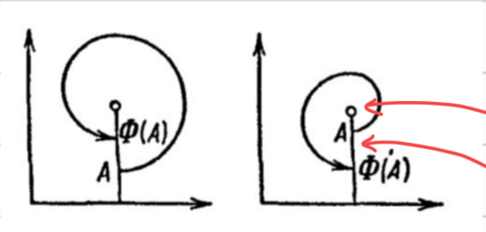
An Example p. 28

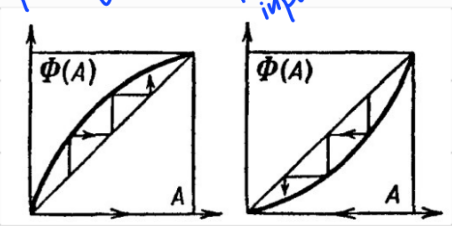
Phase space (2-dim.) and the phase curves



equilibrium point

return value is greater than input

return value is smaller than input



corresponding graph of the return function

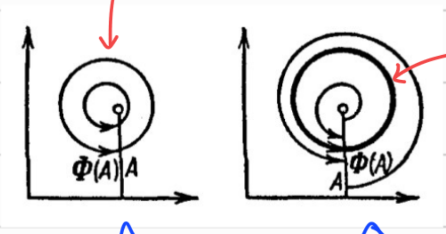
A is the initial condition with fixed x-value.

$\Phi(A)$ gives the value of y when the phase curve first returns back to the same x-value.

* $\Phi(A)$ is position periodic.

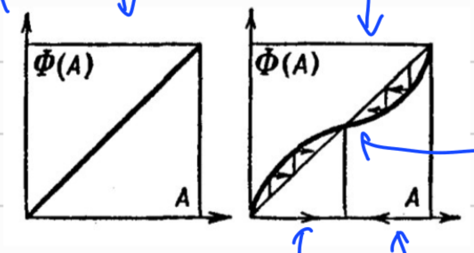
Note: could have fixed the y-value instead and varied the x-value.

closed phase curve



stable limit phase curve

return function returns input



closed limit curve

notice phase curves with initial condition above converges this point

phase curves with initial condition below also converges to this point.

p. 44-47: Study of the change in the qualitative properties of the phase curves under a small perturbation

$$\begin{cases} \dot{x} = x(k - ay + \varepsilon f(x,y)) \\ \dot{y} = y(-l + bx + \varepsilon g(x,y)) \end{cases}$$

small corrections

1) Is there still an equilibrium position? How does it change under a small perturbation?

$\hookrightarrow$ Theorem on p. 45

equilibrium positions are solutions of $\begin{cases} F(x,y,\varepsilon) = k - ay + \varepsilon f = 0 \\ G(x,y,\varepsilon) = -l + bx + \varepsilon g = 0 \end{cases}$

know: $(x_0, y_0, 0)$ is a solution

Important: for every ε fixed,

$\Phi_\varepsilon : (x,y) \longmapsto (F_\varepsilon, G_\varepsilon)$ is a distinct mapping

$\hookrightarrow$ change of view

we can think of ε as determining a family of $\Phi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$

mappings

Claim: for $\varepsilon = 0$, if $D\Phi_{\varepsilon=0} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}_{\varepsilon=0}$ is invertible

at (x_0, y_0) (i.e. (x_0, y_0) is a non-degenerate point under $\Phi_{\varepsilon=0}$)

follows from the implicit function thm

then, we can find

$$\Phi^{-1}_{\varepsilon=0} : (0,0) \longmapsto (x_0, y_0)$$

for ε small,

we can find a family of local inverses

$$\{ \Phi_\varepsilon^{-1} : \Phi_\varepsilon^{-1} \circ \Phi_\varepsilon (x,y) = (x,y) \text{ for } (x,y) \text{ near } (0,0) \}$$

and $\{(x,y) : \Phi_\varepsilon(x,y) = (0,0)\}$ can be locally parameterized by $x(\varepsilon)$, $y(\varepsilon)$ w/ $x(0) = x_0$ and $y(0) = y_0$

