VV156 RC1

Functions and Limits

Li Junhao

UM-SJTU Joint Institute

September 22, 2022

RC Overview

About Honors Calculus

VV156 (FA2022)

- Limits
- Derivatives and Integrals
- Series
- Polar Coordinates
- Simple Differential Equations

VV255 (SU2023)

- Vectors & Simple Linear Algebra
- Differential of Multivariable Functions
- Multiple Integrals
- Curve Integrals and Surface Integrals

About Honors Calculus

VV256 (FA2023)

- Differential Equations
- More About Linear Algebra
- Fourier Transform and Laplace Transform

Other courses might contribute to Honors Calculus:
VV214, VE203, etc.

Number Systems

Number Systems (Memorize)

$\mathbb{R}$, real numbers.

$\mathbb{Q}$, rational numbers.

$\mathbb{N}$, natural numbers.

$\mathbb{Z}$, integers.

$\mathbb{C}$, complex numbers.

Relationships and Operations on Sets

Relationships of Sets

If every element of A is an element of B, then A is a subeset of B, denoted by $A \subset B$, or $A \subseteq B, A \subseteqq B$.
If there's an element in A that is not in B, then A is a proper subset of B, denoted by $A \subsetneqq B, A \subsetneq B$.

Operations on Sets

(1)The union of A and B is denoted $A \cup B$, i.e., $A \cup B := \{x | x \in A$ or $x \in B\}$.
(2) The intersection of A and B is denoted $A \cap B$, i.e., $A \cap B := \{x | x \in A$ and $x \in B\}$.
(3) The difference of A and B is denoted $A - B$ or $A \setminus B$, i.e., $A \setminus B := \{x | x \in A$ and $x \notin B\}$

Theorem

Relationships of Sets

$A = B$ **if and only if** $A \subset B$ **and** $B \subset A$**.**

This is widely used in showing or proving that two sets are equal. In linear algebra, you can also use similar method to prove that the two spaces are equal.

Basic definitions

Definition

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Domain and range

- Domain: D
- Range: E

Basic properties: Symmetry

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Reminder: if you need to determine the parity of a function, the very first step is to check if its domain is symmetric about the origin.

Example

Show that $f(x) = \dfrac{1}{1+a^x} - \dfrac{1}{2}(a > 0, a \neq 1)$ is an odd function.

Basic properties

Increasing & Decreasing property

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Upper Bound and Lower Bound

If there're two constant m and M satisfying that for any $x \in D$, $m \leq f(x) \leq M$, then $f(x)$ is a limited function (i.e. bounded function), where m is called the lower bound and M is the upper bound.

Exercise 1

Find the domain of these functions

(1) $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

(2) $f(u) = \dfrac{u+1}{1 + \frac{1}{u+1}}$

(3) $F(p) = \sqrt{2 - \sqrt{p}}$

Exercise 1

Solution

(1) $h(x) = 1/\sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \Leftrightarrow x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

(2) $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$ is defined when $u+1 \neq 0[u \neq -1]$ and $1 + \frac{1}{u+1} \neq 0$. Since $1 + \frac{1}{u+1} = 0 \Rightarrow \frac{1}{u+1} = -1 \Rightarrow 1 = -u-1 \Rightarrow u = -2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

(3) $F(p) = \sqrt{2 - \sqrt{p}}$ is defined when $p \geq 0$ and $2 - \sqrt{p} \geq 0$. Since $2 - \sqrt{p} \geq 0 \Rightarrow 2 \geq \sqrt{p} \Rightarrow \sqrt{p} \leq 2 \Rightarrow 0 \leq p \leq 4$, the domain is $[0, 4]$.

Exercise 1

How to solve these questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x, y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- $y = \tan x$, $x \neq k\pi + \pi/2, y = \cot x$, $x \neq k\pi$, k is integer

Exercise 2

Prove or Disprove

- If f and g are both even functions, is $f+g$ even? If f and g are both odd functions, is $f+g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Exercise 2

Solution for 1

(i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an even function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an odd function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is neither even nor odd. (Exception: if f is the zero function, then $f+g$ will be odd. If g is the zero function, then $f+g$ will be even.)

