

- §1.1 Error Analysis
- **Normalization:** 32 bit - Sign bit: 1, Sign exp: 1, Exp: 7, Normalized mantissa: 23.
- Absolute Error = $|p - p^*|$, Relative Error = $\frac{|p - p^*|}{|p|}$
- Significant Digits: $RE < 5 \times 10^{-t}$
- **f(x):** Machine representation
- **Cancellation Error:** Subtracting nearly equal numbers
- **Example:** $p = 0.54617$, $q = 0.54601$, true $r = p - q = 0.00016$
- 4-digit: $p^* = 0.5462$, $q^* = 0.5460$, $r^* = 0.002$ (RE=25%)
- **Nested Multiplication:** Reduces error
- $f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99 = (((1.01z - 4.62)z - 3.11)z + 12.2)z - 1.99$

- $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$
- Remainder: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$, $\xi \in (a, x)$
- Linear Approx: $f(x_0 + h) \approx f(x_0) + h f'(x_0)$

- **Example:** $\xi \in (0, \pi/2)$, $\sin \xi \leq 1 \Rightarrow R_n \leq \frac{(x)^n}{n!}$

§1.3 Convergence

- $\alpha = \lim_{n \rightarrow \infty} (\alpha_n)$
- Rate: $\alpha_n = \alpha + O(\beta_n)$ if $|\alpha_n - \alpha| \leq K |\beta_n|$
- Find largest p where $\alpha_n - \alpha = O(1/n^p)$

§1.4 Matrix Operations

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$
- $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

§6.1 Gaussian Elimination

- $O(n^3)$ complexity
- **Pivoting:**
 - Partial (PP): Max element in column
 - Scaled PP: $s_i = \max_j |a_{ij}|$, pivot $\max(a_{ik}/s_i)$
 - Complete (CP): Full matrix search ($O(n^3)$)
- **LU Decomposition:** $PA = LU$ through GE steps and $LUx = Pb$.
- **LU Algorithm:** $L = E_{n-a,m-b}^{-1} E_{n-a+1,m-b+1}^{-1} \dots U = GE$.
- **Choleski Factorization:** If a matrix is symmetric and positive definite, it may be factored to the form LDL^T

§6.2 Special Matrices

- **Inverse Matrix:** An inverse matrix of A is A^{-1} such that $AA^{-1} = I$
- Properties: $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^T = (A^T)^{-1}$
- **Singular:** A matrix is singular iff its det is 0.
- **Diagonal:** $d_{ij} = 0$ for $i \neq j$: All non-diagonal entries are 0.
- **Symmetric:** $A = A^T$, $(AB)^T = B^T A^T$
- **Permutation:** Row swaps of I_n , PA reorders rows: $P^T = P^{-1}$
- **Diagonally Dominant:** $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ (nonsingular)
- **Positive Definite:** $x^T A x > 0 \Rightarrow A = LDL^T$, $a_{ii} > 0$, $a_{ij}^2 < a_{ii}a_{jj}$
- **Minor:** M_{ij} is a submatrix of A with the row i deleted and column j deleted.
- **Band:** an $n \times n$ matrix is a band matrix if $p, q \in \mathbb{Z} : 1 \leq p, q \leq n$ exist with $a_{ij} = 0$ for $i + p \leq j$ or $j + q \geq i$ The bandwidth is defined as $w = p + q - 1$. For a diagonal matrix, $p = 1, q = 1, w = 1$
- **Tridiagonal:** Band with $p = 2, q = 2$. It exhibits the following properties
 - $a_{ii} = l_{ii}$ - $a_{i,i+1} = l_{ii}u_{i,i+1} : i = 1 \dots n - 1$
 - $a_{i,i-1} = l_{i,i-1} : i = 2, 3, \dots, n$ - $a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} : i = 2 \dots n$
- Crout Factorization: This factorization happens in $O(n)$ time

