VV156 Regular RC3/RC4

Integrals

Li Junhao

UM-SJTU Joint Institute

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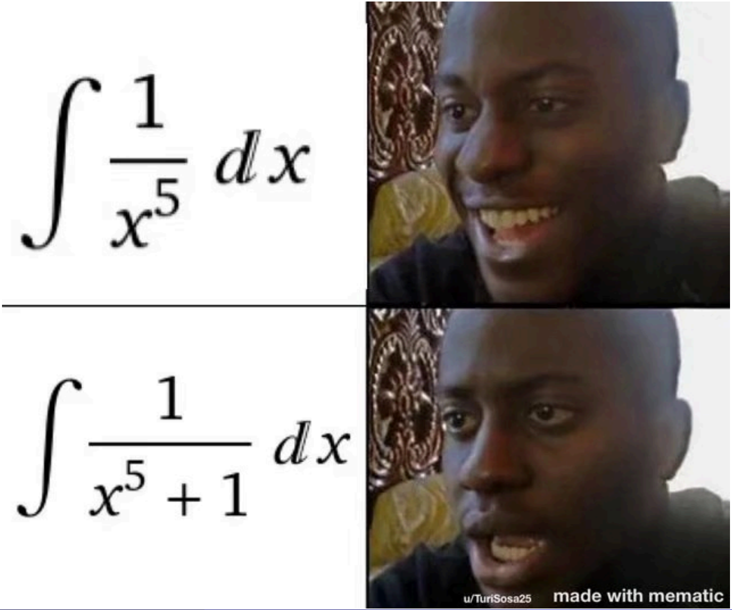
RC Overview

Integral memes



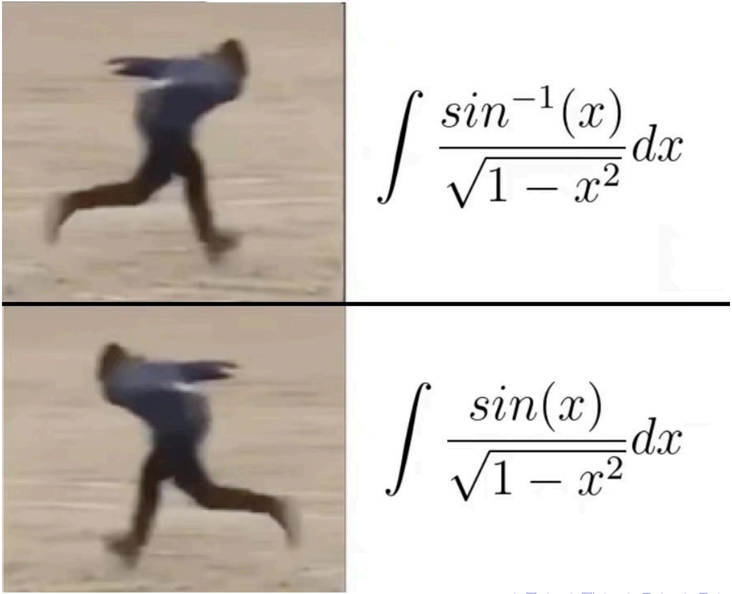
Integral memes

$$\begin{aligned}\int \frac{1}{1+x}\,dx &= \int \left(\frac{1}{x}+\frac{1}{1}\right)\,dx \\ &= \int \frac{1}{x}\,dx + \int \frac{1}{1}\,dx \\ &= \log(x)+\log(1) \\ &= \log(x+1)+C.\end{aligned}$$

Integral memes

$$\int \ln(x)\,dx \qquad\qquad \int \frac{1}{\ln(x)}\,dx$$

Integral memes



Integral memes

“你应该尊重其他人的观点！”

他们的观点:

$$\sum_{k=0}^{\infty}\int_0^{\infty}\frac{(-x)^k}{k!}\,dx=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,dx$$

$$=\int_0^{\infty}e^{-x}\,dx$$

$$=1$$

Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivative Table

Function	Antiderivative	Function	Antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec x \tan x$	$\sec x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

Linearity of Antiderivatives

Theorem

If the antiderivatives of functions $f(x)$ and $g(x)$ exist, then for any constant k_1 and k_2, the antiderivate of $k_1 f(x) + k_2 fg(x)$ also exists, and we have

$$\int [k_1 f(x) + k_2 g(x)] dx = k_1 \int f(x) dx + k_2 \int g(x) dx$$

Type 1: Direct Substitution

Substitution Rule for Direct Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(g(x))dg(x) = \int f(u)du = F(u) + C$$

This method can also be called "gather together differential".
In this method, the substitution process is in the form of

$$u = g(x)$$

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Direct Substitution: Example

$$\text{Calculate } \int \tan x dx$$

First we write tangent in terms of sine and cosine:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

This suggests that we should substitute $u = \cos x, \text{since} \, then du = -\sin x dx$ and so $\sin x dx = -du$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du$$
$$= -\ln |u| + C = -\ln |\cos x| + C$$

Since $-\ln |\cos x| = \ln \left(|\cos x|^{-1} \right) = \ln(1/|\cos x|) = \ln |\sec x|$, the result of Example can also be written as

$$\int \tan x dx = \ln |\sec x| + C$$

Exercises (Practice these problems as many as possible!)

