VV156 Regular RC6

Series

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December 1, 2022

RC Overview

Limits of Sequences

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Limits of Sequences: Precise Definition

Definition

Suppose that there is a sequence a_n. If for any fixed positive number ε, there exits a positive integer N such that for any $n > N$, we have

$$|a_n - L| < \varepsilon$$

Then we say that the sequence a_n has the limit L.

Note: the precise definition (i.e. $\varepsilon - N$ language) is optional.

Theorems

1. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$
2. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$
3. (Squeeze theorem) If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$
4. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$
5. Every bounded, monotonic sequence is convergent.

6*. (Bolzano Weierstrass Theorem) A Bounded sequence must have a convergent subsequence.

Properties

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}(a_n-b_n)=\lim_{n\to\infty}a_n-\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}ca_n=c\lim_{n\to\infty}a_n \qquad \lim_{n\to\infty}c=c$$

$$\lim_{n\to\infty}(a_nb_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n} \qquad \text{if } \lim_{n\to\infty}b_n\neq 0$$

$$\lim_{n\to\infty}a_n^p=\left[\lim_{n\to\infty}a_n\right]^p \text{ if } p>0 \text{ and } a_n>0$$

Series: Definition

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

P-Series

Definition of P-Series

P-series is defied as

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p = 1$, then we also call it harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

It is divergent.

Fourier Series (Optional)

Fourier Series

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right) \right)$$

Fundamental Properties of Series

Requirement for Convergent Series

Suppose the series $\sum_{n=1}^{\infty} x_n$ is convergent, then the sequence x_n is an infinitesimal, which means

$$\lim_{n\to\infty} x_n = 0$$

This can be used to test if a series is divergent.

Linearity for Convergent Series

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and ξ, η are two constants, then

$$\sum_{n=1}^{\infty} (\xi a_n + \eta b_n) = \xi A + \eta B$$

Exercise

Sequences and Series

Let

$$a_n = \frac{2n}{3n+1}$$

1. Determine whether $\{a_n\}$ is convergent.
2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

Exercise

Solution

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the sequence $\{a_n\}$ is convergent
2. Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

Exercise

Convergence and Divergence

Find the values of x for which the series converges. Find the sum of the series for those values of x

1.
$$\sum_{n=0}^{\infty}(-4)^n(x-5)^n$$

2.
$$\sum_{n=0}^{\infty}\frac{\sin^n x}{3^n}$$

Tips: regard x as a parameter.

Exercise

Solution

1. $\sum_{n=0}^{\infty}(-4)^n(x-5)^n = \sum_{n=0}^{\infty}[-4(x-5)]^n$ is a geometric series with $r = -4(x-5)$, so the series converges $\Leftrightarrow$ $|r| < 1 \Leftrightarrow |-4(x-5)| < 1 \Leftrightarrow |x-5| < \frac{1}{4} \Leftrightarrow -\frac{1}{4} < x-5 < \frac{1}{4} \Leftrightarrow \frac{19}{4} < x < \frac{21}{4}$. In that case, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-[-4(x-5)]} = \frac{1}{4x-19}$.

2. $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n} = \sum_{n=0}^{\infty} \left(\frac{\sin x}{3}\right)^n$ is a geometric series with $r = \frac{\sin x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1 \Leftrightarrow \left|\frac{\sin x}{3}\right| < 1 \Leftrightarrow |\sin x| < 3$, which is true for all x. Thus, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-(\sin x)/3} = \frac{3}{3-\sin x}$

Several Methods

- Divergence Test Theorem (Requirement for Convergent Series)
- Integral Test
- Comparison Test
- Cauchy Test (Root Test)
- d'Alembert Test (Ratio Test)
- Leibniz Test
- Absolute Convergence Test

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:
(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Integral Test: Important Conclusions for p-series

For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION: If $p < 0$, then $\lim_{n\to\infty}(1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty}(1/n^p) = 1$. In either case $\lim_{n\to\infty}(1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7) If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

Integral Test: Important Conclusions for p-series

Conclusion

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Exercise

Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

Ex 6

Solution

$f(x) = \frac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \frac{x\left(2-x^3\right)}{(x^3+1)^2} < 0$ for $x \geq 2$ so we can use the Integral Test [note that f is not decreasing on $[1, \infty)$].

$\int_2^\infty \frac{x^2}{x^3+1} dx = \lim_{t\to\infty} \left[\frac{1}{3}\ln\left(x^3+1\right)\right]_2^t = \frac{1}{3}\lim_{t\to\infty}\left[\ln\left(t^3+1\right) - \ln 9\right] = \infty$, so the series $\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\sum_{n=1}^\infty \frac{n^2}{n^3+1}$

Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.
In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:
- A p-series $[\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1]$
- A geometric series $\left[\sum ar^{n-1}\right.$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1]$

Comparison Test: Expressed by Limits

Theorem

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Usually, this is more convenient to use.