§Strategies

- **RoC With inf limit:** set $h = 1/n$ and solve accordingly.
- $D \cdot (L+U)$: given D has ONLY diagonal entries and L+U has NO diagonal entries, the resulting matrix A is composed of entries $a_{ij} = d_{ii} \cdot (l + u)_{ij}$
- **Verification of Bisection:** To verify bisection can be applied, make sure that f(a) and f(b) are of different signs.
- **Error of Bisection:** To compute the accuracy of bisection to an ϵ , we use $\frac{b-a}{2^n} \leq \epsilon$
- **Failure of Newton's Method:** NM Fails if $f'(x) = 0$ for some x.
- **Triangle Inequality:** $|x + y| \leq |x| + |y|$

§Key Definitions & Identities

- **Continuity:** $f \in C^n[a, b]$ reads: the nth derivative of f on [a,b] is continuous.
- **Series Expansions**
 - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ ($\forall x \in \mathbb{R}$)
 - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
 - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
 - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ ($|x| \leq 1$)
 - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ ($|x| < 1$)
 - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots : |x| < 1$
 - $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots : |x| < 1$

• Core Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

• Angle Transformations

- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
- $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$, $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

• Vector Norms

- $\|x\| > 0$
- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

§Useful Examples

- Suppose $g(x) = \frac{5}{x^2} + 2$ Show $p_n = g(p_{n-1})$ will converge to g for $\forall p_0 [2.5, 3]$. Since this is a decreasing function, the max of g(x) is g(2.5) and the min of g(x) is g(3).
 - First compute the max, $g(2.5) = 14/5 < 3$
 - Second compute the min, $g(3) = 5/9 + 2 > 2.5$
 - Last compute $|g'(x)| = -10/x^3 \leq \max_{x \in [2.5, 3]} |g'(x)| = 16/25 < 1$
- Given $\|A\|$ is a natural matrix norm of matrix A. show $\| \lambda \| \leq \|A\|$ for any nonsingular A and any λ of A. $\|A\| = \max_{\|x\|=1} \|Ax\| \geq \| \lambda x \| : x$ is an e-vec s.t $x = \frac{1}{\lambda} \lambda x = 1 = \| \lambda x \| = \| \lambda \| \|x\| = \| \lambda \|$
- When performing Jacobi or GS, when computing L+U, flip the signs of all entries.
- To determine convergence for fixed point, compute $g'(p_0) \leq 1$, which gives a,b. Prove $g(x)$ cts on [a,b], $g(x) \in [a, b]$, $g'(x)$ exists on (a,b), $|g'(x)| \leq k : \forall x \in (a, b)$, $0 < k < 1$

§LA Determinants

- 2x2: $|A| = ad - bc$
- nxn: $|A| = \sum_{j=1}^n a_{ij} A_{ij}$ via cofactors $A_{ij} = (-1)^{i+j} M_{ij}$
- **Properties**
 - Identical rows: $|A| = 0$
 - Swap rows: $|\tilde{A}| = -|A|$
 - Scale row: $|\tilde{A}| = \lambda |A|$
 - $|A^{-1}| = \frac{1}{|A|}$

§7 Norms & Eigen

- $\|x\|_2 = \sqrt{\sum x_i^2}$, $\|x\|_\infty = \max |x_i|$
- $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ Basically sum all rows together and determine the largest one.
- $\forall x \in \mathbb{R}^n : \|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
- A distance between matrices A and B wrt a matrix norm $\| \cdot \|$ is $\|A - B\|$
- Theorem: For any vector $x \neq 0$, matrix Am and abt natural norm $\| \cdot \|$ we have $\|Ax\| \leq \|A\| \cdot \|x\|$
- Cauchy-Schwarz: $\|x + y\|_2 \leq \|x\|_2 \|y\|_2$
- Eigen: λ is an eigenvalue if $A\lambda = v \cdot \lambda$
- Finding Eigenthings: $\det(A - \lambda I) = 0 : \forall \lambda$.
- Spectral Radius: $\rho(A) = \max |\lambda_i|$, $\rho(A) \leq \|A\|$
- Theorem: If A is $n \times n$:
 - $\|A\|_2 = [\rho(A^t A)]^{1/2}$
 - $\rho(A) \leq \|A\| : \forall \| \cdot \|$
- Convergent: $\lim_{k \rightarrow \infty} A^k = 0 \Leftrightarrow \rho(A) < 1$
- Matrix Norms: matrix norms have the following properties
 - $\|A\| \geq 0$ with $\|A\| = 0 \Leftrightarrow A = 0$
 - $\|\alpha A\| = |\alpha| \cdot \|A\|$
 - $\|A + B\| \leq \|A\| + \|B\|$