Solution

1.Let $u = 1 + x^2$. Then $du = 2xdx$, so

$$\begin{aligned}\int \frac{1+x}{1+x^2}dx &= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx \\ &= \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} \\ &= \tan^{-1} x + \frac{1}{2}\ln|u| + C \\ &= \tan^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= \tan^{-1} x + \frac{1}{2}\ln\left(1+x^2\right) + C \quad \left[\text{ since } 1+x^2>0\right]\end{aligned}$$

Exercises (Practice these problems as many as possible!)

Solution

2.Let $u = 2x+5$. Then $du = 2dx$ and $x = \frac{1}{2}(u-5)$, so

$$\begin{aligned}
\int x(2x+5)^8 dx &= \int \frac{1}{2}(u-5)u^8\left(\frac{1}{2}du\right) \\
&= \frac{1}{4}\int \left(u^9 - 5u^8\right) du \\
&= \frac{1}{4}\left(\frac{1}{10}u^{10} - \frac{5}{9}u^9\right) + C \\
&= \frac{1}{40}(2x+5)^{10} - \frac{5}{36}(2x+5)^9 + C
\end{aligned}$$

Exercises (Practice these problems as many as possible!)

Solution

3.Let $u = \sin x$, so $du = \cos x dx$. When $x = 0, u = 0$; when $x = \frac{\pi}{2}, u = 1$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \cos x \sin(\sin x) dx &= \int_0^1 \sin u du \\ &= [-\cos u]_0^1 \\ &= -(\cos 1 - 1) \\ &= 1 - \cos 1\end{aligned}$$

Type 2: Inverse Substitution

Substitution Rule for Inverse Substitution

If $x = \varphi(t)$ is an invertible function, then

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = f(\varphi(t))\varphi'(t)dt = \widetilde{F}(t) = \widetilde{F}(\varphi^{-1}(x)) + C$$

In this method, the substitution process is in the form of

$$x = \varphi(t)$$

You can compare it with the 1st type of substitution.

Application: Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Application: Trigonometric Substitutions

Example

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx$$

Example Solution

Let $x = 2\tan\theta, -\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\left(\tan^2\theta+1\right)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

Thus we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$;

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

Example Solution

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \frac{du}{u^2}$$
$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$$
$$= -\frac{\csc\theta}{4} + C$$

We determine that $\csc\theta = \sqrt{x^2+4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

Integration by Parts

For any differentiable functions $u(x)$ and $v(x)$, we have

$$d[u(x)v(x)] = v(x)d[u(x)] + u(x)d[v(x)]$$

Then, we take the antiderivatives of both sides

$$u(x)v(x) = \int v(x)d[u(x)] + \int u(x)d[v(x)]$$

Formula

Therefore we have

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Integration by Parts

Example

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

Integration by Parts

Example Solution

Let $u = xe^{2x}, dv = \frac{1}{(1+2x)^2}dx \Rightarrow du = \left(x \cdot 2e^{2x} + e^{2x} \cdot 1\right)dx = e^{2x}(2x+1)dx, v = -\frac{1}{2(1+2x)}$ Then

$$\begin{aligned}\int \frac{xe^{2x}}{(1+2x)^2}dx &= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int \frac{e^{2x}(2x+1)}{1+2x}dx \\ &= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}dx \\ &= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{4}e^{2x} + C\end{aligned}$$

The answer could be written as $\frac{e^{2x}}{4(2x+1)} + C$

Ex 5

Evaluate the integral

$$\int_0^{\pi} e^{\cos t} \sin 2t dt$$

Ex 5

Solution

Let $x = \cos t$, so that $dx = -\sin t dt$. Thus,
$\int_0^\pi e^{\cos t} \sin 2t dt = \int_0^\pi e^{\cos t}(2 \sin t \cos t) dt = \int_1^{-1} e^x \cdot 2x(-dx) = 2 \int_{-1}^1 x e^x dx$.
Now use parts with $u = x$, $dv = e^x dx, du = dx, v = e^x$ to get
$2 \int_{-1}^1 x e^x dx = 2 \left([x e^x]_{-1}^1 - \int_{-1}^1 e^x dx \right) = 2 \left(e^1 + e^{-1} - [e^x]_{-1}^1 \right) =$
$2 \left(e + e^{-1} - \left[e^1 - e^{-1} \right] \right) = 2 \left(2e^{-1} \right) = 4/e$

Basic Integration Table

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C(\alpha \neq -1)$$

$$\int \ln x dx = x(\ln x - 1) + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C, \int \cos x dx = \sin x + C$$

$$\int \tan x dx = -\ln|\cos x| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C$$

Basic Integration Table

$$\int \frac{dx}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} dx = \ln |x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{x^2 - a^2} dx = \frac{1}{2a} \ln |\frac{x - a}{x + a}| + C$$

$$\int \frac{dx}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} (x \sqrt{x^2 \pm a^2} \pm a^2 \ln |x + \sqrt{x^2 \pm a^2}|) + C$$

A Useful Model

Model

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

A Useful Model

Model

$$I_n = \int \frac{dx}{(x^2+a^2)^n}$$

When $n=1$, $\int \frac{dx}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$.
When $n=2$, we have

$$LHS = \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(x^2+a^2)^n}dx = \frac{I_{n-1}}{a^2}+\frac{1}{a^2}\int\frac{-x^2}{(x^2+a^2)^n}dx$$

By using integration by parts method on the last part, I_n is equal to

$$\frac{I_{n-1}}{a^2}+\frac{1}{2a^2(n-1)}\int xd[\frac{1}{(x^2+a^2)^{n-1}}] = \frac{I_{n-1}}{a^2}+\frac{1}{2a^2(n-1)}[\frac{x}{(x^2+a^2)^{n-1}}-I_{n-1}]$$

A Useful Model

Formula

For

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

We have

$$I_1 = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$I_n = \frac{2n-3}{2a^2(n-1)} I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2 + a^2)^{n-1}}$$

For $n = 2$, we have

$$I_2 = \frac{1}{2a^3} \arctan \frac{x}{a} + \frac{1}{2a^2} \cdot \frac{x}{x^2 + a^2}$$

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(a) If the power of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x \left(\cos^2 x\right)^k \cos x dx \\ &= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x dx\end{aligned}$$

Then substitute $u = \sin x$.

