第 二 章

1.(1) $z=1+iy$，则 $w=\frac{1}{1+iy}=\frac{1-iy}{1+y^2}=\frac{1}{1+y^2}-\frac{y}{1+y^2}i$ 圆.

(2) $z=x$，$w=\frac{1}{x}$，u轴，但不包括原点.

(3) $z=x(1+i)$，$w=\frac{1}{z}=\frac{1}{x}\frac{1}{1+i}=\frac{1}{2x}(1-i)$，

不包括原点的直线

(4) $z=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{4}-i\frac{y}{4}$，圆.

(5) $x=1+\sqrt{5}\cos\theta$，$y=\sqrt{5}\sin\theta$.

$\Rightarrow z=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{4+2x}-i\frac{y}{4+2x}$ 由

或：$x^2+y^2-2x=4 \Rightarrow |z|^2-2x=4$

$\left(\frac{x}{4+2x}+\frac{1}{4}\right)^2+\left(\frac{y}{4+2x}\right)^2=\left(\frac{\sqrt{5}}{4}\right)^2$

$\Rightarrow \frac{1}{|w|^2}-\frac{2u}{u^2+v^2}=4 \Rightarrow \frac{1-2u}{u^2+v^2}=4$

$\Rightarrow 4u^2+4v^2+2u-1=0$，结论相同.

知是以 $(-\frac{1}{4},0)$ 为圆心，$\frac{\sqrt{5}}{4}$ 为半径的圆.

2. $f(z)=\begin{cases}\frac{xy}{x^2+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$

$\lim\limits_{z\to 0}f(z)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$，令 $y=kx$，则

$\lim\limits_{z\to 0}f(z)=\frac{kx^2}{x^2+k^2x^2}=\frac{k}{1+k^2}$，当 k 发生变化，即 $f(z)$ 以不同方向

靠近原点，极限值不同. 故不连续在 $z=0$ 处.

3. $p_n(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_2z^2+a_1z+a_0$

$$=z^n\left(a_n+\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right)$$

当 $z\to\infty$，括号内各项均 $\to 0$（除 a_n 外），故

$p_n(z)\to a_nz^n$，令 $z=re^{i\varphi}$，故 $|z^n|=r^n$，当 $z\to\infty$，$r\to\infty$，故 $z^n\to\infty$

a_n 是个常数，故 $a_nz^n\to\infty$

综上，$z\to\infty$ 时 $p_n(z)\to\infty$。

4. 考虑 C-R 方程，

(1) 令 $f(z)=u(x,y)+iv(x,y)$，若可导即满足 ① 当 $z\neq 0$

$\begin{cases}u_x=v_y\\u_y=-v_x\end{cases}$，$f(z)=|z|=\sqrt{x^2+y^2}$，故 $v(x,y)=0$，即 $v_x=v_y=0$

而 $u_x, u_y\neq 0$ 故不可导

② 当 $z=0$ 时 $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$ 分母为 0 也不可导，

综上 $f(z)=|z|$ 不可导，

(2) 类(1)，$u(x,y)=x+y$，$v(x,y)=0$

不满足 C-R 方程，处处不可导。

(3) 类(1)，$f(z)=\frac{1}{z}=\frac{1}{x-iy}=\frac{x+iy}{|z|^2}=\frac{x}{x^2+y^2}+i\frac{y}{x^2+y^2}$，

$\frac{\partial u}{\partial x}=\frac{y^2-x^2}{|z|^4}$，$\frac{\partial v}{\partial y}=\frac{x^2-y^2}{|z|^4}$，不满足 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$。

故 $f(z)=\frac{1}{z}$ 处处不可导。

5、(1) $\frac{\partial u}{\partial x}=y$，$\frac{\partial u}{\partial y}=x$，$\frac{\partial v}{\partial x}=0$，$\frac{\partial v}{\partial y}=1$．由C-R方程

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}\Rightarrow\begin{cases}y=1\\ x=-0\end{cases}\Rightarrow\begin{cases}y=1\\ x=0\end{cases}$$

只在 $z=i$ 这一个点可导，那么处处不解析.

