§1.1 Error Analysis

- **Normalization**: 32 bit - Sign bit: 1, Sign exp: 1, Exp: 7, Normalized mantissa: 23.
- Absolute Error $= |p - p^*|$, Relative Error $= \frac{|p-p^*|}{|p|}$
- Significant Digits: RE $< 5 \times 10^{-t}$
- **fl(x)**: Machine representation
- **Cancellation Error**: Subtracting nearly equal numbers
- **Example**: $p = 0.54617$, $q = 0.54601$, true $r = p - q = 0.00016$
  4-digit: $p^* = 0.5462$, $q^* = 0.5460$, $r^* = 0.002$ (RE=25%)
- **Nested Multiplication**: Reduces error
  $f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99 = (((1.01z - 4.62)z - 3.11)z + 12.2)z - 1.99$
- $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k$
- Remainder: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$, $\xi \in (a, x)$
- Linear Approx: $f(x_0 + h) \approx f(x_0) + hf'(x_0)$
- **Example**: $\xi \in (0, \pi/2)$, $\sin\xi \le 1 \Rightarrow R_n \le \frac{(x)^n}{n!}$

§1.3 Convergence

- $\alpha = \lim_{n\to\infty}(\alpha_n)$
- Rate: $\alpha_n = \alpha + \mathcal{O}(\beta_n)$ if $|\alpha_n - \alpha| \le K|\beta_n|$
- Find largest p where $\alpha_n - \alpha = \mathcal{O}(1/n^p)$

§1.4 Matrix Operations

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$
- $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

§6.1 Gaussian Elimination

- $\mathcal{O}(n^3)$ complexity
- **Pivoting**:
  - Partial (PP): Max element in column
  - Scaled PP: $s_i = \max_j |a_{ij}|$, pivot $\max(a_{ik}/s_i)$
  - Complete (CP): Full matrix search ($\mathcal{O}(n^3)$)
- **LU Decomposition**: $PA = LU$ through GE steps and $LUx = Pb$.
- **LU Algorithm**: $L = E^{-1}_{n-a,m-b}E^{-1}_{n-a+1,m-b+1}\cdots$ $U =$ GE.
- **Choleski Factorization**: If a matrix is symmetric and positive definite, it may be factored to the form LDL^T

§6.2 Special Matrices

- **Inverse Matrix**: An inverse matrix of A is A^{-1} such that $AA^{-1} = I$
- Properties: $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^T = (A^T)^{-1}$
- **Singular**: A matrix is singular iff its det is 0.
- **Diagonal**: $d_{ij} = 0$ for $i \ne j$: All non-diagonal entries are 0.
- **Symmetric**: $A = A^T$, $(AB)^T = B^TA^T$
- **Permutation**: Row swaps of I_n, PA reorders rows: $P^T = P^{-1}$
- **Diagonally Dominant**: $|a_{ii}| > \sum_{j\ne i}|a_{ij}|$ (nonsingular)
- **Positive Definite**: $x^TAx > 0 \Rightarrow A = LDL^T$, $a_{ii} > 0$, $a_{ij}^2 < a_{ii}a_{jj}$
- **Minor**: M_{ij} is a submatrix of A with the row i deleted and column j deleted.
- **Band**: an $n \times n$ matrix is a band matrix if $p, q \in \mathbb{Z} : 1 \le p, q \le n$ exist with $a_{i_j} = 0$ for $i + p \le j$ or $j + q \ge i$ The bandwidth is defined as $w = p + q - 1$. For a diagonal matrix, $p = 1, q = 1, w = 1$
- **Tridiagonal**: Band with $p = 2, q = 2$. It exhibits the following properties
  - $a_{ii} = l_{ii}$
  - $a_{i,i+1} = l_{ii}u_{i,i+1} : i = 1\cdots n-1$
  - $a_{i,i-1} = l_{i,i-1} : i = 2, 3, \cdots, n$
  - $a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} : i = 2\cdots n$
- Crout Factorization: This factorization happens in $O(n)$ time

