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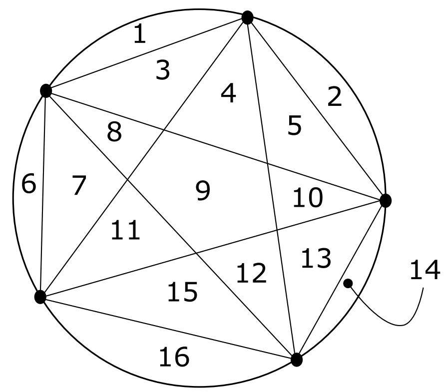
Homework 6

Problem #1
Place n points on a circle and connect every pair of points with a line segment. What is the maximum number of regions determined by n points with corresponding line segments in place? As an example, the illustration below shows that 16 is the number of regions obtained from connecting all pairs of 5 points placed on a circle. Note that, as with the pizza problem, the line segments must satisfy certain conditions to ensure that the number of regions they create is maximized.



For this problem, I assumed the solution could be expressed as a polynomial, so I took multiple derivatives of my function. In order to do this, I first calculated the number of regions I could make based on various numbers of points. My table of results was, then:

	0	1	2	3	4	5	6	7
R(n)	1	1	2	4	8	16	31	57

I can take multiple derivatives from this table to determine the degree of my polynomial:

	0	1	2	3	4	5	6	7
R(n)	1	1	2	4	8	16	31	57
Δ R(n)	0	1	2	4	8	15	26	
Δ^2 R(n)	1	1	2	4	7	11		
Δ^3 R(n)	0	1	2	3	4			
Δ^4 R(n)	1	1	1	1				
Δ^5 R(n)	0	0	0					

Since $R(n)$ goes completely to zero after the fifth derivative, I will assume it's a 4th degree polynomial. So, $R(n)$ can be written as: $An^4 + Bn^3 + Cn^2 + Dn + E$. In order to solve for these coefficients, we can use the results we calculated earlier as the initial conditions. So:

$$
\begin{aligned}
R(0) = \quad 1 = \quad & A(0^4) + B(0^3) + C(0^2) + D(0) + E \\
R(1) = \quad 1 = \quad & A(1^4) + B(1^3) + C(1^2) + D(1) + E \\
R(2) = \quad 2 = \quad & A(2^4) + B(2^3) + C(2^2) + D(2) + E \\
R(3) = \quad 4 = \quad & A(3^4) + B(3^3) + C(3^2) + D(3) + E \\
R(4) = \quad 8 = \quad & A(4^4) + B(4^3) + C(4^2) + D(4) + E
\end{aligned}
$$

This gives us the set of equations:

$$\begin{aligned} E &= 1 \\ A+B+C+D+E &= 1 \\ 16A+8B+4C+2D+E &= 2 \\ 81A+27B+9C+3D+E &= 4 \\ 256A+64B+16C+4D+E &= 8 \end{aligned}$$

Which, can be represented using the augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 & 2 \\ 81 & 27 & 9 & 3 & 1 & 4 \\ 256 & 64 & 16 & 4 & 1 & 8 \end{array}\right]$$

Calculating the Reduced Row Echelon form of this matrix gives us:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1/24 \\ 0 & 1 & 0 & 0 & 0 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 23/24 \\ 0 & 0 & 0 & 1 & 0 & -3/4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

So, our function $R(n)$ can be written as:

$$R(n) = \left(\frac{1}{24}\right)n^4 - \left(\frac{1}{4}\right)n^3 + \left(\frac{23}{24}\right)n^2 - \left(\frac{3}{4}\right)n + 1$$

Problem #2
Below is pseudo-code for an algorithm called Pow. Please determine how many times is Pow called to compute Pow(n)?

Algorithm: **Pow**(n), n is a positive integer that means something.
Input: n, or something with n thingys
Output: Something, maybe a number, maybe an apple pie.

