Matrix Analysis Homework

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7.Let T be the linear operator on $\mathcal{R}^4$ defined by

$$\mathbf{T}(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 2x_3 - x_4, x_2 + x_4, 2x_3 - x_4, x_3 + x_4),$$

and let $\mathcal{X} = span\{e_1, e_2\}$ be the subspace that is spanned by the first two unit vectors in $\mathcal{R}^4$.
(a) Explain why X is invariant under $\mathbf{T}$.
(b) Determine$\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{e_1,e_2\}}$.
(c) Describe the structure of $[\mathbf{T}]_{\mathcal{B}}$, where $\mathcal{B}$ is any basis obtained from an extension of $\{e_1, e_2\}$.
Solution:
(a) 对于 $\mathcal{R}^4$ 的标准基 $\{e_1, e_2, e_3, e_4\}$ 而言，如果 $\mathbf{u} = \sum_{k=1}^{n} x_k e_k$, 那么线性变换可以表示为

$$\mathbf{T}(\mathbf{u}) = (x_1 + x_2 + 2x_3 - x_4)e_1 + (x_2 + x_4)e_2 + (2x_3 - x_4)e_3 + (x_3 + x_4)e_4$$

所以代入 $\mathbf{u} = e_1(x_1 = 1, x_2 = x_3 = x_4 = 0)$ 和 $\mathbf{u} = e_2(x_1 = x_3 = x_4 = 0, x_2 = 1)$ 可以得到

$$\begin{aligned}\mathbf{T}(e_1) =& e_1\\ \mathbf{T}(e_2) =& e_1 + e_2\end{aligned}$$

那么对于 $\mathcal{X}$ 空间下的任意一组向量 $\mathbf{u'} = a_1e_1 + a_2e_2$ 有

$$\begin{aligned}\mathbf{T}(\mathbf{u'}) =& \mathbf{T}(a_1e_1 + a_2e_2)\\ =& a_1\mathbf{T}(e_1) + a_2\mathbf{T}(e_2)\\ =& (a_1 + a_2)e_1 + a_2e_2\end{aligned}$$

所以对于任意 $\mathbf{u'} \in \mathcal{X}$, 有 $T(\mathbf{u'}) \in \mathcal{X}$., 所以 $\mathcal{X}$ 是一个不变子空间.
(b) 由 $\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{e_1,e_2\}}$ 的定义可知

$$\begin{aligned}\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{e_1,e_2\}} &= \left[[\mathbf{T}_{/\mathcal{X}}(e_1)]_{\{e_1,e_2\}}, [\mathbf{T}_{/\mathcal{X}}(e_2)]_{\{e_1,e_2\}}\right]\\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

(c) 假设 $\mathcal{B}$ 中的基为 $\{e_1, e_2, e'_3, e'_4\}$ 那么有

$$[\mathbf{T}]_{\mathcal{B}} = [[\mathbf{T}(e_1)]_{\mathcal{B}}, [\mathbf{T}(e_2)]_{\mathcal{B}}, [\mathbf{T}(e'_3)]_{\mathcal{B}}, [\mathbf{T}(e'_4)]_{\mathcal{B}}] \tag{1}$$

对于 $[\mathbf{T}(e_i)]_{\mathcal{B}}, i = 1, 2$ 而言, 其满足

$$\mathbf{T}(e_i) = \sum_{k=1}^{2} a_{ik} e_k, and\ [\mathbf{T}(e_i)]_{\mathcal{B}} = \begin{bmatrix} a_{i1} \\ a_{i2} \\ 0 \\ 0 \end{bmatrix}$$

而对于 $[\mathbf{T}(e_i')]_{\mathcal{B}}, i = 3, 4$ 满足一般情况, 即

$$[\mathbf{T}(e_i')]_{\mathcal{B}} = \begin{bmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \\ b_{i4} \end{bmatrix}$$

所以

$$\begin{aligned}
[\mathbf{T}]_{\mathcal{B}} &= \begin{bmatrix} a_{11} & a_{21} & b_{31} & b_{41} \\ a_{12} & a_{22} & b_{32} & b_{42} \\ 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & b_{34} & b_{44} \end{bmatrix} && (2) \\
&= \begin{bmatrix} \left[\mathbf{T}_{/\mathcal{X}}\right]_{\{e_1, e_2\}} & \mathbf{B}_{2\times 2} \\ \mathbf{O}_{2\times 2} & \mathbf{C}_{2\times 2} \end{bmatrix} && (3)
\end{aligned}$$