Matrix Analysis Homework

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1. Using Housholder reduction and Givens reduction, compute the QR factors of

$$\mathbf{A} = \begin{pmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{pmatrix}$$

Solution:
采用 Householder 约简

$$\mathbf{u}_1 = \mathbf{A}_{*1} - ||\mathbf{A}_{*1}||\mathbf{e}_1 = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \; and \; \mathbf{R}_1 = \mathbf{I} - 2\frac{\mathbf{u}_1\mathbf{u}_1^T}{\mathbf{u}_1^T\mathbf{u}_1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

所以

$$\mathbf{R}_1\mathbf{A} = [\mathbf{R}_1\mathbf{A}_{*1}|\mathbf{R}_1\mathbf{A}_{*2}|\mathbf{R}_1\mathbf{A}_{*3}] = \begin{pmatrix} 3 & 15 & 0 \\ 0 & -9 & 54 \\ 0 & 12 & 3 \end{pmatrix}$$

令 $\mathbf{A}_2 = \begin{pmatrix} -9 & 54 \\ 12 & 3 \end{pmatrix}$，那么

$$\mathbf{u}_2 = [\mathbf{A}_2]_{*1} - ||[\mathbf{A}_2]_{*1}||\mathbf{e}_1 = \begin{pmatrix} -24 \\ 12 \end{pmatrix}, and \hat{\mathbf{R}} = \mathbf{I} - 2\frac{\mathbf{u}_2\mathbf{u}_2^T}{\mathbf{u}_2^T\mathbf{u}_2} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

令 $\mathbf{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ 所以

$$\mathbf{R}_2\mathbf{R}_1\mathbf{A} = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix}$$

所以

$$\mathbf{Q} = \frac{1}{15}\begin{pmatrix} 5 & -10 & 10 \\ 14 & 5 & -2 \\ -2 & 10 & 11 \end{pmatrix}^T = \frac{1}{15}\begin{pmatrix} 5 & 14 & -2 \\ -10 & 5 & 10 \\ 10 & -2 & 11 \end{pmatrix}, R = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix}$$

采用 Givens 约简，首先对第一列一二行约简.

$$\mathbf{P}_{12} = \begin{pmatrix} \frac{\sqrt{5}}{5} & -2\frac{\sqrt{5}}{5} & 0 \\ 2\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{P}_{12}\mathbf{A} = \begin{pmatrix} \sqrt{5} & 29\frac{\sqrt{5}}{5} & -74\frac{\sqrt{5}}{5} \\ 0 & 33\frac{\sqrt{5}}{5} & -48\frac{\sqrt{5}}{5} \\ 2 & 8 & 37 \end{pmatrix}$$

之后对第一列一三行约简.

$$\mathbf{P}_{13} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}, \mathbf{P}_{12}\mathbf{A} = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 33\frac{\sqrt{5}}{5} & -48\frac{\sqrt{5}}{5} \\ 0 & -\frac{6\sqrt{5}}{5} & \frac{111\sqrt{5}}{5} \end{pmatrix}$$

然后对第二列二三行约简.

$$\mathbf{P}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11\sqrt{5}}{25} & -\frac{2\sqrt{5}}{25} \\ 0 & \frac{2\sqrt{5}}{25} & \frac{11\sqrt{5}}{25} \end{pmatrix}, \mathbf{P}_{12}\mathbf{A} = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix}$$

可以得到

$$\mathbf{Q} = (\mathbf{P}_{23}\mathbf{P}_{13}\mathbf{P}_{12})^T = \frac{1}{15}\begin{pmatrix} 5 & 14 & -2 \\ -10 & 5 & 10 \\ 10 & -2 & 11 \end{pmatrix}, R = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix}$$

7.Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathcal{R}^3$ whose respective bases are

$$\mathcal{B}_{\mathcal{X}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\} and \mathcal{B}_{\mathcal{Y}} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

(a) Explain why $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces of $\mathcal{R}^3$.

(b) Determine the projector $\mathbf{P}$ onto $\mathcal{X}$ along $\mathcal{Y}$ as well as the complementary projector $\mathbf{Q}$ onto $\mathcal{Y}$ along $\mathcal{X}$.

(c) Determine the projection of $\mathbf{v} = (2, -1, 1)^T$ onto $\mathcal{Y}$ along $\mathcal{X}$.

(d) Verify that $\mathbf{P}$ and $\mathbf{Q}$ are both idempotent.

(e) Verify that $\mathbf{R(P)} = \mathcal{X} = \mathbf{N(Q)}$ and $\mathbf{N(P)} = \mathcal{Y} = \mathbf{R(Q)}$.

Solution:

(a) 将 $\mathcal{X}$ 和 $\mathcal{Y}$ 的基合并为矩阵 $\mathbf{A}$，求其秩可得

$$rank(\mathbf{A}) = rank\left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}\right) = 3$$

由于 $\mathcal{X}$ 和 $\mathcal{Y}$ 的基的个数分别为 2 和 1，其合并后秩为 3. 所以其一定为 $\mathcal{R}^3$ 的互补集.

