# Метод градиентного спуска.

$$f(x) \to \min_{x \in \mathbb{R}^n}, \quad f \in C^1$$

$$x_{k+1} = x_k + \alpha_k d_k, \quad x_k, d_k \in \mathbb{R}^n$$
$$\alpha_k \in \mathbb{R}_*$$

Требование: $d_k$ - направление спуска.

$$\nabla f(x_k)^T d_k < 0$$

$$\begin{cases} \nabla f(x_k)^T d_k \to \min\limits_{d_k} \\ \|d_k\|^2 \le 1 \end{cases} \Rightarrow d_k = -\frac{\nabla f(x_k)}{\|\nabla f(x_k)\|}$$

GD: $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$

Подбор $\alpha_k$: 1) $\alpha_k = \frac{1}{L}, \quad f \in C_L^{1,1}$

2) $\alpha_k$ удовл. ус-ям Арм./Вульф.

① $\boxed{f \in C_L^{1,1}}$

$$\Rightarrow f(y) \le f(x) + \nabla f(x)^T (y-x) + \frac{L}{2}\|y-x\|^2$$

$$\left[ y = x - \alpha \nabla f(x) \right] = f(x) - \alpha \|\nabla f(x)\|^2 + \frac{L}{2}\alpha^2 \|\nabla f(x)\|^2$$

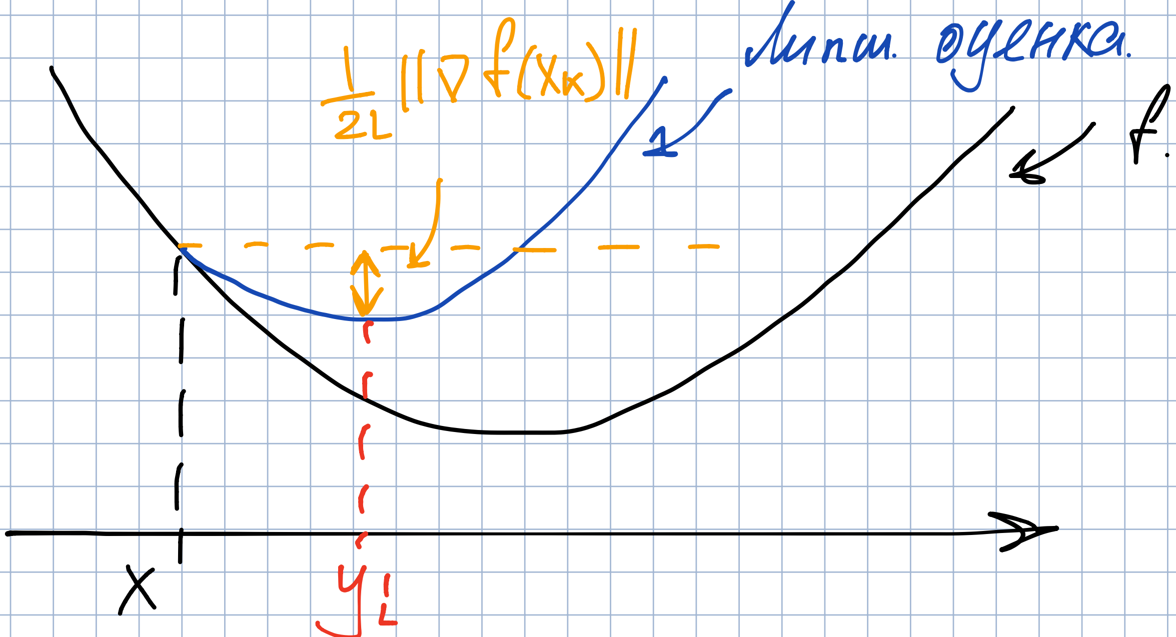
$$= f(x) - \left(\alpha - \frac{L\alpha^2}{2}\right)\|\nabla f(x)\|^2$$

