Solution to Exercises

Exercise 3.1:

Solution:

Exercise 5.1:

Solution: Suppose for a contradiction that L is a regular language, thus there exists pumping length p satisfying the pumping lemma. Now consider the string

$$\omega := 0^p 1^p \in L$$

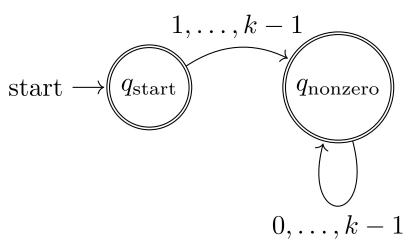
of length $2p$. For this string, in its decomposition $\omega = xyz$, we have xy is made of purely zero's. In particular, y consists of only zero's. It is now easy to see that

$$xy^i z \notin L$$

for, for example, $i = 2$. This contradicts our assumption that L is regular, hence L is not regular.

Exercise 5.2 :

Solution: Since we have the convention that $(0)_k$ is the empty word, we have the following automoton accepting the set:



Exercise 5.3 :

Solution: Recall that we know for $\omega = d_m d_{m-1} \cdots d_0$, we have defined

$$[\omega]_k = d_m k^m + d_{m-1} k^{m-1} + \cdots + d_1 k + d_0$$

Thus we can extend this definition and find that in general, we have

$$\begin{aligned}
[uv^n\omega]_k &= [u]_k \cdot k^{n|v|+|\omega|} + [v]_k \cdot (1 + k^{|v|} + \cdots + k^{(n-1)|v|}) k^{|\omega|} + [\omega]_k \\
&= [u]_k \cdot k^{n|v|+|\omega|} + [v]_k \cdot k^{|\omega|} \cdot \frac{k^{n|v|} - 1}{k^{|v|} - 1} + [\omega]_k \\
&= \underbrace{\left([u]_k \cdot k^{|\omega|} + \frac{[v]_k \cdot k^{|\omega|}}{k^{|v|} - 1} \right) k^{|v|}}_{\alpha} \cdot k^n + \underbrace{\left(-\frac{1}{k^{|v|} - 1} + [\omega]_k \right)}_{\beta}
\end{aligned}$$

the result is now obvious.

Exercise 5.4 :

Solution: Similar to exercise 5.1 with the use of Fermat's Little Theorem. Suppose for a contradiction that the set S is regular, thus it needs to satisfy the pumping lemma for some pumping length p. Find a prime γ such that its base-k expansion has length at least p, and can be written as $\gamma = uvw$. Therefore, by exercise 5.3, we kow that there exists integer k and α and β such that

$$[uvw]_k = \alpha k + \beta$$
$$[uv^n w]_k = \alpha k^n + \beta$$

Subtracting two equations yields us

$$[uv^n w]_k - [uvw]_k = \alpha(k^n - k)$$

We can now pick $n = uvw$. By Fermat's Little Theorem, we know that RHS is divisible by uvm. Thus we can deduce that

$$uvw \mid [uv^n w]_k$$

which implies that $[uv^n w]_k \notin S$, contradiction. ♠

Exercise 5.5 :

Solution: This is similar to exercise 5.1. Suppose for a contradiction that the set of palindromes is regular, then there exists a pumping length p satisfying the pumping lemma. Now consider an arbitrary string ω (palindrome) of length $2p + 1$, we know that in the decomposition ωxyz, xy does not contain the middle character of the palindrome ω. As a result, it is now easy to see that the new string

$$xy^i z$$

wouldn't be a palindrome as desired, which further implies that $xy^i z \notin L$. This contradicts the pumping lemma, which means that that L is not regular. ♠

Exercise 5.6 :

Solution: We denote the set of words over the alphabet $\{0, 1, \ldots, k-1\}$ that represent the base-k expansions of elements of $\{\ell^n : n \geq 0\}$ as L. Suppose for a contradiction that L does form a regular language. Hence for the pumping length p, let $\ell^n \in L$ be an element whose base-k expansion has length at least p, suppose $\ell^n = uvw$, we have

$$[uvw]_k = \alpha k + \beta = \ell^n$$

for some rational numbers α and β. By the pumping lemma, we know that $uv^n w \in L$ for some $n \geq 0$ with $n \neq n'$, thus by exercise 5.3 we have

$$[uv^n w]_k = \alpha k^n + \beta = \ell^{n'}$$

for some n'. Subtracting the two equations we obtain that

$$\ell^{n'} - \ell^n = \alpha k^n - \alpha k$$
$$n' \log \ell - n \log \ell = \log \alpha + n \log k - \log \alpha - \log k$$
$$n' \log \ell - n \log \ell = n \log k - \log k$$
$$(n' - n) \log \ell = (n - 1) \log k$$

This contradicts the fact that ℓ and k are multiplicatively independent, hence the set of words does not form a regular language. ♠

Exercise 5.7a :

Solution: Suppose for a contradiction that L is regular. Hence for a fixed pumping length p, we can find $\ell \in L$ with $|\ell| \geq p$ such that $\ell = xyz$. We assume that $|\ell| = q$ for some prime q. By the pumping lemma, we know that

$$xy^{q+1}z \in L$$

However, it is easy to see that $q \mid |xy^{q+1}z|$, a contradiction. Hence L is not regular. ♠

Exercise 5.7b :

