Calabi conjecture

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Contents

Work distribution:

Shuang-Yen Lee: Section 2, 3, 8, 9.

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1 Introduction

This survey is mainly based on S-T Yau, On The Ricci Curvature of a Compact Kahler Manifold and the Complex Monge-Ampère Equation.

Our first goal is to solve the Calabi conjecture:

Theorem (Calabi conjecture)**.** Let M be a compact Kähler manifold with Kähler metric g. Let

$$\tilde{R}_{\alpha\overline{\beta}}\, dz^{\alpha} \otimes d\overline{z}^{\beta}$$

be a tensor whose associated $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\, dz^{\alpha} \wedge d\overline{z}^{\beta}$ represents $\mathrm{c}_1(M)$. Then we can find a Kähler metric $\tilde{g}$ whose Ricci tensor is given by $\tilde{R}_{\alpha\overline{\beta}}\, dz^{\alpha} \otimes d\overline{z}^{\beta}$.

Furthermore, we can require that $\tilde{g}$ has the same Kähler class as g. In this case, $\tilde{g}$ is unique.

To solve this conjecture, we will see (in Section 4) that it suffices to prove the following theorem:

Theorem. Let $F \in C^{k\geq 3}(M)$ and $⨍_M e^F = 1$. Then there is $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ such that $\tilde{g} = (g_{i\overline{j}} + \varphi_{i\overline{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric and

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = e^F \det(g_{i\overline{j}}).$$

In the first three sections, we are going to use Schauder theory and continuity method to find a solution of this partial differential equation. Hence, we must have the second and third order estimates, which will be completely computed in Section 2 and 3. Similar to what we establish Hodge theory through Gårding's inequality, we can find a solution.

After proving the theorem and the Calabi conjecture, we consider the complex Monge-Ampère equation. In section 5, we will solve the equation

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = |s|^{2k} e^F \det(g_{i\overline{j}}),$$

where s is a nontrivial holomorphic section of a line bundle L. The main difference between these equations is whether the functions on the right-hand side vanish or not. To solve this problem, we consider the equation

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = C_{\varepsilon}(|s|^2 + \varepsilon)^k e^F \det(g_{i\overline{j}}),$$

where C_ε is a suitable constant that will be determined later. Then by estimate the differentiability of φ_ε, we will get a solution when ε tends to zero.

In Section 6 $\sim$ 9, we consider more general right-hand side of the complex Monge-Ampère equation. For instance, we will replace the function $F(x)$ by $F(x,\varphi)$ and apply iteration method to solve it. In the end, we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\varphi)} \det(g_{i\bar{j}}),$$

where $t_i = \sum_{j=1}^{\ell} |s_j|^{2k_j}$ with $k_j \geq 0$ and s_j being a section of some holomorphic line bundle.

2 Estimates up to Second Order

Consider the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}) \tag{2.1}$$

where $F \in C^3(M)$.

We are going to find solutions φ of (2.1) such that $\tilde{g}_{i\bar{j}}\, dz^i \otimes d\overline{z}^j = (g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric on M.

Before proving the existence of φ, we need a priori estimates of φ. Since $F \in C^3(M)$, we assume that $\varphi \in C^5(M)$. We will give second order estimates of φ up to second derivatives under the normalization

$$\int_M \varphi = 0.$$

Differentiating (2.1), we get

$$F_k = \tilde{g}^{i\bar{j}} \left(g_{i\bar{j},k} + \varphi_{i\bar{j}k} \right) - g^{i\bar{j}} g_{i\bar{j},k} = \tilde{g}^{i\bar{j}} \varphi_{;i\bar{j}k}.$$

We differentiate the above equation again and obtain

$$\begin{aligned} F_{k\bar{\ell}} = & -\tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}} \left(g_{t\bar{n},\bar{\ell}} + \varphi_{t\bar{n}\bar{\ell}} \right) \left(g_{i\bar{j},k} + \varphi_{i\bar{j}k} \right) \\ & + \tilde{g}^{i\bar{j}} \left(g_{i\bar{j},k\bar{\ell}} + \varphi_{i\bar{j}k\bar{\ell}} \right) + g^{t\bar{j}} g^{i\bar{n}} g_{t\bar{n},\bar{\ell}} g_{i\bar{j},k} - g^{i\bar{j}} g_{i\bar{j},k\bar{\ell}}. \\ = & \, \tilde{g}^{i\bar{j}} \varphi_{;i\bar{j}k\bar{\ell}} - \tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{;t\bar{n}\bar{\ell}} \varphi_{;i\bar{j}k}. \end{aligned} \tag{2.2}$$

Let $\widetilde{\triangle}$ be the Laplacian associated with the metric $\tilde{g}$. Then

$$\begin{aligned} \widetilde{\triangle}(\triangle\varphi) &= \tilde{g}^{k\bar{\ell}} \partial_k \overline{\partial}_\ell \left(g^{i\bar{j}} \varphi_{i\bar{j}} \right) \\ &= \tilde{g}^{k\bar{\ell}} g^{i\bar{j}} \varphi_{i\bar{j}k\bar{\ell}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,k\bar{\ell}} \varphi_{i\bar{j}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,k} \varphi_{i\bar{j}\bar{\ell}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,\bar{\ell}} \varphi_{i\bar{j}k}. \end{aligned} \tag{2.3}$$

Since M is Kähler, we may take $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j},k} = g_{i\bar{j},\bar{\ell}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$. Then inserting (2.2) into (2.3), we have

$$\widetilde{\triangle}(\triangle\varphi) = \triangle F + \tilde{g}^{k\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{k\bar{n}\ell} \varphi_{i\bar{j}\bar{\ell}} + \tilde{g}^{i\bar{j}} R_{i\bar{j}\ell\bar{\ell}} - R_{i\bar{i}\ell\bar{\ell}} + \tilde{g}^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}}. \tag{2.4}$$

Since $\tilde{g}^{i\bar{j}} = \delta_{ij}(1+\varphi_{i\bar{i}})^{-1}$,

$$\begin{aligned}
\tilde{g}^{i\bar{j}} R_{i\bar{j}\ell\bar{\ell}} - R_{i\bar{i}\ell\bar{\ell}} + \tilde{g}^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}} &= -R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} + R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} \\
&= -R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \\
&= \frac{1}{2}\left(-R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} - R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{\ell\bar{\ell}}(\varphi_{i\bar{i}} - \varphi_{\ell\bar{\ell}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \right) \\
&= \frac{1}{2} R_{i\bar{i}\ell\bar{\ell}} \frac{(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \\
&\geq \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1}{2} \cdot \frac{(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \right) \\
&= \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right).
\end{aligned}$$

Combining (2.4) and the above equation, we see that

$$\widetilde{\triangle}(\triangle\varphi) \geq \triangle F + \tilde{g}^{k\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{k\bar{n}\bar{\ell}} \varphi_{i\bar{j}\ell} + \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right). \tag{2.5}$$

Let C be a positive constant. We want to estimate $e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi))$. Using (2.5) and Schwarz inequality, we have

$$\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &= \widetilde{\triangle}(\triangle\varphi) + C^2|\widetilde{\nabla}\varphi|^2(m+\triangle\varphi) \\
&\quad - C\left(2\langle \widetilde{\nabla}\varphi, \widetilde{\nabla}(\triangle\varphi)\rangle + (\widetilde{\triangle}\varphi)(m+\triangle\varphi) \right). \\
&\geq \widetilde{\triangle}(\triangle\varphi) - \frac{|\widetilde{\nabla}(\triangle\varphi)|^2}{m+\triangle\varphi} - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi) \\
&\geq \triangle F + \frac{\varphi_{k\bar{i}\bar{j}}\varphi_{ik\bar{j}}}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})} - \frac{1}{m+\triangle\varphi} \sum_i \frac{|\sum \varphi_{k\bar{k}i}|^2}{1+\varphi_{i\bar{i}}} \\
&\quad + \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right) - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi).
\end{aligned} \tag{2.6}$$

By Schwarz inequality,

$$\begin{aligned}
\frac{1}{m+\triangle\varphi} \sum_i \frac{|\sum \varphi_{k\bar{k}i}|^2}{1+\varphi_{i\bar{i}}} &\leq \frac{1}{m+\triangle\varphi} \left(\sum \frac{\varphi_{k\bar{k}i}\varphi_{\bar{k}k\bar{i}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \right) \sum (1+\varphi_{k\bar{k}}) \\
&\leq \frac{\varphi_{k\bar{i}\bar{j}}\varphi_{i\bar{k}j}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})}.
\end{aligned} \tag{2.7}$$

Inserting the above equation into (2.6), we obtain

$$e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) \geq \triangle F + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right) - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi).$$

Note that

$$\widetilde{\triangle}\varphi = \sum \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} = m - \sum \frac{1}{1+\varphi_{i\bar{i}}}.$$

So, we get

$$
\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \cdot \left(\sum \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}}\right) \\
&\quad - Cm(m+\triangle\varphi) + C(m+\triangle\varphi)\sum \frac{1}{1+\varphi_{i\bar{i}}}. \\
&= \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi)\sum \frac{1}{1+\varphi_{i\bar{i}}}. \qquad (2.8)
\end{aligned}
$$

By AM-GM inequality,

$$
\sum \frac{1}{1+\varphi_{i\bar{i}}} \geq \left(\frac{\sum(1+\varphi_{i\bar{i}})}{\prod(1+\varphi_{i\bar{i}})}\right)^{1/(m-1)} = (m+\triangle\varphi)^{1/(m-1)} e^{-F/(m-1)}. \qquad (2.9)
$$

Choose C so that

$$
C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq 1.
$$

Then

$$
\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{-F/(m-1)}(m+\triangle\varphi)^{1+1/(m-1)}. \qquad (2.10)
\end{aligned}
$$

By maximum principle, at some point x that $e^{-C\varphi}(m+\triangle\varphi)$ achieve its maximum, we have

$$
\begin{aligned}
0 &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{-F/(m-1)}(m+\triangle\varphi)^{1+1/(m-1)}.
\end{aligned}
$$

Hence $(m+\triangle\varphi)(x)$ has an upper bound C_1 depending only on $\sup(-\triangle F)$, $\sup|\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}|$, Cm and $\sup F$.

Since $e^{-C\varphi}(m+\triangle\varphi)$ achieves its maximum at x, we have the following inequality

$$
0 < m + \triangle\varphi \leq C_1 e^{C(\varphi - \inf\varphi)}. \qquad (2.11)
$$

We want to estimate $\sup|\varphi|$. Since

$$
m + \triangle\varphi = \sum_i (1+\varphi_{i\bar{i}}) = g^{i\bar{j}}\tilde{g}_{i\bar{j}} > 0,
$$

we can estimate $\sup\varphi$ by using the Green's function.

Let $G(p,y)$ be the Green's function of the operator $\triangle$ on M. Let A be a constant (depending only on M) such that $G(p,y)+A\geq 0$. Then

$$\varphi(p) = -\int_M G(p,y)\triangle\varphi(y)\,dy = -\int_M (G(p,y)+A)\triangle\varphi(y)\,dy$$

by the normalization of φ (which gives $\varphi \in \operatorname{Im}\triangle$). Therefore,

$$\sup\varphi \leq m\sup_p \int_M (G(p,y)+A)\,dy.$$

The inequality and the normalization also imply

$$\begin{aligned}
\fint_M |\varphi| &\leq \fint_M |\sup\varphi - \varphi| + \fint_M |\sup\varphi| \\
&\leq 2m\sup_p \int_M (G(p,y)+A)\,dy. \qquad (2.12)
\end{aligned}$$

Let us now give an estimate of $-\inf\varphi$. Choose N large enough so that $N+\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq N/2$. Then, by (2.9),

$$\left(N+\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right) \geq \frac{N}{2}e^{-F/(m-1)}(m+\triangle\varphi)^{m/(m-1)}.$$

There is a constant C_1 depending only on $\sup F$ and m such that

$$\frac{N}{2}e^{-F/(m-1)}(m+\triangle\varphi)^{m/(m-1)} \geq 2Nm(m+\triangle\varphi) - NC_1.$$

Inserting above inequalities into (2.7) with C replaced by N, we get

$$e^{N\varphi}\widetilde{\triangle}(e^{-N\varphi}(m+\triangle\varphi)) \geq \triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_1 + Nm(m+\triangle\varphi).$$

Therefore,

$$\begin{aligned}
&e^{N\varphi+F}\widetilde{\triangle}(e^{-N\varphi}(m+\triangle\varphi)) \\
&\geq e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3\right) + Ne^{\inf F}m(m+\triangle\varphi) \\
&= e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3 + m^2Ne^{\inf F - F}\right) + mNe^{\inf F}\triangle\varphi \\
&= e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3 + m^2Ne^{\inf F - F}\right) + me^{\inf F}(-e^{N\varphi}\triangle e^{-N\varphi} + N^2|\nabla\varphi|^2) \\
&\geq me^{\inf F}(-e^{N\varphi}\triangle e^{-N\varphi} + N^2|\nabla\varphi|^2) - C_2,
\end{aligned}$$

where C_2 depends only on N, F and M. Multiplying the above inequality by $e^{-N\varphi}$ and integrating, we get the inequality

$$\int_M |\nabla e^{-N\varphi/2}|^2 = \frac{N^2}{4}\int_M e^{-N\varphi}|\nabla\varphi|^2 \leq \frac{C_2}{4m}e^{-\inf F}\int_M e^{-N\varphi}.$$

Claim. We have an estimate of $\int_M e^{-N\varphi}$ (depending on N, F and M).

Proof of Claim. We are going to prove this statement by contradiction. Suppose there exists a sequence $\{\varphi_i\}$ satisfying the above inequality and (2.12) such that

$$\lim \int_M e^{-N\varphi_i} = \infty.$$

Then we define

$$e^{-N\tilde{\varphi}_i} = e^{-N\varphi_i}\left(\int_M e^{-N\varphi_i}\right)^{-1} \tag{2.13}$$

so that $\int_M e^{-N\tilde{\varphi}_i} = 1$.

It follows that $\int_M |\nabla e^{-N\tilde{\varphi}_i/2}|^2$ is uniformly bounded from above by a constant. Since $W^{1,2} \subset\subset L^2(M)$, there exists a subsequence of $e^{-N\tilde{\varphi}_i/2}$, which we may assume is itself, converges to $f \in L^2(M)$.

