VV156 Final RC Part 2

Series

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RC Overview

Several Methods

- Divergence Test Theorem (Requirement for Convergent Series)
- Integral Test
- Comparison Test
- Cauchy Test (Root Test)
- d'Alembert Test (Ratio Test)
- Leibniz Test
- Absolute Convergence Test

Divergence Test Theorem

Requirement for Convergent Series

Suppose the series $\sum_{n=1}^{\infty} x_n$ is convergent, then the sequence x_n is an infinitesimal, which means

$$\lim_{n\to\infty} x_n = 0$$

This can be used to test if a series is divergent.

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:
(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Integral Test: Important Conclusions for p-series

Conclusion

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Exercise

Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

Exercise

Solution

$f(x) = \frac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \frac{x\left(2-x^3\right)}{(x^3+1)^2} < 0$ for $x \geq 2$ so we can use the Integral Test [note that f is not decreasing on $[1, \infty)$].

$\int_2^\infty \frac{x^2}{x^3+1} dx = \lim_{t\to\infty} \left[\frac{1}{3} \ln\left(x^3+1\right)\right]_2^t = \frac{1}{3} \lim_{t\to\infty} \left[\ln\left(t^3+1\right) - \ln 9\right] = \infty$, so the series $\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\sum_{n=1}^\infty \frac{n^2}{n^3+1}$

Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.
In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:
- A p-series $[\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1]$
- A geometric series $\left[\sum ar^{n-1}\right.$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1]$

Comparison Test: Expressed by Limits

Theorem

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Usually, this is more convenient to use.

Exercise

Determine whether the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$$

Exercise

Solution

$\frac{1}{(n^2+2n+2)^2} < \frac{1}{(n^2)^2} = \frac{1}{n^4}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^4}$, which converges because it is a p-series with $p = 4 > 1$.

Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Determine whether the series converges or diverges

1.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

2.
$$\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$$

Exercise

Solution

1.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1, \text{ so the series } \sum_{n=1}^{\infty}\frac{n!}{n^n}$$

converges absolutely by the Ratio Test.

Exercise

Solution

2.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \frac{2^5 n^5}{(n+1)^5}$$

$$= 32 \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^5} = 32 \lim_{n\to\infty} \frac{1}{(1+1/n)^5}$$

$$= 32 > 1,$$

so the series $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

Alternating Series

Definition

If the series satisfies

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{u+1} u_n$$

Then we call it an alternating series.

Leibniz Series

Further, if the series

$$\sum_{n=1}^{\infty}(-1)^{u+1}u_n$$

satisfies

$$\text{(i) } u_{n+1} \leqslant u_n \text{ for all } n$$

$$\text{(ii) } \lim_{n\to\infty} u_n = 0$$

Then the series is convergent.
Actually, the condition (i) can be weakened to

$$u_{n+1} \leqslant u_n \text{ for } n \leqslant N$$

Example

Show that the series

$$\sum_{n=1}^{\infty} \sin(\sqrt{n^2+1}\pi)$$

is convergent.
Hint:

$$\sin(x+n\pi) = (-1)^n \sin x$$

We know that

$$\sin(\sqrt{n^2+1}\pi) = (-1)^n \sin(\sqrt{n^2+1}-n)\pi = (-1)^n \sin\frac{\pi}{\sqrt{n^2+1}+n}$$

Obviously, $\{\sin\dfrac{\pi}{\sqrt{n^2+1}+n}\}$ $(n \geqslant 1)$ is a decreasing sequence, and

$$\lim_{n\to\infty} \sin\frac{\pi}{\sqrt{n^2+1}+n} = 0$$

Example

Example

What if

$$\sum_{n=0}^{\infty} \sin(\sqrt{n^2+1}\pi)$$

Now $\{\sin \frac{\pi}{\sqrt{n^2+1}+n}\}$ $(n \geqslant 0)$ is not a decreasing sequence. But we can see the series is still convergent. Why?

An Important Conclusion

Conclusion

The convergence and divergence property of a series has nothing to do with the first N terms, where N is a finite number.

So we can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N}^{\infty} a_n$$

Then if

$$\sum_{n=N}^{\infty} a_n$$

satisfies the conditions of Leibniz Series, we can still conclude that the series is convergent.