(2) $$f(z)=\begin{cases}x\sqrt{x^2+y^2}+iy\sqrt{x^2+y^2}, & |z|<1\\ x^2-y^2+i2xy, & |z|\geq1\end{cases}$$

① $|z|<1$ 时，$u_x=\frac{2x^2+y^2}{|z|^2}$，$u_y=\frac{xy}{|z|^2}$

$$v_x=\frac{xy}{|z|^2},\quad v_y=\frac{x^2+2y^2}{|z|^2}$$

$$\begin{cases}\frac{2x^2+y^2}{|z|^2}=\frac{x^2+2y^2}{|z|^2}\\ \frac{xy}{|z|^2}=\frac{-xy}{|z|^2}\end{cases}\Rightarrow x=y=0\Rightarrow z=0.$$

只有一点可导故 $|z|<1$ 不解析.

② $|z|>1$ 时 $\begin{cases}u_x=2x=v_y\\ u_y=-2y=-v_x\end{cases}$，处处解析.

③ $|z|=1$ 时，考察是否连续，若连续则

$$\lim_{|z|\to1^-}f(z)=\lim_{|z|\to1^-}z=\lim_{|z|\to1^+}f(z)=\lim_{|z|\to1^+}z^2=f(z)\Big|_{|z|=1}.$$

只有 $z=1$ 时满足条件，所以 $z=1$ 处不解析

综上，解析区域是 $|z|>1$.

6、计算即可略

7. $f'(z_0)=\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$，$g'(z_0)=\lim\limits_{z\to z_0}\dfrac{g(z)-g(z_0)}{z-z_0}$

$\dfrac{f'(z_0)}{g'(z_0)}=\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim\limits_{z\to z_0}\dfrac{f(z)}{g(z)}$ 证毕.

8. (1) $f'(z)=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}=0$.

又由C-R方程可得 $u_x=u_y=v_x=v_y=0$.

故 $f(z)=C_1+iC_2$

(2) $\overline{f(z)}=u(x,y)-iv(x,y)$，类(1)可证.

(3) $\mathrm{Re}\,f(z)=u(x,y)=C$，则由C-R方程类(1)可证.

(4) 类(1)

(5) $|f(z)|=\sqrt{u^2+v^2}=C$，即 $u^2(x,y)+v^2(x,y)=C^2$.

则 $\begin{cases}2uu_x+2vv_x=0\\2uu_y+2vv_y=0\end{cases}\Rightarrow\begin{cases}uu_x+vv_x=0\\uu_y+vv_y=0\end{cases}$

又由C-R方程 $u_x=v_y$，$u_y=-v_x$，得.

$\begin{cases}uu_x+vv_x=0\\vu_x-uv_x=0\end{cases}$，视为一个齐次线性方程组，有

$\begin{vmatrix}u & v\\ v & -u\end{vmatrix}=-(u^2+v^2)$，要使此方程组有非平凡解

$u^2+v^2=|f(z)|^2=0$，也即当 $|f(z)|\neq0$ 时，$u_x=v_x=0$，$f(z)$为常数

当 $|f(z)|=0$ 时显然 $f(z)=0$ 为常数.

(6) $f(z)=u(x,y)+iv(x,y)=\sqrt{u^2+v^2}\left(\frac{u}{\sqrt{u^2+v^2}}+i\frac{v}{\sqrt{u^2+v^2}}\right)$

$\arg f(z)=\arcsin\frac{v}{\sqrt{u^2+v^2}}=C$. 故

$$\frac{\partial\frac{v}{\sqrt{u^2+v^2}}}{\partial v}=\frac{u^2}{(u^2+v^2)^{3/2}}=0\ ,\ \frac{\partial\frac{v}{\sqrt{u^2+v^2}}}{\partial u}=\frac{-uv}{(u^2+v^2)^{3/2}}=0$$

$$\Rightarrow\begin{cases}u^2=0\\-uv=0\end{cases}\Rightarrow u=0\Rightarrow v=0$$

$$\Rightarrow f(z)=0$$

9. $\zeta=f(z)=\xi(x,y)+i\eta(x,y)$

由 $f(z)$ 是解析函数 $\frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial y}$, $\frac{\partial\xi}{\partial y}=-\frac{\partial\eta}{\partial x}$