For any $\lambda > 0$,

$$\mathrm{Vol}\{x \mid \lambda \leq e^{-N\tilde{\varphi}_i/2}\} = \mathrm{Vol}\left\{x \;\middle|\; \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i} \leq -\varphi_i\right\}.$$

Since $\lim \int_M e^{-N\varphi_i/2} = \infty$, we conclude that, for i large enough,

$$\begin{aligned}\mathrm{Vol}\{x \mid \lambda \leq e^{-N\tilde{\varphi}_i/2}\} &\leq \mathrm{Vol}\left\{x \;\middle|\; \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i} \leq |\varphi_i|\right\} \\ &\leq \left(\frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i}\right)^{-1}\int_M |\varphi_i|.\end{aligned}$$

By (2.12), $\int_M |\varphi_i|$ is uniformly bounded and thus,

$$\mathrm{Vol}\{x \mid \lambda \leq e^{-N\tilde{\varphi}_i/2}\} \to 0$$

for all $\lambda > 0$. For all $\lambda > 0$, we get

$$\begin{aligned}\mathrm{Vol}\{x \mid \lambda \leq f\} &\leq \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \leq |f - e^{-N\tilde{\varphi}_i/2}|\right\} + \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \leq e^{-N\tilde{\varphi}_i/2}\right\} \\ &\leq \frac{4}{\lambda^2}\int_M |f - e^{-N\tilde{\varphi}_i/2}|^2 + \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \leq e^{-N\tilde{\varphi}_i/2}\right\} \to 0.\end{aligned} \tag{2.14}$$

Since f is the L^2-limit of $e^{-N\tilde{\varphi}_i/2}$, f is zero almost everywhere. This is a contradiction because $\int_M f^2 = 1$. ∎

Using (2.11) and the Schauder estimate, there are constants C_3 and C_4 depending only on M such that

$$\sup |\nabla\varphi| \leq C_3\left(e^{-C\inf\varphi} + \int_M |\varphi|\right) \leq C_4(e^{-C\inf\varphi} + 1). \tag{2.15}$$

We introduce the geodesic ball trick. Let q be a point in M where $\varphi(q) = \inf \varphi$. Then in the geodesic ball, with center q and radius

$$\frac{-\frac{1}{2}\inf \varphi}{C_4(e^{-C \inf \varphi} + 1)},$$

φ is not greater than $\frac{1}{2}\inf\varphi$. Since we may assume $-\inf\varphi$ to be large (otherwise we get an upper bound), we may assume that the radius is smaller than $\mathrm{inj}(M)$. Then we choose N larger so that $N \geq 4mC$. Since

$$\int_B e^{-N\varphi} \geq e^{-N \inf \varphi/2}\,\mathrm{Vol}(B) \gtrsim e^{-N\inf\varphi/2}\left(\frac{-\frac{1}{2}\inf\varphi}{C_4(e^{-C\inf\varphi}+1)}\right)^{2m},$$

we have an estimate of $-\inf\varphi$.

Together with the estimate of $\sup\varphi$, we get an estimate of $\sup|\varphi|$. The inequalities (2.15) and (2.11) then give estimates of $\sup|\nabla\varphi|$ and $\sup(m + \triangle\varphi)$. Since $(\delta_{ij} + \varphi_{i\bar{j}})$ is positive definite, we can find upper estimates of $(1 + \varphi_{i\bar{i}})$ for each i. The equation $\prod_i (1 + \varphi_{i\bar{i}}) = e^F$ then gives a positive lower estimate of $(1 + \varphi_{i\bar{i}})$ for each i. Hence, the metric $\tilde{g}$ is uniformly equivalent to g.

So we get

Proposition 1. Let M be a compact Kähler manifold with metric g. Let φ be a real-valued function in $C^4(M)$ such that $\int_M \varphi = 0$ and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\,dz^i \otimes d\bar{z}^j$ defines another metrix tensor on M. Suppose

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Then there are positive constants $C_1 \sim C_4$, depending on $\inf F$, $\sup F$, $\inf \triangle F$ and M such that $\sup|\varphi| \leq C_1$, $\sup|\nabla\varphi| \leq C_2$ and $C_3 \cdot g \leq \tilde{g} \leq C_4 \cdot g$.

3 Third-Order Estimates

We now estimate the third derivatives $\varphi_{;i\bar{j}k}$ assuming φ solves the equation (2.1) and F is $C^3(M)$. Consider the function

$$S = \sum \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{;i\bar{j}k}\varphi_{;\bar{r}s\bar{t}} \geq 0.$$

We are going to compute $\widetilde{\triangle}S$. We say that

- $A \simeq B$ if $|A-B| \lesssim \sqrt{S}+1$,
- $A \cong B$ if $|A-B| \lesssim S+\sqrt{S}+1$.

Since $\tilde{g}$ is uniformly equivalent to g, we see that $\varphi_{;i\bar{j}k} \simeq 0$.

Claim. Take $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j},k} = g_{i\bar{j},\bar{\ell}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$ at a point. We have the following estimate:

$$\widetilde{\triangle}S \cong \frac{\left|\varphi_{;i\bar{j}k\alpha} - \dfrac{\varphi_{;\bar{i}p\bar{k}}\varphi_{;\bar{p}j\alpha}}{1+\varphi_{;p\bar{p}}}\right|^2 + \left|\varphi_{;i\bar{j}k\alpha} - \dfrac{\varphi_{;\bar{p}ia}\varphi_{;p\bar{j}k} + \varphi_{;\bar{p}ik}\varphi_{;p\bar{j}\alpha}}{1+\varphi_{;p\bar{p}}}\right|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{\alpha\bar{\alpha}})} \tag{3.1}$$

Proof of Claim. Since $\tilde{g}$ is uniformly equivalent to g,

$$\begin{aligned}
\widetilde{\triangle}S &= \tilde{g}^{\alpha\bar{\beta}}S_{\bar{\beta}\alpha} \\
&= \tilde{g}^{\alpha\bar{\beta}}\Big(-\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}\bar{\beta}}\tilde{g}^{k\bar{t}}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad -\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{p}}\tilde{g}^{q\bar{t}}\varphi_{\bar{p}q\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}}\Big)_{\alpha} \\
&\simeq \tilde{g}^{\alpha\bar{\beta}}\Big(-2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{p\bar{q}\beta}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad +\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}})\Big)_{\alpha} \\
&\simeq \tilde{g}^{\alpha\bar{\beta}}\Big(2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad - 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{q\bar{p}\bar{\beta}\alpha}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\alpha}) \\
&\qquad + \tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}p}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\left(\varphi_{p\bar{q}\bar{\beta}\alpha}\varphi_{i\bar{j}k}\varphi_{\bar{r}st} + \varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\alpha}\right) \\
&\qquad - (2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha})(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \varphi_{i\bar{j}k}\varphi_{\bar{r}st\bar{\beta}}) \\
&\qquad +g^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}\alpha} + \varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}\bar{\beta}} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}\alpha})\Big).
\end{aligned} \tag{3.2}$$

From the commutation formula, we have

$$\begin{aligned}\varphi_{i\bar{j}k\bar{\beta}\alpha} &= \varphi_{i\bar{j}\bar{\beta}k\alpha} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{j}\bar{\beta}k} - \varphi_{p\bar{j}}R^{p}_{ik\bar{\beta}}\right)_{\alpha} \\ &= \varphi_{i\bar{\beta}\alpha\bar{j}k} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{\beta}\bar{j}\alpha} - \varphi_{p\bar{\beta}}R^{p}_{i\alpha\bar{j}}\right)_{k} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{j}\bar{\beta}k} - \varphi_{p\bar{j}}R^{p}_{ik\bar{\beta}}\right)_{\alpha} \\ &\simeq \varphi_{i\bar{\beta}\alpha\bar{j}k}. \end{aligned}\tag{3.3}$$

We can see from (2.2) that

$$\tilde{g}^{i\bar{j}}\varphi_{;i\bar{j}k\bar{\ell}} = F_{k\bar{\ell}} + \tilde{g}^{t\bar{j}}\tilde{g}^{i\bar{n}}\varphi_{;t\bar{n}\bar{\ell}}\varphi_{;i\bar{j}k}.\tag{3.4}$$

Differentiating this one more time, we get

$$\tilde{g}^{i\bar{j}}\varphi_{;i\bar{j}k\bar{\ell}s} = \tilde{g}^{i\bar{t}}\tilde{g}^{n\bar{j}}\varphi_{;n\bar{t}s}\varphi_{;i\bar{j}k\bar{\ell}} + F_{k\bar{\ell}s} + \left(\tilde{g}^{t\bar{j}}g^{i\bar{n}}\varphi_{;t\bar{n}\bar{\ell}}\varphi_{;i\bar{j}k}\right)_{s}.$$

By (3.3),

$$\begin{aligned}\tilde{g}^{\alpha\beta}\varphi_{i\bar{j}k\bar{\beta}\alpha} &\simeq \tilde{g}^{\alpha\beta}\varphi_{\alpha\bar{\beta}i\bar{j}k} \\ &= \tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}i\bar{j}} + F_{i\bar{j}k} + \left(\tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i}\right)_{k} \\ &= \tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}i\bar{j}} + F_{i\bar{j}k} - \tilde{g}^{p\bar{a}}\tilde{g}^{b\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} \\ &\quad - \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{a}}\tilde{g}^{b\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}k}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}ik}.\end{aligned}\tag{3.5}$$

Using (3.2), (3.4) and (3.5), we get

$$\begin{aligned}\widetilde{\triangle}S \cong{}& 2\tilde{g}^{\alpha\bar{\beta}}\Big(\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ &\qquad + \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}}\Big) \\ &- 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\Big(F_{q\bar{p}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{t\bar{\beta}}\tilde{g}^{\alpha\bar{n}}\varphi_{t\bar{n}\bar{p}}\varphi_{\alpha\bar{\beta}q}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ &\qquad + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}st} + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}st\alpha}\Big) \\ &+ \tilde{g}^{\alpha\bar{\beta}}\Big(\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{q\bar{s}}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ &\qquad + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}}\Big) \\ &- \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\Big(F_{q\bar{p}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{t\bar{\beta}}\tilde{g}^{\alpha\bar{n}}\varphi_{t\bar{n}\bar{p}}\varphi_{\alpha\bar{\beta}q}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ &\qquad + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}st} + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}st\alpha}\Big) \\ &- \tilde{g}^{\alpha\bar{\beta}}(2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha})(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}}) \\ &+ 2\,\mathrm{Re}\Big(\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{r}s\bar{t}}(\tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}ij} + F_{i\bar{j}k} - \tilde{g}^{p\bar{a}}\tilde{g}^{b\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} \\ &\qquad - \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{a}}\tilde{g}^{b\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}k}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}ik})\Big) \\ &+ \tilde{g}^{\alpha\bar{\beta}}\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}\alpha} + \varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}\bar{\beta}}).\end{aligned}$$

Take a coordinate such that at some point, $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j}k} = g_{i\bar{j}\ell} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$. We get

$$\begin{aligned}
\widetilde{\triangle} S \cong\ & \frac{2\,\varphi_{\bar{i}p\alpha}\varphi_{q\overline{p\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{q}j\bar{k}} + 2\,\varphi_{\bar{k}p\alpha}\varphi_{q\bar{i}\bar{\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{q}j\bar{p}} + \varphi_{\bar{p}q\alpha}\varphi_{j\overline{q\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{i}p\bar{k}}}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{p\bar{p}})(1+\varphi_{q\bar{q}})} \\
& - 2\,\mathrm{Re}\left(\frac{\varphi_{p\bar{i}\bar{\alpha}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{p}j\bar{k}} + \varphi_{j\overline{p\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{i}p\bar{k}\alpha} + \varphi_{\bar{i}p\alpha}\varphi_{i\bar{j}k}\varphi_{\bar{p}j\bar{k}\bar{\alpha}}}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{p\bar{p}})}\right) \\
& + \frac{\left|\varphi_{i\bar{j}k\alpha}\right|^2 + \left|\varphi_{i\bar{j}k\bar{\alpha}}\right|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})} \\
=\ & \frac{\left|\varphi_{\bar{i}j\bar{k}\alpha} - \frac{\varphi_{\bar{i}p\bar{k}}\varphi_{\bar{p}j\alpha}}{1+\varphi_{p\bar{p}}}\right|^2 + \left|\varphi_{i\bar{j}k\alpha} - \frac{\varphi_{\bar{p}i\bar{\alpha}}\varphi_{p\bar{j}k} + \varphi_{\bar{p}i\bar{k}}\varphi_{p\bar{j}\alpha}}{1+\varphi_{p\bar{p}}}\right|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})}.
\end{aligned}$$

□

By (2.5) and Proposition 1,

$$\widetilde{\triangle}(\triangle\varphi) \geq \sum \frac{\left|\varphi_{ki\bar{j}}\right|^2}{(1+\varphi_{k\bar{k}})\,(1+\varphi_{i\bar{i}})} - C_1,$$

where C_1 is a constant that can be estimated. Take C_2 large enough, we get

$$\widetilde{\triangle}\,(S + C_2\triangle\varphi) \geq -C_3(S + \sqrt{S} + 1) + C_2(C_4 S - C_1) \geq C_5 S - C_6,$$

where $C_2 \sim C_6$ are positive constants that can be estimated.

Using maximum principle, we see that

$$C_5\,(S + C_2\triangle\varphi) \leq C_6 + C_5 C_2 \triangle\varphi.$$

The estimate on $\triangle\varphi$ then gives an estimate of $\sup\,(S + C_2\triangle\varphi)$ and hence of $\sup S$. Finally, we get the estimates of $\varphi_{;i\bar{j}k}$ for all i, j, k.

Proposition 2. Let M be a compact Kähler manifold with metric g. Let φ be a real-valued function in $C^5(M)$ such that $\int_M \varphi = 0$ and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\,dz^i \otimes d\overline{z}^j$ defines another metric on M. Suppose

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Then there is an estimate of $\varphi_{;i\bar{j}k}$ in terms of g, $\sup|F|$, $\sup|\nabla F|$, $\sup\left(\sup_{i,j}\left|F_{i\bar{j}}\right|\right)$ and $\sup\left(\sup_{i,j,k}\left|F_{i\bar{j}k}\right|\right)$.

4 Solutions of the Equation

So we are going to solve the equation

$$\det(\tilde{g}_{i\bar{j}}) = e^F \det(g_{i\bar{j}}), \tag{4.1}$$

where F satisfies

$$⨍ e^F = 1. \tag{4.2}$$

With the estimates of Section 2 and Section 3, we shall now prove that if $F \in C^k(M)$ with $k \geq 3$ and F satisfies (4.2), then we can find a solution φ of (4.1) where $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$. ($C^{k+1,\alpha}(M)$ are the functions whose $(k+1)$-derivatives are Hölder continuous with exponent α.) We are going to use the continuity method. Consider the set

$$S = \left\{ t \in [0,1] \;\middle|\; \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{e^{tF}\det(g_{i\bar{j}})} = \left(⨍_M e^{tF} \right)^{-1} \text{ has a solution in } C^{k+1,\alpha}(M) \right\}.$$

Since $0 \in S$, we need only to show that S is both closed and open in $[0,1]$.

