COMP 361: Elementary Numerical Methods

Assignment 1

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COMP 361 - Section DD

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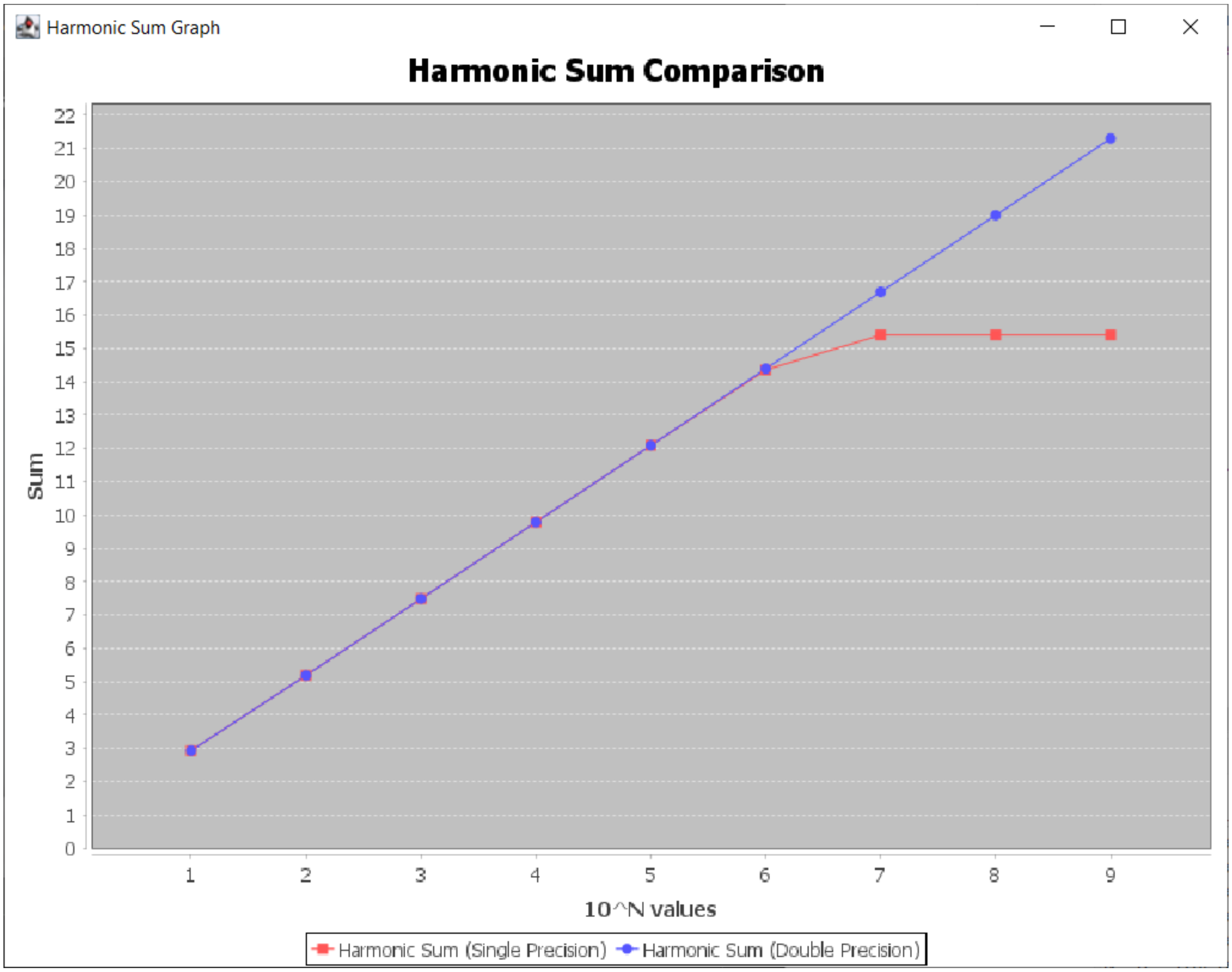
“I certify that this submission is my original work and meets the Faculty’s Expectations of Originality.”

Problem 1

I wrote a program in Java to compute the Harmonic sum $\sum_{k=1}^{N}\frac{1}{k}$ numerically. The partial sums were plotted on the graph below using JFreeChart in terms of 10^N values and I also printed the output in the console as shown below.




```
N: 1, Single: 2.9289684296, Double: 2.9289682540, Difference: 0.0000001756
N: 2, Single: 5.1873779297, Double: 5.1873775176, Difference: 0.0000004120
N: 3, Single: 7.4854784012, Double: 7.4854708606, Difference: 0.0000075406
N: 4, Single: 9.7876129150, Double: 9.7876060360, Difference: 0.0000068790
N: 5, Single: 12.0908508301, Double: 12.0901461299, Difference: 0.0007047002
N: 6, Single: 14.3573579788, Double: 14.3927267229, Difference: 0.0353687440
N: 7, Single: 15.4036827087, Double: 16.6953113659, Difference: 1.2916286571
N: 8, Single: 15.4036827087, Double: 18.9978964139, Difference: 3.5942137051
N: 9, Single: 15.4036827087, Double: 21.3004815023, Difference: 5.8967987936
```

