

- §1.1 Error Analysis
- **Normalization:** 32 bit - Sign bit: 1, Sign exp: 1, Exp: 7, Normalized mantissa: 23.
- Absolute Error =  $|p - p^*|$ , Relative Error =  $\frac{|p - p^*|}{|p|}$
- Significant Digits:  $RE < 5 \times 10^{-t}$
- **f(x):** Machine representation
- **Cancellation Error:** Subtracting nearly equal numbers
- **Example:**  $p = 0.54617$ ,  $q = 0.54601$ , true  $r = p - q = 0.00016$
- 4-digit:  $p^* = 0.5462$ ,  $q^* = 0.5460$ ,  $r^* = 0.002$  (RE=25%)
- **Nested Multiplication:** Reduces error
- $f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99 = (((1.01z - 4.62)z - 3.11)z + 12.2)z - 1.99$

- $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$
- Remainder:  $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$ ,  $\xi \in (a, x)$
- Linear Approx:  $f(x_0 + h) \approx f(x_0) + h f'(x_0)$

- **Example:**  $\xi \in (0, \pi/2)$ ,  $\sin \xi \leq 1 \Rightarrow R_n \leq \frac{(x)^n}{n!}$

### §1.3 Convergence

- $\alpha = \lim_{n \rightarrow \infty} (\alpha_n)$
- Rate:  $\alpha_n = \alpha + O(\beta_n)$  if  $|\alpha_n - \alpha| \leq K |\beta_n|$
- Find largest  $p$  where  $\alpha_n - \alpha = O(1/n^p)$

### §1.4 Matrix Operations

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ ,  $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$
- $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

### §6.1 Gaussian Elimination

- $O(n^3)$  complexity
- **Pivoting:**
  - Partial (PP): Max element in column
  - Scaled PP:  $s_i = \max_j |a_{ij}|$ , pivot  $\max(a_{ik}/s_i)$
  - Complete (CP): Full matrix search ( $O(n^3)$ )
- **LU Decomposition:**  $PA = LU$  through GE steps and  $LUx = Pb$ .
- **LU Algorithm:**  $L = E_{n-a,m-b}^{-1} E_{n-a+1,m-b+1}^{-1} \dots U = GE$ .
- **Choleski Factorization:** If a matrix is symmetric and positive definite, it may be factored to the form  $LDL^T$

### §6.2 Special Matrices

- **Inverse Matrix:** An inverse matrix of A is  $A^{-1}$  such that  $AA^{-1} = I$
- Properties:  $(AB)^{-1} = B^{-1}A^{-1}$ ,  $(A^{-1})^T = (A^T)^{-1}$
- **Singular:** A matrix is singular iff its det is 0.
- **Diagonal:**  $d_{ij} = 0$  for  $i \neq j$ : All non-diagonal entries are 0.
- **Symmetric:**  $A = A^T$ ,  $(AB)^T = B^T A^T$
- **Permutation:** Row swaps of  $I_n$ ,  $PA$  reorders rows:  $P^T = P^{-1}$
- **Diagonally Dominant:**  $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$  (nonsingular)
- **Positive Definite:**  $x^T A x > 0 \Rightarrow A = LDL^T$ ,  $a_{ii} > 0$ ,  $a_{ij}^2 < a_{ii}a_{jj}$
- **Minor:**  $M_{ij}$  is a submatrix of A with the row i deleted and column j deleted.
- **Band:** an  $n \times n$  matrix is a band matrix if  $p, q \in \mathbb{Z} : 1 \leq p, q \leq n$  exist with  $a_{ij} = 0$  for  $i + p \leq j$  or  $j + q \geq i$  The bandwidth is defined as  $w = p + q - 1$ . For a diagonal matrix,  $p = 1, q = 1, w = 1$
- **Tridiagonal:** Band with  $p = 2, q = 2$ . It exhibits the following properties
  - $a_{ii} = l_{ii}$       -  $a_{i,i+1} = l_{ii}u_{i,i+1} : i = 1 \dots n - 1$
  - $a_{i,i-1} = l_{i,i-1} : i = 2, 3, \dots, n$       -  $a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} : i = 2 \dots n$
- Crout Factorization: This factorization happens in  $O(n)$  time

