VV156 RC For Midterm 1

Derivatives and Differentiation

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RC Overview

Definition

Derivative

The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

The slope of the tangent line of a function is the corresponding derivative.

Definition

Derivative: Another Definition

The **derivative of a function *f* at a number *a***, denoted by $f'(x)$, is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

if this limit exists.

The slope of the tangent line of a function is the corresponding derivative.

Definition

Notations

Newton:

$$\dot{y}$$

Leibniz:

$$\frac{dy}{dx}$$

Lagrange:

$$f'(x)$$

Jacobi: (Partial Derivatives)

$$\frac{\partial f}{\partial x}$$

Definition

Exercise

Suppose that the derivative of function $f(x)$ at $x = x_0$ is $f'(x_0)$. calculate the limit

$$\lim_{h\to 0} \frac{f(x_0 + \alpha h) - f(x_0 - \beta h)}{h}$$

Definition

Exercise

Suppose that the derivative of function $f(x)$ at $x = x_0$ is $f'(x_0)$. calculate the limit

$$\lim_{h\to 0} \frac{f(x_0+\alpha h)-f(x_0-\beta h)}{h}$$

Solution

$\lim_{h\to 0} \frac{f(x_0+\alpha h)-f(x_0-\beta h)}{h} =$

$\lim_{h\to 0} \frac{[f(x_0+\alpha h)-f(x_0)]-[f(x_0-\beta h)-f(x_0)]}{h} = \alpha f'(x_0)+\beta f'(x_0)$

Definition of Differentiation

Increment and Difference

Give a function $y = f(x)$, if there is an increment Δx at $x = x_0$, the increment of y should be

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Definition (not required)

Suppose x_0 is in the domain of the function $y = f(x)$. If there exist a number $g(x_0)$, which is only dependent on x_0, such that when $x \to 0$, we have

$$\Delta y = g(x_0)\Delta x + o(\Delta x)$$

Then we say that $f(x)$ is differentiable on x_0.

Definition of Differentiation: Example

Example

Suppose $y = f(x) = x^2$, at any point $x \in D$, we take an increment Δx and we have

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2$$

Suppose that $x \sim 1$ and $\Delta x \sim 10^{-10}$, what is Δy? Are there any terms can be ignored?

Definition of Differentiation: Example

Example

Suppose $y = f(x) = x^2$, at any point $x \in D$, we take an increment Δx and we have

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Suppose that $x \sim 1$ and $\Delta x \sim 10^{-10}$, what is Δy? Are there any terms can be ignored?

When $\Delta x \to 0$, we have

$$dy = d(x^2) = 2xdx$$

This is equivalent to

$$\frac{dy}{dx} = 2x$$

Differentiable

Differentiable

A function f is differentiable at $\boldsymbol{a}$ if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Differentiable and Continuity

If f is differentiable at a, then f is continuous at a.
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

Differential formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

Differential formulas

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Differential formulas

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$$

Differential formulas

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Exercise 1

Calculate the Derivative

1.
$$f(x) = \frac{1 - xe^x}{x + e^x}$$

2.
$$f(x) = xe^x \csc x$$

3.
$$f(x) = x \ln x - x$$

Exercise 1

Solutions

1.

$$
\begin{aligned}
f(x) &= \frac{1 - xe^x}{x + e^x} \\
&= \frac{(x + e^x)(-xe^x)' - (1 - xe^x)(1 + e^x)}{(x + e^x)^2} \\
&= \frac{(x + e^x)\left[-(xe^x + e^x \cdot 1)\right] - \left(1 + e^x - xe^x - xe^{2x}\right)}{(x + e^x)^2} \\
&= \frac{-x^2 e^x - xe^x - xe^{2x} - e^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x + e^x)^2} \\
&= \frac{-x^2 e^x - e^{2x} - e^x - 1}{(x + e^x)^2}
\end{aligned}
$$

Exercise 1

Solutions

2.

$$(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = \left(f'g + fg'\right)h + (fg)h'$$
$$= f'gh + fg'h + fgh'$$

$$f'(x) = (x)'e^x \csc x + x(e^x)' \csc x + xe^x(\csc x)'$$
$$= 1e^x \csc x + xe^x \csc x + xe^x(-\cot x \csc x)$$
$$= e^x \csc x(1 + x - x\cot x)$$

