sum_i b_i(x) h_i - q(x) \pmod{a^{m+2}} \end{aligned}$$

Since $\mathfrak{a}^{m+1} = b_1 \mathfrak{a}^m + \cdots + b_n \mathfrak{a}^m$, there are elements $h_1, \dots, h_n$ in $\mathfrak{a}^m$ with

$$b_1(x) h_1 + \cdots + b_n(x) h_n = q(x).$$

■

Proposition 24.4. Let $(A, \mathfrak{m})$ be a complete local ring, $Q(x_1, \dots, x_n)$ a nondegenerate quadratic form, $a, a_1, \dots, a_n$ elements in $\mathfrak{m}$, and $F \in A[[x_1, \dots, x_n]]$ of the form

$$F(x) = a + \sum_i a_i x_i + Q(x) \pmod{(x_1, \dots, x_n)^3}.$$

Then there is an element $a' \in \mathfrak{m}$ and a nondegenerate quadratic form Q' with

$$A[[x]]/(F) \cong_A A[[x]]/(Q'(x) + a)$$

Proof. By (24.3), it is enough to show that there is an n-tuple α of elements in $\mathfrak{m}$ with

$$F(x+\alpha) = a' + Q'(x) \pmod{(x_1, \dots, x_n)^3}.$$

This congruence is equivalent to the system of equations

$$\frac{\partial F}{\partial x_i}(\alpha) = 0.$$

The existence of a solution α follows from the implicit function theorem, since the matrix

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j}(0)\right) = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(0)\right)$$

is nondegenerate. ∎

Let $\varphi : A \to B$ be a locally finitely generated local homomorphism.

Case 1. Let $A = k = \bar{k}$. Then B/k resp. φ is called nondegenerately quadratically singular when the completion $\widehat{B}$ of B over k is isomorphic to

$$k[[x_1, \dots, x_n]]/(f(x_1, \dots, x_n)),$$

where f is a power series that is congruent modulo $(x_1, \dots, x_n)^3$ to a nondegenerate quadratic form Q.

Case 2. Let k be the algebraic closure of the residue field of A. Then B/A resp. φ is called nondegenerately quadratically singular if φ is flat and there is a localization $(B \otimes_A k)_{\mathfrak{p}}$ for which $B \to (B \otimes_A k)_{\mathfrak{p}}$ is local and $k \to (B \otimes_A k)$ is nondegenerately quadratically singular.

Definition 24.5. Let $f : X \to S$ be a morphism of finite type. A point a of X is called a nondegenerate double point of f if

$$\mathcal{O}_{S,f(a)} \to \mathcal{O}_{X,a}$$

is nondegenerately quadratically singular.

Here is our main result.

Proposition 24.6. Let $A \to B$ be a nondegenerately quadratically singular homomorphism. Assume that A is strictly Henselian with maximal ideal $\mathfrak{m}$. Let $Q(x_0, \dots, x_n)$, $n = \dim B - \dim A$, be a nondegenerate quadratic form over A. Then there is an element

$a \in \mathfrak{m}$ with the following property: Let $\mathfrak{n}$ be the maximal ideal generated by $x_0, \ldots, x_n$ and $\mathfrak{m}$, and let

$$R = (A[x_0, \ldots, x_n]/(Q(x) + a))_{\mathfrak{n}}.$$

The Henselizations of B and R are isomorphic over A, i.e.,

$$\widetilde{B} \cong_A \widetilde{R}$$

Here is the geometric form of this theorem.

Proposition 24.7. Let $S = \operatorname{Spec} A$ be the spectrum of a strictly Henselian ring with maximal ideal $\mathfrak{m}$, and

$$f : X \to S$$

a finitely generated morphism. Let the point a in X be a nondegenerate double point over $\mathfrak{m}$ of f. Let $n = \dim_a f^{-1}(f(a)) + 1$. Let $\Omega = (A/\mathfrak{m})^{\text{sep}}$, and let

$$\alpha : \operatorname{Spec} \Omega \to X$$

over $\operatorname{Spec} A$ be a geometric point of X centered at a. Consider a nondegenerate quadratic form $Q(x_1, \ldots, x_n)$, an element u in $\mathfrak{m}$, and the S-scheme

$$Y = \operatorname{Spec}(A[x]/(Q + u)).$$

Let $\beta : \operatorname{Spec} \Omega \to Y$ over $\operatorname{Spec} A$ be the geometric point of Y under which the elements $x_1, \ldots, x_n$ sends to 0. For a suitable choice of the element u in $\mathfrak{m}$, there are étale neighborhoods U of α and V of β that are isomorphic over S.

25 Monodromy theory

Let $(R, \mathfrak{m})$ be a strictly Henselian DVR with residue field k and quotient field K, and let

$$s : \operatorname{Spec} k \to \operatorname{Spec} R, \ \eta : \operatorname{Spec} \bar{K} \to \operatorname{Spec} R$$

be the geometric points of $S = \operatorname{Spec} R$.

Let $f : X \to \operatorname{Spec} R$ be a quasi-projective, flat scheme over $\operatorname{Spec} R$ with fibers $X_s = X \times_S \operatorname{Spec} k$ (called the special fiber) and $X_\eta = X \times_S \operatorname{Spec} \bar{K}$ (called the generic fiber), we have the diagram:

$$\begin{tikzcd}
X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\
\operatorname{Spec} \bar{K} \arrow[r, "\eta"] & S & \operatorname{Spec} k. \arrow[l, "s"']
\end{tikzcd}$$

We always assume that all fibers of X at all points have dimension $n = n(X/S)$.

Let $r \in \mathbb{N}$ such that $\operatorname{char} k \nmid r$, and set

$$\Lambda = \mathbb{Z}/r\mathbb{Z}.$$

Let $\mathscr{F}$ be a sheaf (of Λ-modules) on X. We are interested above all in the following case:

(a) f is proper and smooth apart from one singularity in the special fiber X_s. This singularity is an ordinary double point.

(b) $\mathscr{F} = \Lambda_X$.

Because of the proper base change theorem, we have the canonical isomorphisms

$$(\mathsf{R}^\bullet f_! \mathscr{F})_s \cong H_c^\bullet(X_s, \mathscr{F}_s), \quad (\mathsf{R}^\bullet f_! \mathscr{F})_\eta \cong H_c^\bullet(X_\eta, \mathscr{F}_\eta).$$

The specialization mapping

$$(\mathsf{R}^\bullet f_! \mathscr{F})_s \to (\mathsf{R}^\bullet f_! \mathscr{F})_\eta$$

induces a mapping

$$H_c^\bullet(X_s, \mathscr{F}_s) \to H_c^\bullet(X_\eta, \mathscr{F}_\eta).$$

For an arbitrary sheaf $\mathscr{G}$ on X_η, we have

$$\mathsf{R}\Gamma(X_\eta, \mathscr{G}) = \mathsf{R}\Gamma(X, \mathsf{R}j_* \mathscr{G}).$$

Together with the restriction mapping to the fiber, this gives us

$$\mathsf{R}\Gamma(X_\eta, \mathscr{G}) \to \mathsf{R}\Gamma(X_s, i^* \mathsf{R}j_* \mathscr{G}).$$

This mapping is an isomorphism for proper f. To compute the cohomology with compact support, we consider a compactification

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "f"] & \bar{X} \arrow[d, "\bar{f}"] \\
& S.
\end{tikzcd}$$

We get a diagram

$$\begin{tikzcd}
\bar{X}_\eta \arrow[r, "\bar{j}"] & \bar{X} & \bar{X}_s \arrow[l, "\bar{i}"'] \\
X_\eta \arrow[u, "\varphi_\eta"] \arrow[r, "j"] \arrow[d, "f_\eta"] & X \arrow[u, "\varphi"] \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[u, "\varphi_s"] \arrow[d, "f_s"] \\
\operatorname{Spec} \bar{K} \arrow[r, "\eta"] & S & \operatorname{Spec} k. \arrow[l, "s"']
\end{tikzcd}$$

By definition,

$$H_c^\bullet(X_\eta, \mathscr{G}) = H^\bullet(\bar{X}_\eta, (\varphi_\eta)_!\mathscr{G}).$$

Checking on stalks, there's a canonical isomorphism

$$\mathsf{R}j_*\mathscr{G} \cong \varphi^*\mathsf{R}\bar{j}_*(\varphi_\eta)_!\mathscr{G}$$

in the derived category. Adjoint to this is a mapping

$$\varphi_!\mathsf{R}j_*\mathscr{G} \to \mathsf{R}\bar{j}_*(\varphi_\eta)_!\mathscr{G}.$$

We restrict this mapping to the special fiber $\bar{X}_s$. Since extension by zero is compatible with formation of inverse images, we get

$$(\varphi_s)_!i^*\mathsf{R}j_*\mathscr{G} \to \bar{i}^*\mathsf{R}\bar{j}_*(\varphi_\eta)_!\mathscr{G},$$

and from this

$$H_c^p(X_s, i^*\mathsf{R}j_*\mathscr{G}) = H^p(\bar{X}_s, (\varphi_s)_!i^*\mathsf{R}j_*\mathscr{G}) \to H^p(\bar{X}_s, \bar{i}^*\mathsf{R}\bar{j}_*(\varphi_\eta)_!\mathscr{G}).$$

Since $\bar{f}$ is proper,

$$H^p(\bar{X}_\eta, (\varphi_\eta)_!\mathscr{G}) \to H^p(\bar{X}_s, i^*\mathsf{R}j_*(\varphi_\eta)_!\mathscr{G})$$

is an isomorphism. We compose its inverse with the map just constructed and finally get

$$H_c^p(X_s, i^*\mathsf{R}j_*\mathscr{G}) \to H_c^p(X_\eta, \mathscr{G}).$$

The special case $\mathscr{G} = \mathscr{F}_\eta = j^*\mathscr{F}$ gives

$$H^p(X_\eta, \mathscr{F}_\eta) \to H^p(X_s, i^*\mathsf{R}j_*\mathscr{F}_\eta)$$

$$H_c^p(X_\eta, \mathscr{F}_\eta) \leftarrow H_c^p(X_s, i^*\mathsf{R}j_*\mathscr{F}_\eta)$$

that are isomorphisms inverse to each other for proper X/S.

Let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution. From this we get a resolution $\mathscr{F}_\eta \to \mathscr{I}_\eta^\bullet$. In fact, this is an acyclic resolution of $\mathscr{F}_\eta$:

Since X_η and therefore U is a inverse limit of schemes étale over X. The restriction of an injective sheaf to an étale scheme remains injective. The assertion

$$H^i(U, \mathscr{I}_\eta|_U) = 0$$

for $i > 0$ follows then from the limit theorems.

Let

$$\mathscr{A}^\bullet = \mathscr{A}^\bullet(\mathscr{F}, f) = i^* j_* \mathscr{I}_\eta^\bullet, \quad \mathscr{A}^\bullet(f) = \mathscr{A}^\bullet(\Lambda_X, f).$$

Then $\mathscr{A}^\bullet$ represents $i^* \mathsf{R} j_* \mathscr{F}_\eta$ in the derived category.

Functoriality of $\mathscr{A}^\bullet$: The adjunction mapping

$$\mathscr{I}^\bullet \to j_* j^* \mathscr{I}^\bullet = j_* \mathscr{I}_\eta^\bullet$$

induces a mapping $\varphi : \mathscr{F}_s \to \mathscr{A}^\bullet$. The pair $(\mathscr{A}^\bullet, \varphi)$ is unique in the following sense.

Let

$$\mathscr{F} \to \widetilde{\mathscr{I}^\bullet}$$

be another injective resolution, and $(\widetilde{A}, \widetilde{\varphi})$ the associated pair. It is well-known that there exists a mapping of complexes $h : \mathscr{I}^\bullet \to \widetilde{\mathscr{I}^\bullet}$ such that the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{I}^\bullet \arrow[d, "h"] \\ & \widetilde{\mathscr{I}^\bullet} \end{tikzcd}$$

is commutative in the category of complexes. The mapping h is uniquely determined up to homotopy. It induces a commutative diagram

$$\begin{tikzcd} \mathscr{F}_s \arrow[r] \arrow[dr] & \mathscr{A}^\bullet \arrow[d, "h_0"] \\ & \widetilde{\mathscr{A}^\bullet}. \end{tikzcd}$$

The mapping h_0 is a quasi-isomorphism and uniquely determined by this construction up to homotopy.

We have constructed natural mappings

$$H^p(X_\eta, \mathscr{F}_\eta) \to H^p(X_s, \mathscr{A}^\bullet)$$

$$H^p_c(X_\eta, \mathscr{F}_\eta) \leftarrow H^p_c(X_s, \mathscr{A}^\bullet)$$

that are isomorphisms in the proper case. If we compose the second of these mappings with the mapping

$$H_c^p(X_s, \mathscr{F}_s) \xrightarrow{\varphi_\sharp} H_c^p(X_s, \mathscr{A}^\bullet),$$

we get the specialization mapping

$$H_c^p(X_s, \mathscr{F}_s) \to H_c^p(X_\eta, \mathscr{F}_\eta).$$

25.1 The action of the Galois group $G = \mathrm{Gal}(\bar{K}/K)$

Each element $\sigma \in \mathrm{Gal}(\bar{K}/K)$ defines an automorphism $\operatorname{Spec} \bar{K} \to \operatorname{Spec} \bar{K}$, again denoted by σ, and hence defines an automorphism

$$\mathrm{id}_X \times \sigma : X \times_S \operatorname{Spec} \bar{K} \to X \times_S \operatorname{Spec} \bar{K}$$

For simplicity we once again denote this by

$$\sigma : X_\eta \to X_\eta.$$

The Galois group acts on the scheme X_η on the right. We describe the action of G on the cohomology $H^*(X_\eta, \mathscr{F}_\eta)$.

Since $j \circ \sigma = j$. We have a canonical isomorphism

$$\sigma^* \mathscr{F}_\eta \to \mathscr{F}_\eta.$$

This induces

$$\mathscr{F}_\eta \to \sigma_* \mathscr{F}_\eta$$

and hence a homomorphism

$$H^*(X_\eta, \mathscr{F}_\eta) \to H^*(X_\eta, \mathscr{F}_\eta).$$

The Galois group acts on the cohomology groups on the left. Our goal is to describe this representation of G precisely.

Let G be a profinite group. A G-sheaf is a sheaf $\mathscr{F}$ of Λ-modules where G acts on every module of sections $\mathscr{F}(U)$, in such a way that this action is compatible with restriction. We require the usual continuity property:

> The stabilizer of an element $s \in \mathscr{F}(U)$ is open in G (and hence has finite index).

Back to our situation. The Galois group acts trivially on X. We show that $j_*\mathscr{F}_\eta$ has a natural G-sheaf structure. We have already assigned a mapping

$$H^p(X_\eta, \mathscr{F}_\eta) \to H^p(X_\eta, \mathscr{F}_\eta)$$

to each $\sigma \in G$. This mapping is defined more generally for any étale scheme U/X. In the special case $p = 0$ we get a family of mappings

$$(j_*\mathscr{F}_\eta)(U) = \mathscr{F}_\eta(U_\eta) \to \mathscr{F}_\eta(U_\eta).$$

This family represents a sheaf mapping

$$j_*\mathscr{F}_\eta \to j_*\mathscr{F}_\eta.$$

The assignment

$$\mathscr{F} \mapsto j_*\mathscr{F}_\eta$$

is a functor from the category of sheaves on X to the category of G-sheaves on X. The adjunction mapping

$$\mathscr{F} \to j_*j^*\mathscr{F}$$

is G-equivariant, if one lets G act trivially on $\mathscr{F}$.

We recall the construction of the complex $\mathscr{A}^\bullet$. First we consider an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$. Applying the functor j_*j^* to this, we get a morphism

$$j_*\mathscr{F}_\eta \to j_*\mathscr{I}_\eta^\bullet$$

This is an equivariant morphism from the G-sheaf $j_*\mathscr{F}_\eta$ to a complex of G-sheaves. In particular, the composite morphism

$$\mathscr{F} \to j_*j^*\mathscr{F} = j_*\mathscr{F}_\eta \to j_*\mathscr{I}_\eta^\bullet$$

is equivariant. We restrict this mapping to the special fiber X_s. We let the Galois group G act trivially on X_s. The restriction $i^*\mathscr{G}$ of a G-sheaf on X is a G-sheaf on X_s. The morphism

$$i^*\mathscr{F} = \mathscr{F}_s \to i^*j_*\mathscr{I}_\eta^\bullet = \mathscr{A}^\bullet(\mathscr{F}, f)$$

is an equivariant morphism of the trivial G-sheaf $\mathscr{F}_s$ into a complex of G-sheaves.

The assignment

$$\mathscr{F} \mapsto (\mathscr{F}_s, \varphi, \mathscr{A}^\bullet)$$

is a functor from the category of sheaves to an "equivariant triple category": The objects are the equivariant homomorphisms

$$\varphi : \mathscr{G} \to \mathscr{K}^\bullet$$

of G-trivial sheaf $\mathscr{G}$ on X_s to positive complex $\mathscr{K}^\bullet$ of G-sheaves. The morphisms are the commutative diagrams

$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & \mathscr{K}^\bullet \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta^\bullet} \\ \widetilde{\mathscr{G}} & \longrightarrow & \widetilde{\mathscr{K}}^\bullet \end{array}$$

where $\beta^\bullet$ is an equivariant homomorphism of complexes uniquely determined modulo G-equivariant homotopy.

The natural mapping

$$H^*(X_\eta, \mathscr{F}_\eta) \to H^*(X_s, \mathscr{A}^\bullet)$$

is G-equivariant. If one applies this construction to a compactification of X/S, one sees that

$$H_c^*(X_\eta, \mathscr{F}_\eta) \leftarrow H_c^*(X_s, \mathscr{A}^\bullet)$$

is also in a natural way G-equivariant.

We now consider the mapping cone of the mapping of complexes

$$\varphi : \mathscr{F}_s \to \mathscr{A}^\bullet.$$

Definition 25.1. Let the complex $\mathscr{B}^\bullet(\mathscr{F}, f)$ be the mapping cone of φ, given as the complex

$$\cdots \to 0 \to \mathscr{F}_s \xrightarrow{\varphi} \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots$$

We set $\mathscr{B}^\bullet(f) = \mathscr{B}^\bullet(\Lambda_X, f)$. In accordance with its later significance, we call $\mathscr{B}^\bullet$ the complex of vanishing cycles.

The distinguished triangle

$$\mathscr{F}_s \to \mathscr{A}^\bullet \to \mathscr{B}^\bullet \to \mathscr{F}_s[1]$$

gives a long exact cohomology sequence. If X/S is proper, this runs

$$\cdots \to H^p(X_s, \mathscr{F}_s) \to H^p(X_\eta, \mathscr{F}_\eta) \to H^p(X_s, \mathscr{B}^\bullet) \to H^{p+1}(X_s, \mathscr{F}_s) \to \cdots$$

The hypercohomology of the complex $\mathscr{B}^\bullet$ thus describes the derivation between the cohomology of the generic fiber and the special fiber. We investigate the complex $\mathscr{B}^\bullet = \mathscr{B}^\bullet(f)$ in the case of the constant sheaf $\mathscr{F} = \Lambda_X$.

By the smooth base change theorem for the diagram

$$\begin{array}{ccc} X_\eta & \xrightarrow{j} & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \bar{K} & \xrightarrow{\eta} & S, \end{array}$$

we have

$$\Lambda_X = f^*\eta_*\Lambda_{\operatorname{Spec}\bar{K}} = f^*\mathsf{R}\eta_*\Lambda_{\operatorname{Spec}\bar{K}} \to \mathsf{R}j_*\Lambda_{X_\eta} = j_*\mathscr{I}_\eta^\bullet$$

is a quasi-isomorphism at all smooth points of X/S. In particular,

$$\Lambda_{X_s} = i^*\Lambda_X \to i^*j_*\mathscr{I}_\eta^\bullet = \mathscr{A}^\bullet(f)$$

is a quasi-isomorphism on the smooth locus of X_s. So the restriction of $\mathscr{B}^\bullet = \mathscr{B}^\bullet(f)$ to the smooth locus is exact.

We assume that the mapping $f : X \to S$ is smooth apart from a single point $a \in X_s$.

Lemma 25.2. Let the mapping $f : X \to S$ be smooth apart from a single point $a \in X_s$. Then the complex $\mathscr{B}_{\bar{x}}^\bullet$ is exact for all geometric points x apart from a. In particular, there are canonical isomorphisms

$$H_c^p(X, \mathscr{B}^\bullet) = H^p(X, \mathscr{B}^\bullet) = H_{\{a\}}^p(X, \mathscr{B}^\bullet) = H^p(\mathscr{B}_{\bar{a}}^\bullet).$$

25.2 The variation

The Galois group $G = \operatorname{Gal}(\bar{K}/K)$ acts on the complex $\mathscr{A}^\bullet$. Since G acts trivially on $\mathscr{F}_s$, for every $\sigma \in G$ the mappings

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathscr{F}_s & \longrightarrow & \mathscr{A}^0 & \longrightarrow & \mathscr{A}^1 & \longrightarrow & \cdots \\ & & \downarrow 0 & & \downarrow \sigma-1 & & \downarrow \sigma-1 & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathscr{A}^0 & \longrightarrow & \mathscr{A}^1 & \longrightarrow & \cdots \end{array}$$

form a chain map. We will denote it by

$$\sigma - 1 : \mathscr{B}^\bullet \to \mathscr{A}^\bullet.$$

We apply to the mapping $\mathscr{B}^\bullet \to \mathscr{A}^\bullet$ the functor $H^p_{\{a\}}(X_s, \cdot)$. If a is the only singularity of f, and thus $\mathscr{B}^\bullet$ is exact apart from a, we have a canonical isomorphism

$$H^p_{\{a\}}(X_s, \mathscr{B}^\bullet) \cong H^p(\mathscr{B}^\bullet_{\bar{a}}).$$

In addition,

$$H^p(\mathscr{B}^\bullet_{\bar{a}}) = H^p(\mathscr{A}^\bullet_{\bar{a}})$$

for $p \neq 0$. Moreover, there are canonical mappings

$$H^p_{\{a\}}(X_s, \mathscr{A}^\bullet) \to H^p_c(X_s, \mathscr{A}^\bullet) \to H^p_c(X_\eta, \mathscr{F}_\eta).$$

Definition 25.3. Let $n = \dim X/S$. We denote by $\operatorname{var}\sigma$ the mapping

$$H^n(\mathscr{B}^\bullet_{\bar{a}}) = H^n(\mathscr{A}^\bullet_{\bar{a}}) \to H^n_{\{a\}}(X_s, \mathscr{A}^\bullet)$$

induced by $\sigma - 1$.

Let σ_a be the map induced on $H^n_{\{a\}}(X_s, \mathscr{A}^\bullet)$ by $\sigma : \mathscr{A}^\bullet \to \mathscr{A}^\bullet$. Then

$$\operatorname{var}(\sigma\tau) = \operatorname{var}\sigma + \operatorname{var}\tau + (\sigma_a - 1)(\operatorname{var}\tau).$$

There is a natural pairing

$$H^n(\mathscr{A}^\bullet_{\bar{a}}) \times H^n_{\{a\}}(X_s, \mathscr{A}^\bullet) \to \Lambda(-n)$$

For the construction we apply the usual duality pairing

$$H^n(U_\eta, \Lambda) \times H^n_c(U_\eta, \Lambda) \to \Lambda(-n)$$

for arbitrary étale neighborhoods U of a. Using the restriction mapping

$$H^n_c(X_\eta, \Lambda) \to H^n_c(U_\eta, \Lambda)$$

and the canonical isomorphism

$$H^n(\mathscr{A}^\bullet_{\bar{a}}) \cong \varinjlim H^n(U_\eta, \Lambda),$$

we get the desired pairing by passage to limit.

26 The Picard-Lefschetz formulas

Definition 26.1. We define the fundamental character

$$\chi : G \to \Lambda(1)$$

by the formula

$$\chi(\sigma) = \frac{\sigma(\sqrt[r]{\pi})}{\sqrt[r]{\pi}}.$$

Here π is a generator of $\mathfrak{m}$ and $\sqrt[r]{\pi}$ is any root of $x^r - \pi = 0$ in $\bar{K}$. More generally, if $\lambda \in R$ is of order v, we set

$$\chi_\lambda = \chi^v.$$

Proposition 26.2. every homomorphism φ of G into an Λ-module M factors uniquely through χ, i.e., there is an element $u \in M(-1) = M \otimes_\Lambda \Lambda(-1)$ with

$$\varphi(\sigma) = \chi(\sigma) \cdot u.$$

Proof. Let ω be a generator of $\Lambda(1)$. Since χ is clearly surjective, there exists $\sigma \in G$ such that $\chi(\sigma) = \omega$. Note that every element in $M = \Lambda(1) \otimes M(-1)$ can be written in the form $\omega \cdot u$. We have $\varphi(\sigma) = \chi(\sigma) \cdot u$ for some $u \in M(-1)$. It suffices to show that $\varphi(\tau) = \chi(\tau) \cdot u$ for any $\tau \in G$. Since $\chi(\tau)$ is a multiple of ω, we only need to show that

$$\chi(\tau) = 1 \implies \varphi(\tau) = 0.$$

Consider the exact sequence

$$0 \to P \to I \to I_t \to 0$$

in ramification theory. We have $I = G$ since R is strict Henselian,

$$I_t \cong \varprojlim_{\operatorname{char} k \nmid n} \mu_n(k) \cong \prod_{\ell \neq \operatorname{char} k} \mathbb{Z}_\ell$$

and the map $I \to I_t$ is the limit of the χ's for each r in this case. Let $\chi(\sigma) = (a_\ell) \in \prod_{\ell \neq \operatorname{char} k} \mathbb{Z}_\ell$, since $a_\ell = 0$ for $\ell \mid r$, we can find $b_\ell \in \mathbb{Z}_\ell$ for each $\ell \neq \operatorname{char} k$ such that $(a_\ell) = r(b_\ell)$. Since χ is surjective, there exists $\rho \in I$ such that $\chi(\rho) = (b_\ell)$. Then $\tau' = \tau\rho^{-r} \in P$.

If $\operatorname{char} k = 0$, then $P = \{1\}$, so $\tau = \rho^r$ and

$$\varphi(\tau) = r\varphi(\rho) = 0.$$

If $\operatorname{char} k = p$, then P is a pro-p-group, say $P = \varprojlim_{\lambda} P_\lambda$. Let $\tau' = (\tau'_\lambda)$ and let $H_\lambda = \langle \tau'_\lambda \rangle \leq P_\lambda$, then

$$\varprojlim_{\lambda} H_\lambda \cong \mathbb{Z}/p^N\mathbb{Z} \text{ or } \mathbb{Z}_p.$$

In either case we can find $\rho' \in P$ such that $(\rho')^r = \tau'$. So

$$\varphi(\tau) = r(\varphi(\rho) + \varphi(\rho')) = 0. \qquad \blacksquare$$

If R is a strict Henselization of a Henselian valuation ring R_0 with quotient field K_0 and residue field k_0,

$$\widetilde{G} = \operatorname{Gal}(\bar{K}/K_0), \quad \Gamma := \widetilde{G}/G = \operatorname{Gal}(K/K_0) \cong \operatorname{Gal}(k/k_0).$$

Then Γ acts on the group $\Lambda(1)$ of r^{th} roots of unity, and an element $\widetilde{\tau}$ in $\widetilde{G}$ with image τ in Γ satisfies

$$\chi(\widetilde{\tau}\sigma\widetilde{\tau}^{-1}) = \tau(\chi(\sigma))$$

for $\sigma \in G$.

Theorem 26.3 (Picard-Lefschetz Formulas)**.** Let $S = \operatorname{Spec} R$ be the spectrum of a strictly Henselian valuation ring R. Let $f : X \to S$ be a proper flat morphism of odd fiber dimension $n = 2m+1$ with special fiber X_s and generic fiber X_η. Let f be singular at just one point, a, and let this point be an ordinary double point. Then

(0) The Henselization of $\mathcal{O}_{X,a}$ is R-isomorphic to

$$R\{X_0, \ldots, X_n\}/\left(X_0X_{m+1} + \cdots + X_mX_{2m+1} + \lambda\right)$$

with $0 \neq \lambda \in \mathfrak{m}$.

(1)

$$H^p(X_s, \Lambda) \cong H^p(X_\eta, \Lambda)$$

for $p \neq n, n+1$.

(2) There is a vanishing cycle

$$\delta \in H^n(X_\eta, \Lambda)(m) = H^n(X_\eta, \Lambda) \otimes \Lambda(m)$$

and a covanishing cycle

$$\delta^* \in H^{n+1}(X_s, \Lambda(n-m))$$

such that the sequence

$$0 \longrightarrow H^n(X_s,\Lambda) \longrightarrow H^n(X_\eta,\Lambda) \xrightarrow{x\mapsto\langle x,\delta\rangle} \Lambda(m-n)$$

$$\xrightarrow{x\mapsto x\cdot\delta^*} H^{n+1}(X_s,\Lambda) \longrightarrow H^{n+1}(X_\eta,\Lambda) \longrightarrow 0$$

is exact.