§Iterative Methods

- **General Iteration:** $x^{k+1} = T x^k + c$
- **Jacobi:** $x^{k+1} = D^{-1}(L + U)x^k + D^{-1}b$
- **Gauss-Seidel:** $x^{k+1} = (D - L)^{-1}Ux^k + (D - L)^{-1}b$
- **Stein-Rosenberg:** For matrices with positive diagonals: $\rho_{GS} \leq \rho_J < 1$
- **Speed of Convergence:** given matrices $T_{GS} = (D - L)^{-1}U$ and $T_J = D^{-1}(L + U)$, compare ρ . The bigger the ρ , the faster the convergence.
- Error: $\|x - x^k\| \leq \frac{\|T\|}{1 - \|T\|} \|x^1 - x^0\|$

- Stopping: $\frac{\|x^k - x^{k-1}\|}{\|x^k\|} < \epsilon$

§2 Nonlinear Equations

- **Bisection:**
 - While $f(p_n) \neq 0$ or $< T$: $p_n = \frac{a_1 + b_1}{2}$
 - Error: $\frac{b_n - a_n}{2} < T$, $p = a + \frac{b-a}{2}$
- **Fixed-Point:**
 - $p_n = g(p_{n-1})$, converges if $|g'(x)| \leq K < 1$
 - Algorithm: For $i < N_0$: $p = g(p_0)$, check $|p - p_0| < T$
 - A fixed point is defined as a point in which $p = f(p)$
- **Newton:**
 - $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
 - Quadratic convergence if $f'(p) \neq 0$
- **Secant:**
 - $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$
 - Approx derivative: $\frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$
 - Algorithm: Store $q_0 = f(p_0)$, $q_1 = f(p_1)$, SET $p = p_1 - \frac{q_1(p_1 - p_0)}{q_1 - q_0}$ IF STOPPING CONDITION: RETURN p; i++, $p_0 = p_1$, $q_0 = q_1$, $p_1 = p$, $q_1 = f(p)$ ENDWHILE OUTPUT FAILURE.

§Theorems

- **Bisection:** Suppose $f \in C[a, b] : f(a) \cdot f(b) < 0$. Bisection generates $\{p_n\}$ approximating a zero p with $|p_n - p| \leq \frac{b-a}{2^n} : n \geq 1$
- **Fixed Point:** If $g \in C[a, b]$, $g([a, b]) \subseteq [a, b]$ g has a fixed point in $[a, b]$, additionally if $|g'| \leq K < 1$, then the fixed point is unique.
- **Fixed Point Theorem:** Let $g \in C[a, b]$ and $g(x) \in [a, b] : \forall x \in [a, b]$. Suppose as well that g' exists on (a,b) and positive $K < 1$ exists with $|g'(x)| \leq K : \forall x \in (a, b)$. Then for any number $p_0 \in [a, b]$ the sequence defined by $p_n = g(p_{n-1}) : n \geq 1$ converges to the unique point $p \in [a, b]$
- **Corollary:** If g satisfies the hypothesis of the above theorem, $|p_n - p| \leq k^n \max(p_0 - a, b - p_0)$ and $|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0| : \forall n > 1$
- **Newton:** For $f \in C^2[a, b]$ with simple root, $\exists \delta > 0 : p_0 \in [p - \delta, p + \delta]$ converges.
- **Matrix Invertibility:** $|A| \neq 0 \Leftrightarrow$ unique solution $Ax = b \Leftrightarrow A^{-1}$ exists
- **Taylor:** With $R_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$, $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} : \xi \in (x, x_0)$
- **Existence of Inverse:** if A is square, $\det A \neq 0 \Leftrightarrow Ax = 0$ has soln $x = 0 \Leftrightarrow Ax = b$ has a unique soln for any n-vector b. $\Leftrightarrow A^{-1}$ exists.
- **Diagonally Dominant Matrices:** dd matrices are nonsingular. A being dd means LU can be performed without P. A matrix is positive definite if $x^t A x > 0$. PD matrices are nonsingular, $\forall i = 1, \dots, n : a_{ii} > 0$, $\max |a_{kj}| > \max |a_{ii}|$, $(a_{ii})^2 < a_{ii}a_{jj} : \forall i \neq j$.
- **Convergence and Spectral Radii:** equivalent statements: A is convergent, $\rho < 1$, $\lim A^n x = 0 : \forall x$, $\lim \|A^n\| = 0 : \forall$ norms.
- **Convergence of DD:** If A is strictly DD, Jacobi and GS converge to the unique soln. $Ax=b$.
- **Positive Definitive Check:** A matrix is PD if the leading principle submatrix determinants are positive.
- **Positive Definitive Check:** A matrix is PD iff it may be factored into LL^T
- **Non-singularity Check:** A matrix A has an inverse iff $\det A \neq 0$.
- **Determinant of Triangular Matrices:** The determinant of a triangular matrix is Πa_{ii} .

