MATH 3310 – Midterm Experience

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David E Brown

Matt Bishop
Phil Brannon
Daniel Oliveros
Molly Robinson
Nicole Westfall

✠ Sub-Experience One: PIE, Recurrences, The Mad Mail Carrier, City Block Metric

(*No outside resources, please.*) Suppose Sue is a Mail Carrier who is crazy. He likes to ensure that none of the n houses on his delivery route get the mail they are supposed to. Your goal, should you choose to accept it, for this sub-experience is to determine the number of ways Sue can deliver mail so that no one gets their mail in two ways. Also, to assess the validity of Sue defending his actions as an accident, determine the probability that a random distribution of the mail results in the circumstance where no one gets the correct mail; that is, compute the limit as n grows without bound of $\frac{D_n}{n!}$, where D_n is the number of ways Sue can distribute mail in his twisted anti-social reprehensible way (and $n!$ is the total number of ways to distribute the mail).

Assume that every house gets one packet of mail each and that all mail is delivered somehow; that is, Sue is effectively a bijection from the set of n packets of mail to the set of n houses.

1.1. Prove the recurrence $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$, for $n \geq 2$, and $D_0 = 1, D_1 = 0$.

1.2. Deduce, from the above recurrence $D_n = nD_{n-1} + (-1)^n$, for $n \geq 1$, and $D_0 = 1$.

1.3. Use the Principle of Inclusion-Exclusion to compute a closed formula for D_n.

1.4. Compute $\lim_{n\to\infty} \dfrac{D_n}{n!}$. This limit may be interpreted as the asymptotic probability of dispersing mail, say, via a cannon so that everyone gets a piece of mail, so that no one gets their correct mail.

Recall that $\left\{ {n \atop k} \right\}$ is the number of ways to partition an n-set into k blocks; equivalently, the number of distributions of people (with names and snowflake-like in their uniqueness) into rooms (seen-one-seen-em-all type rooms) so that no room is empty.

1.5. Please use the Principle of Inclusion-Exclusion to determine a closed formula for $\left\{ {n \atop k} \right\}$

Number of shortest paths using the *city-block metric*. Consider only the points of the Cartesian coordinate system with nonnegative integer coordinates. The distance between two points (x_1, y_1) and (x_2, y_2) using the *city block metric* is $|x_1 - x_2| + |y_1 - y_2|$.

1.6. For yuks, determine π in the city block metric. Remember that π is defined to be the ratio of the circumference to the diameter of a circle.

1.7. Count the number of shortest paths from $(0, 0)$ to (X, Y), where $X, Y \in \mathbb{N}$. An alternative perspective to this problem is to count paths from $(0, 0)$ to (X, Y) using steps of the form $R : (x, y) \mapsto (x + 1, y)$ or $U : (x, y) \mapsto (x, y + 1)$.

Problem

In order to get a deeper understanding of the ways that crazy Sue can deliver mail we are asked to tackle some of the different sides of this scenario. We will tackle this problem step-by-step, following the parts of the assignment outlined previously.

Problem 1.1 Prove the recurrence $\mathbf{D_n = (n-1)D_{n-1} + (n-1)D_{n-2}}$, for $\mathbf{n \geq 2}$, and $\mathbf{D_0 = 1, D_1 = 0}$

Combinatorial Proof:
To deliver n people the wrong mail, Sue must first deliver n-2 people the wrong mail and then n-1 people the wrong mail. But to deliver n-2 people the wrong mail, Sue must start by giving 1 person the wrong mail and then there are n-1 ways to deliver the rest of the mail so that no one gets their own. The same goes for n-1 people. Therefore, $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ is the recurrence relation for distributing n people the wrong mail.

Problem 1.2 Deduce, from the above recurrence $\mathbf{D_n = nD_{n-1} + (-1)^n}$, for $\mathbf{n \geq 1}$, and $\mathbf{D_0 = 1}$.

$\rightarrow$ Expanding out the original recurrence,

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2} = nD_{n-1} - D_{n-1} + nD_{n-2} - D_{n-2}$$

$\rightarrow$ For the original recurrence to equal this new form of the recurrence given in the prompt for 1.2,

$$-D_{n-1} + nD_{n-2} - D_{n-2} = (-1)^n$$

$\rightarrow$ Using a table of recurrence values to deduce from the above recurrence the new recurrence,

n	original recurrence : D_n	nD_{n-1}	$nD_{n-1} + (-1)^n$
0	1	0	$0 + 1 = 1 = D_0$
1	0	1	$1 + (-1) = 0 = D_1$
2	1	0	$0 + 1 = 1 = D_2$
3	2	3	$3 + (-1) = 2 = D_3$
4	9	8	$8 + 1 = 9 = D_4$
5	44	45	$45 + (-1) = 44 = D_5$
6	265	264	$264 + 1 = 265 = D_6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore from the original recurrence, $D_n = nD_{n-1} + (-1)^n$ can be deduced.

