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Homework 3

Problem #1
Please prove the identity $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$ using mathematical induction.

To prove this using mathematical induction, we will perform the three following steps:

1. The **base case** corresponds to showing $0 \in A$;
2. The **induction assumption** corresponds to $x \in A \Rightarrow x + 1 \in A$, in particular the assumption is often that $x \in A$ for $x \in \mathbb{N}$;
3. The **induction step** corresponds also to $x \in A \Rightarrow x + 1 \in A$, but this is where the implication is proved.

So, first we will prove the **base case**. For the set of Fibonacci numbers this isn't all that difficult. Recall that $F_0 = 0$ and $F_1 = 1$. Just from simple observation, it is clear that 0 is part of the set A, which contains all Fibonacci numbers and is itself contained within the realm of all natural numbers.

Now, let's look at the **induction assumption**. Just to check that the result of inputting some number, like 0, falls within the natural numbers, we will plug it into the formula and see.

$$\begin{aligned} F_{n+2} &= 1 + \textstyle\sum_{i=0}^{n} F_i \\ F_{0+2} &= 1 + \textstyle\sum_{i=0}^{0} F_i \\ F_2 &= 1 + 0 \\ F_2 &= 1 \end{aligned}$$

Since the resulting number is within the realm of $\mathbb{N}$, we can determine there exist some elements within the field $A \in \mathbb{N}$.

Finally, to prove the **induction step**, we will show how, logically, the existence of $F_n \in \mathbb{N}$ entails that $F_{n+1} \in \mathbb{N}$. So:

$$F_{n+2} = 1 + \sum_{i=0}^{n} F_i$$

From this equation, we would expect $F_{(n+1)+2}$ to equal $1 + \sum_{i=0}^{n+1} F_i$ to show this:

$$F_{(n+1)+2} = 1 + \sum_{i=0}^{n} F_i + F_{n+1}$$

ThisF_{n+1} can be absorbed into the summation, leaving us with:

$$F_{(n+1)+2} = 1 + \sum_{i=0}^{n+1} F_i$$

So, through use of the principle of mathematical induction, we've proven that all elements of F are contained within $\mathbb{N}$.

Problem #2
Suppose $c_1, c_2, \ldots, c_j$ are positive integers serving as capacities of water jugs in some fixed unit. Suppose also that $c_1 \leq c_2 \leq \cdots \leq c_j$ and that$\gcd(c_1, \ldots, c_j) = 1$.

Use the following theorem to prove that n units of water, where n is an integer satisfying $0 \leq n \leq c_1 + c_2 + \cdots + c_j$, can be measured using the jugs and the processes of filling a jug, emptying a jug into another, and completely emptying a jug.
Theorem. *Given the jug capacities as above and an amount $\widehat{n}$ of water with $0 \leq \widehat{n} \leq c_j$, the largest jug can be filled with $\widehat{n}$ units of water.*

In order to prove that we can measure such an amount of water within any subset of jugs that meets these properties, I will show a method that simplifies the problem and allows for a simple, direct proof to be implemented.
So, since we can fill the last jug with any amount of water $\widehat{n}$ such that $0 \leq \widehat{n} \leq c_j$, then we really have two different amounts of water that we can logically find solutions for. All the amounts of water n where n is smaller than, or equal to c_j, and all the capacities greater than this amount. So, for:

- $0 \leq n \leq c_j$, we can easily assert from the theorem that, as long as the conditions specified are met, we will be able to find some sequence of filling jugs, emptying jugs into other jugs, and completely emptying jugs, which will allow us to measure any amount of water found within this range in the biggest jug.

- $c_j < n \leq c_1 + c_2 + \cdots + c_j$; this is a more complicated situation. For this case, there is an algorithm that we can use to calculate these amounts rather easily. Assume n^-, which refers to the amount of water that is yet to be filled within any jugs, and n^+, which refers to the amount of water that has been filled thus far. In this situation $n = n^- + n^+$. So, in order to find a solution for n, we will employ a simple method of filling up jugs, smallest first, such that the value of n^+ increases, which causes the value of n^- to decrease. From here, once we have decreased the value of n^- to a value that falls within the range of $0 \leq n^- \leq c_j$, we will empty all of the jugs we have used thus far. We know, based on the theorem, that we will be able to represent the amount n^- that we found earlier by filling, emptying into one another, and dumping the water between our set of jugs. Once we have represented that amount in the largest jug, we can fill the other jugs, smallest first, until n^+ is equal to n, at which point we will have represented the amount of water n we were looking for.

By showing these two processes, we have proved we can represent n as long as it's within our range and the conditions are met.

Problem #3
Recall that we discussed number systems by writing an ordered triple (X, Y, Z), where X is a set of things we call 'numbers', Y is the notation for an operation we call 'addition', and Z is notation for what we call 'multiplication'. We can do something analogous with *Logical Systems*: we specify the set of statements, the function that determines truth, and the logical operations and operators. For the logical system that we (and essentially everyone) use, let's use the notation$(\mathcal{M}, \Phi, \implies, \wedge, \vee, \neg)$ to mean that $\mathcal{M}$ is the set of statements we work with, Φ is the function that assesses truth, and the others are the logical operations and operator.
Please prove or disprove that the our logical system $(\mathcal{M}, \Phi, \implies, \wedge, \vee, \neg)$ can be replaced with $(\mathcal{M}, \Phi, \nabla)$, where ∇ is defined as follows. For $x, y \in \mathcal{M}$, $x \nabla y$ is equivalent to $\neg(x \vee y)$

To solve this, I will show a the multiple sets of truth tables that help us determine how to represent the different logical operators as different combinations of x, y and ∇. So:

- ¬ x:

x	y	$\neg x$	$x \nabla x$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	T

So, $\neg x$ is equivalent to $x \nabla x$

- $x \vee y$:

x	y	$x \vee y$	$\overbrace{x \nabla y}^{b}$	$b \nabla b$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

So, $x \vee y$ is equivalent to $(x\nabla y)\nabla(x\nabla y)$

- $x \implies y$:

x	y	$x \implies y$	$\overbrace{x \nabla y}^{b}$	$\overbrace{y \nabla b}^{c}$	$c \nabla c$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	F	F	T
F	F	T	T	F	T

So, $x \implies y$ is equivalent to $(y\nabla(x\nabla y))\nabla(y\nabla(x\nabla y))$

- $x \wedge y$:

x	y	$x \wedge y$	$\overbrace{x \nabla x}^{b}$	$\overbrace{y \nabla y}^{c}$	$b \nabla c$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	T	T	F

So, $x \wedge y$ is equivalent to $(x\nabla x)\nabla(y\nabla y)$

So, since we can represent the entire set of Logical systems with the combinations of the operator ∇, then we have shown our logical system $(\mathcal{M}, \Phi, \implies, \wedge, \vee, \neg)$ can be replaced with $(\mathcal{M}, \Phi, \nabla)$.