VV156 Regular RC3/RC4

Integrals

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RC Overview

Parametric Equations

Definition

Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \quad y = g(t)$$

Parametric Equations

$$x = \cos t \quad y = \sin t \quad 0 \leqslant t \leqslant 2\pi$$

If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

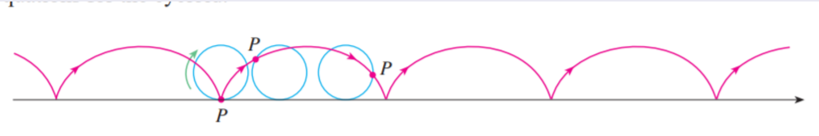
Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians). As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$

A Typical Example: Cycloid

Definition

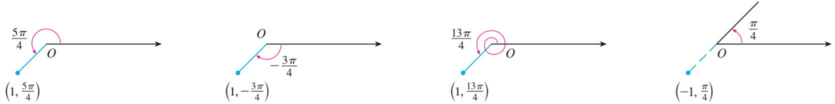
The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a cycloid .
Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta) \quad \theta \in \mathbb{R}$$



Polar Coordinates

We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.
If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1 . Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.



In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n+1)\pi)$$

Polar Coordinates and Cartesian Coordinates

Relationship between Polar Coordinates and Cartesian Coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

Cartesian coordinates, Cylindrical coordinates and Spherical coordinates (Optional, you will see them in VV255)

From cylindrical to Cartesian coordinates:

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
z &= z
\end{aligned}
$$

The inverse relations (from Cartesian to cylindrical coordinates) are

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\phi &= \tan^{-1}\frac{y}{x} \\
z &= z
\end{aligned}
$$

Cartesian coordinates, Cylindrical coordinates and Spherical coordinates (Optional, you will see them in VV255)

From spherical to Cartesian coordinates:

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

The inverse relations (from Cartesian to spherical coordinates) are

$$R = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

CardioidUTF8gbsn

Definition

- parametric representation:

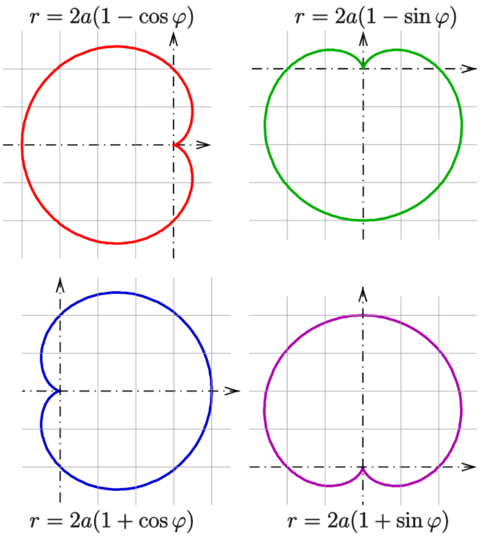
$$x(\varphi) = 2a(1 - \cos\varphi) \cdot \cos\varphi$$
$$y(\varphi) = 2a(1 - \cos\varphi) \cdot \sin\varphi, \quad 0 \leq \varphi < 2\pi$$

- polar coordinates:

$$r(\varphi) = 2a(1 - \cos\varphi)$$

Cardioid is special Cycloid and special Limaçon.

Cardioid

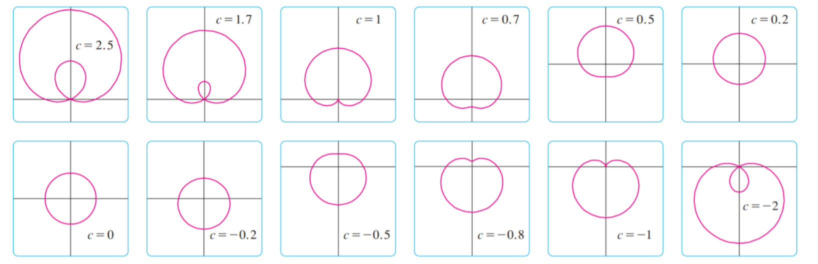


$r = 2a(1 - \cos\varphi)$
$r = 2a(1 - \sin\varphi)$
$r = 2a(1 + \cos\varphi)$
$r = 2a(1 + \sin\varphi)$