Exercise 2

Solution for 2

(i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an even function.
(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an even function.
(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an odd function.

Linear function

The graph of the function is a line:

$$y = f(x) = mx + b$$

m is the slope of the line and b is the y-intercept.

Basic function types

Polynomials

A function P is called a **polynomial** if

$$P(x) = \Sigma_{i=0}^{n} a_i x^i$$

a_i are **coefficients** and n is the **degree** of the polynomial if $a_n \neq 0$.

Quadratic function

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Basic function types

Power function

A function of the form $f(x) = x^a$ is called a **power function**, where a is a constant. Consider an arbitary positive integer n:

- $a = n$:

$$y = x: \text{ line} \qquad y = x^2: \text{ parabola}$$

- $a = \frac{1}{n}$: **root function**
- $a = -1$: **reciprocal function**

Ratio function

A **Ratio function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The domain consists of all values of x such that $Q(x) \neq 0$.

Basic function types

Trigonometric function

$$\sin(x+2\pi) = \sin x \qquad \cos(x+2\pi) = \cos x$$
$$\tan(x+\pi) = \tan x$$

Exponential function

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

Logarithmic function

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions.

Special Functions

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Function Transformations

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Combinations of Functions

Definition

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Question: how would you understand the two denotations?

Exercise 4

Composite Function

(a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

Exercise 4

Solutions

(a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$
(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3\left(x^2 + x\right) + 2 = 3\left(x^2 + x - 1\right) + 5$, so we see that $g(x) = x^2 + x - 1$

Inverse Functions

One-to-one Functions

Suppose x_1 and x_2 are any numbers in the domain of f. If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$, then f is a one-to-one function.

Definition

Let f be a **one-to-one** function with domain A and range B. Then its inverse function $f^{-1}(x)$ has domain B and range A and is defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

Exercise 5

Question

If $y = f(x) = \sqrt[3]{x + \sqrt{x^2+1}} + \sqrt[3]{x - \sqrt{x^2+1}}$, what is $f^{-1}(x)$?

Exercise 5

Question

Let $a = \sqrt[3]{x + \sqrt{x^2+1}}$, $b = \sqrt[3]{x - \sqrt{x^2+1}}$, then we have $y = f(x) = a + b$. $a^3 = x + \sqrt{x^2+1}, b^3 = x - \sqrt{x^2+1}$, $a^3 + b^3 = 2x = (a+b)(a^2 - ab + b^2)$. We have already know that $a + b = y$, $ab = \sqrt[3]{x^2 - (x^2+1)} = -1$, thus

$$2x = (a+b)[(a+b)^2 - 3ab] = y(y^2+3) \Longrightarrow x = \frac{y(y^2+3)}{2}.$$

Rough Definition of Limits

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit if $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

One-sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

When calculating $\lim_{x \to a^-} f(x)$, we consider only $x < a$.

Similarly, we can get the right-hand limit of $f(x)$ as x approaches a.

Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Warning:

$\lim_{x \to a} f(x) = (-)\infty$ does not mean that we are regarding $(-)\infty$ as a number. Nor does it mean that the limit exists!

Limits at Infinity

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x \to \infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.
2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x \to -\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

The Precise Definition: $\varepsilon - \delta$ Language

Neighborhood

Suppose a and δ are two real numbers and $\delta > 0$, then the set

$$\{x | a - \delta < x < a + \delta\}$$

is called an neighborhood of a with radius δ. We can have the denotation like

$$O(a, \delta)$$

Deleted Neighborhood

If a neighborhood is not include $x = a$, then we call it a deleted neighborhood

$$O(a, \delta) \backslash \{a\}$$

The Precise Definition: $\varepsilon-\delta$ Language

Definition

Let f be a function defined on some deleted neighborhoods of x_0 i.e. there exist $\rho > 0$ such that

$$O(x_0,\rho)\backslash\{x_0\} \subset D$$

Then we say that if for every number $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - x_0| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

Then we say that the limit of $f(x)$ as x approaches x_0 is L, and we write

$$\lim_{x\to x_0} f(x) = L$$

You can understand it in this way: no matter how small the small the area is, we can always find a interval, in which the graph of $f(x)$ is located in this area determined by thetwo lines $y = L - \varepsilon$ and $y = L + \varepsilon$.