(3) The Galois group G of the algebraic closure of $\bar{K}$ over K acts trivially on $H^{n+1}(X_\eta,\Lambda)$ and acts on $H^n(X_\eta,\Lambda)$ by

$$\sigma(x) = x + (-1)^{m+1}\chi_\lambda(\sigma)\langle x,\delta\rangle\delta.$$

Proposition 26.4. Let $S = \operatorname{Spec} R$ be the spectrum of a strictly Henselian DVR, and let $f : X \to S$ be a finitely generated flat morphism that is smooth outside a single point $a \in X$. Let the special fiber have the same dimension at all points. Let $\varphi : U \to X$ be an étale mapping over S with $\varphi^{-1}(a) = \{b\}$ and let $g = f \circ \varphi$. Then there exists canonical isomorphisms

$$H^\bullet_{\{b\}}(U_s, \mathscr{A}^\bullet(g)) \cong H^\bullet_{\{a\}}(X_s, \mathscr{A}^\bullet(f))$$
$$H^\bullet(\mathscr{A}^\bullet_b(g)) \cong H^\bullet(\mathscr{A}^\bullet_a(f)).$$

There are compatible with

(a) the variation,

(b) the canonical pairing,

(c) the long exact sequence.

Proof. Consider the diagram

$$\begin{tikzcd}
U_\eta \arrow[r, "j_U"] \arrow[d, "\varphi_\eta"] & U \arrow[d, "\varphi"] & U_s \arrow[l, "i_U"'] \arrow[d, "\varphi_s"] \\
X_\eta \arrow[r, "j"] & X & X_s. \arrow[l, "i"']
\end{tikzcd}$$

We have

$$\varphi_s^*\mathscr{A}^\bullet(f) = \varphi_s^* i^* \mathsf{R}j_*\Lambda_{X_\eta} = i_U^*\varphi^*\mathsf{R}j_*\Lambda_{X_\eta} = i_U^*\mathsf{R}(j_U)_*\Lambda_{U_\eta} = \mathscr{A}^\bullet(g).$$

So we get the second isomorphism immediately. For the first isomorphism, we consider the diagram

$$\begin{array}{ccccccc}
\mathsf{R}\Gamma_{\{a\}}(X_s, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma(X_s, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma(X_s \setminus \{a\}, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma_{\{a\}}(X_s, \mathscr{A}^\bullet(f))[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{R}\Gamma_{\{b\}}(U_s, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma(U_s, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma(U_s \setminus \{b\}, \mathscr{A}^\bullet(f)) & \to & \mathsf{R}\Gamma_{\{b\}}(U_s, \mathscr{A}^\bullet(f))[1]
\end{array}$$

For any sheaf $\mathscr{I}$ on X_s, we have the exact sequence

$$0 \to \Gamma(X_s, \mathscr{I}) \to \Gamma(X_s \setminus \{a\}, \mathscr{I}) \times \Gamma(U_s, \mathscr{I}) \to \Gamma(U_s \setminus \{b\}, \mathscr{I}),$$

this gives the isomorphism

$$\Gamma_{\{a\}}(X_s, \mathscr{I}) \xrightarrow{\sim} \Gamma_{\{b\}}(U_s, \mathscr{I}),$$

and hence the first isomorphism. ■

Remark. Let ℓ be a prime number invertible in R. All constructions in this section are compatible with the homomorphisms induced on cohomology groups by

$$\mathbb{Z}/\ell^{r+1}\mathbb{Z} \to \mathbb{Z}/\ell^r\mathbb{Z}.$$

By passage to the inverse limit, one gets a vanishing cycle δ in

$$H^n(X_\eta, \widehat{\mathbb{Z}}_\ell(m)) = \varinjlim H^n(X_\eta, \mu_{\ell^r}^{\otimes m}),$$

and similarly a covanishing cycle.

26.1 The cohomology of quadrics

By (26.4), we may assume that $X = V(Q) \subset \mathbb{P}_S^{n+1}$ where

$$Q(x_0, \dots, x_{n+1}) = x_0 x_{m+1} + \cdots + x_m x_{2m+1} + \lambda x_{n+1}^2$$

for some $\lambda \in \mathfrak{m} \setminus \{0\}$. Let

$$Y = X \cap V(x_{n+1}) \cong V(x_0 x_{m+1} + \cdots + x_m x_{2m+1}) \subset \mathbb{P}_S^n,$$

which is smooth over S, and let

$$X' = X \setminus Y \cong V(x_0 x_{m+1} + \cdots + x_m x_{2m+1} + \lambda) \subset \mathbb{A}_S^{n+1}$$

with $a = 0 \in X'$.

In order to compute the cohomologies over the generic fiber and the special fiber, we first consider a general smooth quadric Z of dimension N over a (separably closed) field.

Let H be a hyperplane in $\mathbb{P}^{N+1}$,

$$\xi = \mathrm{cl}_{\mathbb{P}^{n+1}}(H) \in H^2(\mathbb{P}^{n+1}, \Lambda(1)) = H^2(\mathbb{P}^{n+1}, \Lambda)(1).$$

The inverse image of this class in $H^2(Z, \Lambda(1))$ is

$$\bar{\xi} = \mathrm{cl}_Z(H \cap Z) \in H^2(Z, \Lambda(1)) = H^2(Z, \Lambda)(1).$$

Here $H \cap Z$ is computed with multiplicities.

The cohomology groups of the projective space satisfy

$$H^p(\mathbb{P}^{N+1}, \Lambda) = 0$$

for $2 \nmid p$ and

$$H^2(\mathbb{P}^{N+1}, \Lambda(s)) = \Lambda \cdot \xi^s \cong \Lambda$$

for $0 \leq s \leq N+1$. Here ξ^s is the s^{th} power of ξ under the cup product.

Proposition 26.5. Let $Z \subset \mathbb{P}^{N+1}$ be a smooth quadric of dimension N over a field. Then

$$H^p(Z, \Lambda) = \begin{cases} 0, & \text{if } 2 \nmid p, \\ \Lambda, & \text{if } 2 \mid p,\ 0 \leq p \leq 2N \text{ and } p \neq N, \\ \Lambda \oplus \Lambda, & \text{if } 2 \mid p \text{ and } p = N. \end{cases}$$

More precisely,

$$H^{2p}(X, \Lambda(p)) = \begin{cases} \Lambda \cdot \bar{\xi}^p, & \text{if } 0 \leq 2p < N \\ \Lambda \cdot \bar{\xi}_p, & \text{if } n < 2p \leq 2N \end{cases}$$

where $2\bar{\xi}_p = \bar{\xi}^p$. For $2 \nmid N = 2m$,

$$H^N(X, \Lambda(m)) = \Lambda \cdot \theta_1 \oplus \Lambda \cdot \theta_2$$

with $\theta_1 + \theta_2 = \bar{\xi}^m$.

Proof. By the weak Lefschetz theorem,

$$H^p(\mathbb{P}^{N+1}, \Lambda) \xrightarrow{\sim} H^p(X, \Lambda)$$

for $p < N$. Since $\deg X = 2$, by Poincaré duality, there's a perfect pairing

$$\begin{array}{rcc} H^{2p}(X, \Lambda(p)) \times H^{2N-2p}(X, \Lambda(n-p)) & \to & \Lambda \\ (\alpha, \beta) & \mapsto & S(\alpha \cup \beta) \end{array}$$

With $S(\bar{\xi}^p \cup \bar{\xi}^{N-p}) = 2$. It remains only to determine $H^n(X, \Lambda)$. By (22.13),

$$\Delta^2 = \sum_{p=0}^{2N} (-1)^p \operatorname{rk}_\Lambda H^p(X, \Lambda).$$

LHS is $c_N(\mathscr{T}_X)$. In order to compute it, consider the exact sequence

$$0 \to \mathscr{T}_X \to \mathscr{T}_{\mathbb{P}^{N+1}}|_X \to \mathscr{N}_{X/\mathbb{P}^{N+1}} = \mathcal{O}_X(2) \to 0.$$

We have $c_t(\mathscr{T}_{\mathbb{P}^{N+1}}|_X) = c_t(\mathscr{T}_X) c_t(\mathcal{O}_X(2))$. Thus, in $A(\mathbb{P}^{N+1}) \cong \mathbb{Z}[H]/[H]^{N+2}$,

$$\begin{aligned} c_t(\mathscr{T}_X) = \frac{(1+[H]t)^{N+2}}{1+2[H]t} &= \left(\sum_{i=0}^{N+1} \binom{N+2}{i} ([H]t)^i \right) \left(\sum_{j=0}^{N+1} (-2)^j ([H]t)^j \right) \\ &= \sum_{k=0}^{N+1} \sum_{i+j=k} \binom{N+2}{i} (-2)^j ([H]t)^k. \end{aligned}$$

So

$$\begin{aligned} \Delta^2 &= \sum_{i=0}^{n} \binom{N+2}{i} (-2)^{N-i} [H]^N \\ &= \frac{1}{2} \cdot \big((1+(-2))^{N+2} - (N+2)(-2) - 1 \big) \\ &= (N+2) - N\%2, \end{aligned}$$

and hence

$$\operatorname{rk}_\Lambda H^n(X, \Lambda) = 2 \cdot ((N+1)\%2).$$

The elements θ_1 and θ_2 is defined by the linear subspace

$$L_1 = V(x_0, \ldots, v_m), \ L_2 = V(x_0, \ldots, x_{m-1}, x_{2m+1}). \qquad \blacksquare$$

For the generic fiber we have the exact sequence

$$\cdots \to H_c^p(X'_\eta, \Lambda) \to H^p(X_\eta, \Lambda) \to H^p(Y_\eta, \Lambda) \to H_c^{p+1}(X'_\eta, \Lambda) \to \cdots$$

By (26.5), let $N = n, n+1$, respectively, we get

$$H^p(X_\eta, \Lambda) \xrightarrow{\sim} H^p(Y_\eta, \Lambda)$$

for $p \neq n-1, 2n$ and the map

$$\begin{array}{ccc} H^{n-1}(X_\eta, \Lambda) & \to & H^{n-1}(Y_\eta, \Lambda) \\ \bar{\xi}^m & \mapsto & \theta_1 + \theta_2 \end{array}$$

is injective. So we get:

Proposition 26.6.

$$\begin{aligned} H_c^p(X', \Lambda) &= 0 \quad \text{ for } p \neq n, 2n \\ H_c^n(X', \Lambda(m)) &\cong \frac{\Lambda \cdot \theta_1 \oplus \Lambda \cdot \theta_2}{\Lambda \cdot (\theta_1 + \theta_2)} \cong \Lambda \cdot (\partial \theta_1) \\ H_c^{2n}(X', \Lambda) &\cong \Lambda(-n). \end{aligned}$$

The last formula comes from duality.

For the special fiber, since X'_s is a cone, we may use "contraction" to determine the cohomology groups.

Proposition 26.7. Let $\mathscr{F}^\bullet$ be a bounded cohomology complex of sheaves on X'_s, and assume $\mathscr{F}^\bullet|_{X'_s \setminus \{a\}}$ are constant. Then

(a) $H^p(X'_s, \mathscr{F}^\bullet) = H^p(\{a\}, \mathscr{F}^\bullet_{\{a\}}) = H^p(\mathscr{F}^\bullet_{\bar{a}})$

(b) $H_c^p(X'_s, \mathscr{F}^\bullet) \cong H^p_{\{a\}}(X'_s, \mathscr{F}^\bullet)$.

Proof. May assume $\mathscr{F}^\bullet = \mathscr{F}$ for some sheaf $\mathscr{F}$. Let $h : X'_s \setminus \{a\} \to X'_s$ be the inclusion. Consider the exact sequence

$$0 \to \mathscr{R} \to \mathscr{F} \to h_* h^* \mathscr{F} \to \mathscr{Q} \to 0.$$

Then $\mathscr{Q}$ and $\mathscr{R}$ are concentrated on $\{0\}$, so we may assume $\mathscr{F} = h_* M = h_* h^* M$, and hence $\mathscr{F} = M$, for some Λ-module M.

For (a), consider the mappings

$$\begin{tikzcd} X'_s \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & X'_s \times \mathbb{A}^1 \arrow[r, "m"] \arrow[d, "\pi"'] & X'_s \\ & X'_s, & \end{tikzcd}$$

where

$$i_0(x) = (x, 0), \;\; i_1(x) = (x, 1), \;\; m(x,t) = tx, \;\; \pi(x,t) = x.$$

Claim. $\pi^* : H^p(X'_s, M) \to H^p(X'_s \times \mathbb{A}^1, M)$ is an isomorphism for each p.

Proof of Claim. It follows from the smooth base change theorem with the diagram

$$\begin{tikzcd} X'_s \times \mathbb{A}^1 \arrow[r, "g'"] \arrow[d, "\pi"] & \mathbb{A}^1 \arrow[d, "\pi_k"] \\ X'_s \arrow[r, "g"] & \operatorname{Spec} k. \end{tikzcd}$$

that

$$\pi_k^* \mathsf{R}\Gamma(X'_s, M) = \mathsf{R}g'_* M.$$

Take $\mathsf{R}\Gamma = \mathsf{R}(\pi_k)_*$ on both side, we get from acyclicity for $\pi_k : \mathbb{A}^1 \to \operatorname{Spec} k$ that

$$\mathsf{R}\Gamma(X'_s, M) = \mathsf{R}\Gamma(\mathbb{A}^1, \pi_k^* \mathsf{R}\Gamma(X'_s, M)) = \mathsf{R}\Gamma(X'_s \times \mathbb{A}^1, M),$$

as desired. □

It follows that the mappings i_0 and i_1, and hence also $m \circ i_0$ and $m \circ i_1$, induces the same mappings on cohomology groups.

For (b), we have the projection map

$$\pi_a : Y_s \times \mathbb{A}^1 \cong X_s \setminus \{a\} \to Y_s$$

from a. As in (a), we know that π_a^* on cohomologies are isomorphism. So by excision and the diagram

$$\begin{tikzcd} \mathsf{R}\Gamma_{\{a\}}(X_s, M) \arrow[r] \arrow[d] & \mathsf{R}\Gamma(X_s, M) \arrow[r] \arrow[d, equal] & \mathsf{R}\Gamma(Y_s \times \mathbb{A}^1, M) \arrow[r] \arrow[d, "\wr"] & \mathsf{R}\Gamma_{\{a\}}(X_s, M)[1] \arrow[d] \\ \mathsf{R}\Gamma_c(X'_s, M) \arrow[r] & \mathsf{R}\Gamma(X_s, M) \arrow[r] & \mathsf{R}\Gamma(Y_s, M) \arrow[r] & \mathsf{R}\Gamma_c(X'_s, M)[1] \end{tikzcd}$$

we get

$$\mathsf{R}\Gamma_{\{a\}}(X'_s, M) = \mathsf{R}\Gamma_{\{a\}}(X_s, M) = \mathsf{R}\Gamma_c(X'_s, M). \qquad \blacksquare$$

Recall that there's a distinguished triangle

$$\Lambda_X \to \mathscr{A}^\bullet \to \mathscr{B}^\bullet \to \Lambda_X[1],$$

where $\mathscr{A}^\bullet$ is quasi-isomorphic to $i^* \mathsf{R} j_* \Lambda_X$ and $\mathscr{B}^\bullet$ is the mapping cone. Since a is the only singularity of the mapping f, the cohomology sheaves of $\mathscr{B}^\bullet$ are concentrated on the point a.

Lemma 26.8. The following mappings are isomorphisms:

(a) $H^p(X'_\eta, \Lambda) \to H^p(X'_s, \mathscr{A}^\bullet) \to H^p(\mathscr{A}^\bullet_{\bar{a}})$.

(b) $H^p_{\{a\}}(X_s, \mathscr{A}^\bullet) \to H^p_{\{a\}}(X'_s, \mathscr{A}^\bullet) \to H^p_c(X'_\eta, \Lambda)$

Proof. Since $\mathscr{A}^\bullet$ is quasi-isomorphic to Λ_X away from a, by (26.7) we know that

$$H^p(X'_s, \mathscr{A}^\bullet) \to H^p(\mathscr{A}^\bullet_{\bar{a}}) \quad \text{and} \quad H^p_{\{a\}}(X'_s, \mathscr{A}^\bullet) \to H^p_c(X'_s, \mathscr{A}^\bullet)$$

are isomorphisms.

By smooth base change theorem for the diagram

$$\begin{array}{ccc} X_\eta & \xrightarrow{j} & X \\ \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} \bar{K} & \xrightarrow{\eta} & S, \end{array}$$

$$\mathscr{F} = f^*\eta_*\Lambda_{\operatorname{Spec} \bar{K}} \to \mathscr{K}^\bullet = \mathsf{R}j_*\Lambda_{X_\eta}$$

is a quasi-isomorphism at all points apart from a. Hence there exists a distinguished triangle

$$\mathscr{F} \to \mathscr{K}^\bullet \to \mathscr{E}^\bullet \to \mathscr{F}[1]$$

such that the cohomology sheaves of $\mathscr{E}^\bullet$ are concentrated on a.

Consider the mappings

$$f' = f|_{X'}, \;\; f_Y = f|_Y.$$

For (a), we must show that

$$H^p(X'_\eta, \Lambda) \cong H^p(X', \mathscr{K}^\bullet) \to H^p(X'_s, \mathscr{A}^\bullet)$$

is an isomorphism. Since the cohomologies of $\mathscr{E}^\bullet$ are concentrated on a,

$$H^p(X', \mathscr{E}^\bullet) \to H^p(X'_s, i^*\mathscr{E}^\bullet)$$

is an isomorphism. Note that $\mathscr{A}^\bullet = i^*\mathscr{K}^\bullet$. It therefore suffices to show that

$$H^p(X', \mathscr{F}) \to H^p(X'_s, i^*\mathscr{F})$$

is an isomorphism. For this we consider the exact sequences

$$\begin{array}{ccccccccc} \cdots \to & H^p(Y, \mathsf{R}\Gamma_Y\mathscr{F}) & \longrightarrow & H^p(X, \mathscr{F}) & \longrightarrow & H^p(X', \mathscr{F}) & \longrightarrow & H^{p+1}(Y, \mathsf{R}\Gamma_Y\mathscr{F}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^p(Y_s, \mathsf{R}\Gamma_Y i^*\mathscr{F}) & \to & H^p(X_s, i^*\mathscr{F}) & \to & H^p(X'_s, i^*\mathscr{F}) & \to & H^{p+1}(Y_s, \mathsf{R}\Gamma_Y i^*\mathscr{F}) & \to \cdots \end{array}$$

The proper base change theorem gives

$$\mathsf{R}\Gamma(X,\mathscr{F})_{\operatorname{Spec}k} = s^*\mathsf{R}f_*\mathscr{F} \xrightarrow{\sim} \mathsf{R}(f_s)_*i^*\mathscr{F} = \mathsf{R}\Gamma(X_s, i^*\mathscr{F})_{\operatorname{Spec}k}$$

and

$$\mathsf{R}\Gamma(Y,\mathsf{R}\Gamma_Y\mathscr{F})_{\operatorname{Spec}k} = s^*\mathsf{R}f_*\mathsf{R}\Gamma_Y\mathscr{F} \xrightarrow{\sim} \mathsf{R}(f_s)_*i^*\mathsf{R}\Gamma_Y\mathscr{F} = \mathsf{R}\Gamma(Y_s, i^*\mathsf{R}\Gamma_Y\mathscr{F})_{\operatorname{Spec}k}.$$

So it suffices to show that

$$i^*\mathsf{R}\Gamma_Y\mathscr{F} \xrightarrow{\sim} \mathsf{R}\Gamma_{Y_s}i^*\mathscr{F}.$$

By purity theorem, we have

$$i^*R^p\Gamma_Y\mathscr{F} = 0 \xrightarrow{\sim} 0 = R^p\Gamma_{Y_s}i^*\mathscr{F}$$

for $p \neq 2$ and

$$\begin{aligned} i^*R^2\Gamma_Y\mathscr{F} &\cong i^*(R^2\Gamma_Y\Lambda_X \otimes_\Lambda \mathscr{F}) \cong i^*R^2\Gamma_Y\Lambda_X \otimes_\Lambda i^*\mathscr{F} \\ &\xrightarrow{\sim} R^2\Gamma_{Y_s}\Lambda_{X_s} \otimes i^{\mathscr{F}} \cong R^2\Gamma_{Y_s}i^*\mathscr{F}. \end{aligned}$$

For (b), by the base change theorem for the direct images with proper support,

$$(\mathsf{R}\Gamma_c(X'_s,\mathscr{A}^\bullet))_{\operatorname{Spec}k} = \mathsf{R}(f'_s)_!i^*\mathscr{K}^\bullet \xrightarrow{\sim} s^*\mathsf{R}f'_!\mathscr{K}^\bullet = (\mathsf{R}\Gamma(S,\mathsf{R}f'_!\mathscr{K}^\bullet))_{\operatorname{Spec}k}.$$

So we need to show that

$$H^p(S,\mathsf{R}f'_!\mathscr{K}^\bullet) \to H^p_c(X'_\eta,\Lambda)$$

is an isomorphism. By smooth base change,

$$\mathscr{K}^\bullet|_Y \xleftarrow{\text{q-iso.}} \mathscr{F}|_Y \xrightarrow{\text{q-iso.}} \mathsf{R}(j|_{Y_\eta})_*\Lambda_{Y_\eta}.$$

Now, consider the exact sequences

$$\cdots \to H^p(S,\mathsf{R}f'_!\mathscr{K}^\bullet) \to H^p(S,\mathsf{R}f_*\mathscr{K}^\bullet) \to H^p(S,\mathsf{R}(f_Y)_*\mathscr{K}^\bullet) \to H^{p+1}(S,\mathsf{R}f'_!\mathscr{K}^\bullet) \to \cdots$$

and the corresponding sequence over $\operatorname{Spec}\bar{K}$. The isomorphism desired follows from

$$H^p(S,\mathsf{R}f_*\mathscr{K}^\bullet) = H^p(X,\mathscr{K}^\bullet) \cong H^p(X_\eta,\Lambda_{X_\eta})$$

and

$$H^p(S,\mathsf{R}(f_Y)_*\mathscr{K}^\bullet) = H^p(Y,\mathscr{K}^\bullet|_Y) \cong H^p(Y,\mathsf{R}(j|_{Y_\eta})_*\Lambda_{Y_\eta}) \cong H^p(Y_\eta,\Lambda_{Y_\eta}). \qquad \blacksquare$$

We denote by

$$\delta \in H^n_{\{a\}}(X_s, \mathscr{A}^\bullet(m))$$

the inverse image of $\partial\theta_1 \in H^n_c(X'_\eta, \Lambda(m))$ in (26.6). It follows from the Poincaré duality theorem for X'_η that the pairing

$$H^n_{\{a\}}(X, \mathscr{A}^\bullet) \times H^n(\mathscr{B}^\bullet_{\bar{a}}) \to \Lambda(-n)$$

is nondegenerate. Hence there is an element

$$\delta^* \in H^n(\mathscr{B}^\bullet_{\bar{a}})(n-m)$$

with $\langle \delta, \delta* \rangle = 1$.

Proposition 26.9. The pencil of quadrics $f : X \to S$ with odd fiber dimension $n = 2m+1$ and the associated complexes $\mathscr{A}^\bullet$ and $\mathscr{B}^\bullet$ satisfy

$$H^p_{\{a\}}(X_s, \mathscr{A}^\bullet) = \begin{cases} 0, & \text{if } p \neq n, 2n \\ \Lambda(-n), & \text{if } p = 2n \\ \Lambda(-m)\delta, & \text{if } p = n, \end{cases} \qquad H^p(\mathscr{B}^\bullet_{\bar{a}}) = \begin{cases} 0, & \text{if } p \neq n \\ \Lambda(m-n)\delta^*, & \text{if } p = n. \end{cases}$$

Proof. The structure of cohomologies $H^\bullet_{\{a\}}(X_s, \mathscr{A}^\bullet)$ follows from (26.6) and (26.8(b)). For the structure of cohomologies $H^\bullet(\mathscr{B}^\bullet_a)$, from (26.8(a)) we have

$$H^p(\mathscr{B}^\bullet_{\bar{a}}) \cong H^p(\mathscr{A}^\bullet_{\bar{a}}) \cong H^p(X'_\eta, \Lambda)$$

for $p \neq 0$ and the result follows from Poincaré duality. For $p = 0$, we have the exact sequence

$$0 \to H^0(\Lambda) \to H^0(\mathscr{A}^\bullet_{\bar{a}}) \to H^0(\mathscr{B}^\bullet_{\bar{a}}) \to H^1(\Lambda) = 0.$$

Since $H^0(\mathscr{A}^\bullet_{\bar{a}}) = \Lambda$ by Poincaré duality, $H^0(\mathscr{B}^\bullet_{\bar{a}}) = 0$. ■

We now come to the proof of (26.3).

For (1), we only need to show that $H^p(X_s, \Lambda) \xrightarrow{\sim} H^p(X_s, \mathscr{A}^\bullet)$ for $p \neq n, n+1$, which follows from the triangle

$$\Lambda_{X_s} \to \mathscr{A}^\bullet \to \mathscr{B}^\bullet \to \Lambda_{X_s}[1]$$

and the fact that

$$H^q(X_s, \mathscr{B}^\bullet) \cong H^q(\mathscr{B}^\bullet_{\bar{a}}) = 0$$

for $q \neq n$.

For (2), we consider the triangle

$$\Lambda_{X_s} \to \mathscr{A}^\bullet \to \mathscr{B}^\bullet \to \Lambda_{X_s}[1]$$

and the resulting long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^n(X_s,\Lambda) \arrow[r] \arrow[dr] & H^n(X_s,\mathscr{A}^\bullet) \arrow[r] \arrow[d, equal] & H^n(X_s,\mathscr{B}^\bullet) \arrow[r, "\partial"] \arrow[d, equal] & H^{n+1}(X_s,\Lambda) \arrow[r] \arrow[dr] & H^{n+1}(X_s,\mathscr{A}^\bullet) \arrow[r] \arrow[d, equal] & 0 \\
 & & H^n(X_\eta,\Lambda) \arrow[r] & \Lambda(m-n)\delta^* \arrow[ur] & & H^{n+1}(X_\eta,\Lambda) &
\end{tikzcd}$$

For (3), we consider the variation mapping $\operatorname{var}\sigma$. For two elements σ, τ in G, we have

$$\operatorname{var}(\sigma\tau) = \operatorname{var}\sigma + \operatorname{var}\tau + (\sigma - 1)\operatorname{var}\tau.$$

Claim. The Galois group G acts trivially on $H^n_{\{a\}}(X_s, \mathscr{A}^\bullet)$.

Proof of Claim. There is a surjective G-equivariant mapping

$$H^{2m}(Y_\eta,\Lambda) \xrightarrow{\partial} H^n_c(X'_\eta,\Lambda) \cong H^n_{\{a\}}(X_s,\mathscr{A}^\bullet).$$

But G acts trivially on $H^{2m}(Y_\eta,\Lambda)$. Indeed, since Y is smooth and proper over S, the specialization mapping

$$\mathsf{R}\Gamma(Y_s,\Lambda) \to \mathsf{R}\Gamma(Y_\eta,\Lambda)$$

is a quasi-isomorphism, and G acts trivially on $H^\bullet(Y_s,\Lambda)$. ☐

Thus

$$\operatorname{var}(\sigma\tau) = \operatorname{var}(\sigma) + \operatorname{var}(\tau).$$

In other words, the mapping

$$\operatorname{var} : G \to \operatorname{Hom}(H^n(\mathscr{B}^\bullet_{\bar{a}}), H^n_{\{a\}}(X_s,\mathscr{A}^\bullet))$$

is a homomorphism. It therefore factors through the canonical character $\chi : G \to \Lambda(1)$. Thus there is an element $c_n \in \Lambda$ such that

$$\operatorname{var}(\sigma)(x) = c_n\chi(\sigma)\langle x,\delta\rangle\delta.$$

Here $\langle\ ,\ \rangle$ is the nondegenerate pairing between $H^n(\mathscr{B}^\bullet_{\bar{a}})$ and $H^n_{\{a\}}(X_s,\mathscr{A}^\bullet)$.

We'll show later that

Lemma 26.10. For our quadric, defined by

$$x_0 x_{m+1} + \cdots + x_m x_{2m+1} + \lambda x_{n+1}^2 = 0,$$

we have

$$c_n = (-1)^{m+1} \cdot v_R(\lambda)$$
$$\mathrm{var}(\sigma) = (-1)^{m+1} \chi_\lambda(\sigma) \langle x, \delta \rangle \delta.$$

27 Algebraic monodromy over $\mathbb{C}$

Let $f : X \to \mathbb{C}$ be the flat holomorphic mapping from a smooth analytic manifold X to the complex plane $\mathbb{C}$. Here X is the relative quadric of odd fiber dimension $n = 2m + 1$ over $\mathbb{C}$

$$X = \{(z, t) \mid z_0^2 + \cdots + z_n^2 = t\}, \quad f(z, t) = t.$$

Then f is smooth apart from one point $a = (0, 0)$ that lies in the fiber $X_0 = f^{-1}(0) \subseteq X$.