$$\alpha - \frac{L\alpha^2}{2} \to \max_\alpha$$

$$\boxed{\alpha_{opt} = \frac{1}{L}}$$

$$\frac{1}{L} - \frac{L}{2L^2} = \frac{1}{2L}$$

$$f(x_{k+1}) \le f(x_k) - \frac{1}{2L}\|\nabla f(x_k)\|^2$$



$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 \leq$$

$$f(x_{k-1}) - \frac{1}{2L} \|\nabla f(x_{k-1})\|^2 \leq \ldots \leq$$

$$f(x_0) - \frac{1}{2L} \sum_{i=0}^{k} \|\nabla f(x_i)\|^2$$

$f(x) > -\infty \ \forall x.$

$$\sum_{i=0}^{k} \|\nabla f(x_i)\|^2 \leq 2L\left(f(x_0) - f(x_{k+1})\right) \leq$$

$$\leq 2L\left(f(x_0) - f_{opt}\right)$$

$$g_k := \min_{0 \leq i \leq k} \|\nabla f(x_i)\|^2$$

$$\Rightarrow g_k \leq \frac{2L\left(f(x_0) - f_{opt}\right)}{k+1}$$

Сублинейная скорость сходимости у функц. с Липш. градиентом.

② $\boxed{f \in C_L^{1,1} \text{ и } \mu \text{ сильно выпуклая}}$

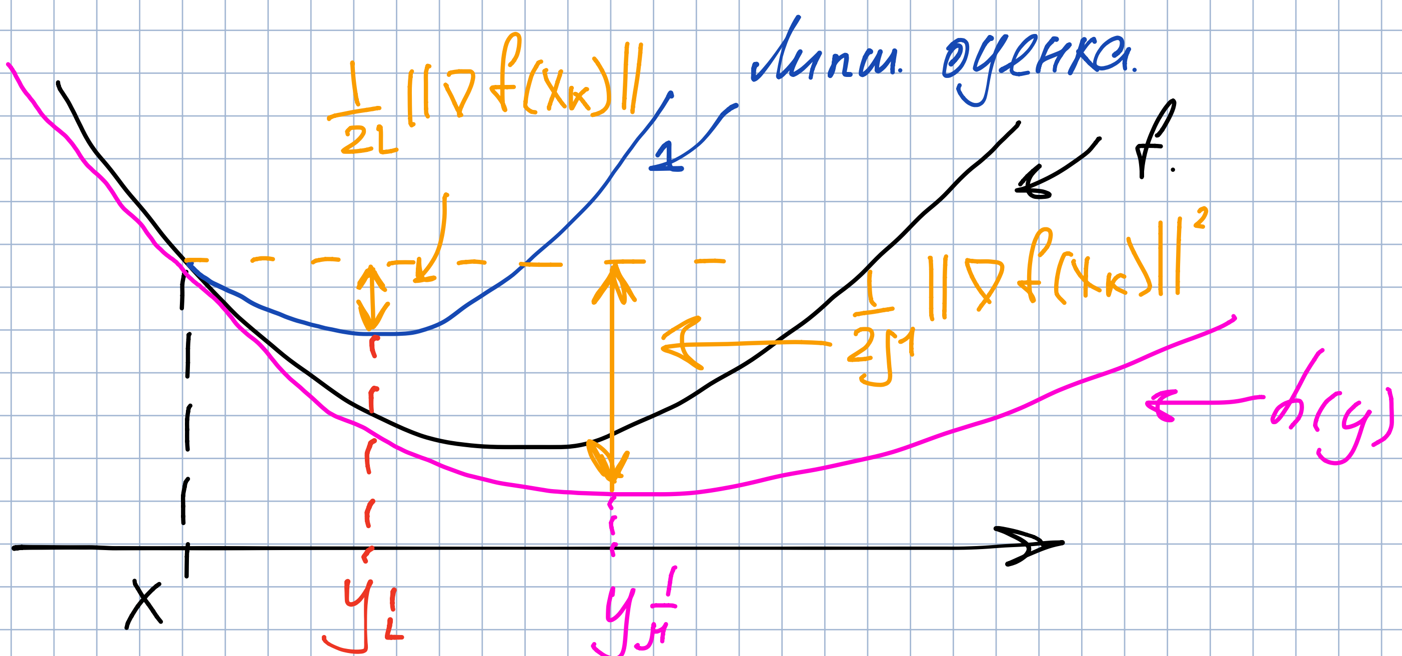
$$f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{\mu}{2}\|y-x\|^2 \;/ =: h(y) \quad \forall y.$$