(b) 投影矩阵 $\mathbf{P}$ 满足

$$\begin{aligned}
\mathbf{P} =& [\mathbf{X}|\mathbf{O}][\mathbf{X}|\mathbf{Y}]^{-1} && (1)\\
=& \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}^{-1} && (2)\\
=& \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} && (3)\\
=& \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} && (4)
\end{aligned}$$

投影矩阵 $\mathbf{Q}$ 满足

$$\begin{aligned}
\mathbf{P} =& [\mathbf{O}|\mathbf{Y}][\mathbf{X}|\mathbf{Y}]^{-1} && (5)\\
=& \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}^{-1} && (6)\\
=& \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} && (7)\\
=& \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} && (8)
\end{aligned}$$

(c)$\mathbf{v}$ 的投影满足

$$\begin{aligned}
\mathbf{v}' =& \mathbf{Q}\mathbf{v} && (9)\\
=& \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} && (10)\\
=& \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} && (11)
\end{aligned}$$

(d) 由于

$$\begin{aligned}
\mathbf{P}^2 &= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} && (12)\\
&= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} = \mathbf{P} && (13)
\end{aligned}$$

$$\mathbf{Q}^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} = \mathbf{Q} \tag{15}$$

所以 $\mathbf{P}$,$\mathbf{Q}$ 均为幂等阵.

(e) 对于任意 $x \in \mathcal{R}^3$

由于 $\{\mathcal{B}_{\mathcal{X}}, \mathcal{B}_{\mathcal{Y}}\}$ 构成 $\mathcal{R}^3$ 的一组基，所以

$$x = a_1x_1 + a_2x_2 + a_3y_1$$

所以

$$\mathbf{Px} = a_1x_1 + a_2x_2 \in \mathcal{X}$$

也即 $\mathbf{R(x)} \subset \mathcal{X}$ 又由于 $rank(\mathbf{R(P)}) = rank(\mathcal{X}) = 2$, 所以

$$\mathbf{R(P)} = \mathcal{X}$$

同理可以证明

$$\mathbf{R(Q)} = \mathcal{Y}$$

然后求解 $\mathbf{N(P)}$，求解

$$\mathbf{Px} = 0 \tag{16}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{17}$$

解得

$$\mathbf{x} = \begin{pmatrix} \frac{x_3}{3} \\ \frac{2x_3}{3} \\ x_3 \end{pmatrix} = \frac{x_3}{3} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

可知 $\mathbf{N(P)}$ 和 $\mathcal{Y}$ 具有相同的基，所以 $N(P) = \mathcal{Y}$

然后求解 $\mathbf{N(Q)}$，求解

$$\mathbf{Qx} = 0 \tag{18}$$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \mathbf{x} = 0 \tag{19}$$

解得

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

又由于

$$\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\1\end{pmatrix}$$

且 $\begin{pmatrix}1\\2\\2\end{pmatrix}$ 与 $\begin{pmatrix}1\\1\\1\end{pmatrix}$ 线性无关，所以 $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$ 也是 $\mathbf{N(Q)}$ 的一组基

可知 $\mathbf{N(Q)}$ 和 $\mathcal{X}$ 具有相同的基，所以 $\mathbf{N(Q)}=\mathcal{X}$

所以 $\mathbf{R(P)}=\mathcal{X}=\mathbf{N(Q)},\mathbf{N(P)}=\mathcal{Y}=\mathbf{R(Q)}$

16. Determine the orthogonal projection of $\mathbf{b}$ onto $\mathcal{M}$, where

$$\mathbf{b}=\begin{pmatrix}5\\2\\5\\3\end{pmatrix}\ and\ \mathcal{M}=span\left\{\begin{pmatrix}-\frac{3}{5}\\0\\\frac{4}{5}\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}\frac{4}{5}\\0\\\frac{3}{5}\\1\end{pmatrix}\right\}$$

Solution:

基构成的矩阵

$$\mathbf{M}=\begin{pmatrix}-\frac{3}{5}&0&\frac{4}{5}\\0&0&0\\\frac{4}{5}&0&\frac{3}{5}\\0&1&1\end{pmatrix}$$

正交投影矩阵 $\mathbf{P}$ 满足

$$\begin{aligned}\mathbf{P}=&\mathbf{M}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T &(20)\\=&\mathbf{M}\mathbf{M}^T &(21)\\=&\begin{pmatrix}1&0&0&\frac{4}{5}\\0&0&0&0\\0&0&0&\frac{3}{5}\\\frac{4}{5}&0&\frac{3}{5}&0\end{pmatrix} &(22)\end{aligned}$$

投影得到向量为 $\mathbf{v}$'，其满足

$$\begin{aligned}\mathbf{v}'=&\mathbf{P}\mathbf{v} &(23)\\=&\begin{pmatrix}1&0&0&\frac{4}{5}\\0&0&0&0\\0&0&0&\frac{3}{5}\\\frac{4}{5}&0&\frac{3}{5}&0\end{pmatrix}\begin{pmatrix}5\\2\\5\\3\end{pmatrix} &(24)\\=&\begin{pmatrix}\frac{37}{5}\\0\\\frac{9}{5}\\7\end{pmatrix} &(25)\end{aligned}$$