Problem 1.3 Use the Principle of Inclusion-Exclusion to compute a closed formula for $\mathbf{D_n}$.

$$\mathbf{PIE:}\ |\bar{\mathbf{A}}_\mathbf{i} \cap ... \cap \bar{\mathbf{A}}_\mathbf{n}| = |\mathcal{U}| - \sum_{\mathbf{i>0}} |\mathbf{A_i}| + \sum_{\mathbf{i>j}} |\mathbf{A_i} \cap \mathbf{A_j}| - ... + (-1)^\mathbf{n} |\mathbf{A_i} \cap ... \cap \mathbf{A_n}|$$

$\rightarrow |\bar{A_i} \cap ... \cap \bar{A_n}| :=$ number of ways to distribute mail such that no one gets the correct mail.

$\rightarrow \mathcal{U} :=$ all possible ways to distribute mail

$\rightarrow |\mathcal{U}| = n!$

$\rightarrow A_i :=$ possible distributions such that house i gets the correct mail

$\rightarrow |A_i| = (n-1)!$

$\rightarrow A_i \cap A_j :=$ possible distributions such that house i and j gets the correct mail

$\rightarrow |A_i \cap A_j| = (n-2)!$

$\vdots$

$\rightarrow A_i \cap ... \cap A_n :=$ possible distributions such that house every house gets the correct mail $= D_n$

$\rightarrow |A_i \cap ... \cap A_n| = (n-n)! = 0! = 1$

Now applying the principle of inclusion-exclusion

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = n! - \sum_{i>0}(n-1)! + \sum_{i>j}(n-2)! - \sum_{i>j>k}(n-3)! + ... + (-1)^n 0!$$

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = \binom{n}{0}n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + ... + \binom{n}{n}(-1)^n$$

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)!$$

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = \sum_{i=0}^{n}(-1)^i\left[\frac{n!}{i!(n-i)!}\right](n-i)!$$

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = \sum_{i=0}^{n}(-1)^i\frac{n!}{i!}$$

$$|\bar{A}_i \cap ... \cap \bar{A}_n| = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

Therefore, by the Principle of Inclusion- Exclusion, $D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$

Problem 1.4 Compute $\lim_{n\to\infty}\frac{\mathbf{D_n}}{\mathbf{n!}}$

$$D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

$$\frac{D_n}{n!} = \frac{n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}}{n!}$$

$$\frac{D_n}{n!} = \sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

$$\lim_{n\to\infty}\frac{D_n}{n!} = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-1)^i}{i!} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}$$

Note: $e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!} \Rightarrow e^{-1} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}$

$$\lim_{n\to\infty}\frac{D_n}{n!} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!} = e^{-1} = \frac{1}{e}$$

Therefore, $\lim_{n\to\infty}\frac{D_n}{n!} = \frac{1}{e}$

Problem 1.5 Please use the Principle of Inclusion-Exclusion to determine a closed formula for $\left\{ {n \atop k} \right\}$

$$\textbf{PIE: } |\bar{A}_i \cap ... \cap \bar{A}_n| = |\mathcal{U}| - \sum_{i>0} |A_i| + \sum_{i>j} |A_i \cap A_j| - ... + (-1)^n |A_i \cap ... \cap A_n|$$

$\rightarrow |\bar{A}_i \cap ... \cap \bar{A}_n| :=$ the number of ways to partition an n-set into k - block so no block is empty $= \left\{ {n \atop k} \right\}$

$\rightarrow \mathcal{U} :=$ all possible ways to distribute a labeled n-set into k, unlabled, blocks

$\rightarrow |\mathcal{U}| = \frac{k^n}{k!}$

$\rightarrow A_i :=$ all possibles distributions such that block i is empty

$\rightarrow |A_i| = \frac{(k-1)^n}{k!}$

$\rightarrow A_i \cap A_j :=$ all possibles distributions such that block i and j is empty

$\rightarrow |A_i \cap A_j| = \frac{(k-2)^n}{k!}$

$\vdots$

$\rightarrow A_i \cap ... \cap A_k :=$ all possibles distributions such that all block are empty