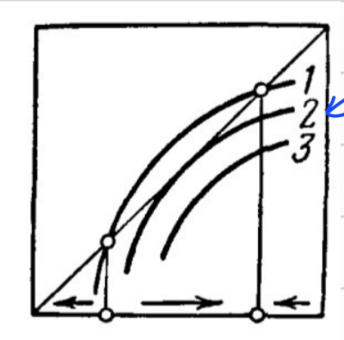
why?

because $\{\Phi_\varepsilon : \varepsilon \text{ is small}\}$ depends smoothly on ε (locally)

and so, the root $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = (0,0)$ also smoothly depends on ε

this is because $\Phi_\varepsilon \sim D\Phi_\varepsilon$ (locally equivalent) and $D\Phi_\varepsilon$ changes smoothly with ε

A degenerate cycle

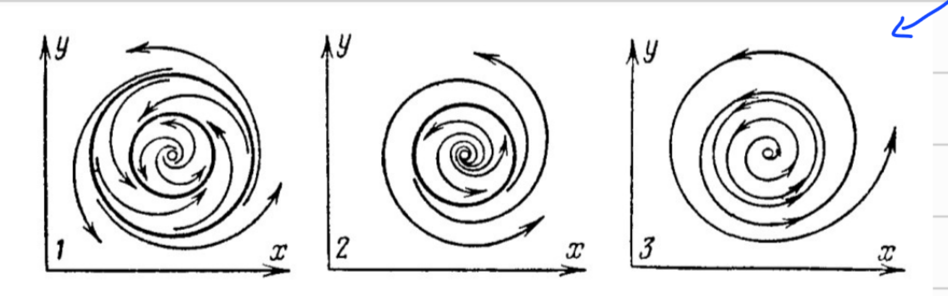


Example of a degenerate cycle

$\Phi(A) = A$ and $\Phi'(A) = 1$

when 2 closed phase curves merge to become 1.

A little more perturbation and the system becomes completely unstable

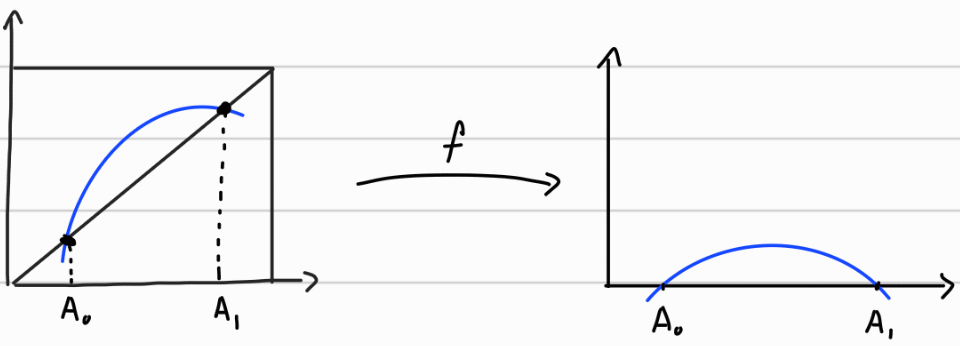


For a non-degenerate cycle:

If a system defined by a vector field v_0 has a nondegenerate limit cycle passing through A_0, then every nearby system (defined by the field v_ε for small ε) has a nearby cycle (passing through a point $A_{(\varepsilon)}$ near A_0).

exp: perturbing system 1 by ε small does not affect the existence of a limit cycle (and in fact, it will be nearby).

Proof: Consider $f(A) = \Phi(A) - A$



The point of the cycles are points such that

$f(A) = \Phi(A) - A = 0$

A family of perturbed phase spaces (parameterized by ε) correspond to a family of return functions $\Phi(A, \varepsilon)$

Suppose for $\varepsilon = 0$, $\Phi(A_0) = A_0$ i.e. $f_0(A_0) = 0$

and $\Phi'(A_0) \neq 1 \iff f_0'(A_0) \neq 0$

then the family of all $f_\varepsilon(A)$ has roots nearby

i.e. $\Phi(A, \varepsilon)$ has a point $A(\varepsilon)$ such that $\Phi(A(\varepsilon), \varepsilon) = A(\varepsilon)$.