Daniel Oliveros A02093272
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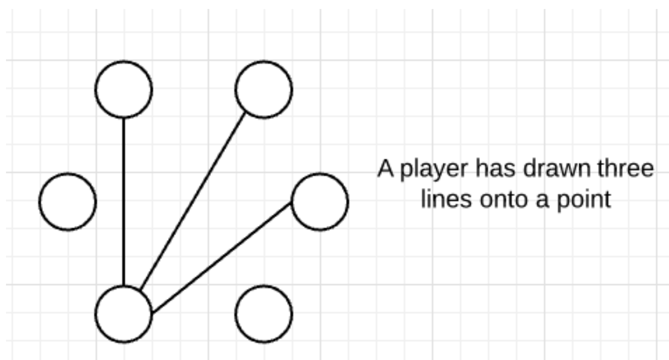
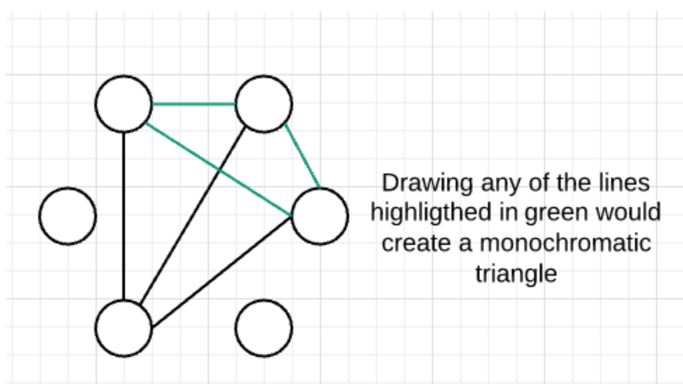
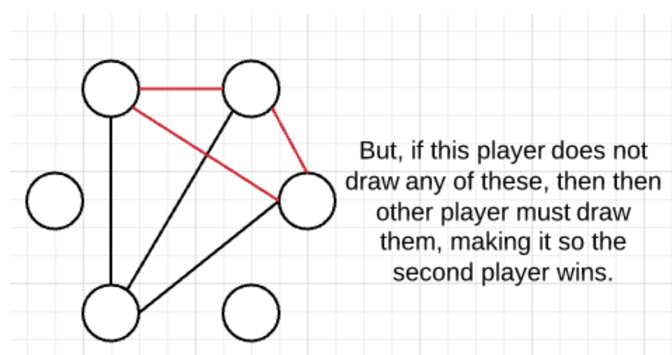
Homework 7

Problem #1
Prove that the following game must have a winner. Place 6 dots equally spaced on a circle on a piece of paper. Player A has a red marker and Player B has a blue marker. Players take turns connecting pairs of dots with line segments using their respective colored markers, only one line segment between any two dots (therefore at most 15 line segments will be drawn). The winner is the first to connect three dots mutually with their respective color, that is, the first to create a monochromatic "triangle" in their marker's color.

In order to show why this game must have a winner, we can look at one particular dot and the connections made towards it. There must be 5 lines connecting onto a dot, meaning that no matter what may happen, either player will end up drawing 3 or four lines that connect to this dot. The issue now is that any line that player draws between these linked dots will result on them making a monochromatic triangle. In the case where that player successfully avoids drawing these lines, if the other player is the one drawing them, then they will be the ones to have drawn the triangle. The following diagram describes this situation:

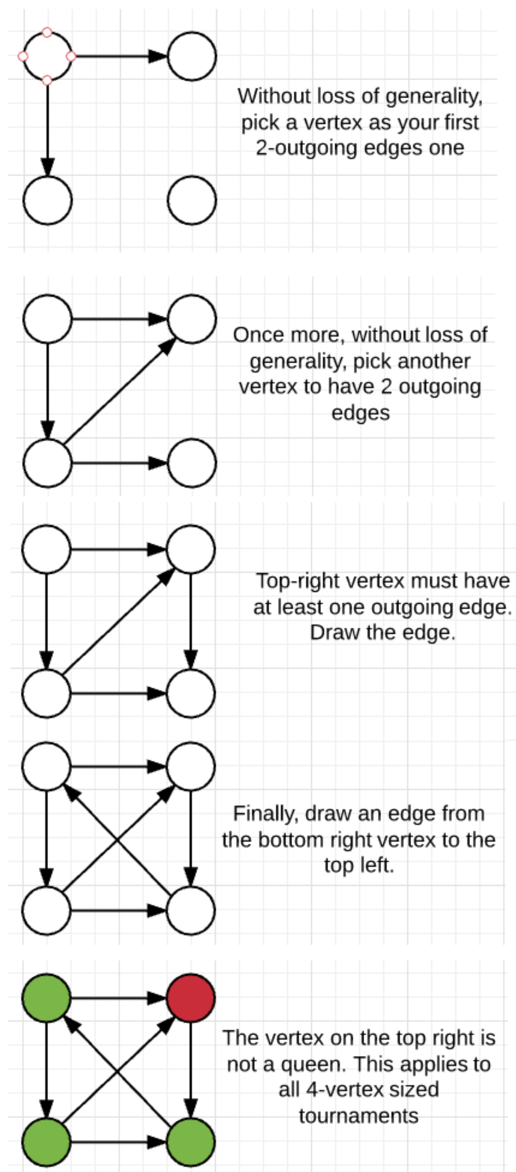






Problem #2
Please prove that no tournament with exactly 4 vertices is such that every vertex is a queen.

Taking advantage of a couple of theorems, we can determine some simple ground rules for this problem. First, no vertex can have 0 outgoing edges, this is because this node would not be a queen by default, so it would instantly be excluded from testing. The second rule, is that we can't have any emperors, because these would also make it impossible for any other vertex to be a queen.
Knowing that, and the fact that there are 6 edges to distribute between all 4 nodes, we can determine that two nodes must have 2 outgoing edges from them, and the other two nodes must have 1. So, in order to determine what would happen here, let's look at the following diagram:



This general proof applies to all 4-vertex tournaments. Since a node will always end up being pointed to from both 2-outgoing-edged vertices, then it will have to have an outgoing edge toward the other 1-outgoing-edged vertex, which, in turn, will point at one of the 2-outgoing-edged vertices.

Problem #3
Please prove that in any tournament, if a vertex is beaten, it is beaten by a queen.

For this proof, let us dive into the relationships of one vertex with all other vertices. For any vertex v, there are two sets of vertices it is related to: v^+, which encompasses all vertices beaten directly by v, and v^-, which encompasses all vertices that beat v directly.
Let us dive more deeply into the set v^- and the relations within it. If we choose to only analyze relations within this set, we can define it as a tournament of vertices in v^-. By following theorem **11.8.1**, we know that every tournament has at least one queen. So, for the set v^-, there must exist at least one vertex that beats all others either directly or indirectly. Remember, this vertex happens to beat our vertex v, which beats all vertices in v^+. This means that any queens in the set v^- are, by definition, queens for the whole tournament.
Or, in other words, whenever a vertex is beaten, it is beaten by at least one queen.