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Homework 1

Problem #1
Let R be the set of positive real numbers and define addition, denoted by $\oplus$, by $a \oplus b = ab$, for every $a, b \in R$, and define multiplication, denoted $\otimes$, by $a \otimes b$ by $a^{\log b}$.
Please prove or disprove $(R, \oplus, \otimes)$ is a field.

Solution: I determined that (R, $\oplus$, $\otimes$) is a field, I will prove it by showing it fulfills all the requirements of a field:

(-1) This field has closure under addition. This is due to the fact that R is comprised of all the positive Real numbers. Since the operation labeled as $\oplus$ can only output positive real values when both its inputs are positive real values, then clearly there is closure under addition.

(0) This field has closure under multiplication. This is due to the fact that the operation $a^{\log b}$ can only output positive real values when both a and b are positive real values.

(1) Addition is commutative. This is true because:

$$a \oplus b = ab = ba = b \oplus a$$

So, by using commutative multiplication between two Real numbers, we can extrapolate that our $\oplus$ operation follows the same principles.

(2) Addition is associative. Meaning $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

$$\begin{aligned} (a \oplus b) \oplus c = &\quad a \oplus (b \oplus c) \\ (ab) \oplus c = &\quad a \oplus (bc) \\ (ab)c = &\quad a(bc) \\ abc = &\quad abc \end{aligned}$$

(3) There exists and additive identity, 0_F, such that $0_F \oplus x = x \oplus 0_F = x$.

$$\begin{aligned} 0_F \oplus x = &\quad x \\ 0_F * x = &\quad x \\ 0_F = &\quad 1 \end{aligned}$$

(4) There is always an additive inverse; For every $x \in F$, there exists an object in F, we denote it by $-x$, for which $x \oplus (-x) = 0_F$

$$\begin{aligned} x \oplus (-x) = &\quad 0_F \\ x * (-x) = &\quad 1 \\ (-x) = &\quad \tfrac{1}{x} \end{aligned}$$

Therefore, the multiplicative inverse of x in this field is $\frac{1}{x}$.

(5) Multiplication is commutative. If $x, y \in F$, then $x \otimes y = y \otimes x$.

$$\begin{aligned} x \otimes y = &\quad y \otimes x \\ x^{\log y} = &\quad y^{\log x} \\ \log(x^{\log y}) = &\quad \log(y^{\log x}) \\ \log y \log x = &\quad \log x \log y \end{aligned}$$

(6) Multiplication is associative.

$$\begin{aligned}(x \otimes y) \otimes z = &\quad x \otimes (y \otimes z)\\ (x^{\log y}) \otimes z = &\quad x \otimes (y^{\log z})\\ (x^{\log y})^{\log z} = &\quad x^{\log(y^{\log z})}\\ x^{\log y \log z} = &\quad x^{\log z \log y}\end{aligned}$$

(7) There exists a multiplicative identity, 1_F, such that $1_F \otimes x = x \otimes 1_F = x$.

$$\begin{aligned}1_F \otimes x = &\quad x \otimes 1_F &&= x\\ {1_F}^{\log x} = &\quad x^{\log 1_F} &&= x\\ \log({1_F}^{\log x}) = &\quad \log(x^{\log 1_F}) &&= \log(x)\\ \log 1_F \log x = &\quad \log x \log 1_F &&= \log x\\ \log 1_F = &\quad \log 1_F &&= 1\\ 1_F = &\quad 1_F &&= e\end{aligned}$$

(8) There exists a multiplicative inverse, x^{-1}, such that $x \otimes x^{-1} = x^{-1} \otimes x = 1_F$.

$$\begin{aligned}x \otimes x^{-1} = &\quad x^{-1} \otimes x &&= 1_F\\ x^{\log x^{-1}} = &\quad {x^{-1}}^{\log x} &&= e\\ \log(x^{\log x^{-1}}) = &\quad \log({x^{-1}}^{\log x}) &&= \log(e)\\ \log x^{-1} \log x = &\quad \log x \log x^{-1} &&= 1\\ \log x^{-1} = &\quad \log x^{-1} &&= \frac{1}{\log x}\\ e^{\log x^{-1}} = &\quad e^{\log x^{-1}} &&= e^{\frac{1}{\log x}}\\ x^{-1} = &\quad x^{-1} &&= \sqrt[\log x]{e}\end{aligned}$$

(9) Multiplication distributes over addition on the left and on the right, so $x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$ and $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$

$$\begin{aligned}x \otimes (y \oplus z) = &\quad x \otimes y \oplus x \otimes z\\ x \otimes (yz) = &\quad (x^{\log y}) \oplus (x^{\log z})\\ x^{\log yz} = &\quad x^{\log y} x^{\log z}\\ x^{\log yz} = &\quad x^{\log y + \log z}\\ x^{\log yz} = &\quad x^{\log yz}\end{aligned}$$

Also:

$$\begin{aligned}(x \oplus y) \otimes z = &\quad x \otimes z \oplus y \otimes z\\ (xy) \otimes z = &\quad (x^{\log z}) \oplus (y^{\log z})\\ xy^{\log z} = &\quad x^{\log z} y^{\log z}\\ xy^{\log z} = &\quad xy^{\log z}\end{aligned}$$

(10) The additive identity and multiplicative identity are distinct $1_F \neq 0_F$.

$$\begin{aligned}1_F \neq &\quad 0_F\\ e \neq &\quad 1\end{aligned}$$

By showing that $(R, \oplus, \otimes)$ meets all the previous properties, we have proven it to be a field.

Problem #2
Denote the set $\{0, 1, 2, 3\}$ by $\mathbb{Z}_4$, and define addition, denoted $+$, and multiplication, denoted by $\cdot$ or juxtaposition,via the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Please prove or disprove $(\mathbb{Z}_4, +, \cdot)$ is a field.

