Matrix Analysis Homework

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3. Evaluate the Frobenius matrix norm, 1-norm,2-norm and ∞-norm for each matrix below.

$$\mathbf{A}=\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{C}=\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

Solution:

$$||\mathbf{A}||_F=\sqrt{10}, ||\mathbf{A}||_1=\max_j \sum_i |a_{ij}|=4, ||\mathbf{A}||_2=\sqrt{\lambda_{\max}}=\sqrt{10}, ||\mathbf{A}||_\infty=\max_i \sum_j |a_{ij}|=3,$$

$$||\mathbf{B}||_F=\sqrt{3}, ||\mathbf{B}||_1=\max_j \sum_i |b_{ij}|=1, ||\mathbf{B}||_2=\sqrt{\lambda_{\max}}=1, ||\mathbf{B}||_\infty=\max_i \sum_j |b_{ij}|=1,$$

$$||\mathbf{C}||_F=9, ||\mathbf{C}||_1=\max_j \sum_i |c_{ij}|=10, ||\mathbf{C}||_2=\sqrt{\lambda_{\max}}=9, ||\mathbf{C}||_\infty=\max_i \sum_j |c_{ij}|=10,$$

10. Let $\mathcal{S}=span\{\mathbf{x}_1=(1;1;1;-1)^T;\mathbf{x}_2=(2;-1;-1;1)^T;\mathbf{x}^3=(-1;2;2;1)^T\}$.
(a) Use the classical GramSchmidt algorithm (with exact arithmetic) to determine an orthonormal basis for $\mathcal{S}$.
(b) Repeat part (a) using the modified GramSchmidt algorithm, and compare the results.
Solution:
(a) 当 $k=1$ 时，

$$\mathbf{u}_1=\frac{\mathbf{x}_1}{||\mathbf{x}_1||}=\frac{1}{2}(1;1;1;-1)^T$$

当 $k=2$ 时，

$$\mathbf{u}_2=\frac{\mathbf{x}_2-\sum_{i=1}^{1}(\mathbf{u}_i^*\mathbf{x}_3)\mathbf{u}_i}{||\mathbf{x}_3-\sum_{i=1}^{1}(\mathbf{u}_i^*\mathbf{x}_3)\mathbf{u}_i||}=\frac{\sqrt{3}}{6}(3;-1;-1;1)^T$$

当 $k=3$ 时，

$$\mathbf{u}_3=\frac{\mathbf{x}_3-\sum_{i=1}^{2}(\mathbf{u}_i^*\mathbf{x}_3)\mathbf{u}_i}{||\mathbf{x}_3-\sum_{i=1}^{2}(\mathbf{u}_i^*\mathbf{x}_3)\mathbf{u}_i||}=\frac{\sqrt{6}}{6}(0;1;1;2)^T$$

得到的一组正交基为 $\mathcal{S}=span\{\mathbf{u}_1=\frac{1}{2}(1;1;1;\ -1)^T;\mathbf{u}_2=\frac{\sqrt{3}}{6}(3;\ -1;\ -1;1)^T;\mathbf{u}^3=\frac{\sqrt{6}}{6}(0;1;1;2)^T\}$
(b) 当 $k=1$ 时，

$$\mathbf{E}_1=\mathbf{I}, \mathbf{u}_1=\frac{\mathbf{E}_1\mathbf{x}_1}{||\mathbf{E}_1\mathbf{x}_1||}=\frac{1}{2}(1;1;1;\ -1)^T$$

当 $k=2$ 时，

$$\mathbf{E}_2=\mathbf{I}-\mathbf{u}_1^*\mathbf{u}_1=\frac{1}{4}\begin{pmatrix}3&-1&-1&1\\-1&3&-1&1\\-1&-1&3&1\\1&1&1&3\end{pmatrix},\mathbf{u}_2=\frac{\mathbf{E}_1\mathbf{x}_2}{||\mathbf{E}_1\mathbf{x}_2||}=\frac{\sqrt{3}}{6}(3;\ -1;\ -1;1)^T$$

当 $k=3$ 时，

$$\mathbf{E}_3=\mathbf{I}-\mathbf{u}_2^*\mathbf{u}_2=\frac{1}{12}\begin{pmatrix}3&3&3&-3\\3&11&-1&1\\3&-1&11&1\\-3&1&1&11\end{pmatrix},\mathbf{u}_2=\frac{\mathbf{E}_2\mathbf{E}_1\mathbf{x}_3}{||\mathbf{E}_2\mathbf{E}_1\mathbf{x}_3||}=\frac{\sqrt{6}}{6}(0;1;1;2)^T$$

得到的一组正交基为 $\mathcal{S}=span\{\mathbf{u}_1=\frac{1}{2}(1;1;1;\ 1)^T;\mathbf{u}_2=\frac{\sqrt{3}}{6}(3;\ -1;\ -1;-1)^T;\mathbf{u}^3=\frac{\sqrt{6}}{6}(0;1;1;2)^T\}$

12. Let $\mathbf{A}=\begin{pmatrix}1&0&-1\\1&2&1\\1&1&-3\\0&1&1\end{pmatrix}$ and $\mathbf{b}=(1,1,1,1)$

(a) Determine the rectangular QR factorization of $\mathbf{A}$.