$\rightarrow |A_i \cap ... \cap A_k| = \frac{(k-k)^n}{k!} = 0$

Now applying the Principle of Inclusion-Exclusion

$$|\bar{A}_i \cap ... \cap \bar{A}_k| = \frac{k^n}{k!} - \sum_{i>0} \frac{(k-1)^n}{k!} + \sum_{i>j} \frac{(k-2)^n}{k!} - \sum_{i>j>k} \frac{(k-3)^n}{k!} + ... + (-1)^k \cdot 0$$

$$|\bar{A}_i \cap ... \cap \bar{A}_k| = \binom{k}{0} \frac{k^n}{k!} - \binom{k}{1} \frac{(k-1)^n}{k!} + \binom{k}{2} \frac{(k-2)^n}{k!} - \binom{k}{3} \frac{(k-3)^n}{k!} + ... + 0$$

$$|\bar{A}_i \cap ... \cap \bar{A}_k| = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

Therefore by the Principle of Inclusion- Exclusion, $\left\{ {n \atop k} \right\} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$

Problem 1.6 determine π in the city block metric. Remember that π is defined to be the ratio of the circumference to the diameter of a circle.

$$\pi = \frac{circumference}{diameter} \text{ such that } circuference = 2\pi r \text{ and } diameter = 2r$$

$\rightarrow$ In the city block metric, a circle is a square oriented at a 45 degree angle. The center of such square is the point (x,y). The radius of this "circle" will be the y component. Multiplying the radius by 2 gives the diameter. The circumference will be the perimeter of the square. Each side of the square will be of length, 2y, because the distance between two points in the city block metric is given by $|x_1 - x_2| + |y_1 - y_2|$. Making the circumference 8 times the y component because the perimeter of a square is 4 times the length a side.

$\rightarrow$ Therefore π in the city block metric is,

$$\pi = \frac{8y}{2y} = 4$$

Problem 1.7 Count the number of shortest paths from $(0,0)$ to (X,Y), where $X,Y \in \mathbb{N}$
$\rightarrow$ Starting with some examples to establish a pattern,

The number of shortest paths from $(0,0) \rightarrow (x,y)$:

$$(0,0) \rightarrow (1,1) \Rightarrow 2 \text{ paths } = \binom{2}{1}$$

$$(0,0) \rightarrow (2,2) \Rightarrow 6 \text{ paths } = \binom{4}{2}$$

$$(0,0) \rightarrow (3,2) \Rightarrow 10 \text{ paths } = \binom{5}{2} \text{ or } \binom{5}{3}$$

Consider: $\binom{x+y}{x} = \binom{x+y}{y}$

$\rightarrow$ This binomial coefficient is comparable to the Vandermonde Inequality that we have seen earlier in the semester this is,

$$\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}$$

$\rightarrow$ For this problem, $r = x$, $s = y$ and $n = x$ or y. This gives

$$\sum_{i=0}^{x} \binom{x}{i}\binom{y}{x-i} = \binom{x+y}{x}$$

Combinatorial Proof:

Suppose there are x steps horizontally and y steps vertically and you want to the shortest path between the origin and the point (x,y) In the city block metric the distance from the origin to that point is $x+y$ and you choose condition the path on the x component, there are $\binom{x+y}{x}$ such paths. But there is another way to count this. To pick the path, you can pick i out of x horizontal steps, and $x-i$ steps in the vertical direction. But you can choose these groups in different ways. You can pick 0 horizontal steps and x steps in the y direction, or you can pick 1 horizontal step and $x-1$ vertical steps or so on and so on. This way there are $\sum_{i=0}^{x} \binom{x}{i}\binom{y}{x-i}$ ways to pick your path.

Therefore, there are $\binom{x+y}{x}$ shortest paths from the origin to point (X,Y)

✠ Sub-Experience Two: Fun and Games

The Game

Two players, A and B, alternately select an edge on the grid graph shown below and color it red. The loser of the game is the player who is forced to select an edge that creates a red C_4 — a red cycle on 4 vertices.

The Fun

Confirm or deny, with proof, whether player A (the first player) can always win if she employs a particular strategy for each move.

Problem

We are asked to confirm or deny with proof, whether player A, the first mover, can always win given a particular strategy. We begin by noting characteristics of the game board.

(1) The game board may be described as a network having 30 nodes, with the potential for 49 distinct edges, 24 of which are oriented vertically and 25 horizontally.

(2) Each players' move consists of selecting a single edge per turn from among 49 edges. The player who is forced to complete the first square, of which there are 20 possibilities, loses the game.

(3) With one exception, each edge of the game board has a symmetric "twin" if we reflect it's image both vertically and horizontally. The center edge is the only move that <u>does not</u> have a twin (see figure 1).