### §Strategies

- **RoC With inf limit:** set  $h = 1/n$  and solve accordingly.
- $D \cdot (L+U)$ : given D has ONLY diagonal entries and L+U has NO diagonal entries, the resulting matrix A is composed of entries  $a_{ij} = d_{ii} \cdot (l + u)_{ij}$
- **Verification of Bisection:** To verify bisection can be applied, make sure that f(a) and f(b) are of different signs.
- **Error of Bisection:** To compute the accuracy of bisection to an  $\epsilon$ , we use  $\frac{b-a}{2^n} \leq \epsilon$
- **Failure of Newton's Method:** NM Fails if  $f'(x) = 0$  for some x.
- **Triangle Inequality:**  $|x + y| \leq |x| + |y|$

### §Key Definitions & Identities

- **Continuity:**  $f \in C^n[a, b]$  reads: the nth derivative of f on [a,b] is continuous.
- **Series Expansions**
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$  ( $\forall x \in \mathbb{R}$ )
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
  - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$  ( $|x| \leq 1$ )
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$  ( $|x| < 1$ )
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots : |x| < 1$
  - $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots : |x| < 1$

- **Core Identities**
  - $\sin^2 \theta + \cos^2 \theta = 1$
  - $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
  - $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- **Angle Transformations**
  - $\sin 2\theta = 2 \sin \theta \cos \theta$
  - $\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
  - $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$ ,  $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

- **Vector Norms**
  - $\|x\| > 0$       -  $\|\alpha x\| = |\alpha| \|x\|$
  - $\|x\| = 0 \Leftrightarrow x = 0$       -  $\|x + y\| \leq \|x\| + \|y\|$

### §Useful Examples

- Suppose  $g(x) = \frac{5}{x^2} + 2$  Show  $p_n = g(p_{n-1})$  will converge to g for  $\forall p_0 [2.5, 3]$ . Since this is a decreasing function, the max of g(x) is g(2.5) and the min of g(x) is g(3).
  - First compute the max,  $g(2.5) = 14/5 < 3$
  - Second compute the min,  $g(3) = 5/9 + 2 > 2.5$
  - Last compute  $|g'(x)| = -10/x^3 \leq \max_{x \in [2.5, 3]} |g'(x)| = 16/25 < 1$
- Given  $\|A\|$  is a natural matrix norm of matrix A. show  $\| \lambda \| \leq \|A\|$  for any nonsingular A and any  $\lambda$  of A.  $\|A\| = \max_{\|x\|=1} \|Ax\| \geq \| \lambda x \| : x$  is an e-vec s.t  $x = \frac{1}{\lambda} Ax \Rightarrow \|x\| = \frac{1}{|\lambda|} \|Ax\| \Rightarrow 1 = \frac{1}{|\lambda|} \| \lambda x \| = \|x\|$
- When performing Jacobi or GS, when computing L+U, flip the signs of all entries.
- To determine convergence for fixed point, compute  $g'(p_0) \leq 1$ , which gives a,b. Prove  $g(x)$  cts on [a,b],  $g(x) \in [a, b]$ ,  $g'(x)$  exists on (a,b),  $|g'(x)| \leq k : \forall x \in (a, b)$ ,  $0 < k < 1$

### §LA Determinants

- 2x2:  $|A| = ad - bc$
- nxn:  $|A| = \sum_{j=1}^n a_{ij} A_{ij}$  via cofactors  $A_{ij} = (-1)^{i+j} M_{ij}$
- **Properties**
  - Identical rows:  $|A| = 0$
  - Swap rows:  $|\tilde{A}| = -|A|$
  - Scale row:  $|\tilde{A}| = \lambda |A|$
  - $|A^{-1}| = \frac{1}{|A|}$