§Strategies

- **RoC With inf limit**: set $h = 1/n$ and solve accordingly.
- $D\cdot(L+U)$: given D has ONLY diagonal entries and L+U has NO diagonal entries, the resulting matrix A is composed of entries $a_{ij} = d_{ii}\cdot(l+u)_{ij}$
- **Verification of Bisection**: To verify bisection can be applied, make sure that f(a) and f(b) are of different signs.
- **Error of Bisection**: To compute the accuracy of bisection to an ε, we use $\frac{b-a}{2^n} \le \varepsilon$
- **Failure of Newton's Method**: NM Fails if $f'(x) = 0$ for some x.
- **Triangle Inequality**: $|x + y| \le |x| + |y|$

§Key Definitions & Identities

- **Continuity**: $f \in C^n[a,b]$ reads: the nth derivative of f on [a,b] is continuous.
- **Series Expansions**
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (\forall x \in \mathbb{R})$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
  - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (|x| \le 1)$
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1)$
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots : |x| < 1$
  - $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots : |x| < 1$

Core Identities
- $\sin^2\theta + \cos^2\theta = 1$
- $\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$
- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

Angle Transformations
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

- **Vector Norms**
  - $||x|| > 0$
  - $||x|| = 0 \Leftrightarrow x = 0$
  - $||\alpha x|| = |\alpha|||x||$
  - $||x + y|| \le ||x|| + ||y||$

§Useful Examples

- Suppose $g(x) = \frac{5}{x^2} + 2$ Show $p_n = g(p_{n-1})$ will converge to g for $\forall p_0[2.5, 3]$.
  Since this is a decreasing function, the max of g(x) is g(2.5) and the min of g(x) is g(3).
  - First compute the max, $g(2.5) = 14/5 < 3$
  - Second compute the min, $g(3) = 5/9 + 2 > 2.5$
  - Last compute $|g'(x)| = -10/x^3 \le max_{x\in[2.5,3]}|g'(x)| = 16/25 < 1$
- Given $||A||$ is a natural matrix norm of matrix A. show $|\lambda| \le ||A||$ for any nonsingular A and any λ of A. $||A|| = \max_{||x||=1}||Ax|| \ge ||Ax||$: x is an e-vec s.t ——x——=1 $= ||\lambda x|| = |\lambda|||x|| = |\lambda|$☐
- When performing Jacobi or GS, when computing L+U, flip the signs of all entries.
- To determine convergence for fixed point, compute $g'(p_0) \le 1$, which gives a,b. Prove $g(x)$ cts on [a,b], $g(x) \in [a,b]$, $g'(x)$ exists on (a,b), $|g'(x)| \le k : \forall x \in (a,b), 0 < k < 1$

§LA Determinants

- 2x2: $|A| = ad - bc$
- nxn: $|A| = \sum_{j=1}^n a_{ij}A_{ij}$ via cofactors $A_{ij} = (-1)^{i+j}M_{ij}$

Properties
- Swap rows: $|\tilde{A}| = -|A|$
- Scale row: $|\tilde{A}| = \lambda|A|$
- Identical rows: $|A| = 0$
- $|AB| = |A||B|$, $|A^T| = |A|$
- $|A^{-1}| = \frac{1}{|A|}$

§7 Norms & Eigen

- $||x||_2 = \sqrt{\sum x_i^2}$, $||x||_\infty = \max|x_i|$
- $||A||_\infty = \max_i\sum_j|a_{ij}|$ Basically sum all rows together and determine the largest one.
- $||A||_2 = \sqrt{\rho(A)}$
- $||A|| = \max_{||x||=1}||Ax||$
- $\forall x \in \mathbb{R}^n : ||x||_\infty \le ||x||_2 \le \sqrt{n}||x||_\infty$
- A distance between matrices A and B wrt a matrix norm $||\cdot||$ is $||A - B||$
- Theorem: For any vector $x \ne 0$, matrix Am and abt natural norm $||\cdot||$ we have $||Ax|| \le ||A||\cdot||x||$
- Cauchy-Schwarz: $||x + y||_2 \le ||x||_2||y||_2$
- Eigen: λ is an eigenvalue if $A\lambda = v\cdot\lambda$
- Finding Eigenthings: $\det(A - \lambda I) = 0 : \forall\lambda$.
- Spectral Radius: $\rho(A) = \max|\lambda_i|$, $\rho(A) \le ||A||$
- Theorem: If A is $n \times n$:
  - $||A||_2 = [\rho(A^tA)]^{1/2}$
  - $\rho(A) \le ||A|| : \forall||\cdot||$
- Convergent: $\lim_{k\to\infty}A^k = 0 \Leftrightarrow \rho(A) < 1$
- Matrix Norms: matrix norms have the following properties
  - $||A|| \ge 0$ with $||A|| = 0 \leftrightarrow A = 0$
  - $||\alpha A|| = |\alpha|\cdot||A||$
  - $||A + B|| \le ||A|| + ||B||$

