Matrix Analysis Homework

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11.Suppose $\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

(a) Use the Sherman-Morrison formula to determine the inverse of the matrix $\mathbf{B}$ that is obtained by changing the $(3, 2)$-entry in $\mathbf{A}$ from 0 to 2.

(b) Let $\mathbf{C}$ be the matrix that agrees with $\mathbf{A}$ except that $c_{32} = 2$ and $c_{33} = 2$. Use the Sherman-Morrison formula to find C^{-1}.

Solution:

(a)

$$
\begin{aligned}
(\mathbf{B})^{-1} &= \left(\mathbf{A} + 2\mathbf{e}_3\mathbf{e}_2^T\right)^{-1} \\
&= \mathbf{A}^{-1} - 2\frac{\mathbf{A}^{-1}\mathbf{e}_3\mathbf{e}_2^T\mathbf{A}^{-1}}{1 + 2\mathbf{e}_2^T\mathbf{A}^{-1}\mathbf{e}_3} \\
&= \mathbf{A}^{-1} - 2\frac{\left[\mathbf{A}^{-1}\right]_{*,3}\left[\mathbf{A}^{-1}\right]_{2,*}}{1 + 2\left[\mathbf{A}^{-1}\right]_{2,3}} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - 2\frac{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}}{1 + 2 * (-1)} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}
\end{aligned}
$$

(b) 易知 $\mathbf{C}$=$\mathbf{B}$+$\mathbf{e}_3\mathbf{e}_3^T$，所以

$$
\begin{aligned}
(\mathbf{C})^{-1} &= \left(\mathbf{B}+\mathbf{e}_3\mathbf{e}_3^T\right)^{-1} \\
&=\mathbf{B}^{-1}-\frac{\mathbf{B}^{-1}\mathbf{e}_3\mathbf{e}_3^T\mathbf{B}^{-1}}{1+\mathbf{e}_3^T\mathbf{B}^{-1}\mathbf{e}_3} \\
&=\mathbf{B}^{-1}-\frac{\left[\mathbf{B}^{-1}\right]_{*,3}\left[\mathbf{B}^{-1}\right]_{3,*}}{1+\left[\mathbf{B}^{-1}\right]_{3,3}} \\
&=\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}-\frac{\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} 1 & 4 & -2 \end{pmatrix}}{1-2} \\
&=\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}+\begin{pmatrix} -1 & -4 & 2 \\ 1 & 4 & -2 \\ -2 & -8 & 4 \end{pmatrix} \\
&=\begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}
\end{aligned}
$$

12.Let $\mathbf{A}=\begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}$

(a) Determine the LU factors of $\mathbf{A}$.

(b) Use the LU factors to solve $\mathbf{A}x_1=b_1$ as well as $\mathbf{A}x_2=b_2$, where $b_1=(6,0,\ 6)^T$ and $b_2=(6,6,12)^T$.

Solution:

(a) 对于 $\mathbf{A}$ 应用高斯消元法

$$
\begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}\begin{matrix} \\ R_2-4R_1 \\ R_3-3R_1 \end{matrix}\rightarrow\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 4 & 15 \end{pmatrix}\begin{matrix} \\ \\ R_3-2R_2 \end{matrix}\rightarrow\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}=\mathbf{U}
$$

设三个初等行变换分别为 $\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3$, 则矩阵 L 满足

$$
\mathbf{L}=\mathbf{G}_1^{-1}\mathbf{G}_2^{-1}\mathbf{G}_3^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}
$$

(b) 对于 $\mathbf{A}x_1=b_1$, 可化为 $\mathbf{LU}x_1=b_1$, 也即 $\mathbf{L}(\mathbf{U}x_1)=b_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{U}x_1 = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$$

$$\mathbf{U}x_1 = \begin{pmatrix} 6 \\ -24 \\ 36 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} x_1 = \begin{pmatrix} 6 \\ -24 \\ 36 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} 138 \\ -48 \\ 12 \end{pmatrix}$$

对于 $\mathbf{A}x_2 = b_2$, 可化为 $\mathbf{LU}x_2 = b_2$, 也即 $\mathbf{L}(\mathbf{U}x_2) = b_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{U}x_1 = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}$$

$$\mathbf{U}x_1 = \begin{pmatrix} 6 \\ -18 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} x_2 = \begin{pmatrix} 6 \\ -18 \\ 30 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 112 \\ -39 \\ 10 \end{pmatrix}$$