Daniel Oliveros A02093272
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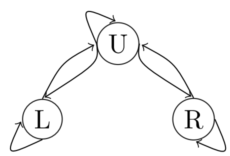
Homework 5

Problem #1
Consider paths in the (x, y)-plane of length n (having n steps taken) starting from $(0, 0)$ with steps $R : (x, y) \mapsto (x + 1, y)$, $L : (x, y) \mapsto (x - 1, y)$, and $U : (x, y) \mapsto (x, y + 1)$. We require that a step R is not followed by a step L and vice versa. Put p_n equal to the number of such paths or length n and determine a closed formula for p_n.

For this problem, I decided to use an adjacency matrix to denote the different paths the nodes in this system could take. Using this knowledge, I was able to derive an easy formula for calculating the number of steps of length n taken. The graph for our system looks like this:



This graph can be represented using the following adjacency matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

By using this matrix to act upon a vector representing the number of paths of length n at a given time, we can get a vector that represents the number of paths of length n+1 in this system. We can simply count this amount using the following closed formula:

$$P_n = (A^n \cdot v_0)^T \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Where v_0 is the vector at time 0. To make future computations easier, I've assumed a motion at time 0 would need to be simply an "up" motion, this is due to the fact that at time 1 we are allowed to make any of the three motions. This means that v_0 is represented as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ This is already a closed formula, however, we can generate an easier to compute version of it by performing an eigendecomposition of our matrix A. Leaving it in $Q\Lambda Q^{-1}$ form. This will allow us to take powers of our matrix much more nicely. After the decomposition:

$$A = Q\Lambda Q^{-1}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{2}/2 & -1/2 \\ 1/2 & \sqrt{2}/2 & -1/2 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1-\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & -1/2 & -\sqrt{2}/2 \end{bmatrix}$$

Here's the process:

$$P_n = (A^n \cdot v_0)^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Perform eigendecomposition on A, leaving it in $Q\Lambda Q^{-1}$ form

$$P_n = ((Q\Lambda Q^{-1})^n \cdot v_0)^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Use properties of $Q\Lambda Q^{-1}$ matrix

$$P_n = (Q\Lambda^n Q^{-1} \cdot v_0)^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

From here, just multiply everything out

$$P_n = (Q\Lambda^n \begin{bmatrix}1/2 & 1/2 & -\sqrt{2}/2\\ -\sqrt{2}/2 & \sqrt{2}/2 & 0\\ -1/2 & -1/2 & -\sqrt{2}/2\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix})^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$P_n = (Q \begin{bmatrix}(1-\sqrt{2})^n & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (1+\sqrt{2})^n\end{bmatrix} \begin{bmatrix}-\sqrt{2}/2\\0\\-\sqrt{2}/2\end{bmatrix})^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$P_n = (\begin{bmatrix}1/2 & -\sqrt{2}/2 & -1/2\\ 1/2 & \sqrt{2}/2 & -1/2\\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2\end{bmatrix} \begin{bmatrix}-\sqrt{2}(1-\sqrt{2})^n/2\\0\\-\sqrt{2}(1+\sqrt{2})^n/2\end{bmatrix})^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$P_n = (\begin{bmatrix}\frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}}\\ \frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}}\\ \frac{(1+\sqrt{2})^n}{2} + \frac{(1-\sqrt{2})^n}{2}\end{bmatrix})^T \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$P_n = \begin{bmatrix}\frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}} & \frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}} & \frac{(1+\sqrt{2})^n}{2} + \frac{(1-\sqrt{2})^n}{2}\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$P_n = \left(\frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}}\right) + \left(\frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}}\right) + \left(\frac{(1+\sqrt{2})^n}{2} + \frac{(1-\sqrt{2})^n}{2}\right)$$

$$P_n = \left(\frac{(1+\sqrt{2})^n}{\sqrt{2}} - \frac{(1-\sqrt{2})^n}{\sqrt{2}}\right) + \left(\frac{(1+\sqrt{2})^n}{2} + \frac{(1-\sqrt{2})^n}{2}\right)$$

$$P_n = \left(\frac{\sqrt{2}(1+\sqrt{2})^n}{2} - \frac{\sqrt{2}(1-\sqrt{2})^n}{2}\right) + \left(\frac{(1+\sqrt{2})^n}{2} + \frac{(1-\sqrt{2})^n}{2}\right)$$

$$P_n = \frac{(1+\sqrt{2})(1+\sqrt{2})^n + (1-\sqrt{2})(1-\sqrt{2})^n}{2}$$

$$P(n) = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2}$$

This is an easier closed formula to compute, and it was created off the adjacency matrix which was originally used to describe the system.

Problem #2: Modified Tower of Hanoi.
Consider the Tower of Hanoi game and label the three pegs L, M, and R, for the left peg, the middle peg, and right peg, respectively. Determine the minimum number of moves required to transfer n disks as in the Tower of Hanoi game, but with the additional constraint that a disk can only be moved to an adjacent peg; that is, a disk can only be moved to M from L or R, and can only be moved to L or R from M.

Through our observations of the increase in number of moves required to move an n number of disks from the left peg to the right one, we determined a recursive relation. This relation is: $H_n = 3H_{n-1} + 2$. We will expand this relation to determine a closed formula:

$$3H_{n-1} + 2 = 3(3H_{n-2} + 2) + 2$$
$$3^2 H_{n-2} + 2 * 3 + 2 = 3^2(3H_{n-3} + 2) + (2 * 3) + 2$$
$$3^2(3H_{n-3} + 2) + 2 * 3 + 2 = 3^3 H_{n-3} + (2 * 3^2) + (2 * 3) + 2$$

Expanding this out entirely

$$(3^n H_{n-n}) + (3^{n-1} * 2) + \ldots + (3^2 * 2) + (3^1 * 2) + (3^0 + 2)$$

From this, we can clearly identify the relation:

$$\sum_{i=0}^{n-1} 3^i * 2$$

By expanding this, we get:

$$2 * \left(\frac{1 - 3^n}{1 - 3}\right)$$
$$\frac{2 * (1 - 3^n)}{-2}$$
$$3^n - 1$$

Now, we will prove how this relation we found holds up. We will use the PMI to determine this:

First, let us test the base case. So, if $n = 0$:

$$H(n) = 3^n - 1$$
$$H(0) = 3^0 - 1$$
$$H(0) = 1 - 1$$
$$H(0) = 0$$

We have now determined 0 belongs in our set of solutions, now, let's check if $H(n)$ entails $H(n+1)$:

$$\text{Assume: } H_n = 3^n - 1$$
$$\text{Want: } H_{n+1} = 3^{n+1} - 1$$
$$\text{Consider: } H_{n+1} = 3H_n + 2$$
$$H_{n+1} = 3(3^n - 1) + 2$$
$$H_{n+1} = 3 * 3^n - 3 + 2$$
$$H_{n+1} = 3^{n+1} - 1$$

Since we have also proven how H_n entails H_{n+1}. Then we have proved this formula holds up as the correct count of the modified Towers of Hanoi problem.