Consider the universal covering space

$$\begin{array}{ccc} \lambda & & \mathbb{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ e^{2\pi i \lambda} & & \mathbb{C}^\times \end{array}$$

of $\mathbb{C}^\times$. The fundamental group $\pi_1(\mathbb{C}^\times) = \mathbb{Z} \cdot \sigma_0$ where $\sigma_0(z) = z + 1$. We extend the covering mapping π to a continuous mapping

$$\begin{array}{ccccccc} \lambda & & \mathbb{C} & \hookrightarrow & \bar{\mathbb{C}} & & \infty \\ \big\downarrow & & \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \overline{\exp}} & & \big\downarrow \\ e^{2\pi i \lambda} & & \mathbb{C}^\times & \hookrightarrow & \mathbb{C} & & 0 \end{array}$$

where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A neighborhood basis for ∞ in $\bar{\mathbb{C}}$ is given by the sets

$$\bar{U}_r = U_r \cup \{\infty\}, \quad U_r = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > r\}.$$

So maybe we should write $i\infty$ instead of ∞. We extend the action of π_1 on $\mathbb{C}$ to a continuous action on $\bar{\mathbb{C}}$ over $\mathbb{C}$ by

$$\sigma_0(\infty) = \infty.$$

Using the mapping $f : X \to \mathbb{C}$, we define spaces

$$\bar{X} = X \times_{\mathbb{C}} \bar{\mathbb{C}}, \ \bar{X}^\times = \bar{X} - \bar{f}^{-1}(\infty), \ X_0 = \bar{f}^{-1}(\infty)$$

and we get the commutative diagram

$$\begin{tikzcd}
\bar{X}^{\times} \arrow[rr, "\bar{j}"] \arrow[dd, "\bar{f}^{\times}"] & & \bar{X} \arrow[dd, "\bar{f}"] \arrow[ddr] & & X_0 \arrow[ll, "\bar{i}"'] \arrow[dl] \arrow[ddl] \arrow[dd] \\
& & & \{\infty\} \arrow[dl, dashed] \arrow[dr, dashed] & \\
\mathbb{C} \arrow[rr, "j"] \arrow[dr, "\exp"] & & \bar{\mathbb{C}} \arrow[dr, "\overline{\exp}"] & X \arrow[d, "f"] & \{0\} \arrow[dl] \\
& \mathbb{C}^{\times} \arrow[rr] & & \mathbb{C} &
\end{tikzcd}$$

The group π_1 naturally acts on $\bar{X}^{\times}$ and $\bar{X}$, acting trivially on $X_0 \subseteq \bar{X}$.

As before, we start with a sheaf $\mathscr{F}$ on X. The inverse image $\bar{\mathscr{F}}$ on $\bar{X}$ is in a natural way a π_1-sheaf. We consider an equivariant resolution $\mathscr{I}^\bullet$ of $\bar{\mathscr{F}}|_{\bar{X}^{\times}}$ by flabby π_1-sheaves, e.g., the canonical flabby resolution. The complex

$$\mathscr{A}^\bullet = \bar{i}^* \bar{j}_* \mathscr{I}^\bullet$$

is an equivariant complex of π_1-sheaf on X_0, and it represents

$$\bar{i}^* \mathsf{R}\bar{j}_* \bar{\mathscr{F}}|_{\bar{X}^{\times}}$$

in the derived category of π_1-sheaves on X_0. As $\bar{X}$ is locally metrizable, $\mathscr{A}^\bullet$ is actually flabby. There is a natural equivariant mapping

$$\mathscr{F}|_{X_0} = \bar{\mathscr{F}}|_{X_0} \to \mathscr{A}^\bullet.$$

We now assume $\mathscr{F} = M_X$ is a constant sheaf where M is an abelian group. Then for $x \in X_0$, we have

$$H^p(\mathscr{A}^\bullet_x) = H^p((R\bar{j}_* M)_x) = \varinjlim_{U \ni x} H^p(U \cap \bar{X}^{\times}, M),$$

where $H^p(U \cap \bar{X}^*, M)$ is the singular cohomology group with coefficients in M.

Here's an easy fact of smooth morphisms over $\mathbb{C}$.

Lemma 27.1. Assume the mapping $f : X \to \mathbb{C}$ is smooth in a neighborhood of $x \in X_0$. Then there is a neighborhood V of x in X_0, a disc

$$D = \{z \in \mathbb{C} \mid |z| < e^{-2\pi r}\},$$

and a neighborhood of x in X isomorphic to $V \times D$. This isomorphism is compatible with the projection onto $D \subset \mathbb{C}$ and with the mapping f. In particular, there is a neighborhood U of b in $\bar{X}$ and a homeomorphism

$$U \to V \times (\{z \in \mathbb{C} \mid \operatorname{Im} z > r\} \cup \{\infty\})$$

over $\bar{\mathbb{C}}$.

Corollary 27.2. Let $x \in X_0$ be a point of the special fiber different from a. Then $H^p(\mathscr{A}_x^\bullet)$ vanishes for $p \neq 0$. The cohomology sheaves $\mathscr{H}^p(\mathscr{A}^\bullet)$ of the complex $\mathscr{A}^\bullet$ are thus concentrated on a for $p \neq 0$. The mapping $M_X \to \mathscr{H}^0(\mathscr{A}^\bullet)$ is an isomorphism outside a.

Let $\mathscr{B}^\bullet$ be the mapping cone of the mapping $M_X \to \mathscr{A}^\bullet$. It is the extended complex

$$0 \to M_X \to \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots .$$

By (27.2), all its cohomology sheaves are concentrated on a. It is likewise in a natural way an equivariant complex.

As an element $\sigma \in \pi_1(\mathbb{C}^\times)$ acts trivially on M_X,

$$\sigma - 1 : \mathscr{A}^\bullet \to \mathscr{A}^\bullet$$

in a natural way induces a mapping

$$\mathscr{B}^\bullet \to \mathscr{A}^\bullet.$$

We apply the functor $H^p_{\{a\}}(X_0, \cdot)$ to this mapping, getting a mapping

$$\operatorname{var} \sigma : H^p_{\{a\}}(X_0, \mathscr{B}^\bullet) \to H^p_{\{a\}}(X_0, \mathscr{A}^\bullet).$$

Since by (27.2) the cohomology sheaves of $\mathscr{B}^\bullet$ are concentrated on $\{a\}$, we have

$$H^p_{\{a\}}(X_0, \mathscr{B}^\bullet) \cong H^p(\mathscr{B}_a^\bullet)$$

canonically. For $p \neq 0$, $H^p(\mathscr{B}_a^\bullet) = H^p(\mathscr{A}_a^\bullet)$.

Definition 27.3. For an element $\sigma \in \pi_1(\mathbb{C}^\times)$, the mapping

$$\sigma - 1 : H^p(\mathscr{B}_a^\bullet) \to H^p_{\{a\}}(X_0, \mathscr{A}^\bullet)$$

is denoted $\operatorname{var} \sigma$.

Note that

$$X_1 = \left\{ z = x + iy \;\middle|\; |x|^2 - |y|^2 = 1,\ \langle x, y\rangle = 0 \right\},$$

so we consider the isomorphism

$$z = x + iy \mapsto \left(\frac{x}{\sqrt{1+|y|^2}}, y \right)$$

from X_1 to the sphere bundle of $S^n \subset \mathbb{R}^{n+1}$.

Definition 27.4. Let $T = TS^n$ be the tangent bundle of the n-sphere $S^n \subset \mathbb{R}^{n+1}$,

$$\begin{aligned} T &= \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \langle x, x\rangle = 1,\ \langle x, y\rangle = 0\}, \\ T' &= \{(x, y) \in T \mid y \neq 0\}. \end{aligned}$$

For $r > 0$, we set

$$\begin{aligned} T_{\geq r} &= \{(x, y) \in T \mid |y| \geq r\}, \\ T_r &= \{(x, y) \in T \mid |y| = r\}, \\ T_{\leq r} &= \{(x, y) \in T \mid |y| \leq r\}. \end{aligned}$$

Then we get an diffeomorphism

$$\begin{array}{ccc} T_1 & \xrightarrow{\sim} & \{z \in \mathbb{C}^{n+1} \mid z^2 = 0,\ |z|^2 = 2\} \\ (x, y) & \mapsto & z = x + iy. \end{array}$$

Lemma 27.5. We have

(a)

$$\begin{array}{ccc} X_1 = f^{-1}(1) & \xrightarrow{\sim} & T \\ z = x + iy & \mapsto & \left(\dfrac{x}{\sqrt{1+|y|^2}}, y \right). \end{array}$$

(b)

$$\begin{array}{ccc} T' & \xrightarrow{\sim} & T_1 \times \mathbb{R}^+ \\ (x, y) & \mapsto & \left(\left(x, \dfrac{y}{|y|} \right), |y| \right) \end{array}$$

with

$$\begin{aligned} T_{\geq r} &\leftrightarrow T_1 \times \mathbb{R}_{\geq r}, \\ T_r &\leftrightarrow T_1 \times \{r\}, \\ T_{\leq r} &\leftrightarrow T_1 \times \mathbb{R}_{\leq r}. \end{aligned}$$

(c) The multiplicative group $S^1 \subset \mathbb{C}^\times$ acts on T_1 and on T' by

$$w(z) = w \cdot (x + iy)$$

and by $w \times \mathrm{id}_{\mathbb{R}^+}$, respectively.

We want to extend $X_1 \to T$ to something like

$$\bar{X}' \to T' \times \bar{\mathbb{C}} \text{ and } \bar{X}^\times \to T \times \mathbb{C}$$

For $(z, \lambda) \in \bar{X}'$, we want to find

$$(z, \lambda) \mapsto ((X, Y), c, \lambda) \in T_1 \times \mathbb{R}^+ \times \bar{\mathbb{C}}.$$

Similar to the isomorphism $X_1 \cong T$, we consider

$$z^2 = e^{2\pi i\lambda} \implies \left(ze^{-\pi i \operatorname{Re}\lambda}\right)^2 = e^{2\pi i \operatorname{Im}\lambda} = |e^{2\pi i\lambda}|.$$

Write $ze^{-\pi i \operatorname{Re}\lambda} = x + iy$, then the map

$$(z, \lambda) \mapsto \left(w(z, \lambda) \cdot \left(\frac{x}{\sqrt{|e^{2\pi i\lambda}| + |y|^2}}, \frac{y}{|y|}\right), |y|, \lambda\right)$$

for some w seems to be a good choice. We now determine w such that the map is equivariant and continuous at ∞. Note that under this map,

$$(z, \lambda + 1) \mapsto \left(w(z, \lambda + 1) \cdot \left(\frac{-x}{\sqrt{|e^{2\pi i\lambda}| + |y|^2}}, \frac{-y}{|y|}\right), |y|, \lambda + 1\right)$$

So we need $w(z, \lambda + 1) = w(z)$. For the ∞ part, note that

$$x^2 - y^2 = |e^{2\pi i\lambda}| \to 0, \quad x \cdot y = 0$$

as $\lambda \to \infty$. So we set

$$(z, \infty) \mapsto \left(\left(\frac{x}{|y|}, \frac{y}{|y|}\right), |y|, \infty\right)$$

So we need

$$w(z, \lambda) \cdot ze^{-\pi i \operatorname{Re}\lambda} \to z$$

as $\lambda \to \infty$. We simply take

$$w(z, \lambda) = e^{\pi i \operatorname{Re}\lambda}$$

which also satisfies the first condition. Note that

$$\begin{cases} |x|^2 + |y|^2 = |z|^2 \\ |x|^2 - |y|^2 = |e^{2\pi i\lambda}| \end{cases} \implies |y|^2 = \frac{1}{2}(|z|^2 - |e^{2\pi i\lambda}|) =: \rho(z, \lambda)^2$$

Summing up, we get:

Lemma 27.6. Let

$$\bar{X}' = \{(z,\lambda) \in \bar{X} \mid \rho(z,\lambda) \neq 0\}.$$

We have an equivariant homeomorphism $\bar{X}' \to T' \times \bar{\mathbb{C}}$ over $\bar{\mathbb{C}}$ such that the diagram

$$\begin{array}{ccc} \bar{X}' & \longrightarrow & T' \times \bar{\mathbb{C}} \\ \big\uparrow & & \big\uparrow \\ \bar{f}^{-1}(\mathbb{Z}) \cap \bar{X}' = \bigsqcup_{n \in \mathbb{Z}} X_1' & \longrightarrow & T' \times \mathbb{Z} \end{array}$$

commutes and $\bar{X}_r := \{(z,\lambda) \mid \rho(z,\lambda) = r\}$ sends to $T_r \times \bar{\mathbb{C}}$ under this map.

Proof. The map is defined by

$$(z,\lambda) \mapsto \begin{cases} \left(e^{\pi i \operatorname{Re}\lambda} \cdot \left(\dfrac{\operatorname{Re}\zeta}{\sqrt{|e^{2\pi i\lambda}| + |\operatorname{Im}\zeta|^2}}, \dfrac{\operatorname{Im}\zeta}{|\operatorname{Im}\zeta|}\right), |\operatorname{Im}\zeta|, \lambda\right), & \text{if } \lambda \neq \infty \\ \left(\left(\dfrac{\operatorname{Re}(z)}{|\operatorname{Im}(z)|}, \dfrac{\operatorname{Im}(z)}{|\operatorname{Im}(z)|}\right), |\operatorname{Im}(z)|, \infty\right), & \text{if } \lambda = \infty, \end{cases}$$

where $\zeta = e^{-\pi i \operatorname{Re}\lambda} z$. It is equivariant and continuous at ∞ by the discussion above. ■

For $\bar{X}^\times$, we want $\sigma_0 \in \pi_1(\mathbb{C}^\times)$ acts on $T_{\geq 1}$ trivially so that we may calculate the variation on a proper set over $\mathbb{C}$.

Choose $\varepsilon \in (0,1)$ and a monotonically increasing C^∞ function $h : \mathbb{R} \to \mathbb{R}$ with

$$h(t) = \frac{\int_{-\infty}^{t} g(s)\, ds}{\int_{-\infty}^{\infty} g(s)\, ds}, \quad g(t) = \begin{cases} 0, & \text{if } t \notin \left(\frac{\varepsilon}{2}, \varepsilon\right) \\ e^{-1/(\varepsilon - t) - 1/(t - \frac{\varepsilon}{2})}, & \text{if } t \in \left(\frac{\varepsilon}{2}, \varepsilon\right). \end{cases}$$

Lemma 27.7. There is an equivariant homeomorphism over $\mathbb{C}$,

$$\bar{X}^\times \xrightarrow{\sim} T \times \mathbb{C}$$

such that the generating element σ_0 of $\pi_1(\mathbb{C}^\times)$ acts on T by

$$\begin{array}{cccc} \sigma_0 : & T & \to & T \\ & (x,y) & \mapsto & e^{\pi i (1 + h(|y|))} \cdot (x,y) \end{array}$$

In particular,

$$\sigma_0(x,y) = \begin{cases} (-x,-y), & \text{if } |y| \leq \varepsilon/2 \\ (x,y), & \text{if } |y| \geq \varepsilon. \end{cases}$$

Proof. The map is given by

$$(z,\lambda)\mapsto\begin{cases}\left(e^{\pi i\operatorname{Re}\lambda\cdot h(\rho(z,\lambda))}\cdot\left(\dfrac{\operatorname{Re}\zeta}{\sqrt{|e^{2\pi i\lambda}|+|\operatorname{Im}\zeta|^2}},\dfrac{\operatorname{Im}\zeta}{|\operatorname{Im}\zeta|}\right),|\operatorname{Im}\zeta|,\lambda\right), & \text{if } \rho(z,\lambda)>0\\ \left(\left(\dfrac{\operatorname{Re}\zeta}{|e^{\pi i\lambda}|},0\right),\lambda\right), & \text{if } \rho(z,\lambda)=0,\end{cases}$$

where $\zeta=e^{-\pi i\operatorname{Re}\lambda}z$. ∎

Since $\pi_1(\mathbb{C}^\times)$ acts trivially on $T_{\geq\varepsilon}\supset T_{\geq 1}$, we have an equivariant distinguished triangle

$$\mathsf{R}\Gamma_{T_{<1}}(T,\mathscr{F}^\bullet)\to\mathsf{R}\Gamma(T,\mathscr{F}^\bullet)\to\mathsf{R}\Gamma(T_{\geq 1},\mathscr{F}^\bullet)\to\mathsf{R}\Gamma_{T_{<1}}(T,\mathscr{F}^\bullet)[1],$$

where the left exact functor

$$\Gamma_{T_{<1}}(T,\mathscr{F}):=\{s\in\Gamma(T,\mathscr{F})\mid\operatorname{Supp}(s)\subset T_{<1}\},$$

and $\pi_1(\mathbb{C}^\times)$ acts trivially on $\mathsf{R}\Gamma(T_{\geq 1},\mathscr{F}^\bullet)$. So $\sigma-1$ induces a mapping

$$\operatorname{var}_{\mathrm{top}}\sigma:\mathsf{R}\Gamma(T,\mathscr{F}^\bullet)\to\mathsf{R}\Gamma_{T_{<1}}(T,\mathscr{F}^\bullet).$$

We have that the natural mapping

$$H^p_{T_{<1}}(T,M)\to H^p_c(T,M)$$

is an isomorphism. Indeed, there are distinguished triangles

$$\begin{tikzcd}
\mathsf{R}\Gamma_{T_{<1}}(T,M) \arrow[r] \arrow[d] & \mathsf{R}\Gamma(\bar{T},M) \arrow[r] \arrow[d, equal] & \mathsf{R}\Gamma(\bar{T}_{\geq 1},M) \arrow[r] \arrow[d, "\wr"] & \mathsf{R}\Gamma_{T_{<1}}(T,M)[1] \arrow[d] \\
\mathsf{R}\Gamma_c(T,M) \arrow[r] & \mathsf{R}\Gamma(\bar{T},M) \arrow[r] & \mathsf{R}\Gamma(\bar{T}_\infty,M) \arrow[r] & \mathsf{R}\Gamma_c(T,M)[1].
\end{tikzcd}$$

Now we want to show that $\operatorname{var}_{\mathrm{top}}\sigma$ is the same as $\operatorname{var}\sigma$ we defined before, i.e., there are natural isomorphisms such that the diagram

$$\begin{tikzcd}
H^p(T,M) \arrow[r, "\sim"] \arrow[d, "\operatorname{var}_{\mathrm{top}}\sigma_0"] & H^p(\mathscr{A}_0^\bullet) \arrow[d, "\operatorname{var}\sigma_0"] \\
H^p_{T_{<1}}(T,M) \arrow[r, "\sim"] & H^p_{\{0\}}(X_0,\mathscr{A}^\bullet)
\end{tikzcd}$$

is commutative.

We consider a projective quadric $Y\to\mathbb{C}$ defined in homogeneous coordinates by

$$z_0^2+\cdots+z_n^2=tz_{n+1}^2.$$

We denote the infinity part Y^∞. The equation for $Y^\infty \subset \mathbb{P}^n$ is

$$z_0^2 + \cdots + z_n^2 = 0,$$

which is smooth.

We define a complex $\mathscr{D}^\bullet$ by the distinguished triangle

$$M_{\bar{X}} \to \mathsf{R}\bar{j}_* M_{\bar{X}^\times} \to \mathscr{D}^\bullet \to M_{\bar{X}}[1]$$

Then $\mathscr{D}^\bullet|_{X_0} = \mathscr{B}^\bullet$. Hence $\mathscr{D}^\bullet$ has cohomology sheaves concentrated on $\{a\}$.

As $\bar{\mathbb{C}}$ is not locally compact, for a continuous map $g : Y \to \bar{\mathbb{C}}$ we define the functor $g_!$ as follows:

Let U be open in $\bar{\mathbb{C}}$. Let $\Phi(U)$ be the family of closed subsets A of $g^{-1}(U)$ such that $\bar{f} : A \to U$ is proper. They have the following additional property if $\infty \in U$: There is a number $r > 0$ and a constant c with the property that

$$(z, \lambda) \in A, \quad \operatorname{Im} \lambda > r \implies |z| \le c.$$

The functor $g_!$ is now defined as follows:

$$g_! \mathscr{G}(U) = \Gamma_\Phi(g^{-1}(U), \mathscr{G}).$$

Here Γ_Φ is the functor of sections with support in the family Φ, i.e.,

$$\Gamma_\Phi(V, \mathscr{G}) = \{s \in \Gamma(V, \mathscr{G}) \mid \operatorname{Supp}(s) \in \Phi(V)\}.$$

Proposition 27.8. We have natural isomorphisms

(a) $H^p(\mathscr{A}_a^\bullet) \cong H^p(X_0, \mathscr{A}^\bullet) \cong H^p(\bar{X}^\times, M) \cong H^p(X_1, M)$

(b) $H^p_{\{0\}}(X_0, \mathscr{A}^\bullet) \cong H^p_c(X_0, \mathscr{A}^\bullet) \cong H^p_\Phi(\bar{X}^\times, M) \cong H^p_c(X_1, M)$.

Proof. Since X_0 is a cone with vertex a, as in the algebraic case, we have

$$H^p(\mathscr{A}_a^\bullet) \cong H^p(X_0, \mathscr{A}^\bullet), \quad H^p_{\{0\}}(X_0, \mathscr{A}^\bullet) \cong H^p_c(X_0, \mathscr{A}^\bullet).$$

In order to prove the base change theorems for the complex $\mathsf{R}\bar{j}_* M_{\bar{X}^\times}$ and the functor $R^p \bar{f}_*$ for the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\bar{i}} & \bar{X} \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \bar{f}} \\ \{\infty\} & \xrightarrow{i} & \bar{\mathbb{C}}, \end{array}$$

i.e.,

$$\left(R^p\bar{f}_*\mathsf{R}\bar{j}_*M_{\bar{X}^\times}\right)_\infty = H^p(X_0,\mathscr{A}^\bullet),$$

we prove the corresponding theorems for $\mathscr{D}^\bullet$ and for $M_{\bar{X}}$. The case for $\mathscr{D}^\bullet$ is trivial. For M, consider the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\varphi_0"] \arrow[d, "\bar{i}"] & Y_0 \arrow[d, "i_Y"] & Y_0^\infty. \arrow[l, "\psi_0"'] \arrow[d, "i^\infty"] \\
\bar{X} \arrow[r, "\varphi"] \arrow[dr, "\bar{f}"'] & \bar{Y} \arrow[d, "\bar{g}"] & \bar{Y}^\infty \arrow[l, "\psi"'] \arrow[dl, "\bar{h}"] \\
& \mathbb{C}. &
\end{tikzcd}$$

By proper base change, we have

$$(\mathsf{R}\bar{f}_*M)_\infty = (\mathsf{R}\bar{g}_*\mathsf{R}\bar{g}_*M)_\infty = \mathsf{R}\Gamma(Y_0, i_Y^*\mathsf{R}\bar{\varphi}_*M).$$

Now, the base change theorem follows from the morphism between the distinguished triangles

$$\begin{tikzcd}
\mathsf{R}\Gamma_c(X_0, i_Y^*\mathsf{R}\bar{\varphi}_*M) \arrow[r] \arrow[d] & \mathsf{R}\Gamma(Y_0, i_Y^*\mathsf{R}\bar{\varphi}_*M) \arrow[r] \arrow[d] & \mathsf{R}\Gamma_c(Y_0^\infty, i_Y^*\mathsf{R}\bar{\varphi}_*M) \arrow[r] \arrow[d] & \mathsf{R}\Gamma_c(X_0, i_Y^*\mathsf{R}\bar{\varphi}_*M)[1] \arrow[d] \\
\mathsf{R}\Gamma_c(X_0, \mathsf{R}(\varphi_0)_*M) \arrow[r] & \mathsf{R}\Gamma(Y_0, \mathsf{R}(\varphi_0)_*M) \arrow[r] & \mathsf{R}\Gamma_c(Y_0^\infty, \mathsf{R}(\varphi_0)_*M) \arrow[r] & \mathsf{R}\Gamma_c(X_0, \mathsf{R}(\varphi_0)_*M)[1]
\end{tikzcd}$$

since $i_Y^*\mathsf{R}\bar{\varphi} = M = \mathsf{R}(\varphi_0)_*M$ on X_0 and $i_Y^*\mathsf{R}\bar{\varphi} = i_Y^*\mathsf{R}\bar{\varphi}$ on Y_0^∞ by smooth base change theorem. So we get from the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\bar{i}"] \arrow[d, "f_0"] & \bar{X} \arrow[d, "\bar{f}"] & \bar{X}^\times \arrow[l, "\bar{j}"'] \arrow[r, "\sim"] \arrow[d, "\bar{f}^\times"] & X_1 \times \mathbb{C} \arrow[dl, "g'"] \arrow[d, "\pi'"] \\
\{\infty\} \arrow[r, "i"] & \bar{\mathbb{C}} & \mathbb{C} \arrow[l, "j"'] \arrow[d, "\pi"] & X_1 \arrow[dl, "g"] \\
& & \{\mathrm{pt}\} &
\end{tikzcd}$$

that

$$\mathsf{R}\Gamma(X_0,\mathscr{A}^\bullet) = (\mathsf{R}\bar{f}_*\mathsf{R}\bar{j}_*M)_\infty = (\mathsf{R}j_*\mathsf{R}\bar{f}_*^\times M)_\infty = (\mathsf{R}j_*\mathsf{R}g'_*(\pi')^*M)_\infty = (\mathsf{R}j_*\mathsf{R}\Gamma(X_1,M)_{\mathbb{C}})_\infty.$$

and

$$\mathsf{R}\Gamma_c(X_0,\mathscr{A}^\bullet) = (\mathsf{R}\bar{f}_!\mathsf{R}\bar{j}_*M)_\infty = (\mathsf{R}j_*\mathsf{R}\bar{f}_!^\times M)_\infty = (\mathsf{R}j_*\mathsf{R}g'_!(\pi')^*M)_\infty = (\mathsf{R}j_*\mathsf{R}\Gamma_c(X_1,M)_{\mathbb{C}})_\infty.$$

Finally, note that for any abelian group F,

$$(\mathsf{R}j_*F)_\infty = \mathsf{R}\Gamma(\bar{\mathbb{C}}, \mathsf{R}j_*F) = \mathsf{R}\Gamma(\mathbb{C}, F) = F. \qquad \blacksquare$$

In particular, $\mathscr{H}^0(\mathscr{A}^\bullet) = M$, and thus $\mathscr{H}^0(\mathscr{B}^\bullet) = 0$.

Theorem 27.9. The diagram

$$\begin{tikzcd}
H^p(T, M) \arrow[r, "\sim"] \arrow[d, "\mathrm{var}_{\mathrm{top}}\,\sigma_0"] & H^p(X_1, M) \arrow[rr, "\sim"] & & H^p(\mathscr{A}_a^\bullet) \arrow[d, "\mathrm{var}\,\sigma_0"] \\
H^p(T, T_{\geq 1}, M) \arrow[r, "\sim"] & H_c^p(T, M) \arrow[r, "\sim"] & H_c^p(X_1, M) \arrow[r, "\sim"] & H_{\{0\}}^p(X_0, \mathscr{A}^\bullet)
\end{tikzcd}$$

is commutative.

Proof. Consider the homeomorphism

$$\begin{tikzcd}
\bar{X}' \arrow[r, equal, "\sim"] & T' \times \bar{\mathbb{C}} \\
\bar{X}_{\geq r} \arrow[u, hook] \arrow[r, equal, "\sim"] & T_{\geq r} \times \bar{\mathbb{C}}. \arrow[u, hook]
\end{tikzcd}$$

We set

$$\bar{X}^\times_{\geq r(<r)} = \bar{X}^\times \cap \bar{X}_{\geq r(<r)},$$
$$(X_0)_{\geq r} = X_0 \cap \bar{X}_{\geq r}.$$

We choose r such that $\varepsilon < r < 1$. We consider the following equivalence relation on $\bar{X}$: two element $u, v \in \bar{X}$ are equivalent if they both lie in $\bar{X}_{\geq r}$ and their images in $T_{\geq r} \times \bar{\mathbb{C}}$ have the same projection to $T_{\geq r}$. We set

$$\widehat{X} = \bar{X}/\sim, \quad \widehat{X}^\times = \bar{X}^\times/\sim.$$

On $\widehat{X}$ we define a topology by giving a basis: A subset U of $\widehat{X}$ belongs to this basis when its inverse image V in $\bar{X}$ either lies in $\bar{X}_{<r}$ and is open there or when V lies in $\bar{X}' = T' \times \bar{\mathbb{C}}$ and is of the form $W \times \bar{\mathbb{C}}$ with W open in T'.

By (27.7), we have an equivariant continuous mapping

$$\bar{X}^\times \xrightarrow{\sim} T \times \mathbb{C} \to T \cong X_1.$$

This projection is an inverse mapping for the (non-equivariant) embedding

$$T \cong X_1 = \bar{f}^{-1}(0) \subset \bar{X}^\times.$$

The homeomorphism

$$\sigma : \bar{X} \to \bar{X}$$

induces homeomorphisms

$$\widehat{X} \to \widehat{X}, \quad \widehat{X}^\times \to \widehat{X}^\times.$$

The diagram

$$T \leftarrow \bar{X}^\times \hookrightarrow \bar{X} \hookleftarrow X_0$$

induces a commutative diagram of continuous equivariant mappings,

$$\begin{tikzcd}
T \arrow[d, equal] & \bar{X}^\times \arrow[l] \arrow[r, hook, "\bar{j}"] \arrow[d, "q^\times"] & \bar{X} \arrow[d, "q"] & X_0 \arrow[l, hook', "\bar{i}"'] \arrow[d, equal] \\
T & \widehat{X}^\times \arrow[l] \arrow[r, hook, "\widehat{j}"] & \widehat{X} & X_0 \arrow[l, hook', "\widehat{i}"'] \\
T_{\geq 1} \arrow[u, hook] \arrow[r, equal, "\sim"] & \widehat{X}^\times_{\geq 1} \arrow[u, hook] \arrow[r, equal, "\sim"] & \widehat{X}_{\geq 1} \arrow[u, hook] \arrow[r, equal, "\sim"] & (X_0)_{\geq 1} \arrow[u, hook]
\end{tikzcd}$$

Note that $\widehat{i}$ is a closed embedding but $\widehat{j}$ is not an open embedding.