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(b) If the power of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x dx &= \int \left(\sin^2 x\right)^k \cos^n x \sin x dx \\ &= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x dx\end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

(a) If the power of secant is even $(n = 2k, k \geqslant 2)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned} \int \tan^m x \sec^{2k} x dx &= \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x dx \\ &= \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x dx \end{aligned}$$

Then substitute $u = \tan x$.

Strategy for Evaluating $\int \tan^m x \sec^n x dx$

(b) If the power of tangent is odd $(m = 2k+1)$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x dx &= \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x dx \end{aligned}$$

Then substitute $u = \sec x$.

Strategy for Evaluating $\int \sin mx \cos nx dx$

Product-to-sum	Sum-to-product
$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$	$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$	$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$	$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$	$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Ex 8

Evaluate the integral

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$$

Ex 8

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^7\theta\cos^5\theta\,d\theta &= \int_0^{\pi/2} \sin^7\theta\cos^4\theta\cos\theta\,d\theta \\
&= \int_0^{\pi/2} \sin^7\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta \\
&\overset{S}{=} \int_0^1 u^7\left(1-u^2\right)^2 du \\
&= \int_0^1 \left(u^7-2u^9+u^{11}\right) du \\
&= \left[\frac{1}{8}u^8-\frac{1}{5}u^{10}+\frac{1}{12}u^{12}\right]_0^1 \\
&= \left(\frac{1}{8}-\frac{1}{5}+\frac{1}{12}\right)-0=\frac{1}{120}
\end{aligned}$$

Partial Fraction Method

Theorem

The antiderivatives of rational functions are always elementary functions.

Here, rational functions are in the form of $\dfrac{p_m(x)}{q_n(x)}$, where $p_m(x)$ and $q_n(x)$ are polynomials with degree m and n respectively.

Polynomials With Real Roots

Theorem*

If the polynomial $q(x)$ has a real root $x = \alpha$ (i.e. $q(x) = (x-\alpha)^k \widetilde{q}(x)$), then we have

$$\frac{p(x)}{q(x)} = \frac{\lambda}{(x-\alpha)^k} + \frac{\tilde{p}(x)}{(x-\alpha)^{k-1}\tilde{q}(x)}$$

A More Understandable Example

If a proper fraction function $f(x) = \dfrac{p(x)}{(x+\alpha)(x+\beta)(x+\gamma)^2}$, then we can take such transformation:

$$\frac{p(x)}{(x+\alpha)(x+\beta)(x+\gamma)^2} = \frac{A}{x+\alpha} + \frac{B}{x+\beta} + \frac{C}{x+\gamma} + \frac{D}{(x+\gamma)^2}$$

Here A, B, C, D are all constant numbers.

Polynomials With Real Roots

"Proper Fraction Function" means the degree of $p(x)$ should be less than that of $q(x)$. Instead, we need to first take something out:

A More Understandable Example

If we have $f(x) = \dfrac{x^4+x^3+x^2+x}{(x+\alpha)(x+\beta)(x+\gamma)^2}$,then we can take such transformation:

$$\frac{x^4+x^3+x^2+x}{(x+\alpha)(x+\beta)(x+\gamma)^2} = 1 + \frac{A}{x+\alpha} + \frac{B}{x+\beta} + \frac{C}{x+\gamma} + \frac{D}{(x+\gamma)^2}$$

Here A, B, C, D are all constant numbers.

Partial Fraction Example

Example

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$$

Example Solution

$$\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \Rightarrow 4y^2-7y-12$$
$$= A(y+2)(y-3) + By(y-3) + Cy(y+2). \text{ Setting}$$

$y=0$ gives $-12=-6A$, so $A=2$.
Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.
Usually we don't have to expand the polynomial on the right side. Just setting some special points is enough but more convenient.

Example Solution

Now

$$\begin{aligned}
\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy &= \int_1^2 \left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right) dy \\
&= \left[2\ln|y|+\frac{9}{5}\ln|y+2|+\frac{1}{5}\ln|y-3|\right]_1^2 \\
&= 2\ln 2+\frac{9}{5}\ln 4+\frac{1}{5}\ln 1-2\ln 1-\frac{9}{5}\ln 3-\frac{1}{5}\ln 2 \\
&= 2\ln 2+\frac{18}{5}\ln 2-\frac{1}{5}\ln 2-\frac{9}{5}\ln 3 \\
&= \frac{27}{5}\ln 2-\frac{9}{5}\ln 3 \\
&= \frac{9}{5}(3\ln 2-\ln 3) \\
&= \frac{9}{5}\ln\frac{8}{3}
\end{aligned}$$

Polynomials With Imaginary Roots

Theorem*

If the polynomial $q(x)$ has a imaginary root $x = \beta \pm i\gamma$ (i.e. $q(x) = (x^2 + 2\xi x + \eta^2)^k \widetilde{q}(x))$, then we have

$$\frac{p(x)}{q(x)} = \frac{ax+b}{(x^2+2\xi x+\eta^2)^k} + \frac{\tilde{p}(x)}{(x^2+2\xi x+\eta^2)^{k-1}\tilde{q}(x)}$$

A More Understandable Example

If a proper fraction function $f(x) = \dfrac{p(x)}{(x+\alpha)(x^2+\beta)^2} (\beta > 0)$, then we can take such transformation:

$$\frac{p(x)}{(x+\alpha)(x+\beta)(x+\gamma)^2} = \frac{A}{x+\alpha} + \frac{Bx+C}{x^2+\beta} + \frac{Dx+E}{(x^+\beta)^2}$$

Here A, B, C, D, E are all constant numbers.