§Iterative Methods

- **General Iteration**: $x^{k+1} = Tx^k + c$
- **Jacobi**: $x^{k+1} = D^{-1}(L+U)x^k + D^{-1}b$
- **Gauss-Seidel**: $x^{k+1} = (D-L)^{-1}Ux^k + (D-L)^{-1}b$
- **Stein-Rosenberg**: For matrices with positive diagonals: $\rho_{GS} \le \rho_J < 1$
- **Speed of Convergence**: given matrices $T_{GS} = (D-L)^{-1}U$ and $T_J = D^{-1}(L+U)$, compare ρ. The bigger the ρ, the faster the convergence.
- Error: $||x - x^k|| \le \frac{||T||^k}{1-||T||}||x^1 - x^0||$
- Stopping: $\frac{||x^k - x^{k-1}||}{||x^k||} < \varepsilon$

§2 Nonlinear Equations

- **Bisection**:
  - While $f(p_n) \ne 0$ or $< T$: $p_n = \frac{a_1+b_1}{2}$
  - Error: $\frac{b_n-a_n}{2} < T$, $p = a + \frac{b-a}{2}$
- **Fixed-Point**:
  - $p_n = g(p_{n-1})$, converges if $|g'(x)| \le K < 1$
  - Algorithm: For $i < N_0$: $p = g(p_0)$, check $|p - p_0| < T$
  - A fixed point is defined as a point in which $p = f(p)$
- **Newton**:
  - $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
  - Quadratic convergence if $f'(p) \ne 0$
- **Secant**:
  - $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$
  - Approx derivative: $\frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$
  - Algorithm: Store $q_0 = f(p_0)$, $q_1 = f(p_1)$, SET $p = p_1 - \frac{q_1(p_1-p_0)}{q_1-q_0}$ IF STOPPING CONDITION: RETURN p; i++, $p_0 = p_1, q_0 = q_1, p_1 = p, q_1 = f(p)$ ENDWHILE OUTPUT FAILURE.

§Theorems

- **Bisection**: Suppose $f \in C[a,b] : f(a)\cdot f(b) < 0$. Bisection generates $\{p_n\}$ approximating a zero p with $|p_n - p| \le \frac{b-a}{2^n} : n \ge 1$
- **Fixed Point**: If $g \in C[a,b]$, $g([a,b]) \subseteq [a,b]$ g has a fixed point in $[a,b]$, additionally if $|g'| \le K < 1$, then the fixed point is unique.
- **Fixed Point Theorem**: Let $g \in C[a,b]$ and $g(x) \in [a,b] : \forall x \in [a,b]$. Suppose as well that g' exists on (a,b) and positive $K < 1$ exists with $|g'(x)| \le K : \forall x \in (a,b)$. Then for any number $p_0 \in [a,b]$h the sequence defined by $p_n = g(p_{n-1}) : n \ge 1$ converges to the unique point $p \in [a,b]$
- **Corollary**: If g satisfies the hypothesis of the above theorem, $|p_n - p| \le k^n\max(p_0 - a, b - p_0)$ and $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| : \forall n > 1$
- **Newton**: For $f \in C^2[a,b]$ with simple root, $\exists\delta > 0 : p_0 \in [p-\delta, p+\delta]$ converges.
- **Matrix Invertibility**: $|A| \ne 0 \Leftrightarrow$ unique solution $Ax = b \Leftrightarrow A^{-1}$ exists
- **Taylor**: With $R_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$, $R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)^{n+1}$:$\xi\epsilon(x, x_0)$
- **Existence of Inverse**: if A is square, $detA \ne 0 \leftrightarrow Ax = 0$ has soln $x = 0 \leftrightarrow Ax = b$ has a unique soln for any n-vector b. $\leftrightarrow A^{-1}$ exists.
- **Diagonally Dominant Matrices**: dd matrices are nonsingular. A being dd means LU can be performed without P. A matrix is positive definite if $x^tAx > 0$. PD matrices are nonsingular, $\forall i = 1, \cdots, n : a_{ii} > 0$, $max|a_{kj}| > max|a_{ii}|$, $(a_{ij})^2 < a_{ii}a_{jj} : \forall i \ne j$.
- **Convergence and Spectral Radii**: equivalent statements: A is convergent, $\rho < 1$, $limA^nx = 0 : \forall x$, $lim||A^n|| = 0 : \forall norms$.
- **Convergence of DD**: If A is strictly DD, Jacobi and GS converge to the unique soln. Ax=b.
- **Positive Definitive Check**: A matrix is PD if the leading principle submatrix determinants are positive.
- **Positive Definitive Check**: A matrix is PD iff it may be factored into LL^T
- **Non-singularity Check**: A matrix A has an inverse iff $\det A \ne 0$.
- **Determinant of Triangular Matrices**: The determinant of a triangular matrix is Πa_{ii}.

