Network creation game

Introduction

1.1 Problem Statement

The network creation game was introduced in Fabrikant et al, 2003 [1]. The game is defined by the number of players, n, and a parameter α specifying the edge-creation cost. The set of players is {0, 1, . . ., n – 1}. We denote this set [n]. The strategy space of player i is the set $S_i = 2^{[n]-\{i\}}$. A profile is a combination of strategies s=($s_{0,..,}s_{n-1}$) $\in S_0\ X\ S_1\ _{..}X\ S_{n-1}$. For a given profile s we consider the graph G[s], the underlying undirected graph of $G_{0[s]} = ([n], \cup_{i=0}^{n-1}(\{i\} \times s_i))$. The cost incurred by player i under s is defined to be $C_i(s) = \alpha * |s_i| + \sum_{j=0}^{n-1} d_{G(s)}(i,j)$, where $d_{G(s)}(i,j)$ is the distance between nodes i and j in the graph G[s]. The goal of the players is to minimize the cost function. A (pure) Nash equilibrium in this game is an s such that, for each player i, and for all s' that differ from s only in the i'th component,

c_i (s) $\leq$ c_i (s').

Best-Response Dynamics (BRD) is a local search method where in each step some player is chosen and plays its best-response strategy, given the strategies of the others. Our goal is to examine various BRD and the quality of solution.

Best-Response dynamics may differ in the maximum degree and the maximum number of edges in graph. Each node's degree limited to be no greater than the maximum degree and the total number of edges in graph limited to be no greater than the maximum edges.

1.2 Objectives and Motivation

The congestion of a node is the number of shortest paths containing this node. Formally, for each pair of nodes i and j, and a node c, the cong_c (i, j) is the ratio between the number of shortest paths, between i and j containing c, and the total number of shortest paths between i and j. The congestion of c cong_c defined as cong_c = $\sum_{\substack{i,j\in n \\ i\neq j, i\neq c.}}$ cong_c(i, j) . The congestion-gap of graph G is the ratio between the maximum and minimum congestion of two nodes. We aim to determine which heuristics will minimize the cost function and contribute to the reduction of congestion problems, maintaining optimal complexity and solution quality. We will examine the factors that shape each player's decision making. Finally, we will analyze the convergence rate of Best Response Dynamics (BRD). The results will be achieved by experiments in which we measure the above parameters on random graphs.

1.2 related work

Preliminaries

2.1 Definitions and Notations:

- {0, 1..n-1} Set of players.
- s_i The strategy of player i.
- s is a profile. a profile is a set of strategies for all players which fully specifies all actions in a game.
- $G(s)$ Undirected graph generates by s.
- α Fixed price of an edge.
- $d_{G(s)}(i,j)$ the shortest path between node i and node j.
- c_i The cost incurred by player i.
  $c_i = \alpha * |s_i| + \sum_{j=0}^{n-1} d_{G(s)}(i,j)$.
- C is the social cost. the social cost is the $C(s) = \sum_{i=0}^{n-1} c_i$.
- $cong\_i(s)$ congestion of node i in profile s.
- $cong\_G(s)$ congestion-gap of graph G in profile s.
- Outgoing-edge form node i to j, is the edges that node i build to j.
  That is, $j \in s_i$.

Player

Player i represented by node i in the graph. Each player can choose his best strategy in his turn.

Best-Response Dynamics (BRD)

The best response is the strategy (or strategies) which produces the most favorable outcome for a player, taking other players' strategies as given .

Nash equilibria

A profile s is a Nash equilibrium if for each player i, and for all s' that differ from s only in the i-th component, $c_i(s') \geq c_i(s)$.

Congestion

The congestion of node i, denoted by $cong\_i(s)$, is the number of shortest paths containing i.

The congestion-gap of graph G, denoted by $cong\_G(s)$, is the proportion of the maximum congestion and minimum congestion.

$$cong\_G(s) = \frac{\max\ cong\_i(s)}{\min\ cong\_i(s)} .$$

Convergence rate

The convergence rate measures the number of best-response steps performed until a Nash equilibrium is reached. In every step of the dynamics, some player is selected and plays his best-response. If the player is satisfied with his current strategy, he does not change it.

