

- §1.1 Error Analysis
- **Normalization:** 32 bit - Sign bit: 1, Sign exp: 1, Exp: 7, Normalized mantissa: 23.
- Absolute Error =  $|p - p^*|$ , Relative Error =  $\frac{|p - p^*|}{|p|}$
- Significant Digits:  $RE < 5 \times 10^{-t}$
- **f(x):** Machine representation
- **Cancellation Error:** Subtracting nearly equal numbers
- **Example:**  $p = 0.54617$ ,  $q = 0.54601$ , true  $r = p - q = 0.00016$
- 4-digit:  $p^* = 0.5462$ ,  $q^* = 0.5460$ ,  $r^* = 0.002$  (RE=25%)
- **Nested Multiplication:** Reduces error
- $f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99 = (((1.01z - 4.62)z - 3.11)z + 12.2)z - 1.99$

- $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$
- Remainder:  $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$ ,  $\xi \in (a, x)$
- Linear Approx:  $f(x_0 + h) \approx f(x_0) + h f'(x_0)$

- **Example:**  $\xi \in (0, \pi/2)$ ,  $\sin \xi \leq 1 \Rightarrow R_n \leq \frac{(x)^n}{n!}$

### §1.3 Convergence

- $\alpha = \lim_{n \rightarrow \infty} (\alpha_n)$
- Rate:  $\alpha_n = \alpha + O(\beta_n)$  if  $|\alpha_n - \alpha| \leq K|\beta_n|$
- Find largest  $p$  where  $\alpha_n - \alpha = O(1/n^p)$

### §1.4 Matrix Operations

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ ,  $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$
- $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

### §6.1 Gaussian Elimination

- $O(n^3)$  complexity
- **Pivoting:**
  - Partial (PP): Max element in column
  - Scaled PP:  $s_i = \max_j |a_{ij}|$ , pivot  $\max(a_{ik}/s_i)$
  - Complete (CP): Full matrix search ( $O(n^3)$ )
- **LU Decomposition:**  $PA = LU$  through GE steps and  $LUx = Pb$ .
- **LU Algorithm:**  $L = E_{n-a, m-b}^{-1} E_{n-a-1, m-b+1}^{-1} \dots U = GE$ .
- **Choleski Factorization:** If a matrix is symmetric and positive definite, it may be factored to the form  $LDL^T$

### §6.2 Special Matrices

- **Inverse Matrix:** An inverse matrix of A is  $A^{-1}$  such that  $AA^{-1} = I$
- Properties:  $(AB)^{-1} = B^{-1}A^{-1}$ ,  $(A^{-1})^T = (A^T)^{-1}$
- **Singular:** A matrix is singular iff its det is 0.
- **Diagonal:**  $d_{ij} = 0$  for  $i \neq j$ : All non-diagonal entries are 0.
- **Symmetric:**  $A = A^T$ ,  $(AB)^T = B^T A^T$
- **Permutation:** Row swaps of  $I_n$ ,  $PA$  reorders rows:  $P^T = P^{-1}$
- **Diagonally Dominant:**  $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$  (nonsingular)
- **Positive Definite:**  $x^T A x > 0 \Rightarrow A = LDL^T$ ,  $a_{ii} > 0$ ,  $a_{ij}^2 < a_{ii}a_{jj}$
- **Minor:**  $M_{ij}$  is a submatrix of A with the row i deleted and column j deleted.
- **Band:** an  $n \times n$  matrix is a band matrix if  $p, q \in \mathbb{Z} : 1 \leq p, q \leq n$  exist with  $a_{ij} = 0$  for  $i + p \leq j$  or  $j + q \geq i$  The bandwidth is defined as  $w = p + q - 1$ . For a diagonal matrix,  $p = 1, q = 1, w = 1$
- **Tridiagonal:** Band with  $p = 2, q = 2$ . It exhibits the following properties
  - $a_{ii} = l_{ii}$       -  $a_{i, i+1} = l_{ii}u_{i, i+1} : i = 1 \dots n - 1$
  - $a_{i, i-1} = l_{i, i-1} : i = 2, 3, \dots, n$       -  $a_{ii} = l_{i, i-1}u_{i-1, i} + l_{ii} : i = 2 \dots n$
- Crout Factorization: This factorization happens in  $O(n)$  time