1: **If** $n < 0$
  print "Knock it off!"; **exit**
 end If
2: **If** $n = 0$
  return $\aleph_0$; **exit**
 end If
3: **If** $n = 1$
  return $\aleph_1$; **exit**
 end If
4: **If** $n > 1$
 return 3∗Pow($n-1$) − 2∗Pow($n-2$)
 end If

This one was fairly simple to determine a recurrence for. Since the number of operations for the algorithm at n depends on the number of operations of it at $n-1$ and $n-2$, this is due to it calling Pow($n-1$) and Pow($n-2$) in line 4. So, the formula for calculating the number of times Pow is called could be written as:

$$P(n) = P(n-1) + P(n-2) + 1$$

The reason for the 1 is that we are supposed to add one for each time Pow is called

We can transform this by making multiple guesses as to what P_n is. Since this a recursive relation, we know there is a homogeneous side. So, we will solve for this side first. Looking at the function again, we can also show it as:

$$P(n) = P(n-1) + P(n-2) + 1$$
$$P_n = P_{n-1} + P_{n-2} + 1$$
$$P_n = [P_{n-1} + P_{n-2}] + \{1\}$$

The part surrounded with brackets is the homogeneous part.

The part surrounded with curly braces is the non-homogeneous part.

For the homogeneous side:

$$P_n = P_{n-1} + P_{n-2}$$

Guess: $P_n = q^n$

$$q^n = q^{n-1} + q^{n-2}$$
$$\frac{q^n}{q^{n-2}} = \frac{q^{n-1}}{q^{n-2}} + \frac{q^{n-2}}{q^{n-2}}$$
$$q^2 = q + 1$$
$$q^2 - q - 1 = 0$$

Solve for q

$$q = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$q = \frac{1 \pm \sqrt{1+4}}{2}$$
$$q = \frac{1 \pm \sqrt{5}}{2}$$
$$q = \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$$

So, for the homogeneous part $P_n = A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n$

Now, for the non-homogeneous part. Start by guessing $P_n = C$

$$P_n = C = P_{n-1} = P_{n-2}$$
$$P_n = P_{n-1} + P_{n-2} + 1$$
$$C = C + C + 1$$
$$-C = 1$$
$$C = -1$$

So, we can rewrite $P(n)$ as $P(n) = A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n - 1$. To solve for A and B, we can simply use our initial conditions. Simply put, when n is equal to either 0 or 1, the function is only called once since they don't make it down to the recursive statement. So:

$$\begin{aligned}
= P(n) = & \quad A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n - 1 \\
P(0) = \quad 1 = & \quad A\left(\frac{1-\sqrt{5}}{2}\right)^0 + B\left(\frac{1+\sqrt{5}}{2}\right)^0 - 1 = A + B - 1 \\
P(1) = \quad 1 = & \quad A\left(\frac{1-\sqrt{5}}{2}\right)^1 + B\left(\frac{1+\sqrt{5}}{2}\right)^1 - 1 = A\left(\frac{1-\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right) - 1
\end{aligned}$$

From here, we get the system of equations:

$$\begin{aligned} A+B &= 2 \\ A\left(\frac{1-\sqrt{5}}{2}\right)+B\left(\frac{1+\sqrt{5}}{2}\right) &= 2 \end{aligned}$$

Which can be represented using the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ \left(\frac{1-\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right) & 2 \end{array}\right]$$

Calculating the Reduced Row Echelon form of this matrix gives us:

$$\left[\begin{array}{cc|c} 1 & 0 & \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right) \\ 0 & 1 & \left(\frac{\sqrt{5}+1}{\sqrt{5}}\right) \end{array}\right]$$

Which, finally, allows us to write down our complete function $P(n)$ as:

$$P(n) = \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}+1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - 1$$

Problem #3
Please determine, with proof, a closed formula for each of the following recurrences:

1. $a_n = a_{n-1} - 6a_{n-2} + 2^n$ for $n \geq 2$ with $a_0 = 0, a_1 = 1$.

2. OK, fine, I'll modify this: $b_n = 4b_{n-1} - 4b_{n-2} + 4b_{n-3}$ for $n \geq 3$ with $b_0 = 0, b_1 = 1, b_2 = 2$

   to this: $b_n = b_{n-1} - 4b_{n-2} + 4b_{n-3}$ for $n \geq 3$ with $b_0 = 0, b_1 = 1$ and $b_2 = 2$. (If you have already solved the earlier version with all its nastiness, be sure to turn that in too — you'll be rewarded.)

3. $c_n = c_{n-1} + (n+3)(n+2)(n+1)$ for $n \geq 1$ and $c_0 = 6$.

