Phys 122 Notes

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2024 W

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Lecture 3 - Tue - Jan 16 - 2024

Recall Hooke's Law

Definition 0.1

$$F_s = -kx$$

where F_s is the spring force, k is the spring constant, and x is the spring stretch or compression (displacement).

Remark: Recall Newton's Second Law,

$$\sum \vec{F} = m\vec{a} = m\,\frac{d^2x}{dx^2} = -kx$$

From Newton's Second Law, we know that $\frac{d^2x}{dx^2} = -\frac{k}{m}x$, thus we can find that the function for x is either cos, sin, or e^x.

Example: We try:

$$\begin{aligned} x &= A\cos(\omega t + \varphi) \\ \frac{dx}{dt} &= -A\omega\sin(\omega t + \varphi) \\ \frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t + \varphi) \end{aligned}$$

Remark: Notice that $\omega = \sqrt{\frac{k}{m}}$.

We call

$$\begin{aligned} A &\longrightarrow \text{amplitude} \\ \omega &\longrightarrow \text{angular frequency} \\ \varphi &\longrightarrow \text{phase constant} \end{aligned}$$

Therefore, the period T is

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Therefore, our position function is

$$x(t) = A\cos(\omega x + \varphi)$$

which tells us that the maximum speed is

$$\boxed{V_{max} = -\omega A = \sqrt{\frac{k}{m}}A}$$

and the maximum acceleration is

$$\boxed{A_{max} = -\omega^2 A = \frac{k}{m}A}$$

Remark: In SHM, the acceleration is not a contant, it is a function dependent on time (position), so the kinematic equations cannot be applied anymore.

Lecture 4 - Thur - Jan 18 - 2024

1 Mechanical Waves

1.1 Energy in SHM

The total mexchanical energy $E = K + U$ is conserved in SHM:

$$E = \frac{1}{2}m{v_x}^2 + \frac{1}{2}kx^2$$

1.1.1 Kinetic Energy

$$\begin{aligned} K &= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \varphi) \\ &= \frac{1}{2}kA^2 \sin^2(\omega t + \varphi) \end{aligned}$$

1.1.2 Potential Energy

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^x \cos^2(\omega t + \varphi)$$

Result 1.1

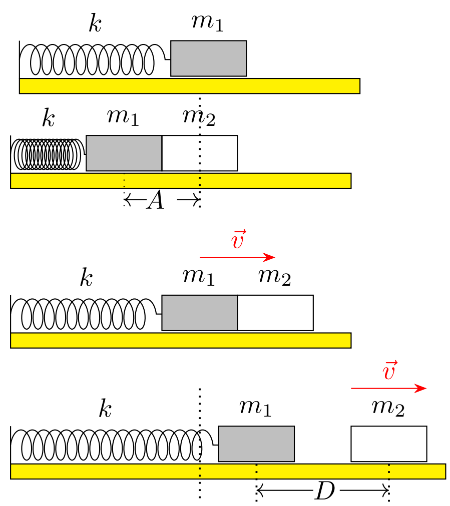
Total energy:

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$

Example 1.1

An object of mass $m_1 = 11.00\ kg$ is in equilibrium when connected to a light spring of constant $k = 100\ N/m$ that is fastened to a wall as shown.

A second object, $m_2 = 6.00\ kg$, is slowing pushed up against m_1, compressing the spring by an amount $A = 0.200\ m$. The system is then released, and both objects start moving to the right on the frictionless surface.



(a) When m_1 reaches the equilibrium point, m_2 loses contact with m_1 and moves to the right with speed v. What is this speed?

(b) How far apart are the objects when the spring is fully stretched for the first time?

1. We know that the initial energy is equal to the final energy

$$U_i + K_i = \frac{1}{2}kA^2 + 0$$
$$\equiv\ U_f + K_f = 0 + \frac{1}{2}\left(m_1 + m_2\right)v^2$$
$$\Rightarrow\ v = \sqrt{\frac{kA^2}{m_1 + m_2}} = \boxed{0.485\ m/s}$$

2. After m_2 decouples, energy of m_1 becomes

$$U_i + K_i = 0 + \frac{1}{2}m_1 v^2 = \frac{1}{2}m_1\left(\sqrt{\frac{kA^2}{m_1 + m_2}}\right)^2$$

When the spring is totally stretched, we have

$$U_f + K_f = \frac{1}{2}kA'^2 + 0$$

We know that the two equation are equivalent, so we have

$$A' = A\sqrt{\frac{m_1}{m_1 + m_2}}$$

Find time occurs at $\frac{1}{4}T = \frac{1}{4} \cdot \frac{2\pi}{\omega} = \frac{1}{4}2\pi\sqrt{\frac{m_1}{k}}$. Hence m_2 travels distance

$$d = vt = v \cdot \frac{\pi}{2}\sqrt{\frac{m_1}{k}}$$

Thus

$$D = d - A' = v \cdot \frac{\pi}{2}\sqrt{\frac{m_1}{k}} - A\sqrt{\frac{m_1}{m_1 + m_2}} = \boxed{0.09m}$$

1.2 The simple pendulum

Definition 1.1

A **simple pendulum** consists of a point mass (the bob) suspended by a massless, unstretchable string.

Remark 1 *If the pendulum swings with a small amplitude θ with the vertical, its motion is simple harmonic.*

Definition 1.2

A **physical pendulum** is any real pendulum that uses an extended body instead of a point-mass bob.

In pysical pendulum

1.2.1 Angular Frequency

$$\omega = \sqrt{\frac{mgd}{I}}$$

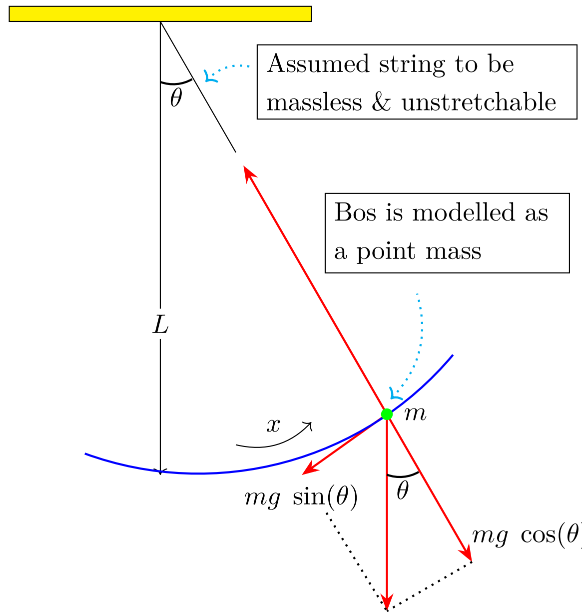
1.2.2 Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Exercise: If you know d, you can find I by measuring the period.

Result 1.2

The restoring force on the bob is proportional to $\sin\theta$, not to θ. However, for small θ, we have $\sin\theta \approx \theta$, so the motion is *approximately* **simple harmonic**.



Example 1.2

What is the period of a pendulum made from a thin rod of length L and mass m that rotates about a point through one end?

Solution: Recall the The Parallel-Axis Theorem:

$$I = I_c + mh^2$$

Hence in our case, we have

$$I = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}$$

Therefore, the period of the pendulum would be

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{mL^2/3}{mgL/2}} = \boxed{2\pi\sqrt{\frac{2L}{3g}}}$$

as desired. □

Remark: Rmbr the period in a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

notice how they differ by just a constant.

1.3 Tortional Pendulum

Assume a rigid object is suspended from a wire attached at its top to a fixed support.

The restoring torque is

$$\tau = -\kappa\theta$$

1.3.1 Angular Frequency

$$\omega = \sqrt{\frac{\kappa}{I}}$$

1.3.2 Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

Remark: No small angle approximation is necessary in this scenario.

Lecture 5 - Tue - Jan 23 - 2024

1.4 Damped and Forced Oscillations

Review SHM, we have

1. Mass-spring
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x, \quad \omega^2 = \frac{k}{m}$$
2. Simple pendulum
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta, \quad \omega^2 = \frac{g}{l}$$
3. Physical pendulum
$$\frac{d^2\theta}{dt^2} = -\left[\frac{mgd}{I}\right]\theta$$
where d is the distance between the pivot point and the centre of mass, and we know that $I \propto mL^2$.
4. Torsional pendulum
$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta, \quad \omega^2 = \frac{\kappa}{I}$$

Remark: Overall, we have $\dfrac{mgd}{I} \propto \dfrac{mgl}{ml^2} \propto \dfrac{g}{l}$.

1.5 Damped Oscillation

The decrease in amplitude is called **damping** and the motion is called **damped oscillation**.

Example: Examples would be a mass attached to a spring oscillating on a horizontal plane that is not frictionless, or a mass attached to a spring vertically (so we have something like air resistance).

Regarding to Newton's Law, we have that

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} = -kx - bv \\
m\frac{d^2x}{dt^2} &= -kx - b\frac{dx}{dt} \\
\frac{d^2x}{dt^2} &= -\frac{k}{m}x - \frac{b}{m}\frac{dx}{dt}
\end{aligned}$$

We define variables $\omega_0 = \sqrt{\frac{k}{m}}$ and $\gamma = \frac{b}{2m}[Hz]$, thus we can rewrite the equation as

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + {\omega_0}^2 x = 0$$

To solve the equation, we try $x = Ae^{-i\omega t}$, so we have that

$$\begin{aligned}
\frac{dx}{dt} &= -A\omega i e^{-i\omega t} \\
\frac{d^2x}{dt^2} &= -A\omega^2 e^{-i\omega t}
\end{aligned}$$

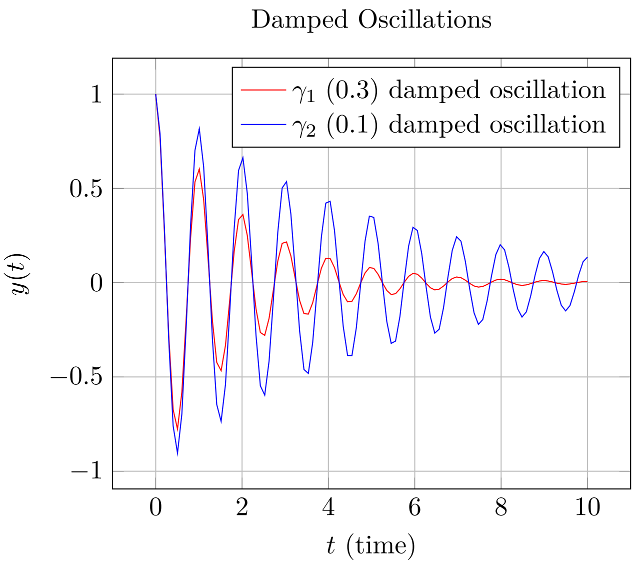
Plug them into the differential equation we have

$$\begin{aligned}
-A\omega^2 e^{-i\omega t} + 2\gamma\left(-A\omega i e^{-i\omega t}\right) + {\omega_0}^2 A e^{-i\omega t} &= 0 \\
\Rightarrow \quad e^{i\omega t}\left[-A\omega^2 - 2\gamma A\omega i + A{\omega_0}^2\right] &= 0 \\
\Rightarrow \quad {\omega_0}^2 - 2\gamma\omega i - \omega^2 &= 0 \\
\Rightarrow \quad \omega = -\gamma i \pm \sqrt{{\omega_0}^2 - \gamma^2}
\end{aligned}$$

Result 1.3

Hence we can obtain the general solution for the differential equation

$$\boxed{x = Ae^{-i\omega_+ t} + Be^{-i\omega_- t}}$$



Theory 1.1

With stronger damping (larger γ):

1. The amplitude decreases more rapidly
2. The period T increases (T_0 = period with zero damping)

Example 1.3: Critically damped

Suppose ${\omega_0}^2 - \gamma^2 = 0$

In this case we have $\omega_+ = \omega_- = -\gamma i$, thus we have

$$x(t) = Ae^{i\omega t} = Ae^{-\gamma t}$$

Example 1.4: Overdamped

Suppose ${\omega_0}^2 - \gamma^2 < 0 \;\Rightarrow\; {\omega_0}^2 < \gamma^2$

In this case we have

$$\begin{aligned} \omega &= -\gamma i \pm \sqrt{{\omega_0}^2 - \gamma^2} \\ &= -\gamma i \pm i\sqrt{\gamma^2 - {\omega_0}^2} \\ &= -i\left(\gamma \pm \sqrt{\gamma^2 - {\omega_0}^2}\right) \end{aligned}$$

Therefore we have

$$
\begin{aligned}
x(t) &= Ae^{-i\omega_+ t} + Be^{-i\omega_- t} \\
&= Ae^{-\left(\gamma - \sqrt{\gamma^2 - \omega_0{}^2}\right)t} + Be^{-\left(\gamma + \sqrt{\gamma^2 - \omega_0{}^2}\right)t}
\end{aligned}
$$

Example 1.5: Underdamp

Suppose $\omega_0{}^2 - \gamma^2 > 0 \;\Rightarrow\; \omega_0{}^2 > \gamma^2$

In this case we have

$$
\begin{aligned}
\omega &= -\gamma i \pm \sqrt{\omega_0{}^2 - \gamma^2} \\
&= -\gamma i \pm \Omega
\end{aligned}
$$

Therefore, we still have

$$
\begin{aligned}
x(t) &= A\exp\Big\{ -i\left(-\gamma i + \Omega\right)t\Big\} + B\exp\Big\{ -i\left(-\gamma i - \Omega\right)\Big\} \\
&= e^{-\gamma t}\Big[A\exp\big\{-i\Omega t\big\} + B\exp\big\{i\Omega t\big\}\Big] \\
&= e^{-\gamma t}\Big[\underbrace{(A+B)\cos(\Omega t)}_{real\,-\,SHO} - \underbrace{i(A-B)\sin(\Omega t)}_{imaginary\ oscillator}\Big]
\end{aligned}
$$

Remark: Generally, we have $x(t) \propto e^{-\gamma t}\cos(\Omega t)$.