Solution: Unmatched number of left and right parentheses. ♠

Exercise 5.8 :

Solution: ♠

Exercise 5.9 :

Solution: ♠

Exercise 7.1 :

Solution: `(a)` $\Rightarrow$ `(b)`: We define

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{d-1} \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(d-1) \end{bmatrix}$$

We prove by induction on n that

$$\begin{bmatrix} f(n) \\ f(1) \\ \vdots \\ f(d-1) \end{bmatrix} = A^n \mathbf{w}$$

For our base case, we know by the definition of $\mathbf{w}$ that

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(d-1) \end{bmatrix} = A^0 \mathbf{w}$$

Suppose the result holds for $m < n$. Now we have

$$A^n \mathbf{w} = A A^{n-1} \mathbf{w} = A \begin{bmatrix} f(n-1) \\ f(n-1+1) \\ \vdots \\ f(n-1+d-1) \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(n+d-1) \end{bmatrix}$$

as desired. For the other direction `(b)` $\Rightarrow$ `(a)`: Recall that from Cayley-Hamilton we know that

$$A^d + c_{d-1} A^{d-1} + \cdots + c_1 A + c_0 I = 0$$

Hence we further know that

$$\mathbf{v} \left(A^{n+d} + c_{d-1} A^{n+d-1} + \cdots + c_1 A^{n+1} + c_0 A^n \right) \mathbf{w} = 0$$

which yields us the recurrence relation of the form in statement (a). ♠

Exercise 7.2 :

Solution: `(a)` $\Rightarrow$ `(c)`: We define

$$Q(x) := 1 + c_{d-1}x + \cdots + c_0 x^d$$

and

$$P(x) := \left(\sum_{n \geq 0} f(x) x^n \right) \cdot Q(x)$$

Thus it now suffices to show that $P(x)$ is a polynomial of degree at most $d-1$. We notice that

$$[x^m]P(x) = \sum_{i=0}^{\min(m,d)} c_{d-i} \cdot f(m-i)$$

Given that $f(n)$ satisfies a linear recurrence:

$$f(n+d) + c_{d-1}f(n+d-1) + \cdots + c_0 f(n) = 0 \quad \text{for all } n \geq 0$$

Then, for any $m \geq d$, we have:

$$f(m) + c_{d-1}f(m-1) + \cdots + c_0 f(m-d) = 0$$

This implies:

$$[x^m]P(x) = f(m) + c_{d-1}f(m-1) + \cdots + c_0 f(m-d) = 0 \qquad \text{for all } m \geq d$$

thus proving that $P(x)$ has degree at most $d-1$. For the other direction, `(c)` $\Rightarrow$ `(a)`: Suppose we know that

$$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{Q(x)}$$

Hence we know

$$[x^i]Q(x) \sum_{n \geq 0} f(n) x^n$$

Suppose

$$Q(x) = c_d x^0 + c_{d-1}x + \cdots + c_0 x^d$$

where $c_d = 1$ since $Q(0) = 1$, we now have that

$$\begin{aligned} [x^i]Q(x) \sum_{n \geq 0} f(n) x^n &= [x^i] \sum_{j=0}^{d} c_{d-j} x^j \sum_{n \geq 0} f(n) x^n \\ &= \sum_{j=0}^{d} c_{d-j} [x^{i-j}] \sum_{n \geq 0} f(n) x^n = \sum_{j=0}^{d} c_{d-j} f(i-j) \end{aligned}$$

Moreover, given that $P(x)$ is a polynomial of degree at most $d-1$, we know that

$$[x^d]P(x) = 0$$

which implies that

$$[x^d]Q(x) \sum_{n \geq 0} f(n) x^n = \sum_{j=0}^{d} c_{d-j} f(d-j) = 0$$

as desired. ♠

Exercise 7.3 :

Exercise 7.4 :

Solution: We first show that $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$. It is easy to see that when $n = 1$, we indeed have $\mu(1) = 1$. For all the other cases, suppose the number n has m distinct prime factors $p_1, \dots, p_m$, thus we know that

$$\sum_{d|n} \mu(d) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} = (1 + (-1))^m = 0$$

Now, we have

$$\begin{aligned}
\sum_{d|n} \mu(d) f(n/d) &= \sum_{d|n} \mu(d) \sum_{m|n/d} g(m) \\
&= \sum_{m|n} g(m) \sum_{d|n/m} \mu(d)
\end{aligned}$$

Notice the only term that is non-zero is when $m = n$, hence we have

$$\sum_{d|n} \mu(d) f(n/d) = g(m) = g(n)$$

as desired. ♠

Exercise 7.5 :

Solution: We notice that every word of length n is built by repeating a primitive word of length d, where $d|n$, hence we have

$$k^n = \sum_{d|n} f(d)$$

Using möbius inversion we obtain that

$$f(n) = \sum_{d|n} \mu(d) k^{n/d}$$

as desired. ♠

Exercise 7.6 :

Solution: We have

$$\begin{aligned}
\sum_{n \geq 0} f(n) x^n &= f(0)x^0 + \sum_{n \geq 1} f(n) x^n \\
&= 1 + \sum_{n \geq 1} \sum_{d|n} \mu(d) k^{n/d} x^n \\
&= 1 + \sum_{d \geq 1} \mu(d) \sum_{j \geq 1} k^j x^{dj}
\end{aligned}$$

♠