S is open: Let

$$U = \left\{ \varphi \in C^{k+1,\alpha}(M) \;\middle|\; \int_M \varphi = 0 \text{ and } (g_{i\bar{j}} + \varphi_{i\bar{j}}) \text{ is positive definite.} \right\}$$

and

$$B = \left\{ f \in C^{k-1,\alpha}(M) \;\middle|\; ⨍_M f = 1 \right\}.$$

Then U is an open subset of a hyperplane in the Banach space $C^{k+1,\alpha}(M)$ and B is a hyperplane in the Banach space $C^{k-1,\alpha}(M)$. We have a map $G : U \to B$:

$$G(\varphi) = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})}.$$

We see that

$$dG_{\varphi_0} = \frac{\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})}{\det(g_{i\bar{j}})} \triangle_0,$$

where $\triangle_0$ is the Laplacian of the metric $(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\, dz^i \otimes d\bar{z}^j$.

It is well-known that the condition for $\triangle_0 \varphi = f$ to have a weak solution on M is that $\int_M f \, d\mathrm{Vol}_{\varphi_0} = 0$. Hence the condition for

$$\frac{\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})}{\det(g_{i\bar{j}})} \triangle_0 \varphi = f$$

to have a weak solution is that $\int_M f = 0$. The Schauder theory makes sure that $\varphi \in C^{k+1,\alpha}(M)$ when $f \in C^{k-1,\alpha}(M)$, which is exactly the tangent space of B. The solution is unique if we assume that $\int_M \varphi = 0$. Hence dG_{φ_0} is invertible. By the inverse function theorem for Banach spaces, G maps an open neighborhood of φ_0 to an open neighborhood of $G(\varphi_0)$ in B. this proves that S is open.

S is closed: Let $\{t_q\}$ be a sequence in S with limit $t_0 \in [0,1]$. Then we have a sequence $\varphi_q \in C^{k+1,\alpha}(M)$ such that

$$\det\left(g_{i\bar{j}} + \varphi_{q,i\bar{j}}\right) = \left(\fint_M e^{t_q F}\right)^{-1} \cdot e^{t_q F} \det(g_{i\bar{j}}) \quad \text{and} \quad \int_M \varphi_q = 0.$$

Differentiating the above equation (in direction ∂_p), we have

$$\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right) \varphi_{q,p} = \left(\fint_M e^{t_q F}\right)^{-1} \cdot \partial_p(e^{t_q F} \det(g_{i\bar{j}})). \tag{4.3}$$

Proposition 1 and Proposition 2 shows that the operator $\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right)$ is uniformly elliptic and the coefficients are Hölder continuous with exponent α for any $0 \leq \alpha \leq 1$.

Using the Schauder estimate, we get an estimate on the $C^{2,\alpha}$-norm of $\varphi_{q,p}$ (and $\varphi_{q,\overline{p}}$ similarly). So the coefficients of $\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right)$ have better differentiability. The Schauder estimate now gives better differentiability of $\varphi_{q,p}$ and $\varphi_{q,\overline{p}}$.

Iterating the process, we get $C^{k+1,\alpha}$-estimates of φ_q (since $F \in C^k(M)$). So the sequence $\{\varphi_q\}$ converges in the $C^{k+1,\alpha}$-norm for $\alpha \in [0,1)$ (by the compact embedding $C^{k+1,1} \to C^{k+1,\alpha}$) to a solution φ_0 of the equation

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{e^{t_0 F} \det(g_{i\bar{j}})} = \left(\fint_M e^{t_0 F}\right)^{-1}.$$

Hence S is closed.

Theorem 1. Assume that M is a compact Kähler manifold with metric g. Let F be $C^k(M)$ with $k \geq 3$ and $\fint_M e^F = 1$. Then there is a function φ in $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ such that $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric and

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Corollary (Calabi conjecture)**.** Let M be a compact Kähler manifold with Kähler metric g. Let $\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$ be a tensor whose associated $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta$

represents $\mathrm{c}_1(M)$. Then we can find a Kähler metric $\tilde{g}$ whose Ricci tensor is given by $\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$. Furthermore, we can require that this Kähler metric has the same Kähler class as the original one. In this case, the required Kähler metric is unique.

Note that

$$R_{\alpha\overline{\beta}} = -\partial_\alpha \overline{\partial}_\beta \log\det(g_{i\overline{j}}). \tag{4.4}$$

Since we assume that $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta$ represents $\mathrm{c}_1(M)$, we see that

$$\tilde{R}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - \partial_\alpha \overline{\partial}_\beta f \tag{4.5}$$

for some smooth real-valued function f.

By Theorem 1, we can find a smooth function φ so that $(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}})\, dz^\alpha \otimes d\overline{z}^\beta$ defines a Kähler metric and that

$$\det(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}) = Ce^f \det(g_{\alpha\overline{\beta}}), \tag{4.6}$$

where C is a constant chosen to satisfy the equation

$$⨍_M Ce^f = 1.$$

From (4.4), (4.5) and (4.6), it is easy to see that $\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$ is the Ricci tensor of $(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}})\, dz^\alpha \otimes d\overline{z}^\beta$. This proves the Calabi conjecture.

Remark. The uniqueness was proved by Calabi and will also be indicated and proved in Theorem 2.

5 Complex Monge-Ampère Equation with Degenerate Right-Hand Side

Let L be a line bundle over M. Let s be a nontrivial holomorphic section of L. Suppose L is equipped with a Hermitian metric. Then we have a globally defined function $|s|^2$ on M.

For $k \geq 0$, we consider the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = |s|^{2k} e^F \det(g_{i\bar{j}}), \tag{5.1}$$

where F is a smooth function such that

$$\fint_M |s|^{2k} e^F = 1.$$

In order to solve (5.1), we approximate the equation by

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon (|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}}), \tag{5.2}$$

where $\varepsilon > 0$ is a small constant and

$$C_\varepsilon = \left(\fint_M (|s|^2 + \varepsilon)^k e^F \right)^{-1} \leq \left(\fint_M |s|^{2k} e^F \right)^{-1} = 1.$$

By Theorem 1, (5.2) has a smooth solution φ_ε such that $\left(g_{i\bar{j}} + \varphi_{\varepsilon, i\bar{j}}\right)$ is positive definite and

$$\int_M \varphi_\varepsilon = 0. \tag{5.3}$$

We are going to prove that when $\varepsilon \to 0^+$, φ_ε tends to a solution of (5.1). So we need some estimates of φ_ε which are independent of ε.

To estimate $\inf \varphi_\varepsilon$ and $\triangle \varphi_\varepsilon$ we notice that, when $s \neq 0$,

$$\triangle \log(|s|^2 + \varepsilon) = \frac{\triangle |s|^2}{|s|^2 + \varepsilon} - \frac{|\nabla |s|^2|^2}{(|s|^2 + \varepsilon)^2} \geq \frac{|s|^2}{|s|^2 + \varepsilon} \cdot \triangle \log |s|^2 \geq - \left| \triangle \log |s|^2 \right|. \tag{5.4}$$

Since $\triangle \log |s|^2$ is the trace of $\mathrm{c}_1(L)$ with respect to g for $s \neq 0$, we see that $\triangle \log(|s|^2 + \varepsilon)$ is uniformly bounded from below. Note that both sides of the above inequality are smooth. By taking limit to the points where $|s|^2$ vanish, we see that the above inequality holds on M.

Let $\triangle_\varepsilon$ be the Laplacian of the metric g_ε. Then according to (2.10), we have

$$e^{C\varphi_\varepsilon}\triangle_\varepsilon(e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)) \geq k\triangle\log(|s|^2+\varepsilon)+\triangle F - m^2\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}} - mC(m+\triangle\varphi_\varepsilon)$$
$$+ C_\varepsilon^{-1/(m-1)}\left(C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{(m+\triangle\varphi_\varepsilon)^{1+1/(m-1)}}{e^{F/(m-1)}(|s|^2+\varepsilon)^{k/(m-1)}}. \quad (5.5)$$

Same as in Section 2, we get

$$m+\triangle\varphi_\varepsilon \lesssim e^{C(\varphi_\varepsilon-\inf\varphi_\varepsilon)}. \quad (5.6)$$

For $s\neq 0$, $\triangle_\varepsilon \log|s|^2$ is dominated from below by the trace of $\mathrm{c}_1(L)$ with respect to g_ε. Hence there is a positive constant C_1 independent of ε such that

$$\triangle_\varepsilon\log|s|^2 \geq -C_1\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}. \quad (5.7)$$

Let p be any non-negative number. Then by Schwarz inequality, when $C>pC_1$,

$$\begin{aligned}
e^{C\varphi_\varepsilon}\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^p\right) &= \triangle_\varepsilon(|s|^2+\varepsilon)^p + 2\left\langle\nabla_\varepsilon(|s|^2+\varepsilon)^p, \nabla_\varepsilon e^{-C\varphi_\varepsilon}\right\rangle\\
&\quad + (|s|^2+\varepsilon)^p\left(\left|\nabla_\varepsilon e^{-C\varphi_\varepsilon}\right|^2 - C\triangle_\varepsilon\varphi_\varepsilon\right)\\
&\geq \triangle_\varepsilon(|s|^2+\varepsilon)^p - \frac{|\nabla_\varepsilon(|s|^2+\varepsilon)^p|^2}{(|s|^2+\varepsilon)^p} - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon\\
&= (|s|^2+\varepsilon)^p\triangle_\varepsilon\log(|s|^2+\varepsilon)^p - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon\\
&\geq -pC_1(|s|^2+\varepsilon)^p\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}} - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon\\
&= (C-pC_1)(|s|^2+\varepsilon)^p\sum_i\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}} - mC(|s|^2+\varepsilon)^p\\
&\geq m(C-pC_1)\frac{(|s|^2+\varepsilon)^{p-k/m}}{C_\varepsilon^{1/m}e^{F/m}} - mC(|s|^2+\varepsilon)^p,
\end{aligned}$$

where the last inequality is due to the AM-GM inequality. Multiplying the above inequality by $(|s|^2+\varepsilon)^k e^{F-C\varphi_\varepsilon}$ and integrating, we get

$$\begin{aligned}
Ce^{\sup F}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p} &\geq C\int_M e^{F-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p}\\
&\geq (C-pC_1)C_\varepsilon^{-1/m}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{(m-1)k/m+p}e^{(m-1)F/m}\\
&\gtrsim (C-pC_1)\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{(m-1)k/m+p}.
\end{aligned}$$

By the above inequality, we see that, for all $q\in\left[\frac{m-1}{m}k+p, k+p\right]$, there exists a positive constant C_2 such that

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^q \leq C_2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p}.$$

Hence, for $n \in \mathbb{N}$ such that $p - \frac{(n-1)k}{m} \geq 0$,

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{nk}{m}} \lesssim \cdots \lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{k}{m}} \lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p},$$

Let n be the largest integer so that $p - \frac{(n-1)k}{m} \geq 0$. Then we have $k \in \big[k+p-\frac{nk}{m}, k+p-\frac{(n-1)k}{m}\big]$ and hence,

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k} \leq C_3' \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{nk}{m}} \leq \cdots \leq C_3 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p}. \quad (5.8)$$

for some $C_3, C_3' > 0$. By (5.5), we can find positive constants C_4 and C_5 such that

$$e^{C\varphi_\varepsilon} \triangle_\varepsilon \left(e^{-C\varphi_\varepsilon}\left(m + \triangle\varphi_\varepsilon\right)\right) \geq C_4\left(m+\triangle\varphi_\varepsilon\right) - C_5.$$

Multiplying the above inequality by $(|s|^2+\varepsilon)^k e^{F-C\varphi_\varepsilon}$ and integrating, we obtain

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k\left(m+\triangle\varphi_\varepsilon\right) \leq \frac{C_5\, e^{\sup F - \inf F}}{C_4} \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.$$

Since $m + \triangle\varphi_\varepsilon > 0$, it follows from the above inequality that we can find a positive constant C_6 independent of ε (for ε small) such that

$$\begin{aligned}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\triangle\varphi_\varepsilon &\leq \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}(m+\triangle\varphi_\varepsilon)\\ &\leq C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

Integrating by parts in the above inequality, we derive

$$\begin{aligned}C\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\left|\nabla\varphi_\varepsilon\right|^2 &\leq (k+1)\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k\left\langle\nabla\varphi_\varepsilon, \nabla|s|^2\right\rangle\\ &\quad + C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k\\ &\leq \frac{(k+1)^2}{C}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2\\ &\quad + \frac{1}{4}C\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\left|\nabla\varphi_\varepsilon\right|^2\\ &\quad + C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{3}{4}C^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\left|\nabla\varphi_\varepsilon\right|^2 &\leq (k+1)^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2\\ &\quad + CC_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

On $|s| \neq 0$,

$$\left|\nabla|s|^2\right|^2 = |s|^2\triangle|s|^2 - |s|^4\triangle\log|s|^2.$$

Note that $\triangle|s|^2$ and $|s|^2$ are upper bounded and $\triangle\log|s|^2$ is lower bounded. So we see that

$$\left|\nabla|s|^2\right|^2 \leq \left(\sup\triangle|s|^2 + \max\{\sup|s|^2\cdot\sup(-\triangle\log|s|^2), 0\}\right)\cdot|s|^2.$$

Since both side are smooth on M, we see that $\left|\nabla|s|^2\right|^2$ is dominated by $|s|^2$ on M. Together with (5.8), we see that

$$\begin{aligned}
&\int_M \left|\nabla\left(e^{-C\varphi_\varepsilon/2}(|s|^2+\varepsilon)^{(k+1)/2}\right)\right|^2 \\
&\leq \frac{1}{2}(k+1)^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2 + \frac{1}{2}C^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\left|\nabla\varphi_\varepsilon\right|^2 \\
&\leq \frac{7}{6}(k+1)^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2 + \frac{2}{3}CC_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k} \\
&\lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k} \\
&\lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}.
\end{aligned} \tag{5.9}$$

Using the Green's function as before, we get an estimate of $\int_M|\varphi_\varepsilon|$ that is independent of ε, we apply the normalization trick in Section 2 that (5.9) gives an estimate of

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}$$

independent of ε. (Suppose there is no estimate. Then we can find a sequence $\varepsilon_j \to 0$ such that $\int_M e^{-C\varphi_{\varepsilon_j}}(|s|^2+\varepsilon_{\varepsilon_j})^{k+1}$ tends to infinity. Then we define

$$e^{-C\tilde{\varphi}_j} = e^{-C\varphi_{\varepsilon_j}}\left(\int_M e^{-C\varphi_{\varepsilon_j}}\left(|s|^2+\varepsilon_j\right)^{k+1}\right)^{-1}.$$

By (5.9), $(|s|^2+\varepsilon_j)^{(k+1)/2}e^{-\frac{1}{2}C\tilde{\varphi}_j}$ converges to some f in $L^2(M)$. Using the L^1-estimate of $|\varphi_\varepsilon|$ on the set $\{x\in M \mid |s| \geqslant 1/n\}$, we see that $f\equiv 0$ a.e. and get a contradiction.)

