1. (1)

$z = 2-2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}e^{-i\frac{\pi}{4}}$

$\mathrm{Arg}z = -\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

(2)

$z = -\sqrt{3}i = \sqrt{3}(0-i) = \sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = \sqrt{3}e^{-i\frac{\pi}{2}}$

$\mathrm{Arg}z = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$

(3)

$z = \frac{\sqrt{13}}{2}\left(-\frac{1}{\sqrt{13}} - \frac{2\sqrt{3}}{\sqrt{13}}\right) = \frac{\sqrt{13}}{2}\left(\arccos\left(-\frac{1}{\sqrt{13}}\right) + \arcsin\left(\frac{-2\sqrt{3}}{\sqrt{13}}\right)\right) = \frac{\sqrt{13}}{2}e^{i(-\pi+\arctan 2\sqrt{3})+2k\pi}$

$\mathrm{Arg}z = (2k-1)\pi + \arctan 2\sqrt{3},\ k \in \mathbb{Z}.$

(4) 略

2. (1)

令 $z = re^{i\varphi}$，则 $z^3 = r^3(\cos 3\varphi + i\sin 3\varphi)$

$r^3 = \sqrt{1+3} = 2 \Rightarrow r = \sqrt[3]{2}$，

$\begin{cases} \cos 3\varphi = -\frac{1}{2} \\ \sin 3\varphi = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow 3\varphi = \frac{2\pi}{3} + 2k\pi \Rightarrow \varphi = \frac{\frac{2\pi}{3}+2k\pi}{3}.$

$\Rightarrow z = \sqrt[3]{2}\left[\cos\frac{\frac{2\pi}{3}+2k\pi}{3} + i\sin\frac{\frac{2\pi}{3}+2k\pi}{3}\right],\quad k=0,1,2$

(2) 略

(3) $z^4 = r^4(\cos 4\varphi + i\sin 4\varphi) = -1$

$\Rightarrow \begin{cases} \cos 4\varphi = -1 \\ \sin 4\varphi = 0 \end{cases} \Rightarrow 4\varphi = \pi + 2k\pi \Rightarrow \varphi = \frac{\pi}{4} + \frac{k\pi}{2}.$

$\Rightarrow z = \cos\frac{\pi+2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4},\quad k=0,1,2,3.$

3. $z^3 = 1$，则 $r^3(\cos 3\varphi + i\sin 3\varphi) = 1$

$\Rightarrow \begin{cases} \cos 3\varphi = 1 \\ \sin 3\varphi = 0 \end{cases} \Rightarrow 3\varphi = 2k\pi \Rightarrow \varphi = \frac{2k\pi}{3}.$

$\Rightarrow z = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$

$(w-1)(w^2+w+1) = w^3 - 1 = 0$. 故 $w^3 = 1$，由 w 是复根，$w \neq 1$. 故

4. $x^2-y^2+2xyi = a+bi$

$$\Rightarrow \begin{cases} a=x^2-y^2 \\ b=2xy \end{cases} \Rightarrow \begin{cases} x=\pm\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} \\ y=\pm\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}} \end{cases}$$

$b>0$, xy 同号. $b<0$, xy 异号.

5. 令 $S_n=\sum\limits_{k=1}^{n}e^{ik\theta}=\sum\limits_{k=1}^{n}(\cos k\theta+i\sin k\theta)$

等比数列首项 $a=e^{i\theta}$. 公比为 $r=e^{i\theta}$.

$$S_n=a\frac{1-r^n}{1-r}=e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}$$

$$1-e^{in\theta}=e^{in\frac{\theta}{2}}\left(e^{-in\frac{\theta}{2}}-e^{in\frac{\theta}{2}}\right)=e^{in\frac{\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)=-2ie^{in\frac{\theta}{2}}\sin\frac{n\theta}{2}$$

$$1-e^{i\theta}=e^{in\frac{\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)\Big|_{n=1}=-2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$$

$$\Rightarrow S_n=e^{i\theta}\frac{-2ie^{in\frac{\theta}{2}}\sin\frac{n\theta}{2}}{-2ie^{\frac{i\theta}{2}}\sin\frac{\theta}{2}}=e^{i\left(\frac{n+1}{2}\right)\theta}\left(\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)=\left[\cos\left(\frac{n+1}{2}\right)\theta+i\sin\left(\frac{n+1}{2}\right)\theta\right]\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