### §Strategies

- **RoC With inf limit:** set  $h = 1/n$  and solve accordingly.
- $D \cdot (L+U)$ : given D has ONLY diagonal entries and L+U has NO diagonal entries, the resulting matrix A is composed of entries  $a_{ij} = d_{ii} \cdot (l + u)_{ij}$
- **Verification of Bisection:** To verify bisection can be applied, make sure that f(a) and f(b) are of different signs.
- **Error of Bisection:** To compute the accuracy of bisection to an  $\epsilon$ , we use  $\frac{b-a}{2^n} \leq \epsilon$
- **Failure of Newton's Method:** NM Fails if  $f'(x) = 0$  for some x.
- **Triangle Inequality:**  $|x + y| \leq |x| + |y|$

### §Key Definitions & Identities

- **Continuity:**  $f \in C^n[a, b]$  reads: the nth derivative of f on [a, b] is continuous.
- **Series Expansions**

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (\forall x \in \mathbb{R})$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad (|x| \leq 1)$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (|x| < 1)$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots : |x| < 1$
- $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots : |x| < 1$
- $f(x_0 - h) = f(x_0) - h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots$

### Core Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

### Vector Norms

- $\|x\| > 0$       -  $\| \alpha x \| = |\alpha| \|x\|$
- $\|x\| = 0 \Leftrightarrow x = 0$       -  $\|x + y\| \leq \|x\| + \|y\|$

### Useful Examples

- Suppose  $g(x) = \frac{5}{x^2} + 2$  Show  $p_n = g(p_{n-1})$  will converge to g for  $\forall p_0 [2.5, 3]$ .
- Since this is a decreasing function, the max of g(x) is g(2.5) and the min of g(x) is g(3).
- First compute the max,  $g(2.5) = 14/5 < 3$
- Second compute the min,  $g(3) = 5/9 + 2 > 2.5$
- Last compute  $|g'(x)| = -10/x^3 \leq \max_{x \in [2.5, 3]} |g'(x)| = 16/25 < 1$
- Given  $\|A\|$  is a natural matrix norm of matrix A. show  $|\lambda| \leq \|A\|$  for any nonsingular A and any  $\lambda$  of A.  $\|A\| = \max_{\|x\|=1} \|Ax\| \geq \|Ax\| : x$  is an e-vec s.t  $x = \frac{1}{\lambda} Ax \Rightarrow \|x\| = \frac{1}{|\lambda|} \|Ax\| \Rightarrow 1 = \frac{1}{|\lambda|} \|Ax\| \Rightarrow \|Ax\| = |\lambda|$
- When performing Jacobi or GS, when computing L+U, flip the signs of all entries.
- To determine convergence for fixed point, compute  $g'(p_0) \leq 1$ , which gives a, b. Prove  $g(x)$  cts on [a, b],  $g(x) \in [a, b]$ ,  $g'(x)$  exists on (a, b),  $|g'(x)| \leq k : \forall x \in (a, b), 0 < k < 1$
- In general, Aitken's method only accelerates convergence of linear sequences. So not bisection, Newton's, secant...
- Truncation of IVPs is defined as the error made in one step. (assuming all previous steps executed without error.)

### §LA Determinants

- 2x2:  $|A| = ad - bc$
- nxn:  $|A| = \sum_{j=1}^n a_{ij} A_{ij}$  via cofactors  $A_{ij} = (-1)^{i+j} M_{ij}$
- **Properties**
  - Identical rows:  $|A| = 0$
  - Swap rows:  $|\tilde{A}| = -|A|$
  - Scale row:  $|\tilde{A}| = \lambda |A|$
  - $|A^{-1}| = \frac{1}{|A|}$