The strategy of player A is therefore, not to lose. Supposing that the best game is played by both players, the first player to lose will be the one that executes an odd move given 49 possibilities. Is there a strategy that player A, as the first mover, may reliably employ such that she guarantees her opponents moves are always unbalanced and ultimately lead to a loss?

Strategy

If we view the game as a choice between odd and even moves, and we consider that each move, albeit the center edge, has a symmetric reflective twin, then player A may force a losing move by player B by selecting the center edge on the first move and then selecting the reflective twin for each of player B's moves for the remainder of the game. The execution of this strategy is depicted in figures 2 through 5.

Proof

Player A selects the center edge on the first turn. By doing so, she assures that the remaining moves on the board all have a reflective twin. By selecting the reflective twin of each move made by player B, player A assures that she will have a move in each subsequent turn, until the options run out. Moreover, we know that since the game board presents options for 20 squares, when each of the 20 squares has 3 sides completed out of 4 we are left with an even turn which also happens to be a losing turn as it will complete the 4th side for at least one square.

(20 boxes X 4 sides) - (20 boxes X 3 sides) = 80 - 60 = 20 sides

Our proof for problem 2 is complete.

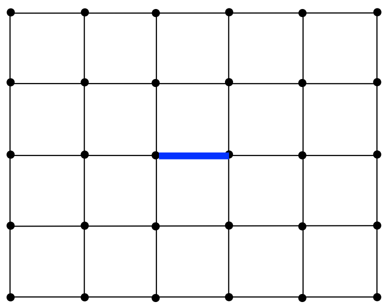


figure 1

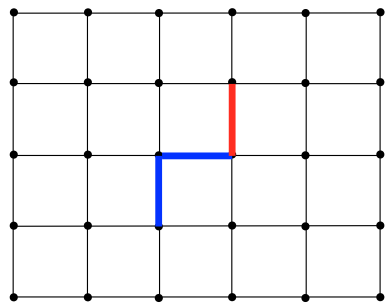


figure 2

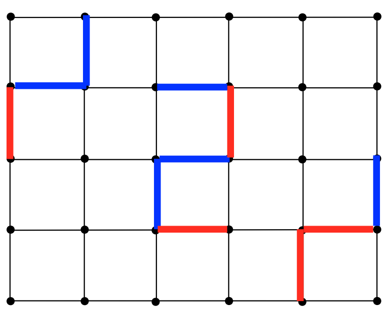


figure 3

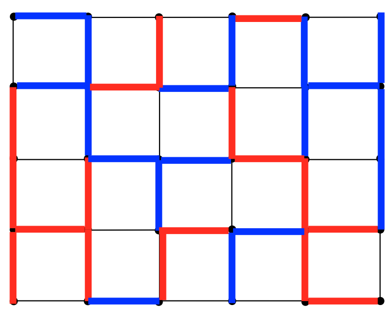
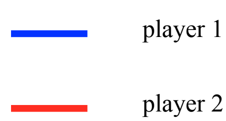


figure 4: game point, turn = 35
no moves for red



✠ **Sub-Experience Three: The Master Table of Distributions.**

Consider functions $f : A \longrightarrow B$, where A is a finite set with n elements and B is a finite set with x elements. The table below has 12 entries for each of the possibilities corresponding to various properties f, A, and B have. The function f may be injective, surjective, or unrestricted. The sets A and B may consist of elements that are distinguishable or indistinguishable (like socks versus shoes).

Notation: If A and B are sets, the notation B^A stands for the set of all functions that map A into B; $f : A \overset{1-1}{\longrightarrow} B$ denotes that f is a function from A into B that is injective; $f : A \overset{\text{onto}}{\longrightarrow} B$ denotes that f is a surjective function mapping A into B.

The goal is count the number of functions with the properties indicated by the row and column headings in the table. For example, in entry number 5 should be the number of functions that are injective that map n indistinguishable objects to x distinguishable objects. In Math-porn, this is

$$\left|\left\{f \in B^A : |A| = n, B = \{b_1, b_2, \ldots, b_x\}, f : A \overset{1-1}{\longrightarrow} B\right\}\right|.$$

While entry 12 should house the number

$$\left|\left\{f \in B^A : |A| = n, |B| = x, f : A \overset{\text{onto}}{\longrightarrow} B\right\}\right|.$$

A	B	unrestricted	injective	onto
distinguishable	distinguishable	1.	2.	3.
indistinguishable	distinguishable	4.	5.	6.
distinguishable	indistinguishable	7.	8.	9.
indistinguishable	indistinguishable	10.	11.	12.