As in (2.15), inequality (5.6) and the estimate of $\sup\varphi_\varepsilon$ give an estimate of

$$\frac{|\nabla\varphi_\varepsilon|}{e^{-C\inf\varphi_\varepsilon}+1}$$

independent of ε. Now we use the geodesic ball trick. For some geodesic ball B of radius

$$R = \frac{C_7(-\inf\varphi_\varepsilon)}{e^{-C\inf\varphi_\varepsilon}+1},$$

φ_ε is not greater than $\frac{1}{2}\inf\varphi_\varepsilon$. (Here C_7 is a positive constant independent of ε, and R is less than the injectivity radius of M.) We see that

$$\begin{aligned}\int_B e^{-N\inf\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1} &\geq e^{-N\inf\varphi_\varepsilon/2}\int_B |s|^{2(k+1)} \\ &\gtrsim e^{-N\inf\varphi_\varepsilon/2}\int_0^R r^{a(k+1)}dr \\ &\geq \frac{1}{2a(k+1)}e^{-N\inf\varphi_\varepsilon/2}\left(\frac{C_7(-\inf\varphi_\varepsilon)}{e^{-C\inf\varphi_\varepsilon}+1}\right)^{ak+a+1}.\end{aligned}$$

By choosing $N > 2C(ak+a+1)$, we get an estimate of $-\inf\varphi_\varepsilon$ independent of ε and (5.6) gives an upper estimate of $m+\triangle\varphi_\varepsilon$ independent of ε.

Now we want to find the third-order estimate. Let $\rho \geq 0$ be a smooth function in M with $\operatorname{supp}\rho \subseteq K$. Since $(|s|^2+\varepsilon)^k e^F$ has a uniform lower bound over K, the metric g_ε is uniformly equivalent to g.

As in Section 3, we define

$$S_\varepsilon = g_\varepsilon^{i\bar r} g_\varepsilon^{\bar j s} g_\varepsilon^{k\bar t}\varphi_{\varepsilon;i\bar j k}\varphi_{\varepsilon;\bar r s\bar t}.$$

From (2.6), we can find positive constants C_8 and C_9 independent of ε such that

$$\rho\,\triangle_\varepsilon\left(\triangle\varphi_\varepsilon\right) \geq C_8\rho S_\varepsilon - C_9\rho$$

Integrating the above inequality with respect to the volume form $(|s|^2+\varepsilon)^k e^F\,d\mathrm{Vol}$, we see that

$$C_8\int_M \rho S_\varepsilon(|s|^2+\varepsilon)^k e^F \leq C_9\int_M \rho(|s|^2+\varepsilon)^k e^F + \int_M \triangle_\varepsilon\rho\cdot\triangle\varphi_\varepsilon\cdot(|s|^2+\varepsilon)^k e^F.$$

Note that the RHS can be estimated. Since $\inf|s|>0$ on K, we can find an estimate of $\int_M \rho S_\varepsilon$ independent of ε.

Since the compact set K and the function ρ are chosen arbitrary, we see that we have found an L^1-estimate of S_ε over any compact subset K of M which is disjoint from the divisor of s. Say

$$\int_K S_\varepsilon < C_K, \tag{5.10}$$

where C_K is independent to ε.

Let

$$B(R) = \left\{(z_1,\cdots,z_m)\;\middle|\;\sum_i |z_i|^2 \leq R\right\} \subseteq K$$

be a coordinate chart. We want to estimate $S_\varepsilon(0)$ by the L_1-norm of S_ε over $B(R)$.

Using the computations of Section 3, we know that there are positive constants C_{10} and C_{11} independent of ε and φ_ε, such that on $B(R)$,

$$\triangle_\varepsilon \left(S_\varepsilon + C_{10}\triangle\varphi_\varepsilon + C_{11}\sum_i |z_i|^2 \right) \geq C_{12}S_\varepsilon - C_{13} + C_{11}\triangle_\varepsilon \left(\sum_i |z_i|^2 \right) > 0.$$

We may also assume that the function $\overline{S}_\varepsilon = S_\varepsilon + C_{10}\triangle\varphi_\varepsilon + C_{11}\left(\sum_i |z_i|^2 + 1\right) > 0$.

The Dirichlet problem

$$\begin{cases} \triangle_\varepsilon \psi = 0 & \text{on } B(R), \\ \psi = \overline{S}_\varepsilon & \text{on } \partial B(R). \end{cases}$$

has a smooth solution $\tilde{S}_\varepsilon$. By the maximum principle, $\tilde{S}_\varepsilon \geq \overline{S}_\varepsilon > 0$ in $B(R)$.

Since g_ε is uniformly equivalent to g on $B(R)$, we know that $\tilde{S}_\varepsilon$ is a solution of a uniform elliptic equation of divergence form whose elliplicity is estimated (this means that the eigenvalues have a uniform bound).

By Moser's Harnack inequality

$$\sup_{B(R)} \tilde{S}_\varepsilon \lesssim \inf_{B(R)} \tilde{S}_\varepsilon,$$

we get

$$\tilde{S}_\varepsilon(0) \lesssim \int_{B(R)} \tilde{S}_\varepsilon. \tag{5.11}$$

Let σ be a non-decreasing C^∞-function defined on $\mathbb{R}$ such that

(i) $\sigma(t) = 0$ for $t \leq 0$,

(ii) $\sigma(t) = 1$ for $t \geq \delta$ and

(iii) $\sigma'(t) \leq \frac{2}{\delta}$ for all t.

For $\tau < R$, we define $\psi_\tau(s) = \int_s^\infty t\sigma(\tau - t)\,dt$. We see that $\psi_\tau(r) = \psi_\tau((\sum_i |z_i|^2)^{1/2})$ vanishes outside a compact subset of the interior of $B(R)$.

By direct computation, we have

$$\begin{aligned} \triangle_\varepsilon \psi_\tau(r) = g_\varepsilon^{i\bar{j}} \partial_i \overline{\partial}_j \psi_\tau(r) &= r\sigma'(\tau - r) g_\varepsilon^{i\bar{j}} (\partial_i r)(\overline{\partial}_j r) - \frac{1}{2}\sigma(\tau - r) g_\varepsilon^{i\bar{j}} \partial_i \overline{\partial}_j r^2 \\ &= r\sigma'(\tau - r) g_\varepsilon^{i\bar{j}} (\partial_i r)(\overline{\partial}_j r) - \frac{1}{2}\sigma(\tau - r) g_\varepsilon^{i\bar{i}}. \end{aligned}$$

Multiplying the above equation by $\tilde{S}_\varepsilon \det(g_{\varepsilon p\bar{q}})$ and integrating with respect to the Euclidean volume form dE, we obtain (by integration by parts)

$$
\begin{aligned}
0 &= \int_{B(R)} (\triangle_\varepsilon \tilde{S}_\varepsilon)\psi_\tau(r)\det(g_{\varepsilon p\bar{q}})\,dE = \int_{B(R)} \tilde{S}_\varepsilon\,(\triangle_\varepsilon \psi_\tau(r))\det(g_{\varepsilon p\bar{q}})\,dE \\
&= \int_{B(R)} \tilde{S}_\varepsilon r\sigma'(\tau - r) g_\varepsilon^{i\bar{j}} (\partial_i r)(\overline{\partial}_j r)\det(g_{\varepsilon p\bar{q}})\,dE - \frac{1}{2}\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r) g_\varepsilon^{i\bar{j}} \det(g_{\varepsilon p\bar{q}})\,dE.
\end{aligned}
$$

Since $\sigma \geq 0$, and $\sigma' \geq 0$, it follows from the above equation that

$$
\begin{aligned}
&\frac{1}{2}\inf_{B(R)}\left(g_\varepsilon^{i\bar{j}}\det(g_{\varepsilon p\bar{q}})\right)\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r)\,dE \\
&\leq \sup_{B(R)}\left(r g_\varepsilon^{i\bar{j}}(\partial_i r)(\overline{\partial}_j r)\det(g_{\varepsilon p\bar{q}})\right)\int_{B(R)} \tilde{S}_\varepsilon \sigma'(\tau - r)\,dE.
\end{aligned}
$$

Therefore, by the uniform bound of g_ε, we can find a positive constant C_{14} independent of σ, τ, ε such that

$$
\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r)\,dE \leq C_{14}\int_{B(R)} \tilde{S}_\varepsilon \sigma'(\tau - r)\,dE.
$$

Letting $\tau \to R^-$, we may replace τ by R in the above inequality. Then

$$
\int_{B(R-\delta)} \tilde{S}_\varepsilon dE \leq \frac{2C_{14}}{\delta}\int_{B(R)\backslash B(R-\delta)} \tilde{S}_\varepsilon dE.
$$

Letting $\delta \to 0^+$, we see that $\int_{B(R)} \tilde{S}_\varepsilon dE$ can be estimated by $\int_{\partial B(R)} \tilde{S}_\varepsilon$. Since $\overline{S}_\varepsilon|_{\partial B(R)} = \tilde{S}_\varepsilon|_{\partial B(R)}$ and $\tilde{S}_\varepsilon > 0$, we conclude from (5.11) that there is a positive constant C_{15} independent of φ_ε and ε such that

$$
\overline{S}_\varepsilon(0) \leq \tilde{S}_\varepsilon(0) \leq C_{15}\int_{\partial B(R)} \overline{S}_\varepsilon.
$$

Since C_{15} can be chosen to be independent of R when $B(R)$ lies in K, we can integrate the above inequality (over R) to find an estimate of $\overline{S}_\varepsilon(0)$ in terms of the L^1-norm of $\overline{S}_\varepsilon$ over K. Together with the L^1-estimate (5.10) of $\overline{S}_\varepsilon$, we get an estimate of $\overline{S}_\varepsilon$ on K.

Using the method in Section 4, we can estimate the higher derivatives of φ_ε. Differentiate

$$
\det\left(g_{i\bar{j}} + \varphi_{\varepsilon, i\bar{j}}\right) = C_\varepsilon(|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}})
$$

in direction ∂_k. Then

$$
\left(g_\varepsilon^{i\bar{j}}\partial_i\overline{\partial}_j\right)\varphi_{\varepsilon,k} = \partial_k\left(\log\left(C_\varepsilon(|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}})\right)\right).
$$

Since we have Lipschitz estimates (Hölder exponent 1) of these coefficients over K, the Schauder estimate shows that all higher derivatives of φ_ε can be estimated over these sets.

Since K is arbitrary, by letting $\varepsilon \to 0^+$, we can now conclude that $\{\varphi_\varepsilon\}$ has a subsequence converging to a solution φ of (5.1) such that φ is smooth outside of the divisor of s and $\left\{|\varphi_{i\bar{j}}|\right\}$ is bounded for all i, j.

Theorem 2. Let L be a holomorphic line bundle over a compact Kähler manifold M. Let s be a holomorphic section of L. Let g be the Kähler metric of M. Then, for any $k \geq 0$ and any smooth function F with $⨍_M |s|^{2k} e^F = 1$, we can find a solution φ of the equation

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = |s|^{2k} e^F \det(g_{i\bar{j}})$$

with the following properties:

(i) φ is smooth outside the divisor of s, and

(ii) $\triangle\varphi$ is bounded over M

Furthermore, any function ψ satisfying the above properties must be equal to φ plus a constant.

Proof. We only need to prove the last statement. We claim that, if f is a function such that $\left\{|f_{i\bar{j}}|\right\}$ is bounded over M for all i, j, then

$$\int_M (\widetilde{\triangle} f)|s|^{2k} e^F = 0. \tag{5.12}$$

Indeed, if we let $c(g_\varepsilon)_{i\bar{j}}$ be the $(i, \bar{j})$-th cofactor of the matrix $\left(g_{\varepsilon, i\bar{j}}\right)$, we have

$$\int_M (\triangle_\varepsilon f)\,(|s|^2 + \varepsilon)^k e^F = \int_M c(g_\varepsilon)_{i\bar{j}} f_{i\bar{j}}\, dz^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^m = 0. \tag{5.13}$$

Since $c(g_\varepsilon)_{i\bar{j}}$ and $f_{i\bar{j}}$ are bounded independent of ε, we can use the Lebesgue dominated convergence theorem to obtain (5.12) from (5.13).

Now let ψ be another solution of (5.1) satisfying the properties mentioned in the theorem. Then we have

$$\frac{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi - \varphi)_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)} = 1.$$

Using the AM-GM inequality, we have

$$\widetilde{\triangle}(\psi - \varphi) = \frac{1}{m}\left(m + \widetilde{\triangle}(\psi - \varphi)\right) - 1 \geq 0.$$

Since $\left|\psi_{i\bar{j}}\right|$ and $\left|\varphi_{i\bar{j}}\right|$ are both bounded, $\left|(\psi-\varphi)^2_{i\bar{j}}\right|$ is also bounded over M and $\psi - \varphi \in C^1(M)$. We may assume that $\psi - \varphi \geq 0$ by adding a constant to $\psi - \varphi$. Then applying (5.12) to $f = (\psi - \varphi)^2$, we obtain

$$2\int_M (\psi-\varphi)\widetilde{\triangle}(\psi-\varphi) + 2\int_M \left|\widetilde{\nabla}(\psi-\varphi)\right|^2 = \int_M \widetilde{\triangle}\left((\psi-\varphi)^2\right) = 0.$$

Since $(\psi - \varphi) \geq 0$ and $\widetilde{\triangle}(\psi - \varphi) \geq 0$, we conclude that $\widetilde{\nabla}(\psi - \varphi) = 0$ and $\psi - \varphi$ is a constant. ■

6 Complex Monge-Ampère Equation with More General Right-Hand Side

Consider the following equation:

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{6.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

If such φ exists, then integrating (6.1), the integral of the RHS is equal to the volume of M. So we assume that there exists a smooth function ψ such that

$$\fint_M e^{F(x,\psi)} = 1.$$

We are going to use an iteration method to solve (6.1).

Lemma 1 (Uniqueness of the solution of (6.1))**.** Let φ and ψ be two smooth solutions of (6.1) such that both $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$ and $\left(g_{i\bar{j}} + \psi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$ define Kähler metrics on M. Then $\varphi - \psi$ is a constant.

Proof. Note that

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi - \varphi)_{i\bar{j}})}{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})} = e^{F(x,\psi) - F(x,\varphi)}.$$

Let $\triangle_\varphi$ be the normalized metric Laplacian of the metric $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$. Then it follows from the AM-GM inequality and the above equation that we have the inequality

$$m + \triangle_\varphi(\varphi - \psi) \geq m e^{(F(x,\psi) - F(x,\varphi))/m}.$$

By the mean value theorem we have

$$F(x,\psi) - F(x,\varphi) = \int_{\varphi(x)}^{\psi(x)} F_t(x,\tau)\, d\tau = F_t(x,\bar{t}(x))(\psi(x) - \varphi(x)),$$

where $\bar{t}(x)$ is a number between $\inf\{\varphi(x), \psi(x)\}$ and $\sup\{\varphi(x), \psi(x)\}$.