1.6 Forced Oscillation

1. A damped oscillator left to itself will eventually stop moving
2. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic way
3. We call this additional force a **driving force**

   Therefore, we have

$$\sum \vec{F} = m\vec{a} = -kx - bv + F_{app}(t)$$

since we know that $F_{app}(t)$ is a periodic function, we may have

$$\sum \vec{F} = m\vec{a} = -kx - bv + F_{max}\sin(\omega_d t) = 0$$

In general, the solution will be

$$A = \frac{F_{max}/m}{\sqrt{(\omega_0{}^2 - \omega_d{}^2)^2 + 4\gamma^2\omega_d{}^2}} \approx \frac{F_{max}/m}{2\gamma\omega_d}$$

the approx sign works when the damping is small ($\omega_d \approx \omega_0$).

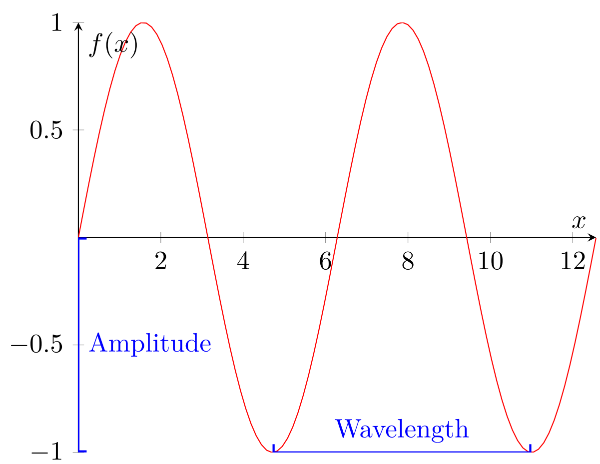
1.7 Waves

1. Almost nothing is more central to physics than waves
2. Light and sounds are waves
3. It is an **oscillation** of something in both **space and time**!
4. Types of waves
   (a) A wave on a string is a type of **mechanical wave**
   (b) A travelling wave that causes the elements of the disturbed medium to move perpendicular to the direction of propagation is called a **transverse wave**
   (c) A travelling wave that causes the elements of the disturbed medium to move parallel to the direction of propagation is called a **longitudinal wave**
   Example: Sound waves are an example of **longitudinal wave**
   Remark: Mechanical longitudinal waves, also called **compressional** or **compression waves**
   (d) A series of drops falling into water produces a **periodic wave** that spreads radially outward. The wavelength λ is the **radial distance** between the **crests** (max) or the **troughs** (min).
   (e) In **sinusoidal wave**, to create a series of pulses, the string can be attached to an oscillating blade.

At a fixed location x_0, we have $y = A\cos(\omega x + \varphi)$, where $\omega = 2\pi/T$.

Lecture 6 - Thu - Jan 25 - 2024

Recall a siusoidal wave on a string, It is important to note that particles only move up and down!



In general, the equation for a fixed position x_0 at time t is given by

$$y(x_0, t) = A\cos(kx_0 \pm \omega t + \varphi)$$

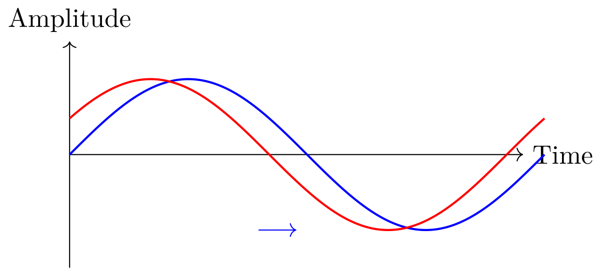
Therefore, we can eliminate the phase constant by choosing our time of origin :

$$y(x_0, t) = A\cos(kx_0 \pm \omega t)$$

Definition 1.3: What is k and ω ?

Similarly as what we did for period T, we have

$$k = \frac{2\pi}{\lambda} \quad \omega = \frac{2\pi}{T}$$



Example 1.6: In which way is the wave moving?

Still, it is important to remember that all the fixed point on a wave only moves up and down. But we can imagine that the wave is "pushing" at a horizontal direction.

1. Wave moving right
$$y(x,t) = A\cos(kx - \omega t)$$
2. Wave moving left
$$y(x,t) = A\cos(kx + \omega t)$$

1.8 Velocity of Waves

Result 1.4: Velocity of the Wave

Exercise: We can also obtain that the velocity of the wave is

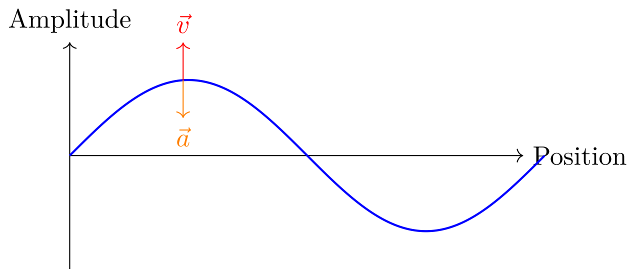
$$\frac{\lambda}{T} = f\lambda = f\frac{2\pi}{k} = \frac{\omega}{k}$$

Remark: The velocity of any wave is set by the physical state of the medium.

Result 1.5

For an ideal string,

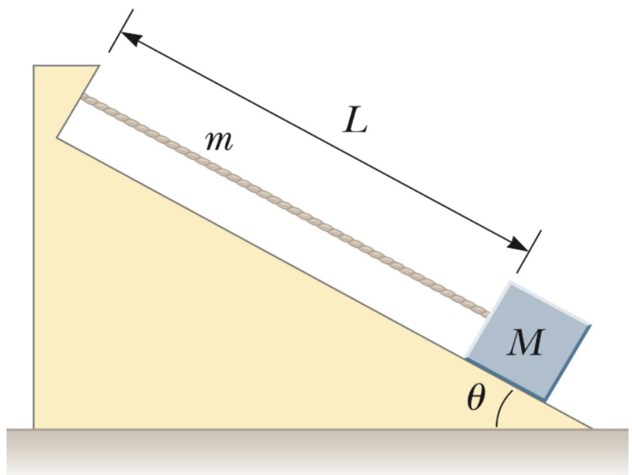
$$v = \sqrt{\frac{T}{\mu}}, \qquad \text{where } T \text{ is tension and } \mu = \frac{m}{L}$$



Refering to the diagram above, we notice that the transverse velocity and acceleration is dependent on time and position.

Theory 1.2

1. The speed of the wave v, is a constant for a uniform medium.
2. Transverse velocity, v_y of a point on the string varies sinusoidally.



Example 1.7

A block of mass M, supported by a string, rests on a frictionless incline making an angle of θ with the horizontal. The length of the string is L and its mass is $m << M$. Derive an expression for the time interval required for a transverse wave to travel from one end of the string to the other.

Proof: We have

$$\sum \vec{F} = Mg\sin\theta - T = \max = 0$$
$$\Rightarrow \quad T = Mg\sin\theta$$

Therefore,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg\sin\theta}{m/L}} = \sqrt{\frac{M}{m}}\sqrt{Lg\sin\theta}$$

and thus the time interval required is

$$t = \frac{L}{v} = \sqrt{\frac{m}{M}} \frac{L}{\sqrt{Lg \sin\theta}} = \sqrt{\frac{mL}{Mg \sin\theta}}$$

as desired. □

Example 1.8

A $1.40m$ string of weight $0.0128N$ is tied to the ceiling at its upper end, and the lower end supports a weight W. When you pluck the string slightly, the waves travelling up the string obey the equation

$$y(x,t) = (8.50mm)\cos(172\text{rad} \cdot m^{-1}x - 2730\text{rad} \cdot s^{-1}t)$$

Assume that the tension of the string is constant and equal to W.

1. How much time does it take to for the wave to travel the full length of the string
2. What is the weight, W
3. How many wavelengths are on the string at any instant of time

Proof:

1. We have
$$t = \frac{L}{v} = \frac{Lk}{\omega}$$
2. Since we have $T = W$ and $v = \sqrt{\frac{T}{\mu}}$, so
$$\mu = \frac{m}{L}$$
Hence we can find
$$T = \mu v^2 = \mu \left(\frac{\omega}{k}\right)^2 = W$$
3. Recall we have $\lambda = \frac{2\pi}{k}$, so
$$n = \frac{L}{\lambda} = \frac{Lk}{2\pi}$$

as desired. □

Now we have

Theory 1.3

With respect to time t:

$$
\begin{aligned}
y &= A\cos(kx - \omega t) \\
v_y &= \frac{\partial y}{\partial t} = \omega A \sin(kx - \omega t) \\
a_y &= \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \cos(kx - \omega t)
\end{aligned}
$$

With respect to position x:

$$
\begin{aligned}
v_y &= \frac{\partial y}{\partial x} = -kA \sin(kx - \omega t) \\
a_y &= \frac{\partial^2 y}{\partial x^2} = -k^2 A \cos(kx - \omega t)
\end{aligned}
$$

Result 1.6

As a result, we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial x^2}$$

which yields us

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial x^2}}$$

Lecture 7 - Tue - Jan 30 - 2024

Theory 1.4

The speef of all mechanical waves follows a general form

$$v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$$

For instance, the speed of sound depends on the characteristic of the medium. The speed of the sound also depends on the temperature of the medium.

Example 1.9

For air, the relationship between the speed and temperature is

$$v = (331\ m/s)\sqrt{\frac{T_K}{273}}$$

where T_K is the air temperature in Kelvin.

1.9 Power in Waves

Recall that work done is defined as

$$W = \vec{F} \cdot \vec{ds}$$

Definition 1.4: Power

Therefore, power is

$$\begin{aligned}\mathcal{P} = \vec{F} \cdot \frac{\vec{ds}}{dt} &= \vec{F} \cdot \vec{v} \\ &= T_y \cdot v_y \\ &= \left(T\,\frac{\partial y}{\partial x}\right)\frac{\partial t}{\partial t}\end{aligned}$$

Remark: Note that there is no work being done in the x direction.
Therefore, as a conclusion

$$\begin{aligned}\mathcal{P} &= T\Big[kA\sin(kx-\omega t)\Big]\Big[\omega A\sin(kx-\omega t)\Big] \\ &= \boxed{\omega^2\sqrt{\mu T}A^2\sin^2(kx-\omega t)}\end{aligned}$$

Remark: $\mathcal{P} > 0$, $\max\mathcal{P} = \omega^2 A^2\sqrt{\mu T}$

Result 1.7: Average Power

As a result,

$$\langle \mathcal{P} \rangle \;=\; \omega^2 A^2 \sqrt{\mu T}\,\langle \sin^2(kx-\omega t)\rangle$$

where $\langle \sin^2\theta \rangle \;=\; \frac{1}{2}$ (given), and thus,

$$\langle \mathcal{P} \rangle \;=\; \frac{1}{2}\omega^2 A^2\sqrt{\mu T} \quad \propto \quad \omega^2 A^2$$

1.10 Wave Intensity

Definition 1.5: Intensity

The **intensity** of a wave is the average power it carries per unit area.