Please determine, with proof, the entries in the table.

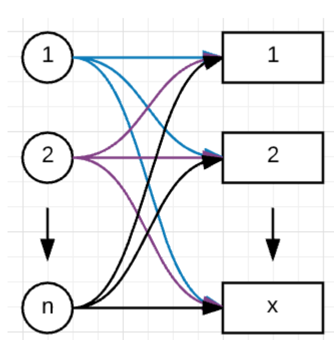
Problem

For this sub-experience, we are asked to find formulas for how to group up elements with each other based on certain restrictions. These restrictions are centered around the ways the elements themselves are cataloged, and around the ways they can be grouped with one another.

Solution

For all of these different solutions, we hand-grouped varying numbers of balls (representing set A) in more and more "boxes" (which represented set B). By looking at patterns in the number of ways these balls can be put in the containers, we found formulas that matched these behaviors. Here are our results and methods:

1. **Distinguishable balls, Distinguishable boxes, Unrestricted:**

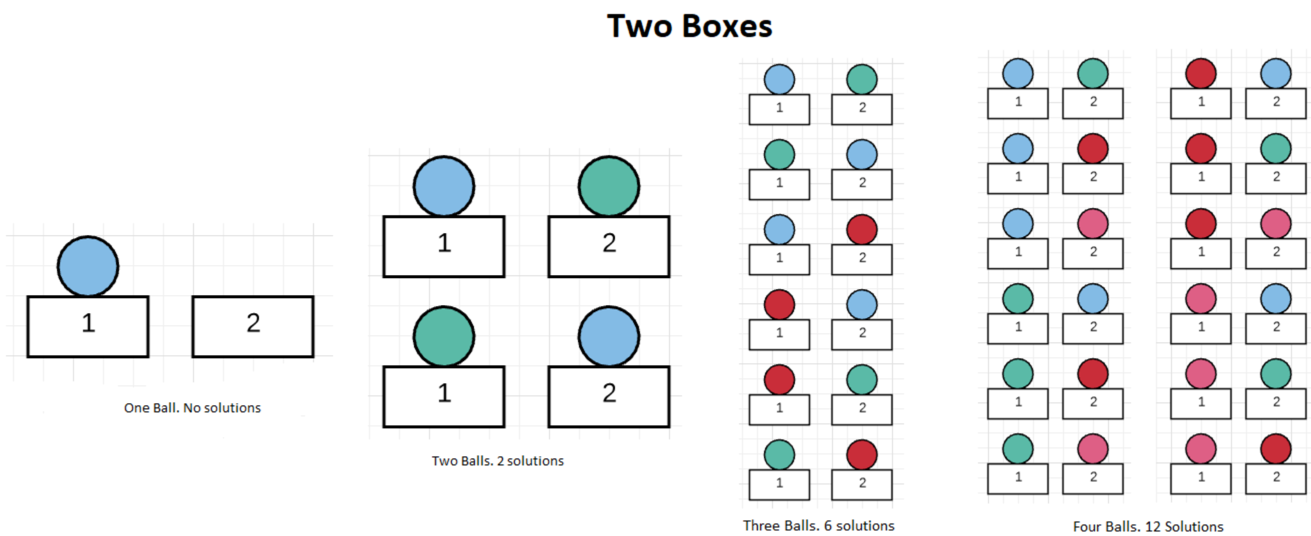


From charting the way the elements in A can be placed within elements in B, we can determine that whenever an element from A is grouped with elements in B there are always x ways of grouping them. This gives us the formula:

$$f(x, n) = \overbrace{x * x * x * \ldots * x}^{n}$$
$$f(x, n) = x^n$$

So, we can determine entry 1 to be x^n, where x is the number of "balls", and n is the number of "boxes"

2. Distinguishable balls, Distinguishable boxes, 1 to 1:
We begin by using two boxes and an increasing number of balls:



One Ball. No solutions
Two Balls. 2 solutions
Three Balls. 6 solutions
Four Balls. 12 Solutions

From these results we can determine various things. First, once we see the results for four balls, we can tell the solution is not equal to a factorial, as would be hinted at by the prior tests. Instead, it looks like the solution is more of a "bounded" factorial, where we have x ways of placing balls in the first box, and from there we have $x - 1$ ways of placing them in the second one. From here, we can extrapolate from these results that, based on the observation that the number of ways we can place our x balls is bounded bounded by our n number of boxes, determine that our formula is: $x^{\underline{n}}$

3. Distinguishable balls, Distinguishable boxes, Onto:
For this case, and plenty of the *onto* ones, we determined we could base ourselves off of the partition relation. So, when we normally partition an x-set of indistinguishable into n-sized containers, we use the notation $\left\{ {x \atop n} \right\}$. We can base ourselves off this, and by determining that we are performing the same operation with the one difference being that all of our elements are now distinguishable, we can count this slightly different approach by multiplying by $x!$. The resulting function $f(n, x) = \left\{ {x \atop n} \right\} x!$ counts the number of entries in this case.