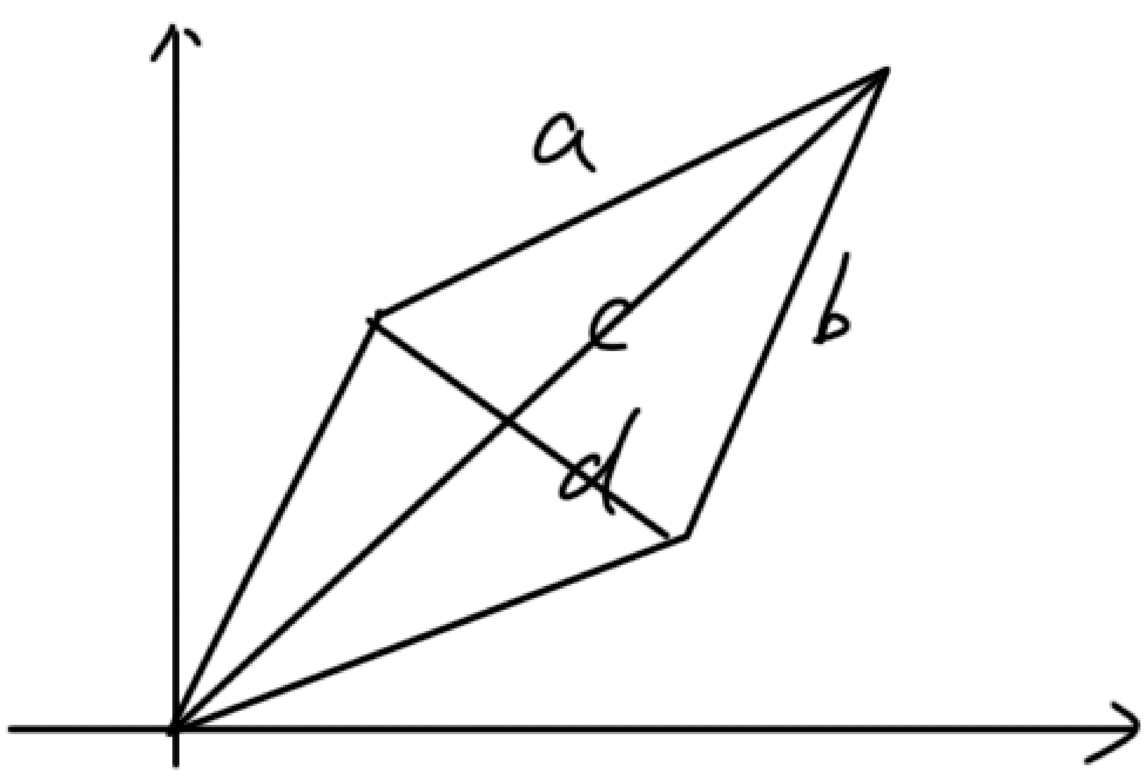
$$\Rightarrow \mathrm{Re}(S_n)=\frac{\cos\left(\frac{n+1}{2}\right)\theta\left(\sin\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}=-\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

即 (1) 证毕. 对 (2)、$\mathrm{Im}(S_n)=\sin\left(\frac{n+1}{2}\theta\right)\dfrac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$，类 (1) 易证 (2).

6. $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\bar{z}_2)$

$|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\mathrm{Re}(z_1\bar{z}_2)$

$\Rightarrow LHS=2(|z_1|^2+|z_2|^2)=RHS.$



平行四边形 $2(a^2+b^2)=c^2+d^2$

7.
(1) $\left|\dfrac{z-a}{1-\bar{a}z}\right|=\dfrac{|z-a|}{|1-\bar{a}z|}$

$|z-a|^2=|z|^2+|a|^2-2\mathrm{Re}(z\bar{a})=1+|a|^2-2\mathrm{Re}(z\bar{a})$

$|1-\bar{a}z|^2=1^2+|\bar{a}z|^2-2\mathrm{Re}(a\bar{z})=1+|a|^2-2\mathrm{Re}(\overline{z\bar{a}})$

$$\Rightarrow\left|\frac{z-a}{1-\bar{a}z}\right|=\sqrt{\frac{|z-a|^2}{|1-\bar{a}z|^2}}=1$$

(2) $|z-a|^2=|z|^2+|a|^2-2\mathrm{Re}(z\bar{a})$

$|1-\bar{a}z|^2 = 1^2 + |a|^2|z|^2 - 2\mathrm{Re}(\overline{z\bar{a}})$

由 $|z|,|a|<1$，知 $\dfrac{|z-a|}{|1-\bar{a}z|} = \sqrt{\dfrac{|z-a|^2}{|1-\bar{a}z|^2}} < \sqrt{\dfrac{1+|a|^2-2\mathrm{Re}(z\bar{a})}{1+|a|^2-2\mathrm{Re}(z\bar{a})}} = 1$.