Since $F_t \geq 0$, we can combine the inequality and the equation above to conclude that whenever $\psi(x) - \varphi(x)$ is strictly positive, $\triangle_\varphi(\psi - \varphi)(x)$ is nonnegative.

Suppose $\sup\,(\psi - \varphi)(x) > 0$. By the maximal principle we see that $\psi - \varphi$ is locally constant on the set $\{x \in M \mid (\psi - \varphi)(x) > 0\}$. Interchanging φ and ψ, we see that $\psi - \varphi$ must be a constant function. ■

We now introduce the iteration method. By Theorem 1, we can find a smooth function φ_0 such that $(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric and

$$\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}}) = e^{F(x,\psi)} \det(g_{i\bar{j}}). \tag{6.2}$$

If we define

$$\varphi_0^{\pm} = \varphi_0 \pm \sup |\varphi_0 - \psi|,$$

then both φ_0^+ and φ_0^- satisfy the equation.

The set $A = \{(x,t) \mid x \in M, \varphi_0^+(x) \geq t \geq \varphi_0^-(x)\}$ is a compact subset of $M \times \mathbb{R}$. Hence we can define

$$k = \sup_{(x,t)\in A} F_t(x,t) + 1 > 0.$$

For each $i \geq 1$, we define φ_i^+ and φ_i^- as the smooth solutions of the following equations:

$$\det\big(g_{\alpha\overline{\beta}} + \varphi^{\pm}_{i,\alpha\overline{\beta}}\big) = e^{k(\varphi_i^{\pm} - \varphi_{i-1}^{\pm}) + F(x,\varphi_{i-1}^{\pm})} \det(g_{\alpha\overline{\beta}}) \tag{6.3}$$

so that $g_i^{\pm} = \big(g_{\alpha\overline{\beta}} + \varphi^{\pm}_{i,\alpha\overline{\beta}}\big)\, dz^{\alpha} \otimes d\overline{z}^{\beta}$ define Kähler metrics.

Lemma 2 (Existence of $\varphi_i^{\pm}$)**.** Let M be a compact Kähler manifold with Kähler metric g. Let $F(x)$ be any smooth function defined on M. Then, for any constant $\overline{k} > 0$, there exists a unique smooth function φ such that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\overline{k}\varphi + F} \det(g_{i\bar{j}})$$

and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric.

Proof. As in Theorem 1, we can use the continuation method where the one parameter family (with parameter t) of equations is

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\overline{k}\varphi + tF} \det(g_{i\bar{j}}).$$

By maximum principle and AM-GM inequality, when φ achieves its maximum at a point x_0, we must have

$$e^{\overline{k}\varphi(x_0) + tF(x_0)} = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} \leq 1.$$

This implies immediately $\sup \varphi \leq -(t/\overline{k})F(x_0)$. Similarly one can draw an estimate of $\inf \varphi$. Since $\overline{k} > 0$, the uniqueness part follows from Lemma 1. ■

Claim. For all $i \geq 0$, $\varphi_i^- \leq \varphi_{i+1}^- \leq \varphi_{i+1}^+ \leq \varphi_i^+$.

Proof of Claim. The proof is almost based on the maximum principle and AM-GM inequality. We induction on i. For $i=0$, we see that

$$\begin{aligned}\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{1,\alpha\overline{\beta}}\big) &= e^{k\varphi_1^+-k\varphi_0^+}e^{F(x,\varphi_0^+)}\det(g_{\alpha\overline{\beta}})\\ &\geq e^{k(\varphi_1^+-\varphi_0^+)}e^{F(x,\psi)}\det(g_{\alpha\overline{\beta}}) = e^{k(\varphi_1^+-\varphi_0^+)}\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{0,\alpha\overline{\beta}}\big).\end{aligned}$$

At the point where $\varphi_1^+-\varphi_0^+$ achieves its maximum, by AM-GM inequlity,

$$\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{1,\alpha\overline{\beta}}\big)\leq\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{0,\alpha\overline{\beta}}\big).$$

Hence $\sup(\varphi_1^+-\varphi_0^+)\leq 0$. Similarly, $\sup(\varphi_0^- -\varphi_1^-)\leq 0$.

To show that $\varphi_1^-\leq\varphi_1^+$, by (6.3) we see that

$$\frac{\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{1,\alpha\overline{\beta}}\big)}{\det\big(g_{\alpha\overline{\beta}}+\varphi^-_{1,\alpha\overline{\beta}}\big)} = e^{k(\varphi_1^+-\varphi_1^-)+F(x,\varphi_0^+)-F(x,\varphi_0^-)-k(\varphi_0^+-\varphi_0^-)}.$$

Since $\varphi_0^+\geq\varphi_0^-$, by mean value theorem we get

$$F(x,\varphi_0^+)-F(x,\varphi_0^-)-k(\varphi_0^+-\varphi_0^-)\leq 0.$$

Therefore

$$\frac{\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{1,\alpha\overline{\beta}}\big)}{\det\big(g_{\alpha\overline{\beta}}+\varphi^-_{1,\alpha\overline{\beta}}\big)}\leq e^{k(\varphi_1^+-\varphi_1^-)}.$$

At the point where $\varphi_1^+-\varphi_1^-$ achieves its minimum, (by maximum principle and AM-GM inequality,) the RHS of the above inequality is greater than or equal to 1 and hence $\varphi_1^+\geq\varphi_1^-$.

For general i. Applying (6.3) twice, we have

$$\frac{\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{i+1,\alpha\overline{\beta}}\big)}{\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{i,\alpha\overline{\beta}}\big)} = e^{k(\varphi_{i+1}^+-\varphi_i^+)+F(x,\varphi_i^+)-F(x,\varphi_{i-1}^+)-k(\varphi_i^+-\varphi_{i-1}^+)}\geq e^{k(\varphi_{i+1}^+-\varphi_i^+)},$$

where the inequality is due to MVT. Hence the maximal principle shows that $\varphi_i^+\geq\varphi_{i+1}^+$. Similarly one can show that $\varphi_i^-\leq\varphi_{i+1}^-$.

To prove that $\varphi_{i+1}^+\geq\varphi_{i+1}^-$, by (6.3) we see that

$$\frac{\det\big(g_{\alpha\overline{\beta}}+\varphi^+_{i+1,\alpha\overline{\beta}}\big)}{\det\big(g_{\alpha\overline{\beta}}+\varphi^-_{i+1,\alpha\overline{\beta}}\big)} = e^{k(\varphi_{i+1}^+-\varphi_{i+1}^-)+F(x,\varphi_i^+)-F(x,\varphi_i^-)-k(\varphi_i^+-\varphi_i^-)}.$$

Using $\varphi_i^+\geq\varphi_i^-$, one can repeat the above argument to show that $\varphi_i^+\geq\varphi_i^-$. □

Therefore both φ_i^+ and φ_i^- are uniformly bounded. Again, we want to find a uniform estimate of $\varphi^+_{i,\alpha\overline{\beta}}$. As in Section 2, it suffices to estimate $\triangle\varphi_i^+$.

Let $\triangle_i^+$ be the Laplacian operator associated with the metric g_i^+. Let C be any positive constant such that $C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} > 1$. Then by same computation as in (2.8), we have

$$\begin{aligned}
e^{C\varphi_i^+}\triangle_i^+(e^{-C\varphi_i^+}(m+\triangle\varphi_i^+)) = {} & k(\triangle\varphi_i^+ - \triangle\varphi_{i-1}^+) + g^{\alpha\overline{\beta}}F_{\alpha\overline{\beta}}(x,\varphi_{i-1}^+) \\
& + g^{\alpha\overline{\beta}}F_{t\alpha}(x,\varphi_{i-1}^+)\varphi_{i-1,\overline{\beta}}^+ + g^{\alpha\overline{\beta}}F_{t\overline{\beta}}(x,\varphi_{i-1}^+)\varphi_{i-1,\alpha}^+ \\
& + F_{tt}(x,\varphi_{i-1}^+)|\nabla\varphi_{i-1}^+|^2 + F_t(x,\varphi_{i-1}^+)\triangle\varphi_{i-1}^+ \\
& - Cm(m+\triangle\varphi_i^+) \\
& + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi_i^+)\sum\frac{1}{1+\varphi_{i,\alpha\overline{\alpha}}^+}.
\end{aligned}$$

Since $\sup|\varphi_i^+|$ has been estimated, it follows from Schauder's estimate that

$$\sup|\nabla\varphi_i^+| \lesssim \left(\sup|\triangle\varphi_i^+| + 1\right).$$

As in (2.9),

$$\begin{aligned}
\sum\frac{1}{1+\varphi_{i,\alpha\overline{\alpha}}^+} &\geq (m+\triangle\varphi_i^+)^{1/(m-1)}e^{(-k(\varphi_i^+-\varphi_{i-1}^+)+F(x,\varphi_{i-1}^+))/(m-1)} \\
&\gtrsim (m+\triangle\varphi_i^+)^{1/(m-1)}.
\end{aligned}$$

Noting again that $\sup|\varphi_i^+|$ has been estimated, it follows from above inequalities that there are positive constants C_1, C_2, independent of i, such that

$$\begin{aligned}
& e^{C\varphi_i^+}\triangle_i^+(e^{-C\varphi_i^+}(m+\triangle\varphi_i^+)) \\
& \geq C_1(m+\triangle\varphi_i^+)^{1+1/(m-1)} - C_2\left((m+\triangle\varphi_i^+) + (m+\sup\triangle\varphi_{i-1}^+) + 1\right)
\end{aligned}$$

At the point where $e^{-C\varphi_i^+}(m+\triangle\varphi_i^+)$ achieves its maximum, the RHS must be non-positive and so

$$C_1\left(m+\sup\triangle\varphi_i^+\right)^{1+1/(m-1)} \leq e^{\frac{mC}{m-1}\sup\varphi_i^+}C_2\left((m+\sup\triangle\varphi_i^+) + (m+\sup\triangle\varphi_{i-1}^+) + 1\right)$$

Then we can find a positive constant C_3, independent of i, such that

$$\left(m+\sup\triangle\varphi_i^+\right) \leq \frac{1}{2}\left(m+\sup\triangle\varphi_{i-1}^+\right) + C_3.$$

By iteration, this gives

$$m+\sup\triangle\varphi_i^+ \leq \frac{m+\sup\triangle\varphi_0^+}{2^i} + 2C_3.$$

Therefore we have found estimates for $\varphi_{i,\alpha\overline{\beta}}^+$. To find uniform estimate of $\varphi_{i,\alpha\overline{\beta}\gamma}^+$, let

$$S_i = g_i^{+\alpha\overline{\ell}}g_i^{+\overline{\beta}p}g_i^{+\gamma\overline{q}}\varphi_{i,\alpha\overline{\beta}\gamma}^+\varphi_{i,\overline{\ell}p\overline{q}}^+.$$

By a computation similar to that of (3.1), we have

$$\triangle_i^+ (S_i + C_4 \triangle\varphi_i^+) \geq C_5 S_i - C_6 \sqrt{S_i}\sqrt{S_{i-1}} - C_7, \tag{6.4}$$

where $C_4 \sim C_7$ are positive constants independent of i.

Since $|\triangle\varphi_i^+|$ has been estimated, it follows from the maximum principle that

$$\sup S_i \leq \frac{C_6}{C_5}\sqrt{\sup S_i}\sqrt{\sup S_{i-1}} + \frac{C_7}{C_5} + C_4 \sup|\triangle\varphi_i^+|.$$

It should be noted that in (6.4), we can choose C_5 to be arbitrarily large if we are allowed to increase C_4 and C_7. In particular, we may assume that $2C_6 \leq C_5$. By AM-GM inequality,

$$\frac{C_6}{C_5}\sqrt{\sup S_i}\sqrt{\sup S_{i-1}} \leq \frac{3}{4}\sup S_i + \frac{1}{12}\sup S_{i-1}.$$

Then we get

$$\sup S_i \leq \frac{1}{3}\sup S_{i-1} + \frac{4C_7}{C_5} + 4C_4 \sup|\triangle\varphi_i^+|.$$

By iteration, we can find a uniform estimate of S_i and hence a uniform estimate of $\varphi^+_{i,\alpha\bar{\beta}\gamma}$.

Letting $i \to \infty$, we can then obtain a solution of (6.1). The Schauder estimate guarantees the solution to be smooth.

Theorem 3. Let M be a compact Kähler manifold with Kähler metric g. Let $F(x,t)$ be a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$. Suppose that, for some smooth function ψ defined on M,

$$⨍_M e^{F(x,\psi(x))} = 1.$$

Then there exists a smooth function φ on M such that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{F(x,\varphi(x))}\det(g_{i\bar{j}})$$

and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric. Furthermore, any other smooth function satisfying the same property differs from φ by only a constant.

Corollary. Let M be a Kähler manifold with ample canonical line bundle. Then there is a Kähler-Einstein metric whose Ricci tensor is the negative of the metric tensor. Furthermore, a metric of this form is unique and depends only on the complex structure of M.

By hypothesis, $-\,c_1(M)$ is represented by some positive $(1,1)$-form $\sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$. Take this form as our Kähler form. Since the closed $(1,1)$-form $-\partial\overline{\partial}\log\det(g_{i\bar{j}})$ also represents $c_1(M)$, we can find a smooth function f such that

$$\partial\overline{\partial}\log\det(g_{i\bar{j}}) = \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j + \partial\overline{\partial} f.$$

Now by Theorem 3, we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\varphi - f}\det(g_{i\bar{j}})$$

so that $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric. By these equations we have

$$\begin{aligned} -\partial\overline{\partial}\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) &= -\partial\overline{\partial}\varphi + \partial\overline{\partial} f - \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j - \partial\overline{\partial} f \\ &= -\sqrt{-1}(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \wedge d\bar{z}^j. \end{aligned}$$

This is indeed the metric we want.

For the uniqueness. Suppose that $\tilde{g}_{i\bar{j}}$ is another such metric. Then its Kähler form must represent $-\,c_1(M)$. Hence we can find a smooth function ψ defined on M such that $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \psi_{i\bar{j}}$. Together with the fact that $-\tilde{R} = \tilde{g}$, we get

$$\begin{aligned} -\partial\overline{\partial}\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) &= -\sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j - \partial\overline{\partial}\psi \\ &= -\partial\overline{\partial}\log\det(g_{i\bar{j}}) + \partial\overline{\partial} f - \partial\overline{\partial}\psi, \end{aligned}$$

which is equivalent to

$$\partial\overline{\partial}\log\left(\frac{\det(g_{i\bar{j}} + \psi_{i\bar{j}})}{\det(g_{i\bar{j}})} e^{f-\psi}\right) = 0$$

Therefore,

$$\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = e^{\psi + c - f}\det(g_{i\bar{j}})$$

for some c. The function $\psi + c$ then satisfies the equation. Lemma 1 shows that $\varphi - \psi$ is a constant. Hence,

$$(g_{i\bar{j}} + \psi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j = (g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j.$$

7 Degenerate Complex Monge-Ampère Equation with General Right-Hand Side

In this section, we combine the main results of the last two sections.