Lemma 27.10. We have

$$R^p q_* M_{\bar{X}} = \begin{cases} 0, & \text{if } p > 0 \\ M_{\widehat{X}}, & \text{if } p = 0 \end{cases} \quad \text{and} \quad R^p q_*^\times M_{\bar{X}^\times} = \begin{cases} 0, & \text{if } p > 0 \\ M_{\widehat{X}^\times}, & \text{if } p = 0. \end{cases}$$

Proof. We treat the case of the sheaf M on $\bar{X}$. The proof for the case on $\bar{X}^\times$ is similar. It suffices to show that for every point $v \in \widehat{X}$ there exists a neighborhood basis of neighborhoods U with

$$\varinjlim_U H^p(q^{-1}(U), M) = \begin{cases} 0, \text{ if } p > 0 \\ M, \text{ if } p = 0. \end{cases}$$

The case $v = q(u)$ with $u \in \bar{X}_{<r}$ is trivial, as q maps the space $\bar{X}_{<r}$ topologically onto an open subset of $\widehat{X}$. In the other case

$$v = q(u), u = (z, \lambda) \in \bar{X}_{\geq r} = T_{\geq r} \times \bar{\mathbb{C}},$$

there exists a basis of neighborhoods U of the form

$$q^{-1}(U) = V \times \bar{\mathbb{C}} \subset \bar{X}' = T' \times \bar{\mathbb{C}}.$$

Here V runs through a neighborhood basis of contractible neighborhoods of z in T'. One sees easily that

$$H^p(V \times \bar{\mathbb{C}}, M) = \begin{cases} 0, \text{ if } p > 0 \\ M, \text{ if } p = 0. \end{cases} \qquad \square$$

We've constructed the equivariant complex $\mathscr{A}^\bullet = \bar{\imath}^*\mathsf{R}\bar{\jmath}_*M$ on X_0. We can carry out the same construction with the mappings $\widehat{\imath}$, $\widehat{\jmath}$: we choose a flabby equivariant resolution $\widehat{\mathscr{I}^\bullet}$ of M on $\widehat{X}^\times$. We shall actually assume that $\pi_1(\mathbb{C}^\times)$ acts trivially on $\widehat{\mathscr{I}^\bullet}|_{\widehat{X}_{\geq r'}}$ for some $r < r' < 1$. We can for instance use the canonical flabby Godement resolution. Let $\widehat{\mathscr{A}^\bullet} = \widehat{\imath}^*\mathsf{R}\widehat{\jmath}_*M$. It follows from (27.10) and cohomology sheaves of the mapping cone of

$$M \to \mathsf{R}\bar{\jmath}_*M$$

is concentrated on $\{a\}$ that

$$\widehat{\mathscr{A}^\bullet} = \widehat{\imath}^*\mathsf{R}\widehat{\jmath}_*M = \bar{\imath}^*q^*\mathsf{R}\widehat{\jmath}_*\mathsf{R}q^\times_*M = \bar{\imath}^*q^*\mathsf{R}q^\times_*(\mathsf{R}\widehat{\jmath}_*M) \to \bar{\imath}^*\mathsf{R}\widehat{\jmath}_*M = \mathscr{A}^\bullet$$

is an isomorphism in the derived category of $\pi_1(\mathbb{C}^\times)$-sheaves on X_0. Thus $\operatorname{var}\sigma_0$ can be constructed using $\widehat{\mathscr{A}^\bullet}$ in place of $\mathscr{A}^\bullet$. By the assumption on $\widehat{\mathscr{I}^\bullet}$ we have $\pi_1(\mathbb{C}^\times)$ acting trivially on $\widehat{\mathscr{A}^\bullet}|_{(X_0)_{\geq r'}}$ and indeed trivially on the restriction to a neighborhood $(X_0)_{\geq r'}$ of $(X_0)_{\geq 1}$. As X_0 is a cone, we further have

$$\begin{aligned} H^p(X_0, \widehat{\mathscr{A}^\bullet}) &= H^p(\widehat{\mathscr{A}^\bullet_0}), \\ H^p_{\{0\}}(X_0, \widehat{\mathscr{A}^\bullet}) &= H^p_{(X_0)_{<1}}(X_0, \widehat{\mathscr{A}^\bullet}) = H^p_c(X_0, \widehat{\mathscr{A}^\bullet}). \end{aligned}$$

With this identification, $\operatorname{var}\sigma_0$ is a mapping

$$\operatorname{var}\sigma_0 : H^p(X_0, \widehat{\mathscr{A}^\bullet}) \to H^p_{(X_0)_{<1}}(X_0, \widehat{\mathscr{A}^\bullet}), \quad p \neq 0.$$

It is easy to think through the fact that

$$\operatorname{var}\sigma_0 = \operatorname{var}_{\mathrm{top}}\sigma_0 : H^p(X_0, \widehat{\mathscr{A}^\bullet}) \to H^p_{(X_0)_{<1}}(X_0, \widehat{\mathscr{A}^\bullet}).$$

We have the commutative diagram

$$\begin{tikzcd}
H^p(T,M) \arrow[r, "\sim"] \arrow[d, equal] & H^p(\bar{X}^\times, M) & H^p(\bar{X}, \mathsf{R}\bar{\jmath}_*M) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H^p(X_0, \mathscr{A}^\bullet) \arrow[d, equal] \\
H^p(T,M) \arrow[r, "\sim"] \arrow[d, "\operatorname{var}_{\mathrm{top}}\sigma_0"] & H^p(\widehat{X}^\times, M) \arrow[u, "\wr"'] \arrow[d, "\operatorname{var}_{\mathrm{top}}\sigma_0"] & H^p(\widehat{X}, \mathsf{R}\widehat{\jmath}_*M) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\wr"'] \arrow[d, "\operatorname{var}_{\mathrm{top}}\sigma_0"] & H^p(X_0, \widehat{\mathscr{A}^\bullet}) \arrow[d, "\operatorname{var}_{\mathrm{top}}\sigma_0"', "=\operatorname{var}\sigma_0"] \\
H^p_{T_{<1}}(T,M) \arrow[r, "\sim"] \arrow[d, equal] & H^p_{\widehat{X}^\times_{<1}}(\widehat{X}^\times, M) \arrow[d, "\wr"] & H^p_{\widehat{X}_{<1}}(\widehat{X}, \mathsf{R}\widehat{\jmath}_*M) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & H^p_{(X_0)_{<1}}(X_0, \widehat{\mathscr{A}^\bullet}) \arrow[d, equal] \\
H^p_{T_{<1}}(T,M) \arrow[r, "\sim"] & H^p_{\bar{X}^\times_{<1}}(\bar{X}^\times, M) & H^p_{\bar{X}_{<1}}(\bar{X}, \mathsf{R}\bar{\jmath}_*M) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H^p_{(X_0)_{<1}}(X_0, \mathscr{A}^\bullet).
\end{tikzcd}$$

The isomorphisms of the top and bottom lines come from (27.8). The vertical isomorphisms come from (27.10) because of

$$\bar{X}_{\geq 1} = \widehat{X}_{\geq 1} \times \mathbb{C},$$

other homeomorphisms, and the long exact sequence for a pair of spaces. ■

We now want to determine

$$\mathrm{var}_{\mathrm{top}}\, \sigma_0 : H^p(T, \mathbb{Z}) \to H^p_{T_{<1}}(T, \mathbb{Z}) = H^p_c(T, \mathbb{Z}), \quad p \neq 0$$

for the "universal" coefficient system $\mathbb{Z}$.

We consider the projection

$$\pi : T \to S^n.$$

It induces isomorphisms

$$\pi^* : H^p(S^n, \mathbb{Z}) \xrightarrow{\sim} H^p(T, \mathbb{Z}),$$

and hence

$$H^p(T, \mathbb{Z}) = \begin{cases} 0, & \text{if } p \neq 0, n \\ \mathbb{Z}, & \text{if } p = n. \end{cases}$$

It thus suffices to study the cases $p = n$. The natural mapping

$$H^p(T, \mathbb{Z}) \to H^p(T, \mathbb{C})$$

is injective. Since σ_0 is the identity in a neighborhood of $T_{\geq 1}$, we can use the de Rham complex Ω_T for the computation of $\mathrm{var}_{\mathrm{top}}\, \sigma_0$.

We look for some orientation on S^n and T. One is defined on T by the isomorphism

$$T \cong X_1$$

and the complex orientation

$$i^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

on $X_1 \subset \mathbb{C}^{n+1}$. It differs from the natural orientation on the sphere and on the tangent bundle T by $(-1)^{n(2n-1)}$. Let ω be a differential form of degree n on S^n with

$$\int_{S^n} \omega = 1.$$

It represents a generator of $H^n(S^n, \mathbb{Z})$. Hence $\pi^*\omega$ represents one of the two generating elements δ of $H^n(T, \mathbb{Z})$, and $\operatorname{var}_{\text{top}} \sigma_0(\delta)$ is represented by

$$\sigma_0^* \pi^* \omega - \pi^* \omega.$$

We want to compute

$$\int_T (\sigma_0^* \pi^* \omega - \pi^* \omega) \wedge \pi^* \omega.$$

By Fubini's theorem,

$$\int_T \sigma_0^* \pi^* \omega \wedge \pi^* \omega = \int_{S^n} \left(\int_{T_x} (\pi \circ \sigma_0)^* \omega \right) \omega =: \int_{S^n} \varepsilon(x) \omega.$$

We consider the restriction of $\pi \circ \sigma_0$ to T_x,

$$\alpha = \pi \circ \sigma_0|_{T_x} : T_x \to S^n.$$

α is a C^∞-isomorphism outside $\alpha^{-1}(\pm x)$. Consequently,

$$\varepsilon(x) = \int_{T_x} \alpha^* \omega = \pm \int_{S^n} \omega = \pm 1.$$

By continuity, the sign does not depend on x. We get:

$$\int_{S^n} \varepsilon(x) \omega = \pm \int_{S^n} \omega = \pm 1,$$

$$\int_T (\sigma_0^* \pi^* \omega - \pi^* \omega) \wedge \pi^* \omega = \pm 1.$$

Theorem 27.11. Let $\langle \cdot\,, \cdot \rangle$ be the duality pairing between $H^n(T, \mathbb{Z})$ and $H_c^n(T, \mathbb{Z})$ for some orientation. Let δ be a generator of

$$H^n(T, \mathbb{Z}) \cong \mathbb{Z}.$$

Let δ' be the generator of $H_c^n(T, \mathbb{Z})$ with $\langle \delta, \delta' \rangle = 1$. Then

$$\operatorname{var}_{\text{top}} \sigma_0(x) = \pm \langle x, \delta' \rangle \delta'.$$

Computing the orientation yields

$$\operatorname{var}_{\text{top}} \sigma_0(x) = (-1)^{m+1} \langle x, \delta' \rangle \delta'$$

where $n = 2m + 1$.

We now want to compare the algebraic variation with that defined analytically.

Let $S = \operatorname{Spec} \mathbb{C}[t]$ be the affine line and $0 \in S$ the origin. Then $S_{\mathrm{an}} = \mathbb{C}$ is the complex plane. We consider again the logarithmically ramified covering space

$$\begin{array}{ccc} \mathbb{C} & \hookrightarrow & \bar{\mathbb{C}} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ \mathbb{C}^\times & \hookrightarrow & \mathbb{C}. \end{array}$$

The polynomial ring $\mathbb{C}[t]$ is contained in $\mathscr{O}_{\mathbb{C},0}$, the ring of germs of holomorphic functions at the origin of $S_{\mathrm{an}} = \mathbb{C}$. We have a natural injection of $\mathscr{O}_{\mathbb{C},0}$ to the field $\mathscr{M}$ of meromorphic functions in an arbitrary small connected pointed neighborhood U_r of ∞ in $\bar{\mathbb{C}}$.

$$\begin{array}{rccc} \mathbb{C}[t] \subset & \mathscr{O}_{\mathbb{C},0} & \hookrightarrow & \mathscr{M} \\ & f(t) & \mapsto & f(e^{2\pi i\lambda}). \end{array}$$

Let K be the field of those functions in $\mathscr{M}$ that are algebraic over the quotient field of $\mathbb{C}[t]$. We consider the family generated étale $\mathbb{C}[t]$-algebras A that are contained in $\mathscr{O}_{\mathbb{C},0}$.

The residue field mapping $\mathscr{O}_{\mathbb{C},0} \to \mathbb{C}$ defines a geometric point with values in $\mathbb{C}$, and the spectrum $V = \operatorname{Spec} A$,

$$\begin{array}{ccc} & & V \\ & \nearrow & \downarrow \\ 0 & \longrightarrow & S, \end{array}$$

is an etale neighborhood of $0 \to S$. The ring

$$\varinjlim_{A} A = \bigcup A = R \subseteq \mathscr{O}_{\mathbb{C},0}$$

is the strict Henselization of $\mathscr{O}_{S,0}$ at 0. For given A and a natural number k, we consider further the A-subalgebra B_k of K given by

$$B_k = A[t^{1/k}] = A[e^{2\pi i\lambda/k}] \subset K.$$

The mapping $\operatorname{Spec} B_k \to \operatorname{Spec} A$ is ramified only over 0. Over this point of V there lies exactly one point of $\operatorname{Spec} B$. Let

$$V_k = \operatorname{Spec} B_k, \quad V_k^\times = V_k \setminus \{0\} = \operatorname{Spec} B_k[t^{-1}].$$

Then

$$\varinjlim_{A,k} B_k[t^{-1}] = \bigcup_{A,k} B_k[t^{-1}] = K.$$

The element $\sigma_0 \in \pi_1(\mathbb{C}^\times)$ acts on $\mathscr{M}$ by:

$$f(\lambda) \mapsto f(\lambda+1).$$

It acts trivially on the subring $\mathcal{O}_{\mathbb{C},0}$. This action carries K to itself:

In fact,

$$\sigma_0(f) = f$$

for $f \in R \subset \mathcal{O}_{\mathbb{C},0}$ and

$$\sigma(t^{1/k}) = \sigma(e^{2\pi i\lambda/k}) = (e^{2\pi i/k})^m\, t^{1/k}.$$

Thus we have a natural injection

$$\pi_1(\mathbb{C}^\times) \to \mathrm{Gal}(K/R) = \varprojlim_k \mu_k(\mathbb{C})$$

by $\sigma_0 \mapsto (e^{2\pi i/k})_k$. We have a natural diagram

$$\begin{tikzcd}
 & \widetilde{S} = \operatorname{Spec} R \arrow[d] & \operatorname{Spec} K \arrow[l] \arrow[d] \\
 & V \arrow[d] & V_k^\times \arrow[l] \\
0 \arrow[r] \arrow[uur] \arrow[ur] & S, &
\end{tikzcd}$$

We consider the corresponding analytic situation:

$$U_r = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > r\}, \quad \bar{U}_r = U_r \cup \{\infty\}$$

$$\begin{tikzcd}
 & V_{\mathrm{an}} \arrow[d] \\
0 \arrow[r] \arrow[ur] & \mathbb{C}.
\end{tikzcd}$$

Since $V_{\mathrm{an}} \to \mathbb{C}$ is unramified, there is a number $r > 0$ and an open neighborhood D of 0 in V_{an} that $V_{\mathrm{an}} \to \mathbb{C}$ is mapped isomorphically onto the open circle of radius $e^{-2\pi r}$ around 0,

$$D \cong \{t \in \mathbb{C} \mid |t| < e^{-2\pi r}\}.$$

The mapping $\bar{\mathbb{C}} \to \mathbb{C}$ induces a mapping

$$\bar{U}_r \to D.$$

The embedding of the algebra of $V_k^\times$ induces then a holomorphic mapping

$$U_r \to (V_k^\times)_{\mathrm{an}}.$$

The image D' is a finite unramified cover of $D \setminus \{0\}$ of degree k.

We have the diagram

$$
\begin{tikzcd}
 & U_r \arrow[rr] \arrow[dl] \arrow[dd] & & \bar{U}_r \arrow[dl] \arrow[dd] & & \{\infty\} \arrow[ll] \arrow[dd] \\
\mathbb{C} \arrow[rr, crossing over] \arrow[dd] & & \bar{\mathbb{C}} & & & \\
 & (V_k^\times)_{\mathrm{an}} \arrow[rr] \arrow[dl] & & (V_k)_{\mathrm{an}} \arrow[dl] & & \{0\} \arrow[ll] \\
\mathbb{C}^\times \arrow[rr] & & \mathbb{C}. \arrow[from=uu, crossing over] & & &
\end{tikzcd}
$$

Now we consider the affine algebraic quadric $g : Z \to S$,

$$
Z = \operatorname{Spec}\left(\mathbb{C}[t, z_0, \ldots, z_n]/(z_0^2 + \cdots + z_n^2 - t)\right).
$$

Then

$$
f = g_{\mathrm{an}} : X = Z_{\mathrm{an}} \to \mathbb{C}
$$

is the analytic quadric treated before. We set

$$
\begin{aligned}
\bar{X}_{\bar{U}_r} &= \bar{X} \times_{\bar{\mathbb{C}}} \bar{U}_r \subset \bar{X}, \\
\bar{X}^\times_{U_r} &= \bar{X}^\times \times_{\mathbb{C}} U_r \subset \bar{X}^\times, \\
\bar{X}^\times_{V_k} &= X \times_{\mathbb{C}} (V_k^\times)_{\mathrm{an}}, \\
X_V &= X \times_{\mathbb{C}} V_{\mathrm{an}}
\end{aligned}
$$

and get a commutative diagram

$$
\begin{tikzcd}
\bar{X}^\times \arrow[r, "\bar{j}"] & \bar{X} & X_0 \arrow[l, "\bar{i}"'] \\
\bar{X}^\times_{U_r} \arrow[r] \arrow[u, hook] \arrow[d] & \bar{X}_{\bar{U}_r} \arrow[u, hook] \arrow[d] & X_0 \arrow[l] \arrow[u, equal] \arrow[d, equal] \\
\bar{X}^\times_{V_k} \arrow[r, "\bar{j}_{V_k}"] & X_V & X_0. \arrow[l, "\bar{i}_V"']
\end{tikzcd}
$$

In the algebraic case, we set

$$
\widetilde{Z} = Z \times_S \widetilde{S}, \quad Z_\eta = Z \times_S \operatorname{Spec} K, \quad Z_0 = g^{-1}(0),
$$

$$
Z^\times_{V_k} = Z \times_S V_k^\times, \quad Z_V = Z \times_S V
$$

and get a commutative diagram

$$\begin{tikzcd}
Z_\eta \arrow[r, "j"] \arrow[d] & \widetilde{Z} \arrow[d] & Z_0 \arrow[l, "i"'] \arrow[d, equal] \\
Z^\times_{V_k} \arrow[r, "j_{V_k}"] & Z_V \arrow[d] & Z_0 \arrow[l, "i_V"'] \arrow[d, equal] \\
 & Z & Z_0. \arrow[l]
\end{tikzcd}$$

Passing to the analytic version of the middle line here gives the lower line of the analytic diagram,

$$X^\times_{V_k} \to X_V \leftarrow X_0.$$

Let M be a finite abelian group, $\mathscr{I}^\bullet$ an injective resolution of the constant sheaf M on Z, and $\mathscr{J}^\bullet$ an injective resolution of $\mathscr{I}^\bullet_{\mathrm{an}}$ on $X = Z_{\mathrm{an}}$,

$$\mathscr{I}^\bullet_{\mathrm{an}} \to \mathscr{J}^\bullet.$$

Then $\mathscr{J}^\bullet$ is also an injective resolution of $M_X = (M_Z)_{\mathrm{an}}$. The inverse image $\mathscr{I}^\bullet_\eta$ of $\mathscr{I}^\bullet$ on Z_η is then in a natural way an equivariant complex. It is an injective resolution of M_{Z_η}. The corresponding statement holds for the inverse image $\mathscr{I}^\bullet_{V_k}$ of $\mathscr{I}^\bullet$ on $Z^\times_{V_k}$.

In the analytic case, the inverse image $\bar{\mathscr{J}}^\bullet$ of $\mathscr{J}^\bullet$ on $\bar{X}^\times$ is in a natural way an equivariant injective resolution of M on $\bar{X}^\times$. The corresponding properties hold for the inverse image $\mathscr{J}^\bullet_{V_k}$ of $\mathscr{J}^\bullet$ on $\bar{X}^\times_{V_k}$.

The quasi-isomorphism

$$\mathscr{I}^\bullet_{\mathrm{an}} \to \mathscr{J}^\bullet$$

gives us a natural equivariant quasi-isomorphism

$$(\mathscr{I}^\bullet_{V_k})_{\mathrm{an}} \to \mathscr{J}^\bullet_{V_k}$$

on $\bar{X}^\times_{V_k} = (Z_{V_k})_{\mathrm{an}}$.

We set

$$\mathscr{A}^\bullet = i^* j_* \mathscr{I}^\bullet_\eta, \quad \mathscr{A}^\bullet_{V_k} = i^\bullet_V (j_{V_k})_* \mathscr{I}^\bullet_{V_k}$$

and get a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathscr{A}}^\bullet_{V_k} \arrow[r] & \widetilde{\mathscr{A}}^\bullet \\
M. \arrow[u] \arrow[ur] &
\end{tikzcd}$$

From the quasi-isomorphism

$$(\mathscr{I}^\bullet_{V_k})_{\text{an}} \to \mathscr{J}^\bullet_{V_k}$$

we get an equivariant comparison mapping

$$(\mathscr{A}^\bullet_{V_k})_{\text{an}} \to \widetilde{\mathscr{A}_{V_k}}.$$

It is a quasi-isomorphism. Apply the comparison theorems for étale and analytic cohomology to $\mathsf{R}(j_{V_k})*$.

Summing up, we get an equivariant diagram on X_0,

$$\begin{tikzcd}
\mathscr{A}^\bullet_{\text{an}} & (\mathscr{A}^\bullet_{V_k})_{\text{an}} \arrow[l] \arrow[r, "\sim"] & \widetilde{\mathscr{A}^\bullet_{V_k}} \arrow[r] & \widetilde{\mathscr{A}}^\bullet \\
& M_{\text{an}} \arrow[ul] \arrow[u] \arrow[r, equal] & M \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Proposition 27.12. We have

(a) $\varinjlim\limits_{V_k} \mathscr{A}^\bullet_{V_k} = \mathscr{A}^\bullet$, and

(b) the mapping of complexes $\varinjlim\limits_{V_k} \widetilde{\mathscr{A}^\bullet_{V_k}} = \widetilde{\mathscr{A}}^\bullet$ is a quasi-isomorphism.

Proof. (a) comes from the theorems on the étale cohomology of direct limits of rings. For (b),

$$\varinjlim \mathscr{H}^0(\widetilde{\mathscr{A}^\bullet_{V_k}}) = \big(\varinjlim \mathscr{H}^0(\mathscr{A}^\bullet_{V_k})\big)_{\text{an}} = M_{\text{an}} = M = \mathscr{H}^0(\widetilde{\mathscr{A}}^\bullet).$$

For $p \neq 0$, $b \neq 0$,

$$\varinjlim H^p(\widetilde{\mathscr{A}^\bullet_{V_k,b}}) = \varinjlim H^p(\mathscr{A}^\bullet_{V_k,b}) = 0 = H^p(\widetilde{\mathscr{A}^\bullet_b}).$$

It remains to show that

$$\varinjlim H^p(\widetilde{\mathscr{A}^\bullet_{V_k,0}}) = H^p(\widetilde{\mathscr{A}^\bullet_0})$$

for $p \neq 0$. For this, we consider the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] \arrow[d, "\exp"'] & \bar{\mathbb{C}} \arrow[dd] & \{\infty\} \arrow[l, hook'] \arrow[dd] \\
\mathbb{C}^\times \arrow[d, "{[k]}"'] & & \\
\mathbb{C}^\times \arrow[r, hook] & \mathbb{C} & \{0\}. \arrow[l, hook']
\end{tikzcd}$$

By extension with analytic quadric, we get

$$\begin{tikzcd}
\bar{X}^\times \arrow[r, hook, "\bar{j}"] \arrow[d, "\exp"'] \arrow[ddr, "j"] & \bar{X} \arrow[dd] & X_0 \arrow[l, hook', "\bar{i}"'] \arrow[dd, equal] \\
X_k^\times \arrow[d, "{[k]}"'] & & \\
X_k^\times \arrow[r, hook, "j_k"] & X & X_0. \arrow[l, hook', "i"']
\end{tikzcd}$$

We must show that

$$\varinjlim_{W \ni 0, k} H^p(j_k^{-1}(W), M) = \varinjlim_{W \ni 0} H^p(j^{-1}(W), M).$$

For $\rho, c > 0$, we set

$$W = \{(z,t) \in X \mid |t| < e^{-2\pi r}, |z|^2 - |t| < 2\rho^2\}.$$

Using the identification $\bar{X}^\times \cong T \times \mathbb{C}$, we get

$$j^{-1}(W) \cong \{(x,y) \in T \mid |y| < \rho\} \times U_r, \quad j_k^{-1}(W) = j^{-1}(W)/G_k$$

with $G_k = k\mathbb{Z} \subset \mathbb{Z} = \pi_1(\mathbb{C}^\times)$. The cohomology groups are thus finite. There is a spectral sequence

$$H^p(G_k, H^q(j^{-1}(W), M)) \Rightarrow H^{p+q}(j_k^{-1}(W), M).$$

The assertion now follows from

Claim. Set $G = \mathbb{Z}$, $G_k = kG \subset G$. Let F be a finite abelian group on which G acts. Then

$$\varinjlim_k H^p(G_k, F) = \begin{cases} F, & \text{if } p = 0 \\ 0, & \text{if } p \neq 0. \end{cases}$$

Proof of Claim. Since $|\operatorname{Aut}(F)| < \infty$, $\mathbb{Z} \to \operatorname{Aut}(F)$ has non-trivial kernel $d\mathbb{Z}$. ■

This proposition yields the desired comparison theorem for variation.

Proposition 27.13. There is a natural equivariant quasi-isomorphism

$$\begin{tikzcd}
\mathscr{A}^\bullet_{\mathrm{an}} \arrow[r] & \widetilde{\mathscr{A}^\bullet} \\
M_{\mathrm{an}} \arrow[u] \arrow[r, equal] & M. \arrow[u]
\end{tikzcd}$$

in $D(X_0)$. As the cohomology sheaves of $\mathscr{A}^\bullet$ are constructible, this yields a commutative diagram

$$\begin{array}{ccc} H^p(\mathscr{A}_0^\bullet) & \xrightarrow{\sim} & H^p(\widetilde{\mathscr{A}_0^\bullet}) \\ \downarrow{\scriptstyle \operatorname{var}\sigma} & & \downarrow{\scriptstyle \operatorname{var}\sigma} \\ H^p_{\{0\}}(Z_0, \mathscr{A}^\bullet) & \xrightarrow{\sim} & H^p_{\{0\}}(X_0, \widetilde{\mathscr{A}^\bullet}) \end{array}$$

The left var is algebraically defined while the right var is analytically defined. Here we use the natural mapping

$$\pi_1(\mathbb{C}^\times) \to \mathrm{Gal}(K/R).$$

Lemma 27.14. The fundamental lemma (26.10) holds for the projective quadric defined by the equation

$$x_0x_{m+1} + \cdots + x_mx_{2m+1} + tx_{n+1}^2 = 0$$

where $n = 2m+1$ over the ring $\mathbb{C}\{t\}$: The algebraic variation mapping is of the form

$$\operatorname{var}\sigma(x) = (-1)^{m+1}\chi_\lambda(\sigma)\langle x, \delta\rangle\delta.$$

28 The behavior of the monodromy mapping under change of base ring

Consider an injective local homomorphism

$$\varphi : R \hookrightarrow R_1$$

of strictly Henselian DVR with residue fields $k = R/\mathfrak{m}$, $k_1 = R_1/\mathfrak{m}_1$, quotient fields K, K_1, and an R-homomorphism

$$\varphi : \bar{K} \to \bar{K}_1$$

from the separable closure $\bar{K}$ of K to the separable closure $\bar{K}_1$ of K_1, $\mathfrak{m} = \langle\pi\rangle_R$, $\mathfrak{m}_1 = \langle\pi_1\rangle_{R_1}$, $\varphi(\pi) \in \pi_1^e \cdot R_1^\times$.