8. (1) 假设

$|z_1+\cdots+z_{n-1}| \geq |z_1|-|z_2|-\cdots|z_{n-1}|$ 成立，则

$|z_1+z_2+\cdots+z_{n-1}|-|z_n| \geq |z_1|-|z_2|-\cdots-|z_{n-1}|-|z_n|$.

下证 $|z_1+z_2+\cdots+z_{n-1}|-|z_n| \leq |z_1+z_2+\cdots+z_n|$.

令 $\sum_{k=1}^{n-1} z_k = S_{n-1}$ $|S_{n-1}+z_n-z_n| = |S_{n-1}| \leq |S_{n-1}+z_n|+|z_n|$

即 $|S_{n-1}+z_n| \geq |S_{n-1}|-|z_n|$

即 $|z_1+z_2+\cdots+z_n| \geq |z_1+z_2+\cdots+z_{n-1}|-|z_n|$,

显然 $n=2$ 时有 $|z_1+z_2| \geq |z_1|-|z_2|$ 证毕.

(2) 设 $|z_0|<1$ 且 $p_n(z_0)=0$. 且 $z_0\neq 0$.

由 $|z_0|<1$. 对 $(1-z)p_n(z)$. 由 $1-z\neq 0$. $(1-z_0)p_n(z_0)=0$

$$(1-z_0)p_n(z_0) = a_0z_0^n + a_1z_0^{n-1}+\cdots+a_{n-1}z_0+\underline{a_n}$$
$$-a_0z_0^{n+1}-a_1z_0^n-\cdots-a_{n-1}z_0^2-a_nz_0 = 0.$$

$$\Rightarrow a_n = a_0z_0^{n+1}+(a_1-a_0)z_0^n+\cdots+(a_n-a_{n-1})z_0$$

$$|a_n| = |a_0z_0^{n+1}+(a_1-a_0)z_0^n+\cdots+(a_n-a_{n-1})z_0|.$$

$$\leq |a_0z_0^{n+1}| + |(a_1-a_0)z_0^n|+\cdots+|(a_n-a_{n-1})z_0|$$

$$= |a_0||z_0^{n+1}| + |a_1-a_0||z_0^n|+\cdots+|a_n-a_{n-1}||z_0|.$$

$$< a_0+(a_1-a_0)+(a_2-a_1)+\cdots+(a_n-a_{n-1}) = a_n.$$

$a_n<a_n$? 相悖了. 故 $\nexists |z_0|<1$ s.t $p_n(z_0)=0$.

9. (0) z_1, z_2, z_3 共线：

(0) $\Rightarrow$ (1) 是显然的

(1) $\Rightarrow$ (2)：

令 $\frac{z_1-z_2}{z_2-z_3}=t$，则 $z_1-z_2=tz_2-tz_3$.

$$(x_1-x_2)+i(y_1-y_2)=(tx_2-tx_3)+i(ty_2-ty_3)$$

$$\Rightarrow \begin{cases} x_1-x_2=tx_2-tx_3 \\ y_1-y_2=ty_2-ty_3 \end{cases}$$

则 $\bar{z}_1z_2+\bar{z}_2z_3+\bar{z}_3z_1=(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_1x_3+y_1y_3)+i(x_1y_2-x_2y_1+x_2y_3-x_3y_2+(x_3y_1-x_1y_3))$

$=(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_1x_3+y_1y_3)$

$+i\left[x_1(y_2-y_3)-(x_2-x_3)y_1+y_2(x_2-x_3)-x_2(y_2-y_3)\right]=$

$(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_1x_3+y_1y_3)\in\mathbb{R}$.

(2) $\Rightarrow$ (3)

已知

$(x_1-x_2)(y_2-y_3)=(x_2-x_3)(y_1-y_2)$，证有 $\lambda_1\ \lambda_2\ \lambda_3$ s.t.

