§1.1 Error Analysis

- **Normalization**: 32 bit - Sign bit: 1, Sign exp: 1, Exp: 7, Normalized mantissa: 23.
- Absolute Error $= |p - p^*|$, Relative Error $= \frac{|p-p^*|}{|p|}$
- Significant Digits: RE $< 5 \times 10^{-t}$
- **fl(x)**: Machine representation
- **Cancellation Error**: Subtracting nearly equal numbers
- **Example**: $p = 0.54617$, $q = 0.54601$, true $r = p - q = 0.00016$
  4-digit: $p^* = 0.5462$, $q^* = 0.5460$, $r^* = 0.002$ (RE=25%)
- **Nested Multiplication**: Reduces error
  $f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99 = (((1.01z - 4.62)z - 3.11)z + 12.2)z - 1.99$
- $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
- Remainder: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$, $\xi \in (a, x)$
- Linear Approx: $f(x_0 + h) \approx f(x_0) + hf'(x_0)$
- **Example**: $\xi \in (0, \pi/2)$, $\sin\xi \le 1 \Rightarrow R_n \le \frac{(x)^n}{n!}$

§1.3 Convergence

- $\alpha = \lim_{n\to\infty}(\alpha_n)$
- Rate: $\alpha_n = \alpha + \mathcal{O}(\beta_n)$ if $|\alpha_n - \alpha| \le K|\beta_n|$
- Find largest p where $\alpha_n - \alpha = \mathcal{O}(1/n^p)$

§1.4 Matrix Operations

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$
- $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

§6.1 Gaussian Elimination

- $\mathcal{O}(n^3)$ complexity
- **Pivoting**:
  - Partial (PP): Max element in column
  - Scaled PP: $s_i = \max_j |a_{ij}|$, pivot $\max(a_{ik}/s_i)$
  - Complete (CP): Full matrix search ($\mathcal{O}(n^3)$)
- **LU Decomposition**: $PA = LU$ through GE steps and $LUx = Pb$.
- **LU Algorithm**: $L = E^{-1}_{n-a,m-b}E^{-1}_{n-a+1,m-b+1}\cdots$ U = GE.
- **Choleski Factorization**: If a matrix is symmetric and positive definite, it may be factored to the form LDL^T

§6.2 Special Matrices

- **Inverse Matrix**: An inverse matrix of A is A^{-1} such that $AA^{-1} = I$
- Properties: $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^T = (A^T)^{-1}$
- **Singular**: A matrix is singular iff its det is 0.
- **Diagonal**: $d_{ij} = 0$ for $i \ne j$: All non-diagonal entries are 0.
- **Symmetric**: $A = A^T$, $(AB)^T = B^TA^T$
- **Permutation**: Row swaps of I_n, PA reorders rows: $P^T = P^{-1}$
- **Diagonally Dominant**: $|a_{ii}| > \sum_{j\ne i}|a_{ij}|$ (nonsingular)
- **Positive Definite**: $x^TAx > 0 \Rightarrow A = LDL^T$, $a_{ii} > 0$, $a_{ij}^2 < a_{ii}a_{jj}$
- **Minor**: M_{ij} is a submatrix of A with the row i deleted and column j deleted.
- **Band**: an $n \times n$ matrix is a band matrix if $p, q \in \mathbb{Z} : 1 \le p, q \le n$ exist with $a_{i_j} = 0$ for $i + p \le j$ or $j + q \ge i$ The bandwidth is defined as $w = p + q - 1$. For a diagonal matrix, $p = 1, q = 1, w = 1$
- **Tridiagonal**: Band with $p = 2, q = 2$. It exhibits the following properties
  - $a_{ii} = l_{ii}$
  - $a_{i,i+1} = l_{ii}u_{i,i+1} : i = 1\cdots n-1$
  - $a_{i,i-1} = l_{i,i-1} : i = 2, 3, \cdots, n$
  - $a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} : i = 2\cdots n$
- Crout Factorization: This factorization happens in $O(n)$ time

§Strategies

- **RoC With inf limit**: set $h = 1/n$ and solve accordingly.
- $D\cdot(L+U)$: given D has ONLY diagonal entries and L+U has NO diagonal entries, the resulting matrix A is composed of entries $a_{ij} = d_{ii}\cdot(l+u)_{ij}$
- **Verification of Bisection**: To verify bisection can be applied, make sure that f(a) and f(b) are of different signs.
- **Error of Bisection**: To compute the accuracy of bisection to an ε, we use $\frac{b-a}{2^n} \le \varepsilon$
- **Failure of Newton's Method**: NM Fails if $f'(x) = 0$ for some x.
- **Triangle Inequality**: $|x + y| \le |x| + |y|$

§Key Definitions & Identities

- **Continuity**: $f \in C^n[a, b]$ reads: the nth derivative of f on [a,b] is continuous.
- **Series Expansions**
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (\forall x \in \mathbb{R})$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
  - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (|x| \le 1)$
  - $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1)$
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots : |x| < 1$
  - $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots : |x| < 1$

Core Identities
- $\sin^2\theta + \cos^2\theta = 1$
- $\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$
- $\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$

