Lecture 6

Burr and Erdős conjectured that, given a natural number Δ, there exists a $c(\Delta)$ such that any graph H with n vertices and maximum degree Δ satisfies $r(H) \leq c(\Delta)n$. That is, provided the maximum degree is fixed, the Ramsey number should grow linearly with the number of vertices. This conjecture was resolved by Chvátal, Rödl, Szemerédi and Trotter, in what was one of the first applications of Szemerédi's regularity lemma. In this lecture, we will give a more modern treatment due to Graham, Rödl and Ruciński, avoiding any use of the regularity lemma.

The basic idea is as follows. We first attempt to find a large subset in which red is very dense. If such a set can be found, we can easily embed the required graph. If, on the other hand, this is not the case, then the blue edges must be quite well distributed. Again, this allows us to prove an embedding lemma. This unbalanced approach is quite useful in Ramsey theory, particularly when there are only two colours.

We begin with a very basic lemma which allows us to embed graphs with maximum degree Δ into very dense graphs.

Lemma 1 *Let G be a graph with at least $4n$ vertices and density at least $1 - \frac{1}{8\Delta}$. Then G contains a copy of every graph H on n vertices with maximum degree Δ.*

Proof: If G has density greater than $1 - \frac{1}{8\Delta}$, its complement $\overline{G}$ has density at most $\frac{1}{8\Delta}$. There are therefore at most $\frac{1}{2}V$ vertices of $\overline{G}$ which have degree greater than $\frac{V}{4\Delta}$. Otherwise $\overline{G}$ would contain more than $\frac{1}{8\Delta}\frac{V^2}{2}$ edges, a contradiction. If we remove these vertices (and more if necessary) we have an induced subgraph G_0 of G on $V' = \frac{1}{2}V$ vertices such that the degree of every vertex in the complement $\overline{G_0}$ is at most $\frac{V}{4\Delta} = \frac{V'}{2\Delta}$. We now show how to embed H vertex by vertex in G_0.

Suppose that we have an embedding f of the set $\{v_1, v_2, \cdots, v_i\}$ and we would like to embed v_{i+1}. Since v_{i+1} has at most $l \leq \Delta$ neighbours $\{u_1, u_2, \cdots, u_l\}$ preceding it in the ordering, the total number of vertices in V' which are not connected to all of the $f(u_j)$ in G_0 is at most $\Delta\frac{V'}{2\Delta} \leq \frac{V'}{2}$. If we choose $f(v_{i+1})$ to be any element in the complement of this set which is not already the image of some vertex, we are done. This is possible since $\frac{V'}{2} \geq \frac{V}{4} \geq n$. $\square$

On the other hand, we would like to prove an embedding lemma if we know that the edges of G are well distributed. To state this lemma, we need to define two notions of what it means for the edges in a graph to be well distributed. Let G be a graph on vertex set V and let X, Y be two subsets of V. Define $e(X, Y)$ to be the number of edges between X and Y. The density of the pair (X, Y) is

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

Similarly, given a subset X of V, let $e(X)$ be the number of edges in X. The edge density of X is

$$d(X) = \frac{e(X)}{\binom{|X|}{2}}.$$

Our two local density properties are now as follows. Firstly, a graph is said to be (ρ, δ)-*dense* if, for all $X \subset V$ with $|X| \geq \rho|V|$, we have $d(U) \geq \delta$. Secondly, a graph is said to be *bi*-(ρ, δ)-*dense* if, for all $X, Y \subset V$ with $X \cap Y = \emptyset$ and $|X|, |Y| \geq \rho|V|$, we have $d(X, Y) \geq \delta$.

It will turn out that these two notions are closely related and that a dense graph necessarily contains a large subset which is bi-dense. We will return to this. For now, we simply note that if a graph is sufficiently bi-dense, it contains all graphs with a fixed maximum degree.

Lemma 2 *Let $\delta > 0$ be a real number. If G is a bi-$(\frac{1}{4}\delta^{\Delta}\Delta^{-2}, \delta)$-dense graph on at least $4\delta^{-\Delta}\Delta n$ vertices then G contains a copy of any graph H on n vertices with maximum degree Δ.*

Proof: Let V be the vertex set of G and suppose without loss of generality that $|V| = (\Delta+1)N$, where $N \geq 2\delta^{-\Delta}n$. Split V into $\Delta+1$ pieces $V_1, V_2, \cdots, V_{\Delta+1}$, each of size N. Since the chromatic number of H is at most $\Delta+1$, we may split its set of vertices W into $\Delta+1$ independent sets $W_1, W_2, \cdots, W_{\Delta+1}$. We will give an embedding f of H in G so that $f(W_i) \subset V_i$ for all $1 \leq i \leq \Delta+1$.