Let n be the number of players. In order to have a measure independent of n, the convergence rate is defined as the number of individual steps divided by n. Note that a step, doesn't mean actual strategy changes but rather mean just the opportunities to change.

Maximum degree

Imposes a limitation on the number of edges of a node. Degree of a node is the number of edges incident to the node, than is, all out-going and in-going edges.

Maximum edges

Imposes a limitation on the number of total edges in the graph. Defined by *n* and *k*, where *n* is the number of players, *k* is constant and Maximum edge= n-1+k. where n-1 is the minimum number of edges such that the graph is connected.

Deviation rule

The BRD is not limited to activate the players in round-robin fashion. The **Deviation rule** determined the order according to which the players perform best-response. The order can be determined by one of the following:

1. Round robin
2. Max cost: the next to play is the player with maximum cost.
3. Max degree: the next to play is the player with maximum degree
4. Max gain: the next to play is the player that can achieve the most significant improvement of his cost.

2.2 Implemented Algorithms

2.2.1 general

Each player plays in his turn, he can delete existing edges outgoing from his node and add new outgoing edges. A player can act in any way as long he resists the limitations of the game (Maximum degree, Maximum edges). Each player chooses the best strategy considering only is own good. The BRD halts when no player has a beneficial move. Previous work has shown that for Trees convergence guarantee [3]. However, even one non-tree edge suffices to destroy the convergence guarantee [2]. Thus, for non-tree graph Nash equilibrium does not necessarily can be achieved.

2.2.2 Algorithms

Network Creation

- Start with graph of nodes with no edges.
- Until no node wants to change his set of neighbors:
  - For each node:
    - Find all possible set of neighbors- all groups of nodes that the player can build an edge to them under game limitation.
    - Find the group of nodes that building an edge to them minimizes the cost function.
    - Build the chosen edges.
- Return the graph

Find all possible set of neighbors

- The possible neighbors are
  - Those whose degree does not exceed the *Maximum degree.*
  - Those who don't have out-going edge to current player.
- A possible set of neighbors:
  - Belong to the set of all subsets of possible neighbors (not include player himself).
  - Is one whose size, in addition to the size of existing player's neighbors does not exceed the *Maximum degree*.

- Is one whose size, in addition to the number of the other edges, does not exceed *Maximum edges*.

- Return a set of all possible set neighbors

Deviation rule

Return the next deviating player according to the implemented deviator rule. The Deviation rule is pre-selection, and the deviation function dynamically determine the next order according to which the players perform.

Maximum degree

Pre-selected variable.

Maximum edges

Pre-selected variable.

Compute graph congestion

Compute congestion-gap $cong\_G(s)$ of graph. *Find all node's congestion* $cong\_i(s)$, find the node with maximum $cong\_i(s)$ and the one with minimum $cong\_i(s)$ and returns $\frac{maximum\ cong_i\,(s)}{\text{minimum}\ cong_i\,(s)}$.

Find node's congestion

For each pair of nodes search for all shortest path between those nodes. For each node that participant in one of the shortest path ,add to his congestion $cong\_i(s)$ the ratio between the number of shortest paths containing this node and the total number of shortest paths.

[1]Fabrikant, A., Luthra, A., Maneva, E., Papadimitriou, C. H., & Shenker, S. (2003, July). On a network creation game. In *Proceedings of the twenty-second annual symposium on Principles of distributed computing* (pp. 347-351). ACM.

[2]Kawald, B., & Lenzner, P. (2013, July). On dynamics in selfish network creation. In *Proceedings of the twenty-fifth annual ACM symposium on Parallelism in algorithms and architectures* (pp. 83-92). ACM.

[3]Lenzner, P. (2011, October). On dynamics in basic network creation games. In *International Symposium on Algorithmic Game Theory* (pp. 254-265). Springer, Berlin, Heidelberg.