### §7 Norms & Eigen

- $\|x\|_2 = \sqrt{\sum x_i^2}$ ,  $\|x\|_\infty = \max |x_i|$  •  $\|A\|_2 = \sqrt{\rho(A)}$
- $\|A\|_\infty = \max_i \sum_j |a_{ij}|$  Basically •  $\|A\| = \max_{\|x\|=1} \|Ax\|$
- sum all rows together and determine the largest one.
- $\forall x \in \mathbb{R}^n : \|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
- A distance between matrices A and B wrt a matrix norm  $\|\cdot\|$  is  $\|A - B\|$
- Theorem: For any vector  $x \neq 0$ , matrix Am and abt natural norm  $\|\cdot\|$  we have  $\|Ax\| \leq \|A\| \cdot \|x\|$
- Cauchy-Schwarz:  $\|x + y\|_2 \leq \|x\|_2 \|y\|_2$
- Eigen:  $\lambda$  is an eigenvalue if  $A\lambda = v \cdot \lambda$
- Finding Eigenthings:  $\det(A - \lambda I) = 0 : \forall \lambda$ .
- Spectral Radius:  $\rho(A) = \max |\lambda_i|$ ,  $\rho(A) \leq \|A\|$
- Theorem: If A is  $n \times n$ :
  - $\|A\|_2 = [\rho(A^t A)]^{1/2}$
  - $\rho(A) \leq \|A\| : \forall \|\cdot\|$
- Convergent:  $\lim_{k \rightarrow \infty} A^k = 0 \Leftrightarrow \rho(A) < 1$
- Matrix Norms: matrix norms have the following properties
  - $\|A\| \geq 0$       -  $\|A\| = 0 \Leftrightarrow A = 0$
  - $\|\alpha A\| = |\alpha| \cdot \|A\|$       -  $\|AB\| = \|A\| \|B\|$
  - $\|A + B\| \leq \|A\| + \|B\|$

### §Iterative Methods

- **General Iteration:**  $x^{k+1} = T x^k + c$
- **Jacobi:**  $x^{k+1} = D^{-1}(L + U)x^k + D^{-1}b$
- **Gauss-Seidel:**  $x^{k+1} = (D - L)^{-1}Ux^k + (D - L)^{-1}b$
- **Stein-Rosenberg:** For matrices with positive diagonals:  $\rho_{GS} \leq \rho_J < 1$
- **Speed of Convergence:** given matrices  $T_{GS} = (D - L)^{-1}U$  and  $T_J = D^{-1}(L + U)$ , compare  $\rho$ . The smaller the  $\rho$ , the faster the convergence.

- Error:  $\|x - x^k\| \leq \frac{\|T\|^k}{1 - \|T\|} \|x^1 - x^0\|$

- Stopping:  $\frac{\|x^k - x^{k-1}\|}{\|x^k\|} < \epsilon$

### §2 Nonlinear Equations

- **Bisection:**
  - While  $f(p_n) \neq 0$  or  $< T$ :  $p_n = \frac{a_1 + b_1}{2}$
  - Error:  $\frac{b_n - a_n}{2} < T$ ,  $p = a + \frac{b-a}{2}$
- **Fixed-Point:**
  - $p_n = g(p_{n-1})$ , converges if  $|g'(x)| \leq K < 1$
  - Algorithm: For  $i < N_0$ :  $p = g(p_0)$ , check  $|p - p_0| < T$
  - A fixed point is defined as a point in which  $p = f(p)$
- **Newton:**
  - $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
  - Quadratic convergence if  $f'(p) \neq 0$
- **Secant:**
  - $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$
  - Approx derivative:  $\frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$
  - Algorithm: Store  $q_0 = f(p_0)$ ,  $q_1 = f(p_1)$ , SET  $p = p_1 - \frac{q_1(p_1 - p_0)}{q_1 - q_0}$  IF STOPPING CONDITION: RETURN p; i++,  $p_0 = p_1, q_0 = q_1, p_1 = p, q_1 = f(p)$  ENDWHILE OUTPUT FAILURE.