Theory 1.5: Intensity

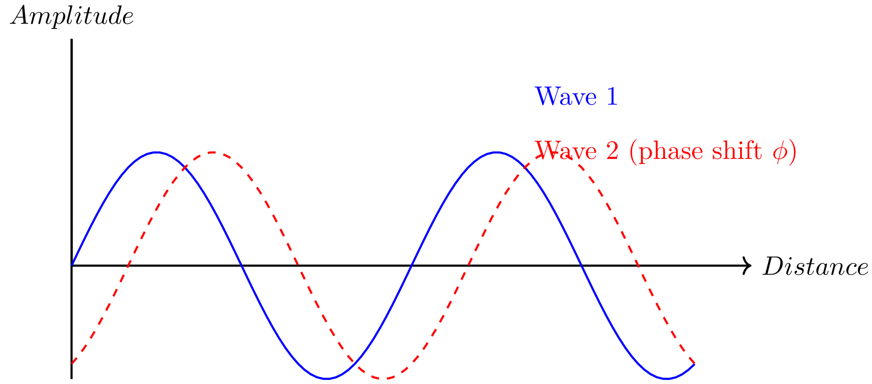
$$I = \frac{power}{area} \;\to\; I = \frac{power}{4\pi r^2}$$

1.10.1 Principle of Superposition

Theory 1.6: Principle of Superposition

Suppose we have two waves such that

$$y(x,t) = y_1(x,t) + y_2(x,t)$$



Example 1.10

Consider two identical waves travelling in the same direction, but offset by a phase angle ϕ, thus

$$y = A\sin(kx - \omega t) + A\sin(kx - \omega t + \phi)$$

Remark:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)$$

Therefore

$$y = 2A\sin\left(kx - \omega t + \frac{\phi}{2}\right)\underbrace{\cos\left(\frac{\phi}{2}\right)}_{constant}$$

$$= \boxed{\left[2A\cos\left(\frac{\phi}{2}\right)\right]\sin\left(kx - \omega t + \frac{\phi}{2}\right)}$$

Exercise:

1. If $\phi = 0$, then $A' = 2A$.
2. If $\phi = \pi$, then $A' = 0$.

1.11 Reflection and Introduction to Standing Waves

1. Require two identical waves travelling in opposite directions that have the right wavelength
2. Occurs when a wave with the right wavelength interferes with its reflection
3. The resultant wave does not appear to move
4. The end of the string can be fixed or loose

Consider two waves travelling in opposite direction offset by $\phi = \pi$,

$$\begin{aligned} y(x,t) &= A\cos(kx-\omega t) + A\cos(kx-\omega t+\pi) \\ &= A\cos(kx-\omega t) - A\cos(kx-\omega t) \end{aligned}$$

Recall the trigonometric identity to span out each cos, we have

$$y(x,t) = \boxed{2A\sin kx \sin \omega t}$$

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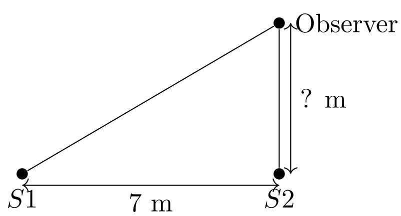
On a string which is doing wave-like motion, the tention in the x-direction, T_x, remains constant. On the other hand, the tension in the y-direction, T_y keeps changing. And we have

$$\vec{F} = T_y = -\frac{\partial y}{\partial x}$$

Example 1.11

Two identical sound sources $S1$ and $S2$ emit sound at $f = 520\ Hz$ and are in phase. The sources are seperated by $7\ m$. If an observer starts walking from $S2$ along a line perpendicular to the line joining $S1$ and $S2$, how far from $S2$ will the observer be when they here the first sound minimum?



Proof: Suppose Δs is the difference between the distance from the two different sound sources, and we know that it has to be a multiple of $\frac{1}{2}\lambda$ for the two sound to cancel each other out. Moreover, we know that the wavelength is

$$\lambda = \frac{v}{f} = \frac{340\ m/s}{520\ Hz} \approx 0.654\ m$$

We also have $D_1 = D_2 + 10.5\lambda$ with $D_1^2 = D_2^2 + 7^2$, combining the two equations and solving for D_2 we get $D_2 \approx 0.136\ m$. $\square$

2 Sound and Hearing

2.1 Beats

Definition 2.1: Beats

When two waves with different frequency interfere, the result is an inteference pattern with **BEAT**s

$$f_{beat} = f_a - f_b$$

Result 2.1

(a) 2 waves, same direction, same λ, f, $\implies$, travelling with same λ, different A

(b) 2 waves, same direction, same λ, different f, $\implies$, travelling with beats

(c) 2 waves, opposite direction, same λ, f and $\Delta\varphi = \pi$, $\implies$,

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) + A\cos(kx + \omega t + \pi) \\ &= \underbrace{2A\sin(kx)}_{A'(x)}\underbrace{\sin(\omega t)}_{SHO} \end{aligned}$$

which is not a travelling wave.

For L is the length between the two nodes, our requirement is that we need to have $\sin(k \cdot L) = 0$, which means that $k \cdot L$ is a multiple of π. Moreover

$$k \cdot L = n\pi = \frac{2\pi}{\lambda_n} \cdot L$$

Hence we have

$$\lambda_n = \frac{2L}{n} \qquad \text{for } n = 1, 2, 3, \ldots$$

Therefore, we have that the nodes occur at

$$x_m = \frac{m\lambda}{2} \qquad \text{for } m = 1, 2, 3, \ldots$$

Remark: The number of n also determines the shape of the wave:

1. $n = 1$: Fundamental harmonic (aka first harmonic)
2. $n = 2$: second harmonic
3. $n = 3$: third harmonic
4. $\cdots$

Theory 2.1: String with tension T

In this case, we have

$$\lambda_n = \frac{2L}{n}, \quad f_n = \frac{v}{\lambda_n} = \sqrt{\frac{T}{\mu}} \cdot \frac{1}{\lambda_n} = \sqrt{\frac{T}{\mu}} \cdot \frac{n}{2L}$$

(i) Keep f fixed, reduce T
We have $n \propto \frac{f_n}{\sqrt{T}}$, which implies that the node with increase

(ii) Keep T fixed, increase f, n increases

Example 2.1

A 1kg mass is attached to a 0.13kg string, the string is fixed to the wall at one end, extends horizontally for 1m, then hangs 30cm vertically via a massless, frictionless pulley. A m=3 standing wave of amplitude 10cm is excited between the wall and the pulley. Find the equation of the wave length travelling to the right.

Proof: We have

$$L = 1m = \frac{n\lambda}{2} = \frac{3\lambda}{2}$$

which means that

$$\lambda_3 = \frac{2}{3}\, m \qquad k_3 = \frac{2\pi}{\lambda_3} = 3\pi$$

Moreover, since we know that $v = \sqrt{T/\mu}$, and we have that $\mu = 0.13\ kg/1.3\ m = 0.1\ kg/m$, and the tension $T = 1\ kg \cdot 9.8\ N/kg = 9.8\ N$, so

$$v = \sqrt{\frac{9.8}{0.1}} = 9.9 = \frac{\omega}{k_3} \quad \Rightarrow \quad \omega = 93.3 \text{ rad}/s$$

Therefore, we have the equation

$$\begin{aligned} y &= A\cos(kx - \omega t) \\ &= 0.05\cos(3\pi - 93.3t) \end{aligned}$$

as desired. □

2.2 Doppler effect

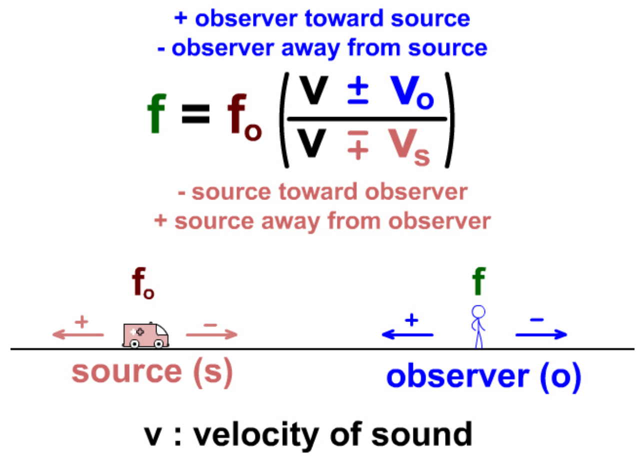
The Doppler effect explains the observed change in pitch of the siren on a fire engine or ambulance.

Theory 2.2

(a) The frequency shift higher ($f_L > f_S$) when the source is approaching you.
(b) The frequency shift lower ($f_L < f_S$) when the source is moving away from you.

Remark:

f_L = listener
f_S = source



2.3 Summary

1. Travelling wave

$$y = A\sin(kx - \omega t) + A\sin(kx - \omega t + \varphi) = 2A\sin\left(\frac{\varphi}{2}\right)\sin\left(kx - \omega t + \frac{\varphi}{2}\right)$$

2. Standing wave

$$y = A\sin(kx - \omega t) + A\cos(kx + \omega t) = 2A\sin(kx)\sin(\omega t)$$

3. Travelling × Travelling

$$y = A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x \pm \omega_2 t) = 2A\sin\left[\frac{k_1 + k_2}{2}\cdot x + \frac{\omega_1 \mp \omega_2}{2}\cdot t\right]\cos\left[\frac{k_1 - k_2}{2}\cdot x - \frac{\omega_1 \pm \omega_2}{2}\cdot t\right]$$

2.4 The Decibel Scale

Because the ear is sentitive over a broad range of intensities, a logarithmic measure of intensity called **sound intensity level** is often used

$$\beta = (10\ db) \log \left(\frac{I}{I_0} \right)$$

where I and I_0 are the intensity of the two sounds.

3 Electric Charge and Electric Field

3.1 Electric Charge

Definition 3.1: Charge

There is another property of matter, and that is what we called: **charge**.

Like mass, charge is a fundamental property of matter.

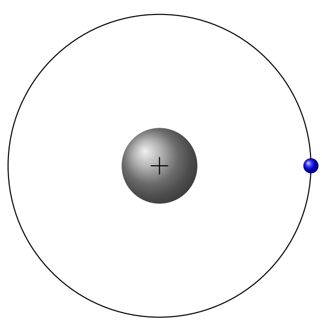
Theory 3.1: Charge has two types

There are two types of charge: **negative and positive**.

Plastic rods, glass, silk and fur are particularly goof for demonstrating electrostatics, the interactions between electric charges that are at rest (or nearly so).

Theory 3.2

Like charges repel, opposive charges attract



- *Image of an atom.*

The atom is made of negative electrons, the postive protons and uncharged neutrons. Protons and neutrons make up the tiny dense nucleus, which is surrounded by electrons.

Result 3.1

Total chartge on an object is the number of extra electrtons or protons (n) times the charge

$$q_{total} = nq$$

1. Electrons: $q_e = -1.6 \cdot 10^{-19}\ C$
2. Protons: $q_e = 1.6 \cdot 10^{-19}\ C$

where C is short for **Coulomb**, the standard unit of charge

Theory 3.3

The magnitude of chatge if the elecvton or proton is a natutral unit of chatge

$$e = 1.6 \times 10^{-10}\ Coulombs$$

All observable charge is quantized in this unit.

3.2 Principle of Charge Conservation

Theory 3.4: Principle of Charge Conservation

The principle of charge conservation states that the algebraic sum of all the electric charges in any closed system is constant.

3.3 Conductors, Insulators, and Induced Charges

Definition 3.2: Insulators, Conductors

Conductor allows current to flow easily through it.
Insulators don't allow current to flow through it.

Definition 3.3: Polarization

The negatively charged plastic comb causes a slight shifting of charge within the molecules of the neutral insulator, an effect called **Polarization**.

3.4 Charging by Induction

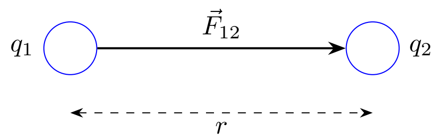
1. Bring a Charged Object Near a Conductor: A charged object (e.g., a negatively charged rod) is brought near a neutral conductor (e.g., a metal sphere), causing electrons in the conductor to move away from the rod, creating a separation of charges within the conductor.

2. Ground the Conductor: Connecting the conductor to the ground allows excess electrons to leave (if the rod is negatively charged) or enter (if positively charged), making the conductor acquire a net charge opposite to that of the rod.

3. Remove the Ground Connection: Once the ground is disconnected, the conductor retains its net charge.

4. Withdraw the Charged Object: Removing the charged object leaves the conductor permanently charged, without ever having been in direct contact with the charge source.