Let G be the Galois group of $\bar{K}$ over K and G_1 the Galois group of $\bar{K}_1$ over K_1. The embedding $\varphi : \bar{K} \to \bar{K}_1$ induces a homomorphism

$$\psi : G_1 \to G.$$

The fundamental characters

$$\chi_\lambda : G \to \Lambda(1), \quad \chi_{\lambda_1} : G_1 \to \Lambda(1)$$

satisfy

$$\chi_\lambda(\psi(\sigma)) = \chi_{\varphi(\lambda)}(\sigma) \quad \forall \sigma \in G_1.$$

In particular,

$$\chi(\psi(\sigma)) = e \cdot \chi(\sigma).$$

We consider the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Spec} \bar{K}_1 & \xrightarrow{\eta_1} & S_1 = \operatorname{Spec} R_1 & \xleftarrow{s_1} & \operatorname{Spec} k_1 \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} \bar{K} & \xrightarrow{\eta} & S = \operatorname{Spec} R & \xleftarrow{s} & \operatorname{Spec} k.
\end{array}$$

We again study our quadric

$$f : X \to S$$

with homogeneous equation

$$x_0 x_{m+1} + \cdots + x_m x_{2m+1} + \lambda x_{n+1}^2 = 0$$

where $0 \neq \lambda \in \mathfrak{m}$. By extending the base scheme, we get a quadric

$$f_1 : X_1 = X \times_S S_1 \to S_1$$

over S_1 with equation

$$x_0 x_{m+1} + \cdots + x_m x_{2m+1} + \varphi(\lambda) x_{n+1}^2 = 0.$$

The double point a_1 of X_1 lies over the double point a of X. We get a commutative diagram

$$\begin{array}{ccccccc}
X_{1,\eta} & \xrightarrow{j_1} & X_1 & \xleftarrow{i_1} & X_{s,1} & \qquad & a_1 \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_s} & & \downarrow \\
X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s & & a
\end{array}$$

Using the base change mapping, we get a commutative diagram, equivariant with respect to the homomorphism $G_1 \to G$:

$$\begin{array}{ccccccc}
p_s^* \Lambda & \longrightarrow & p_s^* \mathscr{A}^\bullet(f) & \longrightarrow & p_s^* \mathscr{B}^\bullet(f) & \longrightarrow & p_s^* \Lambda[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Lambda & \longrightarrow & \mathscr{A}^\bullet(f_1) & \longrightarrow & \mathscr{B}^\bullet(f_1) & \longrightarrow & \Lambda[1]
\end{array}$$

With the isomorphisms of (26.8), we have a commutative diagram

$$\begin{array}{ccc} H^\bullet(X'_\eta, \Lambda) & \xrightarrow{\sim} & H^\bullet(\mathscr{A}^\bullet(f)_{\bar{a}}) = H^\bullet(p_s^*\mathscr{A}^\bullet(f)_{\bar{a}_1}) \\ \downarrow & & \downarrow \\ H^\bullet(X'_{1,\eta}, \Lambda) & \xrightarrow{\sim} & H^\bullet(\bar{\mathscr{A}}(f_1)_{\bar{a}}) \end{array}$$

On the other hand, the étale cohomology groups are invariant under the base field extension $\bar{K} \to \bar{K}_1$, even though $X_{1,\eta}$ is not proper over $\bar{K}$:

$$H^\bullet(X'_\eta, \Lambda) \xrightarrow{\sim} H^\bullet(X'_\eta \times_{\operatorname{Spec} \bar{K}}) \operatorname{Spec} \bar{K}_1, \Lambda) = H^\bullet(\bar{X}'_{1,\eta}, \Lambda).$$

This follows from the smooth base change theorem and the invariance theorem for purely inseparable field extensions. Indeed $\bar{K}_1$ is a purely inseparable extension of an (infinite) separable field extension L of $\bar{K}$. The latter is a direct limit of smooth k-algebras. Thus we have:

Proposition 28.1. The complexes $\mathscr{A}^\bullet(f_1)$ and $p_s^*\mathscr{A}^\bullet(f)$, and hence also the mapping cones $\mathscr{B}^\bullet(f_1)$ and $p_s^*\mathscr{B}^\bullet(f)$ are quasi-isomorphic.

The same considerations, applied to the schemes X_s and $X_s \setminus \{a\}$ and the base field extension $k \to k_1$, show that

$$H^\bullet_{\{a\}}(X_s, \mathscr{A}^\bullet(f)) \xrightarrow{\sim} H^\bullet_{\{a_1\}}(X_{1,s}, p_s\mathscr{A}^\bullet(f)).$$

With the isomorphisms of (26.8) and the element $\theta = \partial\theta_1$ of (26.6) we have a commutative diagram

$$\begin{array}{ccc} H^n_{\{a\}}(X_s, \mathscr{A}^\bullet(f))(m) & \xrightarrow[\delta \mapsto \theta]{\sim} & H^n_c(X'_\eta, \Lambda)(m) \\ \downarrow \wr & & \downarrow \\ H^n_{\{a_1\}}(X_{1,s}, \mathscr{A}^\bullet(f_1))(m) & \xrightarrow[\delta \mapsto \theta]{\sim} & H^n_c(X'_{1,\eta}, \Lambda)(m). \end{array}$$

We obtain:

Proposition 28.2. The behavior of the variation under the base ring extension

$$R \to R_1$$

is described by a natural commutative diagram

$$\begin{array}{ccc} H^n(\mathscr{B}^\bullet(f)_{\bar{a}}) & \xrightarrow{\operatorname{var} \psi(\sigma)} & H^n_{\{a\}}(X_s, \mathscr{A}^\bullet(f)) \\ \downarrow \wr & & \downarrow \wr \\ H^n(\mathscr{B}^\bullet(f_1)_{\bar{a}_1}) & \xrightarrow{\operatorname{var} \sigma} & H^n_{\{a_1\}}(X_{1,s}, \mathscr{A}^\bullet(f_1)). \end{array}$$

The vanishing cycle δ of X is mapped to that of X_1.

Corollary 28.3. For our quadrics

$$X : x_0x_{m+1} + \cdots + x_mx_{2m+1} + \lambda x_{n+1}^2 = 0,$$
$$X_1 : x_0x_{m+1} + \cdots + x_mx_{2m+1} + \varphi(\lambda)x_{n+1}^2 = 0,$$

a constant c in Λ will satisfy

$$\operatorname{var}\sigma(x) = c \cdot \chi_\lambda(\sigma)\langle x, \delta\rangle\delta, \quad \forall\sigma \in G,$$

if and only if it satisfies

$$\operatorname{var}\sigma(x) = c \cdot \chi_{\varphi(\lambda)}(\sigma)\langle x, \delta\rangle\delta, \quad \forall\sigma \in G_1.$$

In the previous section we determined the variation constant for the universal quadric

$$x_0x_{m+1} + \cdots + x_mx_{2m+1} + tx_{n+1}^2 = 0$$

over $\mathbb{C}\{t\}$. Together with the corollary yields a proof of the fundamental lemma for every base ring with residue field of characteristic zero.

Now we want the study the behavior of the variation under specialization. For this we need to modify the construction of the complex $\mathscr{A}^\bullet(f)$. We restrict ourselves to the coefficient rings $\Lambda = \mathbb{Z}/\ell^r\mathbb{Z}$, where ℓ is a prime different from the characteristic of the residue field k. We want to replace the field $\bar{K}$ by the subfield

$$K_0 = \bigcup_q K(\pi^{1/\ell^q}) \subseteq K.$$

Let $G_0 \subseteq G$ be the Galois group of $\bar{K}$ over K_0, and $\Gamma - G/G_0$ the Galois group of K_0 over K. Γ is canonically isomorphic to $\widehat{Z}_\ell(1) = \varprojlim_q \mu_{\ell^q}(R)$. Our character $\chi : G \to \Lambda(1)$ is the composite of the projection $G \to \Gamma$ and the natural mapping $\Gamma = \widehat{Z}_\ell(1) \to \Lambda(1)$. The group G_0 is a projective limit of finite groups with order prime to ℓ.

Let $\eta_0 : \operatorname{Spec} K_0 \to S$ be the mapping induced by $R \subseteq K_0$. We get a commutative diagram

$$\begin{tikzcd}
X_\eta \arrow[rr, "j"] \arrow[d, "q"] & & X \arrow[d, equal] & \\
X \times_S \operatorname{Spec} K_0 = X_{\eta_0} \arrow[rr, "j_0"] & & X & X_s. \arrow[l, "i"']
\end{tikzcd}$$

In the construction of $\mathscr{A}^\bullet(f)$, we replace X_η by X_{η_0}: Let $\Lambda \to \mathscr{I}^\bullet$ be an injective resolution of Λ. Then

$$\lambda \to \mathscr{I}^\bullet_{\eta_0} = j_0^* \mathscr{I}^\bullet$$

is a Γ-equivariant flabby resolution of Λ and

$$\Lambda \to j^* \mathscr{I}^\bullet$$

is a G-equivariant flabby resolution of Λ. In analogy with $\mathscr{A}^\bullet(f) = i^* j_* \mathscr{I}^\bullet_\eta$ we construct the complex of Γ-sheaves $\mathscr{A}_0^\bullet(f) = i^*(j_0)_* \mathscr{I}^\bullet_{\eta_0}$, and we get a commutative diagram of equivariant mappings of complexes

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathscr{A}_0^\bullet(f) \\ \| & & \downarrow \\ \Lambda & \longrightarrow & \mathscr{A}^\bullet(f). \end{array}$$

One shows

$$(q_* q^* \mathscr{I}^\bullet_{\eta_0})^{G_0} = \mathscr{I}^\bullet_{\eta_0},$$

whence

$$\mathscr{A}_0^\bullet(f) = \mathscr{A}^\bullet(f)^{G_0}.$$

As G_0 is a profinite group with order relatively prime to ℓ, the cohomology sheaves of the complex $\mathscr{A}^\bullet(f)$, $\mathscr{A}_0^\bullet(f)$ satisfy

$$\mathscr{H}^p(\mathscr{A}_0^\bullet(f)) \xrightarrow{\sim} \mathscr{H}^p(\mathscr{A}^\bullet(f)).$$

29 Global monodromy theory

Consider a smooth irreducible projective variety $X \subseteq \mathbb{P}^r$ over an algebraically closed field k of characteristic p and a Lefschetz embedding $X \subseteq \mathbb{P}^N$. Let

$$\dim X = n+1, \quad n = 2m+1.$$

Let $F \subset \check{\mathbb{P}}^r$ be the dual variety of X, all the hyperplanes $H \subseteq \mathbb{P}^r$ that touch X at any point. We now consider a Lefschetz pencil of X

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & \mathbb{P}^1 = D \\ \downarrow & & \\ X & & \end{array}$$

Then f is singular only over the points of $S = F \cap D$ and over every point $s \in S$ there lies exactly one singular point in the fiber $f^{-1}(s)$. It is an ordinary double point, actually nondegenerate since $2 \mid n$. We identify the points of S with geometric points $\operatorname{Spec} k \to D$.

Let Ω be the separable closure of the function field of D and let

$$\omega : \operatorname{Spec} \Omega \to B = D \setminus F$$

be the corresponding geometric point. For a point $s \in S$, let $R(s)$ be the strict Henselization $\widetilde{\mathcal{O}}_{D,s}$ of the local ring $\mathcal{O}_{D,s}$ with respect to s, and $D(s) = \operatorname{Spec} \widetilde{\mathcal{O}}_{D,s}$. There is then always a geometric point $\widetilde{\omega}$ in $D(s)$ over ω induced by a homomorphism $\varphi : R(s) \to \Omega$ that identifies Ω with the separable closure of the quotient field K of $R(s)$. Let $D(s)^{\times} = D(s) \setminus \{s\}$. We consider the natural mapping

$$\pi_1(D(s)^{\times}, \widetilde{\omega}) = \operatorname{Gal}(\Omega/K) \to \widehat{\mathbb{Z}}^{(p)}(1) = \varprojlim_{p \nmid r} \mu_r(k).$$

Let $\mathfrak{R}_s$ be the kernel. We have the following characterization of the tame fundamental group $\pi_1^t(B, \omega)$:

$$\begin{tikzcd}
\pi_1(D(s)^{\times}, \widetilde{\omega}) \arrow[r] \arrow[d] & \pi_1(B, \omega) \arrow[d] \\
\widehat{\mathbb{Z}}^{(p)}(1) \arrow[r, "\gamma_{s,\widetilde{\omega}}"] & \pi_1^t(B, \omega)
\end{tikzcd}$$

where π_1^t is the quotient group by the smallest closed normal subgroup of $\pi_1(B, \omega)$ that contains the images of all the kernels $\mathfrak{R}_s$ (for all $s \in S$ and $\widetilde{\omega}$). The induced homomorphism $\gamma_{s,\widetilde{\omega}}$ depends only on the point $s \in S$ up to conjugation. We also write

$$\gamma_s : \widehat{\mathbb{Z}}^{(p)}(1) \to \pi_1^t(B, \omega).$$

A representation of $\pi_1(B, \omega)$ on a finite dimensional vector space V over $\mathbb{Q}_\ell$, resp. the $\pi_1(B, \omega)$-module V, is called tamely ramified if the representation factors through $\pi_1^t(B, \omega)$. That is equivalent to saying that the kernel acts trivially on V for all $s \in S$ (and $\widetilde{\omega}$). A locally constant constructible sheaf $\mathscr{F}$ (of $\mathbb{Q}_\ell$-vector spaces) on B is called tamely ramified when the stalk $\mathscr{F}_\omega$ is a tame $\pi_1(B, \omega)$-module.

Every point $s \in S$ gives rise to a local Lefschetz pencil of the type studied in the previous sections:

$$\begin{tikzcd}
 & \widetilde{X}_{D(s)} = \widetilde{X} \times_D D(s) \arrow[d, "f_{D(s)}"] & \\
\operatorname{Spec} \Omega \arrow[r, "\widetilde{\omega}"] & D(s) & \operatorname{Spec} k \arrow[l, "s"']
\end{tikzcd}$$

The generic resp. special geometric fibers of $f_{D(s)}$ are canonically isomorphic to the generic fiber of f over ω, $\widetilde{X}_\omega = \widetilde{X} \times_D \operatorname{Spec}\Omega$, resp. the fiber $\widetilde{X}_s = f^{-1}(s)$ of f over s.

We have the diagram:

$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & & \uparrow & & \\ \widetilde{X}_\omega & \longrightarrow & \widetilde{X}_{D(s)} & \longleftarrow & \widetilde{X}_s \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\Omega & \xrightarrow{\widetilde{\omega}} & D(s) & \xleftarrow{s} & \operatorname{Spec} k \end{array}$$

By proper base change theorem,

$$(R^i f_* \mathbb{Q}_\ell)_\omega \cong H^i(\widetilde{X}_\omega, \mathbb{Q}_\ell)$$

and likewise

$$(R^i f_* \mathbb{Q}_\ell)_s \cong H^i(\widetilde{X_s}, \mathbb{Q}_\ell).$$

Once again we formulate the Picard-Lefschetz formulas:

Theorem 29.1.

(i) The sheaves $R^i f_* \mathbb{Q}_\ell$ are locally constant for $i \neq n, n+1$, hence constant on D, since the fundamental group of D vanishes.

(ii) For each $s \in S$ there is a "vanishing cycle" δ_s in $(R^n f_* \mathbb{Q}_\ell)_\omega(m)$ that depends up to conjugation (and up to sign) only on s and not on the choice of $\widetilde{\omega}$, a cohomology class δ_s^* in $(R^{n+1} f_* \mathbb{Q}_\ell)_s(n-m)$, and an exact sequence between the specialization mappings with respect to $\widetilde{\omega}$,

$$0 \to (R^n f_* \mathbb{Q}_\ell)_s \to (R^n f_* \mathbb{Q}_\ell)_\omega \to \mathbb{Q}_\ell(m-n) \to (R^{n+1} f_* \mathbb{Q}_\ell)_s \to (R^{n+1} f_* \mathbb{Q}_\ell)_\omega \to 0$$

(iii) The sheaves $R^n f_* \mathbb{Q}_\ell$ and $R^{n+1} f_* \mathbb{Q}_\ell$ are locally constant on B. $\pi_1(B, \omega)$ acts trivially on $(R^{n+1} f_* \mathbb{Q}_\ell)_\omega$ and tamely on $(R^n f_* \mathbb{Q}_\ell)_\omega$. For $x \in (R^n f_* \mathbb{Q}_\ell)_\omega$ and $u \in \widehat{\mathbb{Z}}^{(p)}(1)$,

$$\gamma_s(u)x = x + (-1)^{m+1} u \langle x, \delta_s \rangle \delta_s.$$

The latter u is the natural image of u in $\mathbb{Z}_\ell(1) \subseteq \mathbb{Q}_\ell(1)$. Obviously δ_s is uniquely determined up to sign by the choice of γ_s.

Proof. Since f is smooth and proper over B, all the sheaves $R^p f_* \mathbb{Q}_\ell$ are locally constant on B. For $p \neq n, n+1$, the specialization mappings of these sheaves with respect to $s \in S$ and associated ω are isomorphisms. Hence they are locally constant on D. The groups $\pi_1(D(s)^\times, \widetilde{\omega})$ act trivially on $H^{n+1}(\widetilde{X}_\omega, \mathbb{Q}_\ell)$. As they and their conjugates generate the fundamental group $\pi_1(B, \omega)$, this group also acts trivially. By (26.3), the action of $\pi_1(D(s)^\times, \widetilde{\omega})$ on $H^n(\widetilde{X}_\omega, \mathbb{Q}_\ell)$ factors through

$$\chi : \pi_1(D(s)^\times, \widetilde{\omega}) \to \mathbb{Z}_\ell(1).$$

Thus $\pi_1(B, \omega)$ acts tamely. Since $\widetilde{X}$ is regular at all points, $\chi_\lambda = \chi$ in formula (3) of (26.3). ■

Let V be the $\pi_1(B, \omega)$-module $(R^n f_* \mathbb{Q}_\ell)_\omega = H^n(\widetilde{X}_\omega, \mathbb{Q}_\ell)$.

Definition 29.2. The space

$$E = \sum_{\substack{s \in S \\ \sigma \in \pi_1^t(B, \omega)}} \mathbb{Q}_\ell(-m) \sigma(\delta_s) \subseteq V$$

is called the space of vanishing cycles in V.

Proposition 29.3. All the vanishing cycles δ_s are conjugate up to sign. In particular, E vanishes iff one vanishing cycle δ_s is zero.

Proof. We embed the Lefschetz pencil $\widetilde{X} \to D$ in the family of all hyperplane sections of X:

$$\begin{array}{ccccc} & & \widetilde{X} & \longrightarrow & Z \subseteq X \times \check{\mathbb{P}}^N \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} \Omega & \xrightarrow{\omega} & D & \hookrightarrow & \check{\mathbb{P}}^N \end{array}$$

Here $Z = \{(x, H) \in X \times \check{\mathbb{P}}^N \mid x \in H\}$. The mapping g is smooth outside the dual variety $F \subset \check{\mathbb{P}}^N$ of hyperplanes $H \in \check{\mathbb{P}}^N$ that touch X in at least one point. We can assume that the codimension of F is equal to 1, as otherwise by our assumptions S is empty. F meets D only at smooth points, and there transversally. By the definition of a Lefschetz embedding, there is a closed subset F' of codimension ≥ 2 in $\check{\mathbb{P}}^N$ with the following property: For $x \in F \setminus F'$, g has exactly one singular point in $g^{-1}(x)$. It is an ordinary double point and hence actually nondegenerate. By considering the corresponding local Lefschetz pencil over the strict Henselization $\widetilde{\mathcal{O}}_{\check{\mathbb{P}}^N, a}$ of the local ring $\mathcal{O}_{\check{\mathbb{P}}^N, a}$ at the generic point a of F, we get: The stalk $(R^p g_* \mathbb{Q}_\ell)_\omega$ is a tame $\pi_1(\check{\mathbb{P}}^N \setminus F, \omega)$-module. We consider the surjective mapping

$$q : \pi_1^t(B, \omega) \to \pi_1^t(\check{\mathbb{P}}^N \setminus F, \omega).$$

It follows from the proper base change theorem that

$$V = (R^n f_* \mathbb{Q}_\ell)_\omega = (R^n g_* \mathbb{Q}_\ell)_\omega.$$

Therefore the representation of $\pi_1^t(B, \omega)$ factors through q. On the other hand, all the homomorphisms

$$q \circ \gamma_s : \widehat{\mathbb{Z}}^{(p)}(1) \to \pi_1^t(\check{\mathbb{P}}^N \setminus F, \omega)$$

are conjugate in $\pi_1^t(\check{\mathbb{P}}^N \setminus F, \omega)$. Thus for s, $s' \in S$, there is an element $\sigma \in \pi_1^t(B, \omega)$ with

$$\gamma_{s'}(u)x = \sigma \gamma_s(u) \sigma^{-1} x \quad \forall u \in \widehat{\mathbb{Z}}^{(p)}(1),\ x \in V.$$

It follows that

$$\gamma_{s'}(u)x = x + (-1)^{m+1} u \langle x, \sigma(\delta_s) \rangle \sigma(\delta_s)$$

for all $u \in \widehat{\mathbb{Z}}^{(p)}(1)$, $x \in V$. Thus:

$$\sigma \delta_s = \pm \delta_{s'}. \qquad \blacksquare$$

The representation of $\pi_1^t(B, \omega)$ on V is compatible with the Poincaré pairing

$$\langle\ ,\ \rangle : V \times V \to \mathbb{Q}_\ell(-n),$$

i.e., $\langle \sigma(u), \sigma(v) \rangle = \langle u, v \rangle$. The form $\langle\ ,\ \rangle$ is alternating, since n is odd. We thus actually have a representation in the symplectic group $\mathrm{Sp}(V)$ with respect to $\langle\ ,\ \rangle$:

$$\rho : \pi_1^t(B, \omega) \to \mathrm{Sp}(V).$$

The form $\langle\ ,\ \rangle$ induces a nondegenerate alternating bilinear form again on $E/(E \cap E^\perp)$.

Corollary 29.4. The induced representation of $\pi_1^t(B, \omega)$ on $E/(E \cap E^\perp)$ is absolutely irreducible.

Proof. Let L be an extension field of $\mathbb{Q}_\ell$ and W a $\pi_1^t(B, \omega)$-stable vector subspace of $E \otimes L$ that is not contained in $(E \cap E^\perp) \otimes L$. Then there is an element x in W and a point s in S with $\langle x, \delta_s \rangle \neq 0$,

$$\gamma_s(u)x - x = (-1)^{m+1} \widetilde{u} \langle x, \delta_s \rangle \delta_s \in W$$

for all $u \in \widehat{\mathbb{Z}}^{(p)}(1)$. It follows from this that δ_s lies in W, and so $E \otimes L \subseteq W$ by (29.3). $\blacksquare$

Proposition 29.5 (Kazhdan-Margulis). The image of the group $\pi_1^t(B,\omega)$ under the induced mapping

$$\pi_1^t(B,\omega) \to \mathrm{Sp}(E/(E\cap E^\perp))$$

is open in the symplectic group $\mathrm{Sp}(E/(E\cap E^\perp))$.

Proof. For simplicity we select an isomorphism $\mathbb{Z}_\ell \xrightarrow{\sim} \mathbb{Z}_\ell(1)$. Let $M = E/(E\cap E^\perp) \neq 0$, G the image of $\pi_1^t(B,\omega)$ in the symplectic group $\mathrm{Sp}(M)$, and $\mathfrak{g}$ the Lie algebra of G. We must show that $\mathfrak{g} = \mathfrak{sp}(M)$. For each $\theta \in M$ consider the linear mapping

$$\begin{array}{cccc} N(\theta): & M & \to & M \\ & x & \mapsto & \langle x,\theta\rangle\theta \end{array}.$$

We know that $\mathfrak{sp}(M)$ is generated by $N(\theta)$. Let W be the set of vectors $\theta \in M$ for which the $N(\theta)$ already lies in $\mathfrak{g}$, we show that $W = M$. Note that W is not empty, as it contains all the vanishing cycles (differentiate $\gamma_s(u)$ with u). Let U be a maximal nonzero vector space contained entirely in W, we want to prove $U = W$. We have

$$\mathbb{Q}_\ell W \subseteq W.$$

Consider the equation

$$\langle u,v\rangle N(u+v) = \langle u,v\rangle(N(u)+N(v)) + [N(v),N(u)] \in \mathfrak{g},$$

we get

$$u,v\in W \implies u\perp v \quad\text{or}\quad u+v\in W.$$

Let θ an element of W that is not orthogonal to all of U. Then $(U\setminus\theta^\perp)+\mathbb{Q}_\ell\theta$ is contained in W. Since W is Zariski-closed, the vector space $U+\mathbb{Q}_\ell\theta \subseteq W$, and hence θ is an element of U. Thus we have

$$W = U \cup (U^\perp \cap W).$$

But it follows from this that U is stable under the transformation

$$x \mapsto x + \lambda\langle x,\theta\rangle\theta$$

for all $\theta \in W$ and $\lambda \in \mathbb{Q}_\ell$. But such transformations generate the group G topologically, i.e., the subgroup generated by these transformations is dense in G, since the image of the homomorphisms γ_s together with their conjugates generate $\pi_1^t(B,\omega)$ topologically. As M is an irreducible G-module, we thus have:

$$M = U = W. \qquad \blacksquare$$

Theorem 29.6. Let $j : B \to D$ be the embedding. The direct image sheaves $R^p f_* \mathbb{Q}_\ell$ for $p \neq n, n+1$ are locally constant, and hence constant on D. For the two remaining direct images, we have:

Case 1. If $E = 0$. Then $R^n f_* \mathbb{Q}_\ell$ is constant. The exact sequence in (29.1) gives a natural exact sequence

$$0 \to \bigoplus_{s \in S} \mathbb{Q}_\ell(m-n)_{\{s\}} \to R^{n+1} f_* \mathbb{Q}_\ell \to j_*((R^{n+1} f_* \mathbb{Q}_\ell)_\omega)_B \to 0.$$

Here $(\cdot)_B$ is the functor from the category of continuous representations of the fundamental group $\pi_1(D, \omega)$ on finite-dimensional $\mathbb{Q}_\ell$ vector spaces to the category of locally constant sheaves of $\mathbb{Q}_\ell$-vector space. The sheaf $j_*((R^{n+1} f_* \mathbb{Q}_\ell)_\omega)_B$ is constant.

Case 2. If $E \neq 0$. Then $R^{n+1} f_* \mathbb{Q}_\ell$ is constant. We have

$$R^n f_* \mathbb{Q}_\ell = j_* j^* R^n f_* \mathbb{Q}_\ell = j_* V_B.$$

(a) Suppose $E \subseteq E^\perp$. Then we have an exact sequence

$$0 \to j_* E_B^\perp \to R^n f_* \mathbb{Q}_\ell \to j_*(V/E^\perp)_B \to \bigoplus_{s \in S} \mathbb{Q}_\ell(m-n)_{\{s\}} \to 0.$$

The sheaves $j_* E_B^\perp$ and $j_*(V/E^\perp)_B$ are constant.

(b) Suppose $E \not\subseteq E^\perp$. Then we have two exact sequences

$$0 \to j_* E_B \to R^n f_* \mathbb{Q}_\ell \to j_*(V/E)_B \to 0,$$

$$0 \to j_*(E \cap E^\perp)_B \to j_* E_B \to j_*(E/(E \cap E^\perp))_B \to 0.$$

The sheaves $j_*(V/E)_B$, $j_*(E \cap E^\perp)_B$ and $j_*(E/(E \cap E^\perp))_B$ are constant.

Proof. The images of the homomorphisms γ_s and their conjugate homomorphisms generate $\pi_1^t(B, \omega)$ topologically. Therefore it follows from the Picard-Lefschetz formulas that $\pi_1^t(B, \omega)$ acts trivially on $E^\perp$ and V/E. The associated sheaves $E_B^\perp$ and $(V/E)_B$ are therefore constant on B, and their direct images $j_* E_B^\perp$ and $j_*(V/E)_B$ are constant on D.

For the second case, by (29.3), $\delta_s \neq 0$ for any $s \in S$, so we get the exact sequence

$$0 \to (R^n f_* \mathbb{Q}_\ell)_s \to (R^n f_* \mathbb{Q}_\ell)_\omega \to \mathbb{Q}_\ell(m-n) \to 0$$

and $R^{n+1} f_* \mathbb{Q}_\ell$ is constant. The Picard-Lefschetz formula

$$\sigma(x) = x + (-1)^{m+1} \chi(\sigma) \langle x, \delta_s \rangle \delta_s$$

shows that

$$(j_*j^*R^nf_*\mathbb{Q}_\ell)_s = V^{\pi_1(D(s)^\times,\widetilde{\omega})} = \delta_s^\perp = (R^nf_*\mathbb{Q}_\ell)_s.$$

(a) Suppose $E \subseteq E^\perp$, we need to show that

$$(R^nf_*\mathbb{Q}_\ell)_s \to V/E^\perp \to \mathbb{Q}_\ell(m-n) \to 0$$

is exact, which follows from the fact that $V \to \mathbb{Q}_\ell(m-n)$ factors through $V \to V/E^\perp$.
(b) Suppose $E \not\subseteq E^\perp$. By (29.3), then,

$$\mathbb{Q}_\ell(-m)\delta_s \not\subseteq E^\perp \implies E \not\subseteq \delta_s^\perp = (R^nf_*\mathbb{Q}_\ell)_s.$$

As $\operatorname{codim}(R^nf_*\mathbb{Q}_\ell)_s = 1$ in $V = (R^nf_*\mathbb{Q}_\ell)_\omega$, the mapping

$$(R^nf_*\mathbb{Q}_\ell)_s \to V/E = (j_*(V/E)_B)_s$$

is surjective. The mapping

$$j_*E_B \to j_*(E/(E \cap E^\perp))_B$$

is surjective, since for every point $s \in S$ we have

$$(j_*E_B)_s = E^{\pi_1(D(s)^\times,\widetilde{\omega})} = E \cap \delta_s^\perp,$$

$$(j_*(E/(E\cap E^\perp))_B)_s = (E \cap \delta_s^\perp)/(E \cap E^\perp). \qquad \blacksquare$$

Let k be the separable closure of a subfield k_0. We want to suppose that the Lefschetz pencil $\widetilde{X} \to D$ is already defined over k_0:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[r] \arrow[d, "f"] & \widetilde{X}_0 \arrow[d, "f_0"] \\
\operatorname{Spec}\Omega \arrow[r, "\omega"] & D \arrow[r] \arrow[d] & D_0 = \mathbb{P}^1_{k_0} \arrow[d] \\
 & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k_0
\end{tikzcd}$$

We still denote by ω the geometric points

$$\operatorname{Spec}\Omega \to D_0 \quad \text{and} \quad \operatorname{Spec}\Omega \to \operatorname{Spec} k_0$$

induced by ω. Let A be the inverse image of a set A_0 of points

$$s : \operatorname{Spec} k_0 \to D_0$$

with coordinates in k_0. We identify S_0 with S. The singular point of f_0 are then rational over over k_0. We denote by $D_0(s)$ the spectrum of the (not strict) Henselization of the local ring $\mathcal{O}_{D_0,s}$ at $s \in S_0$.

$$\begin{tikzcd}
\operatorname{Spec}\Omega \arrow[r] \arrow[dr] & D(s) \arrow[r] \arrow[d] & D \arrow[d] \\
 & D_0(s) \arrow[r] & D_0
\end{tikzcd}$$

We likewise use s to denote the special points $\operatorname{Spec} k \to D(s)$ and $\operatorname{Spec} k_0 \to D_0(s)$. We consider the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(D(s)^\times, \widetilde{\omega}) \arrow[r] \arrow[d] & \pi_1(D_0(s)^\times, \widetilde{\omega}) \arrow[r] \arrow[d] & \pi_1(\operatorname{Spec} k_0, \omega) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1(B, \omega) \arrow[r] & \pi_1(B_0, \omega) \arrow[r] & \pi_1(\operatorname{Spec} k_0, \omega) \arrow[r] & 0.
\end{tikzcd}$$

The image G_s of $\pi_1(D_0(s)^\times, \widetilde{\omega})$ in $\pi_1(B_0, \widetilde{\omega})$ thus satisfies

$$\pi_1(B, \omega) \cdot G_s = G_s \cdot \pi_1(B, \omega) = \pi_1(B_0, \omega).$$

For the homomorphism

$$\gamma_s : \widehat{\mathbb{Z}}^{(p)}(1) = \widehat{\mathbb{Z}}^{(p)}(1)(k) = \varprojlim_{p \nmid r} \mu_r(k) \to \pi_1^t(B, \omega),$$

an element σ in G_s satisfies

$$(\spadesuit) \qquad \sigma\gamma_s(u)\sigma^{-1} = \gamma_s(\bar{\sigma}(u)).$$

$V = (R^n f_* \mathbb{Q}_\ell)_\omega$ is canonically isomorphic to $(R^n (f_0)_* \mathbb{Q}_\ell)_\omega$ and is thus actually a $\pi_1(B_0, \omega)$-module. For the vanishing cycle δ_s, equation ($\spadesuit$) implies

$$\gamma_s(u)x = x + (-1)^{m+1}\widetilde{u}\langle x, \sigma^{-1}\delta_s\rangle \sigma^{-1}\delta_s$$

and thus

$$\pi_1(B_0, \omega)\delta_s = \pi_1(B, \omega) \cdot G_s\delta_s \subseteq E.$$

E, $E^\perp$, and all other vector spaces occurring in (29.6) are actually $\pi_1(B_0, \omega)$-modules, and thus the associated locally constant sheaves are already defined in a natural way on B_0.