### §7 Norms & Eigen

- $\|x\|_2 = \sqrt{\sum x_i^2}$ ,  $\|x\|_\infty = \max |x_i|$  •  $\|A\|_2 = \sqrt{\rho(A)}$
- $\|A\|_\infty = \max_i \sum_j |a_{ij}|$  Basically sum all rows together and determine the largest one.
- $\forall x \in \mathbb{R}^n : \|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
- A distance between matrices A and B wrt a matrix norm  $\| \cdot \|$  is  $\|A - B\|$
- Theorem: For any vector  $x \neq 0$ , matrix Am and abt natural norm  $\| \cdot \|$  we have  $\|Ax\| \leq \|A\| \cdot \|x\|$
- Cauchy-Schwarz:  $\|x + y\|_2 \leq \|x\|_2 \|y\|_2$
- Eigen:  $\lambda$  is an eigenvalue if  $A\lambda = v \cdot \lambda$
- Finding Eigenthings:  $\det(A - \lambda I) = 0 : \forall \lambda$ .
- Spectral Radius:  $\rho(A) = \max |\lambda_i|$ ,  $\rho(A) \leq \|A\|$
- Theorem: If A is  $n \times n$ :
  - $\|A\|_2 = [\rho(A^t A)]^{1/2}$
  - $\rho(A) \leq \|A\| : \forall \| \cdot \|$
- Convergent:  $\lim_{k \rightarrow \infty} A^k = 0 \Leftrightarrow \rho(A) < 1$
- Matrix Norms: matrix norms have the following properties
  - $\|A\| \geq 0$  with  $\|A\| = 0 \Leftrightarrow A = 0$
  - $\|\alpha A\| = |\alpha| \cdot \|A\|$
  - $\|A + B\| \leq \|A\| + \|B\|$

### §Iterative Methods

- **General Iteration:**  $x^{k+1} = T x^k + c$
- **Jacobi:**  $x^{k+1} = D^{-1}(L + U)x^k + D^{-1}b$
- **Gauss-Seidel:**  $x^{k+1} = (D - L)^{-1}Ux^k + (D - L)^{-1}b$
- **Stein-Rosenberg:** For matrices with positive diagonals:  $\rho_{GS} \leq \rho_J < 1$
- **Speed of Convergence:** given matrices  $T_{GS} = (D - L)^{-1}U$  and  $T_J = D^{-1}(L + U)$ , compare  $\rho$ . The bigger the  $\rho$ , the faster the convergence.
- Error:  $\|x - x^k\| \leq \frac{\|T\|}{1 - \|T\|} \|x^1 - x^0\|$

- Stopping:  $\frac{\|x^k - x^{k-1}\|}{\|x^k\|} < \epsilon$

### §2 Nonlinear Equations

- **Bisection:**
  - While  $f(p_n) \neq 0$  or  $< T$ :  $p_n = \frac{a_1 + b_1}{2}$
  - Error:  $\frac{b_n - a_n}{2} < T$ ,  $p = a + \frac{b-a}{2}$
- **Fixed-Point:**
  - $p_n = g(p_{n-1})$ , converges if  $|g'(x)| \leq K < 1$
  - Algorithm: For  $i < N_0$ :  $p = g(p_0)$ , check  $|p - p_0| < T$
  - A fixed point is defined as a point in which  $p = f(p)$
- **Newton:**
  - $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
  - Quadratic convergence if  $f'(p) \neq 0$
- **Secant:**
  - $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$
  - Approx derivative:  $\frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$
  - Algorithm: Store  $q_0 = f(p_0)$ ,  $q_1 = f(p_1)$ , SET  $p = p_1 - \frac{q_1(p_1 - p_0)}{q_1 - q_0}$  IF STOPPING CONDITION: RETURN p; i++,  $p_0 = p_1$ ,  $q_0 = q_1$ ,  $p_1 = p$ ,  $q_1 = f(p)$  ENDWHILE OUTPUT FAILURE.