### §Theorems

- **Bisection:** Suppose  $f \in C[a, b] : f(a) \cdot f(b) < 0$ . Bisection generates  $\{p_n\}$  approximating a zero p with  $|p_n - p| \leq \frac{b-a}{2^n} : n \geq 1$
- **Fixed Point:** If  $g \in C[a, b]$ ,  $g([a, b]) \subseteq [a, b]$  g has a fixed point in  $[a, b]$ , additionally if  $|g'| \leq K < 1$ , then the fixed point is unique.
- **Fixed Point Theorem:** Let  $g \in C[a, b]$  and  $g(x) \in [a, b] : \forall x \in [a, b]$ . Suppose as well that  $g'$  exists on (a, b) and positive  $K < 1$  exists with  $|g'(x)| \leq K : \forall x \in (a, b)$ . Then for any number  $p_0 \in [a, b]$  the sequence defined by  $p_n = g(p_{n-1}) : n \geq 1$  converges to the unique point  $p \in [a, b]$
- **Corollary:** If g satisfies the hypothesis of the above theorem,  $|p_n - p| \leq k^n \max(p_0 - a, b - p_0)$  and  $|p_n - p| \leq \frac{1}{1-K} |p_1 - p_0| : \forall n > 1$
- **Newton:** For  $f \in C^2[a, b]$  with simple root,  $\exists \delta > 0 : p_0 \in [p - \delta, p + \delta]$  converges.
- **Matrix Invertibility:**  $|A| \neq 0 \Leftrightarrow$  unique solution  $Ax = b \Leftrightarrow A^{-1}$  exists
- **Taylor:** With  $R_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$ ,  $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} : \xi \in (x, x_0)$
- **Existence of Inverse:** if A is square,  $\det A \neq 0 \Leftrightarrow Ax = 0$  has soln  $x = 0 \Leftrightarrow Ax = b$  has a unique soln for any n-vector b.  $\Leftrightarrow A^{-1}$  exists.
- **Diagonally Dominant Matrices:** dd matrices are nonsingular. A being dd means LU can be performed without P. A matrix is positive definite if  $x^t A x > 0$ . PD matrices are nonsingular,  $\forall i = 1, \dots, n : a_{ii} > 0$ ,  $\max |a_{kj}| > \max |a_{ii}|$ ,  $(a_{ij})^2 < a_{ii}a_{jj} : \forall i \neq j$ .
- **Convergence and Spectral Radii:** equivalent statements: A is convergent,  $\rho < 1$ ,  $\lim A^n x = 0 : \forall x$ ,  $\lim \|A^n\| = 0 : \forall$  norms.
- **Convergence of DD:** If A is strictly DD, Jacobi and GS converge to the unique soln.  $Ax=b$ .
- **Positive Definitive Check:** A matrix is PD if the leading principle submatrix determinants are positive.
- **Positive Definitive Check:** A matrix is PD iff it may be factored into  $LL^T$
- **Non-singularity Check:** A matrix A has an inverse iff  $\det A \neq 0$ .
- **Determinant of Triangular Matrices:** The determinant of a triangular matrix is  $\Pi a_{ii}$ .

### §Proofs

- **Bisection (THM1):**  $\forall n \geq 1 : b_n - a_n = (b - a) \cdot \frac{1}{2^{n-1}} : p \in (a_n, b_n)$ . Since  $p_n = \frac{1}{2}(a_n + b_n) : \forall n \geq 1, |p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$ . □
- **Fixed Point:** Part i: If  $g \in [a, b]$ ,  $g(x) \in [a, b] : \forall x \in [a, b]$  then g(x) has a fixed point in [a, b]:  
If  $g(a) = a$  or  $g(b) = b$ , g has a fixed point at an endpoint. Suppose for contradiction that it does not.  $g(a) > a$  and  $g(b) < b$ . Define  $h(x) = g(x) - x$ . Then h is cts on [a, b] and  $h(a) = g(a) - a > 0$  and  $h(b) = g(b) - b < 0$  IVT states that  $\exists p \in (a, b) : h(p) = 0$  Thus  $g(p) - p = 0 \Rightarrow p$  is a fixed point of g.  
Part ii: Suppose as well  $|g'(x)| \leq k < 1 : \forall x \in (a, b)$  and that  $p, q \in [a, b] : p \neq q$ . By MVT,  $\exists \zeta : \frac{g(p) - g(q)}{p - q} = g'(\zeta)$ .  $|p - q| = |g(p) - g(q)| = |g'(\zeta)| |p - q| \leq k |p - q| < |p - q|$  contradiction.