$$|f'(z)|^2=\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2=\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$\frac{\partial H}{\partial x}=\frac{\partial H}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial H}{\partial\eta}\frac{\partial\eta}{\partial x}\ ,\ \frac{\partial H}{\partial y}=\frac{\partial H}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial H}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$\frac{\partial^2H}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\xi}\right)\frac{\partial\xi}{\partial x}+\frac{\partial H}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\eta}\right)\frac{\partial\eta}{\partial x}+\frac{\partial H}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}.$$

$$=\frac{\partial^2H}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+\frac{\partial H}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}$$
$$+\frac{\partial^2H}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+\frac{\partial H}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}$$

$$\frac{\partial^2H}{\partial y^2}=\frac{\partial^2H}{\partial\xi^2}\left(\frac{\partial\xi}{\partial y}\right)^2+\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}+\frac{\partial H}{\partial\xi}\frac{\partial^2\xi}{\partial y^2}$$
$$+\frac{\partial^2H}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2+\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}+\frac{\partial H}{\partial\eta}\frac{\partial^2\eta}{\partial y^2}$$

$$LHS=\frac{\partial^2H}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{\partial^2H}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{\partial^2H}{\partial\xi^2}\left(\frac{\partial\xi}{\partial y}\right)^2+\frac{\partial^2H}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2$$
$$+2\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+2\frac{\partial^2H}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}+$$
$$\frac{\partial H}{\partial\xi}\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\xi^2}{\partial y^2}\right)+\frac{\partial H}{\partial\eta}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)$$

第一行=RHS. 由 C-R 方程. 第二、三行=0.

故 LHS=RHS.

10. x-y 中, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

令 $x=r\cos\theta$, $y=r\sin\theta$.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}\quad.\ \frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y}.$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}\quad.\ \frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$dx = \cos\theta\, dr - r\sin\theta\, d\theta \Rightarrow dr = \cos\theta\, dx + \sin\theta\, dy$

$dy = \sin\theta\, dr + r\cos\theta\, d\theta \quad d\theta = \frac{-\sin\theta}{r} dx + \frac{\cos\theta}{r} dy$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} \\ \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} \\ \frac{\partial v}{\partial x} = \cos\theta \frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta} \\ \frac{\partial v}{\partial y} = \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta} \end{cases} \quad 又 \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \end{cases}$$

可得 $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$ ， $\frac{1}{r}\frac{\partial u}{\partial\theta} = -\frac{\partial v}{\partial r}$

令 $z = re^{i\theta}$. 则 $f(z) = z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$

$u(r,\theta) = r^n\cos n\theta$, $v(r,\theta) = r^n \sin n\theta$、

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$. $\frac{\partial u}{\partial\theta} = nr^{n-1}\sin n\theta$, $\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$, $\frac{\partial v}{\partial\theta} = nr^n\cos n\theta$.

满足条件.

对 $f(z) = \ln z = \ln r + i\theta$ ，有

$u(r,\theta) = \ln r$ ， $v(r,\theta) = \theta$.

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial u}{\partial\theta} = 0$, $\frac{\partial v}{\partial r} = 0$. $\frac{\partial v}{\partial\theta} = 1$.

满足条件.

11. (1)

$\frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$ ，只要 $z \neq 1$ 且 $z \neq 2$ 即可.

(2) $\frac{1}{z^3+a}$ ，只要 $z^3 + a \neq 0$. 令 $z = re^{i\theta}$ ，即 $r^3 e^{i3\theta} = r^3(\cos 3\theta + i\sin 3\theta) = -a$

$$\Rightarrow \begin{cases} r^3 = a \\ \cos 3\theta = -1 \\ \sin 3\theta = 0 \end{cases} \Rightarrow \begin{cases} r = \sqrt[3]{a} \\ \theta = \frac{\pi + 2k\pi}{3} \end{cases}, k = 0,1,2.$$

$$\Rightarrow z \neq \sqrt[3]{a}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), -\sqrt[3]{a}, \sqrt[3]{a}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