Solution: I determined that $(\mathbb{Z}_4, +, \cdot)$ is not a field. In order to disprove the claim that it is I will show the steps I took to determine it fails to meet some requirements that are needed:

1. I started by determining the value of 0_F, this felt like one of the easiest things to check quickly, and from analyzing the addition table, I determined 0_F to be equal to 0. This is fairly easy to deduce since any number that has 0 added to it in the table remains the same.

2. After this, I decided I'd check for 1_F. This also felt like an easy check, and it was. By checking all the rows I determined that, in this set, 1_F is equal to 1.

3. The next thing I decided to check for was the additive inverse. Once more, by inspecting the left table it was fairly easy to determine all numbers in the set have one. 1 and 3 are each other's additive inverses, and 2 is its own additive inverse.

4. Finally, I checked for the multiplicative inverse. From inspecting the table, I found that 2 has no multiplicative inverse. This is shown by observing that multiplying it by any other number in the set does not give 1 in any way.

So, since $(\mathbb{Z}_4, +, \cdot)$ does not meet the multiplicative inverse requirement, then I determined it is not a field.

Problem #3
Suppose x is a positive integer with n digits, say $x = d_1 d_2 d_3 \cdots d_n$. In other words, $d_i \in \{0, 1, 2, \ldots, 9\}$ for $1 \leq i \leq n$, but$d_1 \neq 0$. Please prove the following. Recall that, for $a, b \in \mathbb{Z}$, a is a **divisor** of b if $b = ak$, for some $k \in \mathbb{Z}$.

a) If 9 is a divisor of $d_1 + d_2 + \cdots + d_n$, then 9 is a divisor of x.

b) If $d_n = 0$ or $d_n = 5$, then 5 is a divisor of x.

Solution: In order to prove these points, I will decompose x into a sum of all digits multiplied by their respective powers of 10, performing slightly different analyses on this model will help prove both claims.

$$\begin{aligned} x = &\quad d_1 d_2 ... d_n \\ x = &\quad d_1(10^{n-1}) + d_2(10^{n-2}) + ... + d_n(10^{n-n}) \\ x = &\quad d_1(10^{n-1}) + d_2(10^{n-2}) + ... + d_n \end{aligned}$$

a) By pulling out one instance of each digit from their original powers of 10, I can learn two things about the resulting sets of numbers. All of the powers of 10 from which a single copy of their respective digits has been removed are multiples of 9, and the remaining sum of digits on the right side may or may not be. However, only when the sum of digits is a multiple of 9, can the whole right side be a multiple of 9, which entails the left side also being a multiple of 9. More descriptively:

$$
\begin{aligned}
x \quad & = d_1(10^{n-1}) + d_2(10^{n-2}) + ... + d_n \\
d_1d_2...d_n \quad & = d_1(10^{n-1} - 1 + 1) + d_2(10^{n-2} - 1 + 1) + ... + d_n \\
d_1d_2...d_n \quad & = d_1(10^{n-1} - 1) + d_1 + d_2(10^{n-2} - 1) + d_2 + ... + d_n \\
d_1d_2...d_n \quad & = d_1(10^{n-1} - 1) + d_2(10^{n-2} - 1) + ... + d_{n-1}(9) + d_1 + d_2 + ... + d_n \\
A \quad & = d_1(10^{n-1} - 1) + d_2(10^{n-2} - 1) + ... + d_{n-1}(9) \\
B \quad & = d_1 + d_2 + ... + d_n \\
x = d_1d_2...d_n \quad & = A + B
\end{aligned}
$$

To show how every number belonging to A must be a multiple of A we can separate the values within it even further. For the general number in A we can see:

$$
\begin{gathered}
d_i(10^{n-i} - 1) \\
d_i(9(10^{n-i-1}) + 9(10^{n-i-2}) + ... + 9) \\
d_i(9((10^{n-i-1}) + (10^{n-i-2}) + ... + 1)) \\
9d_i((10^{n-i-1}) + (10^{n-i-2}) + ... + 1)
\end{gathered}
$$

To reiterate, A in this scenario is always a multiple of 9 since each of the numbers that add up to it must a multiple of 9. And only when B, which is the sum of all of x's digits, is a multiple of 9, can we say that x itself is also a multiple of 9.

b) Every digit except d_n is multiplied by some non-unitary power of 10, which means that they are also multiples of 5 by definition. So in reality, the only number that needs to be a multiple of 5 for x to be a multiple of 5 is d_n. More descriptively:

$$
\begin{aligned}
x = & \quad d_1(10^{n-1}) + d_2(10^{n-2}) + ... + d_n \\
x = & \quad d_1((5*2)^{n-1}) + d_2((5*2)^{n-2}) + ... + d_n \\
A = & \quad d_1((5*2)^{n-1}) + d_2((5*2)^{n-2}) + ... \\
B = & \quad d_n \\
x = & \quad A + B
\end{aligned}
$$

To reiterate, A in this scenario is always a multiple of 5 since it's made up of a sum of numbers that are necessarily multiples of 5. So, in order for x to be a multiple of 5, B must be either 5 or 0.

Problem # 4
Please prove or disprove: *If $n \in \mathbb{Z}^+$, then $n^2 + n + 41$ is prime.*

Solution: This can be easily disproven by noticing that by substituting 41 into n, the resulting number will not be prime. To elaborate:

$$
\begin{aligned}
x = & \quad n^2 + n + 41 \\
x = & \quad (41)^2 + (41) + 41 \\
x = & \quad (41 * 41) + (2 * 41) \\
x = & \quad (43 * 41)
\end{aligned}
$$

So, the number resulting from substituting n with 41 is a multiple of both 41 and 43, making it a non-prime number.