Polynomials With Imaginary Roots

Example

$$\int \frac{x^4 + x^3 + 3x^2 - 1}{(x^2+1)^2(x-1)}\,dx$$

Polynomials With Imaginary Roots

Example

$$\int \frac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)}dx$$

Solution:

Suppose $\dfrac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$, then

$$x^4+x^3+3x^2-1 = A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1)$$

(1) Let $x=1$, we get $A=1$; (2) Compare the coefficients of x^4, we get $B=1-A=0$; (3) Compare the coefficients of x^3, we get $C=1$; (4) Let $x=0$, we get $E=1$; (5) Let $x=2$ and we get $D=2$. So

$$LHS = \int [\frac{1}{x-1} + \frac{1}{x^2+1} + \frac{2x+1}{(x^2+1)^2}]dx$$

Polynomials With Imaginary Roots

Solution (continued):

Now we need to calculate $\int[\frac{1}{x-1}+\frac{1}{x^2+1}+\frac{2x+1}{(x^2+1)^2}]dx$. The first two parts is apparent. So we only need to know how to calculate $\int\frac{2x+1}{(x^2+1)^2}dx$.

$$\int\frac{2x+1}{(x^2+1)^2}dx=\int[\frac{2x}{(x^2+1)^2}+\frac{1}{(x^2+1)^2}]dx=\int\frac{d(x^2+1)}{(x^2+1)^2}dx+\int\frac{1}{(x^2+1)^2}dx$$

$=-\frac{1}{x^2+1}+\frac{1}{2}\arctan x+\frac{x}{2(x^2+1)}+C$. (Hint: use the useful formula covered above)

So the result is $ln|x-1|+\frac{3}{2}\arctan x-\frac{1}{1+x^2}+\frac{x}{2(x^2+1)}+C$

Summary

(1) You need to first guarantee that the degree of the numerator is less than the degree of the denominator, otherwise you need to take something out.
(2) If the denominator has already been factorized, like $\int \frac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)}dx$, then go to (3), otherwise do factorize the denominator first.
(3) Transform the fraction using methods mentioned above. (refer to two "more understandable examples")
(4) Determine the coefficients (A,B,C,...).
(5) Calculate the integrals one by one.

A Useful Model*

Model

How to calculate $I_n = \int \frac{ax+b}{(x^2+2\xi x+\eta^2)^n}dx$? ($\xi^2 < \eta^2$, which means there's no real roots.)

$$I_n = \frac{a}{2}\int \frac{2x+2\xi}{(x^2+2\xi x+\eta^2)^n}dx + (b-a\xi)\int \frac{1}{(x^2+2\xi x+\eta^2)^n}dx$$

$$= \frac{a}{2}\int \frac{d(x^2+2\xi x+\eta^2)}{(x^2+2\xi x+\eta^2)^n} + (b-a\xi)\int \frac{1}{[(x+\xi)^2+(\eta^2-\xi^2)]^n}dx$$

The first part can be regarded as $\int \frac{1}{t^n}dt$. And for the second part you can refer to the useful formula mentioned above (at the end of Integration by Parts).

A Useful Model*

How to calculate $I_n = \int \frac{ax+b}{(x^2+2\xi x+\eta^2)^n}dx$? ($\xi^2 < \eta^2$, which means there's no real roots.)

Formula (You can memorize them, but not suggested)

$$I_1 = \frac{1}{\sqrt{\eta^2-\xi^2}}\arctan\frac{x+\xi}{\sqrt{\eta^2-\xi^2}} + C$$

$$I_2 = \frac{1}{2(\sqrt{\eta^2-\xi^2})^3}\arctan\frac{x+\xi}{\sqrt{\eta^2-\xi^2}} + \frac{1}{2(\eta^2-\xi^2)}\cdot\frac{x+\xi}{x^2+2\xi x+\eta^2}$$

$$I_n = \frac{1}{2(\eta^2-\xi^2)(n-1)}[(2n-3)I_{n-1} + \frac{x+\xi}{(x^2+2\xi x+\eta^2)^{n-1}}]$$

Darboux integral (Optional)

A partition of an interval $[a, b]$ is a finite sequence of values x_i such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

Each interval $[x_{i-1}, x_j]$ is called a subinterval of the partition. Let f. $[a, b] \to \mathbf{R}$ be a bounded function, and let

$$P = (x_0, \ldots, x_n)$$

be a partition of $[a, b]$. Let

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$
$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Darboux integral (Optional)