A good way of tackling these types of problems is to start with the homogeneous solution first, then move onto the non-homogeneous once you know your roots. With that out of the way, I will tackle all of these problems:

3.1 First, we handle the homogeneous side. AKA the side with a recurrence relation. So:

$$a_n = a_{n-1} - 6a_{n-2}$$
$$\text{Guess: } a_n = r^n$$
$$r^n = r^{n-1} - 6r^{n-2}$$
$$\text{Divide both sides by smallest exponent}$$
$$\frac{r^n}{r^{n-2}} = \frac{r^{n-1}}{r^{n-2}} - 6\frac{r^{n-2}}{r^{n-2}}$$
$$r^2 = r - 6$$
$$r^2 - r + 6 = 0$$
$$\text{Solve for r}$$
$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(6)}}{2(1)}$$
$$r = \frac{1 \pm \sqrt{1-24}}{2}$$
$$r = \frac{1 \pm \sqrt{23}i}{2}$$
$$r = \left\{\frac{1-\sqrt{23}i}{2}, \frac{1+\sqrt{23}i}{2}\right\}$$
$$\text{So, for the homogeneous part: } a_n = A\left(\frac{1-\sqrt{23}i}{2}\right)^n + B\left(\frac{1+\sqrt{23}i}{2}\right)^n$$

$$\text{Now, for the non-homogeneous part:}$$
$$\text{Guess: } a_n = C2^n$$
$$a_n = a_{n-1} - 6a_{n-2} + 2^n$$
$$C2^n = C2^{n-1} - 6C2^{n-2} + 2^n$$
$$2C2^{n-1} = C2^{n-1} - 3C2^{n-1} + 2 * 2^{n-1}$$
$$\text{divide all by } 2^{n-1}$$
$$2C = C - 3C + 2$$
$$4C = 2$$
$$C = 1/2$$

So, we can rewrite a_n as $a(n) = A\left(\frac{1-\sqrt{23}i}{2}\right)^n + B\left(\frac{1+\sqrt{23}i}{2}\right)^n + \frac{1}{2}2^n$
Now, to solve for the constants, we will use our initial conditions:

$$\begin{aligned}
a(n) = & & A\left(\frac{1-\sqrt{23}i}{2}\right)^n + B\left(\frac{1+\sqrt{23}i}{2}\right)^n + \frac{1}{2}2^n \\
a(0) = & \quad 0 = & A\left(\frac{1-\sqrt{23}i}{2}\right)^0 + B\left(\frac{1+\sqrt{23}i}{2}\right)^0 + \frac{1}{2}2^0 \Rightarrow A + B + \frac{1}{2} \\
a(1) = & \quad 1 = & A\left(\frac{1-\sqrt{23}i}{2}\right)^1 + B\left(\frac{1+\sqrt{23}i}{2}\right)^1 + \frac{1}{2}2^1 \Rightarrow A\left(\frac{1-\sqrt{23}i}{2}\right) + B\left(\frac{1+\sqrt{23}i}{2}\right) + 1
\end{aligned}$$

From here, we get the system of equations:

$$\begin{aligned}
A + B = & \quad -\tfrac{1}{2} \\
A\left(\frac{1-\sqrt{23}i}{2}\right) + B\left(\frac{1+\sqrt{23}i}{2}\right) = & \quad 0
\end{aligned}$$

Which can be represented using the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & -1/2 \\ \left(\frac{1-\sqrt{23}i}{2}\right) & \left(\frac{1+\sqrt{23}i}{2}\right) & 0 \end{array}\right]$$

Calculating the Reduced Row Echelon form of this matrix gives us:

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{4}+\frac{i}{4\sqrt{23}} \\ 0 & 1 & -\frac{1}{4}-\frac{i}{4\sqrt{23}} \end{array}\right]$$

Which, finally, allows us to write down our complete function $a(n)$ as:

$$a(n) = \left(-\frac{1}{4}+\frac{i}{4\sqrt{23}}\right)\left(\frac{1-\sqrt{23}i}{2}\right)^n + \left(-\frac{1}{4}-\frac{i}{4\sqrt{23}}\right)\left(\frac{1+\sqrt{23}i}{2}\right)^n + \frac{1}{2}2^n$$