3.

$$f(x) = x\ln x - x \Rightarrow f'(x) = x \cdot \frac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$$

Chain rule

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain rule

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 2

Calculate the Derivative

1.
$$F(t) = (3t-1)^4(2t+1)^{-3}$$

2.
$$y = \left(\frac{x^2+1}{x^2-1}\right)^3$$

3.
$$y = \sqrt{1+2e^{3x}}$$

Exercise 2

Solutions

1.

$$\begin{aligned}
F(t) &= (3t-1)^4(2t+1)^{-3} \Rightarrow \\
F'(t) &= (3t-1)^4(-3)(2t+1)^{-4}(2) + (2t+1)^{-3} \cdot 4(3t-1)^3(3) \\
&= 6(3t-1)^3(2t+1)^{-4}[-(3t-1)+2(2t+1)] \\
&= 6(3t-1)^3(2t+1)^{-4}(t+3)
\end{aligned}$$

Exercise 2

Solutions

2.

$$
\begin{aligned}
y' &= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2} \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x\left[x^2-1-\left(x^2+1\right)\right]}{\left(x^2-1\right)^2} \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x(-2)}{\left(x^2-1\right)^2} \\
&= \frac{-12x\left(x^2+1\right)^2}{\left(x^2-1\right)^4}
\end{aligned}
$$

Exercise 2

Solutions

3.

$$
\begin{aligned}
y &= \sqrt{1+2e^{3x}} \Rightarrow \\
y' &= \frac{1}{2}\left(1+2e^{3x}\right)^{-1/2}\frac{d}{dx}\left(1+2e^{3x}\right) \\
&= \frac{1}{2\sqrt{1+2e^{3x}}}\left(2e^{3x}\cdot 3\right) \\
&= \frac{3e^{3x}}{\sqrt{1+2e^{3x}}}
\end{aligned}
$$

Find y' if $\sin(x+y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x+y)\cdot\left(1+y'\right) = y^2(-\sin x) + (\cos x)\left(2yy'\right)$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y)\cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Exercise 3

Implicit Differentiation

Find y'' if $x^4 + y^4 = 16$

Exercise 3

Solutions

Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y':

$$y' = -\frac{x^3}{y^3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x :

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3(d/dx)\left(x^3\right) - x^3(d/dx)\left(y^3\right)}{\left(y^3\right)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3\left(3y^2y'\right)}{y^6}$$

Exercise 3

Solutions (Continued)

If we now plug the value of y' into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$
$$= -\frac{3\left(x^2y^4 + x^6\right)}{y^7} = -\frac{3x^2\left(y^4 + x^4\right)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

Derivatives of Inverse Functions

Theorem

If the function $y = f(x)$ is differentiable and its inverse function $x = f^{-1}(y)$ exists, then we have

$$[f^{-1}(y)]' = \frac{1}{f'(x)}$$

Useful Formulas

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1, \frac{d}{dx} f = (\frac{d}{dy} f) \cdot \frac{dy}{dx}$$

Example

If $\frac{dx}{dy} = \frac{1}{y'}$, show that: (i) $\frac{d^2x}{dy^2} = \frac{-y''}{(y')^3}$ (ii) $\frac{d^3x}{dy^3} = \frac{3(y'')^2 - y'y''}{(y')^5}$

Abstract Functions

Exercises

Suppose that $f(x)$ is derivable. Calculate the derivatives of the following abstract functions:

(1) $\arctan f(x)$. (2) $\sin(f(\sin x))$. (3) $f(\dfrac{1}{f(x)})$. (4) $\dfrac{1}{f(f(x))}$.

Abstract Functions

Exercises

Suppose that $f(x)$ is derivable. Calculate the derivatives of the following abstract functions:

(1) $\arctan f(x)$. (2) $\sin(f(\sin x))$. (3) $f(\dfrac{1}{f(x)})$. (4) $\dfrac{1}{f(f(x))}$.

Solutions

(3) $[f(\dfrac{1}{f(x)})]' = f'(\dfrac{1}{x})(\dfrac{1}{x})' = -\dfrac{f'(x)}{f^2(x)}f'(\dfrac{1}{f(x)})$.

