Assignment- 9-9.3-7

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Question: Find the area of the region enclosed by the curves $y^2 = x, x = \frac{1}{4}, y = 0$ and $x = 1$. (12,2022)

Solution: The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{4}$$

for the parabola $y^2 = x$, equation of the directrix is $\begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x} = \frac{1}{4}$

$$\mathbf{V} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{5}$$

$$\mathbf{u} = \begin{pmatrix}\frac{-1}{2}\\0\end{pmatrix} \tag{6}$$

$$f = 0 \tag{7}$$

for the line x-$\frac{1}{4} = 0$, parameters are

$$h_2 = \begin{pmatrix}\frac{1}{4}\\0\end{pmatrix} \tag{8}$$

$$m = \begin{pmatrix}0\\1\end{pmatrix} \tag{9}$$

we know that the points of intersection of the line

$$\mathbf{L} : \mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \tag{10}$$

$$\tag{11}$$

with the conic section $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ is given by $\mathbf{x}_i = \mathbf{h} + \kappa_i\mathbf{m}$ where

$$\kappa_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{U}) + -\sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\right)$$

from this we get $k_i = 1, -1$

That implies that the points of intersections are $a_0 = \begin{pmatrix}\frac{1}{4}\\\frac{1}{2}\end{pmatrix}$ and $a_1 = \begin{pmatrix}\frac{1}{4}\\\frac{-1}{2}\end{pmatrix}$

similarly for line $x = 1$ we get point of intersections as $a_3 = \begin{pmatrix}1\\1\end{pmatrix}$ and $a_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

hence area of the region between the two lines and the parabola and the x axis is

$$\int_{\frac{1}{4}}^{1} \sqrt{x}dx - \int_{\frac{1}{4}}^{1} 0dx = \frac{7}{12}$$

Variables	Description
e	Eccentricity of the conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
$n^T\mathbf{x} = c$	Equation of directrix
n	Slope of normal to directrix
f	$\|\|n\|\|^2 \|\|n\|\|^2 - c^2e^2$
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition
$\mathbf{u}$	Vertex of coniv with same directrix