COMP 361/5611 - Elementary Numerical Methods
Assignment 4 - Due Sunday, December 1, 2024

Problem 1. (34%)

Consider the unique interpolating polynomial $p_n(x)$ of degree n or less that interpolates a function $f(x)$ at $n+1$ *equally spaced interpolation points*

$$\{x_0, x_1, x_2, \cdots, x_n\}$$

on an interval $[a, b]$, taking $x_0 = a$ and $x_n = b$.

Write a program to do the following: Use the Lagrange basis functions $\ell_i(x)$, $i = 0, 1, 2, \cdots n$ on Pages 169-172 of the Lecture Notes to evaluate $p_n(x)$ at $M + 1$ *equally spaced sampling points*

$$\{y_0, y_1, y_2, \cdots, y_M\} ,$$

where $y_0 = a$ and $y_M = b$, and where M is much larger than n. Estimate the maximum interpolation error

$$\max_{[a,b]} |f(x) - p_n(x)| ,$$

by computing the *approximate maximum interpolation error*

$$\max_{0 \le i \le M} |f(y_i) - p_n(y_i)| .$$

Specifically, do the above for each of the following cases :

$$f(x) = \sin(\pi x) , \text{ on the interval } [-1, 1] , \ (i.e., a = -1 \text{ and } b = 1) ,$$

$$f(x) = \frac{1}{1+x^2} , \text{ on the interval } [-2, 2] ,$$

$$f(x) = \frac{1}{1+x^2} , \text{ on the interval } [-5, 5] ,$$

successively using $n = 2, 4, 8, 16$. In each case use $M = 500$.

For each of these 12 cases print the approximate maximum interpolation error.

Also, for each of the three functions $f(x)$, give a graph that shows $f(x)$ and the polynomials $p_n(x)$, $n = 2, 4, 8, 16$.

In addition, for the case $f(x) = \sin(\pi x)$ on the interval $[-1, 1]$, use the Lagrange Interpolation Theorem on Page 176 and the Table on Page 183 of the Lecture Notes to derive the tight upper bound on the maximum interpolation error for $n = 2, 4, 8, 16$. Compare this upper bound to the actual (approximate) maximum interpolation error found above.

Give a concise summary and discussion of your findings.

Problem 2. (22%)

Give complete details on the derivation of the five-point centered approximation to the second derivative of a function $f(x)$ by filling in the gaps in the example on Pages 236-237 of the Lecture Notes.

Also, give complete details on using Taylor expansions to determine the leading error term given on Page 237 of the Notes.

Problem 3. (22%)

Derive the local *Three-point Gauss Quadrature Formula* for integrating a function $f(x)$ over the reference interval $[-1, 1]$. (This formula uses the roots of the Legendre orthogonal polynomial $e_3(x)$.)

Use the corresponding *composite formula* to integrate the function $f(x) = \sin(\pi x)$ over the interval $[0, 1]$, using $N = 2, 4, 8, 16, \cdots$, equally spaced subintervals in $[0, 1]$. List the observed errors (the difference between the numerical integral and the exact integral) in a Table.

What is the minimum number of iterations required for the error to be less than 10^{-7} ?

Present your findings in a concise form and briefly compare them to the corresponding results in Assignment 1.

Problem 4. (22%)

Give complete details on the derivation of the natural cubic spline by solving exercises on Pages 334-338 of the Lecture Notes.

Determine the natural cubic spline that interpolates the function $f(x) = x^6$ over the interval $[0, 2]$ using knots 0,1, and 2.

Problem 5. Bonus (10%)

Give complete details on the derivation of the Backward Differentiation Formula (BDF) on Page 350 of the Lecture Notes.