Limaçon

Definition

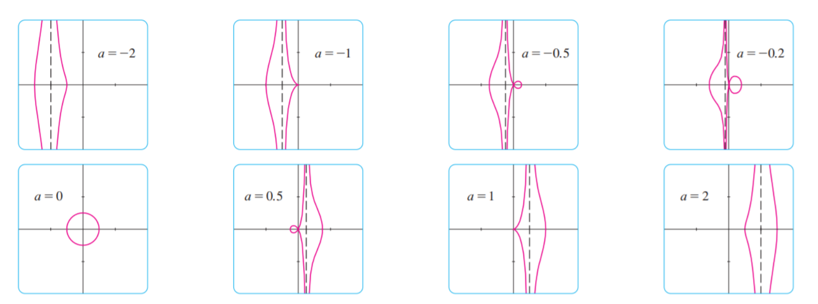
$$r = 1 + c\sin\theta$$



Conchoid

Definition

$$r = 1 + a \sec \theta$$



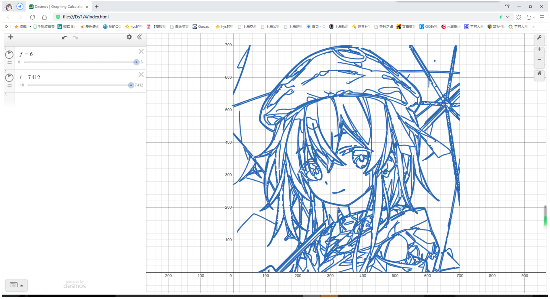
Bézier curves (Optional)

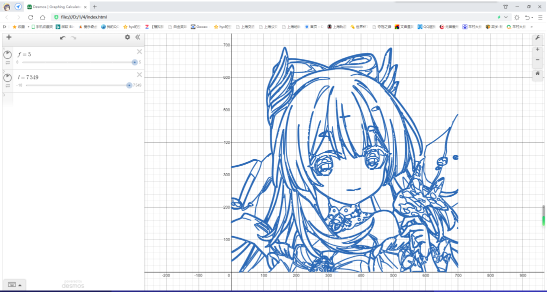
Definition

Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910-1999), who worked in the automotive industry. A cubic Bézier curve is determined by four control points, $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$y = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

Application of Bézier Curves in Desmos





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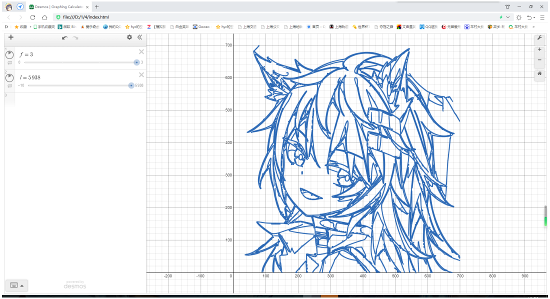