## §2: Error Analysis and Accelerating Convergence

### Basic Methods

- **Newton's Method:** Quadratic convergence if  $f'(p) \neq 0$ . Iteration:  
 $x_{n+1} = x_n - f(x_n)/f'(x_n)$ .
- **Secant Method:** Superlinear convergence (order  $\approx 1.618$ ). Uses two previous points.
- **Newton's Improved Method:**  $p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{f'(p_n)^2 - f(p_n)f''(p_n)}$

### Convergence Analysis

- A soln  $p$  of  $f(x) = 0$  is a zero of multiplicity  $m$  of  $f$  if for  $x \neq p$  we can write  $f(x) = (x - p)^m q(x)$  where  $\lim_{x \rightarrow p} q(x) \neq 0$ . (Simple zeros are multiplicity 1).
- Order of convergence  $\alpha$ :  $\lim_{|p_n - p|^\alpha} = \lambda$
- Linear ( $\alpha = 1$ ), Quadratic ( $\alpha = 2$ )
- Fixed-point: Linear if  $g'(p) \neq 0$ , quadratic if  $g'(p) = 0$  and  $g''$  bounded.

### Special Cases

- Multiple roots: Modify Newton's using  $\mu(x) = f(x)/f'(x)$
- **Aitken's  $\Delta^2$ :** Accelerates linear sequences. Is given by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{(p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)}$$

### Polynomial Methods

- Horner's method: Efficient evaluation ( $n$  mults/adds) Algorithm: ex: evaluate  $P(x) = \sum_{i=0}^n a_i x^i$  and derivative at  $x_0$ . Input  $n, a_j, x_0 : 0 \leq j \leq n$ . Output:  $y = P(x_0)$ ,  $z = P'(x_0)$ . Set  $y = a_n, z = a_n$ . For  $j = n - 1, n - 2, \dots, 1$  set  $y = x_0 y + a_j, z = x_0 z + y$ . Endfor set  $y - x_0 y + a_0$  output  $y, z$ .
- Deflation: Find roots sequentially via  $P(x) \approx Q(x)(x - x_0)$
- Fundamental thm of alg: If  $P(x)$  has a degree  $n \geq 1$ ,  $P(x)$  has at least one root.
- Cor: there also exists unique constants  $x_1, \dots, x_k$  such that  $\sum_{i=1}^k m_i = n, P(x) = a_n \times \prod_{i=1}^k (x - x_i)^{m_i}$
- Cor: these functions are unique.

**Weierstrass:**  $\forall f$  cts on  $[a, b], \forall \varepsilon > 0, \exists$  polynomial  $p(x)$  with  $|f(x) - p(x)| < \varepsilon$   $\forall x \in [a, b]$ .

### Newton's Divided Differences:

$$f[x_i] = f(x_i), \quad f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$P_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

**Hermite:** Given  $(x_j, f(x_j), f'(x_j))$ , unique degree  $\leq 2n + 1$  we have:

- $H(x) = \sum_{j=0}^n f(x_j) H_j(x) + \sum_{j=0}^n f'(x_j) \hat{H}_j(x)$
- $H_j(x) = [1 - 2(x - x_j)L_j'(x_j)]L_j^2(x)$
- $\hat{H}_j(x) = (x - x_j)L_j^2(x)$
- Note:  $L_j(x)$  denotes the  $j$ th Lagrange coefficient polynomial of degree  $n$ .
- Error:  $f(x) - H(x) = \frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$
- Parametric curve Interpolation:

$$\begin{aligned} x(t) &= [2(x_0 - x_1) + 3(a_0 + a_1)]t^2 + [3(x_1 - x_0) - 3(a_1 + 2a_0)]t^2 + 3a_0 t + x_0 \\ y(t) &= [2(y_0 - y_1) + 3(\beta_0 + \beta_1)]t^2 + [3(y_1 - y_0) - 3(\beta_1 + 2\beta_0)]t^2 + 3\beta_0 t + y_0 \end{aligned}$$

### Cubic Splines:

- $S_j^{(n)}(x) = S_{j+1}^{(n)}(x) : n = 0, 1, 2; x$  is an interior point
- Err:  $\max |f(x) - S(x)| \leq \frac{5M}{384} h^4 : h = \max(x_{j+1} - x_j), M = \max |f^{(4)}|$
- $S(x_j) = f(x_j) : \forall j$  provided.  $S \in C^2[a, b]$
- Clamped:  $S'(a) = f'(a), S'(b) = f'(b) : a, b$  are endpoints
- Natural:  $S''(a) = S''(b) = 0 : a, b$  are endpoints