The red line represents the sum using single precision and the blue line represents the sum using double precision. From the graph, we can see that around a large value such as 10^6, the red line starts to deviate from the blue line and around the value 10^7, the red line stops increasing in terms of accumulated sum. This red line's behavior in the graph is probably due to the limitations of single precision. Using single precision might lead to loss of significance since it rounds the value of the accumulated sum of the harmonic series and at some point, the added sums become too small to be significant for single precision representation. Using double precision leads to higher accuracy and it can represent a larger range of sum results. This Harmonic sum is known to diverge, and it can be proven by the integral test as shown below.

$\sum_{k=1}^{N} \frac{1}{k}$ → Harmonic Series

Integral Test : $\sum_{k=1}^{\infty} f(x)$ is convergent if $\int_1^{\infty} f(x)$ is convergent.

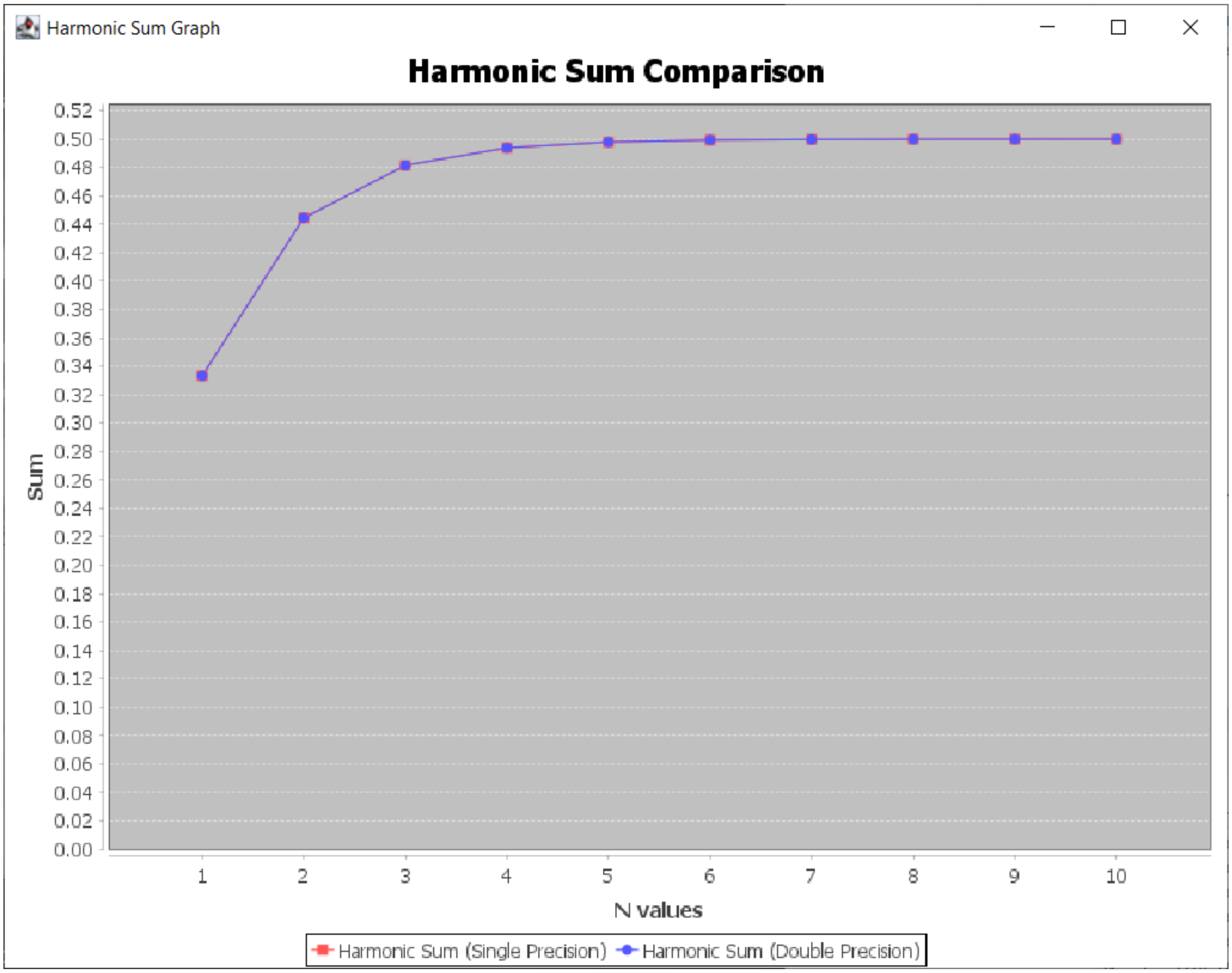
$f(x)$ is continuous on $[1, \infty)$, positive and decreasing.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} dx = \lim_{t \to \infty} \left[\ln(x)\right]_1^t = \lim_{t \to \infty} \left[\ln(t) - \ln(1)\right] = \infty$$