Exercise

Determine whether the series converges or diverges

1.
$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$$

2.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Exercise

Solution

1. $\frac{1}{(n^2+2n+2)^2} < \frac{1}{(n^2)^2} = \frac{1}{n^4}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^4}$, which converges because it is a p-series with $p = 4 > 1$.

2. $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots\cdots (n-1)n}{n \cdot n \cdot n \cdots\cdots n \cdot n} \leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots\cdots 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges $[p = 2 > 1]$, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

d'Alembert Test (Ratio Test)

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Cauchy Test (Root Test)

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Exercise

Determine whether the series converges or diverges

1. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

2. $$\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$$

Exercise

Solution

1.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$
$$= \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1, \text{ so the series } \sum_{n=1}^{\infty}\frac{n!}{n^n}$$
converges absolutely by the Ratio Test.

Exercise

Solution

2.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|}=\lim_{n\to\infty}\frac{2^5n^5}{(n+1)^5}$$

$$=32\lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^5}=32\lim_{n\to\infty}\frac{1}{(1+1/n)^5}$$

$$=32>1,$$

so the series $\sum_{n=2}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

Alternating Series

Definition

If the series satisfies

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{u+1} u_n$$

Then we call it an alternating series.

Leibniz Series

Further, if the series

$$\sum_{n=1}^{\infty} (-1)^{u+1} u_n$$

satisfies

$$\text{(i) } u_{n+1} \leqslant u_n \text{ for all } n$$

$$\text{(ii) } \lim_{n\to\infty} u_n = 0$$

Then the series is convergent. We call it Leibniz series.

Example

Show that the series

$$\sum_{n=1}^{\infty} \sin(\sqrt{n^2+1}\pi)$$

is convergent.
Hint:

$$\sin(x+n\pi) = (-1)^n \sin x$$

We know that

$$\sin(\sqrt{n^2+1}\pi) = (-1)^n \sin(\sqrt{n^2+1}-n)\pi = (-1)^n \sin \frac{\pi}{\sqrt{n^2+1}+n}$$

Obviously, $\{\sin \frac{\pi}{\sqrt{n^2+1}+n}\}$ $(n \geqslant 1)$ is a decreasing sequence, and

$$\lim_{n\to\infty} \sin \frac{\pi}{\sqrt{n^2+1}+n} = 0$$

Example

Example

What if

$$\sum_{n=0}^{\infty} \sin(\sqrt{n^2+1}\pi)$$

Now $\{\sin \dfrac{\pi}{\sqrt{n^2+1}+n}\}$ $(n \geqslant 0)$ is not a decreasing sequence. But we can see the series is still convergent. Why?

An Important Conclusion

Conclusion

The convergence and divergence property of a series has nothing to do with the first N terms, where N is a finite number.

So we can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N}^{\infty} a_n$$

Then if

$$\sum_{n=N}^{\infty} a_n$$

satisfies the conditions of Leibniz Series, we can still conclude that the series is convergent.

Absolute Convergence and Conditional Convergence

Definition

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Then if

$$\sum_{n=1}^{\infty} |x_n|$$

is convergent, $\sum_{n=1}^{\infty} x_n$ is **absolutely convergent**. Else $\sum_{n=1}^{\infty} x_n$ is a **conditionally convergent**.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Absolute Convergence and Conditional Convergence

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Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Method

The convergence and divergence property of $\sum_{n=1}^{\infty} |x_n|$ can be determined by the criterion mentioned before.

Typically, if $\sum_{n=1}^{\infty} |x_n|$ diverges, $\sum_{n=1}^{\infty} x_n$ does not necessarily diverges.
However, if the divergence property is determined by Ratio Test or Root Test, then the series $\sum_{n=1}^{\infty} x_n$ also diverges.
That's because these two criterion are based on the fact that the sequence is not approaches 0 ($x \to \infty$).

Example

Discuss the convergence and divergence property

$$\sum_{n=1}^{\infty} \frac{x^n}{n^p}$$

Solution

Consider

$$\sum_{n=1}^{\infty} \left|\frac{x^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{|x^n|}{n^p}$$

By Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n^p}} = \lim_{n\to\infty} \frac{|x|}{n^{p/n}} = |x|$$

So that when $|x| < 1$, the series is absolutely convergent for every $p \in R$; when $|x| > 1$, the series is divergent for every $p \in R$.

Solution (Continued)

Then let's consider the case when $|x| = 1$
Case 1: $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

We have known that when $p > 1$, the series is absolutely convergent; when $p \leqslant 1$, the series diverges.
Case 2: $x = -1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

This is an alternating series, which is convergent when $p > 0$ (why?).
From the case 1 we know that when $p > 1$, the series is absolutely convergent; when $p \leqslant 1$, the series is conditionally convergent.