$\lambda_2z_2+\lambda_3z_3=(\lambda_2+\lambda_3)z_1$，即

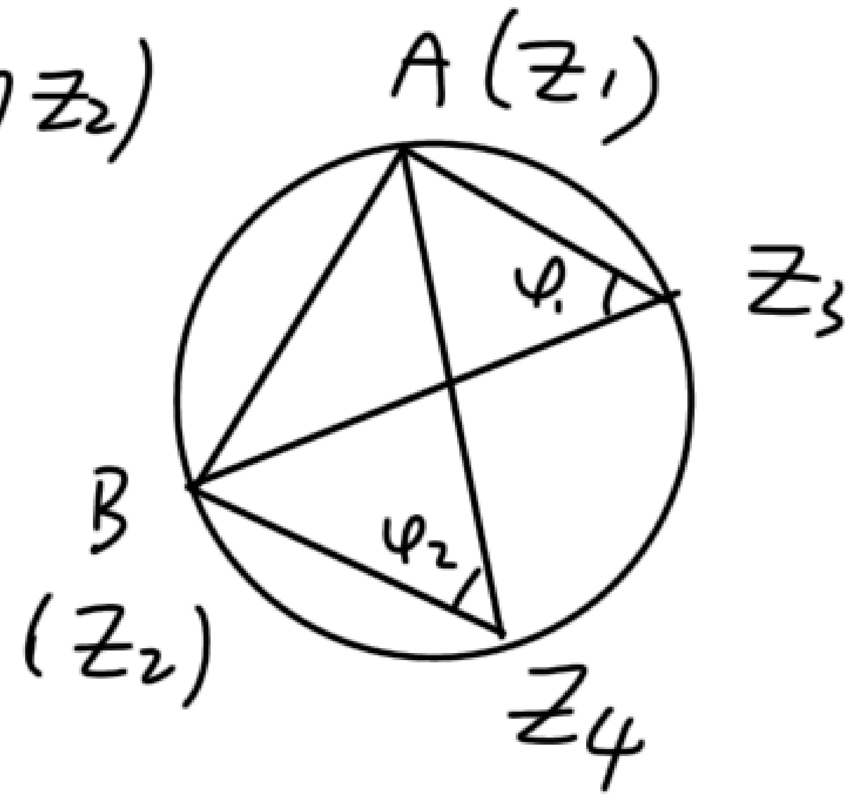
$\begin{cases} \lambda_2x_2+\lambda_3x_3=(\lambda_2+\lambda_3)x_1 \\ \lambda_2y_2+\lambda_3y_3=(\lambda_2+\lambda_3)y_1 \end{cases}$，注意到 $\begin{cases} \lambda_2=y_1-y_3 \\ \lambda_3=y_2-y_1 \end{cases}$ 即可.

(3) $\Rightarrow$ (0)

$$\frac{z_1-z_2}{z_2-z_3}=\frac{\left(\frac{\lambda_2x_2+\lambda_3x_3}{\lambda_2+\lambda_3}-x_2\right)+i\left(\frac{\lambda_2y_2+\lambda_3y_3}{\lambda_2+\lambda_3}-y_2\right)}{(x_2-x_3)+i(y_2-y_3)}$$

$$=\frac{1}{\lambda_2+\lambda_3}\frac{(\lambda_3x_3-\lambda_3x_2)+i(\lambda_3y_3-\lambda_3y_2)}{(x_2-x_3)+i(y_2-y_3)}=\frac{-\lambda_3}{\lambda_2+\lambda_3}\in\mathbb{R}.$$

10. 若四点共圆，则 $\angle AZ_3B = \angle AZ_4B$ （A为 z_1，B为 z_2）



$$\operatorname{Arg}\left(\frac{z_1-z_3}{z_2-z_3}\right) = \angle AZ_3B,$$

$$\operatorname{Arg}\left(\frac{z_1-z_4}{z_2-z_4}\right) = \angle AZ_4B.$$

$\angle AZ_3B + \angle AZ_4B = \pi$ or $0 \Rightarrow \angle AZ_3B - (-\angle AZ_4B) = \pi$ or 0.

$\Rightarrow \operatorname{Arg}$(原式) $= \pi$ or 0. $\Rightarrow$ 原式 = 实数.