4. Indistinguishable balls, Distinguishable boxes, Unrestricted:
Say you are putting an n number of balls into x boxes. Each box can have all or none of the balls. As Dr. Brown showed in class, one way to construct a multiset is with stars and bars. The stars in this example would be the balls, and the bars would be the number of boxes. This works because the boxes are the base set, being distinguished elements, making it possible to show the presence and how many times it is in the multiset. By rearranging the bars, all the possible ways and how many times an element in the multiset could be shown. Thus this entry would be $\left(\!\binom{x}{n}\!\right)$.

5. Indistinguishable balls, Distinguishable boxes, 1-1:
Suppose you are putting n number of balls into an x number of boxes with each box having up to 1 ball. If

$n \geq x$, then there are no ways to put all the balls into boxes, because there will be balls left over. And if $n \leq x$, there would be $\binom{x}{n}$ ways to put the balls into boxes. For example, if $n = 1$, there would be x number of ways that that ball could be placed into the boxes, the boxes being distinguishable. Also if $n = x$ there would be one way to put all the balls into all the boxes. So entry 5 is $\binom{x}{n}$.

6. Indistinguishable balls, Distinguishable boxes, Onto:

Like entry 4, this can be proved by using the starts and bars method shown by Dr. Brown. With the bars being the base set of a distinguishable boxes, and the stars being a number of b indistinguishable balls. With this mapping being surjective, every box has to have at least one ball in it. This means that the solution has to check if $a > b$. If there are the same number of balls and boxes, then there is one multiset where all the elements of a are present. Another example is if there is one more ball than the number of boxes, there are a total of a multisets where each element of a is represented, and where the extra ball is put into a different box each multiset, increasing that box's multiplicity in the multiset. Checking to see if there are enough balls can be done by having the size of the multiset be $b - a$, so $\left(\!\binom{a}{b-a}\!\right)$. For example, when $b = a$,

$$\left(\!\!\binom{a}{a-a}\!\!\right) = \left(\!\!\binom{a}{0}\!\!\right) = 1$$

This corresponds to when there is the same number of balls and boxes, there is one multiset which all the boxes are present. And when $b = a + 1$,

$$\left(\!\!\binom{a}{a+1-a}\!\!\right) = \left(\!\!\binom{a}{1}\!\!\right) = a$$

This corresponds to when there is one more ball than the number of boxes. So this entry would be $\left(\!\binom{a}{b-a}\!\right)$.

7. Distinguishable balls, Indistinguishable boxes, Unrestricted:

Through observation and value checking, we determined that the amount of ways of putting n labeled balls into x unlabeled boxes in an unrestricted manner can be calculated using the roof function:

$$\left\lceil \frac{x^n}{x!} \right\rceil$$

8. Distinguishable balls, Indistinguishable boxes, 1-1:

For putting n labeled balls into x unlabeled boxes injectively, if $n \leq x$, then you can put the balls into boxes one way. But if $n \geq x$, there is no way to put all the balls into their own box. With $f(n, x)$ being the number of ways to count distinguishable set mapped 1-1 with an indistinguishable set, $f(n, x)$ would be

$$f(n, x) = \begin{cases} 1, & \text{if } n \leq x \\ 0, & \text{otherwise} \end{cases}$$

9. Distinguishable balls, Indistinguishable boxes, onto:

The number of functions that are injective that map n distinguishable balls to x indistinguishable boxes is simply, $\left\{ {n \atop x} \right\}$. This expression counts the number of ways to partition a distinguishable n-set into x indistinguishable blocks. This is exactly what we need for this entry. The objective is to count the number of ways we can distribute n labeled balls into exactly x unlabeled boxes. Therefore the solution to this entry is,

$$\left\{ {n \atop x} \right\}$$

10. Indistinguishable balls, Indistinguishable boxes, unrestricted:

The number of functions that have no restriction and map n indistinguishable balls to x indistinguishable boxes is, $P_x(n)$. This expression partitions the n balls into at most x boxes. This is precisely the expression we need. We can put multiple balls into a box and we can leave boxes empty. This integer partition expression counts the number of functions that do exactly that. Therefore, the solution to this entry is,

$$P_x(n)$$

11. Indistinguishable balls, Indistinguishable boxes, 1-1:
The solution to this entry is the same as entry 8. If the number of balls k, is greater than the number of boxes y, there is no way to put the balls into boxes so there are no balls left over. If $k \leq y$ though, there will always be one way to put the balls into boxes. So with $g(k, y)$ being the solution to this entry, $g(k, y)$ is

$$g(k, y) = \begin{cases} 1, & \text{if } k \leq y \\ 0, & \text{otherwise} \end{cases}$$