§Proofs

- **Bisection (THM1):** $\forall n \geq 1 : b_n - a_n = (b - a) \cdot \frac{1}{2^{n-1}} : p \in (a_n, b_n)$. Since $p_n = \frac{1}{2}(a_n + b_n) : \forall n \geq 1$, $|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$. □
- **Fixed Point:** Part i: If $g \in [a, b]$, $g(x) \in [a, b] : \forall x \in [a, b]$ then g(x) has a fixed point in [a,b]. If $g(a) = a$ or $g(b) = b$, g has a fixed point at an endpoint. Suppose for contradiction that it does not. $g(a) > a$ and $g(b) < b$. Define $h(x) = g(x) - x$. Then h is cts on [a,b] and $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$ IVT states that $\exists p \in (a, b) : h(p) = 0$ Thus $g(p) - p = 0 \Rightarrow p$ is a fixed point of g. Part ii: Suppose as well $|g'(x)| \leq k < 1 : \forall x \in (a, b)$ and that $p, q \in [a, b] : p \neq q$. By MVT, $\exists \zeta : \frac{g(p) - g(q)}{p - q} = g'(\zeta)$. $|p - q| = |g(p) - g(q)| = |g'(\zeta)| |p - q| \leq k |p - q| < |p - q|$ contradiction.

§2: Error Analysis and Accelerating Convergence

Basic Methods

- **Newton's Method:** Quadratic convergence if $f'(p) \neq 0$. Iteration:
 $x_{n+1} = x_n - f(x_n)/f'(x_n)$.
- **Secant Method:** Superlinear convergence (order ≈ 1.618). Uses two previous points.
- **Newton's Improved Method:** $p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{f'(p_n)^2 - f(p_n)f''(p_n)}$

Convergence Analysis

- A soln p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$ we can write $f(x) = (x - p)^m q(x)$ where $\lim_{x \rightarrow p} q(x) \neq 0$. (Simple zeros are multiplicity 1).
- Order of convergence α : $\lim_{|p_n - p|^\alpha} = \lambda$
- Linear ($\alpha = 1$), Quadratic ($\alpha = 2$)
- Fixed-point: Linear if $g'(p) \neq 0$, quadratic if $g'(p) = 0$ and g'' bounded.

Special Cases

- Multiple roots: Modify Newton's using $\mu(x) = f(x)/f'(x)$
- **Aitken's Δ^2 :** Accelerates linear sequences. Is given by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{(p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)}$$

Polynomial Methods

- Horner's method: Efficient evaluation (n mults/adds) Algorithm: ex: evaluate $P(x) = \sum_{i=0}^n a_i x^i$ and derivative at x_0 . Input $n, a_j, x_0 : 0 \leq j \leq n$. Output: $y = P(x_0), z = P'(x_0)$. Set $y = a_n, z = a_n$. For $j = n - 1, n - 2, \dots, 1$ set $y = x_0 y + a_j, z = x_0 z + y$. Endfor set $y - x_0 y + a_0$ output y,z.
- Deflation: Find roots sequentially via $P(x) \approx Q(x)(x - x_0)$
- Fundamental thm of alg: If $P(x)$ has a degree $n \geq 1$, $P(x)$ has at least one root.
- Cor: there also exists unique constants x_1, \dots, x_k such that $\sum_{i=1}^k m_i = n, P(x) = a_n \times \prod_{i=1}^k (x - x_i)^{m_i}$
- Cor: these functions are unique.