**Richardson Extrapolation:**  $N_{j+1}(h) = N_j(h/2) + \frac{N_j(h/2) - N_j(h)}{2^j - 1}$

### Numerical Integration:

- **Trapezoid Rule ( $O(h^2)$ ):**
  - Single:  $\int_a^b f(x) dx \approx \frac{h}{2} [f(a) + f(b)]$ , Error:  $-\frac{h^3}{12} f''(\xi)$
  - Composite:  $\int_a^b f(x) dx \approx \frac{h}{2} [f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b)]$
  - Error:  $-\frac{(b-a)h^2}{12} f''(\xi) \approx -\frac{M(b-a)^3}{12n^2}$  where  $M = \max |f''|$
- **Midpoint Rule ( $O(h^2)$ ):**
  - Single:  $\int_a^b f(x) dx \approx (b - a)f(\frac{a+b}{2})$
  - Composite:  $\int_a^b f(x) dx \approx h \sum_{i=1}^n f(a + (i - \frac{1}{2})h)$
- **Simpson's Rules ( $O(h^4)$ ):**
  - 1/3 Rule:  $\int_a^b f(x) dx \approx \frac{h}{3} [f(a) + 4f(\frac{a+b}{2}) + f(b)]$
  - Composite:  $\int_a^b f(x) dx \approx \frac{h}{3} [f(a) + 4 \sum_{i=1,3,5}^{n-1} f(x_i) + 2 \sum_{i=2,4,6}^{n-2} f(x_i) + f(b)]$
  - Error:  $-\frac{h^5}{90} f^{(4)}(\xi)$  (single),  $-\frac{(b-a)h^4}{180} f^{(4)}(\xi)$  (composite)
  - 3/8 Rule:  $\int_a^b f(x) dx \approx \frac{3h}{8} [f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + \dots + f(x_n)]$
  - 3/8 Error:  $-\frac{(b-a)^5}{6480} f^{(4)}(\xi)$

### Romberg - $O(h_k^{2j})$ :

- $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^j - 1 - 1}$ , error  $O(h^{2j})$
- $R_{k,1}$  represents the approximation of the integral using  $m_k = 2^{k-1}$  intervals
- Stop when  $|R_{k,k} - R_{k-1,k-1}| < \varepsilon$ .
- The degree of precision of a quadrature formula is the largest  $n$  such that the formula is exact for  $x^k : \forall k = 0, 1, \dots, n$

### §ODE Initial Value Problems

- **Basic Problem:**  $y'(t) = f(t, y), y(a) = \alpha$
- **Lipschitz Condition:**  $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$
- Existence/uniqueness guaranteed when  $|\frac{\partial f}{\partial y}| \leq L$  over convex domain  $D$

### Numerical Methods

- **Euler's Method:**  $y_{i+1} = y_i + hf(t_i, y_i)$  [Error:  $O(h)$ ]
- truncation error  $>$  round off in Euler for  $h$  large
- **Taylor Methods:**  $y_{i+1} = y_i + hT^{(n)}(t_i, y_i)$  where  $T^{(n)} = f + \frac{h}{2}f' + \dots + \frac{h^{n-1}}{n!}f^{(n-1)}$
- **Runge-Kutta Methods:**
- **Midpoint (RK2)**

$y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{h}{2}f_i)$   
 $k_1 = hf(t_i, y_i)$   
 $k_2 = hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2})$

[Error:  $O(h^4)$ ]
- **Classical RK4:**

$k_3 = hf(t_i + \frac{h}{2}, y_i + \frac{k_2}{2})$   
 $k_4 = hf(t_i + h, y_i + k_3)$   
 $y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

### Modified Euler (RK2)

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f(t_{i+1}, y_i + hf_i))$$

### Error Analysis

- **Local truncation error:**  $\tau_{i+1} = y(t_{i+1}) - w_{i+1}$  given  $w_i = y(t_i)$
- **Global truncation error:** Accumulated error across all steps
- For Euler:  $|\tau_i| \leq \frac{h^2}{2} M$  (local),  $O(h)$  (global) where  $M = \max |y''|$
- For RK4:  $O(h^5)$  (local),  $O(h^4)$  (global)