(4) $[\dfrac{1}{f(f(x))}]' = -\dfrac{f'(f(x))}{f^2(f(x))}[f(x)]' = -\dfrac{f'(f(x))f'(x)}{f^2(f(x))}$.

Calculation Tricks

Exercises

Calculate the derivatives of the following functions:

(1) $f(x) = (x^3 + \sin x)^{\frac{1}{x}}$.

(2) $f(x) = x\dfrac{\sqrt{1-x^2}}{\sqrt{1+x^3}}$.

Calculation Tricks: Logarithmic Transformation

Solutions

(1) $f'(x) = y' = (x^3 + \sin x)^{\frac{1}{x}}[\frac{3x^2 + \cos x}{x(x^3 + \sin x)} - \frac{\ln(x^3 + \sin x)}{x^2}]$. Hint: $\ln y = \frac{1}{x}\ln(x^3 + \sin x)$.

(2) $f'(x) = y' = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^3}}[\frac{1}{x} - \frac{x}{1-x^2} - \frac{3x^2}{2(1+x^3)}]$.

If the given function is $f(x) = x\frac{\sqrt{1-x^2}\sqrt{1-2x^2}\sqrt{1-3x^2}}{\sqrt{1+x^3}}$, can you see the convenience brought by this method?

Problems Related to Definitions

Exercises

1. Suppose $f(x)$ is defined on $\mathbb{R}$. For any $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y) + 2xy$, and $f'(0) = a$. What is $f(x)$?

2. Suppose that $f(x)$ is differentiable. Show that:
(1) If $f(x)$ is an odd function, then $f'(x)$ is an even function.
(2) If $f(x)$ is an even function, then $f'(x)$ is an odd function.
(3) If $f(x)$ is an periodic function, then $f'(x)$ is also an periodic function with the same period.

Exercises

1. Suppose $f(x)$ is defined on $\mathbb{R}$. For any $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y) + 2xy$, and $f'(0) = a$. What is $f(x)$?

Solutions

First let $y = 0$ to get $f(0) = 0$. Then let $y = \Delta x$ and that $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{f(\Delta x) + 2x \cdot \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{f(\Delta x)}{\Delta x} + 2x = f'(0) + 2x$. So $f(x) = f'(0)x + x^2 + C$. Plug in $f(0) = 0$ and we get $f(x) = f'(0)x + x^2$.

Problems Related to Definitions

Exercises

1. Suppose $f(x)$ is defined on $\mathbb{R}$. For any $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y) + 2xy$, and $f'(0) = a$. What is $f(x)$?

Solutions

First let $y = 0$ to get $f(0) = 0$. Then let $y = \Delta x$ and that $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{f(\Delta x) + 2x \cdot \Delta x}{\Delta x} =$ $\lim_{\Delta x \to 0} \dfrac{f(\Delta x)}{\Delta x} + 2x = f'(0) + 2x$. So $f(x) = f'(0)x + x^2 + C$. Plug in $f(0) = 0$ and we get $f(x) = f'(0)x + x^2 = x^2 + ax$.

Problems Related to Definitions

Exercises

2. Suppose that $f(x)$ is differentiable. Show that:
(1) If $f(x)$ is an odd function, then $f'(x)$ is an even function.
(2) If $f(x)$ is an even function, then $f'(x)$ is an odd function.
(3) If $f(x)$ is an periodic function, then $f'(x)$ is also an periodic function with the same period.

Solutions

(1) $f'(-x) = \lim_{\Delta x \to 0} \dfrac{f(-x+\Delta x) - f(-x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{-f(x-\Delta x) + f(x)}{\Delta x}$
$= \lim_{\Delta x \to 0} \dfrac{f(x+(-\Delta x)) - f(x)}{-\Delta x} = f'(x)$.
(2) Similar to (1).
(3) Suppose that the period is T, then $f'(x+T) =$
$\lim_{\Delta x \to 0} \dfrac{f(x+T+\Delta x) - f(x+T)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$.

Q&A

References

[1] Huang, Yucheng. VV156_RC2.pdf. 2021.

[1] Huang, Yucheng. VV156_RC3.pdf. 2021.

[3] Cai, Runze. Chapter02.pdf. 2021.

[4] Cai, Runze. Chapter03.pdf. 2021.