Weierstrass: $\forall f$ cts on $[a, b], \forall \epsilon > 0, \exists$ polynomial $p(x)$ with $|f(x) - p(x)| < \epsilon$
 $\forall x \in [a, b]$.

Lagrange Interpolation (unique!):

$$P(x) = \sum_{m=0}^N f(x_m) L_m(x), \text{ where } L_m(x) = \prod_{\substack{k=0 \\ k \neq m}}^N \frac{x - x_k}{x_m - x_k}$$

Interpolation Error: $f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^n (x - x_k)$ for $f \in C^{n+1}[a, b]$

Newton's Divided Differences:

$$f[x_i] = f(x_i), \quad f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$P_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

Hermite: Given $(x_j, f(x_j), f'(x_j))$, unique degree $\leq 2n + 1$ we have:

- $H(x) = \sum_{j=0}^n f(x_j) H_j(x) + \sum_{j=0}^n f'(x_j) \hat{H}_j(x)$
- $H_j(x) = [1 - 2(x - x_j) L_j'(x_j)] L_j^2(x)$
- $\hat{H}_j(x) = (x - x_j) L_j^2(x)$
- Note: $L_j(x)$ denotes the jth Lagrange coefficient polynomial of degree n.
- Error: $f(x) - H(x) = \frac{(x - x_0)^2 \dots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$
- Parametric curve Interpolation:

$$x(t) = [2(x_0 - x_1) + 3(a_0 + a_1)]t^2 + [3(x_1 - x_0) - 3(a_1 + 2a_0)]t^2 + 3a_0 t + x_0$$

$$y(t) = [2(y_0 - y_1) + 3(\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - 3(\beta_1 + 2\beta_0)]t^2 + 3\beta_0 t + y_0$$

Cubic Splines: $S(x_j) = f(x_j)$ and $S \in C^2[a, b]$; Boundary: Clamped (S' at end-points) or Natural ($S'' = 0$ at endpoints) Error: $\max |f(x) - S(x)| \leq \frac{5M}{384} h^4$, where $h = \max(x_{j+1} - x_j), M = \max |f^{(4)}|$

Richardson Extrapolation: $N_{j+1}(h) = N_j(h/2) + \frac{N_j(h/2) - N_j(h)}{2^j - 1}$

Numerical Integration:

- Composite Trapezoid Rule ($O(h^2)$):
 $\int_a^b f(x) dx \approx \frac{b-a}{2n} [f(a) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(b)]$
- Midpoint Rule ($O(h^2)$): $\int_a^b f(x) dx \approx (b - a) \cdot f(\frac{a+b}{2})$
- Composite Midpoint Rule ($O(h^2)$):
 $\int_a^b f(x) dx \approx h[f(x_1) + f(x_2) + \dots + f(x_n)] : x_i = a + (i - 1/2)h$
- Simpson's Rule ($O(h^4)$): $\int_a^b f(x) dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$
- **Simpson's 3/8 Rule (coefficient pattern goes: 1 3 3 2 ... 3 1)** ($O(h^4)$):
 $\int_a^b f(x) dx \approx \frac{3(b-a)}{8n} [f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3 \dots + f(x_n)]$
- **Simpson's 3/8 Error:** $-\frac{(b-a)^5}{6480} f^{(4)}(\xi)$
- Composite Simpson's Rule:
 $\int_a^b f(x) dx \approx \frac{b-a}{3n} [f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(b)]$
- Single Simpson's Error: $-\frac{h^5}{90} f^{(4)}(\xi)$
- Single Trapezoid error: $-\frac{h^3}{12} f''(\xi)$
- Trapezoidal error: $\frac{M(b-a)^3}{12n^2}$: $M = \max_{x \in [a,b]} \{|f''(x)|\}$
- Composite Simpson's error: $-\frac{(b-a)^5}{180n^4} \max_{x \in [a,b]} \{|f^{(4)}(x)|\}$