After a finite extension of k_0 we can make sure that the mappings

$$\mathbb{Q}_\ell(m - n) \to (R^{n+1} f_* \mathbb{Q}_\ell)_s$$

for each $s \in S$ are $\pi_1(\operatorname{Spec} k_0, \omega)$-equivariant.

Proposition 29.7. Let k be the separable closure of a subfield k_0. After a finite extension of k_0, the following can be guaranteed:

(1) The Lefschetz pencil $\widetilde{X} \to D$ arises by base field extension from a pencil

$$\widetilde{X}_0 \to D_0 = \mathbb{P}^1_{k_0}.$$

(2) The singular points of f_0 are rational over k_0.

(3) All the sheaves and exact sequences occurring in (29.6) are already defined in a natural way on D_0, resp. B_0.

30 Formulation of the Weil Conjecture

Let κ be a finite field of q elements, k the algebraic closure of κ, and ℓ a prime number that does not divide q. We consider algebraic varieties X_0 over κ. These will always be written with the subscript "0", and their extension to k without the subscript "0":

$$X = X_0 \times_{\operatorname{Spec}\kappa} \operatorname{Spec} k.$$

We use the corresponding notation for mappings and sheaves. Let $\mathscr{G}_0$ be a constructible ℓ-adic sheaf on the algebraic variety X_0 over κ and $\mathscr{G}$ its inverse image on X.

For a geometric point

$$\alpha : \operatorname{Spec} k \to X_0$$

over κ with $|\alpha|$ a closed point, we define $d(\alpha) = [\kappa(\alpha) : \alpha]$. The map

$$\begin{array}{rccc} f_\alpha : & k & \to & k \\ & x & \mapsto & x^{\#\kappa(\alpha)} \end{array}$$

is the Frobenius morphism in the Galois group $\mathrm{Gal}(k/\kappa(\alpha))$. This group acts on the stalk $(\mathscr{G}_0)_\alpha$ with the homomorphism

$$f_{\alpha,(\mathscr{G}_0)_\alpha} : (\mathscr{G}_0)_\alpha \to (\mathscr{G}_0)_\alpha$$

corresponding to the element $f_\alpha \in G_\alpha$. We then define the Weil-Grothendieck L-series for the sheaf $\mathscr{G}_0$:

$$Z_{\mathscr{G}_0}(t) = \prod_{\alpha \in X_0} \frac{1}{\det(1 - t^{d(\alpha)} f^{-1}_{\alpha,\mathscr{G}_0})}.$$

The formal power series $Z_{\mathscr{G}_0}(t)$ is multiplicative under short exact sequences of ℓ-adic sheaves and is identically 1 for torsion sheaves. It therefore depends only on the $\mathbb{Q}_\ell$-sheaf associated with $\mathscr{G}_0$. If nothing is explicitly said to the contrary, all sheaves from now on will be constructible sheaves of $\mathbb{Q}_\ell$-vector spaces.

Proposition 30.1. Let $\mathscr{G}_0$ be a constructible $\mathbb{Q}_\ell$-sheaf on X_0. Then

$$Z_{\mathscr{G}_0}(t) = \prod_{i=0}^{\infty} \det(1 - t\,\mathrm{Fr}^* \,|_{H^i(X,\mathscr{G})})^{(-1)^{i-1}}.$$

In particular, $Z_{\mathscr{G}_0}(t)$ is a rational function.

Proof. For an endomorphism φ of a finite dimensional $\mathbb{Q}_\ell$-vector space,

$$\frac{d}{dt}\log\det(1-t\varphi)^{-1} = \sum_{i=1}^{\infty} \mathrm{tr}(\varphi^i)t^{i-1}.$$

So

$$\begin{aligned}
\frac{Z'_{\mathscr{G}_0}(t)}{Z_{\mathscr{G}_0}(t)} &= \sum_{\alpha\in X_0}\sum_{i=1}^{\infty} d(\alpha)\,\mathrm{tr}(f_{\alpha,\mathscr{G}_0}^{-i})t^{d(\alpha)i-1} \\
&= \sum_{a\in X(k)}\sum_{i=1}^{\infty} \mathrm{tr}(f_{a,\mathscr{G}}^{-i})t^{d(a)i-1} \\
&= \sum_{a\in X(k)}\sum_{i=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_{\mathscr{G},a}^{d(a)i})t^{d(a)i-1} \\
&= \sum_{j=1}^{\infty}\sum_{\substack{a\in X(k)\\ d(a)|j}} \mathrm{tr}(\mathrm{Fr}_{\mathscr{G},a}^{j})t^{j-1} \\
&= \sum_{j=1}^{\infty}\sum_{a\in \mathrm{Fix}(\mathrm{Fr}_X^j)} \mathrm{tr}(\mathrm{Fr}_{\mathscr{G},a}^{j})t^{j-1}.
\end{aligned}$$

Also, it follows from (22.11) that

$$\sum_{a\in \mathrm{Fix}(\mathrm{Fr}_X^j)} \mathrm{tr}(\mathrm{Fr}_{\mathscr{G},a}^{j};\mathscr{G}_a) = \sum_i (-1)^i\,\mathrm{tr}(\mathrm{Fr}_{\mathscr{G}}^j; H_c^i(X,\mathscr{G})),$$

and hence

$$\begin{aligned}
\frac{d}{dt}\log Z_{\mathscr{G}_0}(t) &= \sum_{j=1}\sum_i (-1)^i\,\mathrm{tr}(\mathrm{Fr}_{\mathscr{G}}^j; H_c^i(X,\mathscr{G}))t^{j-1} \\
&= \sum_i (-1)^i \sum_{j=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_{\mathscr{G}}^j; H_c^i(X,\mathscr{G}))t^{j-1} \\
&= \frac{d}{dt}\log\left(\prod_i \det(1-t\,\mathrm{Fr}_{\mathscr{G}}^j\,|_{H^i(X,\mathscr{G})})^{(-1)^{i-1}}\right).
\end{aligned}$$

The assertion is thus follows from $Z_{\mathscr{G}_0}(0) = 1$. ∎

We want to look in more detail at the special case of locally constant $\mathbb{Q}_\ell$-sheaves.

Let ω be a geometric point of X over k with values in a separably closed extension field Ω of k. ω induces in an obvious way geometric points of X_0 and of $\operatorname{Spec}\kappa$ over κ, and for simplicity we shall denote them also by ω. Let X be irreducible. Then we have the exact sequence of fundamental groups

$$0 \to \pi_1(X, \omega) \to \pi_1(X_0, \omega) \xrightarrow{d} \pi_1(\operatorname{Spec}\kappa, \omega) = \operatorname{Gal}(k/\kappa) \to 0$$

and a canonical isomorphism

$$\begin{array}{rcl} \operatorname{Gal}(k/\kappa) & \cong & \widehat{\mathbb{Z}} = \varprojlim_{m \in \mathbb{N}} \mathbb{Z}/m\mathbb{Z} \\ [x \mapsto x^q] & \mapsto & 1. \end{array}$$

Suppose we are given a continuous representation of $\pi_1(X_0, \omega)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space V. This representation defines a locally constant sheaf $\mathscr{G}_0 = \mathscr{G}_0(V)$ on X_0. We write

$$Z_V(t) = Z_{\mathscr{G}_0}(t).$$

This L-series can be described as follows using the representation itself. Let

$$\alpha : \operatorname{Spec} k \to X_0$$

over κ be a geometric point with coordinate field $\kappa(\alpha) \subset k$. The embedding $\kappa(\alpha) \subset k$ defines a geometric point $j : \operatorname{Spec} k \to \operatorname{Spec}\kappa(\alpha)$. This induces a homomorphism

$$\operatorname{Gal}(k/\kappa(\alpha)) = \pi_1(\operatorname{Spec}\kappa(\alpha), j) \to \pi_1(X_0, \alpha) \to \pi_1(X_0, \omega).$$

The isomorphism $\pi_1(X_0, \alpha) \to \pi_1(X_0, \omega)$ is determined up to conjugation.

Let the image of the Frobenius isomorphism $[x \mapsto x^{q^{d(\alpha)}}]$ in $\pi_1(X_0, \omega)$ be φ_α; φ_α is determined up to conjugation and acts on V. We have

$$\det(1 - t^{d(\alpha)} f_{\alpha,\mathscr{G}_0}^{-1}) = \det(1 - t^{d(\alpha)} \varphi_\alpha^{-1})$$

and hence

$$Z_V(t) = \prod_{\alpha \in X_0} \frac{1}{\det(1 - t^{d(\alpha)} \varphi_\alpha^{-1})}.$$

The degree $d(\alpha)$ is precisely the image of φ_α in $\widehat{\mathbb{Z}}$.

Theorem 30.2 (Deligne)**.** Let X_0 be smooth, projective and of dimension n over the finite field $\kappa = \mathbb{F}_q$.

(1) The polynomials

$$P_i(t) = \det(1 - t\,\mathrm{Fr}^* |_{H^i(X,\mathbb{Q}_\ell)})$$

in $\mathbb{Q}_\ell[t]$ is in $\mathbb{Z}[t]$. These are independent of ℓ.

(2) The eigenvalues λ of $\mathrm{Fr}^* |_{H^i(X,\mathbb{Q}_\ell)}$, and thus the reciprocal roots of $P_i(t)$, all have complex absolute values

$$|\lambda| = q^{i/2}.$$

The reciprocal roots of $P_i(t)$ are algebraic integers. It follows from (1) and (2) that all conjugates of λ have absolute value $q^{i/2}$.

(3) There is a functional equation for

$$Z_{X_0}(t) = \prod_i P_i(t)^{(-1)^{i-1}},$$

namely

$$Z_{X_0}\left(\frac{1}{q^n t}\right) = \varepsilon \left(q^{\frac{n}{2}} t\right)^{\chi(X)} Z_{X_0}(t).$$

Here

$$\chi(X) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q}_\ell)$$

and

$$\varepsilon = \begin{cases} 1, & \text{if } 2 \nmid n \\ (-1)^\mu & \text{if } 2 \mid n \end{cases}$$

where μ is the multiplicity of the eigenvalue $q^{n/2}$ of $\mathrm{Fr}^* |_{H^n(X,\mathbb{Q}_\ell)}$.

The assertion (1) on the integrality of the coefficients of the polynomials $P_n(t)$, and hence (30.2), is a consequence of the following seemingly weaker theorem.

Theorem 30.3. Assume the hypotheses of (30.2). Then all the eigenvalues of the Frobenius homomorphisms

$$\mathrm{Fr}^* |_{H^i(X,\mathbb{Q}_\ell)}$$

are algebraic numbers. These eigenvalues and all their conjugates λ have complex absolute value

$$|\lambda| = q^{i/2}.$$

Proof of (30.2) assuming (30.3). Since

$$Z_{X_0}(t) = \prod_\alpha \frac{1}{1 - t^{d(\alpha)}} \in \mathbb{Z}[[t]] \subset \mathbb{Q}[[t]],$$

together with

$$Z_{X_0}(t) = \prod_i P_i(t)^{(-1)^{i-1}} \in \mathbb{Q}_\ell(t),$$

we get $Z_{X_0}(t) \in \mathbb{Q}[[t]] \cap \mathbb{Q}_\ell(t) = \mathbb{Q}(t)$.

Let

$$P(t) = \prod_{2 \nmid i} P_i(t), \quad Q(t) = \prod_{2 | i} P_i(t).$$

It follows from (30.3) that $P(t)$ and $Q(t)$ have no common roots and thus relatively prime.

$$P(t) = Q(t) \cdot Z_{X_0}(t).$$

Let $\lambda \in \bar{\mathbb{Q}}_r$ be a root of $Q(t)$. If $|\lambda|_r < 1$, as $Z_{X_0}(t)$ has integer coefficients, $Z_{X_0}(x)$ converges in $\mathbb{C}_r$ for all $x \in \bar{\mathbb{Q}}_r$ with $|x|_r < 1$. Hence

$$P(\lambda) = Q(\lambda) Z_{X_0}(\lambda) = 0.$$

But this is a contradiction. So $|\mu|_r \leq 1$ where $\mu = \lambda^{-1}$. Let $\mu_1, \ldots, \mu_k$ be the roots of the minimal polynomial

$$t^k + a_{k-1} t^{k-1} + \cdots + a_0, \quad a_i \in \mathbb{Q}_{\mathscr{R}}$$

of μ in $\bar{\mathbb{Q}}_r$, then

$$|a_i|_r = \left| \sum \mu_{j_1} \cdots \mu_{j_{k-i}} \right|_r \leq \max |\mu_{j_1} \cdots \mu_{j_{k-i}}|_r = |\mu|_r^{k-i} \leq 1$$

implies that $a_i \in \mathbb{Z}_r$, thus $\mu \in \bar{\mathbb{Z}}_r$. Since the preimage of $\prod_r \bar{\mathbb{Z}}_r$ is $\bar{\mathbb{Z}}$ in the diagonal map

$$\bar{\mathbb{Q}} \longrightarrow \prod_r \bar{\mathbb{Q}}_r,$$

μ is an algebraic integer. Because $Q(0) = 1$, $Q(t) \in \mathbb{Z}[t]$. The same is true for $P(t) = Q(t) Z_{X_0}(t) \in \mathbb{Z}[t]$. As the polynomials $P_i(t)$ are relatively prime to each other, it follows from $P_i(0) = 1$ and the familiar Gauss Lemma that all polynomials $P_i(t) \in \mathbb{Z}[t]$.

For (3), consider the pairing It suffices to prove (30.3) after base field extension. Thus, we can assume that X_0 is absolutely irreducible.

31 The fundamental estimate

We assume that $B_0 = X_0$ is a smooth, absolutely irreducible affine curve over the finite field κ. In addition, we again suppose that there is given a continuous representation

$$\rho : \pi_1(B_0, \omega) \to \mathrm{GL}(V)$$

on a finite-dimensional $\mathbb{Q}_\ell$-vector space V. Let $\mathscr{G}_0 = V_{B_0}$ be the locally constant sheaf associated with V, and $\mathscr{G}$ its inverse image on $B = B_0 \times_{\operatorname{Spec}\kappa} \operatorname{Spec} k$.

We want to study the eigenvalues of the Frobenius homomorphism on $H^i_c(B, \mathscr{G})$ under certain hypotheses on (V, ρ). Here only the cases $i = 1$ and 2 occur; indeed, these cohomology groups vanish for $i > 2$, as B is 1-dimensional, and for $i = 0$ as B is affine. In particular, the L-series for $\mathscr{G}_0$ has the form

$$Z_{\mathscr{G}_0}(t) = \frac{P_1(t)}{P_2(t)}, \quad P_i(t) = \det(1 - t\,\mathrm{Fr}^* \,|_{H^i_c(B,\mathscr{G})}).$$

Since $\mathscr{G}_0 = V_{B_0}$, there is a canonical isomorphism

$$H^0(B, \mathscr{G}) = V^\pi, \quad \pi = \pi_1(B, \omega).$$

The group

$$\mathrm{Gal}(k/\kappa) = \pi_1(B_0, \omega)/\pi_1(B, \omega)$$

acts on the space V^π. The Frobenius elements $\varphi \in \mathrm{Gal}(k/\kappa)$ corresponds to the "arithmetic Frobenius homomorphism"

$$\bar{\rho}(\varphi) : V^\pi \to V^\pi.$$

This is the inverse of the "geometric Frobenius homomorphism"

$$\mathrm{Fr}^* : H^0(B, \mathscr{G}) \to H^0(B, \mathscr{G}).$$

We've assigned to each geometric point

$$\alpha : \operatorname{Spec} k \to B$$

over κ an element

$$\varphi_\alpha \in \pi_1(B_0, \omega)$$

uniquely determined up to conjugacy, and thereby also a "local Frobenius homomorphism"

$$\rho(\varphi_\alpha) : V \to V \quad (V \cong \mathscr{G}_\alpha)$$

likewise determined up to conjugacy. It is immediately clear that the local Frobenius homomorphism $\rho(\varphi_\alpha)$ leaves V^π invariant and acts on V^π as the $d(\alpha)^{\text{th}}$ power of the arithmetic Frobenius homomorphism $\bar{\rho}(\varphi)^{d(\alpha)}$.

Proposition 31.1. Suppose the characteristic polynomial of local Frobenius homomorphisms

$$\rho(\varphi_\alpha) : V \to V$$

have rational coefficients. Then the eigenvalues of

$$\mathrm{Fr}^* : H^i_c(B, \mathscr{G}) \to H^i_c(B, \mathscr{G})$$

are algebraic numbers, i.e., in $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell$.

Proof. We know that

$$\frac{P_1(t)}{P_2(t)} = Z_{\mathscr{G}_0}(t) \in \mathbb{Q}[[t]] \cap \mathbb{Q}_\ell(t) = \mathbb{Q}(t),$$

so it suffices to do the case $i = 2$. By duality,

$$H^2_c(B, \mathscr{G}) = \mathrm{Hom}_{\mathbb{Q}_\ell}(H^0(B, \mathscr{G}), \mathbb{Q}_\ell)(-1) = V_\pi(-1).$$

Since the eigenvalues of $\bar{\rho}(\varphi)$ on V_π are algebraic numbers, and hence that the eigenvalues of

$$\mathrm{Fr}^* : H^2_c(B, \mathscr{G}) \to H^2_c(B, \mathscr{G})$$

are algebraic. ■

Hypotheses.

(1) The characteristic polynomials of the local Frobenius homomorphisms

$$\rho(\varphi_\alpha) : V \to V$$

have rational coefficients.

(2) There is a nondegenerate alternating bilinear form

$$\langle\ ,\ \rangle : V \times V \to \mathbb{Q}_\ell$$

and a number $a \in \mathbb{Z}$ such that

$$\langle \rho(\sigma)x, \rho(\sigma)y\rangle = q^{-ad(\sigma)}\langle x, y\rangle$$

for $\sigma \in \pi_1(B_0, \omega)$, where $d(\sigma)$ is the image of σ in $\widehat{\mathbb{Z}}$ under the mapping

$$\pi_1(B_0, \omega) \to \mathrm{Gal}(k/\kappa) \cong \widehat{\mathbb{Z}}.$$

Since q is an ℓ-adic unit,

$$q^{d(\sigma)} := \lim_{d \to d(\sigma)} q^d \in \mathbb{Q}_\ell$$

is well-defined. The image of $\pi = \ker(\pi_1(B_0, \omega) \to \widehat{\mathbb{Z}})$ in the symplectic group $\mathrm{Sp}(V)$ is Zariski-dense.

In our applications, we will have

$$\mathscr{G}_0 = R^n f_* \mathbb{Q}_\ell,$$

where $f_0 : X_0 \to B_0$ is the smooth part of a Lefschetz pencil of odd fiber dimension $n = a$. The pairing in (2) is the duality pairing.

In what follows, besides (V, ρ) we also need the higher tensor power representations $\rho^k = (V^{\otimes k}, \rho^{\otimes k})$. The locally constant sheaves associated with them are precisely

$$\mathscr{G}_0^{\otimes k} = \mathscr{G}_0 \otimes_{\mathbb{Q}_\ell} \cdots \otimes_{\mathbb{Q}_\ell} \mathscr{G}_0, \quad \mathscr{G}^{\otimes_k}.$$

We denote their L-series by

$$Z^k(t) = \frac{P_1^k(t)}{P_2^k(t)}, \quad P_i^k(t) = \det(1 - t\,\mathrm{Fr}^* |_{H_c^i(B, \mathscr{G}^{\otimes k})}).$$

The roots of $P_i^k(t)$ are algebraic numbers, as the hypothesis of (31.1) carry over tensor powers.

Lemma 31.2. Assume the **Hypotheses**. The eigenvalues λ of

$$\mathrm{Fr}^* : (V^{\otimes k})_\pi \to (V^{\otimes k})_\pi,$$

where

$$(V^{\otimes k})_\pi = V/\{gv - v \mid g \in \pi,\ v \in V\},$$

have complex absolute value

$$|\lambda| = q^{ka/2}.$$

Proof. We write $\mathrm{Sp}(V)$ for the symplectic group of V,

$$\mathrm{A}(V) = \left\{T : V \to V \mid \langle Tx, Ty\rangle = u\langle x, y\rangle \text{ for some } u \in \mathbb{Q}_\ell^\times\right\}$$

for the group of symplectic similarities, and $\mathrm{A}_0(V) \subseteq \mathrm{A}(V)$ for the subgroup generated by $\mathrm{Sp}(V)$ and the "homotheties" $[x \mapsto ux]$, $a \in \mathbb{Q}_\ell^\times$. Note that $\mathrm{A}(V)/\,\mathrm{A}_0(V)$ is a finite abelian group.

By hypothesis, the image of the group π is Zariski-dense in $\mathrm{Sp}(V)$. Hence the group $\mathrm{A}(V)/\,\mathrm{Sp}(V)$ acts on $(V^{\otimes k})_\pi$. For every eigenvalue λ_0 of a fixed element, say of $\rho^k(\varphi_\alpha)$, there exists a character

$$\lambda : \mathrm{A}(V)/\,\mathrm{Sp}(V) \to \bar{\mathbb{Q}}_\ell,$$

where $\lambda(\rho(\varphi_\alpha)) = \lambda_0$. $\lambda(g)$ is an eigenvalue of g on $(V^{\otimes k})_\pi \otimes \bar{\mathbb{Q}}_\ell$ for all $g \in \mathrm{A}(V)/\,\mathrm{Sp}(V)$.

We want to compare the character λ with the determinant character

$$\begin{array}{rccc} \chi: & \mathrm{A}(V)/\,\mathrm{Sp}(V) & \to & \mathbb{Q}_\ell^\times \\ & u & \mapsto & \det u. \end{array}$$

The characters $\lambda^{\dim V}$ and χ^k obviously agree on the subgroup $\mathrm{A}_0(V)/\,\mathrm{Sp}(V)$ generated by the homotheties. As this has finite index in $\mathrm{A}(V)/\,\mathrm{Sp}(V)$, it follows that the character $\lambda^{\dim V}\chi^{-k}$ has finite order. In particular, the $\dim V^{\text{th}}$ power of the eigenvalue λ_0 differs from $\det(\rho(\varphi_\alpha))^k$ only by a root of unity. By hypothesis $\rho(\varphi_\alpha)$ is a symplectic similarity with the distorsion factor $q^{-ad(\alpha)}$. Hence the determinant of $\rho(\varphi_\alpha)$ is $q^{-\dim V \cdot ad(\alpha)/2}$. We get

$$|\lambda_0| = q^{-ad(\alpha)k/2},$$

and the assertion follows from this with $\mathrm{Fr}^* = \rho(\varphi^{-1})$, $\varphi^{d(\alpha)} = \varphi_\alpha$. ■

Since $H_c^2(B, \mathscr{G}^{\otimes k}) = (V^{\otimes k})_\pi \otimes \mathbb{Q}_\ell(-1)$, we get:

Corollary 31.3. The poles of the L-series

$$Z^k(t) = \frac{P_1^k(t)}{P_2^k(t)}$$

all lie on the circle

$$|z| = q^{-(1+ka/2)}.$$

Recall that $\mathbb{Q}_\ell(1) = \left\{x \in k \;\middle|\; x^{\ell^s} = 1 \text{ for some } s\right\}$.

We now use (31.2) to obtain a statement about the eigenvalues of

$$\rho(\varphi_\alpha) : V \to V$$

The L-series Z^k for ρ^k is by definition the product of the local factors

$$\det(1 - t^{d(\alpha)}\rho^{\otimes k}\varphi_\alpha^{-1}|_{V^{\otimes k}})^{-1}.$$

We assume now that k is even. Then these local factors are power series in t with nonnegative rational coefficients in their expansions and leading coefficient 1. This follows from the formulas

$$\det(1 - t\tau)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{tr}(\tau^n)}{n} t^n\right)$$

and

$$\operatorname{tr}(\tau \otimes \tau) = \operatorname{tr}(\tau)^2.$$

Consequently the radius of convergence for each factor is at least as large as that for the product $Z^k(t)$, thus at least $q^{-(1+ka/2)}$. The poles of the factors thus lie outside this circle. Let λ be an eigenvalue of $\rho(\varphi_\alpha)$. Then λ^k is an eigenvalue of $\rho^{\otimes k}(\varphi_\alpha)$. For even k it follows that

$$|\lambda^k| \geq q^{-(1+ka/2)d(\alpha)},$$

thus

$$|\lambda| \geq q^{-ad(\alpha)/2}.$$

It is easy to see that the hypotheses carry over from (V, ρ) to the dual representation $(V^\vee, \rho^\vee)$, with $-a$ in place of a. Hence we get

$$|\lambda^{-1}| \geq q^{ad(\alpha)/2},$$

and thus actually

$$|\lambda| = q^{-ad(\alpha)/2}.$$

We record this:

Lemma 31.4. Under the **Hypothesis**, the eigenvalues of the local Frobenius homomorphisms $\rho(\varphi_\alpha)$ are algebraic numbers whose conjugates have the complex absolute value

$$|\lambda| = q^{-ad(\alpha)/2}.$$

We apply this lemma to estimate the radius of convergence of the infinite product

$$\prod_{\alpha} \det(1 - t^{d(\alpha)}\rho(\varphi_\alpha^{-1})).$$

If we denote the eigenvalues of $\rho(\varphi_\alpha^{-1})$ by

$$\lambda_1(\alpha), \ldots, \lambda_n(\alpha),$$

the product can be written in the form

$$\prod_{\alpha} \prod_{j} (1 - \lambda_j(\alpha) t^{d(\alpha)}).$$

It converges when the series

$$\sum_{j} \sum_{\alpha} |\lambda_j(\alpha)| t^{d(\alpha)} = n \sum_{\alpha} q^{ad(\alpha)/2} t^{d(\alpha)}$$

converges.

Proposition 31.5. The radius of convergence of this series is $q^{-(1+a/2)}$.

We only prove the case when B_0 is the affine line. For the general case, we view B as a cover of the affine line and prove that the number of points for a given degree can be bounded by $O(q^d)$.

Proof. There exist exactly $q^d - q^{d-1}$ points of a given degree $d = d(\alpha)$. The series above is essentially equal to

$$\sum_{d=0}^{\infty} (q^{1+a/2} t)^d,$$

and the assertion is evident. ■

Thus the L-series has no zeros and poles in the circle

$$|z| < q^{-(1+a/2)}.$$

The roots of $P_1(t)$ must therefore lie out side this circle. This gives us the fundamental estimate:

Proposition 31.6. Under the **Hypotheses**, all eigenvalues of

$$\mathrm{Fr}^* : H_c^1(B, \mathscr{G}) \to H_c^1(B, \mathscr{G})$$

are algebraic numbers; the complex conjugates λ satisfy the estimate

$$|\lambda| \leq q^{1+a/2}.$$

32 A rationality proposition

Let B_0 be an absolutely irreducible curve over the finite field $\kappa = \mathbb{F}_q$. Consider the short exact sequence

$$0 \to \pi_1(B,\omega) \to \pi_1(B_0,\omega) \xrightarrow{d} \pi_1(\operatorname{Spec}\kappa,\omega) = \operatorname{Gal}(k/\kappa) \cong \widehat{\mathbb{Z}} \to 0$$

Here ω again denotes a geometric point of B in a separably closed extension field Ω of k and also the induced points of B_0 and $\operatorname{Spec}\kappa$.

