Remark: $dr\cdot n\equiv 0 \Rightarrow d(dr\cdot n)=d^2r\cdot n+dr\cdot dn\equiv 0$

$n(u,v)$ 为单位法向量 $\Rightarrow dn\cdot n\equiv 0 \Rightarrow dn=n_u du+n_v dv$

即 $\mathrm{II}=d^2r\cdot n=-dr\cdot dn=-(r_u du+r_v dv)\cdot(n_u du+n_v dv)$.

$r_u\cdot n\equiv 0 \Rightarrow r_{uv}\cdot n+r_u\cdot n_v=0$

$\Rightarrow r_u n_v=r_v n_u$.

$r_v\cdot n\equiv 0 \Rightarrow r_{uv}\cdot n+r_v\cdot n_u=0$

$L=r_{uu}\cdot n=-r_u\cdot n_u$

$M=r_{uv}\cdot n=-r_u\cdot n_v=-r_v\cdot n_u$

$N=r_{vv}\cdot n=-r_v\cdot n_v$

且 $r_{uu}\cdot n=-r_u n_u$, $r_{vv}\cdot n=-r_v\cdot n_v$.

$\Rightarrow \mathrm{II}=(du\ \ dv)\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}$, $\begin{pmatrix}L & M\\ M & N\end{pmatrix}=-\begin{pmatrix}r_u\\ r_v\end{pmatrix}(n_u,n_v)$

eg: 平面: $r(u,v)=(u,v,0)$.

$dr=(1,0,0)du+(0,1,0)dv$, $d^2r=0 \Rightarrow \mathrm{II}=0$

柱面: $r(u,v)=(a\cos u, a\sin u, v)$

$dr=(-a\sin u\,du, a\cos u\,du, 0)$

$d^2r=(-a\cos u\,du, -a\sin u\,du, 0)$.

$n=r_u\times r_v=(\cos u, \sin u, 0)$

$\Rightarrow \mathrm{II}=-a\,du^2$.

Thm: $\mathrm{II}\equiv 0$, 则 $r(u,v)$ 为平面.