§Proofs

- **Bisection (THM1)**: $\forall n \ge 1$: $b_n - a_n = (b-a)\cdot\frac{1}{2^{n-1}} : p\epsilon(a_n, b_n)$. Since $p_n = \frac{1}{2}(a_n + b_n) : \forall n \ge 1$, $|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$. ☐
- **Fixed Point**: Part i: If $g \in [a,b], g(x) \in [a,b] : \forall x \in [a,b]$ then g(x) has a fixed point in [a,b]:
  If $g(a) = a$ or $g(b) = b$, g has a fixed point at an endpoint. Suppose for contradiction that it does not. $g(a) > a$ and $g(b) < b$. Define $h(x) = g(x) - x$. Then h is cts on [a,b] and $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$ IVT states that $\exists p \in (a,b) : h(p) = 0$ Thus $g(p) - p = 0 \Rightarrow$p is a fixed point of g.
  Part ii: Suppose as well $|g'(x)| \le k < 1 : \forall x \in (a,b)$ and that $p, q \in [a,b] : p \ne q$. By MVT, $\exists\zeta : \frac{g(p)-g(q)}{p-q} = g'(\zeta)$. $|p - q| = |g(p) - g(q)| = |g'(\zeta)||p - q| \le k|p - q| < |p - q|$ contradiction.

§2: Error Analysis and Acelerating Convergence

Basic Methods

- **Newton's Method**: Quadratic convergence if $f'(p) \neq 0$. Iteration: $x_{n+1} = x_n - f(x_n)/f'(x_n)$.
- **Secant Method**: Superlinear convergence (order ≈ 1.618). Uses two previous points.
- **Newton's Improved Method**: $p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{f'(p_n)^2 - f(p_n)f''(p_n)}$

Convergence Analysis

- A soln p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$ we can write $f(x) = (x-p)^m q(x)$ where $\lim_{x\to p} q(x) \neq 0$. (Simple zeros are multiplicity 1).
- Order of convergence α: $\lim \frac{|p_{n+1}-p|}{|p_n - p|^\alpha} = \lambda$
- Linear ($\alpha = 1$), Quadratic ($\alpha = 2$)
- Fixed-point: Linear if $g'(p) \neq 0$, quadratic if $g'(p) = 0$ and g'' bounded.

Special Cases

- Multiple roots: Modify Newton's using $\mu(x) = f(x)/f'(x)$
- **Aitken's** Δ^2: Accelerates linear sequences. Is given by $\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-p_{n+1})-(p_{n+1}-p_n)}$

Polynomial Methods

- Horner's method: Efficient evaluation (n mults/adds) Algorithm: ex: evaluate $P(x) = \sum_{i=0}^n a_i x^i$ and derivative at x_0. Input $n, a_j, x_0 : 0 \leq j \leq n$. Output: $y = P(x_0)$, $z = P'(x_0)$. Set $y = a_n, z = a_n$. For $j = n-1, n-2, ..., 1$ set $y = x_0 y + a_j, z = x_0 z + y$. Endfor set $y - x_0 y + a_0$ output y,z.
- Deflation: Find roots sequentially via $P(x) \approx Q(x)(x - x_0)$
- Fundamental thm of alg: If $P(x)$ has a degree $n \geq 1$, $P(x)$ has at least one root.
- Cor: there also exists unique constants $x_1, ...x_k$ such that $\sum_{i=1}^k m_i = n, P(x) = a_n \times \prod_{i=1}^k (x - x_i)^{m_i}$
- Cor: these functions are unique.