For any ℓ-adic unit $\alpha \in \bar{\mathbb{Q}}_\ell$ and any element m_0 in $\widehat{\mathbb{Z}}$, the limit

$$\alpha^{m_0} = \lim_{m \to m_0} \alpha^m$$

exists.

As before, we consider a finite-dimensional $\mathbb{Q}_\ell$-vector space V together with a nondegenerate alternating bilinear form $\langle\ ,\ \rangle$ with values in $\mathbb{Q}_\ell$ and a continuous representation

$$\rho : \pi_1(B_0,\omega) \to \mathrm{GL}(V)$$

satisfying

$$\langle \rho(\sigma)(x), \rho(\sigma)(y)\rangle = q^{-ad(\sigma)}\langle x, y\rangle.$$

Recall that the image of $\pi_1(B,\omega)$ lies in the symplectic with respect to $\langle\ ,\ \rangle$. If α is a geometric point of B_0 over κ with values in k, then again as before we denote by φ_α the associated "arithmetic" Frobenius element in $\pi_1(B_0,\omega)$, determined up to conjugacy.

Let $\mathrm{A}(V)$ be the symplectic similarity group. We have then the identity

$$\langle g(x), g(y)\rangle = \chi(g)\langle x, y\rangle$$

for an algebraic character $\chi : A(V) \to \bar{\mathbb{Q}}_\ell^\times$. We denote by $\mathrm{Sp}(V)(L)$, resp. $\mathrm{A}(V)(L)$ the groups of geometric points with coordinates in the extension $L \supset \mathbb{Q}_\ell$. These group are in a natural way locally compact groups, if L is finite over $\mathbb{Q}_\ell$.

As q^{-a} is an ℓ-adic unit, we can define the set

$$G = \{(m,g) \in \widehat{\mathbb{Z}} \times \mathrm{A}(V)(\mathbb{Q}_\ell) \mid \chi(g) = q^{-am}\}.$$

Then G is a locally compact group. By hypothesis there is a homomorphism

$$\begin{array}{rccc} d \times \rho : & \pi_1(B_0,\omega) & \to & G \\ & \sigma & \mapsto & (d(\sigma), \rho(\sigma)) \end{array}$$

The image of this mapping is open in G.

Lemma 32.1. Let u be an ℓ-adic unit in $\bar{\mathbb{Q}}_\ell$. Consider the closed, invariant under conjugation subset

$$F = \{\sigma \in \pi_1(B_0, \omega) \mid u^{d(\sigma)} \text{ is an eigenvalue of } \rho(\sigma)\}.$$

Then F is measure zero with respect to the Haar measure.

Proof. Because the image of $\pi_1(B_0, \omega)$ is open in G, it suffices to show that the subset of those elements (m, σ) in G for which u^m is an eigenvalue of σ has measure zero. Using Fubini's theorem, one gets this assertion from the following considerations. Let m be a fixed chosed element in $\widehat{\mathbb{Z}}$ and H the homogeneous space for the algebraic group $\mathrm{Sp}(V)$, defined over $\mathbb{Q}_\ell$, consisting of those symplectic similarities g for which

$$\chi(g) = q^{-am}.$$

This homogeneous space is a smooth absolutely algebraic subvariety of $\mathrm{A}(V)$. The set $H(\mathbb{Q}_\ell)$ of points with coordinates in $\mathbb{Q}_\ell$ is nonempty. Let M be the Zariski-closed subset of H, defined over $\mathbb{Q}_\ell$, of those transformations that have u^m or a $\mathbb{Q}_\ell$-conjugate of it as an eigenvalue.

One sees easily that M is a proper subspace of H:

$$M(\bar{\mathbb{Q}}_\ell) \neq H(\bar{\mathbb{Q}}_\ell).$$

It follows that $M(\mathbb{Q}_\ell)$ is a measure zero set with respect to an invariant measure on $H(\mathbb{Q}_\ell)$. This assertion is "analytically" of local nature and follows from the fact that a proper analytic subset of an open subset of $\mathbb{Q}_\ell^r$ is a "Lebesgue" measure zero set. On the other hand, we have:

$$M(\mathbb{Q}_\ell) \text{ is the set of elements in } H(\mathbb{Q}_\ell) \text{ for which } u^m \text{ is an eigenvalue.}$$ ■

Proposition 32.2 (Rationality criterion)**.** Suppose the image of $\pi_1(B, \omega)$ is open in the symplectic group $\mathrm{Sp}_{\langle\ ,\ \rangle}(V)(\mathbb{Q}_\ell)$. Let $\lambda_1, \dots, \lambda_r$, $\mu_1, \dots, \mu_s$ be nonzero elements in the algebraic closure $\bar{\mathbb{Q}}_\ell$ with

$$\lambda_i \neq \mu_j$$

for all i, j. Suppose the rational function

$$R_\alpha(t) = \frac{\prod(1 - \lambda_i^{-d(\alpha)} t)}{\prod(1 - \mu_j^{-d(\alpha)} t)} \det(1 - \varphi_\alpha^{-1} t) \in \mathbb{Q}(t)$$

for every geometric point α. Then the coefficients of the polynomial $\det(1-\varphi_\alpha^{-1}t)$ are rational numbers for all geometric points α.

Proof. Let $g \in \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$. Since $R_\alpha(t) \in \mathbb{Q}(t)$, we have

$$\det(1-\varphi_\alpha^{-1}t)\frac{\prod(1-\lambda_i^{-d(\alpha)t})}{\prod(1-\mu_j^{-d(\alpha)}t)}\prod(1-g(\mu_j)^{-d(\alpha)}t)$$
$$=\prod(1-g(\lambda_i)^{-d(\alpha)}t)g(\det(1-\varphi_\alpha^{-1}t)) \in \bar{\mathbb{Q}}_\ell[t].$$

By (32.1), the compact group $\pi_1(B_0,\omega)$ contains a closed subset F of measure zero, closed under conjugation, with the following property:

> When $\sigma \in \pi_1(B_0,\omega)$ is not contained in F, then none of the elements $\mu_j^{d(\sigma)}$ for which μ_j is a ℓ-adic unit, is eigenvalue of $\sigma : V \to V$.

We consider also the finite set $E \subset \mathbb{Z}_{>1}$ defined as follows:

> $m \in E$ if λ_i/μ_j is a root of unity of order m,

and

> $m \in E$ if $\mu_i \neq g(\mu_j)$ and $\mu_j/g(\mu_k)$ is a root of unity of order m.

The set $\bigcup\limits_{m\in E} m\cdot\widehat{\mathbb{Z}}$ is then a proper closed subset of $\widehat{\mathbb{Z}}$. Let

$$U = d^{-1}\big(\widehat{\mathbb{Z}} \setminus \bigcup_{m\in E} m\cdot\widehat{\mathbb{Z}}\big) \subseteq \pi_1(B_0,\omega).$$

Then $U' = U \setminus F \subseteq \pi_1(B_0,\omega)$ is an open subset of positive Haar measure, invariant under conjugation. By Čebotarev density theorem for function fields in one variable over finite constant fields, i.e., the lower density of

$$\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_K \mid \mathfrak{p} \text{ unramified in } M \text{ and } \exists\mathfrak{q} \in \operatorname{Spec}\mathcal{O}_M \text{ above } \mathfrak{p} \text{ with } \operatorname{Frob}_\mathfrak{q} \in U'\}$$

is at least $\mu(U') > 0$ (let $K = \kappa(t)$, M the union of the fields that is contained in K^{sep} and B_0 in L is unramified over B_0) implies that there is a geometric point α of B_0 over κ for which φ_α lies in U'. All eigenvalues of the transformations $\rho(\sigma)$, $\sigma \in \pi_1(B_0,\omega)$, are ℓ-adic units, because $\pi_1(B_0,\omega)$ is compact. Since φ_α does not lie in F, the polynomial $\prod(1-\mu_j^{-d(\alpha)}t)$ is relatively prime to $\det(1-\varphi_\alpha^{-1}t)$. Since $d(\alpha) \notin \bigcup\limits_{m\in E} m\cdot\widehat{\mathbb{Z}}$, no number

$m \in E$ divides $d(\alpha)$. Thus the polynomial $\prod(1 - \mu_j^{-d(\alpha)} t)$ is also relatively prime to $\prod(1 - \lambda_i^{-d(\alpha)} t)$. It follows that

$$\prod(1 - \mu_j^{-d(\alpha)} t) = \prod(1 - g(\mu_j)^{-d(\alpha)} t).$$

From this we actually get

$$\prod(1 - \mu_j t) = \prod(1 - g(\mu_j) t).$$

For otherwise there would be elements μ_i, $g(\mu_j)$ with

$$\mu_i \neq g(\mu_j), \quad \left(\frac{\mu_i}{g(\mu_j)}\right)^{d(\alpha)} = 1;$$

and then $d(\alpha)$ would be contained in $\bigcup_{m \in E} m \cdot \widehat{\mathbb{Z}}$, contradicting the choice of α.

Since $\prod(1 - \mu_j t)$ is invariant under the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$, and hence it has rational coefficients. Thus $\prod(1 - \mu_i^m t)$ is contained in $\mathbb{Q}[t]$ for all integers m.

Similarly, the polynomials $\prod(1 - \lambda_i^m t)$ are contained in $\mathbb{Q}[t]$ for all integers m. But by assumption, the rational functions $R_\alpha(t)$ are contained in $\mathbb{Q}(t)$ for all geometric points α in question. This gives us the assertion. ∎

33 THE PROOF

We use the notation of previous sections and we are going to prove (30.3).

For $i < d$, by Lefschetz theorem, we may reduce it to lower dimensional case. By duality, we can prove the case $i > d$. So it suffice to deal with the case $i = d$. In fact, it suffices to do the following estimate:

Lemma 33.1. Let X_0 be a smooth projective algebraic variety of constant dimension over the finite field $\kappa = \mathbb{F}_q$. Let the dimension d of X_0 be even. Let $u \in \bar{\mathbb{Q}}_\ell$ be an eigenvalue of the Frobenius homomorphism

$$\mathrm{Fr}^* : H^d(X, \mathbb{Q}_\ell) \to H^d(X, \mathbb{Q}_\ell).$$

Then u is an algebraic number. The complex absolute value of all complex conjugates u' of u satisfy the estimate

$$q^{(d-1)/2} \leq |u'| \leq q^{(d+1)/2}.$$

Consider the smooth projective variety $X_0^m = X_0 \times \cdots \times X_0$ for even m, by Künneth formula,

$$H^d(X, \mathbb{Q}_\ell) \otimes \cdots \otimes H^d(X, \mathbb{Q}_\ell) \to H^{md}(X^m, \mathbb{Q}_\ell)$$

is injective. For any complex conjugates u' of any eigenvalue of $\mathrm{Fr}^* |_{H^d(X,\mathbb{Q}_\ell)}$, we have

$$q^{(md-1)/2} \leq |u'|^m \leq q^{(md+1)/2}.$$

Take $m \to \infty$, we get

$$|u'| = q^{d/2}.$$

This proves (30.3) assuming (33.1).

Proof. It follows from the Poincaré duality theorem that it suffices to prove the upper estimate. Further, it is clear that the statements of the theorem are unchanged by finite extension of the base field. In particular, after finite extension of the base field and reduction to the connected components, it suffices to assume that X_0 is absolutely irreducible. The proof proceeds now by induction on the dimension d. The initial step $d = 0$ is clear.

Take then $d > 0$, and thus $d \geq 2$. Consider the Lefschetz fibering:

$$\begin{array}{ccc} \widetilde{X}_0 & \xrightarrow{f} & D_0 = S_0 \sqcup B_0 \\ \downarrow & & \\ X_0 & & \end{array}$$

Let $\omega_0 : \operatorname{Spec} \Omega \to B_0$ be a generic geometric point over k with values in the separably closed extension field Ω of k. Since $\widetilde{X} \to X$ is birational,

$$H^d(X, \mathbb{Q}_\ell) \to H^d(\widetilde{X}, \mathbb{Q}_\ell)$$

is injective. So we may assume that $X_0 = \widetilde{X}_0$. We compute $H^d(X, \mathbb{Q}_\ell)$ using the spectral sequence

$$H^r(D, R^s f_* \mathbb{Q}_\ell) \Rightarrow H^{r+s}(X, \mathbb{Q}_\ell).$$

It is compatible with Fr^*. Thus it suffices to prove the estimate on the terms $H^r(D, R^s f_* \mathbb{Q}_\ell)$ for $r + s = d$. Only the case $(r, s) = (0, d)$, $(1, d-1)$, $(2, d-2)$ occur. Let E be the space of vanishing cycles. We denote by $\langle\ ,\ \rangle$ the induced alternating bilinear form on $E/(E \cap E^{\perp})$. Consider the representation

$$\rho : \pi_1(B_0, \omega_0) \to \mathrm{GL}(E),$$

we have

$$\langle \rho(\sigma)(x), \rho(\sigma)(y) \rangle = q^{-nd(\sigma)} \langle x, y \rangle.$$

Let $\mathscr{E}_0 = (E/(E \cap E^{\perp}))_{B_0}$ be the locally constant sheaf defined on B_0, and $\mathscr{E}$ its inverse image on B.

(a) The term $H^2(D, R^{n-1}f_*\mathbb{Q}_\ell)$.

The sheaf $R^{n-1}f_*\mathbb{Q}_\ell$ is constant on D. So

$$H^2(D, R^{n-1}f_*\mathbb{Q}_\ell) = H^2(D, \mathbb{Q}_\ell) \otimes H^{n-1}(X_\alpha, \mathbb{Q}_\ell) = H^{n-1}(X_\alpha, \mathbb{Q}_\ell)(-1).$$

By weak Lefschetz theorem, for any smooth hyperplane section $H \subset X_\alpha$ we have,

$$H^{n-1}(X_\alpha, \mathbb{Q}_\ell) \to H^{n-1}(H, \mathbb{Q}_\ell)$$

is injective. From the induction hypothesis for H_0 of dimension $d-2$ over κ, the eigenvalues of the geometric Frobenius mapping

$$H^{n-1}(X_\alpha, \mathbb{Q}_\ell) \to H^{n-1}(X_\alpha, \mathbb{Q}_\ell)$$

are algebraic numbers whose conjugates γ have complex absolute value satisfying

$$q^{((d-2)-1)/2} \leq |\gamma| \leq q^{((d-2)+1)/2}.$$

The eigenvalues of the Frobenius mapping on $H^2(D, R^{n-1}f_*\mathbb{Q}_\ell)$ arise from these by multiplication by q.

(b) The term $H^0(D, R^{n+1}f_*\mathbb{Q}_\ell)$.

According to (29.6), there is an exact sequence

$$0 \to r \cdot \bigoplus_{s \in S} \mathbb{Q}_\ell(m-n)_{\{s\}} \to R^{n+1}f_*\mathbb{Q}_\ell \to j_*((R^{n+1}f_*\mathbb{Q}_\ell)_\omega)_B \to 0$$

where

$$r = \begin{cases} 0, & \text{if } E \neq 0, \\ 1, & \text{if } E = 0. \end{cases}$$

After some base field extension, we can assume that it is in a natural way defined over κ. Since $j_*((R^{n+1}f_*\mathbb{Q}_\ell)_\omega)_B$ is constant on D, we have

$$H^0(D, j_*((R^{n+1}f_*\mathbb{Q}_\ell)_\omega)_B) = (R^{n+1}f_*\mathbb{Q}_\ell)_\omega = (R^{n+1}f_*\mathbb{Q}_\ell)_\alpha = H^{n+1}(X_\alpha, \mathbb{Q}_\ell).$$

So we get the exact sequence

$$0 \to r \cdot \#S \cdot \mathbb{Q}_\ell(m-n) \to H^0(D, R^{n+1}f_*\mathbb{Q}_\ell) \to H^{n+1}(X_\alpha, \mathbb{Q}_\ell) \to 0.$$

The geometric Frobenius homomorphism acts on $\mathbb{Q}_\ell(m-n)$ by multiplication by

$$q^{n-m} = q^{d/2}.$$

By duality and weak Lefschetz, we get a natural surjective mapping

$$\mathrm{Hom}(H^{n-1}(H, \mathbb{Q}_\ell), \mathbb{Q}_\ell)(-n) \to \mathrm{Hom}(H^{n-1}(X_\alpha, \mathbb{Q}_\ell), \mathbb{Q}_\ell)(-n) \cong H^{n+1}(X_\alpha, \mathbb{Q}_\ell).$$

But the eigenvalues of the geometric Frobenius homomorphism on

$$\mathrm{Hom}(H^{n-1}(H, \mathbb{Q}_\ell), \mathbb{Q}_\ell)(-n)$$

are

$$\gamma^{-1}q^n \in \bar{\mathbb{Q}}_\ell$$

Here γ runs through the geometric Frobenius homomorphism on $H^{n-1}(H, \mathbb{Q}_\ell)$. The desired estimates now follows from the induction hypothesis.

(c) The term $H^1(D, R^n f_*\mathbb{Q}_\ell)$

We have different cases to consider.

Case 1. If $E = 0$, then $R^n f_*\mathbb{Q}_\ell$ is constant on D. As

$$H^1(D, \mathbb{Q}_\ell) = 0, \quad H^1(D, R^n f_*\mathbb{Q}_\ell) = 0.$$

Case 2a. If $E \subseteq E^\perp$. By (29.6) and after some base field extension, we have the exact sequence

$$0 \to j_*E_B^\perp \to R^n f_*\mathbb{Q}_\ell \to j_*(V/E^\perp)_B \to \bigoplus_{s \in S} \mathbb{Q}_\ell(m-n)_{\{s\}} \to 0.$$

We break it into

$$0 \to j_*E_B^\perp \to R^n f_*\mathbb{Q}_\ell \to \mathscr{F} \to 0,$$

$$0 \to \mathscr{F} \to j_*(V/E^\perp)_B \to \bigoplus_{s \in S} \mathbb{Q}_\ell(m-n)_{\{s\}} \to 0.$$

Since $j_*E_B^\perp$ and $j_*(V/E^\perp)_B$ are constant, their H^1 vanishes, and we get

$$0 \to H^1(D, R^n f_*\mathbb{Q}_\ell) \to H^1(D, \mathscr{F})$$

$$\#S \cdot \mathbb{Q}_\ell(m-n) \to H^1(D, \mathscr{F}) \to 0.$$

Thus the estimate follows from the eigenvalue of Fr^* on $\mathbb{Q}_\ell(m-n)$.

Case 2b. If $E \not\subseteq E^\perp$. By (29.6) and after some base field extension, we have the exact sequences

$$0 \to j_* E_B \to R^n f_* \mathbb{Q}_\ell \to j_*(V/E)_B \to 0,$$

$$0 \to j_*(E \cap E^\perp)_B \to j_* E_B \to j_*(E/(E \cap E^\perp))_B \to 0.$$

As the sheaves $j_*(V/E)_B$ and $j_*(E \cap E^\perp)_B$ are constant on D, their H^1 vanishes. Thus, we have

$$H^1(D, j_* E_B) \to H^1(D, R^n f_* \mathbb{Q}_\ell) \to 0,$$

$$0 \to H^1(D, j_* E_B) \to H^1(D, j_* \mathscr{E}).$$

Since $E \neq 0$, $S \neq \varnothing$, so $B = D \setminus S$ is affine. It suffices to estimate the eigenvalues of the Frobenius mapping on $H^1(D, j_* \mathscr{E})$. As the natural mapping

$$H^1_c(B, \mathscr{E}) \to H^1(D, j_* \mathscr{E})$$

is surjective, it suffices to estimate the eigenvalues of the Frobenius mapping on $H^1_c(B, \mathscr{E})$. We want to apply the fundamental estimate (31.6) to $\mathscr{G} = \mathscr{E}$, and for that we must establish the two **Hypotheses**. It follows from Kazhdan-Margulis theorem (29.5) that the image of the fundamental group $\pi_1(B, \omega)$ in $\mathrm{Sp}(E/(E \cap E^\perp))$ is open.

It remains still to establish the rationality hypothesis, i.e.: For every geometric point

$$\beta : \operatorname{Spec} k \to B_0$$

over κ, the polynomial $\det(1 - \varphi_\beta^{-1} t) \in \mathbb{Q}[t]$, not just in $\mathbb{Q}_\ell[t]$.

We now deduce the desired rationality property easily from the rationality criterion (32.2): The first hypothesis of (32.2) follows again from the Kazhdan-Margulis theorem. The L-series

$$Z_{X_{0,\beta}}(t) = \prod_i \det(1 - \mathrm{Fr}^* |_{H^i(X_\beta, \mathbb{Q}_\ell)} t)^{(-1)^{i-1}}$$

of the variety $X_{0,\beta}$ over $k(\beta)$ lies in $\mathbb{Q}(t)$. The second hypothesis then follows from the multiplicative of determinants in short exact sequence, the constancy of the sheaves $j_*(V/E)_B$, $j_*(E \cap E^\perp)_B$, and $R^i f_* \mathbb{Q}_\ell$ for $i \neq n$, and the following lemma.

Lemma 33.2. Let $\mathscr{G}_0$ be a locally constant sheaf on B_0 whose inverse image $\mathscr{G}$ on B becomes constant. Then there are units $\lambda_1, \ldots, \lambda_r \in \bar{\mathbb{Q}}_\ell$ such that every geometric point

$$\beta : \operatorname{Spec} k \to B_0$$

over k satisfies

$$\det(1 - f_{\beta,\mathscr{G}_0}^{-1} t) = \prod (1 - \lambda_i^{-d(\beta)} t).$$

Here

$$f_{\beta,\mathscr{G}_0} : (\mathscr{G}_0)_\beta \to (\mathscr{G}_0)_\beta$$

is again the arithmetic Frobenius element.

Proof. Write $\mathscr{G}_0 = W_{B_0}$ for some $\pi_1(B_0, \omega_0)$-module W, on which $\pi_1(B, \omega)$ acts trivially. Then the factor group $\operatorname{Gal}(k/\kappa)$ acts on W. We have

$$\det(1 - f_{\beta,\mathscr{G}_0}^{-1} t) = \det(1 - f^{-d(\beta)} t).$$

Here $f : W \to W$ is the mapping induced by the Frobenius isomorphism of k over κ, and $f^{d(\beta)}$ is thus the mapping induced by the Frobenius isomorphism of k over $\kappa(\beta)$. This finishes the proof. ■

34 Čebotarev theorem

We complete the detail of Čebotarev theorem that we used before.

34.1 Valuations, places, and valuation rings

Definition 34.1. A valuation v of a field F is a surjective map $v : F \to \Gamma \cup \{\infty\}$, where Γ is a nontrivial ordered abelian group, called the value group, with these properties:

- $v(ab) = v(a) + v(b)$
- $v(a + b) \geq \min\{v(a), v(b)\}$
- $v(a) = \infty \iff a = 0$.

Here $\alpha < \infty$ for each $\alpha \in \Gamma$.

The set

$$\mathcal{O}_v = \{a \in F \mid v(a) \geq 0\}$$

is the valuation ring of v. It has a unique maximal ideal

$$\mathfrak{m}_v = \{a \in F \mid v(a) > 0\}.$$

$\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$ is called the residue field of F at v.

Two valuations v, v' with value group Γ, Γ' are equivalent if there exists an isomorphism $f : \Gamma \to \Gamma'$ with $v' = f \circ v$.

Definition 34.2. A place of a field F is a surjective map $\varphi : F \to \kappa(\varphi) \cup \{\infty\}$, where $\kappa(\varphi)$ is a field, called the residue field of F at φ, with these properties:

- $\varphi(a + b) = \varphi(a) + \varphi(b)$
- $\varphi(ab) = \varphi(a)\varphi(b)$
- There exist a, $b \in F$ with $\varphi(a) = \infty$ and $\varphi(b) \neq 0, \infty$.

Neither $\infty + \infty$ nor $0 \cdot \infty$ are defined.

We call an element $x \in F$ with $\varphi(x) \neq \infty$ finite at φ. The ring

$$\mathcal{O}_\varphi = \{a \in F \mid \varphi(a) \neq \infty\}$$

is the valuation ring of φ. It has a unique maximal ideal

$$\mathfrak{m}_\varphi = \{a \in \mathcal{O}_\varphi \mid \varphi(a) = 0\}.$$

We have

$$\mathcal{O}_\varphi/\mathfrak{m}_\varphi \cong \{\varphi(a) \mid a \in \mathcal{O}_\varphi\} = \kappa(\varphi)$$

. We say φ is a E-place if E is a subfield of F and $\varphi(a) = a$ for each $a \in E$.

Two places φ and φ' of F are equivalent if there exists an isomorphism $\alpha : \kappa(\varphi) \to \kappa(\varphi')$ with $\varphi' = \alpha \circ \varphi$.

Definition 34.3. A valuation ring of a field F is a proper subring $\mathcal{O}$ of F such that if $x \in F^\times$, then $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

The subset

$$\mathfrak{m} = \{x \in \mathcal{O} \mid x^{-1} \notin \mathcal{O}\}$$

is the unique maximal ideal of $\mathcal{O}$. The map $\varphi : F \to \mathcal{O}/\mathfrak{m} \cup \{\infty\}$ defines a place of F with valuation $\mathcal{O}$. $F^\times/\mathcal{O}^\times$ is a multiplicative group ordered by the rule $x\mathcal{O}^\times \le y\mathcal{O}^\times$ iff $yx^{-1} \in \mathcal{O}$. The map $x \mapsto x\mathcal{O}^\times$ defines a valuation of F with $\mathcal{O}$ being its valuation ring.

These definitions easily give a bijective correspondence between the valuation classes, the place classes and the valuation rings of a field F.

Definition 34.4. A prime divisor $\mathfrak{p}$ of F/k is an equivalence class of k-places of F.

34.2 Global fields

Definition 34.5. Call a field extension K/k an (algebraic) function field (of one variable) if

- The transcendence degree of K/k is 1.
- K/k is finitely generated.
- K/k is regular, i.e., k is algebraically closed in K.

So there exists $t \in K$, transcendental over k, with $K/k(t)$ a finite separable extension. All valuations of $k(t)$ trivial on K are discrete, so there extensions to K are also discrete. Also, since the residue fields of the valuations of $k(t)$ are finite extensions of k, so are the residue fields of the valuation.

For a prime divisor $\mathfrak{p}$ of K/k, choose a place $\varphi_\mathfrak{p}$ in $\mathfrak{p}$. Then $\varphi_\mathfrak{p}$ fixes the element of k and $\kappa(\varphi_\mathfrak{p})$ is a finite extension of degree $\deg(\mathfrak{p}) = [\kappa(\varphi_\mathfrak{p}) : k]$, which we call the degree of $\mathfrak{p}$. Also, choose a valuation $v_\mathfrak{p}$ corresponding to $\mathfrak{p}$ and normalize it so that $v_\mathfrak{p}(K^\times) = \mathbb{Z}$.

Definition 34.6. We say a field K is a number field if K is a finite extension of $\mathbb{Q}$, K is a global function field if K is a function field of a finite field $\mathbb{F}_q$. We say K is a global field is K is either a number field or a global function field.

For K a number field (resp. global function field), we denote the integral closure of $\mathbb{Z}$ (resp. $\mathbb{F}_q[t]$, we fix t here) in K by $\mathcal{O}_K$. The local ring of $\mathcal{O}_K$ at a prime ideal $\mathfrak{p} \neq (0)$

is a valuation ring of K. Denote its residue field by $\mathbb{F}_{\mathfrak{p}}$. We define the norm of $\mathfrak{p}$ by $N(\mathfrak{p}) = |\mathbb{F}_{\mathfrak{p}}|$. If we regard $\mathfrak{p}$ as a prime divisor, then $N(\mathfrak{p}) = q^{\deg(\mathfrak{p})}$.

Let L be a finite Galois extension of K. Suppose $\mathfrak{p}$ is unramified in L. If $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_L$ over $\mathfrak{p}$, then reduction module $\mathfrak{q}$ gives a canonical isomorphism of the decomposition group

$$D_{\mathfrak{q}} = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}$$

and $\mathrm{Gal}(\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}})$, as $|I_{\mathfrak{q}}| = e_{\mathfrak{q}/\mathfrak{p}} = 1$. The latter group is cyclic, generated by the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{q}}$. It acts on $\mathbb{F}_{\mathfrak{q}}$ by

$$\mathrm{Frob}_{\mathfrak{q}}(x) = x^{N(\mathfrak{p})}.$$

The element of $D_{\mathfrak{q}}$ that corresponds to $\mathrm{Frob}_{\mathfrak{q}}$ is called the Frobenius automorphism at $\mathfrak{q}$ and denote it by $\left[\frac{L/K}{\mathfrak{q}}\right]$. It is uniquely determined in $\mathrm{Gal}(L/K)$ by the condition

$$\left[\frac{L/K}{\mathfrak{q}}\right](x) = x^{N(\mathfrak{p})} \pmod{\mathfrak{q}} \quad \text{for all } x \in \mathcal{O}_L.$$

If $\sigma \in \mathrm{Gal}(L/K)$, then $\left[\frac{L/K}{\sigma(\mathfrak{q})}\right] = \sigma\left[\frac{L/K}{\mathfrak{q}}\right]\sigma^{-1}$. Therefore, as $\mathfrak{q}$ ranges over all prime ideals of $\mathcal{O}_L$ lying over $\mathfrak{p}$, $\left[\frac{L/K}{\mathfrak{q}}\right]$ ranges over some conjugacy class in $\mathrm{Gal}(L/K)$ that depends on $\mathfrak{p}$. This conjugacy class is the Artin symbol, denoted by $\left(\frac{L/K}{\mathfrak{p}}\right)$. Let L/K be a Galois extension of number fields or function field of over finite field and let X be a conjugacy class of $\mathrm{Gal}(L/K)$. Write

$$S_X = \left\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_K \;\middle|\; \left(\frac{L/K}{\mathfrak{p}}\right) = X\right\}.$$

Definition 34.7. Let K be a global field. If A is a subset of $\operatorname{Spec}\mathcal{O}_K$, the (Dirichlet) upper (resp. lower) density of A, denoted by $d_{\sup}(A)$ (resp. $d_{\inf}(A)$), is defined by

$$d_{\sup}(A) = \limsup_{s\to1^+}\frac{\sum_{\mathfrak{p}\in A}N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in\operatorname{Spec}\mathcal{O}_K}N(\mathfrak{p})^{-s}}$$
$$d_{\inf}(A) = \liminf_{s\to1^+}\frac{\sum_{\mathfrak{p}\in A}N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in\operatorname{Spec}O_K}N(\mathfrak{p})^{-s}}.$$

If $d_{\sup}(A) = d_{\inf}(A)$, then the density of A is defined by

$$d(A) = d_{\sup}(A) = d_{\inf}(A).$$

Theorem 34.8 (Čebotarev)**.** Let K be a global field, L/K a finite Galois extension, and X a conjugacy class in $\mathrm{Gal}(L/K)$. Then

$$d(S_X) = \frac{\#X}{[L:K]}.$$

We prove the case where K is a function field over $\mathbb{F}_q$.