Absolute Convergence and Conditional Convergence

Definition

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Then if

$$\sum_{n=1}^{\infty} |x_n|$$

is convergent, $\sum_{n=1}^{\infty} x_n$ is **absolutely convergent**. Else $\sum_{n=1}^{\infty} x_n$ is a **conditionally convergent**.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Absolute Convergence and Conditional Convergence

Method

The convergence and divergence property of $\sum_{n=1}^{\infty} |x_n|$ can be determined by the criterion mentioned before.

Typically, if $\sum_{n=1}^{\infty} |x_n|$ diverges, $\sum_{n=1}^{\infty} x_n$ does not necessarily diverges.
However, if the divergence property is determined by Ratio Test or Root Test, then the series $\sum_{n=1}^{\infty} x_n$ also diverges.
That's because these two criterion are based on the fact that the sequence is not approaches 0 $(x \to \infty)$.

Example

Discuss the convergence and divergence property

$$\sum_{n=1}^{\infty} \frac{x^n}{n^p}$$

Solution

Consider

$$\sum_{n=1}^{\infty} \left|\frac{x^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{|x^n|}{n^p}$$

By Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n^p}} = \lim_{n\to\infty} \frac{|x|}{n^{p/n}} = |x|$$

So that when $|x| < 1$, the series is absolutely convergent for every $p \in R$; when $|x| > 1$, the series is divergent for every $p \in R$.

Solution (Continued)

Then let's consider the case when $|x| = 1$
Case 1: $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

We have known that when $p > 1$, the series is absolutely convergent; when $p \leqslant 1$, the series diverges.
Case 2: $x = -1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

This is an alternating series, which is convergent when $p > 0$ (why?).
From the case 1 we know that when $p > 1$, the series is absolutely convergent; when $p \leqslant 1$, the series is conditionally convergent.

Shanks Transformation

For each series $\sum_{n=0}^{\infty} a_n$, we can form the sequence of partial sums

$$A_n = \sum_{k=0}^{n} a_n$$

and

$$S_n = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}.$$

This new sequence, called the Shanks transformation of the series, will usually converge faster than the original series. It is denoted by $S(A_n)$, and works particular well on alternating series.

Power Series

Power Series is a special kind of function series.

Definition

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

This kind of function series is called power series.

Radius of Convergence

Cauchy-Hadamard Theorem: for General Cases

The power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is absolutely convergent when $|x-x_0|<R$, and it is divergent when $|x-x_0|>R$ $(R>0)$.

Radius of Convergence: Root Test

Root Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = A$$

Then the radius of convergence of this power series is $R = \dfrac{1}{A}$.
Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Radius of Convergence: Ratio Test

Ratio Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = A$$

Then the radius of convergence of this power series is $R = \frac{1}{A}$. Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Exercise

Exercise

Find the domain of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\ln(n+1)}{n+1}(x+1)^n$$

Solution

First, we need to find the radius of convergence. By using Ratio Test, we have

$$\lim_{n\to\infty}\left|\frac{n+1}{n+2}\cdot\frac{\ln(n+2)}{\ln(n+1)}\right|=1$$

So its radius of convergence is $R=1$.

Then let $x=-1+1=0$, so the series becomes $\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n+1)}{n+1}$, which is Leibniz Series.

Let $x=-1-1=2$, so the series becomes

$$\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}=\sum_{n=2}^{\infty}\frac{\ln n}{n}>\frac{\ln 2}{2}+\sum_{n=3}^{\infty}\frac{1}{n}$$

Which is divergent.

Take the Integrals Term by Term

We can take the integrals of a power series term by term, if the interval lies in its domain of convergence.

That means, if $a, b \in D$ (D is the domain of convergence), then

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx$$

If we take $a = 0$ and $b = x$, then

$$\int_0^x \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Take the Derivatives Term by Term

Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence R. Then we can take the derivatives term by term on $(-R, R)$.