12. Indistinguishable balls, Indistinguishable boxes, onto:
The number of functions that are surjective that map n indistinguishable balls to x indistinguishable boxes is, $P_{=x}(n)$. This expression counts the number of partitions of n indistinguishable balls into exactly x indistinguishable boxes. For these functions to be surjective, no box can be empty but there can be multiple balls in any given box. Therefore, this integer partition into exactly x boxes is the expression we need. Thus the solution to this entry is,

$$P_{=x}(n)$$

✠ Sub-Experience Four: Non-Standard Dice

(*No outside resources, please.*) A 6-sided die labeled with the integers $1, 2, 3, 4, 5, 6$ will be called a *standard die*. The goal for this part of the Midterm Experience is to determine all ways to label a pair of dice with positive integers so that the probabilities of rolling the usual sums $2, 3, \dots, 12$ are the same, but the labels are non-standard.

Step 1. Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, and explain why $(p(x))^2$ is the generating function for the probabilities of outcomes in rolling a pair of standard dice.

> **Note:** $(p(x))^2 = x^{12} + 2x^{11} + 3x^{10} + 4x^9 + 5x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2$

Each coefficient represents the number of ways possible the pair of dice can roll so that the sum of the two dice adds up to a number between 2 and 12. The sum of the two dice is represented by the exponent and $p(x)$ is squared because $p(x)^2$ represents the rolling of one die and then another. Thus $(p(x))^2$ is the generating function for the probabilities of the outcomes when rolling pair of standard dice.

Step 2. Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two lists of positive integers. Put $p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$ and $p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$. Explain why finding a_is and b_is such that $p_A(x)p_B(x) = (p(x))^2$ is relevant to this part of the Experience.

$p_A(x)$ is the general form of a general generating function for a non-standard, 6 sided die. $p_B(x)$ is also a the general form for a generating functions for a non-standard 6 sided die, but with different positive integers on its faces. If we multiply $p_A(x)$ and $p_B(x)$ together, it will yield an equation with 36 terms, in the form $x^{a_1+b_1} + x^{a_2+b_1} + x^{a_1+b_2} + x^{a_1+b_3} + x^{a_2+b_2} + ... + x^{a_4+a_6} + x^{a_5+b_5} + x^{a_5+b_6} + x^{a_6+b_5} + x^{a_6+b_6}$. This gives us the same probability density function of the two standard die, but now for two non-standard die. If we look at the sum of the subscripts of the exponents, there is one sum that adds up to two, two that adds up to three, three that add up to four and so on. The sum of the subscripts of the exponents represent a number between 2 and 12 and the amount of the same sum is the frequency distribution for that number.

Step 3. Factor $p(x)$ into irreducible polynomials and use this factorization to help solve for the a_is and b_is. Specifically, the factorization will force the form of $p_A(x)$ to be something like $p_1(x)^q p_2(x)^r p_3(x)^s p_4(x)^t$, where $0 \le q, r, s, t \le 2$ and $p_i(x)$, for $1 \le i \le 4$, is a factor of $p(x)$. In your solution to this step, you must motivate why you take this step.

We can factor $(p(x))^2$ into the irreducible polynomial $(x)^2(x+1)^2(X^2+x+1)(x^2-1+1)^2$. This is the factorization for a pair of standard die. We want different faces then what is on a standard die, so we need to factor $(p(x))^2$ using different combinations then the one above. But, to ensure we will get exactly six sides for our new dice, we must always have only one x in any of our possible factorizations. The table below is all the possible irreducible polynomials with the above constraint.