3.2 In this case, there is only a homogeneous part to this. So, we will solve this recursive relation:

$$b_n = 4b_{n-1} - 4b_{n-2} + 4b_{n-3}$$
$$\text{Guess: } b_n = q^n$$
$$q^n = 4q^{n-1} - 4q^{n-2} + 4q^{n-3}$$
$$\text{Divide by lowest exponent}$$
$$\frac{q^n}{q^{n-3}} = 4\frac{q^{n-1}}{q^{n-3}} - 4\frac{q^{n-2}}{q^{n-3}} + 4\frac{q^{n-3}}{q^{n-3}}$$
$$q^3 = 4q^2 - 4q + 4$$
$$q^3 - 4q^2 + 4q - 4 = 0$$
$$\text{The roots of this polynomial are: } \{1, -2i, 2i\}$$

So, we can rewrite b_n as:

$$b_n = A + B(-2i)^n + C(2i)^n$$

To solve for these constants, we will use our initial conditions:

$$\begin{aligned} b(n) &= & & A + B(-2i)^n + C(2i)^n \\ b(0) &= & 0 \quad & A + B(-2i)^0 + C(2i)^0 \Rightarrow A \\ b(1) &= & 1 \quad & A + B(-2i)^1 + C(2i)^1 \Rightarrow A - 2Bi + 2Ci \\ b(2) &= & 2 \quad & A + B(-2i)^2 + C(2i)^2 \Rightarrow A - 4B - 4C \end{aligned}$$

From here, we get the system of equations:

$$\begin{aligned} A + B + C &= \quad 0 \\ A - 2Bi + 2Ci &= \quad 1 \\ A - 4B - 4C &= \quad 2 \end{aligned}$$

Which can be represented using the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2i & 2i & 1 \\ 1 & -4 & -4 & 2 \end{array}\right]$$

By calculating the Reduced Row Echelon form of this matrix we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & -\frac{1}{5}-\frac{3}{20} \\ 0 & 0 & 1 & -\frac{1}{5}+\frac{3}{20} \end{array}\right]$$

Which, finally, allows us to write down our complete function $b(n)$ as:

$$b_n = \left(\frac{2}{5}\right) + \left(-\frac{1}{5} - \frac{3}{20}\right)(-2i)^n + \left(-\frac{1}{5} + \frac{3}{20}\right)(2i)^n$$

3.3 For this problem, we will use the identity discussed in class $c(n) = \sum_{k\geq 0} \Delta^{(k)} c(0)\binom{n}{k}$ So, for our function $c(n)$:

$$\begin{aligned}
c(n) &= c(n-1) + (n+3)(n+2)(n+1) \\
\Delta c(n) &= n^3 + 9n^2 + 26n + 24 \\
\Delta^2 c(n) &= 3n^2 + 21n + 36 \\
\Delta^3 c(n) &= 6n + 24 \\
\Delta^4 c(n) &= 6 \\
\Delta^5 c(n) &= 0
\end{aligned}$$

So, using this information, we can determine that:

$$\begin{aligned}
c(n) &= \sum_{k\geq 0} \Delta^{(k)} c(0)\binom{n}{k} \\
c(n) &= \Delta^{(0)} c(0)\binom{n}{0} + \Delta^{(1)} c(0)\binom{n}{1} + \Delta^{(2)} c(0)\binom{n}{2} + \Delta^{(3)} c(0)\binom{n}{3} + \Delta^{(4)} c(0)\binom{n}{4}
\end{aligned}$$

To calculate these values, we can use the formulas we found earlier, so:

$$\begin{aligned}
c(0) &= 6 \\
\Delta c(0) &= 0^3 + 9(0)^2 + 26(0) + 24 \Rightarrow 24 \\
\Delta^2 c(0) &= 3(0)^2 + 21(0) + 36 \Rightarrow 36 \\
\Delta^3 c(0) &= 6(0) + 24 \Rightarrow 24 \\
\Delta^4 c(0) &= 6
\end{aligned}$$

So, our equation is equal to:

$$c(n) = 6\binom{n}{0} + 24\binom{n}{1} + 36\binom{n}{2} + 24\binom{n}{3} + 6\binom{n}{4}$$