Angle Transformations
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

- **Vector Norms**
  - $||x|| > 0$
  - $||x|| = 0 \Leftrightarrow x = 0$
  - $||\alpha x|| = |\alpha|||x||$
  - $||x + y|| \le ||x|| + ||y||$

§Useful Examples

- Suppose $g(x) = \frac{5}{x^2} + 2$ Show $p_n = g(p_{n-1})$ will converge to g for $\forall p_0[2.5, 3]$. Since this is a decreasing function, the max of g(x) is g(2.5) and the min of g(x) is g(3).
  - First compute the max, $g(2.5) = 14/5 < 3$
  - Second compute the min, $g(3) = 5/9 + 2 > 2.5$
  - Last compute $|g'(x)| = -10/x^3 \le max_{x\in[2.5,3]}|g'(x)| = 16/25 < 1$
- Given $||A||$ is a natural matrix norm of matrix A. show $|\lambda| \le ||A||$ for any nonsingular A and any λ of A. $||A|| = \max_{||x||=1}||Ax|| \ge ||Ax||$: x is an e-vec s.t ——x——=1 $= ||\lambda x|| = |\lambda|||x|| = |\lambda|\square$
- When performing Jacobi or GS, when computing L+U, flip the signs of all entries.
- To determine convergence for fixed point, compute $g'(p_0) \le 1$, which gives a,b. Prove $g(x)$ cts on [a,b], $g(x) \in [a, b]$, $g'(x)$ exists on (a,b), $|g'(x)| \le k : \forall x \in (a, b), 0 < k < 1$

§LA Determinants

- 2x2: $|A| = ad - bc$
- nxn: $|A| = \sum_{j=1}^{n} a_{ij}A_{ij}$ via cofactors $A_{ij} = (-1)^{i+j}M_{ij}$

Properties
- Swap rows: $|\tilde{A}| = -|A|$
- Scale row: $|\tilde{A}| = \lambda|A|$
- Identical rows: $|A| = 0$
- $|AB| = |A||B|$, $|A^T| = |A|$
- $|A^{-1}| = \frac{1}{|A|}$

§7 Norms & Eigen

- $||x||_2 = \sqrt{\sum x_i^2}$, $||x||_\infty = \max|x_i|$
- $||A||_\infty = \max_i \sum_j |a_{ij}|$ Basically sum all rows together and determine the largest one.
- $||A||_2 = \sqrt{\rho(A)}$
- $||A|| = \max_{||x||=1}||Ax||$
- $\forall x \in \mathbb{R}^n : ||x||_\infty \le ||x||_2 \le \sqrt{n}||x||_\infty$
- A distance between matrices A and B wrt a matrix norm $||\cdot||$ is $||A - B||$
- Theorem: For any vector $x \ne 0$, matrix Am and abt natural norm $||\cdot||$ we have $||Ax|| \le ||A||\cdot||x||$
- Cauchy-Schwarz: $||x + y||_2 \le ||x||_2||y||_2$
- Eigen: λ is an eigenvalue if $A\lambda = v\cdot\lambda$
- Finding Eigenthings: $\det(A - \lambda I) = 0 : \forall\lambda$.
- Spectral Radius: $\rho(A) = \max|\lambda_i|$, $\rho(A) \le ||A||$
- Theorem: If A is $n \times n$:
  - $||A||_2 = [\rho(A^tA)]^{1/2}$
  - $\rho(A) \le ||A|| : \forall||\cdot||$
- Convergent: $\lim_{k\to\infty} A^k = 0 \Leftrightarrow \rho(A) < 1$
- Matrix Norms: matrix norms have the following properties
  - $||A|| \ge 0$ with $||A|| = 0 \leftrightarrow A = 0$
  - $||\alpha A|| = |\alpha|\cdot||A||$
  - $||A + B|| \le ||A|| + ||B||$

§Iterative Methods

- **General Iteration**: $x^{k+1} = Tx^k + c$
- **Jacobi**: $x^{k+1} = D^{-1}(L + U)x^k + D^{-1}b$
- **Gauss-Seidel**: $x^{k+1} = (D - L)^{-1}Ux^k + (D - L)^{-1}b$
- **Stein-Rosenberg**: For matrices with positive diagonals: $\rho_{GS} \le \rho_J < 1$
- **Speed of Convergence**: given matrices $T_{GS} = (D-L)^{-1}U$ and $T_J = D^{-1}(L + U)$, compare ρ. The bigger the ρ, the faster the convergence.
- Error: $||x - x^k|| \le \frac{||T||^k}{1-||T||}||x^1 - x^0||$
- Stopping: $\frac{||x^k - x^{k-1}||}{||x^k||} < \varepsilon$