若原式为实数则 $\arg\left(\frac{z_1-z_3}{z_2-z_3} \Big/ \frac{z_1-z_4}{z_2-z_4}\right) = k\pi$.

则显然四点共圆.

11. $z_1 + z_2 = -z_3$

$\Rightarrow |z_1+z_2| = |-z_3| = 1$

$\Rightarrow |z_1+z_2|^2 = 1^2 + 1^2 + 2\operatorname{Re}(z_1\bar{z}_2) = 1 \Rightarrow 2\operatorname{Re}(z_1\bar{z}_2) = -1$.

$\Rightarrow |z_1-z_2|^2 = 1^2 + 1^2 - 2\operatorname{Re}(z_1\bar{z}_2) = 3 \Rightarrow |z_1 - z_2| = \sqrt{3}$

同理 $|z_1 - z_3| = |z_2 - z_3| = \sqrt{3}$. 故为等边三角形.

12、$z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$

$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

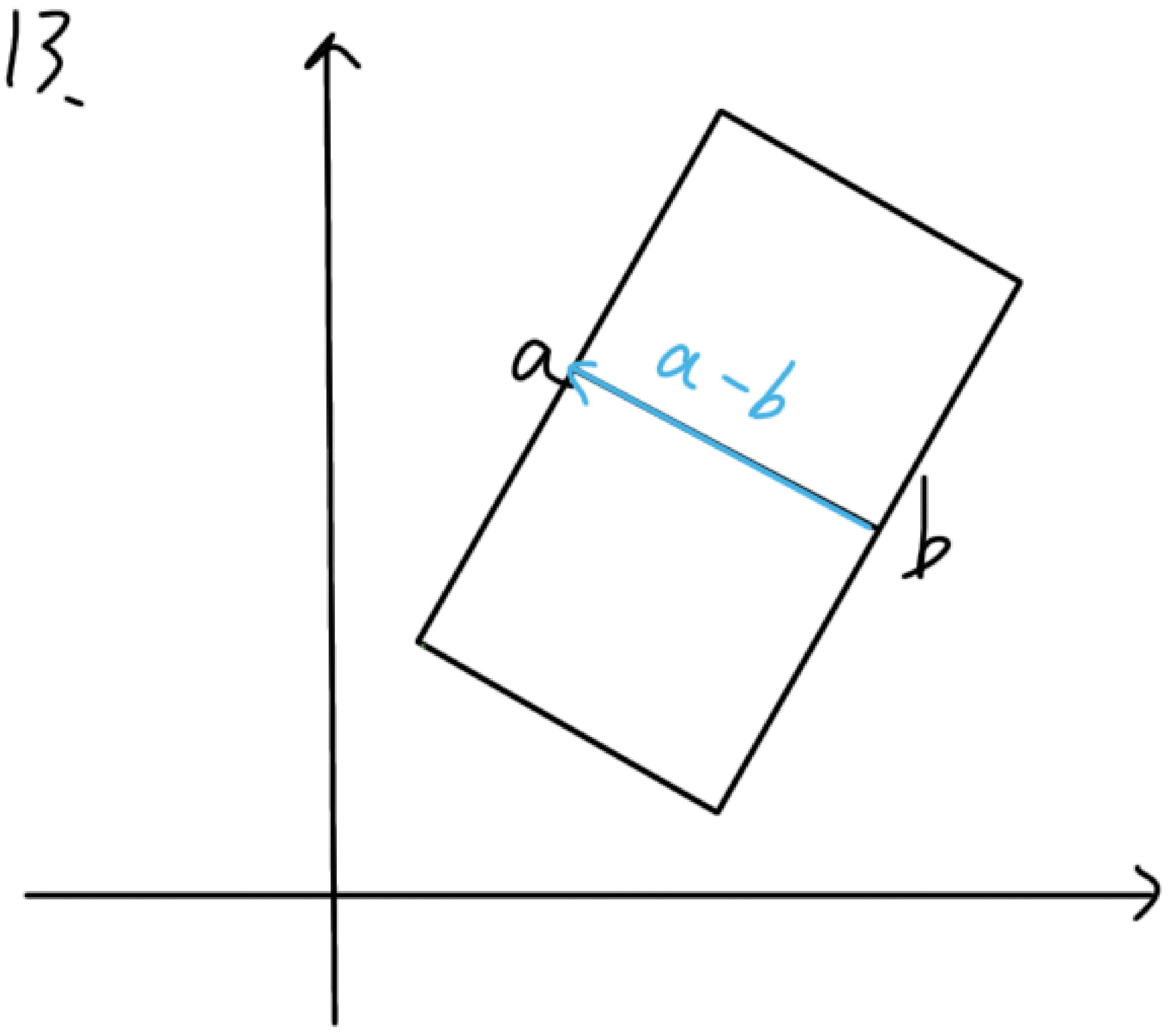
$$\begin{cases} y_1 + y_2 = 0 \\ x_1y_2 + x_2y_1 = 0 \end{cases}$$