3.5 Coulomb's Law



Theory 3.5: Magnitude of Electrostatic Force

The magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

$$F = \frac{k|q_1 q_2|}{r^2}$$

where r is the distance between the two charges, q_1 and q_2 are the amount of charges on object 1 and 2 respectively.

$$k = 8.99 \times 10^9 \, \frac{N \cdot m^2}{C^2} = \frac{1}{4\pi\varepsilon_0}.$$

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Example 3.1

Two fixed particles, of charges $q_a = +1.0\ \mu C$ and $q_b = -5.0\ \mu C$, are 8 cm apart. At what coordinates should a third charge be located so that no net electrostatic force acts on it?

Proof: T-word □

4 Electric Fields and Electric Forces

Definition 4.1

Electric field is a measure of the direction and relative strength of the electrostatic force that would be exerted on a positive charge places at a given location.
The electric force on a charged body is exerted by the electric field created by other charged bodies.
Since force is a vector, field is also a vector.

Recall that suppose we have two charges q_{source} and q_{target}, then the force between the two charges are

$$\vec{F}_{12} = k\frac{q_{\text{source}}\, q_{\text{target}}}{r^2}\hat{r}$$

Now in terms of electric field, its definition is electric force per unit charge:

Result 4.1: electric field of a point charge

$$\vec{E} = \frac{\vec{F}}{q_{\text{target}}} = k\frac{q_{\text{source}}}{r^2}\hat{r}$$

therefore, as a consequence, we have

$$\vec{F} = q_{\text{target}}\,\vec{E} = m\vec{a}$$
$$\Rightarrow \quad \vec{a} = \boxed{\frac{q_i}{m_i}\,\vec{E}}$$

4.1 Electric Field Lines

Definition 4.2: Electric Field Lines

At any given point, the electric field vector is tangent to the **electric field line**.

At any point, the electric field has a unique direction, so the electric field lines never intersect.

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4.2 Electric Dipoles

An electric dipole is a pair of point charges with equal magnitude and opposite sign (a positive charge q and a negative charge $-q$) separated by a distance d.

4.3 Force and Torque on an Electric Dipole

When a dipole is placed in a uniform electric field, the net force is always zero, but there can be a net torque on the dipole

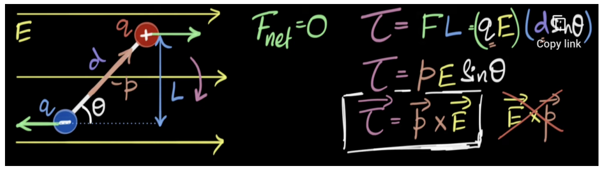
Definition 4.3: Force and Torque on a Dipole

Two equal and opposite charges separated by some distance constitute a dipole. The product of the charge and distance between them is called the dipole moment.

$$\boxed{\vec{p} = q\vec{d}} \qquad \text{Dipole Moment (vector)}$$
$$\vec{d} \qquad \text{Distance from negative to positive}$$

Theory 4.1

When an electric dipole is placed in a uniform field, it experiences a torque.



thus we have

$$\boxed{\sum \vec{\tau} = \vec{p} \times \vec{E}}$$

4.4 Potential Energy of an Electric Dipole

Definition 4.4: Potential Energy

We also define the potential energy to be

$$\boxed{U = -\vec{p} \cdot \vec{E} \cos\phi}$$

Theory 4.2: Work done by displacing dipole through andgle ϕ

We have

$$dW = \vec{\tau} d\phi = (\vec{p} \times \vec{E}) d\phi = -pE \sin\phi \, d\phi$$

Because the torque is in the direction of decreasing ϕ, we must write the torque as $\tau = -pE \sin\phi$

4.5 Field of an Electric Dipole

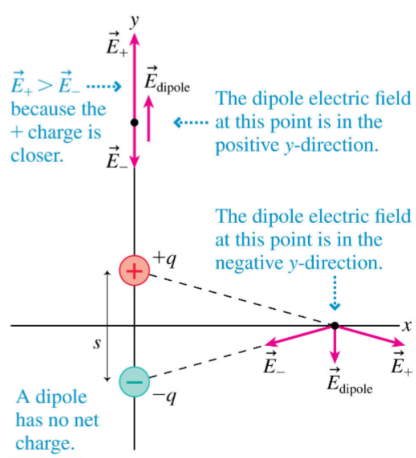
In written HW 1 we found the force associated with the electric dipole and a charge Q "on axis" with the dipole.

It is useful to define the **dipole moment** $\vec{p} = q\vec{s}$ where $\vec{s}$ is a displacement vector which points from negative q to positive q. The direction of $\vec{p}$ identifies the orientation of the dipole and the dipole moment magnitude determines the electric field strength. The dipole moment can be used to write a succinct approximate expression for the fields in both cases (see below).

$$\vec{E}_{axis} \approx \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{r^3}$$

$$\vec{E}_{bisect} \approx \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}}{r^3}$$

In this problem we will explore the electric potential of an electric dipole.



4.6 Continuous Charge Distribtions

When the charges are packed densely in a volumn, suppose we want to find the field at a point P:

1. Divide the charge distribution into infinitesimal elements dq.
2. Consider each element as a point charge and calculate the vector field $d\vec{E}$ due to each at the observation point.
3. Add the individual vector fields to obtain the resultant field $\vec{E}$.

Definition 4.5: Charge (uniform) in a Volumn

In a uniform distribution, we define

$$\text{Charge densty} : \rho = \frac{Q}{V} = \frac{dq}{dV}$$

thus the total charge is given by

$$\int_V dq = \int_V \rho \, dV$$

Definition 4.6: Charge (uniform) in a Area

In a uniform distribution, we define

$$\text{Charge densty} : \sigma = \frac{Q}{A} = \frac{dq}{dA}$$

thus the total charge is given by

$$\int_V dq = \int_V \sigma \, dA$$

Definition 4.7: Charge (uniform) in a Line

In a uniform distribution, we define

$$\text{Charge densty} : \lambda = \frac{Q}{L} = \frac{dq}{dL}$$

thus the total charge is given by

$$\int_V dq = \int_V \lambda \, dL$$

Example 4.1: Electric Field due to a uniformly charged ring

We have

$$\lambda = \frac{dq}{dL} = \frac{Q}{L} = \frac{Q}{2\pi a}$$

and

$$\begin{aligned}
dq &= \lambda \, dl \\
&= \frac{Q}{2\pi a} \, dl \\
&= \frac{Q}{2\pi a} a \, d\phi \\
&= Q \frac{d\phi}{2\pi}
\end{aligned}$$

Thus

$$|d\vec{E}| = \frac{k \, dq}{r^2} = \frac{kQ \, d\phi}{2\pi(x^2 + a^2)}$$

However, we only care about the x-direction since y-direction will all cancel out,

$$|d\vec{E}_x| = |d\vec{E}| \cos(\alpha) = \frac{kQx \, d\phi}{2\pi(x^2 + a^2)^{3/2}}$$

Therefore,

$$\begin{aligned}
E_x &= \int \, dE_x \\
&= \int_0^{2\pi} \frac{kQx}{2\pi(x^2 + a^2)^{3/2}} \, d\phi \\
&= kQ \cdot \frac{x}{(x^2 + a^2)^{3/2}} \cdot \hat{i}
\end{aligned}$$

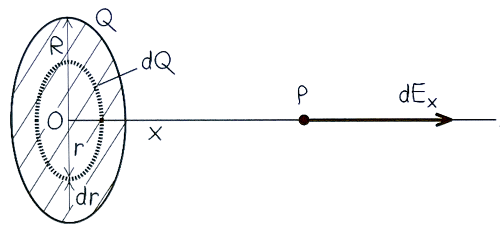
For $x \gg a$, we would have=

$$\vec{E} \approx \frac{kQ}{x^2} \cdot \hat{i}$$

Example 4.2: Electric Field due to a uniformly charged disk

Electric Field to due a uniformly charged disk

What is the electric field a distance x from the centre of the disk?



Surface charge density is

$$\sigma = \frac{Q}{\pi R^2}$$

From last time, we obtained the electric field due to a charged ring, in this case of a disk, we can view it as a bunch of rings, thus we have

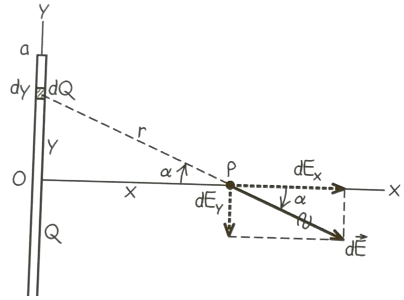
$$\begin{aligned}
dQ &= \sigma\, dA = \sigma \cdot 2\pi r\, dr \\
&= \frac{2\pi Q r\, dr}{\pi R^2} \\
&= \frac{2Qr}{R^2}\, dr \\
\Rightarrow \quad d\vec{E} &= \frac{2\pi k Q r x\, dr}{\pi R^2 (x^2+r^2)^{3/2}} \cdot \hat{i} \\
\Rightarrow \quad \vec{E} = E_x \hat{i} &= \int_0^R \frac{2kQrx}{R^2(x^2+r^2)^{3/2}}\, dr \\
&= \frac{2kQx}{R^2} \int_0^R \frac{r}{(x^2+r^2)^{3/2}}\, dr \\
&= -\frac{2kQx}{R^2\sqrt{x^2+r^2}}\bigg|_0^R \\
&= \frac{2kQ}{R^2}\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right] \cdot \hat{i}
\end{aligned}$$

Therefore, for $x \ll R$, we have

$$\vec{E} = 2\pi k \sigma \hat{i}$$

Example 4.3: Electric Field due to a Uniformly Charged Line

What is the electric field a distance x from the centre of the line?



Hence we have the charge density

$$\lambda = \frac{dq}{dy} = \frac{Q}{2a}$$

hence

$$d\vec{E} = \frac{k\ dq}{r^2} = \frac{kQ\ dy}{2a(x^2+y^2)}$$

$$dE_x = d\vec{E}\cos(\alpha)$$

$$= \frac{kQx\ dy}{2a(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad E_x = \int_{-a}^{a} \frac{kQx}{2a(x^2+y^2)^{3/2}}\ dy$$

$$= \frac{kQ}{x\sqrt{x^2+a^2}}$$

$$= \frac{2ka\lambda}{x\sqrt{x^2+a^2}} \cdot \hat{i}$$

thus when $x \gg a$, then

$$\vec{E} = \frac{2k\lambda}{x} \cdot \hat{i}$$

5 Gauss's Law

5.1 Charge and Electric Flux

Definition 5.1: Flux

Flux is the amount of something that flows into or out of a surface area or volume.
We define fluc to be positive for flow out of a volume.
Additionally:

1. Whether there is a net outward or inward electric flux through a closed sur- face depends on the sign of the enclosed charge.
2. Charges outside the surface do not give a net electric flux through the sur- face.
3. The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.

5.2 Calculating Electric Flux

5.2.1 Flux of a Uniform Electric Field

Definition 5.2: Electric Flux: uniform field

Consider a flat area tilted to a uniform electric field with angle ϕ, we would then have

$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos(\phi)$$

Example 5.1

A sylinder of radius 3 meters is placed in a uniform electric field of $E = 4\ N/C$ directed to the right. The electric flux through the right surface would then be

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

5.2.2 Flux of a Nonuniform Electric Field

Definition 5.3: Electric Flux: non-uniform field

In this case, we would have

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} = \int_S E \cdot (\phi)\, dA$$

where S implies that we are integrating over the whole surface.
This is the general definition of electric flux.

5.3 Gauss's Law

Gauss's law is an alternative to Coulomb's law. While completely equivalent to Coulomb's law, Gauss's law provides a different way to express the relationship between electric charge and electric field.

Theory 5.1: Gauss' Law

Let A_{encl} be the total charge enclosed by a surface,

Gauss's law states that the total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ε_0, no matter how weird-looking that enclosed shape is:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

where ε_0 is the electric constant.

Remark: The notation $\oint$ implies that the surface has to be closed, and note that Gauss's law only works when the surface is closed.

Gauss's law is equivalent to the Coulomb's law.

Example 5.2

As for fun, we can sketch the derivation of the law (for a sphere):

$$\Phi_E = \oint_S E_\perp \, dA = \oint_S \left(\frac{q}{4\pi\varepsilon_0 r^2} \right) dA = \frac{q}{4\pi\varepsilon_0 r^2} \oint_S dA = \frac{q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

Result 5.1

An insulating, hollow sphere has inner radius a and outer radius b. Within the insulating material, the volume charge density is given by

$$\rho(r) = \frac{\alpha}{r}$$

where $\alpha > 0$ is a positive constant. Find the magnitude of the electric field at a distance r from the centre of the shell, for $a < r < b$.

