Solution to Exercises

Exercise 3.1:

Solution:

Exercise 5.1:

Solution: Suppose for a contradiction that L is a regular language, thus there exists pumping length p satisfying the pumping lemma. Now consider the string

$$\omega := 0^p 1^p \in L$$

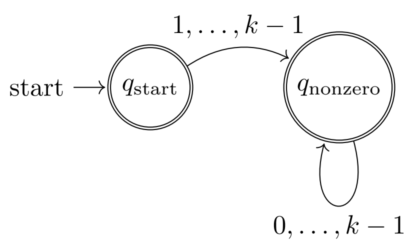
of length $2p$. For this string, in its decomposition $\omega = xyz$, we have xy is made of purely zero's. In particular, y consists of only zero's. It is now easy to see that

$$xy^i z \notin L$$

for, for example, $i = 2$. This contradicts our assumption that L is regular, hence L is not regular.

Exercise 5.2 :

Solution: Since we have the convention that $(0)_k$ is the empty word, we have the following automoton accepting the set:



Exercise 5.3 :

Solution: Recall that we know for $\omega = d_m d_{m-1} \cdots d_0$, we have defined

$$[\omega]_k = d_m k^m + d_{m-1} k^{m-1} + \cdots + d_1 k + d_0$$

Thus we can extend this definition and find that in general, we have

$$\begin{aligned}
[uv^n\omega]_k &= [u]_k \cdot k^{n|v|+|\omega|} + [v]_k \cdot (1 + k^{|v|} + \cdots + k^{(n-1)|v|}) k^{|\omega|} + [\omega]_k \\
&= [u]_k \cdot k^{n|v|+|\omega|} + [v]_k \cdot k^{|\omega|} \cdot \frac{k^{n|v|} - 1}{k^{|v|} - 1} + [\omega]_k \\
&= \underbrace{\left([u]_k \cdot k^{|\omega|} + \frac{[v]_k \cdot k^{|\omega|}}{k^{|v|} - 1} \right) k^{|v|}}_{\alpha} \cdot k^n + \underbrace{\left(-\frac{1}{k^{|v|} - 1} + [\omega]_k \right)}_{\beta}
\end{aligned}$$

the result is now obvious.

Exercise 5.4 :

Solution: Similar to exercise 5.1 with the use of Fermat's Little Theorem. Suppose for a contradiction that the set S is regular, thus it needs to satisfy the pumping lemma for some pumping length p. Find a prime γ such that its base-k expansion has length at least p, and can be written as $\gamma = uvw$. Therefore, by exercise 5.3, we kow that there exists integer k and α and β such that

$$[uvw]_k = \alpha k + \beta$$
$$[uv^n w]_k = \alpha k^n + \beta$$

Subtracting two equations yields us

$$[uv^n w]_k - [uvw]_k = \alpha(k^n - k)$$

We can now pick $n = uvw$. By Fermat's Little Theorem, we know that RHS is divisible by uvm. Thus we can deduce that

$$uvw \mid [uv^n w]_k$$

which implies that $[uv^n w]_k \notin S$, contradiction. ♠

Exercise 5.5 :

Solution: This is similar to exercise 5.1. Suppose for a contradiction that the set of palindromes is regular, then there exists a pumping length p satisfying the pumping lemma. Now consider an arbitrary string ω (palindrome) of length $2p + 1$, we know that in the decomposition ωxyz, xy does not contain the middle character of the palindrome ω. As a result, it is now easy to see that the new string

$$xy^i z$$

wouldn't be a palindrome as desired, which further implies that $xy^i z \notin L$. This contradicts the pumping lemma, which means that that L is not regular. ♠

Exercise 5.6 :

Solution: We denote the set of words over the alphabet $\{0, 1, \ldots, k-1\}$ that represent the base-k expansions of elements of $\{\ell^n : n \geq 0\}$ as L. Suppose for a contradiction that L does form a regular language. Hence for the pumping length p, let $\ell^n \in L$ be an element whose base-k expansion has length at least p, suppose $\ell^n = uvw$, we have

$$[uvw]_k = \alpha k + \beta = \ell^n$$

for some rational numbers α and β. By the pumping lemma, we know that $uv^n w \in L$ for some $n \geq 0$ with $n \neq n'$, thus by exercise 5.3 we have

$$[uv^n w]_k = \alpha k^n + \beta = \ell^{n'}$$

for some n'. Subtracting the two equations we obtain that

$$\ell^{n'} - \ell^n = \alpha k^n - \alpha k$$
$$n' \log \ell - n \log \ell = \log \alpha + n \log k - \log \alpha - \log k$$
$$n' \log \ell - n \log \ell = n \log k - \log k$$
$$(n' - n) \log \ell = (n - 1) \log k$$

This contradicts the fact that ℓ and k are multiplicatively independent, hence the set of words does not form a regular language. ♠

Exercise 5.7a :

Solution: Suppose for a contradiction that L is regular. Hence for a fixed pumping length p, we can find $\ell \in L$ with $|\ell| \geq p$ such that $\ell = xyz$. We assume that $|\ell| = q$ for some prime q. By the pumping lemma, we know that

$$xy^{q+1}z \in L$$

However, it is easy to see that $q \mid |xy^{q+1}z|$, a contradiction. Hence L is not regular. ♠

Exercise 5.7b :

Solution: Unmatched number of left and right parentheses. ♠

Exercise 5.8 :

Solution: ♠

Exercise 5.9 :

Solution: ♠

Exercise 7.1 :

Solution: `(a)` $\Rightarrow$ `(b)`: ♠