Let L be a line bundle over M. Let s be a nontrivial holomorphic section of L. Suppose L is equipped with a Hermitian metric so that the function $|s|^2$ is globally defined on M. For $k \geq 0$, we consider the equation

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = |s|^{2k} e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{7.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

As in Section 6, we assume that there is a function ψ whose partial derivatives $\psi_{i\bar{j}}$ are uniformly bounded on M so that

$$\fint_M |s|^{2k} e^{F(x,\psi)} = 1.$$

We approximate (7.1) by

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon(|s|^2 + \varepsilon)^k e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{7.2}$$

where $\varepsilon > 0$ is a smooth constant and

$$C_\varepsilon = \left(\fint_M (|s|^2 + \varepsilon)^k e^{F(x,\psi_\varepsilon)}\right)^{-1}.$$

Consider a sequence of smooth functions $\{\psi_\varepsilon\}$ such that $\psi_\varepsilon \to \psi$ uniformly on M and that $\sup\left|\psi_{\varepsilon,i\bar{j}}\right|$ is uniformly bounded on every coordinate chart.

By Theorem 3, we can find smooth solutions φ_ε of (7.2) such that $(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a metric. As in the proof of Theorem 3, we get an estimate of $\sup|\varphi_\varepsilon|$ in the following way.

Let φ_ε^+ and φ_ε^- be two smooth solutions of the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon(|s|^2 + \varepsilon)^k e^{F(x,\psi_\varepsilon)} \det(g_{i\bar{j}}) \tag{7.3}$$

such that $\varphi_\varepsilon^+ \geq \psi_\varepsilon \geq \varphi_\varepsilon^-$. Then the arguments of Theorem 3 show that $\varphi_\varepsilon^+ \geq \varphi_\varepsilon \geq \varphi_\varepsilon^-$.

On the other hand, for the unique solution of (7.3) with $\int_M \varphi = 0$, we can find an estimate of $\sup|\varphi|$ which is independent of ε. (This is seen in the proof of Theorem 3. Note that boundedness of $\triangle\psi_\varepsilon$ is needed.) In particular,

$$\sup|\varphi_\varepsilon| \leq \max\{\sup|\varphi_\varepsilon^-|, \sup|\varphi_\varepsilon^+|\} \leq \sup|\varphi| + \sup|\varphi - \psi_\varepsilon| \leq 2\sup|\varphi| + \sup|\psi_\varepsilon|.$$

is bounded from above by a constant independent of ε.

Let us now proceed to estimate $\triangle\varphi_\varepsilon$ from above. Then, as in (5.5), we have

$$e^{C\varphi_\varepsilon}\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}\left(m+\triangle\varphi_\varepsilon\right)\right)\geq g^{i\bar{j}}F_{i\bar{j}}+g^{i\bar{j}}F_{it}\varphi_{\varepsilon,\bar{j}}+g^{i\bar{j}}F_{t\bar{j}}\varphi_{\varepsilon,i}+g^{i\bar{j}}F_{tt}\varphi_{\varepsilon,i}\varphi_{\varepsilon,\bar{j}}-mF_t$$
$$+k\triangle\log(|s|^2+\varepsilon)-m^2\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}-mC\left(m+\triangle\varphi_\varepsilon\right)$$
$$+C_\varepsilon^{-1/(m-1)}\left(C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{\left(m+\triangle\varphi_\varepsilon\right)^{m/(m-1)}}{e^{F/(m-1)}(|s|^2+\varepsilon)^{k/(m-1)}}.$$

Choose C so that $C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\geq\frac{1}{2}C\geq1$. Then noting (5.4) and the fact that $\sup|\varphi_\varepsilon|$ is bounded, we can find positive constants C_1 and C_2 independent of ε such that

$$\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}\left(m+\triangle\varphi_\varepsilon\right)\right)\geq C_1\left(m+\triangle\varphi_\varepsilon\right)^{m/(m-1)}-C_2\left(\left(m+\triangle\varphi_\varepsilon\right)+|\nabla\varphi_\varepsilon|+1\right). \quad (7.4)$$

On the other hand, by Schauder's estimate and the estimate of $\sup|\varphi_\varepsilon|$, we have

$$\sup|\nabla\varphi_\varepsilon|\lesssim\left(\sup|\triangle\varphi_\varepsilon|+\sup|\varphi_\varepsilon|\right)\lesssim\left(\sup\left(m+\triangle\varphi_\varepsilon\right)+1\right).$$

By the maximum principle, we get an upper estimate of $m+\triangle\varphi_\varepsilon$. Therefore, we have uniform estimates of $\left|\varphi_{\varepsilon,i\bar{j}}\right|$ on every coordinate chart of M.

Using the uniform estimate of $\varphi_{\varepsilon,i\bar{j}}$, we follow the arguments of Section 5 to provide higher derivative estimates of φ_ε on compact subsets of the complement of the divisor of s. Letting $\varepsilon\to0^+$, we have then proved the following theorem.

Theorem 4. Let L be a holomorphic line bundle over a compact Kähler manifold M whose Kähler metric is given by g. Let s be a holomorphic section of L. Let $F(x,t)$ be a smooth function defined on $M\times\mathbb{R}$ such that $F_t\geq0$. Suppose, for some function ψ with $\left|\psi_{i\bar{j}}\right|$ bounded on every coordinate chart of M, we have $⨍_M|s|^{2k}e^{F(x,\psi(x))}=1$. Then we can find a solution φ of the equation

$$\det\left(g_{i\bar{j}}+\varphi_{i\bar{j}}\right)=|s|^{2k}e^{F(x,\varphi(x))}\det(g_{i\bar{j}})$$

with the following properties:

(i) φ is smooth outside the divisor of s, and

(ii) $\triangle\varphi$ is bounded over M.

Furthermore, any solution satisfying the above properties must be equal to φ plus a constant.

Proof. We have only to prove the last statement. Let $\widetilde{\triangle}$ be the normalized Laplacian of the metric $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$. Then we claim that if f is a C^1-function on M such that, for all i, j, $\left|f_{i\bar{j}}\right|$ is bounded on every coordinate chart of M, then

$$\int_{\{x|f(x)>0\}} \widetilde{\triangle}(f^2)|s|^{2k} e^{F(x,\varphi(x))} = 0. \tag{7.5}$$

Approximating f by a sequence of smooth functions, we may assume that f is smooth.

For all $\delta > 0$ such that the boundary of $\{x \mid f(x) \geq \delta\}$ is a C^1-manifold (which is true for $\delta \notin E$, where E the set of critical values, whose measure is zero by Sard's theorem), we know that by Stoke's theorem,

$$\int_{\{x|f(x)\geq\delta\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))}$$

can be expressed in terms of the boundary integral of $2f\partial_n f$. Here ∂_n is the normal of the sets $\{x \mid f(x) = \delta\}$ taken with respect to our metric $(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}})\, dz^i \otimes d\bar{z}^j$. It is clear that

$$\int_{\{x|\delta>f(x)>0\}} (|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))}\, d\mathrm{Vol} \to 0 \qquad \text{as } \delta \to 0^+$$

So we can find a sequence $\delta_i \to 0^+$ such that

$$\delta_i \cdot \mathrm{Vol}(\{x \mid f(x) = \delta_i\})$$

tends to zero as δ_i tends to zero. Otherwise, for some $c > 0$,

$$\int_{[0,\delta]\setminus E} \mathrm{Vol}(\{x \mid f(x) = \eta\})\, d\eta \geq \int_0^\delta \frac{c}{\eta}\, d\eta = \infty,$$

a contradiction.

Combining this with the boundary integral, we conclude that

$$\int_{\{x|f(x)\geq\delta_i\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))} \to 0 \qquad \text{as } i \to \infty.$$

Hence we have

$$\int_{\{x|f(x)>0\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))} = 0.$$

Letting $\varepsilon \to 0^+$ as in Theorem 3, we see that (7.5) follows from the above formula.

Suppose now that ψ is another solution of (7.1) with all the properties described in the theorem. Then

$$\frac{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi-\varphi)_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)} = e^{F(x,\psi)-F(x,\varphi)}.$$

Consider the set $\Omega = \{x \in M \mid \psi(x) - \varphi(x) > 0\}$; if it is nonempty, then the AM-GM inequality shows that the inequality

$$\widetilde{\triangle}(\psi - \varphi) \geq m \cdot e^{(F(x,\psi(x)) - F(x,\varphi(x)))/m} - m \geq 0 \tag{7.6}$$

holds on Ω. (Note that $F_t \geq 0$ is used here.)

Applying (7.5) to $f = \psi - \varphi$, we get

$$\begin{aligned} &2\int_\Omega (\psi - \varphi)\widetilde{\triangle}(\psi - \varphi)|s|^{2k} e^{F(x,\varphi(x))} + 2\int_\Omega \left|\widetilde{\nabla}(\psi - \varphi)\right|^2 |s|^{2k} e^{F(x,\varphi(x))} \\ &= \int_\Omega \widetilde{\triangle}(\psi - \varphi)^2 |s|^{2k} e^{F(x,\varphi(x))} = 0. \end{aligned}$$

Combining (7.6) and the above equality, we see that $\widetilde{\nabla}(\psi - \varphi) = 0$ on Ω and $\psi - \varphi$ is a constant on each component of $\Omega = \{x \mid \psi(x) - \varphi(x) > 0\}$. Since $\psi - \varphi$ is continuous, this is possible only if Ω is empty or $\Omega = M$. In the first case, $\psi(x) \leq \varphi(x)$ for all $x \in M$. In the second case, $\psi - \varphi$ is a constant. Interchanging ψ and φ, we conclude easily that, in any case, $\psi - \varphi$ is a constant. ■

8 Complex Monge-Ampère Equations with Meromorphic Right-Hand Side

Let L_1 and L_2 be two holomorphic line bundles over a compact Kähler manifold M. Let s_1 and s_2 be two (non-trivial) holomorphic sections of L_1 and L_2 that are equipped with Hermitian metrics so that we have globally defined functions $|s_1|^2$ and $|s_2|^2$ on M. Then, for $k_1 \geq 0$ and $k_2 \geq 0$, we shall study equations of the form

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F \det(g_{i\bar{j}}),$$

where F is a smooth function such that

$$⨍_M \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F = 1. \tag{8.1}$$

As before we approximate the PDE by the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon \frac{(|s_1|^2 + \varepsilon)^{k_1}}{(|s_2|^2 + \varepsilon)^{k_2}} e^F \det(g_{i\bar{j}})$$

where

$$C_\varepsilon = \left(⨍_M \frac{(|s_1|^2 + \varepsilon)^{k_1}}{(|s_2|^2 + \varepsilon)^{k_2}} e^F \right)^{-1}.$$

In order to prove that the normalized solutions φ_ε of the above equation converge on the complement of the divisors of s_1 and s_2, we consider the expression $(|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}(m + \triangle\varphi_\varepsilon)$ with $p \geq 0$.

We compute the Laplacian of the above expression as follows:

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon \left((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}(m + \triangle\varphi_\varepsilon) \right) \\
&= \frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon})(m + \triangle\varphi_\varepsilon) + \triangle_\varepsilon(\triangle\varphi_\varepsilon) \\
&\quad + \frac{2e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \left\langle \nabla_\varepsilon((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}), \nabla_\varepsilon(\triangle\varphi_\varepsilon) \right\rangle \\
&\geq \frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon})(m + \triangle\varphi_\varepsilon) + \triangle_\varepsilon(\triangle\varphi_\varepsilon) \\
&\quad - |\nabla_\varepsilon(p \log(|s_2|^2 + \varepsilon) - C\varphi_\varepsilon)|^2 (m + \triangle\varphi_\varepsilon) - \frac{|\nabla_\varepsilon(\triangle\varphi_\varepsilon)|^2}{m + \triangle\varphi_\varepsilon} \\
&\geq (m + \triangle\varphi_\varepsilon)(p\triangle_\varepsilon \log(|s_2|^2 + \varepsilon) - C\triangle_\varepsilon\varphi_\varepsilon) - \frac{|\nabla_\varepsilon(\triangle\varphi_\varepsilon)|^2}{m + \triangle\varphi_\varepsilon} + \triangle_\varepsilon(\triangle\varphi_\varepsilon).
\end{aligned}$$

By applying the same reasoning as in (2.5), (2.6) and (2.7), we have

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2+\varepsilon)^p}\triangle_\varepsilon\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right)\\
&\geq (m+\triangle\varphi_\varepsilon)(p\triangle_\varepsilon\log(|s_2|^2+\varepsilon)-C\triangle_\varepsilon\varphi_\varepsilon)+\triangle F\\
&\quad + k_1\triangle\log(|s_1|^2+\varepsilon)-k_2\triangle\log(|s_2|^2+\varepsilon)+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\left(\sum\frac{1+\varphi_{\varepsilon,i\bar{i}}}{1+\varphi_{\varepsilon,\ell\bar{\ell}}}-m^2\right).
\end{aligned}$$

As in (5.7), we have a positive constant C_1 which is independent of ε such that

$$p\triangle_\varepsilon\log(|s_2|^2+\varepsilon)\geq -pC_1\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}.$$

Note that

$$\triangle_\varepsilon\varphi_\varepsilon = m-\sum\frac{1}{1+\varphi_{i\bar{i}}}.$$

Combining the above inequalities and equation and computing as before, we can find positive constant C_2 and C_3 which are independent of ε such that

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2+\varepsilon)^p}\triangle_\varepsilon\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}\left(m+\triangle\varphi_\varepsilon\right)\right)\\
&\geq\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi_\varepsilon)\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}\\
&\quad -C_2-mC\left(m+\triangle\varphi_\varepsilon\right)-k_2\triangle\log(|s_2|^2+\varepsilon)\\
&\geq C_3\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{(|s_2|^2+\varepsilon)^{k_2/(m-1)}}{(|s_1|^2+\varepsilon)^{k_1/(m-1)}}\left(m+\triangle\varphi_\varepsilon\right)^{m/(m-1)}\\
&\quad -C_2-mC\left(m+\triangle\varphi_\varepsilon\right)-k_2\triangle\log(|s_2|^2+\varepsilon).
\end{aligned}\tag{8.2}$$