Proof: We have

$$Q = \int_a^r \rho \cdot 4\pi r^2 \, dr$$

as desired. $\square$

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5.4 Charges on Conductors

Example 5.3

A solid conductor with a cavity carries a total charge of $+7\ nC$. Within the cavity, insulated from the conductor, is a point charge of $-5\ nC$. How much charge is on each surface (inner and outer) of the conductor?

Proof: If the charge in the cavity is $q = -5\ nC$, the charge on the inner cavity surface must be $-q = -(-5\ nC) = +5\ nC$. The conductor carries a total charge of $+7\ nC$, none of which is in the interior of the material. If $+5\ nC$ is on the inner surface of the cavity, then there must be $+7\ nC - +5\ nC = +2\ nC$ on the outer surface of the conductor. □

Charge Distribution	**Point in Electric Field**	**Electric Field Magnitude**
Single point charge q	Distance r from q	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
Charge q on surface of conducting sphere with radius R	Outside sphere, $r > R$ Inside sphere, $r < R$	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $E = 0$
Infinite wire, charge per unit length λ	Distance r from wire	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
Infinite conducting cylinder with radius R, charge per unit length λ	Outside cylinder, $r > R$ Inside cylinder, $r < R$	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$ $E = 0$
Solid insulating sphere with radius R, charge Q distributed uniformly throughout volume	Outside sphere, $r > R$ Inside sphere, $r < R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ $E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
Infinite sheet of charge with uniform charge per unit area σ	Any point	$E = \frac{\sigma}{2\epsilon_0}$
Two oppositely charged conducting plates with surface charge densities $+\sigma$ and $-\sigma$	Any point between plates	$E = \frac{\sigma}{\epsilon_0}$
Charged conductor	Just outside the conductor	$E = \frac{\sigma}{\epsilon_0}$

Table 1: Electric field of various symmetric charge distributions.

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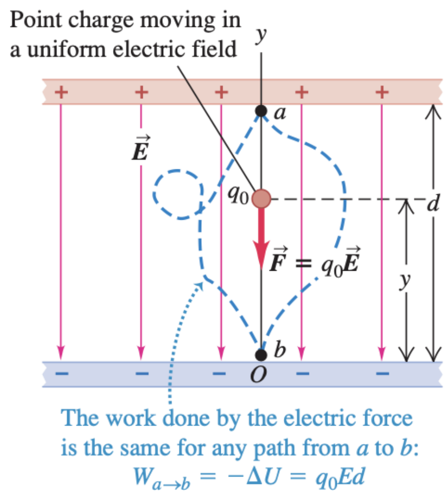
6 Electric Potential (Energy)

6.1 Electric Potential Energy in a Uniform Field

So the work done by the electric field is the product of the force magnitude and the component of displacement in the (downward) direction of the force:

$$W_{a\to b} = Fd = q_0 Ed = -\Delta U$$



6.2 Electric Potential Energy of Two Point Charges

Result 6.1: electric potential energy of point charges q and q_0

Therefore, as a result, the potential energy for point charges is

$$U = \frac{kqq_0}{r} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r}$$

where q_0 is the test charge.

Remark: Don't confuse equation for the potential energy of two point charges with the similar expression in equatoin for the radial component of the electric force that one charge exerts on the other. Potential energy U is proportional to $1/r$, while the force component F is proportional to $1/r^2$.

6.3 Electric Potential Energy with Several Point Charges

Theory 6.1: point charge q_0 and collection of charges q_i

We conclude that the potential energy associated with the test charge q_0 at point a with other charges q_i is the algebraic sum (not a vector sum):

$$U = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots\right) = \frac{q_0}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

6.3.1 Interpreting Electric Potential Energy

Interpretation 6.1: Interpreting Electric Potential Energy

1. When a particle moves from point a to point b, the work done on it by the electric field is $W_{a\to b} = U_a - U_b$. Thus the potential-energy difference $U_a - U_b$ equals the work that is done by the electric force when the particle moves from a to b. When U_a is greater than U_b, the field does positive work on the particle as it "falls" from a point of higher potential energy (a) to a point of lower potential energy (b).

2. An alternative but equivalent viewpoint is to consider how much work we would have to do to "raise" a particle from a point b where the potential energy is U_b to a point a where it has a greater value U_a (pushing two positive charges closer together, for example). To move the particle slowly (so as not to give it any kinetic energy), we need to exert an additional external force $\vec{F}_{ext}$ that is equal and opposite to the electric-field force and does positive work. The potential- energy difference $U_a - U_b$ is then defined as the work that must be done by an external force to move the particle slowly from b to a against the electric force.

6.4 Electric Potential

Definition 6.1: Electric Potential

Now we want to describe this potential energy on a "per unit charge" basis, just as electric field describes the force per unit charge on a charged particle in the field. We define the potential to be

$$V = \frac{U}{q_0} \qquad [J/C] \equiv [\text{volts}]$$

6.4.1 Calculating Electric Potential

Result 6.2

To find the potential V due to a single point charge q, we devide U by q_0:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Similarly, potential due to a collection of point charge is

$$V = \frac{U}{q_0} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

When we have a continuous distribution of charge:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

6.4.2 Finding Electric Potential from Electric Field

Recall that we have

$$W_{a\to b} = U_a - U_b = \int_a^b \vec{F} \cdot d\vec{s}$$

If we divide this by q_0, we find

Definition 6.2: potential difference as an integral of $\vec{E}$

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{s}$$

6.4.3 Electron Volts

Definition 6.3: Electron Volts

When a particle with charge q moves from a point where the potential is V_b to a point where it is V_a, the change in the potential energy U is

$$U_a - U_b = q(V_a - V_b) = qV_{ab}$$

If the charge q equals the magnitude e of the electron charge, $1.602 \cdot 10^{-19}\ C$, and the potential difference is $V_{ab} = 1\ V$, the change in energy is

$$U_a - U_b = (11.602 \cdot 10^{-19}\ C)(1\ V) = 1.602 \cdot 10^{-19}\ J$$

This quantity of energy is defined to be **1 electron volt** ($1\ eV$):

$$1\ eV = 1.602 \cdot 10^{-19}\ J$$

The multiples meV, keV, MeV, GeV, and TeV are often used.

6.5 Potential Gradient

Recall that we have the potential difference as an integral of $\vec{E}$:

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$$

Moreover, $V_a - V_b$ is the potential of a with respect to b, that is, the change of potential encountered on a trip from b to a. We can write this as

$$V_a - V_b = \int_b^a dV = -\int_a^b dV$$

Result 6.3

Therefore, comparing two equations, we can obtain the following:

$$\begin{aligned} -\int_a^b dV &= \int_a^b \vec{E}\cdot d\vec{l} \\ -dV &= \vec{E}\cdot d\vec{l} \\ -dV &= E_x\,dx + E_y\,dy + E_z\,dz \end{aligned}$$

Result 6.4

Therefore, as a result, we have

$$\begin{aligned} \vec{E} &= -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V \\ &= \vec{\nabla}\vec{V} \end{aligned}$$

Theory 6.2: Conservation of Mechanical Energy

In general, for a mass moving from A to B due to a conservative force,

$$K_i + U_i = K_f + U_f$$

For the electric force we have $U = qV$, so

$$\begin{aligned} K_i + qV_i &= K_f + qV_f \\ \frac{1}{2}mv_i^2 - \frac{1}{2}mv_f^2 &= q(V_f - V_i) \end{aligned}$$

6.6 Potential from a Uniformly Charged Disk

Suppose the point is z distance above the disk, and the radius of the disk is R, then

$$V = k\int \frac{dq}{r} \;\Rightarrow\; V_{\text{ring}} = \frac{k}{r}\int dq$$

where we know that

$$\begin{aligned} \int dq_{\text{ring}} &= \sigma\cdot dA_{\text{ring}} \\ &= \sigma\cdot 2\pi R' dR' \end{aligned}$$

which then yields us

$$V_{\text{ring}} = \frac{k\cdot 2\pi\sigma R' dR'}{r}$$

Therefore, do find the potential from the disk, we need to integrate again

$$\begin{aligned} V_{\text{disk}} &= \int_0^R dV \\ &= 2\pi k\sigma \int_0^R \frac{R' dR'}{r} \\ &= 2\pi k\sigma \int_0^R \frac{R' dR'}{\sqrt{z^2 + R^2}} \end{aligned}$$

Result 6.5

Now we can do a u-substitution with $u^2 = z^2 + R^2$, so

$$\begin{aligned} V_{\text{disk}} &= 2\pi k\sigma \int_z^{\sqrt{z^2+R^2}} \frac{u\,du}{d} \\ &= 2\pi k\sigma \left[\sqrt{z^2 + R^2} - z\right] \end{aligned}$$

6.7 Charged Conductor: Electric potential and field

A solid conducting sphere of radius R has a total charge q. Notice that the electric field in the centre of the sphere is simply zero, however, potential in the centre is not zero:

$$V = k \int \frac{dq}{r} = \frac{kq}{R} \qquad r < R$$

Remember that we know that $\vec{E_r} = \frac{dV}{dr} = 0$.

In the case when $r > R$, we can establish a Gaussian surface and get

$$\begin{aligned} 4\pi r^2 E(r) &= \frac{q}{\epsilon_0} \\ E(r) &= \frac{q}{4\pi\epsilon_0 r^2} = \frac{kq}{r^2} \\ \Rightarrow \quad V &= \frac{1}{4\pi\epsilon_0} \end{aligned}$$

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7 Capacitance and Dielectrics

7.1 Capacitors and Capacitance

A capacitor is a device that stores electric potential energy and electric charge. To make a capacitor, just insulate two conductors from each other. Any two conductors separated by an insulator (or a vacuum) form a capacitor.

Definition 7.1: definition of capacitance

We define capacitance to be

$$C = \frac{Q}{V} \qquad [F \equiv 1\ C/V]$$

We want to explore the relationship between voltage and charge in capacitors.

7.1.1 Calculating Capacitance: Capacitors in Vacuum

Result 7.1

Suppose we have two capacitors with area A and each with charge with magnitude of Q but opposite signs, and assume they are separated by a distance d. We know that the electric field from an infinite sheet of charge with charge density σ is given by $E_{sheet} = \sigma/\varepsilon_0$, where $\sigma = \frac{Q}{A}$. Suppose we have a charge of $1C$, then the difference of potential between the the two sheet is given by

$$V_a - V_b = 1C \cdot E_{sheet} \cdot d\ \ [V]$$

Thereforem the difference in potential of the two points is

$$\Delta V = \frac{\sigma}{\varepsilon_0} \cdot d$$
$$\Rightarrow \quad \frac{Q}{V} = \varepsilon_0 \cdot \frac{A}{d}$$

7.2 Capacitors in Series and Parallel

Theory 7.1: Capacitors in series and parallel

1. Capacitor in parallel:

   $$C_{eq} = C_1 + C_2 + C_3 + \cdots$$

   where C_{eq} is the equivalent capacitance of parallel combination and C_i are capacitances of individual series.

2. Capacitor in series:

   $$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

   where C_{eq} is the equivalent capacitance of parallel combination and C_i are capacitances of individual series.

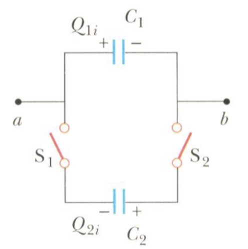
Example 7.1

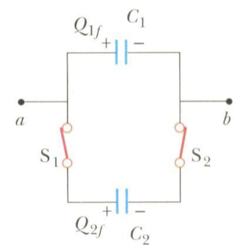
Denote V_i to be the initial voltage, thus we have

$$Q_{1i} = C_1 V_i \qquad \& \qquad Q_{2i} = C_2 V_i$$

Thus the total charge would be $Q = Q_{1i} - Q_{2i}$, and since the circuit is parallel, we can find that the equivalent capacitance is $C = C_1 + C_2$. Therefore,

$$\begin{aligned} V_f = \frac{Q}{C} &= \frac{Q_{1i} - Q_{2i}}{C_1 + C_2} \\ &= \left(\frac{C_1 - C_2}{C_1 + C_2}\right) V_i \end{aligned}$$





Electric potential energy stored is equal to tbe amount of work done to charge the capacitor $\rightarrow$ to seperate charges and place them onto the oposite plates.