### Stability & Step Size

- Well-posed problem requires: unique solution exists + small input changes  $\rightarrow$  small output changes
- Step size formula:  $h < \frac{2\varepsilon}{M(b-a)}$  for error  $\varepsilon$ , where  $M = \max |y''|$
- Example:  $y' = y \cos t$  has Lipschitz constant  $L = 1$  since  $|\frac{\partial f}{\partial y}| = |\cos t| \leq 1$

### Chapter 3: Interpolation Lagrange Interpolation

- $\Delta^k: \Delta^2 f_i = \Delta(\Delta f_i) = \Delta(f_{i+1} - f_i) = f_{i+2} - 2f_{i+1} + f_i$

- **Interpolation Error:**  $f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^n (x - x_k)$  for  $f \in C^{n+1}[a, b]$
- $P(x) = \sum_{m=0}^N f(x_m) L_m(x)$ , where  $L_m(x) = \prod_{k \neq m}^N \frac{x - x_k}{x_m - x_k}$

- Reuse computations with **Neville's Method:**  
for  $i = 1, 2, \dots, n$  do: for  $j = 1, 2, \dots, i$  do  $Q_{i,j} \leftarrow \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}$
- **Cubic splines:** Solve tridiagonal system ( $O(n)$  ops) with:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1})$$

### Adaptive Quadrature

- Error estimate:  $\frac{1}{15}|S(a, b) - S(a, c) - S(c, b)|, c = (a + b)/2$
- Subdivide intervals where error  $> \epsilon/2$ .
- Step 1: Apply Simpson's with  $h = (b - a)/2$ .  
 $\int_a^b f(x) = h/3 [f(a) + 4f(a + h) + f(b)] - \frac{h^5}{90} f^{(4)}(\mu) \mu \in (a, b)$
- Step 2: Find error using Simpson's on  $h = (b - a)/4$   
 $\int_a^b f(x) = h/6 [f(a) + 4f(a + h/2) + 2f(a + b) + 4f(a + 3h/2) + f(b)] - (\frac{h}{2})^4 \cdot \frac{(b-a)}{180} f^{(4)}(\bar{\mu}) : \bar{\mu} \in (a, b)$
- Note: We assume  $f^4(\mu) = f^4(\bar{\mu})$ : true for small  $h$ .
- Step 3: Calculate error as  $1/10 \left| S(a, b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) \right| < \varepsilon$
- Step 4: If true, RETURN. Else, GOTO step 1.

### Gaussian Quadrature

- $\int_a^b w(x)f(x)dx = \sum_{i=1}^n w_i f(x_i)$  where  $w(x)$ : weight functions,  $w_i$ : weight at  $i$ ,  $x_i$  node at  $i$ .
- Nodes: Roots of Legendre polynomials  $P_n(x)$ . Weights:  $c_i = \int_{-1}^1 \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx$
- Exact for polynomials of degree  $\leq 2n - 1$ .
- Step 1: Transform to  $[-1, 1]$ :  $x = \frac{(b-a)t + (a+b)}{2}, dx = \frac{b-a}{2} dt$
- Step 2: substitute  $x$  into integrand
- Step 3: use the formulae to get the answer:
  - 1-point:  $\int_{-1}^1 f(x)dx = 2f(0)$
  - 2-point:  $\int_{-1}^1 f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) : w_1 = w_2; x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$
  - 3-point:  $\int_{-1}^1 f(x)dx = \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$

### Legendre Polynomials

- $P_n(x)$  denotes the  $n$ 'th degree Legendre polynomial
- $\int_{-1}^1 P(x)P_n(x)dx = 0 : P(x)$  is of degree  $\neq n$
- $P_0(x) = 1, P_1(x) = x, P_2(x) = x^2 - 1/3, P_3(x) = x^3 - 3/5x, P_4(x) = x^4 - 6/7x^2 + 3/35$

### Tips:

- Use Gaussian quadrature with  $n = c \rightarrow$  apply  $c$ -point Gauss-Legendre Rule.
- When asked for the degree of precision, plug in  $x^0, x^1, x^2, \dots, x^n$  until failure. The  $n$  before it fails is the degree of precision. A formula is **Quadrature** if it has a degree of precision  $\geq 3$
- Polynomial interpolation. When asked for the degree, apply  $\Delta x^k x_{i+j} - x_i \rightarrow \Delta x^{k+1}$  until  $\Delta x^k : x_i = c : \forall i$  Return  $k$ .
- When asked to approximate  $x$  for  $f(x)$  on  $(\hat{x}, f(\hat{x}))$ , construct iteration such that  $f = d/dx((\hat{x} - x)^2 + (f(\hat{x}) - f(x))^2)$ . Use this function as the iteration function.