Clearly, for any fixed p, we can choose C large enough so that

$$C_3\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\left(|s_1|^2+\varepsilon\right)^{-k_1/(m-1)}\geq 1$$

With this choice of C, we consider the point where $(|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)$ achieves its maximum. At this point,

$$(|s_2|^2+\varepsilon)^{k_2/(m-1)}(m+\triangle\varphi_\varepsilon)^{m/(m-1)}\lesssim\max\left\{C_2, mC(m+\triangle\varphi_\varepsilon), k_2\triangle\log(|s_2|^2+\varepsilon)\right\}.$$

It follows easily from the above inequality and

$$\triangle\log(|s_2|^2+\varepsilon)\leq\frac{\triangle|s_2|^2}{|s_2|^2+\varepsilon}$$

that

$$\begin{aligned}
\sup\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right)\lesssim (C^{m-1}+1)\max\Big\{&\sup\left((|s_2|^2+\varepsilon)^{p-k_2/m}e^{-C\varphi_\varepsilon}\right),\\
&\sup\left((|s_2|^2+\varepsilon)^{p-k_2}e^{-C\varphi_\varepsilon}\right),\\
&\sup\left((|s_2|^2+\varepsilon)^{p-(m-1)/m-k_2/m}e^{-C\varphi_\varepsilon}\right)\Big\}.
\end{aligned}$$

From (8.1), $k_2 < 1$. Hence the third term in the RHS of the above inequality will be the dominating term. If we choose $p = \frac{m-1+k_2}{m} + Cq$ with $q \geq 0$, we see that

$$\sup\left((|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+Cq} e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right) \lesssim (C^{m-1}+1)\left(\sup(|s_2|^2+\varepsilon)^q e^{-\varphi_\varepsilon}\right)^C. \quad (8.3)$$

We are going to estimate $\sup|\varphi_\varepsilon|$. As in (2.12), we have an estimate of $\sup\varphi_\varepsilon$. Hence it remains to found an estimate of $\inf\varphi$. Integrating (8.2) with respect to the volume form $\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^{F-C\varphi_\varepsilon}\, d\mathrm{Vol}$, we have

$$\begin{aligned}
& C_3\left(C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{\inf F} \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{\frac{(m-2)k_1}{m-1}}}{(|s_2|^2+\varepsilon)^{\frac{(m-2)k_2}{m-1}-p}} (m+\triangle\varphi_\varepsilon)^{m/(m-1)} \\
& - k_2 \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F \triangle\log(|s_2|^2+\varepsilon) \\
& \leq C_2 e^{\sup F} \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \\
& \quad + mCe^{\sup F} \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} (m+\triangle\varphi_\varepsilon). \qquad (8.4)
\end{aligned}$$

We can find a positive constant C_4 which is independent of ε such that

$$\triangle\log(|s_2|^2+\varepsilon) \geq \frac{|s|^2}{|s|^2+\varepsilon}\triangle\log|s|^2 \geq -C_4.$$

Hence,

$$\begin{aligned}
& \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F \triangle\log(|s_2|^2+\varepsilon) \\
& \leq \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F (\triangle\log(|s_2|^2+\varepsilon) + C_4) \\
& \lesssim \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \triangle\log(|s_2|^2+\varepsilon) + \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}. \qquad (8.5)
\end{aligned}$$

By AM-GM inequality, we know that, for any $\delta > 0$,

$$\begin{aligned}
& m\int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} (m+\triangle\varphi_\varepsilon) \\
& \leq (m-1)\delta^{\frac{m}{m-1}} \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{\frac{(m-2)k_1}{m-1}}}{(|s_2|^2+\varepsilon)^{\frac{(m-2)k_2}{m-1}-p}} (m+\triangle\varphi_\varepsilon)^{\frac{m}{m-1}} \\
& \quad + \delta^{-m} \int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}. \qquad (8.6)
\end{aligned}$$

For any $p \geq 0$, we choose C large enough so that $C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq \frac{1}{2}C \geq 1$. Then we choose δ so that

$$\left((m-1)\delta^{\frac{m}{m-1}}\right) Ce^{\sup F} = \frac{1}{2} C_3 \left(C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{\inf F}.$$

Substituting (8.6) into (8.4) and keeping (8.5) in mind, we see that we can find positive constant C_5 and C_6 which are independent of ε and C for which

$$\begin{aligned}&\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}(m+\triangle\varphi_\varepsilon)-\frac{C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\triangle\log(|s_2|^2+\varepsilon)\\ &\lesssim C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}+C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}.\end{aligned}\tag{8.7}$$

In order to make use of the above inequality to derive an integral estimate $e^{-C\varphi_\varepsilon}$, we shall assume that the integral $\int_M |s_2|^{-2mk_2}$ is finite. Choose $p=C_5+k_2$.

Claim. We have

$$\begin{aligned}&\int_M\left|\nabla\left(e^{-C\varphi_\varepsilon/2}\frac{(|s_1|^2+\varepsilon)^{k_1/2}}{(|s_1|^2+\varepsilon)^{(k_2-p)/2}}\right)\right|^2\\ &\lesssim C\left(\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}.\end{aligned}\tag{8.8}$$

Proof of Claim. Integrating by parts in (8.7), we have

$$\begin{aligned}&-k_1\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla|s_1|^2,\nabla\varphi_\varepsilon\rangle\\ &-(p-k_2)\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_2|^2,\nabla\varphi_\varepsilon\rangle\\ &+C\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}|\nabla\varphi_\varepsilon|^2-C_5\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla\varphi_\varepsilon,\nabla|s_2|^2\rangle\\ &+\frac{k_1C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle\\ &+\frac{(p-k_2)C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}}\left|\nabla|s_2|^2\right|^2\\ &\le C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}+C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}.\end{aligned}$$

By (5.4), we have

$$\begin{aligned}C_7\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{p-k_2}}&\ge\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{p-k_2}}\triangle\log(|s_1|^2+\varepsilon)\\ &\ge\int_M e^{-C\varphi_\varepsilon}\Bigg((p-k_2)\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle\\ &\qquad+k_1\frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}}\left|\nabla|s_1|^2\right|^2\\ &\qquad-C\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla\varphi_\varepsilon,\nabla|s_1|^2\rangle\Bigg).\end{aligned}$$

Hence, using the above inequalities and the assumption that $p = C_5 + k_2$, we get

$$
\begin{aligned}
&\int_M \left| \nabla \left(e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \right)^{1/2} \right|^2 \\
&= \int_M \frac{e^{-C\varphi_\varepsilon}}{4} \Bigg((k_2-p)^2 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}} \left|\nabla|s_2|^2\right|^2 + k_1^2 \frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}} \left|\nabla|s_1|^2\right|^2 \\
&\quad + C^2 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} |\nabla\varphi_\varepsilon|^2 + 2k_1(p-k_2) \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}} \langle \nabla|s_1|^2, \nabla|s_2|^2 \rangle \\
&\quad -2C(p-k_2) \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}} \langle \nabla\varphi_\varepsilon, \nabla|s_2|^2 \rangle - 2k_1 C \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \langle \nabla\varphi_\varepsilon, \nabla|s_1|^2 \rangle \Bigg) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4} \Bigg(C_5^2 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}} \left|\nabla|s_2|^2\right|^2 + k_1^2 \frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}} \left|\nabla|s_1|^2\right|^2 \\
&\quad + C^2 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} |\nabla\varphi_\varepsilon|^2 + 2k_1 C_5 \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}} \langle \nabla|s_1|^2, \nabla|s_2|^2 \rangle \\
&\quad + \frac{2C(p-k_2)}{(p-k_2+C_5)} \Bigg[k_1 \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \langle \nabla|s_1|^2, \nabla\varphi_\varepsilon \rangle - C \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} |\nabla\varphi_\varepsilon|^2 \\
&\quad - \frac{k_1 C_5}{C} \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}} \langle \nabla|s_1|^2, \nabla|s_2|^2 \rangle - \frac{C_5^2}{C} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}} \left|\nabla|s_2|^2\right|^2 \\
&\quad + C_6 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + C_6 \frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}} \Bigg] - 2k_1 C \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \langle \nabla\varphi_\varepsilon, \nabla|s_1|^2 \rangle \Bigg) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4} \Bigg(-k_1 C \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \langle \nabla\varphi_\varepsilon, \nabla|s_1|^2 \rangle + k_1^2 \frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}} \left|\nabla|s_1|^2\right|^2 \\
&\qquad + k_1(p-k_2) \frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}} \langle \nabla|s_1|^2, \nabla|s_2|^2 \rangle \\
&\qquad + CC_6 \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + CC_6 \frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}} \Bigg) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4} \left((CC_6 + k_1 C_7) \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + CC_6 \frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}} \right) \\
&\lesssim C \left(\int_M \left(e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \right)^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}},
\end{aligned}
$$

where the last inequality is due to the Hölder inequalities

$$
\int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \le \left(\int_M 1 \right)^{\frac{1}{m}} \left(\int_M \left(e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \right)^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}},
$$

$$
\int_M e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}} \le \left(\int_M \frac{(|s_1|^2+\varepsilon)^{mk_1}}{(|s_2|^2+\varepsilon)^{mk_2}} \right)^{\frac{1}{m}} \left(\int_M \left(e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \right)^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}}
$$

and the assumption $\int_M |s_2|^{-2mk_2} < \infty$. ■

Since we have an estimate of $\int_M |\varphi_\varepsilon|$, we can use the method of Section 5 to find an estimate of

$$
\int_M \left(e^{-C\varphi_\varepsilon} \frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \right)^{\frac{m}{m-1}}
$$

which is independent of ε.

From the above inequality, we conclude that when $p = C_5 + k_2$ and N is a large constant, we can find a positive constant C_8 independent of ε such that

$$\int_M e^{-N\varphi_\varepsilon}(|s_1|^2+\varepsilon)^{k_1}(|s_2|^2+\varepsilon)^{C_5} = \int_M e^{-N\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \le C_8. \tag{8.9}$$

From (8.3) and the estimate of $\sup\varphi_\varepsilon$, we derive that, for any $q \ge 0$,

$$\sup\Big((|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+Cq}(m+\triangle\varphi_\varepsilon)\Big) \lesssim (C^{m-1}+1)e^{C\sup\varphi_\varepsilon}(\sup e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q)^C, \tag{8.10}$$

where C is any positive constant so that

$$C_3\left(C - \left(\tfrac{m-1+k_2}{m}+Cq\right)C_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) \ge \sup(|s_1|^2+\varepsilon)^{\frac{k_1}{m-1}}.$$

Using (8.9) and (8.10) we shall show that, for any $q>0$, $e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q$ has an upper bound which is independent of ε. Note that we may assume q is small enough.

Suppose not, we could find $\varepsilon_i \to 0^+$ and $x_i \to x_0$ in M such that

$$e^{-\varphi_{\varepsilon_i}(x_i)}(|s_2|^2(x_i)+\varepsilon_i)^q = \sup\left(e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right) \to \infty.$$

Suppose the sequence $\{\varepsilon_i^{-1}|s_2|^2(x_i)\}$ is bounded. Then $\varepsilon_i^q e^{-\varphi_{\varepsilon_i}(x_i)} \to \infty$. On the other hand, using (8.10) and the L^1-estimate of φ_ε, we can apply the Schauder estimate to get

$$\sup|\nabla\varphi_{\varepsilon_i}| \lesssim (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+1.$$

It follows from the above estimate, $|\nabla|s_2|^2| \lesssim |s_2|$ and AM-GM inequality that

$$\begin{aligned}\sup\left|\nabla\left(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i}\right)\right| &\lesssim \frac{|\nabla|s_2|^2|}{|s_2|^2+\varepsilon} + (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+1\\ &\lesssim (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+\varepsilon_i^{-1/2}.\end{aligned}$$

Clearly we may assume that $\sup(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i}) \ge 0$. Then proceeding geodesic ball trick as in Section 2, we can now conclude that

$$\begin{aligned}C_8 &\ge \int_M e^{-N\varphi_{\varepsilon_i}}(|s_1|^2+\varepsilon_i)^{k_1}(|s_2|^2+\varepsilon_i)^{C_5} \ge \int_M e^{N(\log(\varepsilon_i(|s_2|^2+\varepsilon_i))^q-\varphi_{\varepsilon_i})}\varepsilon_i^{k_1+C_5}(\varepsilon_i(|s_2|^2+\varepsilon_i))^{-qN}\\ &\gtrsim \left(\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}\frac{\sup(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i})}{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C+1}\right)^{2m}\cdot\varepsilon_i^{k_1+C_5-qN}\cdot\left(\varepsilon_i^q\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^{N/2}.\end{aligned}$$

Taking $N>4mC$, the above inequality shows that $\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q$ is bounded.

So we may assume that $\varepsilon_i^{-1}|s_2|^2(x_i) \to \infty$. For each x_i, let $B_i = B(x_i, \delta_i)$ be a geodesic ball around x_i such that, for each $x \in B_i$,

$$\frac{3}{2}|s_2|^2(x_i) \geq |s_2|^2(x) \geq \frac{1}{2}|s_2|^2(x_i). \tag{8.11}$$

Let $C_9 \geq \sup|\nabla|s_2|^2|$ sufficiently large enough. Then we may assume

$$\delta_i = \frac{1}{2C_9}|s_2|^2(x_i),$$

and is smaller than the injectivity radius of M. It is easy to derive from (8.10) that, over the ball B_i,

$$0 < m + \triangle\varphi_{\varepsilon_i} \lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C}{\left(\frac{1}{2}|s_2|^2(x_i)\right)^{(m-1+k_2)/m+Cq}}.$$

By applying the Schauder estimate on the balls B_i and $B_i' = B(x_i, \frac{\delta_i}{2})$, we get

$$\begin{aligned}\sup_{x\in B_i'}|\nabla\varphi_{\varepsilon_i}|(x) &\lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{\int_{B_i}|\varphi_{\varepsilon_i}|}{\delta_i^{2m+1}} \\ &\lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{\int_{B_i}|\varphi_{\varepsilon_i}|}{(|s_2|^2(x_i))^{2m+1}}.\end{aligned}$$

Since we have an estimate of $\int_M|\varphi_{\varepsilon_i}|$, it follows from the above inequality that

$$\sup_{x\in B_i'}|\nabla(\log(|s_2|^2+\varepsilon_i)^q - \varphi_{\varepsilon_i})| \lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{1}{(|s_2|^2(x_i))^{2m+1}}. \tag{8.12}$$

Since s_1 is holomorphic, one can find positive constant a such that, for any small $r > 0$ and $x \in M$,

$$\int_{B(x,r)}|s_1|^{2k_1} \gtrsim r^a.$$

As before, we may assume that $\sup(q\log(|s_2|^2+\varepsilon_i) - \varphi_{\varepsilon_i}) \geq 0$. Then proceeding the geodesic ball trick as above, we can now conclude from (8.11), (8.12), the above inequality and $\varepsilon^{-1}|s_2|^2(x_i) \to \infty$ that

$$\begin{aligned}C_8 &\geq \int_M e^{-N\varphi_{\varepsilon_i}}(|s_1|^2+\varepsilon_i)^{k_1}(|s_2|^2+\varepsilon_i)^{C_5} \gtrsim \int_{B'} e^{N(\log(|s_2|^2+\varepsilon)^q-\varphi_{\varepsilon_i})}|s_1|^{2k_1}(|s_2|^2(x_i))^{C_5-qN} \\ &\gtrsim \left(\left(\frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{\frac{m-1+k_2}{m}+Cq}} + \frac{1}{(|s_2|^2(x_i))^{2m+1}}\right)^{-1}\cdot\sup(\log(|s_2|^2+\varepsilon)^q - \varphi_{\varepsilon_i})\right)^a \\ &\quad\cdot(|s_2|^2(x_i))^{C_5-qN}\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^{N/2}\end{aligned}$$

Take N large enough, we see that the quantity $\sup e^{-C\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q$ can be estimated by a constant independent of i.