### §Theorems

- **Bisection:** Suppose  $f \in C[a, b] : f(a) \cdot f(b) < 0$ . Bisection generates  $\{p_n\}$  approximating a zero p with  $|p_n - p| \leq \frac{b-a}{2^n} : n \geq 1$
- **Fixed Point:** If  $g \in C[a, b]$ ,  $g([a, b]) \subseteq [a, b]$  g has a fixed point in  $[a, b]$ , additionally if  $|g'| \leq K < 1$ , then the fixed point is unique.
- **Fixed Point Theorem:** Let  $g \in C[a, b]$  and  $g(x) \in [a, b] : \forall x \in [a, b]$ . Suppose as well that g' exists on (a,b) and positive  $K < 1$  exists with  $|g'(x)| \leq K : \forall x \in (a, b)$ . Then for any number  $p_0 \in [a, b]$  the sequence defined by  $p_n = g(p_{n-1}) : n \geq 1$  converges to the unique point  $p \in [a, b]$
- **Corollary:** If g satisfies the hypothesis of the above theorem,  $|p_n - p| \leq k^n \max(p_0 - a, b - p_0)$  and  $|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0| : \forall n > 1$
- **Newton:** For  $f \in C^2[a, b]$  with simple root,  $\exists \delta > 0 : p_0 \in [p - \delta, p + \delta]$  converges.
- **Matrix Invertibility:**  $|A| \neq 0 \Leftrightarrow$  unique solution  $Ax = b \Leftrightarrow A^{-1}$  exists
- **Taylor:** With  $R_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$ ,  $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} : \xi \in (x, x_0)$
- **Existence of Inverse:** if A is square,  $\det A \neq 0 \Leftrightarrow Ax = 0$  has soln  $x = 0 \Leftrightarrow Ax = b$  has a unique soln for any n-vector b.  $\Leftrightarrow A^{-1}$  exists.
- **Diagonally Dominant Matrices:** dd matrices are nonsingular. A being dd means LU can be performed without P. A matrix is positive definite if  $x^t A x > 0$ . PD matrices are nonsingular,  $\forall i = 1, \dots, n : a_{ii} > 0$ ,  $\max |a_{kj}| > \max |a_{ii}|$ ,  $(a_{ii})^2 < a_{ii}a_{jj} : \forall i \neq j$ .
- **Convergence and Spectral Radii:** equivalent statements: A is convergent,  $\rho < 1$ ,  $\lim A^n x = 0 : \forall x$ ,  $\lim \|A^n\| = 0 : \forall$  norms.
- **Convergence of DD:** If A is strictly DD, Jacobi and GS converge to the unique soln.  $Ax=b$ .
- **Positive Definitive Check:** A matrix is PD if the leading principle submatrix determinants are positive.
- **Positive Definitive Check:** A matrix is PD iff it may be factored into  $LL^T$
- **Non-singularity Check:** A matrix A has an inverse iff  $\det A \neq 0$ .
- **Determinant of Triangular Matrices:** The determinant of a triangular matrix is  $\Pi a_{ii}$ .