Weierstrass: $\forall f$ cts on $[a,b]$, $\forall \varepsilon > 0$, $\exists$ polynomial $p(x)$ with $|f(x) - p(x)| < \varepsilon$ $\forall x \in [a,b]$.

Lagrange Interpolation (unique!):
$P(x) = \sum_{m=0}^N f(x_m)L_m(x)$, where $L_m(x) = \prod_{\substack{k=0\\k\neq m}}^N \frac{x - x_k}{x_m - x_k}$

Interpolation Error: $f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{k=0}^n (x - x_k)$ for $f \in C^{n+1}[a,b]$

Newton's Divided Differences:

$$f[x_i] = f(x_i), \quad f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$P_n(x) = \sum_{k=0}^n f[x_0, \ldots, x_k]\prod_{j=0}^{k-1}(x - x_j)$

Hermite: Given $(x_j, f(x_j), f'(x_j))$, unique degree $\leq 2n+1$ we have:

- $H(x) = \sum_{j=0}^n f(x_j)H_j(x) + \sum_{j=0}^n f'(x_j)\hat{H}_j(x)$
- $H_j(x) = [1 - 2(x - x_j)L_j'(x_j)]L_j^2(x)$
- $\hat{H}_j(x) = (x - x_j)L_j^2(x)$
- Note: $L_j(x)$ denotes the jth Lagrange coefficient polynomial of degree n.
- Error: $f(x) - H(x) = \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$
- Parametric curve Interpolation:

$$x(t) = [2(x_0 - x_1) + 3(a_0 + a_1)]t^2 + [3(x_1 - x_0) - 3(a_1 + 2a_0)]t^2 + 3a_0 t + x_0$$
$$y(t) = [2(y_0 - y_1) + 3(\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - 3(\beta_1 + 2\beta_0)]t^2 + 3\beta_0 t + y_0$$

Cubic Splines: $S(x_j) = f(x_j)$ and $S \in C^2[a,b]$; Boundary: Clamped (S' at endpoints) or Natural ($S'' = 0$ at endpoints) Error: $\max|f(x) - S(x)| \leq \frac{5M}{384}h^4$, where $h = \max(x_{j+1} - x_j)$, $M = \max|f^{(4)}|$

Richardson Extrapolation: $N_{j+1}(h) = N_j(h/2) + \frac{N_j(h/2) - N_j(h)}{2^j - 1}$

Numerical Integration:

- **Trapezoid Rule** ($O(h^2)$):
  - Single: $\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + f(b)]$, Error: $-\frac{h^3}{12}f''(\xi)$
  - Composite: $\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)]$
  - Error: $-\frac{(b-a)h^2}{12}f''(\xi) \approx -\frac{M(b-a)^3}{12n^2}$ where $M = \max|f''|$
- **Midpoint Rule** ($O(h^2)$):
  - Single: $\int_a^b f(x)\,dx \approx (b-a)f(\frac{a+b}{2})$
  - Composite: $\int_a^b f(x)\,dx \approx h\sum_{i=1}^n f(a + (i - \frac{1}{2})h)$
- **Simpson's Rules** ($O(h^4)$):
  - 1/3 Rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$
  - Composite: $\int_a^b f(x)\,dx \approx \frac{h}{3}[f(a) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) + f(b)]$
  - Error: $-\frac{h^5}{90}f^{(4)}(\xi)$ (single), $-\frac{(b-a)h^4}{180}f^{(4)}(\xi)$ (composite)
  - 3/8 Rule: $\int_a^b f(x)\,dx \approx \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + \ldots + f(x_n)]$
  - 3/8 Error: $-\frac{(b-a)^5}{6480}f^{(4)}(\xi)$