7.3 Total work, Stored energy, and Energy density

Result 7.2: Total work, Stored energy, and Energy density

Total work:

$$W = \int_0^Q V(q)\, dq = \int_0^Q \frac{q}{C}\, dq = \frac{Q^2}{2C}$$

and the stored energy is

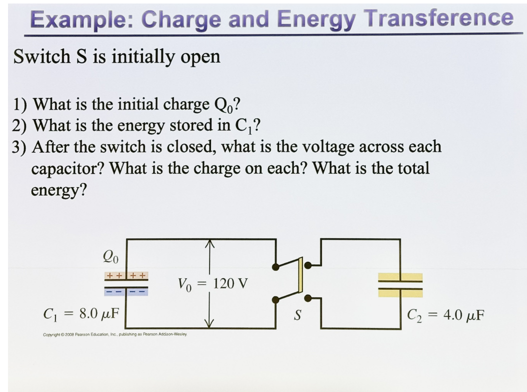
$$U = \frac{Q^2}{2C} = \frac{1}{2} QV = \frac{CV^2}{2}$$

Remark: Recall the the energy stored in a stretched/compressed spring is also in the form of $\frac{1}{2}kx^2$.

Energy density:

$$u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2}{Ad} = \frac{1}{2}\epsilon_0 E^2$$

Example 7.2



1) The initial charge is

$$Q_0 = C_1V_0 = (8.0\mu F)(120V) = 960\mu C$$

2) The energy stored is

$$U_i = \frac{1}{2}Q_0V_0 = 5.76 \cdot 10^{-2}J$$

3) After the switch is closed, then we have

$$Q_1 + Q_2 = Q_0$$
$$C_{eq} = C_1 + C_2$$

thus

$$V_f = \frac{Q_0}{C_{eq}} = \frac{C_1V_0}{C_1 + C_2} = 80V$$

Therefore,

$$Q_1 = C_1V_f = 640\mu C$$
$$Q_2 = C_2V_f = 320\mu C$$

Total energy would then be

$$U_f = \frac{1}{2}Q_1V_f + \frac{1}{2}Q_2V_f = 3.84 \times 10^{-2}J$$

Remark: Notice that we have lost some energy.

7.4 Dielectric

Definition 7.2: What are Dielectrics

1. Dielectric
   An insulating material placed between plates of a capacitor to increase capacitance
2. Dielectric Constant
   A dimentionless factor that determines how much the capacitance is increased by a dieletric. It is a property of the dielectric and varies from one material to another.
3. Dielectric Strength
   Maximum E field before dielectric breaks down and acts as a conductor between the plates (sparks)
4. Breakdown potential
   Maximum potential difference before sparking

Definition 7.3: definition of dielectric constant

The original capacitance C_0 is given by $C_0 = Q/V_0$, and the capacitance C with the dielectric present is $C = Q/V$. The charge Q is the same in both cases, and V is less than V_0, so we conclude that the capacitance C with the dielectric present is greater than C_0. When the space between plates is completely filled by the dielectric, the ratio of C to C_0 (equal to the ratio of V_0 to V) is called the dielectric constant of the material, K:

$$K = \frac{C}{C_0} \quad \equiv \quad V = \frac{V_0}{K}$$

Interpretation 7.1

Just multiply ϵ_0 with the dielectric constant, and change all ϵ_0 with that value and everything works.

Result 7.3

When an insulating material is inserted between the plates of a capacitor whose original capacitance is C_0, the new capacitance is greater by a factor K, where K is the dielectric constant of the material.

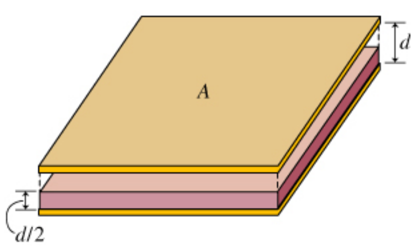
$$C = KC_0 = K\epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d}$$

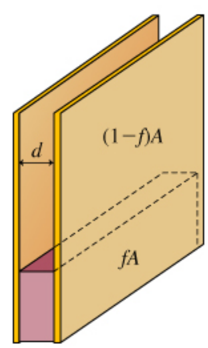
The energy density in the capacitor also increases:

$$u = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2$$

Example 7.3

Consider two parallel-plate capacitors identical in shape, one aligned so that the plates are horizontal (Figure 1), and the other with the plates vertical (Figure 2).





Proof: For Figure 1, the capacitance of the air-filled part is:

$$C_{air} = \frac{\varepsilon_0 A}{d/2} = \frac{2\varepsilon_0 A}{d}$$

The capacitance of the dielectric-filled part is:

$$C_{dielectric} = \frac{\varepsilon_0 K A}{d/2} = \frac{2\varepsilon_0 K A}{d}$$

Therefore

$$\frac{1}{C_h} = \frac{1}{C_{air}} + \frac{1}{C_{dielectric}} \quad \Rightarrow \quad C_h = \frac{2\varepsilon_0 K A}{d(K+1)}$$

Similarly, in the figure to the right, the capacitance of the air-filled part, C_{air}, with area $A_{air} = A(1-f)$, is:

$$C_{air} = \frac{\varepsilon_0 A(1-f)}{d}$$

The capacitance of the dielectric-filled part, $C_{dielectric}$, with area $A_{dielectric} = Af$, is:

$$C_{dielectric} = \frac{\varepsilon_0 K A f}{d}$$

Therefore

$$C_v = \frac{A\varepsilon_0(Kf - f + 1)}{d}$$

which gives us the equation,

$$\frac{2\varepsilon_0 K A}{d(K+1)} = \frac{A\varepsilon_0(Kf - f + 1)}{d}$$

Solving this equation for f will give us

$$f = \frac{1}{K+1}$$

as desired. $\square$

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8 Current, Resistance, and electromotive Force

8.1 Circutis

Definition 8.1: Current

The symbol for current is I or i. The SI unit of current is Ampere (A), $1\ A = 1\ C/s$.

Example 8.1: Two Light Bulbs in Series

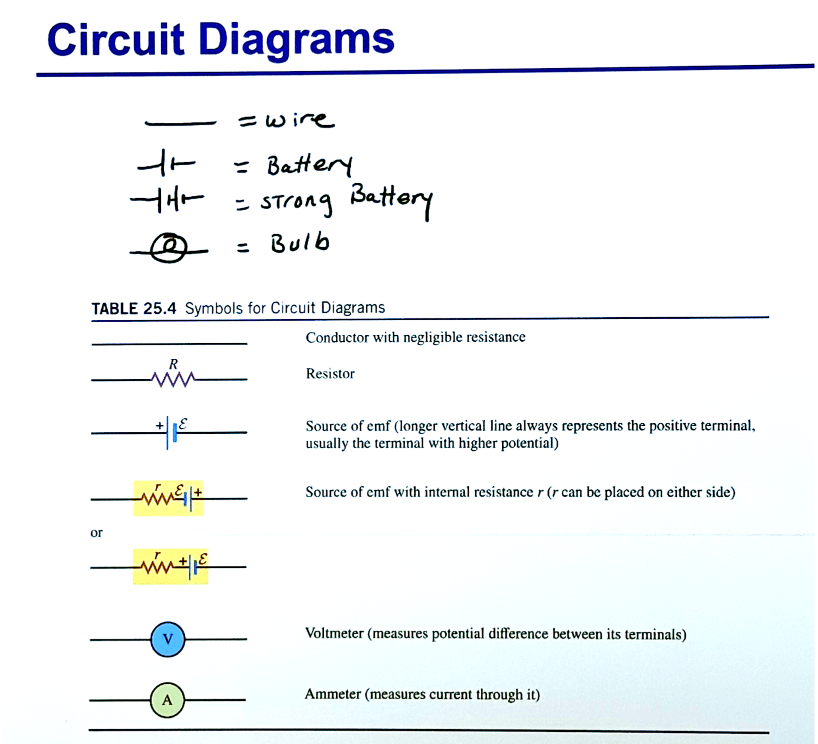
When two identical light bulbs are in series, they have the same brightness, but each of them is less bright than a single light bulb.

Example 8.2: Two Light Bulbs in Parallel

When two identical light bulbs are in series, they have the same brightness, and each of them is the same bright as a single light bulb.

Theory 8.1

Bulbs and other electrical devices DO NOT USE UP CURRENT.
The amount of current is the same everywhere in a single loop circuit.



8.2 Drift Velocity and Current

Interpretation 8.1: drift velocity

A charged particle moving in a conductor undergoes frequent collisions with the massive, nearly stationary ions of the material. In each such collision the particle's direction of motion undergoes a random change. The net effect of the electric field $\vec{E}$ is that in addition to the random motion of the charged particles within the conductor, there is also a very slow net motion or drift of the moving charged particles as a group in the direction of the electric force $\vec{F} = q\vec{E}$. This motion is described in terms of the drift velocity $\vec{v}_d$ of the particles. As a result, there is a net current in the conductor.

8.2.1 Direction of Current

Definition 8.2: Direction of Current

The direction of current is the direction of the "positive charges" are moving, however, in reality, it is the opposite of the direction the negatively charged electrons are moving.

Definition 8.3: Formal Definition for Current

In net charge ΔQ passes throught a plane in time Δt, then the average current is defined as

$$I_{avg} \equiv \frac{\Delta Q}{\Delta t}$$

total charge ΔQ is charge per particle (q_e) $\times$ number of particles (n).

8.2.2 Current Density

Definition 8.4: Current Density

Current density is the amount of charge per unit time that flows through a unit area of a chosen cross section.

$$dQ = nq \cdot A\Delta t v_d$$
$$\frac{dQ}{dt} = I = nqv_d A$$

We can define a vector current density that includes the direction of the drift velocity:

$$\boxed{\vec{J} = \frac{I}{A} = nq\vec{v_d}}$$

The vector current density is always in the same direction as the electric field, no matter what the signs of the charge carriers are.

$$I = \frac{dQ}{dt} = n|q|v_d A$$

where we call n the **concentration** of particles; its SI unit is m^{-3}.

Definition 8.5: Non-uniform Current Density

$$I = \int \vec{J} \cdot \, d\vec{A}$$

Example 8.3

Consider a wire whose current density caries with distance from the centre as $J = ar^2$, where $a = 3.0 \times 10^{11}\ A/m^2$. What current will flow if the radius of the wire is 2.0 mm? How much flows through the core of radius 1.0 mm.

Proof: We have

$$\begin{aligned} I &= \int \vec{J} \cdot d\vec{A} \\ &= \int_0^R ar^2 \cdot 2\pi r \, dr \\ &= 2\pi a \int_0^R r^3 \, dr \\ &= \left(\frac{2\pi a r^4}{4}\right)\Big|_0^R \end{aligned}$$

□

8.3 Resistivity (ρ)

Definition 8.6: Resistivity

The **Resistivity**, ρ, of a resistor is defined as:

$$\rho = \frac{E}{J} \quad \Rightarrow \quad \vec{E} = \rho \vec{J}$$

The **conductivity** of a material is the reciprocal of its resistivity:

$$\sigma = \frac{1}{\rho}$$

8.3.1 Resistivity and Temperature

Remark: The resistivity of a *metallic* conductor nearly always increases with increasing room temperature.

Theory 8.2: Temperature dependence of resistivity

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)]$$

Some material show superconductivity. As the tempreature decreases, the resistivity at first decreases smoothly, like that of any metal. However, below a certain critical temperature T_c a phase transition occurs and the resisticity suddenly drops to zero.

8.4 Resistance (R)

Definition 8.7: Resistance

Light bulbs and all components of electrical circuits, present obstacles or blockages to the flow of current. We call this "electrical obstruction" to the flow of current **Resistance**.
The "constant" of proportionality is the **resistance** of the conductor.

$$R \equiv \frac{\Delta V}{I}$$

Interpretation 8.2: relationship between resistance and resistivity

Since we have $V = EL = \rho JL$ and $I = JA$, we can also see that the resistance R of a particular conductor is related to the resistivity ρ of its material by

$$R = \frac{\rho L}{A}$$

8.4.1 Ohm's Law

Theory 8.3: Ohm's Law

Voltage is equal to the current multiplied by the resistance

$$V = IR$$

Example 8.4

A rectangular lead block of dimension $15cm \times 1.2cm \times 1.2cm$ has a potential difference applied to two of its paralll sides so that they are equipotential surfaces. What is the resistance of the blovk if

(a) the parallel sides are the square ends and

(b) two rectangular sides?