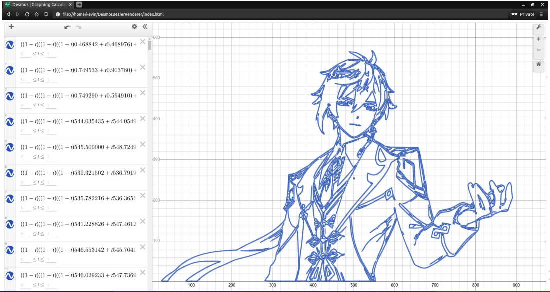

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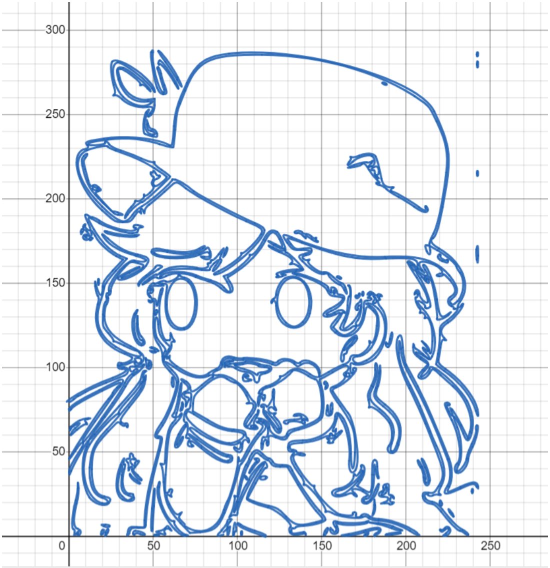
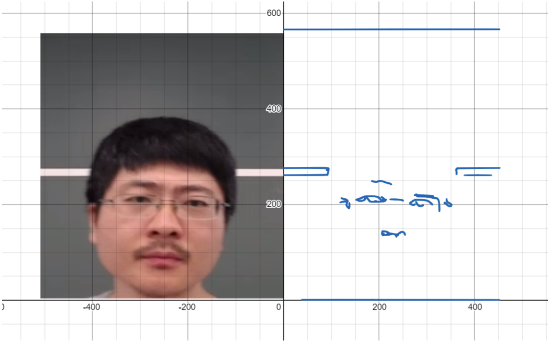


Application of Bézier Curves in desmos



Tangents

Slope of the Tangent Line with Parametric Curves

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

For the second order derivative, we have:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Area and Arc Length

Area

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt \quad \left[\text{or } \int_\beta^\alpha g(t) f'(t) dt \right]$$

Arc Length

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t), \alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area

In the same way as for arc length, we can adapt to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \leqslant t \leqslant \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geqslant 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Ex 1

Tangents

Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$

Ex 1

Solution

$x=3t^2+1, y=2t^3+1, \frac{dx}{dt}=6t, \frac{dy}{dt}=6t^2$, so $\frac{dy}{dx}=\frac{6t^2}{6t}=t$

So at the point corresponding to parameter value t, an equation of the tangent line is $y-\left(2t^3+1\right)=t\left[x-\left(3t^2+1\right)\right]$.

If this line is to pass through $(4,3)$, we must have

$3-\left(2t^3+1\right)=t\left[4-\left(3t^2+1\right)\right] \Leftrightarrow 2t^3-2=3t^3-3t \Leftrightarrow$
$t^3-3t+2=0 \Leftrightarrow (t-1)^2(t+2)=0 \Leftrightarrow t=1$ or -2.

Hence, the desired equations are $y-3=x-4$, or $y=x-1$, tangent to the curve at $(4,3)$, and $y-(-15)=-2(x-13)$, or $y=-2x+11$, tangent to the curve at $(13,-15)$.

Ex 2

Area and Arc Length

1. Find the exact length of the curve

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$$

2. Find the area enclosed by the x-axis and the curve

$$x = 1 + e^t, y = t - t^2$$

3. Find the exact area of the surface obtained by rotating the given curve about the x-axis.

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$$

Ex 2

Solution

1.

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \le t \le 1 . dx/dt = 6t \text{ and}$$

$$dy/dt = 6t^2, \text{ so } (dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$$

$$\text{Thus, } L = \int_0^1 \sqrt{36t^2 + 36t^4}\, dt = \int_0^1 6t\sqrt{1+t^2}\, dt$$

$$= 6\int_1^2 \sqrt{u}\left(\frac{1}{2} du\right) \quad \left[u = 1 + t^2, du = 2t dt\right]$$

$$= 3\left[\frac{2}{3}u^{3/2}\right]_1^2 = 2\left(2^{3/2} - 1\right) = 2(2\sqrt{2} - 1)$$