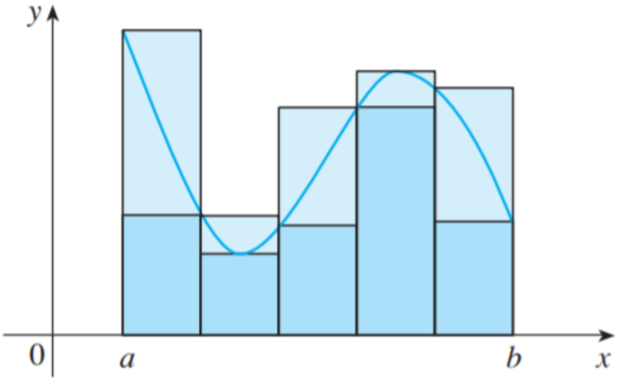
The upper Darboux sum of f with respect to P is

$$U_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1})\, M_i$$

The lower Darboux sum of f with respect to P is

$$L_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i$$

Darboux integral (Optional)



Definite integral

Definition

If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the definite integral of $\boldsymbol{f}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Properties of the Integral

Property 1: Linearity

$$\int_a^b [k_1 f(x) + k_2 g(x)]dx = k_1 \int_a^b f(x)dx + k_2 \int_a^b g(x)dx$$

Property 2: Interval Additivity

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Properties of the Integral

Property 3: Integral of Products (Optional)

If $f(x)$ and $g(x)$ are integrable on $[a, b]$, then $f(x) \cdot g(x)$ is also integrable on $[a, b]$, but generally

$$\int_a^b f(x)g(x)dx \neq (\int_a^b f(x)dx) \cdot (\int_a^b g(x)dx)$$

Property 4: Order-preserving Property

If $f(x)$ and $g(x)$ are integrable on $[a, b]$, and $f(x) \leq g(x)$, then we have

$$\int_a^b f(x)dx) \leq \int_a^b g(x)dx$$

Properties of the Integral

Property 5: Absolute Integrability

If $f(x)$ is integrable on $[a, b]$, then $|f(x)|$ is also integrable on $[a, b]$, and we have

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Property 6

If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

Ex 1

Calculate the definite integral by definition

1.
$$\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$$

2.
$$\int_{\pi}^{\pi} \sin^2 x \cos^4 x dx$$

Ex 1

Solution

1. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3 , plus the area of the rectangle, so

$$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$$

2.

$$\int_{\pi}^{\pi}\sin^2 x\cos^4 x dx=0$$

since the limits of intergration are equal.

The Fundamental Theorem of Calculus

Newton-Leibniz Formula

Suppose f is continuous on $[a, b]$.
1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If F is any antiderivative of f (i.e. $F' = f$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Application of Newton-Leibniz Formula

Useful Formulas to Calculate Derivatives

1. If $F(x) = \int_x^b f(t)dt$, then $F'(x) = -\int_b^x f(t)dt = -f(x)$.
2. If $F(x) = \int_a^x f(g(t))dt$, then $F'(x) = f(g(x))$.
2. If $F(x) = \int_a^{g(x)} f(t)dt$, then $F'(x) = f(g(x)) \cdot g'(x)$.
3. If $F(x) = \int_{g(x)}^{h(x)} f(t)dt$, then $F'(x) = f(h(x)) \cdot h'(x) - f(g(x) \cdot g'(x)$.

Can you prove the last formula using interval additivity of definite integrals?

Application of Newton-Leibniz Formula

Excercise

Calculate the limit

$$\lim_{x \to +\infty} \frac{\left(\int_0^x e^{u^2}\, du\right)^2}{\int_0^x e^{2u^2}\, du}$$

Application of Newton-Leibniz Formula

Excercise

Calculate the limit

$$\lim_{x \to +\infty} \frac{\left(\int_0^x e^{u^2} du\right)^2}{\int_0^x e^{2u^2} du}$$

Solution:

By using L'Hospital's rule, we have

$$LHS = \lim_{x \to +\infty} \frac{2\left(\int_0^x e^{u^2} du\right) \cdot e^{x^2}}{e^{2x^2}} = \lim_{x \to +\infty} \frac{2e^{x^2}}{2xe^{x^2}} = 0$$

This question is from The Chinese Mathematics Competitions (2015 Final)

Application of Newton-Leibniz Formula

Excercise: Concave function and integral

On what interval is the curve

$$y = \int_0^x \frac{t^2}{t^2+t+2} dt$$

concave downward?

Application of Newton-Leibniz Formula

Solution

$$y = \int_0^x \frac{t^2}{t^2+t+2}\,dt \Rightarrow y' = \frac{x^2}{x^2+x+2} \Rightarrow$$

$$\begin{aligned} y'' &= \frac{\left(x^2+x+2\right)(2x) - x^2(2x+1)}{\left(x^2+x+2\right)^2} \\ &= \frac{2x^3+2x^2+4x-2x^3-x^2}{\left(x^2+x+2\right)^2} \\ &= \frac{x^2+4x}{\left(x^2+x+2\right)^2} \\ &= \frac{x(x+4)}{\left(x^2+x+2\right)^2} \end{aligned}$$

The curve y is concave downward when $y'' < 0$; that is, on the interval $(-4, 0)$.