In conclusion we have proved that, for any $q > 0$, $\log(|s_2|^2 + \varepsilon)^q - \varphi_\varepsilon$ is bounded from above by a constant independent of ε. In particular, $-\varphi_\varepsilon$ is uniformly bounded over any compact subset K of the complement of the divisor of s_2. From (8.10) and the estimate of $\sup \varphi_\varepsilon$, we see that both $|\varphi_\varepsilon|$ and $|\triangle\varphi_\varepsilon|$ are uniformly bounded over K. The arguments of Section 5 now show that one can find uniform estimates of $\{\varphi_{\varepsilon;i\bar{j}k}\}$ over K.

Theorem 5. Let L_1 and L_2 be two holomorphic line bundles over a compact Kähler manifold M whose Kähler metric is given by g. Let s_1 and s_2 be two holomorphic sections of L_1 and L_2, respectively, and let F be a smooth function defined on M such that

$$\fint_M \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F = 1 \quad \text{and} \quad \int_M |s_2|^{2mk_2} < \infty.$$

where k_1 and k_2 are two non-negative integers. Then we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F \det(g_{i\bar{j}})$$

so that

(i) φ is smooth outside the divisors of s_1 and s_2 with $\sup \varphi < \infty$,

(ii) $(\varphi_{i\bar{j}})$ is a bounded matrix outside the divisor of s_2 and, for any $q > 0$,

$$|s_2|^{2(m-1+k_2)/m+q} \triangle\varphi$$

is bounded on M,

(iii) for any $q > 0$, the function $\varphi - q \log |s_2|^2$ is bounded from below,

(iv) the matrix $(g_{i\bar{j}} + \varphi_{i\bar{j}})$ is positive definite outside the complement of the divisors of s_1 and s_2.

Furthermore, if we assume that

$$\int_M \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} < \infty$$

for some $q > 0$, the any two solutions of the equation which has the above properties (i), (ii) and (iv) must differ from each other by a constant. If we also know that $(|s_2|^{2(m-1+k_2)/m+q})^{-1}$ is integrable over every analytic disc of M, then the unique solution φ is bounded from below on M.

Proof. We have only to prove the last part. Suppose ψ is another solution of the equation with (i), (ii) and (iv). Then the AM-GM inequality shows that

$$\triangle_\varepsilon(\psi - \varphi_\varepsilon) \geq m \left(C_\varepsilon^{-1} \cdot \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} \cdot \frac{(|s_2|^2 + \varepsilon)^{k_2}}{(|s_1|^2 + \varepsilon)^{k_1}} \right)^{1/m} - m. \tag{8.13}$$

Let k be any constant. We claim that, over $\Omega_{\varepsilon,k} = \{x \in M \mid (\psi - \varphi_\varepsilon)(x) \geq k\}$,

$$\int_{\Omega_{\varepsilon,k}} (\psi - \varphi_\varepsilon - k) \triangle_\varepsilon(\psi - \varphi_\varepsilon) \leq 0. \tag{8.14}$$

In fact, for $\delta > 0$, $\Omega_{\varepsilon,k,\delta} = \{x \in M \mid (\psi - \varphi_\varepsilon)(x) \geq k - \delta \log |s_2|^2\}$ is disjoint from the divisor of s_2 by property (i). Hence both $(\psi_{i\bar{j}})$ and $(\varphi_{\varepsilon;i\bar{j}})$ are bounded on $\Omega_{\varepsilon,k,\delta}$ and we can integrate by parts on $\Omega_{\varepsilon,k,\delta}$ to find

$$\begin{aligned} &\int_{\Omega_{\varepsilon,k,\delta}} (\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2) \triangle_\varepsilon(\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2) \\ &= - \int_{\Omega_{\varepsilon,k,\delta}} |\nabla_\varepsilon(\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2)|^2. \end{aligned} \tag{8.15}$$

Using property (ii) and the assumption $\int_M \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} < \infty$, we can find a constant $C(\varepsilon)$ independent of δ such that

$$\begin{aligned} \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle_\varepsilon(\psi - \varphi_\varepsilon)| &\lesssim \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle \psi| + \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle \varphi_\varepsilon| \\ &\leq \left(\int_{\Omega_{\varepsilon,k,\delta}} |s_2|^{2(m-1+k_2)/m+q} (\log |s_2|^2 |\triangle \psi|)^2 \right)^{1/2} \\ &\quad \cdot \left(\int_{\Omega_{\varepsilon,k,\delta}} \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} \right)^{1/2} + \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle \varphi_\varepsilon| \\ &\leq C(\varepsilon). \end{aligned} \tag{8.16}$$

It follows easily from (8.15), (8.16) and the boundedness of $|\triangle_\varepsilon \log |s_2|^2|$ that

$$\lim_{\delta \to 0^+} \int_{\Omega_{\varepsilon,k,\delta}} (\psi - \varphi_\varepsilon - k) \triangle_\varepsilon(\psi - \varphi_\varepsilon) \leq 0.$$

Using the definition of $\Omega_{\varepsilon,k,\delta}$, we see that, over $\Omega_{\varepsilon,k,\delta}$, $(\psi - \varphi_\varepsilon - k)$ is bounded by a constant independent of δ when δ is small. The function $(\psi - \varphi_\varepsilon - k)\triangle_\varepsilon(\psi - \varphi_\varepsilon)$ is therefore uniformly integrable and we can apply Lebesgue's dominated convergence theorem to prove (8.14).

Applying (8.13) and (8.14), we can now prove the following inequality:

$$\int_{\Omega_{\varepsilon,k}} (\psi - \varphi_\varepsilon - k) \left(m - m \left(C_\varepsilon^{-1} \cdot \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} \cdot \frac{(|s_2|^2 + \varepsilon)^{k_2}}{(|s_1|^2 + \varepsilon)^{k_1}} \right)^{1/m} \right) \geq 0. \tag{8.17}$$

When $\varepsilon \to 0^+$, the integral on the LHS tends to zero. Let K be a compact subset of the complement of the divisor of s_2. By (8.13) and the above inequality, we have

$$\lim_{\varepsilon\to 0^+}\int_{K\cap\Omega_{\varepsilon,k}}(\psi-\varphi_\varepsilon-k)\left(m-m\left(C_\varepsilon^{-1}\cdot\frac{|s_1|^{2k_1}}{|s_2|^{2k_2}}\cdot\frac{(|s_2|^2+\varepsilon)^{k_2}}{(|s_1|^2+\varepsilon)^{k_1}}\right)^{1/m}+\triangle_\varepsilon(\psi-\varphi_\varepsilon)\right)=0.$$

Let $\Omega_k=\{x\in M \mid (\psi-\varphi)(x)\geq k\}$. Then the above equation gives

$$\int_{K\cap\Omega_k}(\psi-\varphi-k)\widetilde{\triangle}(\psi-\varphi)=0.$$

As in (8.13), we know that $\widetilde{\triangle}(\psi-\varphi)\geq 0$ and hence, $\widetilde{\triangle}(\psi-\varphi)=0$ on Ω_k. The AM-GM inequality now becomes equality, so $\psi_{i\bar{j}}=\varphi_{i\bar{j}}$ on $K\cap\Omega_k$. Since k and K are arbitrary, $\psi_{i\bar{j}}=\varphi_{i\bar{j}}$ on the complement of the divisor of s_2. Letting first $\delta\to 0^+$ and then $\varepsilon\to 0^+$ in (8.15). We get

$$\int_{K\cap\Omega_k}|\widetilde{\nabla}(\psi-\varphi)|^2\leq -\lim_{\varepsilon\to 0^+}\int_{\Omega_{\varepsilon,k}}(\psi-\varphi_\varepsilon-k)\triangle_\varepsilon(\psi-\varphi_\varepsilon),$$

which is equal to zero by (8.17). Hence, $\psi-\varphi$ is constant.

It remains to prove that $-\inf\varphi<\infty$. From (8.3) and the estimate of $e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q$ for any $q>0$, we know that, for any $q>0$,

$$\sup\left((m+\triangle\varphi_\varepsilon)(|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+\frac{q}{2}}\right)\lesssim 1 \tag{8.18}$$

Let x be any point on the divisor of s_2. Let D_x be an analytic disc passing through x such that s_2 is not zero on ∂D_x. Then $|\varphi_\varepsilon|$ is uniformly bounded on ∂D_x when $\varepsilon\to 0^+$. It follows from (8.18) that when we restrict φ_ε to D_x, the absolute value of its Laplacian is estimated by $(|s_2|^2+\varepsilon)^{-(\frac{m-1+k_2}{m}+\frac{q}{2})}$ over D_x. Cauchy integral formula gives

$$2\pi\sqrt{-1}\partial\varphi_\varepsilon(p)=\int_{\partial D_x}\frac{\partial\varphi_\varepsilon(z)}{z-p}dz+\int_{D_x}\frac{\triangle\varphi}{z-p}$$

Integrate over the curve $\gamma(t)=tp+(1-t)\overline{p}$, where $\overline{p}=\frac{p}{|p|}$, we get

$$\begin{aligned}|\varphi_\varepsilon(p)|\lesssim |\varphi_\varepsilon(\overline{p})|&+\int_{\partial D_x}|\partial\varphi(z)|\cdot|\log(z-p)-\log(z-\overline{p})|\;dz\\&+\int_{D_x}\frac{1}{|s|^{2(\frac{m-1+k_2}{m})+q}}\cdot|\log(z-p)-\log(z-\overline{p})|\end{aligned}$$

Using Hölder inequality and taking a smaller q, we obtain an estimate of $|\varphi_\varepsilon|$ on D_x. Since x is arbitrary, we can conclude the boundedness of φ. ∎

9 The General Case

Let $t_1, t_2, \ldots, t_{n_1+n_2}$ be non-zero non-negative functions defined on M such that $t_i = \sum_{j=1}^{\ell} |s_j|^{2k_j}$, where $k_j \geq 0$ for each j and $s_1, s_2, \ldots, s_\ell$ are holomorphic sections of some holomorphic line bundle.

Then we consider

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{9.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

Then we assume t_i's satisfying the following properties:

- there exists a smooth function ψ defined on M such that
$$\fint_M \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\psi)} = 1.$$
- $(t_{n_1+1} \cdots t_{n_1+n_2})^{-m}$ is integrable over M.
- for some $q > 0$,
$$\frac{|\triangle \log(t_{n_1+1} \cdots t_{n_1+n_2})|^{(m-1)/m}}{(t_{n_1+1} \cdots t_{n_1+n_2})^{q/m}}$$
is integrable over M and over every analytic disk of M.

As before, we have

Theorem 6. Let M be a compact Kähler manifold. Suppose that, in equation (9.1), the t_i are functions satisfying the above mentioned properties. Then we can find a solution φ of (9.1) such that

(i) φ is smooth outside the divisors of the t_i's and $\sup |\varphi| < \infty$,

(ii) $\sup \dfrac{(t_{n_1+1} \cdots t_{n_1+n_2})^{q+1/m} (\triangle \varphi)}{(|\triangle \log t_{n_1+1} \cdots t_{n_1+n_2}| + 1)^{(m-1)/m}} < \infty$, and

(iii) $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$ defines a Kähler metric outside the divisors of the t_i's.

Furthermore, any solution of (9.1) satisfying the above three properties differs from φ by a constant.

Corollary 1. Let M be a compact Kähler variety with log terminal singularity such that the canonical line bundle is ample. Then there is a Kähler-Einstein metric whose Ricci tensor is the negative of the metric tensor on the smooth part of M.

Take a resolution of singularities $\pi : \widetilde{M} \to M$ so that

$$K_{\widetilde{M}} = \pi^* K_M + \sum_{E \in \mathcal{E}} a_E E$$

and $a_E > -1$ for all $E \in \mathcal{E}$. We know that there exists $c_E \in \mathbb{Q}^+$ such that

$$L = \pi^* K_M - \sum_{E \in \mathcal{E}} c_E E$$

is ample. Then

$$K_{\widetilde{M}} = L + \sum_{E \in \mathcal{E}} (a_E + c_E) E$$

gives

$$-\operatorname{c}_1(\widetilde{M}) = \operatorname{c}_1(L) + \sum (a_E + c_E) \operatorname{c}_1(E)$$

Since $\operatorname{c}_1(L)$ is represented by some positive $(1,1)$-form $\sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\overline{z}^j$. Take this form as our Kähler form on $\widetilde{M}$. Then $-\operatorname{c}_1(\widetilde{M})$ is represented by

$$\sqrt{-1} h_{i\bar{j}}\, dz^i \wedge d\overline{z}^j - \sum (a_E + c_E) \partial\overline{\partial} \log |s_E|^2.$$

Since the closed $(1,1)$-form $-\partial\overline{\partial} \log \det(g_{i\bar{j}})$ also represents $\operatorname{c}_1(\widetilde{M})$, we can find a smooth function f such that

$$\partial\overline{\partial} \log \det(g_{i\bar{j}}) = \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\overline{z}^j - \sum (a_E + c_E) \partial\overline{\partial} \log |s_E|^2 + \partial\overline{\partial} f.$$

Now by Theorem 6, we can solve the equation (since $a_E + c_E > -1$)

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \prod_E |s_E|^{2(a_E + c_E)} \cdot e^{\varphi - f} \det(g_{i\bar{j}})$$

so that $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric outside $\bigcup_{E \in \mathcal{E}} E$. By these equations we have

$$\begin{aligned} -\partial\overline{\partial} \log \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) &= -\partial\overline{\partial}\varphi - \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\overline{z}^j \\ &= -\sqrt{-1} (g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \wedge d\overline{z}^j \end{aligned}$$

on the $\widetilde{M}$. Since the smooth part of M is isomorphic to some open subset of $\widetilde{M}$, we get the metric we want.