Romberg - $O(h_k^{2j})$:

- $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$, error $O(h^{2j})$
- $R_{k,1}$ represents the approximation of the integral using $m_k = 2^{k-1}$ intervals

§ODE Initial Value Problems

- **Basic Problem**: $y'(t) = f(t,y)$, $y(a) = \alpha$
- **Lipschitz Condition**: $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$
- Existence/uniqueness guaranteed when $|\frac{\partial f}{\partial y}| \leq L$ over convex domain D

Numerical Methods

- **Euler's Method**: $w_{i+1} = w_i + hf(t_i, w_i)$ [Error: $O(h)$]
- **Taylor Methods**: $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ where $T^{(n)} = f + \frac{h}{2}f' + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}$
- **Runge-Kutta Methods**:

Midpoint (RK2)
$w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f_i)$

Modified Euler (RK2)
$w_{i+1} = w_i + \frac{h}{2}(f_i + f(t_{i+1}, w_i + hf_i))$

- **Classical RK4**:

$$\begin{aligned} k_1 &= hf(t_i, w_i)\\ k_2 &= hf(t_i + \tfrac{h}{2}, w_i + \tfrac{k_1}{2})\\ k_3 &= hf(t_i + \tfrac{h}{2}, w_i + \tfrac{k_2}{2})\\ k_4 &= hf(t_i + h, w_i + k_3)\\ w_{i+1} &= w_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\end{aligned}$$

[Error: $O(h^4)$]

Error Analysis

- **Local truncation error**: $\tau_{i+1} = y(t_{i+1}) - w_{i+1}$ given $w_i = y(t_i)$
- **Global truncation error**: Accumulated error across all steps
- For Euler: $|\tau_i| \leq \frac{h^2}{2}M$ (local), $O(h)$ (global) where $M = \max|y''|$
- For RK4: $O(h^5)$ (local), $O(h^4)$ (global)

Stability & Step Size

- Well-posed problem requires: unique solution exists + small input changes $\rightarrow$ small output changes
- Step size formula: $h < \frac{2\varepsilon}{M(b-a)}$ for error ε, where $M = \max|y''|$
- Example: $y' = y\cos t$ has Lipschitz constant $L = 1$ since $|\frac{\partial f}{\partial y}| = |\cos t| \leq 1$

Chapter 3: Interpolation Lagrange Interpolation

- Δ^k: $\Delta^2 f_i = \Delta(\Delta(f_i)) = \Delta(f_{i+1} - f_i) = f_{i+2} - 2f_{i+1} + f_i$
- Reuse computations with **Neville's Method**:
  for $i = 1, 2, ..., n$ do: for $j = 1, 2, ..., i$ do $Q_{i,j} \leftarrow \frac{(x - x_{i-j}Q_{i,j-1}) - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}$

Hermite Interpolation

- Error term: $\frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!}\max|f^{(2n+2)}|$
- Handle derivatives via **divided differences** with repeated nodes: $z_{2i} = z_{2i+1} = x_i$
- *Warning*: Noisy derivatives $\Rightarrow$ amplified errors.

Splines

- **Cubic splines**: Solve tridiagonal system ($O(n)$ ops) with:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1})$$

- **Natural splines** ($S'' = 0$): Stable but less accurate. **Clamped splines**: Require $f'(a), f'(b)$ but higher accuracy.
- Error: $O(h^4)$ for $f \in C^4$, $O(h^2)$ for linear splines.

Trapezoidal & Simpson's Rules

- **Romberg Integration**: Accelerate Trapezoidal Rule via: $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$
- Stop when $|R_{k,k} - R_{k-1,k-1}| < \epsilon$.
- The degree of precision of a quadrature formula is the largest n such that the formula is exact for $x^k : \forall k = 0, 1, \cdots, n$