That means

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

Shift the Index of Summation

We can shift the "starting point" of summation. General Case:

$$\sum_{n=m}^{\infty} a_n(x-x_0)^n = \sum_{n=m+k}^{\infty} a_{n-k}(x-x_0)^{n-k}$$

$$\sum_{n=m}^{\infty} a_n(x-x_0)^n = \sum_{n=m-k}^{\infty} a_{n+k}(x-x_0)^{n+k}$$

For example:

$$\sum_{n=1}^{\infty}(-1)^n(n+1)x^n = \sum_{n=0}^{\infty}(-1)^{n+1}(n+2)x^{n+1}$$

Exercise

Exercise

The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

Taylor Expansion of Elementary Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ..., x \in R$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ..., x \in R$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!} + ..., x \in R$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ..., x \in [-1, 1]$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1} \frac{x^n}{n} + ..., x \in (-1, 1]$$

Taylor Expansion of Elementary Functions

When α is not zero or positive integer, we have

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} x^n$$

The domain of convergence is $x \in (-1,1)$ when $\alpha \leqslant -1$, $x \in (-1,1]$ when $-1 < \alpha < 0$ and $x \in [-1,1]$ when $\alpha > 0$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ..., x \in (-1,1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - ... + (-1)^n x^n + ..., x \in (-1,1)$$

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \dfrac{1}{3+5x-2x^2}$ at $x = 0$

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \dfrac{1}{3+5x-2x^2}$ at $x = 0$

Solution:

$$f(x) = \frac{1}{3+5x-2x^2} = \frac{1}{(3-x)(1+2x)} = \frac{1}{7}(\frac{1}{3-x} + \frac{2}{1+2x})$$

$$= \frac{1}{7}(\frac{1}{3}\cdot\frac{1}{1-x/3} + 2\cdot\frac{1}{1+2x}) = \frac{1}{7}[\frac{1}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n + 2\sum_{n=0}^{\infty}(-2x)^n]$$

$$= \frac{1}{7}\sum_{n=0}^{\infty}[\frac{1}{3^{n+1}} - (-2)^{n+1}]x^n, x \in (-\frac{1}{2}, \frac{1}{2})$$

Don't forget to write the domain of convergence!

Determine Taylor Expansion

Example

Determine the Power Series Expansion of $f(x) = \ln \dfrac{\sin x}{x}$ at $x = 0$

Determine Taylor Expansion

Solution:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ..., x \in R$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ... + (-1)^n \frac{x^{2n}}{(2n+1)!} + ..., x \in R$$

Plug in

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1} \frac{x^n}{n} + ..., x \in (-1, 1]$$

We get

$$\ln \frac{\sin x}{x} = (-\frac{x^2}{3!} + \frac{x^4}{5!} - ...) - \frac{1}{2}(-\frac{x^2}{3!} + \frac{x^4}{5!} - ...)^2 + ... = -\frac{x^2}{6} - \frac{x^4}{180} - ...$$

Determine the Sum of Series

Example

Show that

$$\sum_{n=1}^{\infty} \frac{2n+1}{3^n} = 2$$

You can use

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, x \in (-1,1)$$

Determine the Sum of Series

Solution:

Take the derivative on both sides:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, x \in (-1,1)$$

So that

$$\sum_{n=1}^{\infty} nx^{n} = \frac{x}{(1-x)^2}, x \in (-1,1)$$

Let $x = \frac{1}{3}$, we have

$$\sum_{n=1}^{\infty} (\frac{1}{3})^n = \frac{1}{2}, \sum_{n=1}^{\infty} n(\frac{1}{3})^n = \frac{3}{4}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{2n+1}{3^n} = 2\sum_{n=1}^{\infty} n(\frac{1}{3})^n + \sum_{n=1}^{\infty} (\frac{1}{3})^n = 2$$

Best Wishes

Thanks for your patience and cooperation! I wish you enjoy your winter break whatever the outcome.

Alfonso Reina Cecco 16:23:20

周四不要做任何疯狂的事情。如果你喜欢喝酒就不要去喝酒， 那天不要和你的女朋友或男朋友分手。

1. Focus 1: Determine the Convergence and Divergence Property of Series
2. Focus 2: Shanks Transformation
3. Focus 3: Determine the Convergence Domain of Power Series
4. Focus 4: Power Series Expansion of Elementary Functions
5. Q&A
6. Reference

Q&A

References

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