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Homework 4

Problem #1
Suppose $r, s, n \in \mathbb{N}$. Please prove the identity (1) in two ways: combinatorially, and algebraically.

$$\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n} \tag{1}$$

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- **Combinatorially:** To prove this identity combinatorially, I will show how both sides of the equation are counting the exact same thing. In the case where the sets r and s represent, respectively, Olympic **R**unners and **S**wimmers, we can determine in how many ways we can make a team of n athletes by using:

  The Left Hand Side: Which is adding up all of the different ways our n-sized crew could be picked. Starting with the case where we have no **R**unners and only **S**wimmers, to the case where we only have **R**unners and no **S**wimmers, and every case in between.

  The Right Hand Side: Which simply adds up both sets and calculates the number of ways an n-set of athletes could be picked from the resulting set of **R**unners **AND S**wimmers.

- **Algebraically:** In order to prove this algebraically, I first wrapped it up in a way which allowed me to use the binomial theorem on the right side. Note that:

  $$(1+x)^n = \sum_{i\geq 0} \binom{n}{i} x^n.$$

  From here, I manipulated my values in a way that would help me display this equality best, it being:

  $$\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}$$

  $$\sum_{n=0}^{r+s} \binom{r+s}{n} x^n$$

  Using the Binomial Theorem

  $$\sum_{n=0}^{r+s} \binom{r+s}{n} x^n = (1+x)^{r+s}$$

  $$\sum_{n=0}^{r+s} \binom{r+s}{n} x^n = (1+x)^{r}(1+x)^{s}$$

  Using the Binomial Theorem, in the opposite direction

  $$\sum_{n=0}^{r+s} \binom{r+s}{n} x^n = \left(\sum_{i\geq 0} \binom{r}{i} x^r\right)\left(\sum_{j\geq 0} \binom{s}{j} x^s\right)$$

  Multiplying out the summations on the right gives us:

  $$\sum_{n=0}^{r+s} \left[\binom{r+s}{n}\right] x^n = \sum_{n=0}^{r+s} \left[\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i}\right] x^n$$

  In order for this equality to hold true, which it must due to the nature of all the steps that took us to it, the terms inside the square braces must equal one another.

By using both a combinatorial and an algebraic proof, we have proven the identity.

Problem #2:

Suppose $n \in \mathbb{N}$. Please prove $n2^{n-1} = \sum_{i\geq 0} \binom{n}{i} i$ in three ways: combinatorially, algebraically, and using the Principle of Mathematical Induction.

- **Combinatorially:** To prove this equality, I will show how both sides of it are counting the same thing in different ways:

  Left Hand Side: Firstly, assume the fact that 2^n counts the amount of unique subsets that can be made from an n-set. Then, the Left Hand Side can be seen as simply the amount of elements that there are in n copies of the unique subsets that can be made out of n-1 elements.

  Right Hand Side: To show this equality, I will slightly change the Right Hand Side. So, the formula $\sum_{i\geq 0} \binom{n}{i} i$ can be rewritten to $\sum_{i\geq 0} \frac{n}{i} \binom{n-1}{i-1} i$ which ,in turn, equals $\sum_{i\geq 0} n \binom{n-1}{i-1}$. This can be represented as n copies of all the different sized combinations that can be made with n-1 elements.

- **Algebraically:** To prove this equality algebraically, we will start from showing how one can go from the left to the right side, then show the process the other way around.

  Left to right:

$$n2^{n-1} = \sum_{i\geq 0} \binom{n}{i} i$$
$$n\left[\sum_{i\geq 0} \binom{n-1}{i-1}\right] = \sum_{i\geq 0} \binom{n}{i} i$$
$$n\left[\sum_{i\geq 0} \frac{i}{n} \binom{n}{i}\right] = \sum_{i\geq 0} \binom{n}{i} i$$
$$\sum_{i\geq 0} i \binom{n}{i} = \sum_{i\geq 0} \binom{n}{i} i$$

  Right to left:

$$n2^{n-1} = \sum_{i\geq 0} \binom{n}{i} i$$
$$n2^{n-1} = \sum_{i\geq 0} \frac{n}{i} \binom{n-1}{i-1} i$$
$$n2^{n-1} = \sum_{i\geq 0} n \binom{n-1}{i-1}$$
$$n2^{n-1} = n \sum_{i\geq 0} \binom{n-1}{i-1}$$
$$n2^{n-1} = n(1+1)^{n-1}$$
$$n2^{n-1} = n2^{n-1}$$

  Since we've proven we can get from either side to the other algebraically, then we've algebraically proven both sides to be equal to one another.

- **Principle of Mathematical Induction:** We will check whether or not the set A of all solutions to this equation belongs in the Natural numbers. To do this, first we will check if 0 belongs in this set:

$$n2^{n-1}$$
$$(0)2^{(0)-1}$$
$$0$$

Since we now know that 0 belongs to this set, we will proceed to check whether or not $F(n)$ entails $F(n+1)$. So, for $F(n)$:

$$n2^{n-1} = \sum_{i \geq 0} \binom{n}{i} i$$

We would expect $F(n+1)$ to equal:

$$(n+1)2^n$$

We will show how we can attain this value on the left from the right side:

$$(n+1)2^n = \sum_{i \geq 0} \binom{n+1}{i} i$$
$$(n+1)2^n = \sum_{i \geq 0} \frac{(n+1)!}{i!(n+1-i)!} i$$
$$(n+1)2^n = \sum_{i \geq 0} \frac{(n+1)!}{(i-1)!(n+1-i)!}$$
$$(n+1)2^n = \sum_{i \geq 0} \frac{(n+1)(n)!}{(i-1)!(n+1-i)!}$$
$$(n+1)2^n = (n+1) \sum_{i \geq 0} \frac{(n)!}{(i-1)!(n+1-i)!}$$
$$(n+1)2^n = (n+1) \sum_{i \geq 0} \binom{n}{i-1}$$

Since the result of a combination with any negative number is equal to 0, the previous equation can be changed to:

$$(n+1)2^n = (n+1) \sum_{i \geq 0} \binom{n}{i}$$

By simply following the binomial theorem now, we get:

$$(n+1)2^n = (n+1)(1+1)^n$$
$$(n+1)2^n = (n+1)2^n$$

So, we have shown that our set A not only contains 0, but also that the function that creates it, $F(n)$, entails the existence of $F(n+1)$. We have, therefore, proved this equality using the Principle of Mathematical Induction.

Problem #3

Please prove the identity $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$.

To prove this, we will use a combinatorial approach. We will show both sides count the same thing.

Left Hand Side: The value of F_{n+1} is also equal to the different combinations of 2's and 1's that can add up to n.

Right Hand Side: The summation of the different ways that we can choose i elements from an n-i set of elements when i increases by one for each set of choices. To show how this relates to the left side, we can write these sets as all the possible sets of 1's and 2's that add up to n, these sets end in a set of 2's and 1's equal to the size of $\lfloor \frac{n}{2} \rfloor$, where if n is an even number, there is only one set made up of 2's, and if n is odd the last group of sets is made up of $\lfloor \frac{n}{2} \rfloor + 1$ elements.