Ex 2

Solution

2. The curve $x = 1 + e^t, y = t - t^2 = t(1-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$. The shaded area is given by

$$\begin{aligned}
\int_{x=2}^{x=1+e} (y_T - y_B)\, dx &= \int_{t=0}^{t=1} [y(t) - 0]x'(t) dt = \int_0^1 \left(t - t^2\right) e^t dt \\
&= \int_0^1 te^t dt - \int_0^1 t^2 e^t dt \\
&= \int_0^1 te^t dt - \left[t^2 e^t\right]_0^1 + 2\int_0^1 te^t dt \\
&= 3\int_0^1 te^t dt - (e - 0) = 3\left[(t-1)e^t\right]_0^1 - e \\
&= 3[0 - (-1)] - e = 3 - e
\end{aligned}$$

Ex 2

Solution

3.

$$x = 3t - t^3, y = 3t^2, 0 \leq t \leq 1.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3 - 3t^2\right)^2 + (6t)^2 = 9\left(1 + 2t^2 + t^4\right)$$

$$= \left[3\left(1 + t^2\right)\right]^2$$

$$S = \int_0^1 2\pi \cdot 3t^2 \cdot 3\left(1 + t^2\right) dt = 18\pi \int_0^1 \left(t^2 + t^4\right) dt$$

$$= 18\pi \left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1$$

$$= \frac{48}{5}\pi$$

Tangents

Slope of the Tangent Line with Polar Coordinates

Since that

$$x = r\cos\theta, y = r\sin\theta$$

where r can be regarded as a function of θ. Hence we have:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Area and Arc Length

Suppose we have the function in polar coordinates:

$$r = f(\theta), a \leqslant \theta \leqslant b$$

For the area enclosed by this function, we have:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

And to calculate the arc length, we have:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Ex 3

Area and Tangents in Polar Coordinates

1. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$r = 2 - \sin\theta, \quad \theta = \pi/3$$

2. Find the area of the region enclosed by one loop of the curve

$$r = 2\sin 5\theta$$

Solution

1.

$$x = r\cos\theta = (2-\sin\theta)\cos\theta, y = r\sin\theta = (2-\sin\theta)\sin\theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2-\sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)}$$

$$= \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

When $\theta = \frac{\pi}{3}, \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$

Ex 3

Solution

2.

$$r = 0 \Rightarrow 2\sin 5\theta = 0 \Rightarrow \sin 5\theta = 0 \Rightarrow 5\theta = \pi n \Rightarrow \theta = \frac{\pi}{5}n.$$

$$A = \int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta \, d\theta$$

$$= 2\int_0^{\pi/5} \frac{1}{2}(1 - \cos 10\theta) d\theta = \left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{5}$$

An Important Reminder

Please do remember these formulas. The application of integral is not hard, but it can have very high discrimination. If you fail to write the correct formula in exam, possibly you will get no more than 1/10 point.

However, if you are not so confident with your memorization skills, I strongly recommend you to be familiar with the derivation process of these formulas. You don't have to memorize all the derivation steps; what I want you to do is to guarantee that you have the ability to derive it by yourself even though it may cause more time in the exam.

Euler Integrals

This part is not required for its on sake in this course. But some of the models may be useful in solving problems of integrals.

If you are interested in this part, you can look through the following slides by yourself.

Beta Function

Definition

Beta function is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Property 1:

$$B(p,q) = B(q,p)$$

Property 2 (when $p > 0$ and $q > 1$):

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1)$$

A conclusion from property 1 and 2 (when $p > 1$ and $q > 1$):

$$B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}B(p-1,q-1)$$

(1) Let $x = \cos^2 \varphi$:

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2p-1}\varphi \cos^{2q-1}\varphi \, d\varphi$$

We usually change the trigonometric integrals to Γ function.

