Lecture 9

A graph H is said to be p-arrangeable if there is an ordering $v_1, \dots, v_n$ of the vertices of H such that, for every vertex v_i, the set of left neighbours of the set of right neighbours of v_i has size at most p. The following theorem, generalising the result about graphs of bounded maximum degree, is due to Chen and Schelp.

Theorem 1 *For every p, there exists $c(p) > 0$ such that, if H is a p-arrangeable graph with n vertices,*

$$r(H) \leq c(p)n.$$

The best known value of $c(p)$, due to Graham, Rödl and Ruciński, is $2^{cp \log^2 p}$. One corollary is that planar graphs have linear Ramsey numbers. This follows because planar graphs are known to be 10-arrangeable.

All of these results on graphs of bounded maximum degree and bounded arrangeability stem from an important conjecture of Burr and Erdős. A graph H is said to be d-degenerate if there is an ordering $v_1, \dots, v_n$ of the vertices of H such that, for every $1 \leq i \leq n$, the vertex v_i has at most d left neighbours. Equivalently, every subgraph of H has a vertex of degree at most d. The Burr-Erdős conjecture states that for every d, there should exist $c(d) > 0$ such that, for every d-degenerate graph on n vertices, $r(H) \leq c(d)n$. This remains open.

The best result, due to Fox and Sudakov, is that, for each d, there exists $c(d) > 0$ such that, if H is a d-degenerate graph with n vertices, then $r(H) \leq 2^{c(d)\sqrt{\log n}} n$. So there is an $n^{1+o(1)}$ bound. Here we prove such a bound when H is bipartite.

Lemma 1 *Let $t, r \geq 2$ and let G be a bipartite graph with N vertices on either side and at least $2N^{2-1/(t^3 r)}$ edges. Then G contains two subsets U_1 and U_2 such that, for $k = 1, 2$, every r-tuple in U_k has at least $m = N^{1-1.8/t}$ common neighbours in U_{3-k}.*

Proof: Let $q = \frac{7}{4}rt$. Note that the density of G is at least $\alpha = 2N^{-1/(t^3 r)}$ and apply the dependent random choice lemma with α, $\beta = N^{-1.8/t^2 r}$, $s = t^2 r$ and q. Then we get a set of size at least $2N^{1-1/t}$ with fewer than

$$4\beta^{sq}\alpha^{-s}\binom{N}{q} \leq 4\beta^{s(q-1)}\frac{N^q}{q!} < 1$$

bad q-tuples, where a q-tuple is bad if it has fewer than $\beta^q N = N^{1-1.8/t}$ common neighbours. For each bad q-tuple remove a vertex. This leaves us with a set of size at least $N^{1-1/t}$ where every q-tuple is good.

Choose a random subset $T \subset U_1$ consisting of $q - r$ (not necessarily distinct) uniformly chosen vertices of U_1. Since $t \geq 2$, we have $q - r = \frac{7}{4}rt - r \geq \frac{5}{4}rt$. Let U_2 be the set of common neighbours of T. The probability that U_2 contains a subset of size r with at most m common neighbours in U_1 is at most

$$\binom{N}{r}\left(\frac{m}{|U_1|}\right)^{q-r} \leq \frac{N^r}{r!}N^{-0.8(q-r)/t} \leq 1/r! < 1,$$

where we use that $m = N^{1-1.8/t}$ and $|U_1| \geq N^{1-1/t}$.

Therefore, there is a choice of T such that every subset of U_2 of size r has at least m common neighbours in U_1. Consider now an arbitrary subset S of U_1 of size at most r. Since $S \cup T$ is a subset of U_1 of size at most q, this set has at least m common neighbours in G. By the definition of U_2 all common neighbours of T in G lie in U_2. Therefore, $N(S \cup T) \subset N(T) \subset U_2$. Hence S has at least m common neighbours in U_2, implying the result. $\square$

The usefulness of this lemma is that the criteria on U_1 and U_2 is easily enough to embed an r-degenerate graph.

Lemma 2 *Let G be a graph with vertex subsets U_1 and U_2 such that, for $k = 1, 2$, every subset of at most r vertices in U_k have at least n common neighbors in U_{3-k}. Then G contains every r-degenerate bipartite graph H with n vertices.*

Proof: Let $v_1, \ldots, v_n$ be an ordering of the vertices of H such that, for $1 \leq i \leq n$, vertex v_i has at most r neighbors v_j with $j < i$. Let A_1 and A_2 be the two parts of H. We find an embedding $f : V(H) \to V(G)$ of H in G such that the image of the vertices in A_k belongs to U_k for $k = 1, 2$. We embed the vertices of H one by one, in the above order. Without loss of generality, suppose that the vertex v_i we want to embed is in A_1. Consider the set $\{f(v_j) : j < i, (v_j, v_i) \in E(H)\}$ of images of neighbors of v_i which are already embedded. Note that this set belongs to U_2, has cardinality at most r and therefore has at least n common neighbors in U_1. All these neighbors can be used to embed v_i and at least one of them is yet not occupied, since so far we embedded less than n vertices. Pick such a neighbor w and set $f(v_i) = w$. $\square$

Putting Lemmas 1 and 2 together gives the following density result.

Corollary 1 *If $r, t \geq 2$ and G is a bipartite graph with N vertices on each side and at least $2N^{2-1/(t^3 r)}$ edges, then G contains every r-degenerate bipartite graph with at most $N^{1-1.8/t}$ vertices.*

The required Ramsey statement is a simple corollary of this result.

Corollary 2 *The Ramsey number of every r-degenerate bipartite graph H with n vertices, n sufficiently large, satisfies*

$$r(H) \leq 2^{8r^{1/3}(\log n)^{2/3}} n.$$

Proof: In every two-colouring of the edges of the complete bipartite graph $K_{N/2,N/2}$, one of the color classes contains at least half of the edges. Let $N = 2^{8r^{1/3}(\log n)^{2/3}} n$ and let $t = \frac{1}{2}(r^{-1} \log n)^{1/3}$. Then $2N^{2-1/(t^3 r)} \leq \frac{1}{2}\left(\frac{N}{2}\right)^2$ and and $N^{1-1.8/t} \geq n$. By Corollary 2, the majority color contains a copy of H. $\square$