Romberg - $O(h_k^{2j})$:

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^j - 1 - 1}, \text{ error } O(h^{2j})$$

- $R_{k,1}$ represents the approximation of the integral using $m_k = 2^{k-1}$ intervals

§ODE Initial Value Problems

- **Basic Problem:** $y'(t) = f(t, y), y(a) = \alpha$
- **Lipschitz Condition:** $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$
- Existence/uniqueness guaranteed when $|\frac{\partial f}{\partial y}| \leq L$ over convex domain D

Numerical Methods

- **Euler's Method:** $w_{i+1} = w_i + hf(t_i, w_i)$ [Error: $O(h)$]
 - **Taylor Methods:** $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ where $T^{(n)} = f + \frac{h}{2} f' + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}$
 - **Runge-Kutta Methods:**
- | | |
|--|--|
| Midpoint (RK2) | Modified Euler (RK2) |
| $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2} f_i)$ | $w_{i+1} = w_i + \frac{h}{2} (f_i + f(t_{i+1}, w_i + hf_i))$ |
| $k_1 = hf(t_i, w_i)$ | |
| $k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2})$ | |
| $k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2})$ | [Error: $O(h^4)$] |
| $k_4 = hf(t_i + h, w_i + k_3)$ | |
| $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | |
- **Classical RK4:**

Error Analysis

- **Local truncation error:** $\tau_{i+1} = y(t_{i+1}) - w_{i+1}$ given $w_i = y(t_i)$
- **Global truncation error:** Accumulated error across all steps
- For Euler: $|\tau_i| \leq \frac{h^2}{2} M$ (local), $O(h)$ (global) where $M = \max |y''|$
- For RK4: $O(h^5)$ (local), $O(h^4)$ (global)
- **Stability & Step Size**
- Well-posed problem requires: unique solution exists + small input changes \rightarrow small output changes
- Step size formula: $h < \frac{2\varepsilon}{M(b-a)}$ for error ε , where $M = \max |y''|$
- Example: $y' = y \cos t$ has Lipschitz constant $L = 1$ since $|\frac{\partial f}{\partial y}| = |\cos t| \leq 1$
- **Chapter 3: Interpolation Lagrange Interpolation**
- Δ^k : $\Delta^2 f_i = \Delta(\Delta(f_i)) = \Delta(f_{i+1} - f_i) = f_{i+2} - 2f_{i+1} + f_i$
- Reuse computations with **Neville's Method:**

$$\text{for } i = 1, 2, \dots, n \text{ do: for } j = 1, 2, \dots, i \text{ do } Q_{i,j} \leftarrow \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}$$

Hermite Interpolation

- Error term: $\frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} \max |f^{(2n+2)}|$
- Handle derivatives via **divided differences** with repeated nodes: $z_{2i} = z_{2i+1} = x_i$
- **Warning:** Noisy derivatives \Rightarrow amplified errors.

Splines

- **Cubic splines:** Solve tridiagonal system ($O(n)$ ops) with:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1})$$

- **Natural splines** ($S'' = 0$): Stable but less accurate. **Clamped splines:** Require $f'(a), f'(b)$ but higher accuracy.

- Error: $O(h^4)$ for $f \in C^4, O(h^2)$ for linear splines.

Trapezoidal & Simpson's Rules

- **Romberg Integration:** Accelerate Trapezoidal Rule via: $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^j - 1 - 1}$

- Stop when $|R_{k,k} - R_{k-1,k-1}| < \epsilon$.