$$\min_y f(y) = f(y_{opt}) \geq \min_y h(y)$$

$$\nabla h(y) = \nabla f(x) + \mu(y-x) = 0$$

$$\Rightarrow \hat{y} = x - \frac{1}{\mu}\nabla f(x)$$

$$f(y_{opt}) \geq h(\hat{y}) = f(x) - \frac{1}{2\mu}\|\nabla f(x)\|^2$$



$$f(x_{k+1}) - f_{opt} \leq f(x_k) - f_{opt} - \frac{1}{2L}\|\nabla f(x_k)\|^2 \leq$$

$$\leq f(x_k) - f_{opt} - \frac{2\mu}{2L}\left(f(x_k) - f_{opt}\right) =$$

$$= \left(1 - \frac{\mu}{L}\right)\left(f(x_k) - f_{opt}\right) \leq \ldots \leq$$

$$\underline{\leq \left(1 - \frac{\mu}{L}\right)^{k+1}\left(f(x_0) - f_{opt}\right).}$$

Линейная скорость сходимости.

# Скорость сходимости град. спуска.

| Класс ф-ций | Невязка | Скорость |
|---|---|---|
| $f \in C_L^{1,1}$ | $\min\limits_{0 \le i \le k} \lVert \nabla f(x_i) \rVert^2$ | $O\left(\frac{1}{k}\right)$ |
| $f \in C_L^{1,1}$ и вып. | $f(x_k) - f_{opt}$ | $O\left(\frac{1}{k}\right)$ |
| $f \in C_L^{1,1}$ и $\mu$ сильно выпукла | $f(x_k) - f_{opt}$ | $O(c^k)$<br>$c = 1 - \frac{\mu}{L}$ |

Пример:

$$f(x) = \frac{1}{2} x^T A x - x^T b \to \min_{x}$$

$$A = A^T \succ 0$$

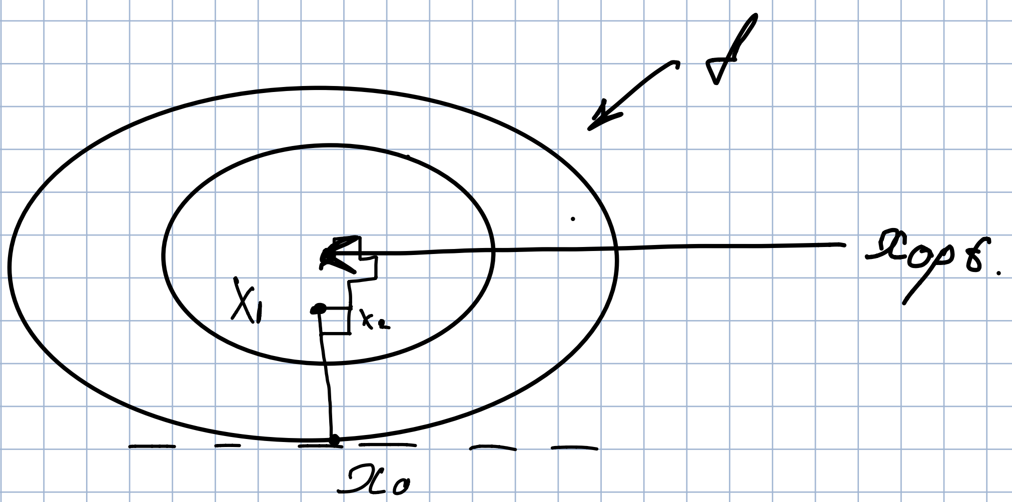
$$\nabla f(x) = \frac{1}{2} 2 A x - b = Ax - b$$