§2 Nonlinear Equations

- **Bisection**:
  - While $f(p_n) \ne 0$ or $< T$: $p_n = \frac{a_1+b_1}{2}$
  - Error: $\frac{b_n-a_n}{2} < T$, $p = a + \frac{b-a}{2}$
- **Fixed-Point**:
  - $p_n = g(p_{n-1})$, converges if $|g'(x)| \le K < 1$
  - Algorithm: For $i < N_0$: $p = g(p_0)$, check $|p - p_0| < T$
  - A fixed point is defined as a point in which $p = f(p)$
- **Newton**:
  - $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
  - Quadratic convergence if $f'(p) \ne 0$
- **Secant**:
  - $p_{n+1} = p_n - \frac{f(p_n)(p_n-p_{n-1})}{f(p_n)-f(p_{n-1})}$
  - Approx derivative: $\frac{f(p_{n-1})-f(p_{n-2})}{p_{n-1}-p_{n-2}}$
  - Algorithm: Store $q_0 = f(p_0)$, $q_1 = f(p_1)$, SET $p = p_1 - \frac{q_1(p_1-p_0)}{q_1-q_0}$ IF STOPPING CONDITION: RETURN p; i++, $p_0 = p_1, q_0 = q_1, p_1 = p, q_1 = f(p)$ ENDWHILE OUTPUT FAILURE.

§Theorems

- **Bisection**: Suppose $f \in C[a, b] : f(a)\cdot f(b) < 0$. Bisection generates $\{p_n\}$ approximating a zero p with $|p_n - p| \le \frac{b-a}{2^n} : n \ge 1$
- **Fixed Point**: If $g \in C[a, b]$, $g([a, b]) \subseteq [a, b]$ g has a fixed point in $[a, b]$, additionally if $|g'| \le K < 1$, then the fixed point is unique.
- **Fixed Point Theorem**: Let $g \in C[a, b]$ and $g(x) \in [a, b] : \forall x \in [a, b]$. Suppose as well that g' exists on (a,b) and positive $K < 1$ exists with $|g'(x)| \le K : \forall x \in (a, b)$. Then for any number $p_0 \in [a, b]$h the sequence defined by $p_n = g(p_{n-1})$: $n \ge 1$ converges to the unique point $p \in [a, b]$
- **Corollary**: If g satisfies the hypothesis of the above theorem, $|p_n - p| \le k^n \max(p_0 - a, b - p_0)$ and $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| : \forall n > 1$
- **Newton**: For $f \in C^2[a, b]$ with simple root, $\exists\delta > 0 : p_0 \in [p - \delta, p + \delta]$ converges.
- **Matrix Invertibility**: $|A| \ne 0 \Leftrightarrow$ unique solution $Ax = b \Leftrightarrow A^{-1}$ exists
- **Taylor**: With $R_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$, $R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)^{n+1}$:$\xi\epsilon(x, x_0)$
- **Existence of Inverse**: if A is square, $detA \ne 0 \leftrightarrow Ax = 0$ has soln $x = 0 \leftrightarrow Ax = b$ has a unique soln for any n-vector b. $\leftrightarrow A^{-1}$ exists.
- **Diagonally Dominant Matrices**: dd matrices are nonsingular. A being dd means LU can be performed without P. A matrix is positive definite if $x^tAx > 0$. PD matrices are nonsingular, $\forall i = 1, \cdots, n : a_{ii} > 0$, $max|a_{kj}| > max|a_{ii}|$, $(a_{ij})^2 < a_{ii}a_{jj} : \forall i \ne j$.
- **Convergence and Spectral Radii**: equivalent statements: A is convergent, $\rho < 1$, $limA^nx = 0 : \forall x$, $lim||A^n|| = 0 : \forall norms$.
- **Convergence of DD**: If A is strictly DD, Jacobi and GS converge to the unique soln. Ax=b.
- **Positive Definitive Check**: A matrix is PD if the leading principle submatrix determinants are positive.
- **Positive Definitive Check**: A matrix is PD iff it may be factored into LL^T
- **Non-singularity Check**: A matrix A has an inverse iff $\det A \ne 0$.
- **Determinant of Triangular Matrices**: The determinant of a triangular matrix is Πa_{ii}.

§Proofs

- **Bisection (THM1)**: $\forall n \ge 1$: $b_n - a_n = (b - a)\cdot\frac{1}{2^{n-1}} : p\epsilon(a_n, b_n)$. Since $p_n = \frac{1}{2}(a_n + b_n) : \forall n \ge 1$, $|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$. $\square$
- **Fixed Point**: Part i: If $g \in [a, b], g(x) \in [a, b] : \forall x \in [a, b]$ then g(x) has a fixed point in [a,b]:
  If $g(a) = a$ or $g(b) = b$, g has a fixed point at an endpoint. Suppose for contradiction that it does not. $g(a) > a$ and $g(b) < b$. Define $h(x) = g(x) - x$. Then h is cts on [a,b] and $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$ IVT states that $\exists p \in (a, b) : h(p) = 0$ Thus $g(p) - p = 0 \Rightarrow$p is a fixed point of g.
  Part ii: Suppose as well $|g'(x)| \le k < 1 : \forall x \in (a, b)$ and that $p, q \in [a, b] : p \ne q$. By MVT, $\exists\zeta : \frac{g(p)-g(q)}{p-q} = g'(\zeta)$. $|p - q| = |g(p) - g(q)| = |g'(\zeta)||p - q| \le k|p - q| < |p - q|$ contradiction.