Adaptive Quadrature

- Error estimate: $\frac{1}{15}|S(a, b) - S(a, c) - S(c, b)|, c = (a + b)/2$
- Subdivide intervals where error $> \epsilon/2$.
- Step 1: Apply Simpson's with $h = (b - a)/2$.
 $\int_a^b f(x) = h/3 [f(a) + 4f(a + h) + f(b)] - \frac{h^5}{90} f^{(4)}(\mu) \mu \in (a, b)$
- Step 2: Find error using Simpson's on $h = (b - a)/4$
 $\int_a^b f(x) = h/6 [f(a) + 4f(a + h/2) + 2f(a + b) + 4f(a + 3h/2) + f(b)] - (\frac{h}{2})^4 \cdot \frac{(b-a)}{180} f^{(4)}(\tilde{\mu}) : \tilde{\mu} \in (a, b)$
- Note: We assume $f^4(\mu) = f^4(\tilde{\mu})$: true for small h.
- Step 3: Calculate error as $1/10 |S(a, b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)| < \epsilon$
- Step 4: If true, RETURN. Else, GOTO step 1.

Gaussian Quadrature

- $\int_a^b w(x)f(x)dx = \sum_{i=1}^n n w_i f(x_i)$ where $w(x)$: weight functions, w_i : weight at i, x_i node at i.
- Nodes: Roots of Legendre polynomials $P_n(x)$. Weights: $c_i = \int_{-1}^1 \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx$
- Exact for polynomials of degree $\leq 2n - 1$.
- Step 1: Transform to $[-1, 1]$: $x = \frac{(b-a)t + (a+b)}{2}, dx = \frac{b-a}{2} dt$
- Step 2: substitute x into integrand
- Step 3: use the formulae to get the answer:
 - 1-point: $\int_{-1}^1 f(x)dx = 2f(0)$
 - 2-point: $\int_{-1}^1 f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) : w_1 = w_2; x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$
 - 3-point: $\int_{-1}^1 f(x)dx = \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$

Legendre Polynomials

- $P_n(x)$ denotes the n'th degree Legendre polynomial
- $\int_{-1}^1 P(x)P_n(x)dx = 0$: P(x) is of degree $\neq n$
- $P_0(x) = 1, P_1(x) = x, P_2(x) = x^2 - 1/3, P_3(x) = x^3 - 3/5x, P_4(x) = x^4 - 6/7x^2 + 3/35$

Tips:

- Use Gaussian quadrature with $n = c \rightarrow$ apply c-point Gauss-Legendre Rule.

Theorems:

- **Thm 2.8:** Let p be a soln of the eq $x = g(x)$ and suppose $g'(p) = 0$ and g'' is cts and strictly bounded by M on an interval I containing p. Then $\exists \delta > 0$ such that $p_0 \in [p - \delta, p + \delta]$ the seq: $p_n = g(p_{n-1}) : n \geq 1$ converges at least quadratically to p. Moreover, for large n, $|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$
- **Thm 2.10:** $f \in C^1[a, b]$ has a simple zero at p in (a,b) iff $f(p) = 0, f'(p) \neq 0$
- **Thm 2.11:** The function $f \in C^m[a, b]$ has a zero of multiplicity m at p iff $0 = f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p)$
- **Thm 3.3:** Suppose x_0, x_1, \dots, x_n are distinct numbers in [a,b] and $f \in C^{n+1}[a, b]$ then for each x in [a, b], a number $\xi(x)$ in (a,b) exists with
 $f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x - x_i)$ with P(x) being the nth Lagrange interpolating polynomial
- **Thm: Err Trapezoid:** Let $f \in C^2[a, b], h = (b - a)/n, x_j = a + jh : 0 \leq j \leq n$. Then $\exists \mu \in (a, b)$ for which the composite trapezoid rule with n subivls has an err term of $\frac{b-a}{12} h^2 f''(\mu)$
- **Thm: Legendre Thm:** suppose x_1, x_2, \dots, x_n are the roots of the nth degree Legendre Polynomial and $\forall i = 1, 2, \dots, n$ are the numbers c_i such that $c_i = \int_{-1}^1 \prod_{j=1, j \neq i}^n \frac{x - x_j}{x_i - x_j} dx$. If $P(x)$ is any polynomial of degree < 2 , then $\int_{-1}^1 P(x)dx = \sum_{i=1}^n c_i P(x_i)$