Trigonometric Integrals (2): Calculate $\int_0^{\pi/2} sin^n x$

Model

$$I_n = \int_0^{\pi/2} \sin^n x dx$$

For $n = 0$ and $n = 1$, the result is very simple.
For $n > 1$, we have:

$$I_n = \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x dx$$

$$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot \cos^2 x dx = (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot (1 - \sin^2 x) dx$$

$$= (n-1)(I_{n-2} - I_n)$$

Model

$$I_n = \int_0^{\pi/2} \sin^n x dx$$

If n is an odd number, we have

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{2}{3} = \frac{(n-1)!!}{n!!}$$

If n is an even number, we have

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}$$

*$n!! = n \cdot (n-2) \cdot (n-4) \ldots \cdot 3 \cdot 1$ or $n!! = n \cdot (n-2) \cdot (n-4) \ldots \cdot 4 \cdot 2$

Symmetry and Periodicity

Suppose $f(x)$ is integrable on $[-a, a]$, then:
(1) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

(2) If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

Suppose $f(x)$ is a function with period T, then for any a, we have

$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$

Type 1: Infinite Intervals

(a) If $\int_a^t f(x)dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).
(b) If $\int_t^b f(x)dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).
The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

Type 1: Infinite Intervals

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

In part (c) any real number a can be used.

Type 2: Discontinuous Integrands

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).
(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number).
The improper integral $\int_a^b f(x)dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

Type 2: Discontinuous Integrands

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Only when the two parts $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are both convergent can we conclude that the improper integral $\int_a^b f(x)dx$ is convergent.

Calculation of Improper Integrals

Evaluate the integral (if it is convergent)

1. $$\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

2. $$\int_0^3 \frac{dx}{x^2-6x+5}$$

Solution

1.

$$\begin{aligned}
\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx &= \lim_{t\to\infty}\int_e^{t} \frac{1}{x(\ln x)^3}\,dx \\
&= \lim_{t\to\infty}\int_1^{\ln t} u^{-3}\,du \qquad \left[\begin{array}{c} u=\ln x, \\ du = dx/x \end{array}\right] \\
&= \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t} \\
&= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2}+\frac{1}{2}\right] \\
&= 0+\frac{1}{2}=\frac{1}{2}. \qquad \text{Convergent}
\end{aligned}$$

Solution

2.

$$I = \int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2$$
$$= \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$$

Now

$$\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \Rightarrow 1 = A(x-5) + B(x-1)$$

Set $x = 5$ to get $1 = 4B$, so $B = \frac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\frac{1}{4}$. Thus

Calculation of Improper Integrals

Solution

2.

$$
\begin{aligned}
I_1 &= \lim_{t\to 1^-} \int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5} \right) dx \\
&= \lim_{t\to 1^-} \left[-\frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x-5| \right]_0^t \\
&= \lim_{t\to 1^-} \left[\left(-\frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t-5| \right) \left(-\frac{1}{4}\ln|-1| + \frac{1}{4}\ln|-5| \right) \right] \\
&= \infty, \quad \text{since } \lim_{t\to 1^-} \left(-\frac{1}{4}\ln|t-1| \right) = \infty
\end{aligned}
$$

Since I_1 is divergent, I is divergent.

Comparison Theorem

Theorem

Suppose that f and g are continuous functions with $f(x) \geqslant Kg(x) \geqslant 0$ for $x \geqslant a$.(K is a positive constant)
(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Comparison Theorem

Corollary of Comparison Theorem

Suppose that f and g are positive functions on $[a, +\infty)$, and we have

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = A$$

Then:
(1) if $0 \leq A < +\infty$ and $\int_a^{+\infty} g(x)dx$ is convergent, then $\int_a^{+\infty} f(x)dx$ is also convergent.
(2) if $0 < A \leq +\infty$ and $\int_a^{+\infty} g(x)dx$ is divergent, then $\int_a^{+\infty} f(x)dx$ is also divergent.
That also means, if $0 < A < +\infty$, the convergence & divergence property of f and g are the same.

Do remember that this is only a corollary of comparison theorem covered on the last page. Sometimes this is more convenient to use.

Comparison Theorem: Exercise

Determine Convergence

$$\int_1^\infty \frac{2+e^{-x}}{x}dx$$

Comparison Theorem: Exercise

Determine Convergence

$$\int_1^\infty \frac{2+e^{-x}}{x} dx$$

Solution

For $x \geq 1$, $\frac{2+e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^\infty \frac{1}{x} dx$ is divergent, so $\int_1^\infty \frac{2+e^{-x}}{x} dx$ is divergent by the Comparison Theorem.

Euler Integrals

This part is not required for its on sake in this course. But some of the models may be useful in solving problems of integrals.

Beta Function

Definition

Beta function is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Property 1:

$$B(p,q) = B(q,p)$$

Property 2 (when $p > 0$ and $q > 1$):

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1)$$

A conclusion from property 1 and 2 (when $p > 1$ and $q > 1$):

$$B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}B(p-1,q-1)$$

Beta Function: Other Expressions

(1) Let $x = \cos^2\varphi$:

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2p-1}\varphi \cos^{2q-1}\varphi d\varphi$$

We usually change the trigonometric integrals to Γ function.