A Challenging Exercise

Discuss the convergence and divergence property

$$\sum_{n=1}^{\infty} \frac{x^n}{n^p \ln^q n}$$

Answer

When $|x| < 1$, the series is absolutely convergent.
When $|x| > 1$, the series is divergent.
When $x = 1$:
If $p > 1$ or ($p = 1$ and $q > 1$), the series is absolutely convergent.
Otherwise it is divergent;
When $x = -1$:
If $p > 1$ or ($p = 1$ and $q > 1$), the series is absolutely convergent.
If ($p = 1$ and $q \leqslant 1$) or ($0 < p < 1$) or ($p = 0$ and $q > 0$), the series is conditionally convergent.
Otherwise the series is divergent.

Note: if you find this too complex, you can just understand the example problem first.

Shanks Transformation

For each series $\sum_{n=0}^{\infty} a_n$, we can form the sequence of partial sums

$$A_n = \sum_{k=0}^{n} a_n$$

and

$$S_n = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}.$$

This new sequence, called the Shanks transformation of the series, will usually converge faster than the original series. It is denoted by $S(A_n)$, and works particular well on alternating series.

Example

Iterated Shanks transformation for the series $\sum_{n=1}^{\infty}(.9)^n/n = \ln(10) \approx 2.302585093$. Shanks transformation improves convergence even though this is not an alternating series.

n	A_n	$S(A_n)$	$S^2(A_n)$	$S^3(A_n)$
1	0.9			
2	1.305	1.9125		
3	1.548	2.052692308	2.245159713	
4	1.712025	2.133803571	2.268413754	2.296053112
5	1.830123	2.184417	2.281042636	2.298856749
6	1.9186965	2.217632063	2.288432590	2.300349676
7	1.987024629	2.240240634	2.292993969	2.301192122
8	2.040833030	2.256066635	2.295924710	
9	2.083879751	2.267394719		
10	2.118747595			

Ex 9

Shanks Transformation

First find A_1 through A_7 for the following sequences. Note that some sums begin at $m = 0$, causing A_1 to be the sum of two terms. Then apply the iterated Shanks transformation to find $S^2(A_n)$ for $n = 3$ to $n = 5$. How many digits of precision does $S^2(A_n)$ give in comparison to the given exact limit?

$$A_n = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m}, \quad \lim_{n\to\infty} A_n = \ln(2)$$

Ex 9

Solution

0.69327731, 0.69310576, 0.69316334; 3 places accuracy

Function Series

Now let's expand the concept of series to functions.

Series with Function Terms

Suppose $u_n(x)$ is a function sequence with common domain E, then the sum of these infinite numbers of function terms

$$u_1(x) + u_2(x) + u_3(x) + ... + u_n(x) + ...$$

is called function series, denoted as

$$\sum_{n=1}^{\infty} u_n(x)$$

Convergence Point and Convergence Domain

Different from series of number terms, function series has the concept of convergence point and convergence domain.

Convergence Point

For a fixed $x_0 \in E$, if the series

$$\sum_{n=1}^{\infty} u_n(x_0)$$

is convergent, then we say that the function series

$$\sum_{n=1}^{\infty} u_n(x)$$

is convergent at x_0.

Convergence Point and Convergence Domain

Convergence Domain

The set that includes all the convergence point of the given function series is called the convergence domain.

Power Series

Power Series is a special kind of function series.

Definition

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x_{x0})^2 + \ldots + a_n(x-x_0)^n + \ldots$$

This kind of function series is called power series.

Radius of Convergence

Cauchy-Hadamard Theorem

The power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is absolutely convergent when $|x| < R$, and it is divergent when $|x| > R$ ($R > 0$). R is called the radius of convergence.

Note: at the endpoints $x = \pm R$, the convergence and divergence property of the function series should be judged by other methods.

Radius of Convergence

Cauchy-Hadamard Theorem: for General Cases

The power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is absolutely convergent when $|x - x_0| < R$, and it is divergent when $|x - x_0| > R$ $(R > 0)$.

Radius of Convergence: Cauchy Test

Cauchy Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = A$$

Then the radius of convergence of this power series is $R = \dfrac{1}{A}$.
Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Radius of Convergence: d'Alembert Test

d'Alembert Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = A$$

Then the radius of convergence of this power series is $R = \dfrac{1}{A}$. Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Exercise

Exercise

Find the domain of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\ln(n+1)}{n+1}(x+1)^n$$

Solution

First, we need to find the radius of convergence. By using Ratio Test, we have

$$\lim_{n\to\infty} \left|\frac{n+1}{n+2} \cdot \frac{\ln(n+2)}{\ln(n+1)}\right| = 1$$

So its radius of convergence is $R = 1$.

