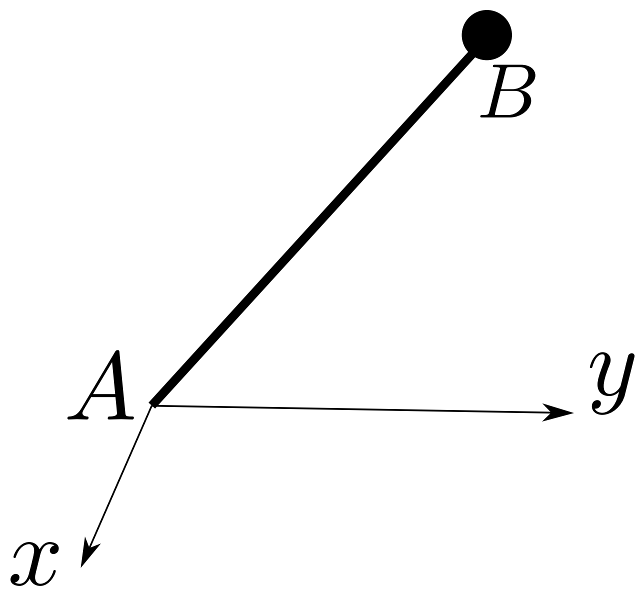

1 Pole/end-effector sub system

Consider a massless pole with a sphere attached to one end, with sphere mass being m_s. The other end of the pole (point A) is attached to the end-effector with mass m_e at point A. The state of the system is the velocity of the end-effector point A $(\dot{x}_A, \dot{y}_A, \dot{z}_A)$, the delta position between the sphere and the end-effector in the horizontal plane $x_{AB} = x_B - x_A, y_{AB} = y_B - y_A$, together with its time derivative $\dot{x}_{AB}, \dot{y}_{AB}$.

The position of the mass is

$$\begin{bmatrix} x_A + x_{AB} \\ y_A + y_{AB} \\ z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2} \end{bmatrix} \tag{1}$$

The velocity of the mass is

$$\begin{bmatrix} \dot{x}_A + \dot{x}_{AB} \\ \dot{y}_A + \dot{y}_{AB} \\ \dot{z}_A - \frac{x_{AB}\dot{x}_{AB} + y_{AB}\dot{y}_{AB}}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}} \end{bmatrix} \tag{2}$$

The total kinetic energy of the system is

$$\begin{aligned} T = 0.5m_e(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2) + 0.5m_s(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2 + \dot{x}_{AB}^2 + \dot{y}_{AB}^2 + \frac{(x_{AB}^2\dot{x}_{AB}^2 + y_{AB}^2\dot{y}_{AB}^2 + 2x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB})}{l^2 - x_{AB}^2 - y_{AB}^2} \\ + 2\dot{x}_A\dot{x}_{AB} + 2\dot{y}_A\dot{y}_{AB} - \frac{2\dot{z}_A(x_{AB}\dot{x}_{AB} + y_{AB}\dot{y}_{AB})}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}}) \end{aligned} \tag{3}$$

The total potential energy is

$$V = m_e g z_A + m_s g(z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2}) \tag{4}$$

Using Lagrangian $L = T - V$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$, we have

$$(m_e + m_s)\ddot{x}_A + m_s\ddot{x}_{AB} = f_x \tag{5}$$

$$(m_e + m_s)\ddot{y}_A + m_s\ddot{y}_{AB} = f_y \tag{6}$$

$$(m_e+m_s)(\ddot{z}_A+g)-m_s\left(\dot{x}_{AB}^2\frac{l^2-y_{AB}^2}{z_{AB}^3}+\frac{x_{AB}}{z_{AB}}\ddot{x}_{AB}+\dot{y}_{AB}^2\frac{l^2-x_{AB}^2}{z_{AB}^3}+\frac{y_{AB}}{z_{AB}}\ddot{y}_{AB}-2\frac{x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB}}{z_{AB}^3}\right)=f_z \tag{7}$$

$$\begin{aligned}m_s(\ddot{x}_A+\ddot{x}_{AB})-m_s x_{AB}(g+\ddot{z}_A)/z_{AB}+m_s(x_{AB}^2\ddot{x}_{AB}+x_{AB}\dot{x}_{AB}^2+x_{AB}y_{AB}\ddot{y}_{AB}+x_{AB}\dot{y}_{AB}^2)/z_{AB}^2\\ +m_s(x_{AB}^3\dot{x}_{AB}^2+2x_{AB}^2y_{AB}\dot{x}_{AB}\dot{y}_{AB}+x_{AB}y_{AB}^2\dot{y}_{AB}^2)/z_{AB}^4=0\end{aligned} \tag{8}$$

$$\begin{aligned}m_s(\ddot{y}_A+\ddot{y}_{AB})-m_s y_{AB}(g+\ddot{z}_A)/z_{AB}+m_s(y_{AB}^2\ddot{y}_{AB}+y_{AB}\dot{y}_{AB}^2+y_{AB}x_{AB}\ddot{x}_{AB}+y_{AB}\dot{x}_{AB}^2)/z_{AB}^2\\ +m_s(y_{AB}^3\dot{y}_{AB}^2+2y_{AB}^2x_{AB}\dot{y}_{AB}\dot{x}_{AB}+y_{AB}x_{AB}^2\dot{x}_{AB}^2)/z_{AB}^4=0\end{aligned} \tag{9}$$