### §Proofs

- **Bisection (THM1):**  $\forall n \geq 1 : b_n - a_n = (b - a) \cdot \frac{1}{2^{n-1}} : p \in (a_n, b_n)$ . Since  $p_n = \frac{1}{2}(a_n + b_n) : \forall n \geq 1$ ,  $|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$ . □
- **Fixed Point:** Part i: If  $g \in [a, b]$ ,  $g(x) \in [a, b] : \forall x \in [a, b]$  then g(x) has a fixed point in [a,b]. If  $g(a) = a$  or  $g(b) = b$ , g has a fixed point at an endpoint. Suppose for contradiction that it does not.  $g(a) > a$  and  $g(b) < b$ . Define  $h(x) = g(x) - x$ . Then h is cts on [a,b] and  $h(a) = g(a) - a > 0$  and  $h(b) = g(b) - b < 0$  IVT states that  $\exists p \in (a, b) : h(p) = 0$  Thus  $g(p) - p = 0 \Rightarrow p$  is a fixed point of g. Part ii: Suppose as well  $|g'(x)| \leq k < 1 : \forall x \in (a, b)$  and that  $p, q \in [a, b] : p \neq q$ . By MVT,  $\exists \zeta : \frac{g(p) - g(q)}{p - q} = g'(\zeta)$ .  $|p - q| = |g(p) - g(q)| = |g'(\zeta)| |p - q| \leq k |p - q| < |p - q|$  contradiction.

## §2: Error Analysis and Accelerating Convergence

### Basic Methods

- **Newton's Method:** Quadratic convergence if  $f'(p) \neq 0$ . Iteration:  $x_{n+1} = x_n - f(x_n)/f'(x_n)$ .
- **Secant Method:** Superlinear convergence (order  $\approx 1.618$ ). Uses two previous points.
- **False Position:** Bracketing method combining bisection and secant approaches.

### Convergence Analysis

- Order of convergence  $\alpha$ :  $\lim_{n \rightarrow \infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$
- Linear ( $\alpha = 1$ ), Quadratic ( $\alpha = 2$ )
- Fixed-point: Linear if  $g'(p) \neq 0$ , quadratic if  $g'(p) = 0$  and  $g''$  bounded.

### Special Cases

- Multiple roots: Modify Newton's using  $\mu(x) = f(x)/f'(x)$
- **Aitken's  $\Delta^2$ :** Accelerates linear sequences. Is given by  $\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-p_{n+1})-(p_{n+1}-p_n)}$
- **Steffensen's:** Combines Aitken's with fixed-point, achieves quadratic convergence

### Polynomial Methods

- Horner's method: Efficient evaluation ( $n$  mults/adds)
- Deflation: Find roots sequentially via  $P(x) \approx Q(x)(x - x_0)$
- Müller's method: Quadratic interpolation using 3 points
- Fundamental thm of alg: If  $P(x)$  has a degree  $n \geq 1$ ,  $P(x)$  has at least one root.
- Cor: there also exists unique constants  $x_1, \dots, x_k$  such that  $\sum_{i=1}^k m_i = n, P(x) = a_n \times \prod_{i=1}^k (x - x_i)^{m_i}$
- Cor: these functions are unique.

**Weierstrass:**  $\forall f$  continuous on  $[a, b]$ ,  $\forall \epsilon > 0$ ,  $\exists$  polynomial  $p(x)$  with  $|f(x) - p(x)| < \epsilon$  on  $[a, b]$ .

**Lagrange Interpolation:**  $P(x) = \sum_{m=0}^N f_m L_m(x)$ , where  $L_m(x) = \prod_{\substack{k=0 \\ k \neq m}}^N \frac{x - x_k}{x_m - x_k}$

**Interpolation Error:**  $f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^n (x - x_k)$  for  $f \in C^{n+1}[a, b]$

### Newton's Divided Differences:

$$f[x_i] = f(x_i), \quad f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$P_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

**Finite Differences** (step  $h$ ): Forward:  $\Delta f_i = f_{i+1} - f_i$ ; Backward:  $\nabla f_i = f_i - f_{i-1}$

**Newton's Formulas:** Forward:  $P(x) = \sum_{k=0}^n \binom{n}{k} \Delta^k f_0$  where  $x = x_0 + sh$

**Hermite:** Given  $(x_j, f(x_j), f'(x_j))$ , unique degree  $\leq 2n + 1$  polynomial:

$$H(x) = \sum_{j=0}^n f(x_j) H_j(x) + \sum_{j=0}^n f'(x_j) \hat{H}_j(x) \quad \text{Error: } f(x) - H(x) = \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

