IIITB COMET
F O U N D A T I O N

36$^{\text{th}}$ International Mathematical Olympiad
First Day – Toronto – July 19, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$.

36$^{\text{th}}$ International Mathematical Olympiad
Second Day – Toronto – July 20, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1, \ldots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \ldots, 1995$,

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

3. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there, the sum of whose elements is divisible by p?

37$^\text{th}$ International Mathematical Olympiad
Mumbai, India
Day I   9 a.m. – 1:30 p.m.
July 10, 1996

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.
   (b) Prove that the task is possible when $r = 73$.
   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB = \angle ACB = \angle APC = \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

3. Let S denote the set of nonnegative integers. Find all functions f from S to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \text{for all } m, n \in S.$$

37$^\text{th}$ International Mathematical Olympiad
Mumbai, India
Day II   9 a.m. – 1:30 p.m.
July 11, 1996

1. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

2. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

3. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$.
   (b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

   Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

   (a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.

   (b) Prove that $f(m, n) \leq \frac{1}{2} \max\{m, n\}$ for all m and n.

   (c) Show that there is no constant C such that $f(m, n) < C$ for all m and n.

2. The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

   and

$$|x_i| \leq \frac{n + 1}{2} \qquad i = 1, 2, \ldots, n.$$

   Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n + 1}{2}.$$

4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.$$

   Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$