Let the vertices of H be $\{w_1, w_2, \cdots, w_n\}$. For each $1 \leq h \leq n$, let $L_h = \{w_1, w_2, \cdots, w_h\}$. For each $y \in W_j \backslash L_h$, let T_y^h be the set of vertices in V_j which are adjacent to all already embedded neighbours of y. That is, letting $N_h(y) = N(y) \cap L_h$, T_y^h is the set of vertices in V_j adjacent to each element of $f(N_h(y))$. We will find, by induction, an embedding of L_h such that, for each $y \in W \backslash L_h$, $|T_y^h| \geq \delta^{|N_h(y)|}N$.

For $h = 0$, there is nothing to prove. We may therefore assume that L_h has been embedded consistent with the induction hypothesis and attempt to embed $w = w_{h+1}$ into an appropriate $v \in T_w^h$. Let Y be the set of neighbours of w which are not yet embedded. We wish to find an element $v \in T_w^h \backslash f(L_h)$ such that, for all $y \in Y$, $|N(v) \cap T_y^h| \geq \delta |T_y^h|$. If such a vertex v exists, taking $f(w) = v$ will then complete the proof.

Let B_y be the set of vertices in T_w^h which are bad for $y \in Y$, that is, such that $|N(v) \cap T_y^h| < \delta |T_y^h|$. Note that, by the induction hypothesis, $|T_y^h| \geq \delta^{\Delta}N \geq \frac{1}{2}\delta^{\Delta}\Delta^{-1}|V|$. Therefore, $|B_y| < \frac{1}{4}\delta^{\Delta}\Delta^{-2}|V| \leq \frac{1}{2}\delta^{\Delta}\Delta^{-1}N$, for otherwise the density between the sets B_y and T_y^h would be less than δ, contradicting the bi-density condition. Hence, since $N \geq 2\delta^{-\Delta}n$,

$$\left| T_w^h \backslash \cup_{y \in Y} B_y \right| > \delta^{\Delta}N - \Delta \frac{1}{2}\delta^{\Delta}\Delta^{-1}N \geq n.$$

Hence, since at most n vertices have already been embedded, an appropriate choice for $f(w)$ exists. □

We are almost ready to finish our proof. On the one hand, we know that if the blue graph is sufficiently bi-dense, then we are done. There is only a problem if the blue graph has small density between two vertex sets. This would imply large red density between the two. We just need to see how to use this to show that there is a set within which the red density is large because then, by Lemma 1, we would be done.

Lemma 3 *For all numbers $0 < s, \sigma, \delta < 1$, $s \geq \lceil \frac{2}{\delta} \rceil$, the following holds. If G is a $((\frac{\sigma}{2})^s, \delta)$-dense graph on N vertices, then there exists $U \subset V$ with $|U| \geq (\frac{\sigma}{2})^s N$ such that $G[U]$ is bi-$(\sigma, \frac{\delta}{4})$-dense.*

Proof: Suppose otherwise. Then there exist two subsets $U_{1,1}, U_{1,2}$ of the vertex set $V_1 = V$ with $|U_{1,1}| \geq \sigma N$, $|U_{1,2}| \geq \sigma N$ such that $d(U_{1,1}, U_{1,2}) \leq \frac{\delta}{4}$. By averaging, there is a subset W_1 of $U_{1,1}$ of size $(\frac{\sigma}{2})^s N$ such that $d(W_1, U_{1,2}) \leq \frac{\delta}{4}$. Let V_2 be the set of vertices within $U_{1,2}$ which have at most $\frac{\delta}{2}W_1$ neighbours in W_1. It is easy to see that $|V_2| \geq \frac{1}{2}|U_{1,2}| \geq \frac{\sigma}{2}N$. Otherwise we would have that the density of edges between W_1 and $U_{1,2}$ was greater than $\frac{\delta}{4}$.