### Theorems:

- **Thm 2.8:** Let  $p$  be a soln of the eq  $x = g(x)$  and suppose  $g'(p) = 0$  and  $g''$  is cts and strictly bounded by  $M$  on an interval  $I$  containing  $p$ . Then  $\exists \delta > 0$  such that  $p_0 \in [p - \delta, p + \delta]$  the seq:  $p_n = g(p_{n-1}) : n \geq 1$  converges at least quadratically to  $p$ . Moreover, for large  $n, |p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$
- **Thm 2.10:**  $f \in C'[a, b]$  has a simple zero at  $p$  in  $(a, b)$  iff  $f(p) = 0, f'(p) \neq 0$
- **Thm 2.11:** The function  $f \in C^m[a, b]$  has a zero of multiplicity  $m$  at  $p$  iff  $0 = f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p)$
- **Thm 3.3:** Suppose  $x_0, x_1, \dots, x_n$  are distinct numbers in  $[a, b]$  and  $f \in C^{n+1}[a, b]$  then for each  $x$  in  $[a, b]$ , a number  $\xi(x)$  in  $(a, b)$  exists with  $f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x - x_i)$  with  $P(x)$  being the  $n$ th Lagrange interpolating polynomial
- **Thm: Err Trapezoid:** Let  $f \in C^2[a, b], h = (b - a)/n, x_j = a + jh : 0 \leq j \leq n$ . Then  $\exists \mu \in (a, b)$  for which the composite trapezoid rule with  $n$  subivs has an err term of  $-\frac{b-a}{12} h^2 f''(\mu)$
- **Thm: Legendre Thm:** suppose  $x_1, x_2, \dots, x_n$  are the roots of the  $n$ th degree Legendre Polynomial and  $\forall i = 1, 2, \dots, n$  are the numbers  $c_i$  such that  $c_i = \int_{-1}^1 \prod_{j=1, j \neq i}^n \frac{x - x_j}{x_i - x_j} dx$ . If  $P(x)$  is any polynomial of degree  $< 2$ , then  $\int_{-1}^1 P(x)dx = \sum_{i=1}^n c_i P(x_i)$
- **Thm: Convexity:** A set is convex if for  $(t_1, y_1) \in D : D \subset \mathbb{R}$ , the point  $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2) \in D : \lambda \in [0, 1]$
- **Thm: Convex-Lipschitz:** Let  $f(t, y)$  be defined on convex  $D \subset \mathbb{R}^2$ . If  $\exists L > 0 : |df/dy(t, y)| \leq L, \forall (t, y) \in D$
- **Thm: Uniqueness-Lipschitz:** Suppose  $D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$  and  $f(t, y)$  cts on  $D$ . If  $f$  satisfies a Lipschitz on  $D$  in  $y$ , then  $y'(t) = f(t, y), a \leq t \leq b, y(a) = \alpha$  has a unique solution  $y(t) : a \leq t \leq b$
- **Thm: IVP well-posed-ness:** IVP is well posed it a unique soln exists and  $\exists \varepsilon_0, k > 0$  st  $\forall \varepsilon, \varepsilon_0 > \varepsilon > 0$ , whenever  $\delta(t)$  is cts with  $|\delta(t)| < \varepsilon : \forall t \in [a, b]$  and  $|\delta_0| < \varepsilon$ , the IVP:  $dz/dt = f(t, z) + \delta(t), a \leq t \leq b, z(0) = \alpha + \delta_0$  has a unique soln with  $dy/dt = f(t, y), a \leq t \leq b, y(0) = \alpha$

### Proofs:

- $\|A\| = \max_{\|x\|=1} \|Ax\| \geq \|\lambda x\| = |\lambda| \|x\| = |\lambda|$