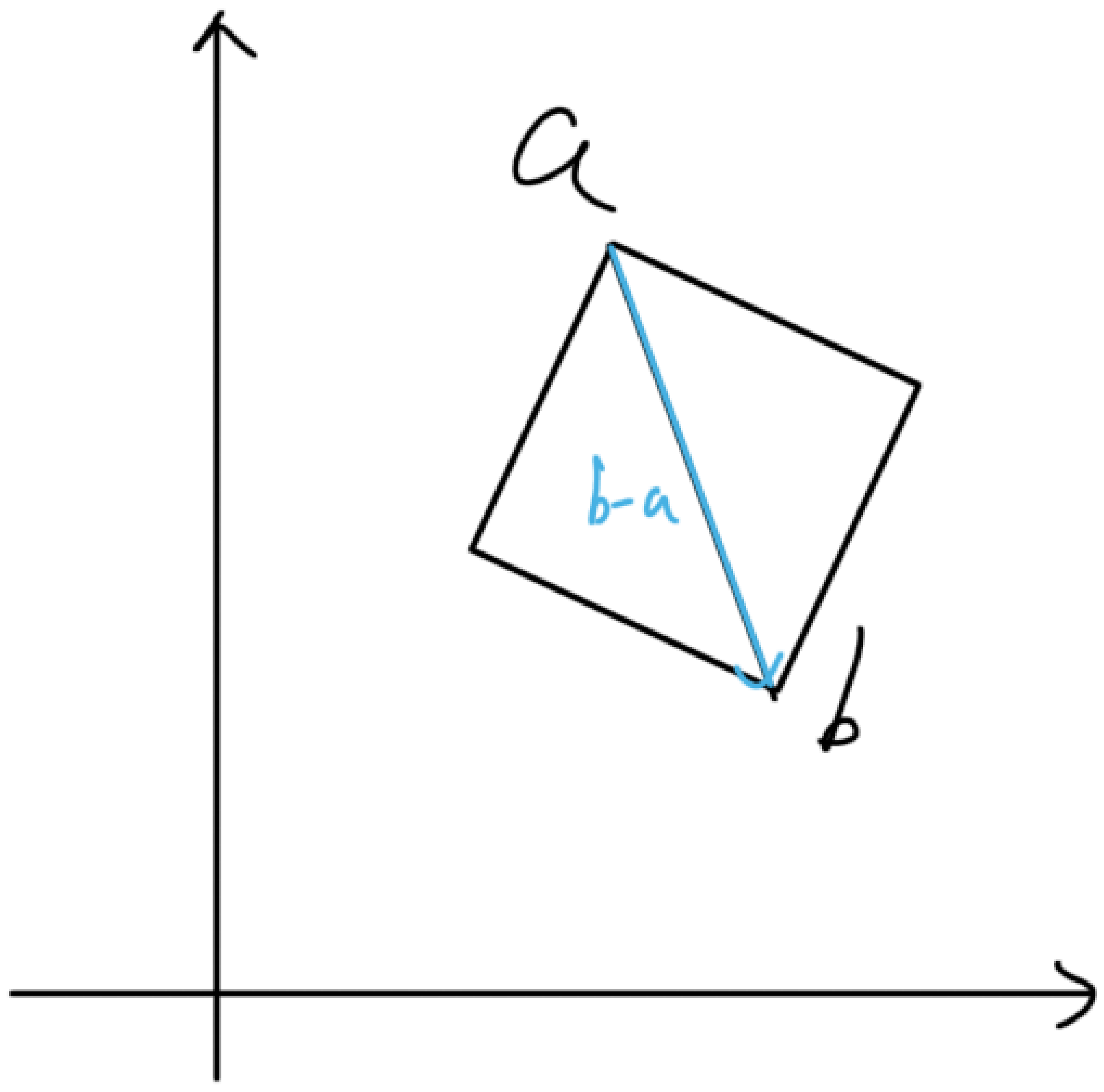
① $y_1 = y_2 = 0$. 都实数

② $y_1 = -y_2 \neq 0$. 则 $x_1y_1 = x_2y_1 \Rightarrow x_1 = x_2$

$\Rightarrow z_1 = \bar{z}_2$. 共轭.