12、$\forall z_1 \neq z_2 \in D$ 有 $f(z_1) \neq f(z_2)$

不妨令 $f(z_1) = z_1^2 + 2z_1 + 3 = z_2^2 + 2z_2 + 3 = f(z_2)$. 则

$(z_1+z_2)(z_1-z_2) + 2(z_1-z_2) = (z_1+z_2+2)(z_1-z_2) = 0$

由 $z_1 \neq z_2$ 知 $z_1 + z_2 + 2 = 0$. 即 $z_1 = -z_2 - 2$

显然若 $|z_2| < 1$, 则 $|-z_2| < 1$ 、则

$|-z_2-2| > 2 - |z_2| \geq 1$. 与 $|z_1| < 1$ 矛盾

故 $f(z)$ 为单叶映射.

13. 上半虚轴为$\frac{\pi}{2}$，则有 $-\frac{3\pi}{2} < \arg z < \frac{\pi}{2}$

令 $z = re^{i\theta}$，则当 $z=1$ 时，即

$$w = \sqrt{r}e^{i\frac{\theta+2k\pi}{2}} = \cos\frac{\theta+2k\pi}{2} + i\sin\frac{\theta+2k\pi}{2} = \cos k\pi + i\sin k\pi$$

由其在正实轴上取正值，知 $k=0$，即 $w = \sqrt{r}e^{\frac{i\theta}{2}}$

对左沿的i，$\arg z \to -\frac{3\pi}{2}$，$w = e^{i(-\frac{3\pi}{4})} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

对右沿的i，$\arg z \to \frac{\pi}{2}$，$w = e^{i(\frac{\pi}{4})} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$w(-1) = e^{i(-\frac{\pi}{2})} = -i$，$w'(z) = \frac{1}{2\sqrt{z}}$，故 $w'(-1) = \frac{1}{-2i} = \frac{i}{2}$.

14、被开方项为零的点是函数的支点，只要令 $(1-z^2)(1-k^2z^2)=0$，可得 $z=\pm1$ 或 $z=\pm\frac{1}{k}$，由 $0<k<1$，知 $\frac{1}{k}>1$，则支点顺序为：$-\frac{1}{k}, -1, 1, \frac{1}{k}$，正好是线段端点，因此能分出单值解析函数

由 $f(z) = \sqrt{|(1-z^2)(1-k^2z^2)|}\exp\left(i\frac{\arg\sqrt{(1-z^2)(1-k^2z^2)}}{2} + ik\pi\right)$

当 $z=0$，$f(0) = \exp\left(i\frac{\arg(1)}{2} + ik\pi\right) > 0$，则 $k=0$

故所求分支为 $f(z) = \sqrt{|(1-z^2)(1-k^2z^2)|}\exp\left(i\frac{\arg\sqrt{(1-z^2)(1-k^2z^2)}}{2}\right)$

15. $w = f(z) = \sqrt[4]{z(1-z)^3}$，其支点满足 $z(1-z)^3=0$，则 $z=0$ 或 $z=1$.

正好在线段两端，故在其外有单值解析分支

对上沿，令 $z = x+i\varepsilon$，其中 $\varepsilon\to0^+$，对 $z=x$，$\arg(z)=0$ 且 $\arg(1-z)=0$

$\arg w = \frac{1}{4}(\arg(z) + 3\arg(1-z)) = 0$ 则 $w(x) = \sqrt[4]{x(1-x)^3} > 0$

$\Rightarrow \arg(w(-1)) = \frac{1}{4}[\arg(-1) + 3\arg(2)] = \frac{\pi}{4}$，$|w(-1)| = \sqrt[4]{8}$

$\Rightarrow w(-1) = \sqrt[4]{8}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$\ln w = \frac{1}{4}[\ln z + 3\ln(1-z)]$ 故

$\frac{w'(z)}{w(z)} = \frac{1-4z}{4z(1-z)} \Rightarrow w'(-1) = \sqrt[4]{2}(1+i)\cdot\left(-\frac{5}{8}\right) = \frac{-5\sqrt[4]{2}}{8}(1+i)$