(b) Use the QR factor from part (a) to determine the least squares solution to $\mathbf{Ax}=\mathbf{b}$.

Solution:

(a)$k=1$ 时，

$$r_{11}=||A_{*1}||=\sqrt{3},\mathbf{q}_1=\frac{\sqrt{3}}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$k=2$ 时，

$$r_{12}=\mathbf{q}_1^*A_{*2}=\sqrt{3},\mathbf{q}_2=\mathbf{x}_2-r_{12}\mathbf{q}_1=\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$$r_{22}=||\mathbf{q}_2||=\sqrt{3},\mathbf{q}_2=\frac{\mathbf{q}_2}{r_{22}}=\frac{\sqrt{3}}{3}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$k=3$ 时，

$$r_{13}=\mathbf{q}_1^*A_{*3}=-\sqrt{3},r_{23}=\mathbf{q}_2^*A_{*3}=\sqrt{3},\mathbf{q}_2=\mathbf{x}_3-r_{13}\mathbf{q}_1-r_{23}\mathbf{q}_2=\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$$

$$r_{33}=||\mathbf{q}_3||=\sqrt{6},\mathbf{q}_2=\frac{\mathbf{q}_2}{r_{33}}=\frac{\sqrt{6}}{6}\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$$

所以 QR 分解的结果为

$$Q=\begin{pmatrix}\frac{\sqrt{3}}{3}&-\frac{\sqrt{3}}{3}&\frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}&\frac{\sqrt{3}}{3}&\frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}&0&-\frac{\sqrt{6}}{3}\\0&\frac{\sqrt{3}}{3}&0\end{pmatrix},R=\begin{pmatrix}\sqrt{3}&\sqrt{3}&\sqrt{3}\\0&\sqrt{3}&\sqrt{3}\\0&0&\sqrt{6}\end{pmatrix}$$

那么方程 **Ax=b** 可以简化为

$$\mathbf{QRx}=\mathbf{b} \tag{1}$$

$$\mathbf{Q}^T\mathbf{QRx}=\mathbf{Q}^T\mathbf{b} \tag{2}$$

$$\mathbf{Rx}=\mathbf{Q}^T\mathbf{b} \tag{3}$$

$$\begin{pmatrix}\sqrt{3}&\sqrt{3}&\sqrt{3}\\0&\sqrt{3}&\sqrt{3}\\0&0&\sqrt{6}\end{pmatrix}\mathbf{x}=\begin{pmatrix}\sqrt{3}\\\frac{\sqrt{3}}{3}\\0\end{pmatrix} \tag{4}$$

$$\mathbf{x}=\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\0\end{pmatrix} \tag{5}$$

16. Let $\mathbf{u}=(-2,1,3,-1)^T$ and $\mathbf{v}=(1,4,0,-1)^T$.
(a) Determine the orthogonal projection of $\mathbf{u}$ onto $span\{v\}$.
(b) Determine the orthogonal projection of $\mathbf{v}$ onto $span\{u\}$.
(c) Determine the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}^{\perp}$?.
(d) Determine the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}^{\perp}$?.

Solution:

(a) 正交投影 $\mathbf{u}_v$ 满足

$$\mathbf{u}_v=\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}}\mathbf{u}=\frac{1}{18}\begin{pmatrix}1&4&0&-1\\4&16&0&-4\\0&0&0&0\\-1&-4&0&1\end{pmatrix}\begin{pmatrix}-2\\1\\3\\-1\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1\\4\\0\\-1\end{pmatrix}$$

(b) 正交投影 $\mathbf{v}_u$ 满足

$$\mathbf{v}_u=\frac{\mathbf{uu}^*}{\mathbf{u}^*\mathbf{u}}\mathbf{v}=\frac{1}{15}\begin{pmatrix}4&-2&-6&2\\-2&1&3&-1\\-6&3&9&-3\\2&-1&-3&1\end{pmatrix}\begin{pmatrix}1\\4\\0\\-1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2\\1\\3\\-1\end{pmatrix}$$

(c) 正交投影 $\mathbf{u}_{v^\perp}$ 满足

$$\mathbf{u}_{v^\perp} = (\mathbf{I} - \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}})\mathbf{u} = \frac{1}{18}\begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -13 \\ 2 \\ 18 \\ -5 \end{pmatrix}$$

(d) 正交投影 $\mathbf{v}_{u^\perp}$ 满足

$$\mathbf{v}_{u^\perp} = (\mathbf{I} - \frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}})\mathbf{v} = \frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ 19 \\ -3 \\ -4 \end{pmatrix}$$