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1 Number-theoretic foundations

Definition 1.1. Let $a, b \in \mathbb{Z}$ with $a \neq 0$. If there exists an integer $q \in \mathbb{Z}$ such that $b = aq$, then we say that **b is divisible by a** (or **a dicides b**), and we write $a|b$; otherwise we write $a \nmid b$.

The integers possess the following properties:

$$a|b \Longleftrightarrow -a|b \Longleftrightarrow a|-b \Longleftrightarrow |a||b| \tag{1.1}$$

$$a|b \wedge b|c \Longrightarrow a|c \tag{1.2}$$

$$a|b \wedge a|c \Longleftrightarrow \forall x, y \in \mathbb{Z}, a|(xb + yc) \tag{1.3}$$

$$a|b \wedge b|a \Longrightarrow b = \pm a \tag{1.4}$$

$$\text{Let } m \neq 0, \text{then } a|b \Longleftrightarrow ma|mb. \tag{1.5}$$

$$\text{Let } b \neq 0, \text{then } a|b \Longrightarrow |a| \leq |b|. \tag{1.6}$$

$$\text{Let } a \neq 0, b = qa + c, \text{then } a| \Longleftrightarrow a|c. \tag{1.7}$$

Theorem 1.1. *If $1 < p < \infty$ and $m > n/p$, or $p = 1$ and $m \geq n$, there exist a constant $C = C(m, n, \tau, p)$,such that*

$$\|R^m u\|_{L^\infty(\Omega)} \leq C d^{m-n/p} |u| W_p^m(\Omega)$$

for all $u \in W_p^m(\Omega)$.

Proof. First, we assume that $u \in C^m(\Omega) \cap W_p^m(\Omega)$. We can use the pointwise representation of $R^m u(x)$.

$$\begin{aligned} |R^m u(x)| &= m \left| \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x, z) D^\alpha u(z) \mathrm{d}z \right| \\ &\leq C \sum_{|\alpha|=m} \int_\Omega |x - z|^{-n+m} |D^\alpha u(z)| \mathrm{d}z \\ &\leq C' d^{m-n/p} |u|_{W_p^m(\Omega)}. \end{aligned}$$

The proof can be completed via a density argument.

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