		p_1^q	p_2^r	p_3^s	p_4^t
1	$p_A(x)$	(x)			
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$	$(x^2-x+1)^2$
2	$p_A(x)$	(x)			(x^2-x+1)
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$	(x^2-x+1)
3	$p_A(x)$	(x)			$(x^2-x+1)^2$
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$	
4	$p_A(x)$	(x)		(x^2+x+1)	
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)	$(x^2-x+1)^2$
5	$p_A(x)$	(x)		(x^2+x+1)	(x^2-x+1)
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)	(x^2-x+1)
6	$p_A(x)$	(x)		(x^2+x+1)	$(x^2-x+1)^2$
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)	
7	$p_A(x)$	(x)	$(x+1)$		
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$	$(x^2-x+1)^2$
8	$p_A(x)$	(x)	$(x+1)$		(x^2-x+1)
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$	(x^2-x+1)
9	$p_A(x)$	(x)	$(x+1)$		$(x^2-x+1)^2$
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$	
10	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)	
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)	$(x^2-x+1)^2$
11	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)	$(x^2-x+1)^2$
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)	
12	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)	(x^2-x+1)
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)	(x^2-x+1)

Step 4. Begin to reduce the possibilities for q, r, s, and t by using information from $p_A(1)$ and $p_A(0)$. Note that, on one hand $p_A(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6$ (since $a_i > 0$), and on the other hand we have $p_A(1) = p_1(1)^q p_2(1)^r p_3(1)^s p_4(1)^t$. Similarly, there are two ways to view $p_A(0)$.

Because we want two six sided dice, when we let $x = 1$, our irreducible polynomials should equal 6, and when we let $x = 0$, $p_A(0) = 0$ and $p_B(0) = 0$. The table below shows what each of our possible polynomials.

		p_1^q	p_2^r	p_3^s	p_4^t	x=0	x=1	
1	$p_A(x)$	(x)				0	1	
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$	$(x^2-x+1)^2$	0	36	
2	$p_A(x)$	(x)			(x^2-x+1)	0	1	
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$	(x^2-x+1)	0	36	
3	$p_A(x)$	(x)			$(x^2-x+1)^2$	0	1	
	$p_B(x)$	(x)	$(x+1)^2$	$(x^2+x+1)^2$		0	36	
4	$p_A(x)$	(x)		(x^2+x+1)		0	3	
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)	$(x^2-x+1)^2$	0	12	
5	$p_A(x)$	(x)		(x^2+x+1)	(x^2-x+1)	0	3	
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)	(x^2-x+1)	0	12	
6	$p_A(x)$	(x)		(x^2+x+1)	$(x^2-x+1)^2$	0	3	
	$p_B(x)$	(x)	$(x+1)^2$	(x^2+x+1)		0	12	
7	$p_A(x)$	(x)	$(x+1)$			0	2	
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$	$(x^2-x+1)^2$	0	18	
8	$p_A(x)$	(x)	$(x+1)$		(x^2-x+1)	0	2	
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$	(x^2-x+1)	0	18	
9	$p_A(x)$	(x)	$(x+1)$		$(x^2-x+1)^2$	0	2	
	$p_B(x)$	(x)	$(x+1)$	$(x^2+x+1)^2$		0	18	
10	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)		0	6	
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)	$(x^2-x+1)^2$	0	6	
11	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)	$(x^2-x+1)^2$	0	6	
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)		0	6	
12	$p_A(x)$	(x)	$(x+1)$	(x^2+x+1)	(x^2-x+1)	0	6	Standard die
	$p_B(x)$	(x)	$(x+1)$	(x^2+x+1)	(x^2-x+1)	0	6	

Step 5. List all possible ways to label a pair of dice so that the probabilities of obtaining the sums $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$, respectively. One such way will be the standard way. In your solution for this step, explain why you have proved that the labels you have found are the only possible ones that give the desired probabilities for roll-outcomes.

From Step 4 we are able to determine that the individual generating functions for $p_A(x)$ and $p_B(x)$ for the new, non-standard dice are $(x)(x+1)(x^2+x+1)$ and $(x)(x+1)(x^2+x+1)(x^2-x+1)^2$. When we expand each of the generating functions, $p_A(x)=x^4+2x^3+2x^2+x$ and $p_B(x)=x^8+x^6+x^5+x^4+x^3+x$ (or vice versa). The exponent represents the number on the new dies face and the coefficient is the recurrence of that number on a face. When we multiply $p_A(x)$ and $p_B(x)$ together, we get $x^{12}+2x^{11}+3x^{10}+4x^9+5x^8+6x^7+5x^6+4x^5+3x^4+2x^3+x^2$, which is equivalent to $(p(x))^2$, which means the probabilities are also the same. Therefore, one of our new dice will have faces of $1, 2, 2, 3, 3, 4$ and our other dice will have faces of $1, 3, 4, 5, 6, 8$, which is all possible ways to label a pair of dice so that the probabilities of obtaining the sums $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$, respectively.