16. (1) 令 $z=x+iy$，则 $\frac{z}{e^z}=\frac{x+iy}{e^x e^{iy}}=\frac{x+iy}{e^x(\cos y+i\sin y)}=\frac{x\cos y+y\sin y+i(y\cos y-x\sin y)}{e^x}$

当 $z\to\infty$，不妨令 $x=0$，$y\to+\infty$ 则

$\lim\limits_{z\to\infty}\frac{z}{e^x}=\lim\limits_{y\to+\infty}y(\sin y+i\cos y)$，故无极限

(2) $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$，则 $z\sin\frac{1}{z}=\frac{z}{2i}\left(e^{\frac{i}{z}}-e^{\frac{-i}{z}}\right)$，不妨令 $z=x+iy$

则 $z\sin\frac{1}{z}=\frac{x+iy}{2i}\left(e^{\frac{y+ix}{x^2+y^2}}-e^{\frac{-y-ix}{x^2+y^2}}\right)=\frac{y-ix}{2}\left[\cos\frac{x}{x^2+y^2}\left(e^{\frac{y}{x^2+y^2}}-e^{\frac{-y}{x^2+y^2}}\right)+i\left(e^{\frac{y}{x^2+y^2}}\sin\frac{x}{x^2+y^2}+e^{\frac{-y}{x^2+y^2}}\sin\frac{x}{x^2+y^2}\right)\right]$

$\Rightarrow\left|z\sin\frac{1}{z}\right|=\frac{\sqrt{x^2+y^2}}{2}\sqrt{e^{\frac{2y}{x^2+y^2}}+e^{\frac{-2y}{x^2+y^2}}-2\cos\frac{2x}{x^2+y^2}}$

不妨令 $x=y=\varepsilon\to0^+$，则 $\left|z\sin\frac{1}{z}\right|>\frac{\varepsilon}{\sqrt{2}}e^{\frac{1}{2\varepsilon}}=+\infty$，故不存在

(3) 令 $z=1+re^{i\varphi}$，$r\to0^+$，

分母 $e^z-1\to e-1$，分子 $z\to1$，另个因子：

$\frac{1}{z-1}=\frac{1}{r}e^{-i\varphi}=\frac{1}{r}(\cos\varphi-i\sin\varphi)$，则 $e^{\frac{1}{z-1}}=e^{\frac{\cos\varphi}{r}}\cdot e^{\frac{-\sin\varphi}{r}i}$

不妨令 $\varphi=0$，则 $e^{\frac{1}{z-1}}=e^{\frac{1}{r}}\to+\infty$，若令 $\varphi=\frac{\pi}{2}$，则 $e^{\frac{1}{z-1}}=e^{-\frac{1}{r}i}$，不定的，故不存

17. 不妨令 $z=x+iy$，则 $y=kx$（射线不垂直于x轴则）

$z+e^z=x+ikx+e^x(\cos kx+i\sin kx)$，若 $k\neq0$，则 $z+e^z\to\infty$

若 $k=0$，则 $z+e^z=x+e^x\to\infty$

若射线垂直于x轴，则 $z=Ai$，$A\to\infty$ 则

$z+e^z=Ai+e^{Ai}=\cos A+i(\sin A+A)\to\infty$

18. $\sin z=2$，即 $\frac{e^{iz}-e^{-iz}}{2i}=2$，即 $e^{iz}-e^{-iz}=4i$，即

$e^{i2z}-1=4ie^{iz}$，即 $e^{i2z}-4ie^{iz}-1=0$，令 $t=e^{iz}$，即

$t^2-4it-1=0$，则 $t=\frac{4i\pm\sqrt{-16+4}}{2}=(2\pm\sqrt{3})i=e^{iz}$

两边取对数，有 $iz=\ln(i(2\pm\sqrt{3}))=\ln(2\pm\sqrt{3})+i\arg(i(2\pm\sqrt{3}))+i2k\pi$