**Cubic Splines:**  $S(x_j) = f(x_j)$  and  $S \in C^2[a, b]$ ; Boundary: Clamped ( $S'$  at endpoints) or Natural ( $S'' = 0$  at endpoints) Error:  $\max |f(x) - S(x)| \leq \frac{5M}{384} h^4$ , where

$$h = \max(x_{j+1} - x_j), \quad M = \max |f^{(4)}|$$

### Numerical Differentiation:

- Forward/Backward ( $O(h)$ ):  $f'(x_0) \approx \frac{f(x_0 \pm h) - f(x_0)}{h}$
- Centered ( $O(h^2)$ ):  $f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$
- Second Derivative:  $f''(x_0) \approx \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$

**Richardson Extrapolation:**  $N_{j+1}(h) = N_j(h/2) + \frac{N_j(h/2) - N_j(h)}{2^j - 1}$

### Numerical Integration:

- Trapezoid Rule ( $O(h^2)$ ):  $\int_a^b f(x) dx \approx \frac{h}{2} [f(a) + f(b)]$
- Simpson's Rule ( $O(h^4)$ ):  $\int_a^b f(x) dx \approx \frac{h}{3} [f(a) + 4f(\frac{a+b}{2}) + f(b)]$
- Composite errors: Trapezoid  $-\frac{(b-a)h^2}{12} f''(\mu)$ ; Simpson  $-\frac{(b-a)h^4}{180} f^{(4)}(\mu)$

**Romberg:**  $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^j - 1}$ , error  $O(h^{2j})$

**Gaussian Quadrature:** Uses Legendre polynomial roots as nodes, exact for degree  $\leq 2n - 1$  Scale to  $[a, b]$  via  $\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{(b-a)t+a+b}{2}\right) dt$

### §ODE Initial Value Problems

- **Basic Problem:**  $y'(t) = f(t, y)$ ,  $y(a) = \alpha$
- **Lipschitz Condition:**  $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$
- Existence/uniqueness guaranteed when  $|\frac{\partial f}{\partial y}| \leq L$  over convex domain  $D$

### Numerical Methods

- **Euler's Method:**  $w_{i+1} = w_i + hf(t_i, w_i)$  [Error:  $O(h)$ ]
- **Taylor Methods:**  $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$  where  $T^{(n)} = f + \frac{h}{2}f' + \dots + \frac{h^{n-1}}{n!}f^{(n-1)}$
- **Runge-Kutta Methods:** **Midpoint (RK2)**  $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f_i)$

**Modified Euler (RK2)**  $w_{i+1} = w_i + \frac{h}{2}(f_i + f(t_{i+1}, w_i + hf_i))$

- **Classical RK4:**  $k_1 = hf(t_i, w_i)$   
 $k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2})$   
 $k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2})$  [Error:  $O(h^4)$ ]  
 $k_4 = hf(t_i + h, w_i + k_3)$   
 $w_{i+1} = w_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

### Error Analysis

- **Local truncation error:**  $\tau_{i+1} = y(t_{i+1}) - w_{i+1}$  given  $w_i = y(t_i)$
- **Global truncation error:** Accumulated error across all steps
- For Euler:  $|\tau_i| \leq \frac{h^2}{2} M$  (local),  $O(h)$  (global) where  $M = \max |y''|$
- For RK4:  $O(h^5)$  (local),  $O(h^4)$  (global)

### Stability & Step Size

- Well-posed problem requires: unique solution exists + small input changes  $\rightarrow$  small output changes
- Step size formula:  $h < \frac{2\varepsilon}{M(b-a)}$  for error  $\varepsilon$ , where  $M = \max |y''|$
- Example:  $y' = y \cos t$  has Lipschitz constant  $L = 1$  since  $|\frac{\partial f}{\partial y}| = |\cos t| \leq 1$