VV156 RC2

Limits and Continuity

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RC Overview

Four fundamental Calculations of Limits

Theorem

Suppose that $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$, then

$$\lim_{x\to x_0}[\alpha f(x) + \beta g(x)] = \alpha A + \beta B$$

$$\lim_{x\to x_0}[f(x)g(x)] = AB$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{A}{B}$$

Other Limit Algorithms

Theorem 1: The sum or product of finite infinitesimals is an infinitesimal.
Theorem 2: If $\lim f(x) = 0$, $g(x)$ is a **bounded function** (i.e. $|g(x)| < M$), then

$$\lim f(x)g(x) = 0$$

Theorem 3: If $\lim f(x)$ exists, and n is a postive integer, then

$$\lim[f(x)]^n = [\lim f(x)]^n$$

Two Important Limits

Formula

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Formula

$$\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}} = e$$

A useful trick:
If $f(x) > 0$ and $\lim_{x \to x_0} f(x) = 0$, then

$$\lim_{x \to x_0} [1 + f(x)]^{g(x)} = \lim_{x \to x_0} [1 + f(x)]^{\frac{1}{f(x)} f(x) g(x)} = e^{f(x) g(x)}$$

Equivalent Infinitesimal

Definition

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$$

Then we called these two infinitesimals $f(x)$ and $g(x)$is equivalent. It can be denoted by

$$f(x) \sim g(x)$$

Example

Since $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, we can denote that $\sin x \sim x$ (when $x \to 0$).

Equivalent Infinitesimal

Some commonly used equivalent infinitesimal substitutions:

$$\sin x \sim x, \tan x \sim x, \arcsin x \sim x, \arctan x \sim x, 1 - \cos x \sim \frac{1}{2}x^2$$

$$e^x \sim x + 1, \ln(1+x) \sim x, (1+kx)^n - 1 \sim 1 + nkx$$

Application

If $f(x)$ is **a multiple or division factor**, then you can sustitute it with its equivalent infinitesimal $g(x)$.

Exercise

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

Equivalent Infinitesimal

Exercise

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$$

Solution

$\tan x - \sin x = \tan x(1-\cos x)$. $\tan x \sim x, 1-\cos x \sim \dfrac{x^2}{2}$. So

$$\tan x - \sin x \sim \frac{1}{2}x^3$$

Equivalent Infinitesimal

Tip: You're highly recommended to remember this part!

When $x \to 0$

$$a^x - 1 \sim x\ln a$$

$$\arcsin(a)x \sim \sin(a)x \sim (a)x$$

$$\arctan(a)x \sim \tan(a)x \sim (a)x$$

$$\ln(1+x) \sim x$$

$$\sqrt{1+x} - \sqrt{1-x} \sim x$$

$$(1+ax)^b - 1 \sim abx$$

$$\sqrt[b]{1+ax} - 1 \sim \frac{a}{b}x$$

$$1 - \cos x \sim \frac{x^2}{2}$$

$$x - \ln(1+x) \sim \frac{x^2}{2}$$

Equivalent Infinitesimal

When $x \to 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Equivalent Infinitesimal

Example

$$\lim_{x \to 0} \frac{\ln(1+x) + (1-\cos x) - x}{x^2}$$

$\ln(1+x)$ and $1-\cos x$ are not multiple factors or division factor. We cannot directly use equivalent infinitesimal substitution.

Explanation (not required)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\ldots$$

Equivalent Infinitesimal

Excercise

$$\lim_{x\to\infty} \frac{x^2+5}{2x+1} \sin \frac{3}{x}$$

$$\lim_{x\to\alpha} \frac{\ln x - \ln a}{x-a} (a>0)$$

Equivalent Infinitesimal

Excercise

$$\lim_{x\to\infty}\frac{x^2+5}{2x+1}\sin\frac{3}{x}$$

$$\lim_{x\to\alpha}\frac{\ln x-\ln a}{x-a}(a>0)$$

Solution

(1) Equivalent infinitesimal: $\sin\frac{3}{x}\sim\frac{3}{x}$.

(2) Hint: $\ln x-\ln a=\ln\frac{x}{a}=\ln(1+\frac{x-a}{a})\sim\frac{x-a}{a}$.

Logarithmic Transformation

Example

$$\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$$

Solution

Let $y=\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$, then $\ln y=\lim_{x\to\infty}\dfrac{1}{x}\ln\ln x=\lim_{x\to\infty}\dfrac{1}{x}\ln\ln x=0$. So $y=e^0=1$.

This may not be a very good example. You will learn how to calculate the limit $\lim_{x\to\infty}\dfrac{1}{x}\ln\ln x$ later in this cource, using L'Hospital's rule.

Logarithmic Transformation

Excercise

(1) Calculate $\lim_{x\to 0^+}[\frac{(1+x)^{\frac{1}{x}}}{e}]^a$, where a is a constant number.

(2) Given that $\ln(1+x) \sim x - \frac{x^2}{2}$, calculate the limit

$$\lim_{x\to 0^+}[\frac{(1+x)^{\frac{1}{x}}}{e}]^{\frac{1}{x}}$$

Solution

(1) $\lim_{x\to 0^+} \frac{(1+x)^{\frac{1}{x}}}{e} = 1$, so the result is also 1^a=1.

(2) let $y = \lim_{x\to 0^+}[\frac{(1+x)^{\frac{1}{x}}}{e}]^{\frac{1}{x}}$, then $\ln y = \lim_{x\to 0^+}[\frac{1}{x}\ln\frac{(1+x)^{\frac{1}{x}}}{e}] =$ $\lim_{x\to 0^+}\frac{1}{x}[\frac{1}{x}\ln(1+x)-1] = \lim_{x\to 0^+}\frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}$. So $y = \frac{1}{\sqrt{e}}$.