即 $z=-i\ln(2\pm\sqrt{3})+\frac{\pi}{2}+2k\pi$

19. (1) 令 $1+e^z=0$，$z=x+iy$，则 $e^xe^{iy}=e^x(\cos y+i\sin y)=-1$

则 $x=0$，$\cos y=-1$，$\sin y=0\Rightarrow y=\pi+2k\pi$

$\Rightarrow D=\{x+iy\mid y\neq\pi+2k\pi,\ k\in\mathbb{Z}\}$

微商：$f'(z)=\dfrac{-e^z}{(1+e^z)^2}$

(2) $\sin z=2$. 即 $z=\frac{\pi}{2}+2k\pi-i\ln(2\pm\sqrt{3})$

$\Rightarrow D=\left\{z \,\middle|\, z\neq\frac{\pi}{2}+2k\pi-i\ln(2\pm\sqrt{3})\right\}$ (见本章18)

微商：$f'(z)=\dfrac{-\cos z}{(\sin z-2)^2}$

(3) 只要 $z\neq1$ 即可. $D=\{z \mid z\neq1\}$

微商：$f'(z)=e^{\frac{1}{z-1}}-\dfrac{ze^{\frac{1}{z-1}}}{(z-1)^2}$

20.(1) $\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$,

$LHS=\dfrac{e^{iz_1+iz_2}+e^{-iz_1-iz_2}}{2}$,

$RHS=\left(\dfrac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\dfrac{e^{iz_2}+e^{-iz_2}}{2}\right)-\left(\dfrac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\dfrac{e^{iz_2}-e^{-iz_2}}{2i}\right)$

$=\dfrac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}+e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}+e^{z_1+z_2}-e^{z_1+z_2}+e^{-(z_1+z_2)}-e^{-(z_1+z_2)}}{4}=LHS$

(2) 类(1). 只需硬算即可. 此处略.

(3) 令 $w=\mathrm{Arccos}\,z$. 则 $z=\cos w=\dfrac{e^{iw}+e^{-iw}}{2}$. 即 $e^{iw}+e^{-iw}-2z=0$. 即

$e^{2iw}-2ze^{iw}+1=0$. 令 $t=e^{iw}$. 即 $t^2-2zt+1=0$. 有

$t=\dfrac{2z\pm\sqrt{4z^2-4}}{2}=z\pm\sqrt{z^2-1}=e^{iw}$. 即

$iw=\ln\left(z\pm\sqrt{z^2-1}\right)$, 即 $w=-i\mathrm{Ln}(z+\sqrt{z^2-1})$

(此处 $\sqrt{z^2-1}$ 是一个双值函数, 替代了"±"的作用, 详见课本)

21. $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$, 令 $z=x+iy$. 则

$\sin z=\dfrac{e^{-y+ix}-e^{y-ix}}{2i}=\dfrac{e^{-y}(\cos x+i\sin x)-e^{y}(\cos x-i\sin x)}{2i}$

$$= \frac{e^{-y}\sin x + e^{y}\sin x}{2} + i\left(\frac{-e^{-y}\cos x + e^{y}\cos x}{2}\right)$$

$\Rightarrow$ 实部：$\frac{\sin x}{2}(e^{y}+e^{-y}) = \sin x\cosh y$

虚部：$\frac{\cos x}{2}(e^{y}-e^{-y}) = \cos x\sinh y$

模：$\sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}$

22、类21. 可求出 $\cos z$ 的虚部为 $-\sinh y\sin x$

只要其为0. 即 $\left(\frac{e^{z}-e^{-z}}{2}\right)\sin x = 0$

① $x=0$，

② $x\neq 0$. $e^{z}-e^{-z}=0$. 即 $e^{x}(\cos y + i\sin y) = e^{-x}(\cos y - i\sin y)$