Proof: We know that $\rho_{Pb} = 22 \times 20^{-6}$, so that

$$R = \rho \frac{L}{A} = \frac{22 \times 20^{-6} \cdot 0.15}{(0.012)^2} = 230\mu\Omega$$

□

8.5 Electromotive Force and Circuits

8.5.1 Internal Resistance

Theory 8.4: terminal voltage, source with internal resistance

When a current is flowing through a source from the negative terminal b to the positive terminal a, the potential difference Vab between the terminals is

$$\boxed{V_{ab} = \varepsilon - Ir}$$

The potential V_{ab}, called the **terminal voltage**, is less than the emf E because of the term Ir representing the potential drop across the internal resistance r.

8.6 Energy and Power in Electric Circuits

8.6.1 Power Input to a Pure Resistance

Definition 8.8: Power Input to a Pure Resistance

The rate at which energy is delivered to or extracted from a circuit element is

$$P = V_{ab} I \qquad [W \equiv J/C \cdot C/s]$$

when applied to a circuit element like a bulb or wire, power is the **rate at which energy is transferred from the battery ot power supply to the device.** Using $V = IR$, we have

$$P = I^2 R = \frac{V^2}{R}$$

Example 8.5

Consider a unform wire made or Nichrome, a nickel-chromium allot, with a resistance $R = 72\Omega$, What is the rate of energy dissipation if

1. a potential difference of $120V$ is applied across the full length of the wire
2. if the wire is cut in half and a $120V$ potential difference is applied across the length of each half

Proof:

1. We have $R = 72\Omega$, thus
$$P = \frac{V^2}{R} = 200W$$
2. Similarly,
$$R = \frac{\rho L}{A} \qquad R' = \frac{R}{72\Omega} \Rightarrow \quad P = \frac{V^2}{R} = 400W$$

wwwww □

8.6.2 Power Output of a Source

Result 8.1

For a source that can be described by an emf ε and an internal resistance r, we may use $V_{ab} = \varepsilon - Ir$. Multiplying this equation by I, we find

$$P = V_{ab}I = \varepsilon I - I^2 r$$

8.6.3 Power Input to a Source

Result 8.2

The current I in the circuit is opposite to what we have above; the lower source is pushing current backward through the upper source. Because of this reversal of current, instead of above equation we have for the upper source

$$V_{ab} = \varepsilon + Ir \qquad \Rightarrow \qquad P = V_{ab}I = \varepsilon I + I^2 r$$

8.7 Real Battery

1. Batteries provide energy (e.g. chemical) to move electrons against the electric field . This energy per unit charge is called ε, the electromotive "force" (emf)
2. Real batteries have interbal resistance r.

$$V_{ab} = \varepsilon - Ir$$

3. For an **ideal bettery**, V_{ab} and emf are the same.

9 Direct-Current Circuits

9.1 Series Circuit

In a series circuit, the resistors are wired one after another

$$I_{series(Total)} = I_1 = I_2 = I_3 = \cdots$$
$$V_{series(Total)} = V_1 + V_2 + V_3 + \cdots$$

Theory 9.1

1. Since they are all part of the same loop they have the same amount of surrent run through them.
2. They all exist between the terminals of the battery, so they share the potential (voltage).
3. The total resistance is the sum of all resistors.

9.2 Parallel Circuit

Theory 9.2

Parallel circuits must have places where the current splits and comes back together. We call these **Junctions**. The current going IN to a junction will always equal to the current going OUT of a junction.

$$V_{parallel(Total)} = V_1 = V_2 = V_3 = \cdots$$
$$I_{parallel(Total)} = I_1 + I_2 + I_3 + \cdots$$

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Result 9.1

Two resistors can be in parallel, series or neither. BOTH is not an option.

9.3 Kirchhoff's Rule

Theory 9.3: Kirchhoff's Rule

A junction is a point where three or more conductors meet. Kirchhoff's rule states that the sum of the currents into any juntion

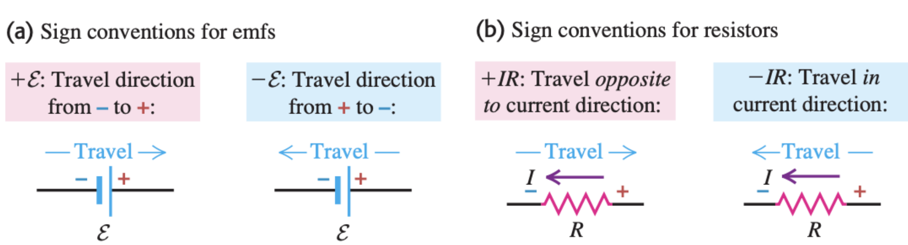
$$\sum I = 0$$

equals zero. Moreover, the sum of the potential differences around any loop

$$\sum V = 0$$

is also zero.

Remark: Sign Conventions for the Loop Rule:



9.4 R-C circuit

Before the switch is closed, the capacitor charge is q_0, and the current is zero. At some initial time $t = 0$, we close the switch, allowing the capacitor to discharge through the resistor. As t increases, the magnitude of the current decreases, while the charge on the capacitor also decreases.

Let i and q be current and charge at time t, by Kirchhoff's rule, we have

$$\Delta V = 0 \quad \Rightarrow \quad iR + \frac{q}{C} = 0$$

and since we know that $i = \frac{dq}{dt}$, so we have

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \Rightarrow \quad \frac{dq}{dt} = -\frac{q}{RC}$$

Integrating on both sides we would have

$$\int_{q_0}^{q} \frac{dq}{q} = -\frac{1}{RC}\int_0^t \, dt \quad \Rightarrow \quad \boxed{q = q_0 e^{-t/RC}}$$

$$\Rightarrow \quad \boxed{i = \frac{dq}{dt} = -\left(\frac{q_0}{RC}\right) e^{-t/RC}}$$

where $i_0 = -\frac{q_0}{RC}$.

Remark: The magnitude of the current through the resistor in a discharging R-C circuit decreases exponentially, with a time constant $\tau = RC$.

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Theory 9.4

The magnitude of the current through the resistor in a discharging R-C circuit decreases exponentially, with a time constant $\tau = RC$.

$$I = \frac{dq}{dt} = -\frac{Q_0}{RC}e^{-\frac{t}{RC}} = I_0 e^{-\frac{t}{RC}}$$

where I is the current, Q_0 is the initial capacitor charge, R is the resistance, and C is the capacitor.

Result 9.2

Capacitor Charging:

$$V = \epsilon\left[1 - e^{-\frac{t}{RC}}\right]$$

Capacitor Discharging:

$$V = \epsilon\, e^{-\frac{t}{RC}}$$

Theory 9.5

The charge on the capacitor in a discharging R-C circuit decreases exponentially, with a time constant $\tau = RC$.

$$q = Q_0 e^{-\frac{t}{RC}}$$

where q is the capacitor charge.

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10 Magnetic Field and Magnetic Forces

10.1 Magnetic Forces on Moving Charges

Definition 10.1: magnetic force on a moving charged particle

The magnetic force is given by

$$|\vec{F}| = q|v||B|\sin\theta$$

where q is the charge, v is the velocity, B is the magnetic field and θ is the angle between v and B vectors.

Remark: The unit for B is $[Ns/Cm] \equiv [T]$, we also define

$$\vec{B}_{earth} \approx 10^{-4}T \equiv 1Gauss$$

Theory 10.1

Largest steady $\vec{B}$ produced in labs is $45T$

Example 10.1

An electron is moving with a velocity of $6 \times 10^6\ m/s$ westwared in a 3.0 T magnetic field that is pointed out of the page. Find the magnitice and the direction of the force acting on the electron.

Proof: We have $\vec{F} = (-1.6 \times 10^{-18}\ C)(-6 \times 10^6\ m/s)(3.0\ T) = 2.88 \times 10^{-12}(\hat{x} \times \hat{z})$. Using the right hand rule we can find that the force is pointind down. (Fingers pointing toward the $\hat{x}$ direction and then curl toward the direction that is pointing out of the page). □

Example 10.2

A proton moving east experience a force of $8.80 \times 10^{-19}\ N$ upward due to the earth's magnetic field. At this location the field has a magnitude of $5.50 \times 10^{-5}\ T$. Find the speed of the particle.

Proof: We have $F = qvB$, thus solving for v we have

$$\vec{v} \approx 100,000\ m/s$$

□

10.2 Lorentz Force

When a charged particle moves through a region of space where both electric and magnetic fields are present, both fields exert forces on the particle. The total force F is the vector sum of the electric and magnetic forces:

Theory 10.2: The Lorentz Force

$$\vec{F}_{em} = q\vec{E} + q(\vec{v} \times \vec{B})$$

10.3 Motion of Charged Particles in a Magnetic Field

When a charged particle moves in a magnetic field, it is acted on by the magnetic force. The force is always perpendicular to the velocity, so it cannot change the speed of the particle because there is not work.

Now suppose there is a particle in a uniform magnetic field, then

Definition 10.2: Cyclotron Frequency

$$\begin{aligned}
\vec{F} &= q\vec{v} \times \vec{B} \\
&= -qvB\hat{r} = ma = \frac{mv^2}{r} \\
\Rightarrow \quad & r = \frac{mv^2}{|q|vB} = \frac{mv}{|q|B} \\
\Rightarrow \quad & \boxed{\omega = \frac{v}{r} = \frac{|q|B}{m}} \\
\Rightarrow \quad & f = \frac{\omega}{2\pi} = \frac{|q|B}{2\pi m}
\end{aligned}$$

Example 10.3

An electron with kinetic energy 22.5 eV moves in a region of uniform magnetic field of magnitude .000455 $Tesla$. The angle between the direction of the field and the electron's velocity is 65.5 degree. What is the pitch of the helical path taken by the electron?

Remark: Pitch: distance travelled parallel to field during one period

Proof: We have $\frac{1}{2}mv^2 = 22.5(1.6 \times 10^{-19})$, since we know that the value of m, thus solving for v we know that $v = 2.8 \times 10^6 \ m/s$, thus the pitch can be found by

$$\text{pitch} = vT = v\cos\theta \cdot \frac{2\pi m}{qB} = 9.16 \times 10^{-2} \ m$$

□

10.4 Magnetic Force on a Current-Carrying Conductor

Theory 10.3: The magnetic force of a current-carrying conductor

The magnetic force on a straight wire segment is

$$\vec{F} = I\vec{l} \times \vec{B}$$

Example 10.4

A wire segment 3 mm long carries a current of 3 A in the $+x$ direction. It lies in a magnetic field of magnitude 0.02 T that is in the xy plane and makes an angle of 30 degrees with the $+x$ direction, as shown. What is the magnetic force exerted on the wire segment?

Proof: We have $\vec{F} = I\vec{l} \times \vec{B}$, thus we have

$$\vec{F} = IlB\sin\theta\hat{z} = 9 \times 10^{-5}\hat{z}\ N$$

□

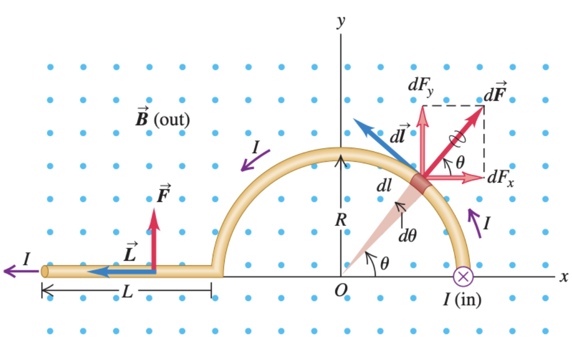
Example 27.8 Magnetic force on a curved conductor

In Fig. 27.30 the magnetic field $\vec{B}$ is uniform and perpendicular to the plane of the figure, pointing out of the page. The conductor, carrying current I to the left, has three segments: (1) a straight segment with length L perpendicular to the plane of the figure, (2) a semicircle with radius R, and (3) another straight segment with length L parallel to the x-axis. Find the total magnetic force on this conductor.

SOLUTION

IDENTIFY and SET UP: The magnetic field $\vec{B} = B\hat{k}$ is uniform, so we find the forces $\vec{F}_1$ and $\vec{F}_3$ on the straight segments (1) and (3) using Eq. (27.19). We divide the curved segment (2) into infinitesimal straight segments and find the corresponding force $d\vec{F}_2$ on each straight segment using Eq. (27.20). We then integrate to find $\vec{F}_2$. The total magnetic force on the conductor is then $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$.

27.30 What is the total magnetic force on the conductor?



EXECUTE: For segment (1), $\vec{L} = -L\hat{k}$. Hence from Eq. (27.19), $\vec{F}_1 = I\vec{L} \times \vec{B} = \mathbf{0}$. For segment (3), $\vec{L} = -L\hat{\imath}$, so $\vec{F}_3 = I\vec{L} \times \vec{B} = I(-L\hat{\imath}) \times (B\hat{k}) = ILB\hat{\jmath}$.