13.





$\begin{cases} a+i(b-a) \\ b+i(b-a) \end{cases}$ $\begin{cases} a+i(a-b) \\ b+i(a-b) \end{cases}$ $\begin{cases} a+(b-a)\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}i} \\ a+(b-a)\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i} \end{cases}$

14. 0, 0, 无

15. (1) 略

(2) $z_0\neq 0$或负数是对的

$z_0<0$时，若z_n从上半面趋近，

$\mathrm{Arg}\,z_n\to-\pi$. 反之$\mathrm{Arg}\,z_n\to\pi$. 等.

$z_0=0$时 $\arg z_n$不收敛

(3) $z_n\to\infty$. 成立. 但$\arg z_n$无极限

16. (1) 垂直平分线

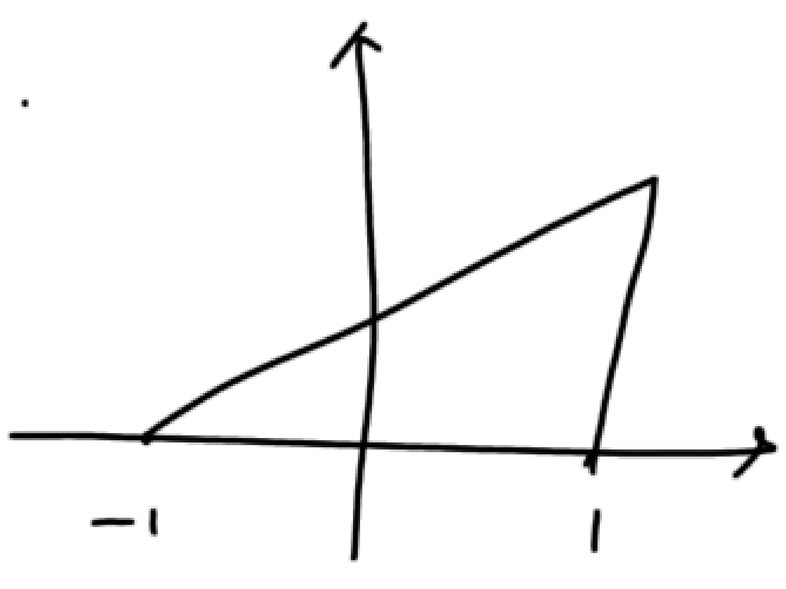
(2) 椭圆 / 圆

(3) $\frac{1}{z}=\frac{1}{r}e^{i(-\varphi)}=\frac{1}{r}e^{-i\varphi}$

$\mathrm{Re}\frac{1}{z}=\frac{\cos\varphi}{r}=\alpha \Rightarrow \alpha r^2=x=\alpha x^2+\alpha y^2 \Rightarrow$ 与y轴相切与(0,0)的圆族及Oy轴

（不包含原点）

(4)



圆族 + Ox轴（不含(-1,0)与(1,0)）

(5) 圆族（以(±1,0)为对称点，加Oy轴）

17. 略

18. $Ax+By=D$ 即

$A\dfrac{z+\bar{z}}{2}+B\dfrac{z-\bar{z}}{2i}=D$

$A(z+\bar{z})-Bi(z-\bar{z})=2D$

$\Rightarrow (A+Bi)\bar{z}+(A-Bi)z=2D=C$

19. (1) $y=x$

(2) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

(3) $y=\dfrac{1}{x}$

(4) $y=\dfrac{1}{x}$ 的一支

20. $x=\dfrac{z+\bar{z}}{2}$, $y=\dfrac{z-\bar{z}}{2i}$

$x^2+y^2=z\bar{z}$, $2x=z+\bar{z}$

$\Rightarrow z\bar{z}+z+\bar{z}=1$

即 $z\bar{z}+z+\bar{z}+1=(z+1)(\bar{z}+1)=(z+1)\overline{(z+1)}=2$

$\Rightarrow |z+1|=\sqrt{2}$