$e^{2x} = \cos 2y - i\sin 2y$，只能 $\sin 2y = 0$

$\Rightarrow 2y = \pi + k\pi \Rightarrow y = \frac{\pi}{2} + \frac{k\pi}{2}$

故在 $x=0$ 与 $y=\frac{\pi}{2}+\frac{k\pi}{2}$，$k\in\mathbb{Z}$ 上取实数

23. 有 $\mathrm{Ln}z = \ln|z| + i(\arg z + 2k\pi)$ 故

$\mathrm{Ln}(-1) = i(\pi + 2k\pi)$，$k\in\mathbb{Z}$ （ arg表主幅角.范围 $(-\pi,\pi]$ ）

$\ln(-1) = \pi$

$\mathrm{Ln}(i) = i\left(\frac{\pi}{2} + 2k\pi\right)$，$k\in\mathbb{Z}$

$\ln(i) = i\frac{\pi}{2}$

$\mathrm{Ln}(3-2i) = \ln\sqrt{13} + i\left(\arctan\frac{-2}{3} + 2k\pi\right)$

$\ln(-2+3i) = \ln\sqrt{13} + i\left(\arctan\frac{-3}{2} + \pi\right)$

$1^{\sqrt{2}} = e^{\sqrt{2}\mathrm{Ln}1} = e^{\sqrt{2}(2k\pi i)} = e^{i\,2\sqrt{2}k\pi} = \cos(2\sqrt{2}k\pi) + i\sin(2\sqrt{2}k\pi)$.

$(-2)^{\sqrt{2}} = e^{\sqrt{2}\mathrm{Ln}(-2)} = e^{\sqrt{2}(\ln 2 + i(\pi+2k\pi))} = e^{\sqrt{2}\ln 2}e^{i(\pi+2k\pi)} = e^{\sqrt{2}\ln 2}(\cos(\pi) + i\sin(\pi)) = -e^{\sqrt{2}\ln 2}$

$2^{i} = e^{i\mathrm{Ln}2} = e^{i(\ln 2 + 2k\pi i)} = e^{-2k\pi + i\ln 2} = e^{-2k\pi}(\cos(\ln 2) + i\sin(\ln 2))$

$\cos(2+i) = \frac{e^{-1+2i} + e^{1-2i}}{2} = \frac{(e^2+1)\cos 2}{2e} + i\frac{(1-e^2)\sin 2}{2e}$

$\sin 2i = \frac{e^{-2} - e^{2}}{2i} = \frac{i}{2}(e^{2} - e^{-2})$

$$\cot\left(\frac{\pi}{4}-i\ln 2\right)=\frac{\cos\left(\frac{\pi}{4}-i\ln 2\right)}{\sin\left(\frac{\pi}{4}-i\ln 2\right)}=\frac{\left(e^{\ln 2+i\frac{\pi}{4}}+e^{-\ln 2-i\frac{\pi}{4}}\right)2i}{e^{\ln 2+i\frac{\pi}{4}}-e^{-\ln 2-i\frac{\pi}{4}}}=$$

$$\frac{\left[e^{\ln 2}\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)+e^{\ln\frac{1}{2}}\left(\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)\right]2i}{e^{\ln 2}\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)-e^{\ln\frac{1}{2}}\left(\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)}=\frac{(2i)(\sqrt2+i\sqrt2)(\sqrt2-i\sqrt2)}{4\left[(\sqrt2+i\sqrt2)-\frac{1}{4}(\sqrt2-i\sqrt2)\right]}=\frac{\sqrt2}{17}(5+3i)$$

使用 $\cosh(x+iy)=\cosh(x)\cos(y)+i\sinh(x)\sin(y)$

$\sinh(x+iy)=\sinh(x)\cos(y)+i\cosh(x)\sin(y)$.

有 $\coth(2+i)=\dfrac{\cosh(2)\cos(1)+i\sinh(2)\sin(1)}{\sinh(2)\cos(1)+i\cosh(2)\sin(1)}=\dfrac{\sinh(4)-i\sin(2)}{\cosh(4)-\cos(2)}$.

使用 $\mathrm{Arcsin}\, z=-i\,\mathrm{Ln}(iz+\sqrt{1-z^2})$ 与 $\mathrm{Arccos}\, z=-i\,\mathrm{Ln}(z+\sqrt{z^2-1})$ 即可得

(前者见书本. 后者见本章习题 20.(3))

24. $(a^b)^c$ 与 a^{bc}

直接举反例即可证不相等: 取 $a=-1$, $b=2$, $c=\frac{1}{2}$,

$(a^b)^c=\sqrt{1}=\pm1$, 而 $a^{bc}=(-1)^1=-1$

故 $(a^b)^c$ 不一定等于 a^{bc}.