(2)Let $x = \dfrac{1}{1+t}$ and $t = \dfrac{1}{u}$:

$$B(p,q) = \int_0^\infty \frac{t^{q-1}}{(1+t)^{p+q}}dt = \int_0^1 \frac{t^{q-1}}{(1+t)^{p+q}}dt + \int_1^\infty \frac{t^{q-1}}{(1+t)^{p+q}}dt$$

$$= \int_0^1 \frac{t^{q-1}}{(1+t)^{p+q}}dt + \int_0^1 \frac{u^{p-1}}{(1+u)^{p+q}}du = \int_0^1 \frac{t^{p-1}+t^{q-1}}{(1+t)^{p+q}}dt$$

Beta Function: Summary

When you need to calculate these three types of integrals:

Beta Functions

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x dx$$

$$B(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx$$

You can regard them as Beta Functions and use the recursion formula

$$B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}B(p-1,q-1)$$

Beta Function: Example

Example

Calculate

$$\int_0^1 \frac{x+x^3}{(1+x)^5}dx$$

Solution:

Let

$$B(2,4)=\int_0^1 \frac{x+x^3}{(1+x)^5}dx$$

then

$$B(2,4)=\frac{1}{2}B(2,3)=\frac{1}{2}\int_0^1 \frac{x+x^2}{(1+x)^4}dx=\frac{1}{2}\int_0^1 \frac{x}{(1+x)^3}dx=\frac{1}{2}\int_0^1 \frac{(1+x)-1}{(1+x)^3}dx$$

$$=\frac{1}{2}[\int_0^1 \frac{1}{(1+x)^2}dx-\int_0^1 \frac{1}{(1+x)^3}dx]$$

Gamma Function

Definition

Gamma function is defined as

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx$$

Property 1:

$$\Gamma(s+1) = s\Gamma(s), s > 0$$

Apply "integration by parts" method:

$$\int_0^{+\infty} x^s e^{-x} dx = -x^s e^{-x}|_0^{+\infty} + s\int_0^{+\infty} x^{s-1} e^{-x} dx = s\int_0^{+\infty} x^{s-1} e^{-x} dx = s\Gamma(s)$$

Property 2 (when $s = n$ is a positive integer):

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!\Gamma(1) = n!$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} = 1$$

Gamma Function

Property 2:
When $s = n$ is a positive integer, we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = ... = n!\Gamma(1) = n!$$

And

$$\Gamma(1) = \int_0^{+\infty} e^{-x} = 1$$

So when n is a positive integer:

Formula

$$\Gamma(n+1) = n!$$

Gamma Function: Other Expressions

Let $x = t^2$, then

$$\Gamma(s) = 2\int_0^{+\infty} t^{2s-1} e^{-t^2} dt$$

Let $x = \alpha t$, then

$$\Gamma(s) = \alpha^s \int_0^{+\infty} t^{s-1} e^{-\alpha t} dt$$

These two forms both satisfy the recursion formula

$$\Gamma(s+1) = s\Gamma(s), \Gamma(n+1) = n!$$

Gamma Function: Example

Example

Given that

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Calculate

$$\int_0^{+\infty} x^{2n} e^{-x^2}\,dx$$

Solution:
We have

$$\Gamma(s) = 2\int_0^{+\infty} x^{2s-1} e^{-x^2}\, dx$$

Here $s = n + \frac{1}{2}$, so

$$LHS = \frac{1}{2}\Gamma(n+\frac{1}{2}) = \frac{1}{2}\cdot(n-\frac{1}{2})\Gamma(n-\frac{1}{2}) = \frac{2n-1}{2^2}\Gamma(n-\frac{1}{2})$$

Keep using the recursion formula, we can get

$$LHS = \frac{(2n-1)!!}{2^{n+1}}\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^{n+1}}$$

Relationship between Gamma function and Beta function

Theorem

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Relationship between Gamma function and Beta function

Previous Exercise

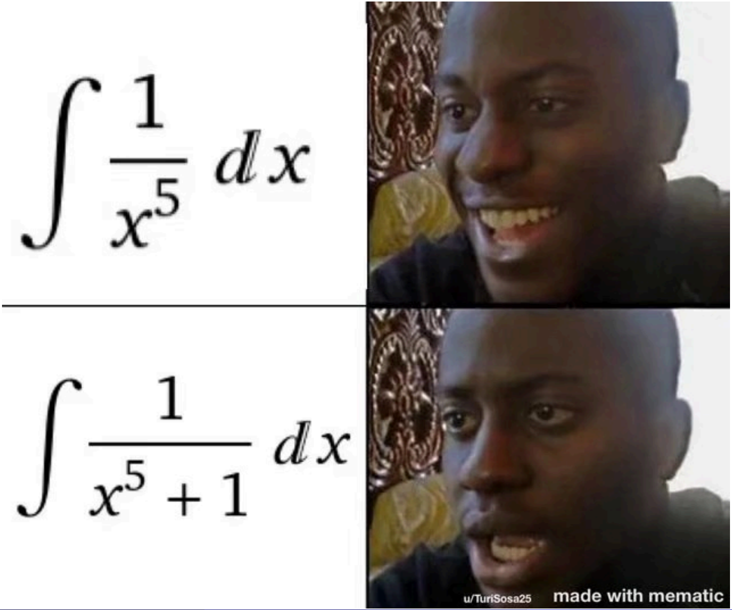
Calculate

$$\int_0^{\pi/2} \sin^7 x \cos^5 x dx$$

$$\text{Transfer } \int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x dx \text{ to } \Gamma \text{ function}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^5 \theta = \frac{1}{2} B(3,4) == \frac{\Gamma(4)\Gamma(3)}{2\Gamma(7)} = \frac{3! \times 2!}{2 \times 6!} = \frac{1}{120}$$

Back to memes



Back to memes

$$\int \frac{1}{x^5} dx = -\frac{1}{4x^4} + C$$

Back to memes

With $\phi_\pm = \frac{1\pm\sqrt{5}}{4}$

$$x^5+1=(1+x)\left(x^2-2\phi_+x+1\right)\left(x^2-2\phi_-x+1\right)$$

and

$$\frac{5}{1+x^5}=\frac{1}{x+1}-\frac{2\phi_+x-2}{x^2-2\phi_+x+1}-\frac{2\phi_-x-2}{x^2-2\phi_-x+1}$$