For the curved segment (2), Fig. 27.20 shows a segment $d\vec{l}$ with length $dl = R\,d\theta$, at angle θ. The right-hand rule shows that the direction of $d\vec{l} \times \vec{B}$ is radially outward from the center; make sure you can verify this. Because $d\vec{l}$ and $\vec{B}$ are perpendicular, the magnitude dF_2 of the force on the segment $d\vec{l}$ is just $dF_2 = I\,dl\,B = I(R\,d\theta)B$. The components of the force on this segment are

$$dF_{2x} = IR\,d\theta\,B\cos\theta \qquad dF_{2y} = IR\,d\theta\,B\sin\theta$$

To find the components of the total force, we integrate these expressions with respect to θ from $\theta = 0$ to $\theta = \pi$ to take in the whole semicircle. The results are

$$F_{2x} = IRB\int_0^{\pi}\cos\theta\,d\theta = 0$$

$$F_{2y} = IRB\int_0^{\pi}\sin\theta\,d\theta = 2IRB$$

Hence $\vec{F}_2 = 2IRB\hat{\jmath}$. Finally, adding the forces on all three segments, we find that the total force is in the positive y-direction:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 + 2IRB\hat{\jmath} + ILB\hat{\jmath} = IB(2R + L)\hat{\jmath}$$

EVALUATE: We could have predicted from symmetry that the x-component of $\vec{F}_2$ would be zero: On the right half of the semicircle the x-component of the force is positive (to the right) and on the left half it is negative (to the left); the positive and negative contributions to the integral cancel. The result is that $\vec{F}_2$ is the force that would be exerted if we replaced the semicircle with a *straight* segment of length $2R$ along the x-axis. Do you see why?

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10.5 Force and Torque on a Current Loop

Theory 10.4: Torque on a current loop

Consider a rectangular loop, which has side lengths a and b, and there is a magnetic field that is perpendicular to the plane, which has a degree φ between the loop. Since the top length is a, so the force on the top is

$$\vec{F}_{top} = I\vec{l} \times \vec{B} = IaB\ \hat{x}$$

Similarly, bottom length is also a, so

$$\vec{F}_{bottom} = I\vec{l} \times \vec{B} = Iab\ \vec{-x}$$

On the two sides, we can find that the force cancel out with each other. Thus we now that $\tau = \vec{r} \times \vec{F}$,

$$\begin{aligned}\tau &= \vec{r} \times \vec{F} \\ &= 2 \cdot \left(\frac{b}{2}\right) IaB\sin(\phi) \\ &= IAB\sin(\phi) = \boxed{I\vec{A} \times \vec{B}}\end{aligned}$$

Remark: Recall dipole moment.

Example 10.5

A circular coil of radious 0.005 m has 30 turns of wire. It carries a conterclockwise current of 7.00 A in a 1.5 Tesla magnetic field. Find the torque on the coil and its magnetic moment.

Definition 10.3: The direct-current motor

Current flows into one side of the rotor and out of the other side, therefore, the magnetic torque causes the rotor to spin counterclockwise. The **rotor** is a wire loop that is free to rotate about an axis, the rotor ends are attached to form the **commutator**.

10.6 Hall Effect

Theory 10.5: The Hall effect: negative charge carriers

When a current is placed in a magnetic field, the **Hall emf** reveals whether the chare carriers are negative or positive.

$$nq = \frac{-JB}{E}$$

Example 10.6

A trip of copper2.00 mm thick and 1.50 cm wide is placed in a magnetic fiwld of 4.0 T as shown. When a 75 A current runs through the strip, the potential difference across it is found to be 0.81 μV. Find the concentration of elections in the material.

Proof: I = I/A, and E = V/d, since we also know q, n = JB / Eq. □

11 Sources of Magnetic Field

11.1 Magnetic Field of a Straight Current-Carrying Conductor

Theory 11.1: near a long, straight, current-carrying conductor

Magnitude and Direction of Magnetic Field Due to Current Carrying Wire

$$|\vec{B}| = \frac{\mu_0 |i|}{2\pi R}$$

direction is determined using Right Hand Rule.

11.2 Biot-Savart Law

Theory 11.2: Biot-Savart Law

Magnetic fields are generated by moving charges,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

Current in a small wire segment $d\vec{l}$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

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Result 11.1

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2}$$

Example 11.1: What is the magnitude of the magnetic field

Suppose we have current I in a wire, and a point charge P at some arbitrary place. We call the wire y-axis while the perpendicular direction to the charge to be the x-axis, hence we have $d\vec{l}$ to be small portions of wire and r to be the distance between the small portion and the point charge, and the angle between r and the wire is denoted as ϕ. Using the right hand rule, we can find that the magnetic field is pointing into the board. Moreover

$$\begin{aligned} d\vec{B} &= \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2} \\ &= \frac{\mu_0}{4\pi} \frac{I d\vec{l} \sin\phi}{r^2} \end{aligned}$$

Since we have $r^2 = x^2 + y^2$, and $\sin\phi = \frac{x}{\sqrt{x^2+y^2}}$, so

$$\begin{aligned}\vec{B} &= \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{x\ dy}{(x^2+y^2)^{3/2}} \\ &= \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2+a^2}}\end{aligned}$$

Therefore, for $a \gg x$, we have

$$B = \frac{\mu_0 I}{2\pi x}$$

Remark: We also calculated the magnitude of electric fields:

$$\vec{E} = \frac{\lambda}{\epsilon_0 2\pi r}\hat{r}$$

Example 11.2: What is the magnitude of the force on the right wire due to the left wire?

Now consider we have two parallel wires with current i in each of the two wires distance r away from each other. What is the magnitude of the force on the right wire due to the left wire? (denote the wire on the left to be wire 1)

We know that wire 1 creats a magnetic field, first we can determine that the direction of the magnetic field on wire 2 due to wire 1 is out of the board. We also have

$$\vec{B_1} = \frac{\mu_0}{2\pi}\frac{i}{r}$$

Therefore we have

$$F_{12} = I\vec{l} \times \vec{B}$$

using right hand rule again, we know that wire 1 is pulling wire 2 towards itself. Hence

$$\begin{aligned}F_{12} &= I_2 L\vec{B} \\ &= I_2 L \frac{\mu_0}{2\pi}\frac{I_1}{r} \\ \Rightarrow \quad \frac{F}{L} &= \frac{\mu_0}{2\pi r} I_1 I_2\end{aligned}$$

Example 11.3: Magnetic field of a current loop

Recall that we have

$$\begin{aligned}d\vec{B} &= \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2} \\ &= \frac{\mu_0}{4\pi} I \frac{dl}{x^2+a^2}\end{aligned}$$

We can find that

$$\begin{aligned}
dB_y &= dB\sin\theta \\
&= \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}\frac{x}{\sqrt{x^2+a^2}} \\
&= \frac{\mu_0 I}{4\pi}I\frac{x}{(x^2+a^2)^{3/2}} \\
dB_x &= \frac{\mu_0 I}{4\pi}I\frac{a}{(x^2+a^2)^{3/2}}
\end{aligned}$$

As a result, we can find that $B_y = 0$ using symmetry, so

$$\begin{aligned}
B &= \oint dB \\
&= B_x \\
&= \frac{\mu_0 I}{4\pi}I\frac{a}{(x^2+a^2)^{3/2}}\int dl \\
&= \frac{\mu_0 I}{4\pi}I\frac{a}{(x^2+a^2)^{3/2}}\cdot 2\pi a \\
&= \frac{\mu_0 I}{2}\frac{a^2}{(x^2+a^2)^{3/2}}
\end{aligned}$$

11.3 Ampere's Law

Theory 11.3: Ampere's Law

Draw any clsoed path, the total current enclosed by the path is equal to the integral of the magnetic field over the path, (only parallel components contribute),

$$\oint \vec{B}\cdot\, d\vec{l} = \mu_o i_{enclosed}$$

Proof: We have

$$\oint \vec{B}\cdot\, d\vec{l} = B\oint dl = \frac{\mu_0 I}{2\pi r}\cdot 2\pi r = \mu_0 I$$

yay. □

Remark: It is equivalent to the Biot-Savart Law.

Example 11.4: Example of a solid conductor with radius a

is supported by insulating disks on the axis of a conducting tube with inner radius b and outer radius c. The central conductor and tube equal currents I in opposite directions. The currents are distributed uniformly over the cross sections of each conductor. Derive an expression for the magnitude of the magnetic field at points outside the central, solid confuctor but inside the tube.

We have

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$
$$\Rightarrow \quad B \oint d\vec{l} = B 2\pi r = \mu_0 I$$
$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

Moreover, the magnetic outside the cable is 0.

Result 11.2: Michael Faraday's Question

If currents produce magnetic fields, then shouldn't magnetic fields produce current?
– Yes, this is an induced current.

Theory 11.4: Magnetic Flux

The unit is $[T/m^2] = Wb \quad \{\text{Weber}\}$

$$\Phi_B = \int \vec{B} \cdot \, d\vec{A}$$

12 Electromagnetic Induction

Theory 12.1: Magnetic Induction

When the flux of magnetic field through a wire loop changes an electric current will be induced.

$$i \propto \frac{d\Phi_B}{dt}$$
$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

Theory 12.2: Lenz's Law

The direction of the induced current will be such that it will produce magnetic flux that is opposite the cahnge that induced it.

Theory 12.3: Faraday's law of induction

The induced emf in a closed loop equals the negative of the time rate of change of magnetic flux through the loop. We have

$$\epsilon_{ind} = -\frac{d\Phi_B}{dt}$$

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13 Magnetic Induction

13.1 Faraday's Law continued

Result 13.1

When the flux of magnetic field through a wire loop changes an electric current will be induced.

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$
$$\varepsilon_{ind} = -\frac{d\Phi_B}{dt}$$

Theory 13.1: Faraday's Law as Generalized by Maxwell

Recall that

$$\int \vec{E} \cdot d\vec{s} = \Delta V = \varepsilon$$

This implies that

$$\int \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

A changing magnetic flux induces an electric field. Similarly, changing electric flux also generate a magnetic field.

$$\int \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

From $\vec{F_E} = q\vec{E}$, we know that the direction of the induced field is the same as the direction of the induced current.

13.2 Maxwell's Equations of Electromagnetism

1. A changing magnetic flux will induce an electric field even if there is no conductor present

Result 13.2: Maxwell's Equation

We have

$$\begin{aligned}
&\oint \vec{E} \cdot d\vec{A} = -\frac{q_{enc}}{\varepsilon_0} \qquad \text{(Gauss's law for } \vec{E}\text{)} \\
&\oint \vec{B} \cdot d\vec{A} = 0 \\
&\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I_{encl} \\
&\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \\
&\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}
\end{aligned}$$

Interpretation 13.1

1. First equation states that the surface integral of $E_\perp$ over any closed surface equals $1/\varepsilon_0$ times the total charge Qencl enclosed within the surface.

2. The second is the analogous relationship for magnetic fields, which states that the surface integral of $B_\perp$ over any closed surface is always zero.

3. The third equation is Ampere's law including displacement current. This states that both conduction current i_C and displacement current $\varepsilon_0 d\Phi_E/dt$, where Φ_E is electric flux, act as sources of magnetic field:

4. The fourth and final equation is Faraday's law. It states that a changing magnetic field or magnetic flux induces an electric field.

13.3 Circuits with Inductors

Definition 13.1: Self-Inductance

If the current i in the coil is changing, the changing flux through the coil induces an emf in the coil

Theory 13.2: Ciucuits with Inductors

$$\varepsilon_{ind} = \int \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$
$$\frac{d\Phi_B}{dt} \propto \frac{dI}{dt}$$
$$V = L\frac{dI}{dt}$$

where L is a constant called **inductance** which is fixed for a particular coil.

Consider a solenoid with N loops of length l, then we define $n := \frac{N}{l}$, it is not hard to work out that

$$B = \mu_0 n I$$

Therefore we have

$$\Phi_B = \mu_0 n I \cdot A = \mu_0 \frac{N}{l} I A$$

Furthermore,

$$\begin{aligned}\varepsilon_{ind} &= -\frac{\Phi_B}{dt} = -\frac{\mu_0 N A}{l}\frac{dI}{dt} \\ &= -\frac{N\mu_0 A}{L}\frac{dI}{dt}\end{aligned}$$

We also define

$$L := \frac{\Phi_B}{I} = \frac{N}{I}\frac{\mu_0 I A}{l} = \frac{N\mu_0 A}{l}$$

thus

$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

and

$$\boxed{\varepsilon_{ind} = -L\frac{dI}{dt}}$$