Define One-sided Limits with $\varepsilon - \delta$ Language

Sometimes the limits of a function will exist only in one side.

Definition

Let f be a function defined on $(x_0 - \rho, x_0)(\rho > 0)$. If for every number $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } -\delta < x - x_0 < 0, \text{ then } |f(x) - L| < \varepsilon$$

Then we say that the **left limit** of $f(x)$ as x approaches x_0 is L, and we write

$$\lim_{x \to x_0 -} f(x) = L$$

The definition of right limit is similar.

Define Infinite Limits with $\varepsilon - \delta$ Language

How to define infinite limits?

Definition

Let f be a function defined on some deleted neighborhoods of x_0. If for every number $M > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - x_0| < \delta, \text{ then } |f(x)| > M$$

Then we say that $f(x)$ is infinite when x approaches x_0.

$$\lim_{x \to x_0} f(x) = \infty$$

Note that the symbol ∞ includes both $+\infty$ and $-\infty$.

Four fundamental Calculations of Limits

Theorem

Suppose that $\lim_{x \to x_0} f(x) = A$, $\lim_{x \to x_0} g(x) = B$, then

$$\lim_{x \to x_0} [\alpha f(x) + \beta g(x)] = \alpha A + \beta B$$

$$\lim_{x \to x_0} [f(x) g(x)] = AB$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Other Limit Algorithms

Theorem 1: The sum or product of finite infinitesimals is an infinitesimal.
Theorem 2: If $\lim f(x) = 0$, $g(x)$ is a **bounded function** (i.e. $|g(x)| < M$), then

$$\lim f(x)g(x) = 0$$

Theorem 3: If $\lim f(x)$ exists, and n is a postive integer, then

$$\lim[f(x)]^n = [\lim f(x)]^n$$

Two Important Limits

Formula

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Formula

$$\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e$$

A useful trick:
If $f(x) > 0$ and $\lim_{x \to x_0} f(x) = 0$, then

$$\lim_{x \to x_0} [1 + f(x)]^{g(x)} = \lim_{x \to x_0} [1 + f(x)]^{\frac{1}{f(x)} f(x) g(x)} = e^{f(x)g(x)}$$

Equivalent Infinitesimal

Definition

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1$$

Then we called these two infinitesimals $f(x)$ and $g(x)$is equivalent. It can be denoted by

$$f(x) \sim g(x)$$

Example

Since $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$, we can denote that $\sin x \sim x$ (when $x \to 0$).

Equivalent Infinitesimal

Some commonly used equivalent infinitesimal substitutions:

$$\sin x \sim x, \tan x \sim x, \arcsin x \sim x, \arctan x \sim x 1 - \cos x \sim \frac{1}{2}x^2$$

$$e^x \sim x + 1, \ln(1+x) \sim x, (1+kx)^n - 1 \sim 1 + nkx$$

Application

If $f(x)$ is **a multiple or division factor**, then you can sustitute it with its equivalent infinitesimal $g(x)$.

Exercise

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

Equivalent Infinitesimal

Exercise

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

Solution

$\tan x - \sin x = \tan x(1 - \cos x)$. $\tan x \sim x, 1 - \cos x \sim \dfrac{x^2}{2}$. So

$$\tan x - \sin x \sim \frac{1}{2}x^3$$

Equivalent Infinitesimal

Example

$$\lim_{x \to 0} \frac{\ln(1+x)+(1-\cos x)-x}{x^2}$$

$\ln(1+x)$ and $1-\cos x$ are not multiple factors or division factor. We cannot directly use equivalent infinitesimal substitution.