In the matrix form, we have

$$M\begin{bmatrix}\ddot{x}_A\\ \ddot{y}_A\\ \ddot{z}_A\\ \ddot{x}_{AB}\\ \ddot{y}_{AB}\end{bmatrix}+C=\begin{bmatrix}f_x\\ f_y\\ f_z\\ 0\\ 0\end{bmatrix} \tag{10}$$

where

$$M=\begin{bmatrix}m_e+m_s & 0 & 0 & m_s & 0\\ 0 & m_e+m_s & 0 & 0 & m_s\\ 0 & 0 & m_e+m_s & -m_s\frac{x_{AB}}{z_{AB}} & -m_s\frac{y_{AB}}{z_{AB}}\\ m_s & 0 & -m_s\frac{x_{AB}}{z_{AB}} & m_s+m_s\frac{x_{AB}^2}{z_{AB}^2} & m_s\frac{x_{AB}y_{AB}}{z_{AB}^2}\\ 0 & m_s & -m_s\frac{y_{AB}}{z_{AB}} & m_s\frac{x_{AB}y_{AB}}{z_{AB}^2} & m_s+m_s\frac{y_{AB}^2}{z_{AB}^2}\end{bmatrix} \tag{11}$$

$$C=\begin{bmatrix}0\\ 0\\ (m_e+m_s)g-m_s\left(\dot{x}_{AB}^2\frac{l^2-y_{AB}^2}{z_{AB}^3}+\dot{y}_{AB}^2\frac{l^2-x_{AB}^2}{z_{AB}^3}-2\frac{x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB}}{z_{AB}^3}\right)\\ -m_s g\frac{x_{AB}}{z_{AB}}+m_s x_{AB}\left(\frac{\dot{x}_{AB}^2+\dot{y}_{AB}^2}{z_{AB}^2}+\frac{(x_{AB}\dot{x}_{AB}+y_{AB}\dot{y}_{AB})^2}{z_{AB}^4}\right)\\ -m_s g\frac{y_{AB}}{z_{AB}}+m_s y_{AB}\left(\frac{\dot{x}_{AB}^2+\dot{y}_{AB}^2}{z_{AB}^2}+\frac{(x_{AB}\dot{x}_{AB}+y_{AB}\dot{y}_{AB})^2}{z_{AB}^4}\right)\end{bmatrix} \tag{12}$$

2 Whole system

Assuming that we construct a controller, that given the current state of the pole/end-effector system, this controller computes the force f_x, f_y, f_z applied from the robot to the end-effector, now we want to compute the robot joint torque τ to apply that force.

We first apply the force f_x, f_y, f_z computed from the controller as input to the pole/end-effector system, from the dynamics equation (10) we can compute the acceleration of the end effector $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$, together with the acceleration of the pole $\ddot{x}_B, \ddot{y}_B, \ddot{z}_B$ (where we use both $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$ and $\ddot{x}_{AB}, \ddot{y}_{AB}, \ddot{z}_{AB}$). We know that for the pole to achieve this acceleration, the end-effector has to apply a force

$$f_B=m_s\begin{bmatrix}\ddot{x}_B\\ \ddot{y}_B\\ \ddot{z}_B+g\end{bmatrix} \tag{13}$$

onto the pole, applied at where the pole makes contact with the end effector. Based on Newton's third law, there is an equal and opposite force $-f_B$ applied on the end-effector at the contact point P between the end-effector and the pole. Hence our goal is to compute the joint torque of the robot arm, such that the end-effector can achieve the desired acceleration $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$ under the external force $-f_B$.

We can write the manipulator equation for the IIWA arm (together with the end-effector welded to the wrist joint)

$$M_{iiwa}\ddot{q}_{iiwa} + C = g(q_{iiwa}) + \tau - \left({}^{W}J^{P}\right)^{T} f_B \tag{14}$$

where ${}^{W}J^{P}$ is the Jacobian of the contact point P written in the world frame. And we also have the constraints on the end-effector acceleration

$${}^{W}J^{E}\ddot{q}_{iiwa} + {}^{W}\dot{J}^{E}\dot{q}_{iiwa} = {}^{W}a^{E}_{des} \tag{15}$$

where ${}^{W}J^{E}$ is the Jacobian of the end-effector written in the world frame. ${}^{W}a^{E}_{des}$ is the desired acceleration of the IIWA end-effector frame E in the world frame W. This acceleration includes both the linear acceleraton $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$, together with the desired angular acceleration. The desired acceleration can be set in two ways

- If our goal is to keep the end-effector to be horizontal, then we can compute the desired angular acceleration using a PD law depending on the orientation error.
- We can also set the desired end-effector angular acceleration to be zero. Note that this won't handle any orientation drift. Namely if currently the end-effector angular velocity is non-zero, then the orientation will continue to drift.

Combining the equations (14) and (15) we have unknown variable $\tau \in \mathbb{R}^7$. The problem is under-constrained, we can also impose the cost as $\min \tau^T\tau$ to get a unique (and optimal) τ. If we ignore the joint torque limit, then this ends up being an equality-constrained QP which can be solved very efficiently in the closed form.