Then let $x = 1 - 1 = 0$, so the series becomes $\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln(n+1)}{n+1}$, which is Leibniz Series.

Let $x = 1 + 1 = 2$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1} = \sum_{n=2}^{\infty} \frac{\ln n}{n} > \frac{\ln 2}{2} + \sum_{n=3}^{\infty} \frac{1}{n}$$

Which is divergent.

Take the Integrals Term by Term

We can take the integrals of a power series term by term, if the interval lies in its domain of convergence.

That means, if $a, b \in D$ (D is the domain of convergence), then

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx$$

If we take $a = 0$ and $b = x$, then

$$\int_0^x \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Take the Derivatives Term by Term

Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence R. Then we can take the derivatives term by term on $(-R, R)$.

That means

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

Shift the Index of Summation

We can shift the "starting point" of summation. General Case:

$$\sum_{n=m}^{\infty} a_n (x - x_0)^n = \sum_{n=m+k}^{\infty} a_{n-k} (x - x_0)^{n-k}$$

$$\sum_{n=m}^{\infty} a_n (x - x_0)^n = \sum_{n=m-k}^{\infty} a_{n+k} (x - x_0)^{n+k}$$

For example:

$$\sum_{n=1}^{\infty} (-1)^n (n+1) x^n = \sum_{n=0}^{\infty} (-1)^{n+1} (n+2) x^{n+1}$$

Taylor Expansion of Elementary Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ..., x \in R$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ..., x \in R$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!} + ..., x \in R$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ..., x \in [-1, 1]$$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1} \frac{x^n}{n} + ..., x \in (-1, 1]$$

Taylor Expansion of Elementary Functions

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ..., x \in (-1,1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - ... + (-1)^n x^n + ..., x \in (-1,1)$$

Determine Taylor Expansion

Example

1. Determine the Power Series of $f(x) = \dfrac{1}{1-x^2}$ at $x = 0$.
2. Determine the Power Series of $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

Determine Taylor Expansion

Example

1. Determine the Power Series of $f(x) = \dfrac{1}{1-x^2}$ at $x = 0$

Solution:

$$f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}, x \in (-1, 1)$$

Determine Taylor Expansion

Example

2. Determine the Power Series of $f(x) = \dfrac{1}{x^2}$ at $x = 1$

Solution:
When $|x-1| < 1$, we have

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Take the derivatives on both sides:

$$-\frac{1}{x^2} = \sum_{n=1}^{\infty} (-1)^n n (x-1)^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)(x-1)^n$$

So that

$$\frac{1}{x^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

Determine Taylor Expansion

Example

2. Determine the Power Series of $f(x) = \dfrac{1}{x^2}$ at $x = 1$

Why this is not correct?

$$f(x) = \frac{1}{x^2} = \frac{1}{1 + (x^2 - 1)} = \sum_{n=0}^{\infty} (-1)^n (x^2 - 1)^n, x \in (-1, 1)$$

Note: the power series expansion should be always in the form

$$f(x) = \sum_{n=k}^{\infty} a_n (x - x_0)^n$$

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \dfrac{1}{3+5x-2x^2}$ at $x=0$

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \dfrac{1}{3+5x-2x^2}$ at $x=0$

Solution:

$$f(x) = \frac{1}{3+5x-2x^2} = \frac{1}{(3-x)(1+2x)} = \frac{1}{7}(\frac{1}{3-x} + \frac{2}{1+2x})$$

$$= \frac{1}{7}(\frac{1}{3}\cdot\frac{1}{1-x/3} + 2\cdot\frac{1}{1+2x}) = \frac{1}{7}[\frac{1}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n + 2\sum_{n=0}^{\infty}(-2x)^n]$$

$$= \frac{1}{7}\sum_{n=0}^{\infty}[\frac{1}{3^{n+1}} - (-2)^{n+1}]x^n$$

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \ln \dfrac{\sin x}{x}$ at $x = 0$

Determine Taylor Expansion

Solution:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots, x \in R$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \ldots, x \in R$$

Plug in

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots, x \in (-1, 1]$$

We get

$$\ln \frac{\sin x}{x} = (-\frac{x^2}{3!} + \frac{x^4}{5!} - \ldots) - \frac{1}{2}(-\frac{x^2}{3!} + \frac{x^4}{5!} - \ldots)^2 + \ldots = -\frac{x^2}{6} - \frac{x^4}{180} - \ldots$$

Q&A

References

[1] Huang, Yucheng. VV156 RC6 Parametric Equations, Polar Coordinates and Series.pdf. 2021.

[2] Chen, Jixiu et al. Mathematical Analysis (3rd Version). 2019