For a function field K, we let C_K be the projective curve associated to $K/\mathbb{F}_q$ (the set of prime divisors of $K/\mathbb{F}_q$). We define

$$C_K(X) = \left\{ \mathfrak{p} \in C_K \;\middle|\; \left(\frac{L/K}{\mathfrak{p}}\right) = X \right\}.$$

Since the embedding $\operatorname{Spec} \mathcal{O}_K \to C_K$ is cofinite, Čebotarev can be restated as follows.

Theorem 34.9. Let K be a function field over $\mathbb{F}_q$, L/K a finite Galois extension, and X a conjugact class in $\mathrm{Gal}(L/K)$. Then

$$d(C_K(X)) = \frac{\#X}{[L:K]}.$$

Proof. Fix $\mathbb{F}_q(t) \subseteq K$ so that the extension is separable. Let $\mathbb{F}_{q^n}$ be the algebraic closure of $\mathbb{F}_q$ in L. Let $d = [K : \mathbb{F}_q(t)]$ and $m = [L : K \cdot \mathbb{F}_{q^n}]$, then we get

$$\begin{tikzcd}
L \arrow[d, dash, "m"] & & \\
K \cdot \mathbb{F}_{q^n} \arrow[d, dash, "n"] \arrow[dr, dash, "d"] & & \\
K \arrow[dr, dash, "d"'] & \mathbb{F}_{q^n}(t) \arrow[d, dash, "n"] & \\
& \mathbb{F}_q(t) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_L \arrow[d, "f"] \\
C_K \arrow[d] \\
\mathbb{P}^1.
\end{tikzcd}$$

We fix some notations:

$$\begin{aligned}
C_K' &= \{\mathfrak{p} \in C_K \mid \ \mathfrak{p} \text{ is unramified over } \mathbb{F}_q(t) \text{ and in } L\ \} \\
C_{K,k} &= \{\mathfrak{p} \in C_K \mid \deg(\mathfrak{p}) = k\} \\
C_{K,k}' &= \{\mathfrak{p} \in C_K' \mid \deg(\mathfrak{p}) = k\} \\
S_k(L/K, X) &= \left\{ \mathfrak{p} \in C_{K,k}' \;\middle|\; \left(\frac{L/K}{\mathfrak{p}}\right) = X \right\} \\
S &= \bigcup_{k=1}^{\infty} S_k(L/K, X) \\
T_k(L/K, \sigma) &= \left\{ \mathfrak{q} \in C_L \;\middle|\; \mathfrak{q} \cap K \in C_{K,k} \text{ and } \left[\frac{L/K}{\mathfrak{q}}\right] = \sigma \right\}.
\end{aligned}$$

Since S and $C_K(X)$ differ by only finitely many elements, they have the same density. To compute this density, we compute $\#S_k(L/K, X)$ for each k.

We deal with the special case first.

Lemma 34.10. Suppose $L = K \cdot \mathbb{F}_{q^n}$, $X = \{\sigma\}$, and $\sigma|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q$. Then

$$|\#S_1(L/K, X) - q| < 2(g_L(\sqrt{q} + 1) + d).$$

Proof. First note that $S_1(L/K, X) = C'_{K,1}$ and each $\mathfrak{p} \in C_K$ is unramified in L. Thus, $C_{K,1} \setminus S_1(L/K, X)$ consists exactly of all prime divisors of the ramification divisor R of the map $C_K \to \mathbb{P}^1$. Since $[K : \mathbb{F}_q(t)] = d$, by Hurwitz formula,

$$2g_K - 2 = d \cdot (-2) + \deg R \implies \deg R = 2(g_K + d - 1).$$

By Riemann Hypothesis for function fields,

$$|\#C_{K,1} - (q + 1)| \le 2g_K\sqrt{q}.$$

Hence,

$$|\#S_1(L/K, X) - q| \le 2g_K\sqrt{q} + 1 + 2(g_K + d - 1).$$

The assertion is thus follows from $g_K = g_L$. □

Lemma 34.11. Let $\sigma \in X$. If A_k is a subset of $S_k(L/K, X)$ and

$$B_k(\sigma) = \{\mathfrak{q} \in T_k(L/K, \sigma) \mid \mathfrak{q} \cap K \in A_k\}$$

the set of primes in $T_k(L/K, \sigma)$ over A_k. Then

$$\#A_k = \frac{\#X \cdot \mathrm{ord}(\sigma) \cdot \#B_k(\sigma)}{[L : K]}.$$

Proof. For each $\tau \in \mathrm{Gal}(L/K)$, $B_k(\tau\sigma\tau^{-1}) = \tau B_k(\sigma)$, so

$$\#B_k(\tau\sigma\tau^{-1}) = \#B_k(\sigma).$$

If $\sigma \neq \sigma' \in X$, then $B_k(\sigma) \cap B_k(\sigma') = \varnothing$. Thus,

$$\bigsqcup_{\sigma \in X} B_i(\sigma)$$

is the set of primes of C_L lying over A_k. Suppose $\mathfrak{q} \in C_L$ lies over $\mathfrak{p} \in S_k(L/K, X)$. Then $[L : K] = e_{\mathfrak{q}/\mathfrak{p}} f_{\mathfrak{q}/\mathfrak{p}} g_{\mathfrak{q}/\mathfrak{p}}$. In our case $e_{\mathfrak{q}/\mathfrak{p}} = 1$ and σ is conjugate to $\left[\frac{L/K}{\mathfrak{q}}\right]$. Thus $f_{\mathfrak{q}/\mathfrak{p}} = \mathrm{ord}(\sigma)$ and we get

$$\#f^{-1}(\mathfrak{p}) = \frac{[L : K]}{\mathrm{ord}(\sigma)}.$$

So

$$\frac{\#A_k \cdot [L : K]}{\mathrm{ord}(\sigma)} = \sum_{\sigma \in X} \#B_k(\sigma) = \#X \cdot \#B_k(\sigma).$$ □

Lemma 34.12. Let $K \subseteq K' \subseteq L$, $\sigma \in \mathrm{Gal}(L/K') \subseteq \mathrm{Gal}(L/K)$, and $\mathbb{F}_{q^r}$ the algebraic closure of $\mathbb{F}_q$ in K'.

$$
\begin{tikzcd}
L \arrow[dr, dash, "d"] \arrow[dd, dash, "m"] & \\
 & \mathbb{F}_{q^n} \arrow[dd, dash, "n"] \\
K' \arrow[dr, dash, "d"] \arrow[dd, dash, "n"] & \\
 & \mathbb{F}_{q^r} \arrow[dd, dash, "n"] \\
K \arrow[dr, dash, "d"] & \\
 & \mathbb{F}_q
\end{tikzcd}
$$

Suppose $k = \ell \cdot r$. Let X and X' be the conjugacy classes of σ in $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L/K')$, respectively, and

$$S'_\ell := S_\ell(L/K', X') \setminus \{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) \leq k/2\}\,.$$

Then

$$\#S_k(L/K, X) = \frac{\#X}{\#X' \cdot [K' : K]} \cdot \#S'_\ell.$$

Proof. We have

$$T'_\ell(\sigma) := T_\ell(L/K', \sigma) \cap \{\mathfrak{q} \in C_L \mid \deg(\mathfrak{q} \cap K) = k\}$$

is the set of primes in $T_\ell(L/K', \sigma)$ lying over

$$S''_\ell = S_\ell(L/K', X') \cap \{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) = k\}.$$

By (34.11),

$$\#T_k(L/K, \sigma) = \frac{[L : K] \cdot \#S_k(L/K, X)}{\#X \cdot \mathrm{ord}(\sigma)}$$
$$\#T'_\ell(\sigma) = \frac{[L : K'] \cdot \#S''_\ell}{\#X' \cdot \mathrm{ord}(\sigma)}$$

Thus, it suffices to show that $S''_\ell = S'_\ell$ and $T_k(L/K, \sigma) = T'_\ell(\sigma)$.

$S''_\ell = S'_\ell$: If $\mathfrak{p}' \in C_{K'}$ is of degree ℓ and $\mathfrak{p} = \mathfrak{p}' \cap K$, then $\mathbb{F}_q \subseteq \mathbb{F}_\mathfrak{p} \subseteq \mathbb{F}_{\mathfrak{p}'}$. Hence $\deg(\mathfrak{p}) \mid k$. Thus, either $\deg(\mathfrak{p}) = k$ or $\deg(\mathfrak{p}) \leq \frac{k}{2}$. Therefore $S''_k = S'_\ell$.

$T_k(L/K, \sigma) = T'_\ell(\sigma)$: Let $\mathfrak{q} \in C_L$, $\mathfrak{p} = \mathfrak{q} \cap K$ and $\mathfrak{p}' = \mathfrak{q} \cap K'$. Suppose $\mathfrak{p} \in C_{K,i}$ and $\mathfrak{p}' \in C_{K',j}$. Then $N(\mathfrak{p}) = q^i$ and $N(\mathfrak{p}') = q^{jr}$. Note that

$$\left[\frac{L/K}{\mathfrak{q}}\right] = \sigma \iff \sigma(x) \equiv x^{q^i} \pmod{\mathfrak{q}} \quad \forall x \in \mathcal{O}_L$$
$$\left[\frac{L/K'}{\mathfrak{q}}\right] = \sigma \iff \sigma(x) \equiv x^{q^{jr}} \pmod{\mathfrak{q}} \quad \forall x \in \mathcal{O}_L.$$

The assertion is now clear from this. $\square$

Lemma 34.13. Let $k \in \mathbb{N}$ such that

$$\sigma|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q^k$$

for every $\sigma \in X$. Let n' be a multiple of n and $L' = L \cdot \mathbb{F}_{q^{n'}}$. Then for each $\sigma \in X$ there exists a unique $\sigma' \in \mathrm{Gal}(L'/K)$ with

$$\sigma'|_L = \sigma, \quad \sigma'|_{\mathbb{F}_{q^{n'}}} = \mathrm{Frob}_q^k.$$

Furthermore,

(a) $\mathrm{ord}(\sigma') = \mathrm{lcm}\left(\mathrm{ord}(\sigma), \dfrac{n'}{\gcd(n', k)}\right)$,

(b) $X' = \{\sigma' \mid \sigma \in X\}$ is a conjugacy class in $\mathrm{Gal}(L'/K)$,

(c) $S_k(L'/K, X') = S_k(L/K, X)$.

Proof. To prove (a), just note that

$$\begin{aligned}\mathrm{ord}(\sigma') &= \mathrm{lcm}\left(\mathrm{ord}(\sigma'|_L), \mathrm{ord}\big(\sigma'|_{\mathbb{F}_{q^{n'}}}\big)\right)\\ &= \mathrm{lcm}\left(\mathrm{ord}(\sigma), \frac{n'}{\gcd(n', k)}\right).\end{aligned}$$

As $\mathbb{F}_{q^{n'}}/\mathbb{F}_q$ is an abelian extension, (b) follows from the uniqueness of σ'.

To prove (c), we show that $\mathfrak{q}' \in C_{L'}$, $\mathfrak{q} = \mathfrak{q}' \cap L$ and $\mathfrak{p} = \mathfrak{q}' \cap K$ so that $\deg(\mathfrak{p}) = k$ imply

$$\left[\frac{L/K}{\mathfrak{q}}\right] = \sigma \iff \left[\frac{L'/K}{\mathfrak{q}'}\right] = \sigma'.$$

Indeed, if $\left[\frac{L/K}{\mathfrak{q}}\right] = \sigma$, then

$$\sigma(x) = x^{q^k} \pmod{\mathfrak{q}} \quad \forall x \in \mathcal{O}_L.$$

By definition, $\mathrm{Frob}_q^k(x) = x^{q^k}$ for each $x \in \mathbb{F}_{q^{n'}}$. Since $\mathcal{O}_{L'} = \mathbb{F}_{q^{n'}}\mathcal{O}_L$,

$$\sigma'(x) = x^{q^k} \pmod{\mathfrak{q}'} \quad \forall x \in \mathcal{O}_{L'}.$$

Hence, $\left[\frac{L'/K}{\mathfrak{q}'}\right] = \sigma'$. The converse is trivial. □

Denote the situation when $a \mid b$ and $a < b$ by $a \mid' b$.

Lemma 34.14. Let K' be a degree $k\ell$ extension of K containing $\mathbb{F}_{q^k}$. Then

$$\#\{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) \mid' k\} = O(\ell \cdot q^{k/2}).$$

Proof. If $j \mid k$ and $\mathfrak{p} \in C_{K,j}$, then $\mathbb{F}_{q^j} \subseteq \mathbb{F}_{q^k}$ and $\mathfrak{p}$ decomposes in $K \cdot \mathbb{F}_{q^j}$ into j prime divisors of degree 1. Each has exactly one extension to $K \cdot \mathbb{F}_{q^k}$. The latter decomposes in K' into at most ℓ prime divisors.

$$
\begin{array}{ccc}
K' & & \mathfrak{p}'_{i1}, \dots, \mathfrak{p}'_{i(\leq \ell)} \\
\Big| \ell & & \Big\downarrow \\
K \cdot \mathbb{F}_{q^k} & \mathfrak{p}_1, \dots, \mathfrak{p}_j & \mathfrak{p}_i \\
\Big| & \Big\downarrow & \\
K \cdot \mathbb{F}_{q^j} & \mathfrak{p}_1, \dots, \mathfrak{p}_j & \\
\Big| & \Big\downarrow & \\
K & \mathfrak{p} & \deg(\mathfrak{p}) = j
\end{array}
$$

Hence,

$$
\begin{aligned}
\#\{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) \mid' k\} &= \sum_{j \mid' k} \#\{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) = j\} \\
&\leq \ell \sum_{j \mid' k} \#\{\mathfrak{q} \in C_{K \cdot \mathbb{F}_{q^k}} \mid \deg(\mathfrak{q} \cap K) = j\} \\
&= \ell \sum_{j \mid' k} \#\{\mathfrak{q} \in C_{K \cdot \mathbb{F}_{q^j}} \mid \deg(\mathfrak{q} \cap K) = j\} \\
&\leq \ell \sum_{j \mid' k} \# C_{K \cdot \mathbb{F}_{q^j}, 1}.
\end{aligned}
$$

By Riemann Hypothesis for function fields,

$$\# C_{K \cdot \mathbb{F}_{q^j}, 1} \leq q^j + 2 g_K q^{j/2} + 1.$$

The assertion is now trivial. $\square$

Proposition 34.15. Let $0 < a \leq n$ be an integer with

$$\sigma|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q^a$$

for each $\sigma \in X$. Let $k \in \mathbb{N}$. If $k \not\equiv a \pmod{n}$, then $S_k(L/K, X) = \varnothing$. If $k \equiv a \pmod{n}$, then

$$\left| \# S_k(L/K, X) - \frac{\# X \cdot q^k}{km} \right| = O(q^{k/2})$$

Proof. Suppose $\mathfrak{q} \in C_L$ lies over $\mathfrak{p} \in S_k(L/K, X)$. Then, $\left[\frac{L/K}{\mathfrak{q}}\right] \in X$, so

$$\mathrm{Frob}_q^a |_{\mathbb{F}_{q^n}} = \left[\frac{L/K}{\mathfrak{q}}\right]\Big|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q^k |_{\mathbb{F}_{q^n}}.$$

Hence, $k \equiv a \pmod{n}$.

Conversely, suppose $k \equiv a \pmod{n}$. Let $\sigma \in X$ and n' some multiple of n. Extend L to $L' = L \cdot \mathbb{F}_{q^{n'}}$. Then

$$[L' : K \cdot \mathbb{F}_{q^{n'}}] = [L : K \cdot \mathbb{F}_{q^n}] = m.$$

Since $k \equiv a \pmod{n}$, $\sigma|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q^k$. By (34.13), there exists $\sigma' \in \mathrm{Gal}(L'/K)$ with $\sigma'|_L = \sigma$ and $\sigma'|_{\mathbb{F}_{q^{n'}}} = \mathrm{Frob}_q^k$. Take $n' = nk \cdot \mathrm{ord}(\sigma)$, then

$$\mathrm{ord}(\sigma') = \mathrm{lcm}\left(\mathrm{ord}(\sigma), \frac{n'}{\gcd(n', k)}\right) = n \cdot \mathrm{ord}(\sigma).$$

Denote the conjugacy class of σ' in $\mathrm{Gal}(L'/K)$ by X', then $S_k(L'/K, X') = S_k(L/K, X)$. Let $K' = (L')^{\sigma'}$. Then $K' \cap \mathbb{F}_{q^{n'}} = \mathbb{F}_{q^k}$ and $K' \cdot \mathbb{F}_{q^{n'}} = L'$. We have the following diagram.

$$\begin{tikzcd}
 & & L' \arrow[dd, dash, "m"] \\
 & K' \arrow[ur, dash, "\mathrm{ord}(\sigma')"] \arrow[dd, dash, "m"] & \\
 & & K \cdot \mathbb{F}_{q^{n'}} \arrow[dd, dash] \\
 & K \cdot \mathbb{F}_{q^k} \arrow[ur, dash] \arrow[dd, dash, "d"] & \\
K \arrow[ur, dash] \arrow[dd, dash, "d"] & & \mathbb{F}_{q^{n'}}(t) \arrow[dd, dash] \\
 & \mathbb{F}_{q^k}(t) \arrow[ur, dash] \arrow[dd, dash] & \\
\mathbb{F}_q(t) \arrow[ur, dash] \arrow[dd, dash] & & \mathbb{F}_{q^{n'}}. \\
 & \mathbb{F}_{q^k} \arrow[ur, dash] & \\
\mathbb{F}_q \arrow[ur, dash] & &
\end{tikzcd}$$

Thus,

$$[K' : K \cdot \mathbb{F}_{q^k}] = [L' : K \cdot \mathbb{F}_{q^{n'}}] = [L : K \cdot \mathbb{F}_{q^n}] = m.$$

Hence, $[K' : \mathbb{F}_{q^k}(t)] = dm$. By (34.12),

$$\#S_k(L/K, X) = \#S_k(L'/K, X') = \frac{\#X' \cdot \#S_1'}{\#\{\sigma'\} \cdot [K' : K]},$$

where

$$S_1' = S_1(L'/K', \{\sigma'\}) \setminus \{\mathfrak{p}' \in C_{K'} \mid \deg(\mathfrak{p}' \cap K) \mid' k\}.$$

Since $[K' : K] = km$, (34.14) implies

$$\left| \#S_k(L/K, X') - \frac{\#X \cdot \#S_1(L'/K', \{\sigma'\})}{km} \right| = \frac{\#X}{km} \cdot O(q^{k/2}).$$

By (34.10) and $g_L = g'_L$,

$$|\#S_1(L'/K', \{\sigma'\}) - q^k| < 2(g_{L'}(q^{k/2}+1)+dm) = 2(g_L(q^{k/2}+1)+dm) = O(q^{k/2}).$$

Finally, we get

$$\left|\#S_k(L/K,X) - \frac{\#X \cdot q^k}{km}\right| = O(q^{k/2}) \qquad \square$$

We now give a final estimate. Recall that $S = \bigcup_{k=1}^{\infty} S_k(L/K,X)$.

Lemma 34.16. Suppose $0 < a \le n$ is an integer with $\sigma|_{\mathbb{F}_{q^n}} = \mathrm{Frob}_q^a|_{\mathbb{F}_{q^n}}$ for each $\sigma \in X$. Then

$$\sum_{\mathfrak{p}\in S} N(\mathfrak{p})^{-s} = -\frac{\#X}{[L:K]}\log(1-q^{1-s}) + O(1).$$

Proof. Apply (34.15), we get

$$\begin{aligned}
\sum_{\mathfrak{p}\in S} N(\mathfrak{p})^{-s} &= \sum_{j=0}^{\infty} \sum_{\mathfrak{p}\in S_{a+jn}(L/K,X)} N(\mathfrak{p})^{-s} \\
&= \sum_{j=0}^{\infty}\left(\frac{\#X\cdot q^{a+jn}}{(a+jn)m} + O(q^{(a+jn)/2})\right) q^{-(a+jn)s} \\
&= \frac{\#X}{m}\sum_{j=0}^{\infty}\frac{q^{(a+jn)(1-s)}}{a+jn} + O(q^{(a+jn)(\frac{1}{2}-s)}) \\
&= \frac{\#X}{mn}\log\left(\frac{1}{1-q^{1-s}}\right) + O(1) \\
&= \frac{\#X}{[L:K]}\log\left(\frac{1}{1-q^{1-s}}\right) + O(1)
\end{aligned}$$

as $s \to 1^+$. $\square$

When $L = K$, (34.16) gives

$$\sum_{\mathfrak{p}\in C_K} N(\mathfrak{p})^{-s} = \log\left(\frac{1}{1-q^{1-s}}\right) + O(1).$$

Thus, we get

$$d(S) = \lim_{s\to 1^+}\frac{\sum_{\mathfrak{p}\in S} N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in C_K} N(\mathfrak{p})^{-s}} = \frac{\#X}{[L:K]}. \qquad \blacksquare$$

34.3 Haar measure and infinite Čebotarev

Let G be a profinite group with $G = \varprojlim_{i\in I} G/G_i$, where I is countable, $[G:G_i] < \infty$ and G_i form a neighborhood basis of $\{1_G\}$. We now define the Haar measure on G explicitly.

Let

$$\mathcal{B} = \left\{ \bigcup_{j=1}^{n} a_j G_{i_j} \;\middle|\; a_j \in G,\ i_j \in I \right\} \cup \{\varnothing\},$$

then $\mathcal{B}$ forms a basis for the topology of G. Every element in $\mathcal{B}$ can be written in the form

$$\bigsqcup_{j=1}^{n} a_j G_i$$

for some i. We define

$$\begin{array}{rccc} \mu: & \mathcal{B} & \to & [0,1] \\ & \displaystyle\bigsqcup_{j=1}^{n} a_j G_i & \mapsto & \dfrac{n}{[G:G_i]}. \end{array}$$

This is well-defined and left invariant. Now, suppose $\displaystyle\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{B}$ and $A_i \in \mathcal{B}$ for each i. Since $\bigsqcup A_i$ is compact and A_i is open for each i, $A_i = \varnothing$ for all but finitely many i. Hence,

$$\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

For $X \subseteq G$, we define the outer measure by

$$\mu^*(X) = \inf_{\substack{X \subseteq \cup A_i \\ A_i \in \mathcal{B}}} \sum_{i=1}^{\infty} \mu(A_i).$$

Then

$$\mu^*\left(\bigcup_{i=1}^{\infty} X_i\right) \leq \sum_{i=1}^{\infty} \mu^*(X_i).$$

Let

$$\mathcal{M} = \{X \subseteq G \mid \mu^*(Y) = \mu^*(Y \cap X) + \mu^*(Y \setminus X)\ \forall Y \subseteq G\}$$

to be the set of "measurable" subsets of G. Note that in general we have

$$\mu^*(Y) \leq \mu^*(Y \cap X) + \mu^*(Y \setminus X).$$

Then $\mathcal{M}$ is a σ-algebra on G. Indeed, if $X = \displaystyle\bigsqcup_{i=1}^{\infty} X_i$ with $X_i \in \mathcal{M}$, then for any $Y \subseteq G$,

$$\sum_{i=1}^{n} \mu^*(Y \cap X_i) + \mu^*(Y \setminus X) \leq \sum_{i=1}^{n} \mu^*(Y \cap X_i) + \mu^*\left(Y \setminus \bigcup_{i=1}^{\infty} X_i\right) = \mu^*(Y),$$

take $n \to \infty$ we get $\mu^*(Y \cap X) + \mu^*(Y \setminus X) \leq \mu^*(Y)$.

We claim that $\mu^*|_{\mathcal{B}} = \mu$. We have $\mu^*(A) \le \mu(A)$ by definition. Suppose $A \subseteq \bigcup_{i=1}^{\infty} A_i$. Let $B_i = A \cap A_i \setminus \bigcup_{j<i} A \cap A_j \in \mathcal{B}$. Then $A = \bigsqcup_{i=1}^{\infty} B_i$ and hence

$$\mu(A) = \sum_{i=1}^{\infty} \mu(B_i) \le \sum_{i=1}^{\infty} \mu(A_i).$$

Thus $\mu(A) \le \mu^*(A)$.

For $A \in \mathcal{B}$, $Y \subseteq G$. Let $Y \subseteq \bigcup_{i=1}^{\infty} A_i$ be a cover, then $Y \cap A \subseteq \bigcup_{i=1}^{\infty} A_i \cap A$ and $Y \setminus A \subseteq \bigcup_{i=1}^{\infty} A_i \setminus A$, So

$$\mu^*(Y \cap A) + \mu^*(Y \setminus A) \le \sum_{i=1}^{\infty} \mu^*(A_i \cap A) + \mu^*(A_i \setminus A) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, $\mu^*(Y \cap A) + \mu^*(Y \setminus A) \le \mu^*(Y)$ and hence $A \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, the Borel σ-algebra $\mathcal{S}$ of G is contained in $\mathcal{M}$. So we may extend μ to $\mathcal{S}$ by $\mu^*|_{\mathcal{S}}$.

For U open in G, we have

$$\mu(U) \ge \sup_{i \in I} \frac{\#\{aG_i \mid aG_i \subseteq U\}}{[G : G_i]}.$$

Define $A_0 = \varnothing$, $A_{i+1} = \bigcup_{aG_i \subseteq U} aG_i \setminus A_i \in \mathcal{B}$, then

$$\begin{aligned} \mu(U) &= \mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sup_n \sum_{i=1}^{n} \mu(A_i) \\ &= \sup_n \mu\left(\bigcup_{aG_i \subseteq U} aG_i\right) = \sup_n \frac{\#\{aG_i \mid aG_i \subseteq U\}}{[G : G_n]}. \end{aligned}$$

We now prove the infinite case of Čebotarev theorem with the finite case.

Theorem 34.17 (Čebotarev)**.** Let K be a global field. Suppose L/K is Galois with Galois group G and let $X \subseteq G$ be closed under conjugation. Suppose the set of primes $\mathfrak{p}$ of K that ramify in L has density 0. Then

$$\mu(\bar{X}) \ge d_{\sup}(X) \ge d_{\inf}(X) \ge \mu(X^\circ).$$

In particular, if $\mu(\partial X) = 0$, then S_X has density $\mu(X)$.

Proof. Let

$$K \subset L_1 \subset L_2 \subset \cdots \subset L$$

be a chain of subextensions of L/K that are finite and Galois over K with $L = \bigcup_i L_i$. Define $X_i \subseteq \mathrm{Gal}(L_i/K)$ to be the image of X through the quotient map

$$\mathrm{Gal}(L/K) \to {}^{\mathrm{Gal}(L/K)}\!\big/_{\mathrm{Gal}(L/L_i)} \cong \mathrm{Gal}(L_i/K).$$

For each i we define

$$S_i = \left\{ \mathfrak{p} \in \mathcal{O}_K \;\middle|\; \left(\frac{L_i/K}{\mathfrak{p}} \right) \subset X_i \right\}.$$

Note that $S_i \supseteq S_{i+1} \supseteq S_X$ for all i. So for all i and $s > 1$ we have

$$\frac{\sum_{\mathfrak{p} \in S_i} N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p} \in \mathrm{Spec}\, O_K} N(\mathfrak{p})^{-s}} \geq \frac{\sum_{\mathfrak{p} \in S_X} N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_K} N(\mathfrak{p})^{-s}}.$$

Taking the limit $s \to 1$ and then letting $i \to \infty$ we see

$$\liminf_{i \to \infty} d(L_i/K, X_i) \geq d_{\mathrm{sup}}(L/K, X).$$

By the finite case and the way μ behaves on closed sets we have

$$\liminf_{i \to \infty} d(L_i/K, X_i) = \liminf_{i \to \infty} \frac{\# X_i}{[L_i : K]} = \mu(\bar{X}).$$

So $\mu(\bar{X}) \geq d_{\mathrm{sup}}(L/K, X)$. Similarly, we get $\mu(X^\circ) \leq d_{\mathrm{inf}}(L/K, X)$. ∎