Adaptive Quadrature

- Error estimate: $\frac{1}{15}|S(a,b) - S(a,c) - S(c,b)|$, $c = (a+b)/2$
- Subdivide intervals where error $> \epsilon/2$.
- Step 1: Apply Simpson's with $h = (b-a)/2$.
  $\int_a^b f(x) = h/3\,[f(a) + 4f(a+h) + f(b)] - \frac{h^5}{90}f^{(4)}(\mu)\mu \in (a,b)$
- Step 2: Find error using Simpson's on $h = (b-a)/4$
  $\int_a^b f(x) = h/6[f(a) + 4f(a + h/2) + 2f(a+b) + 4f(a + 3h/2) + f(b)] - (\frac{h}{2})^4 \cdot \frac{(b-a)}{180}f^{(4)}(\tilde{\mu}) : \tilde{\mu} \in (a,b)$
- Note: We assume $f^4(\mu) = f^4(\tilde{\mu})$: true for small h.
- Step 3: Calculate error as $1/10\left|S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)\right| < \varepsilon$
- Step 4: If true, RETURN. Else, GOTO step 1.

Gaussian Quadrature

- $\int_a^b w(x)f(x)dx = \sum_{i=1} n w_i f(x_i)$ where $w(x)$: weight functions, w_i: weight at i, x_i node at i.
- Nodes: Roots of Legendre polynomials $P_n(x)$. Weights: $c_i = \int_{-1}^1 \prod_{j\neq i}\frac{x - x_j}{x_i - x_j}dx$
- Exact for polynomials of degree $\leq 2n - 1$.
- Step 1: Transform to $[-1,1]$: $x = \frac{(b-a)t + (a+b)}{2}$, $dx = \frac{b-a}{2}dt$
- Step 2: substitute x into integrand
- Step 3: use the formulae to get the answer:
  - 1-point: $\int_{-1}^1 f(x)dx = 2f(0)$
  - 2-point: $\int_{-1}^1 f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) : w_1 = w_2; x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$
  - 3-point: $\int_{-1}^1 f(x)dx = \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$

Legendre Polynomials

- $P_n(x)$ denotes the n'th degree Legrendre polynomial
- $\int_{-1}^1 P(x)P_n(x)dx = 0$: P(x) is of degree ¡ n
- $P_0(x) = 1, P_1(x) = x, P_2(x) = x^2 - 1/3, P_3(x) = x^3 - 3/5x, P_4(x) = x^4 - 6/7x^2 + 3/35$

Tips:

- Use Gaussian quadrature with $n = c \rightarrow$ apply c-point Gauss-Legendre Rule.
- When asked for the degree of precision, plug in $x^0, x^1, x^2, \cdots, x^n$ until failure. The n before it fails is the degree of precision

Theorems:

- **Thm 2.8:** Let p be a soln of the eq $x = g(x)$ and suppose $g'(p) = 0$ and g'' is cts and strictly bounded by M on an interval I containing p. Then $\exists\delta > 0$ such that $p_0 \in [p - \delta, p + \delta]$ the seq: $p_n = g(p_{n-1}) : n \geq 1$ converges at least quadratically to p. Moreover, for large n, $|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$
- **Thm 2.10:** $f \in C'[a,b]$ has a simple zero at p in (a,b) iff $f(p) = 0, f'(p) \neq 0$
- **Thm 2.11:** The function $f \in C^m[a,b]$ has a zero of multiplicity m at p iff $0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p)$
- **Thm 3.3**: Suppose $x_0, x_1, ..., x_n$ are distinct numbers in [a,b] and $f \in C^{n+1}[a,b]$ then for each x in $[a,b]$, a number $\xi(x)$ in (a,b) exists with $f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{i=0}^n (x - x_i)$ with P(x) being the nth Lagrange interpolating polynomial
- **Thm: Err Trapezoid**: Let $f \in C^2[a,b]$, $h = (b-a)/n$, $x_j = a + jh : 0 \leq j \leq n$. Then $\exists\mu \in (a,b)$ for which the composite trapezoid rule with n subivls has an err term of $\frac{b-a}{12}h^2 f''(\mu)$
- **Thm: Legrendre Thm**: suppose $x_1, x_2, \cdots, x_n$ are the roots of the nth degree Legrendre Polynomial and $\forall i = 1, 2, \cdots, n$ are the numbers c_i such that $c_i = \int_{-1}^1 \prod_{j=1, j\neq}^n \frac{x - x_j}{x_i - x_j}dx$. If $P(x)$ is any polynomial of degree < 2, then $\int_{-1}^1 P(x)dx = \sum_{i=1}^n c_i P(x_i)$