The integral for the first term is just $\ln(x+1)$, and for the second and third terms

$$\begin{aligned} I(x,\phi) &= \int \frac{2\phi x-2}{x^2-2\phi x+1}dx = \int \frac{\phi\, d\left[(x-\phi)^2\right]-2\left(1-\phi^2\right)dx}{(x-\phi)^2+(1-\phi^2)} \\ &= \phi \ln\left(x^2-2\phi x+1\right)-2\sqrt{1-\phi^2}\tan^{-1}\frac{x-\phi}{\sqrt{1-\phi^2}} \end{aligned}$$

Back to memes

Thus

$$\int \frac{1}{1+x^5}dx = \frac{1}{5}\left[\ln(x+1) - I(x,\phi_+) - I(x,\phi_-)\right] + C$$

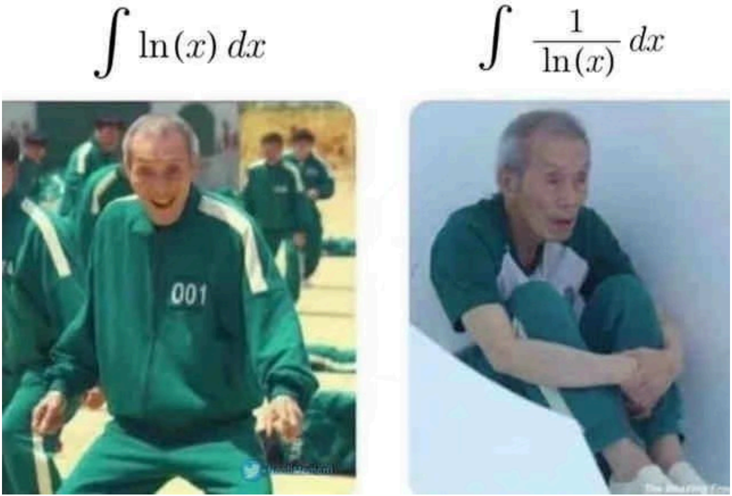
Back to memes

$$\begin{aligned}\int \frac{1}{1+x}\,dx &= \int \left(\frac{1}{x}+\frac{1}{1}\right)dx\\ &= \int \frac{1}{x}\,dx + \int \frac{1}{1}\,dx\\ &= \log(x)+\log(1)\\ &= \log(x+1)+C.\end{aligned}$$

Back to memes

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

Back to memes



Back to memes

$$\int \ln(x)dx = x\ln(x) - x + C$$

Back to memes

$$
\begin{aligned}
\int \frac{dx}{\ln x} &\overset{\text{let } x=e^t}{=} \int \frac{de^t}{\ln t} \\
&= \int \frac{\lim_{n\to+\infty}\sum_{i=0}^{n}\frac{t^i}{i!}}{t}\,dt \\
&= \int \left(\frac{1}{t}+1+\frac{t^1}{2!}+\frac{t^2}{3!}+\cdots\cdots\right)dt \\
&= \ln|t|+\frac{t^1}{1\times 1!}+\frac{t^2}{2\times 2!}+\frac{t^3}{3\times 3!}+\cdots\cdots+C \\
&= \ln|\ln x|+\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{(\ln x)^i}{i\cdot i!}+C
\end{aligned}
$$

It's not an elementary integral. We can only change it by Liouville's theorem.

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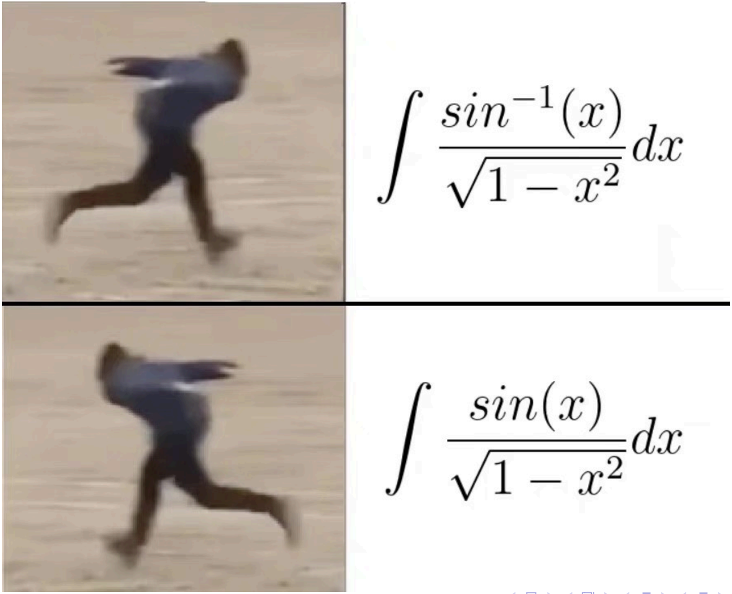

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$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}} dx = \frac{\arcsin^2(x)}{2} + C$$

$$\int \frac{\sin(x)}{\sqrt{1-x^2}} dx = ?\textbf{Cannot solve this}$$

Q&A

References

[1] Huang, Yucheng. VV156_RC4.pdf. 2021.

[2] Chen, Jixiu et al. Mathematical Analysis (3rd Version). 2019