$$\nabla^2 f(x) = A$$

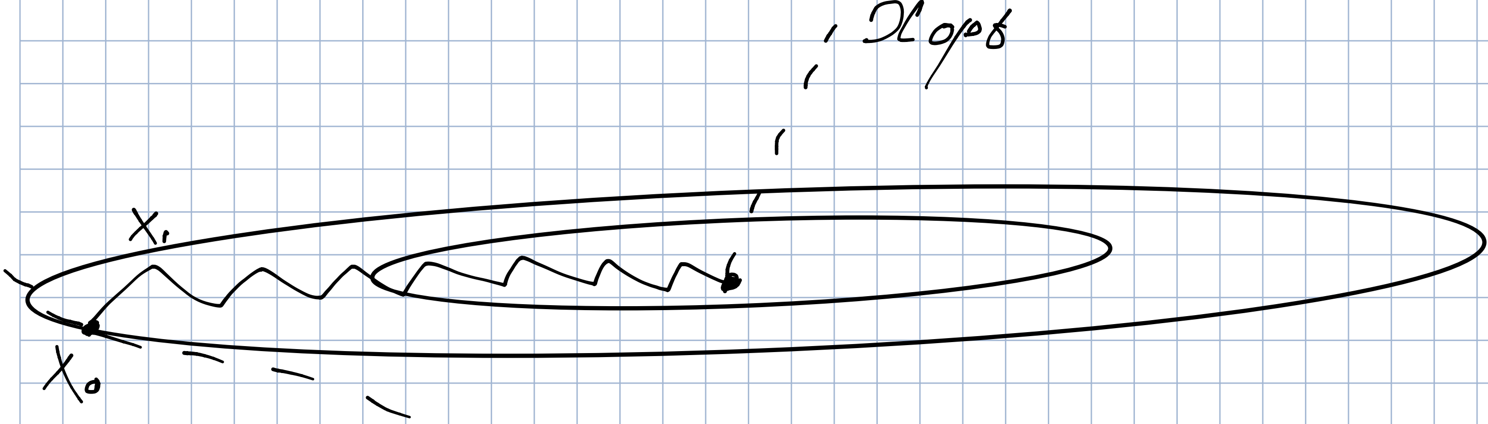
$f \in C_L^{2,1}$ для $L = \lambda_{max}(A)$

$f$ - сильно выпуклая с $\mu = \lambda_{min}(A) > 0$.

① $\mu = L$



② $\mu \le \ell$




$$f(x) \approx f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{\alpha}{2} \|x - x_k\|^2 \to \min_x$$

$$\nabla f(x_k) + \alpha (x - x_k) = 0.$$

$$x = x_k - \frac{1}{\alpha} \nabla f(x_k)$$

## Подбор $L$ = длины шага.

$L_k := 1$

повторять:

$x_{k+1} = x_k - \frac{1}{L_k} \nabla f(x_k)$

если $f(x_{k+1}) \leq f(x_k) - \frac{1}{2L_k} \|\nabla f(x_k)\|^2$, то вых.

$L_k = L_k \cdot \gamma, \ \gamma > 1$

$L_{k+1} = L_k \cdot \rho, \ \rho < 1.$

$$x_{k+1} = x_k + \alpha_k d_k$$

$$\alpha_k : f(x_{k+1}) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T d_k .$$

$$\nabla f(x_{k+1})^T d_k \geq c_2 \nabla f(x_{k+1}) d_k$$