A warm reminder: do not forget that you have taken the logarithm of the final result.

Logarithmic Transformation

A Useful Formula

The Logarithmic Transformation formula $\lim f(x)^{g(x)} = e^{\lim[g(x)\ln f(x)]}$ is coming from the equation $x = e^{\ln x}$.

Exercise

Calculate the limit $\lim_{x\to 0} x^{-3}[(\frac{2+\cos x}{3})^x - 1]$.

Logarithmic Transformation

Exercise

Calculate the limit $\lim_{x\to 0} x^{-3}[(\frac{2+\cos x}{3})^x - 1]$.

Solution

$LHS = \lim_{x\to 0} \dfrac{e^{x\ln\frac{2+\cos x}{3}} - 1}{x^3} = \lim x \to 0 \dfrac{\ln\frac{2+\cos x}{3}}{x^2} =$
$\lim x \to 0 \dfrac{\ln(1+\frac{\cos x - 1}{3})}{x^2} = \lim x \to 0 \dfrac{\cos x - 1}{3x^2} = -\dfrac{1}{6}$.

Squeeze Theorem

Theorem

If there exist a positive number ρ such that when $x \in O(x_0, \rho)$, we have

$$g(x) \leq f(x) \leq h(x)$$

And $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = A$, then $\lim_{x \to x_0} h(x) = A$.

The proof is not required and you can easily understand it by drawing graphs.

Squeeze Theorem

Example

Suppose n is a positive integer. Show that $\lim_{n\to\infty}(\sum_{i=1}^{n}\dfrac{n}{n^2+i\pi})=1$.

Squeeze Theorem

Example

Suppose n is a positive integer. Show that $\lim_{n\to\infty}(\sum_{i=1}^{n} \frac{n}{n^2+i\pi}) = 1$.

Solution

Squeeze theorem. $\frac{n^2}{n^2+n\pi} \leq \sum_{i=1}^{n} \frac{n}{n^2+i\pi} \leq \frac{n^2}{n^2+\pi}$

Squeeze Theorem

A Challenging Problem

Prove that $\lim_{n\to\infty}\prod_{i=1}^{\infty}(1+\frac{i}{n^2})=\sqrt{e}$. (hint: $\log(1+x)>x-\frac{x^2}{2}$)

I will upload the solution on Canvas in the same folder.

Definition of Continuity

Definition

A function f is continuous at a number a if

$$\lim_{x \to a} = f(a)$$

This definition actually implicitly requires three things:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Definition of Continuity

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} = f(a)$$

A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} = f(a)$$

Definition of Continuity

A function is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Theorem 1: Four Fundamental Algorithms

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\dfrac{f}{g}$ (if $g(a) \neq 0$)

Theorem 2

Theorem

All elementary functions are continuous in their domain.

The following types of functions are continuous at every number in their domain(s):

1. polynomials
2. rational functions
3. root functions
4. (inverse) trigonometric functions
5. exponential functions
6. logarithmic functions

Theorem 3: Continuity of Composite Functions

Theorem

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

Theorem

If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

"A continuous function of a continuous function is a continuous function."

Theorem 5: Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Note that the value N can be taken on once or more than once.

Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$, then it's a bounded function on $[a, b]$, which means there exist two number M and m such that

$$m \leq f(x) \leq M$$

Theorem

Further, if $f(x)$ is continuous on the closed interval $[a, b]$, then it has a maximum value and a minimum value on $[a, b]$. This means there exist two number $\xi, \eta [a, b]$ on such that for any $x \in [a, b]$ we have

$$f(\xi) \leq f(x) \leq f(\eta)$$

Theorem 7: Intermediate Value Theorem

We first introduce Zero Existence Theorem.

Zero Existence Theorem

If $f(x)$ is s continuous on the closed interval $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists a value $\xi \in (a, b)$ such that

$$f(\xi) = 0$$

Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that

$$f(c) = N$$

Can you use Zero Existence Theorem to prove Intermediate Value Theorem?

Types of Discontinuities

1. Removable Discontinuity
   The left limit and right limit exist and both are equal, but $f(x)$ is not defined on $x = a$.
2. Jump Discontinuity
   The left limit and right limit exist but are not equal.
3. Infinite Discontinuity
   The left limit or right limit is ∞ (not exist)
4. Oscillation discontinuity
   The function is undefined at this point, and as the independent variable tends to this point, the function changes infinitely many times.

Exercise 1

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$$\sqrt[3]{x} = 1 - x, \quad (0,\ 1)$$

Exercise 2

Let $f(x) = \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. What kind of discontinuity is $x = 0$?

Exercise 3

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x<2 \\ ax^2-bx+3 & 2\leq x<3 \\ 2x-a-b & x\leq 3 \end{cases} \tag{1}$$

Exercise 4

Find a and b that make $f(x) = \lim_{n\to\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n}+1}$ continuous on $(-\infty, \infty)$.

Exercise 5

There are two functions:
$f(x)$ is continuous on $(-\infty, \infty)$, and $f(x) \neq 0$.
$\varphi(x)$ is defined on $(-\infty, \infty)$, but $\varphi(x)$ has discontinuity.

Judge whether the following four statements are correct:

1. $\varphi[f(x)]$ must have discountinuity.
2. $[\varphi(x)]^2$ must have discountinuity.
3. Whether $f[\varphi(x)]$ has discountinuity is uncertain.
4. $\dfrac{\varphi(x)}{f(x)}$ must have discountinuity.

Q&A

References

[1] Huang, Yucheng. VV156_RC2.pdf. 2021.

[2] Cai, Runze. Chapter02.pdf. 2021.