Explanation (not required)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

$$1-\cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\ldots$$

Excercise

$$\lim_{x\to\infty} \frac{x^2+5}{2x+1}\sin\frac{3}{x}$$

$$\lim_{x\to\alpha} \frac{\ln x-\ln a}{x-a}(a>0)$$

Equivalent Infinitesimal

Excercise

$$\lim_{x\to\infty}\frac{x^2+5}{2x+1}\sin\frac{3}{x}$$

$$\lim_{x\to\alpha}\frac{\ln x-\ln a}{x-a}(a>0)$$

Solution

(1) Equivalent infinitesimal: $\sin\frac{3}{x}\sim\frac{3}{x}$.

(2) Hint: $\ln x-\ln a=\ln\frac{x}{a}=\ln(1+\frac{x-a}{a})\sim\frac{x-a}{a}$.

Example

$$\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$$

Solution

Let $y = \lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$, then $\ln y = \lim_{x\to\infty}\dfrac{1}{x}\ln\ln x = \lim_{x\to\infty}\dfrac{1}{x}\ln\ln x = 0$. So $y = e^0 = 1$.

This may not be a very good example. You will learn how to calculate the limit $\lim_{x\to\infty}\dfrac{1}{x}\ln\ln x$ later in this cource, using L'Hospital's rule.

Logarithmic Transformation

Excercise

(1) Calculate $\lim_{x\to 0^+}[\frac{(1+x)^{\frac{1}{x}}}{e}]^a$, where a is a constant number.

(2) Given that $\ln(1+x) \sim x - \frac{x^2}{2}$, calculate the limit

$$\lim_{x\to 0^+}[\frac{(1+x)^{\frac{1}{x}}}{e}]^{\frac{1}{x}}$$

Solution

(1) $\lim_{x\to 0^+} \dfrac{(1+x)^{\frac{1}{x}}}{e} = 1$, so the result is also 1^a=1.

(2) let $y = \lim_{x\to 0^+} [\dfrac{(1+x)^{\frac{1}{x}}}{e}]^{\frac{1}{x}}$, then $\ln y = \lim_{x\to 0^+} [\dfrac{1}{x} \ln \dfrac{(1+x)^{\frac{1}{x}}}{e}] =$
$\lim_{x\to 0^+} \dfrac{1}{x}[\dfrac{1}{x}\ln(1+x) - 1] = \lim_{x\to 0^+} \dfrac{\ln(1+x) - x}{x^2} = -\dfrac{1}{2}$. So $y = \dfrac{1}{\sqrt{e}}$.

A warm reminder: do not forget that you have taken the logarithm of the final result.

Squeeze Theorem

Theorem

If there exist a positive number ρ such that when $x \in O(x_0, \rho)$, we have

$$g(x) \leq f(x) \leq h(x)$$

And $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = A$, then $\lim_{x\to x_0} h(x) = A$.

The proof is not required and you can easily understand it by drawing graphs.

Squeeze Theorem

Example

Suppose n is a positive integer. Show that $\lim_{n\to\infty}(\sum_{i=1}^{n} \dfrac{n}{n^2+i\pi}) = 1$.

Squeeze Theorem

Example

Suppose n is a positive integer. Show that $\lim_{n\to\infty}(\sum_{i=1}^{n} \dfrac{n}{n^2+i\pi}) = 1$.

Solution

Squeeze theorem. $\dfrac{n^2}{n^2+n\pi} \leq \sum_{i=1}^{n} \dfrac{n}{n^2+i\pi} \leq \dfrac{n^2}{n^2+\pi}$

Q&A

References

[1] Huang, Yucheng. VV156_RC1_updated.pdf. 2021.

[2] Cai, Runze. Chapter01.pdf. 2021.