(2)Let $x = \dfrac{1}{1+t}$ and $t = \dfrac{1}{u}$:

$$B(p,q) = \int_0^{\infty} \frac{t^{q-1}}{(1+t)^{p+q}} dt = \int_0^{1} \frac{t^{q-1}}{(1+t)^{p+q}} dt + \int_1^{\infty} \frac{t^{q-1}}{(1+t)^{p+q}} dt$$

$$= \int_0^{1} \frac{t^{q-1}}{(1+t)^{p+q}} dt + \int_0^{1} \frac{u^{p-1}}{(1+u)^{p+q}} du = \int_0^{1} \frac{t^{p-1}+t^{q-1}}{(1+t)^{p+q}} dt$$

Beta Function: Summary

When you need to calculate these three types of integrals:

Beta Functions

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x dx$$

$$B(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx$$

You can regard them as Beta Functions and use the recursion formula

$$B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}B(p-1,q-1)$$

Beta Function: Example

Example

Calculate

$$\int_0^1 \frac{x+x^3}{(1+x)^5}dx$$

Solution:

Let

$$B(2,4) = \int_0^1 \frac{x+x^3}{(1+x)^5}dx$$

then

$$B(2,4) = \frac{1}{2}B(2,3) = \frac{1}{2}\int_0^1 \frac{x+x^2}{(1+x)^4}dx = \frac{1}{2}\int_0^1 \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^1 \frac{(1+x)-1}{(1+x)^3}dx$$

$$= \frac{1}{2}[\int_0^1 \frac{1}{(1+x)^2}dx - \int_0^1 \frac{1}{(1+x)^3}dx]$$

Gamma Function

Definition

Gamma function is defined as

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx$$

Property 1:

$$\Gamma(s+1) = s\Gamma(s), s > 0$$

Apply "integration by parts" method:

$$\int_0^{+\infty} x^s e^{-x} dx = -x^s e^{-x}|_0^{+\infty} + s\int_0^{+\infty} x^{s-1} e^{-x} dx = s\int_0^{+\infty} x^{s-1} e^{-x} dx = s\Gamma(s)$$

Property 2 (when $s = n$ is a positive integer):

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!\Gamma(1) = n!$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} = 1$$

Property 2:
When $s = n$ is a positive integer, we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!\Gamma(1) = n!$$

And

$$\Gamma(1) = \int_0^{+\infty} e^{-x} = 1$$

So when n is a positive integer:

Formula

$$\Gamma(n+1) = n!$$

Gamma Function: Other Expressions

Let $x = t^2$, then

$$\Gamma(s) = 2\int_0^{+\infty} t^{2s-1} e^{-t^2} dt$$

Let $x = \alpha t$, then

$$\Gamma(s) = \alpha^s \int_0^{+\infty} t^{s-1} e^{-\alpha t} dt$$

These two forms both satisfy the recursion formula

$$\Gamma(s+1) = s\Gamma(s), \Gamma(n+1) = n!$$

Gamma Function: Example

Example

Given that

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Calculate

$$\int_0^{+\infty} x^{2n} e^{-x^2}\,dx$$

Gamma Function: Example

Solution:
We have

$$\Gamma(s) = 2\int_0^{+\infty} x^{2s-1} e^{-x^2}\,dx$$

Here $s = n + \frac{1}{2}$, so

$$LHS = \frac{1}{2}\Gamma(n+\frac{1}{2}) = \frac{1}{2}\cdot(n-\frac{1}{2})\Gamma(n-\frac{1}{2}) = \frac{2n-1}{2^2}\Gamma(n-\frac{1}{2})$$

Keep using the recursion formula, we can get

$$LHS = \frac{(2n-1)!!}{2^{n+1}}\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^{n+1}}$$

Relationship between Gamma function and Beta function

Theorem

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Relationship between Gamma function and Beta function

Previous Exercise

Calculate
$$\int_0^{\pi/2} \sin^7 x \cos^5 x dx$$

$$\text{Transfer } \int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x dx \text{ to } \Gamma \text{ function}$$

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta = \frac{1}{2}B(3,4) == \frac{\Gamma(4)\Gamma(3)}{2\Gamma(7)} = \frac{3!\times 2!}{2\times 6!} = \frac{1}{120}$$

Q&A

References

[1] Huang, Yucheng. VV156_RC6.pdf. 2021.

[2] Chen, Jixiu et al. Mathematical Analysis (3rd Version). 2019