Suppose now that for $1 \le i \le s$ we have defined V_i with $|V_i| \ge \left(\frac{\sigma}{2}\right)^{i-1} N$. Since $|V_i| \ge \left(\frac{\sigma}{2}\right)^s N$, our assumption that we cannot find a bi-dense graph of this size tells us that there are subsets $U_{i,1}, U_{i,2}$ of V_i such that $|U_{i,1}| \ge \sigma|V_i|$, $|U_{i,2}| \ge \sigma|V_i|$ and $d(U_{i,1}, U_{i,2}) \le \frac{\delta}{4}$. Again, let W_i be a subset of $U_{i,1}$ of size $\left(\frac{\sigma}{2}\right)^s N$ such that $d(W_i, U_{i,2}) \le \frac{\delta}{4}$. Moreover, as above, we can find a subset V_{i+1} of $U_{i,2}$ such that $|V_{i+1}| \ge \frac{\sigma}{2}|V_i| \ge \left(\frac{\sigma}{2}\right)^i N$ and every vertex in V_{i+1} has degree at most $\frac{\delta}{2}|W_i|$ in W_i. We terminate this process once we have reached the set V_s. Let W_s be a subset of V_s of size $\left(\frac{\sigma}{2}\right)^s N$.

We now claim that $W = W_1 \cup W_2 \cup \cdots \cup W_s$ is a subset of V of size at least $\left(\frac{\sigma}{2}\right)^s N$ with density smaller than δ, contradicting the density assumption on G. The first part of the claim is straightforward, and follows from the definitions of the sets. To prove the density condition, note that the density of edges between W_i and any subset of V_{i+1} is at most $\frac{\delta}{2}$. This is because V_{i+1} was defined exactly so that every vertex had degree at most $\frac{\delta}{2}|W_i|$ in W_i. Moreover, the set $W_{i+1} \cup W_{i+2} \cup \cdots \cup W_s$ is a subset of V_{i+1}. Therefore, for every $1 \le i \le s-1$, the edges between W_i and $W_{i+1} \cup W_{i+2} \cup \cdots \cup W_s$ have density at most $\frac{\delta}{2}$. The only other edges are those lying within each W_i. But these contain

$$\sum_i \binom{|W_i|}{2} = s\binom{\frac{1}{s}|W|}{2} \le \frac{1}{s}\binom{|W|}{2} \le \frac{\delta}{2}\binom{|W|}{2}$$

edges. Overall, therefore, W has at density at most δ. □

We are now ready to put all these ideas together to prove the main theorem.

Theorem 1 *There exists a constant c such that, if H is a graph on n vertices with maximum degree Δ,*

$$r(H) \le \Delta^{c\Delta^2} n.$$

Proof: Let $\delta = \frac{1}{8\Delta}$, $\sigma = \frac{1}{4}\left(\frac{\delta}{4}\right)^\Delta \Delta^{-2}$ and $s = 16\Delta \ge \lceil \frac{2}{\delta} \rceil$. Suppose that the edges of the complete graph on $N = 4\left(\frac{2}{\sigma}\right)^s \left(\frac{4}{\delta}\right)^\Delta \Delta^2 n$ vertices have been 2-coloured.

If the blue graph G is $(\left(\frac{\sigma}{2}\right)^s, \delta)$-dense, then, by Lemma 3, there is a subset U with $|U| \ge \left(\frac{\sigma}{2}\right)^s N \ge 4\left(\frac{4}{\delta}\right)^\Delta \Delta^2 n$ such that $G[U]$ is bi-$(\sigma, \frac{\delta}{4})$-dense. Therefore, by Lemma 2 and the choice of σ, U contains a blue copy of H.

If, on the other hand, G is not $(\left(\frac{\sigma}{2}\right)^s, \delta)$-dense, there is a subset U of size at least $|U| \ge \left(\frac{\sigma}{2}\right)^s N \ge 4n$ such that the density of $G[U]$ is smaller than δ. Therefore, the red density in U is at least $1-\delta = 1-\frac{1}{8\Delta}$. Applying Lemma 1, we see that U must therefore contain a red copy of H. □

A more careful version of Lemma 3 was used by Graham, Rödl and Ruciński to show that $r(H) \le 2^{c\Delta \log^2 \Delta} n$. Very recently, an alternative proof was given by the author, Fox and Sudakov, removing one of the log factors in the exponent.