By Integral Test, $\sum_{k=1}^{N} \frac{1}{k}$ is divergent

I also wrote a program to compute the sum $\sum_{k=1}^{N} \frac{1}{3^k}$ numerically. The partial sums were plotted on the graph as shown below and I also printed the output in the console as shown below.

```
N: 1, Single: 0.3333333433, Double: 0.3333333333, Difference: 0.0000000099
N: 2, Single: 0.4444444478, Double: 0.4444444444, Difference: 0.0000000033
N: 3, Single: 0.4814814925, Double: 0.4814814815, Difference: 0.0000000110
N: 4, Single: 0.4938271642, Double: 0.4938271605, Difference: 0.0000000037
N: 5, Single: 0.4979423881, Double: 0.4979423868, Difference: 0.0000000012
N: 6, Single: 0.4993141294, Double: 0.4993141289, Difference: 0.0000000004
N: 7, Single: 0.4997713864, Double: 0.4997713763, Difference: 0.0000000101
N: 8, Single: 0.4999237955, Double: 0.4999237921, Difference: 0.0000000034
N: 9, Single: 0.4999746084, Double: 0.4999745974, Difference: 0.0000000111
N: 10, Single: 0.4999915361, Double: 0.4999915325, Difference: 0.0000000037
```



The result of this sum for both single and double precisions are overlapping in the graph and their accumulated sum seems to stop increasing around the value of sum 0.5. This can be explained by the fact that this sum is known to converge, and, in our case, it converges to the value 0.5 and its limit when N approaches infinity is 0 as shown below.

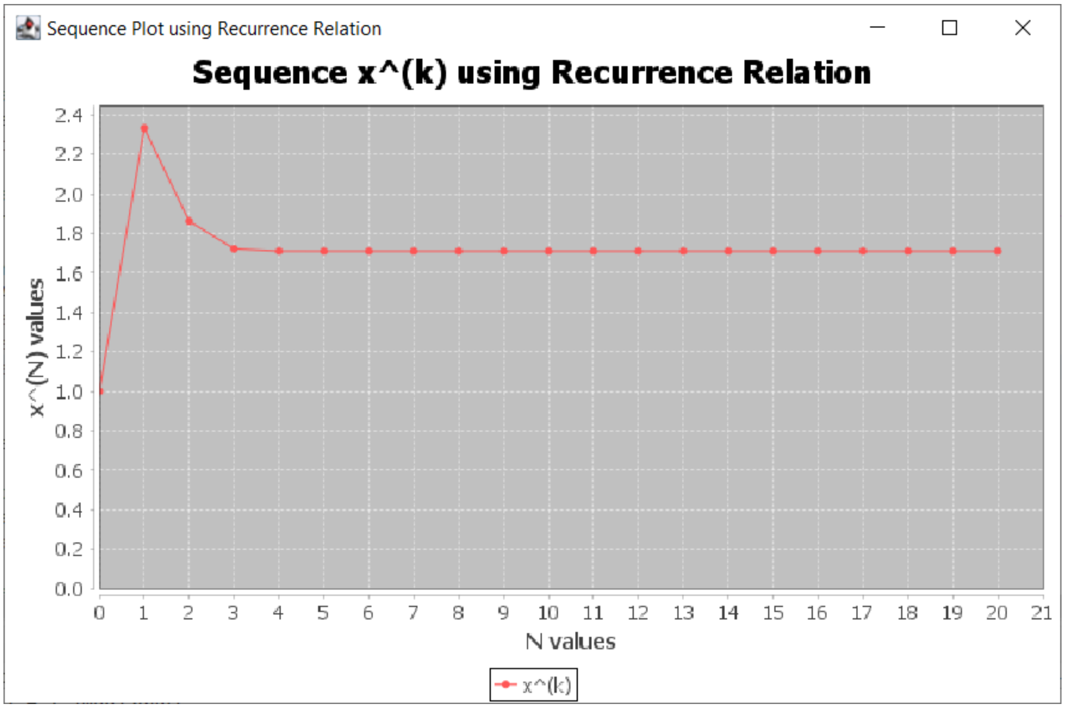
$$\sum_{k=1}^{N} \frac{1}{3^k} = \sum_{k=1}^{N} \left(\frac{1}{3}\right)^k, \text{ using } S_\infty = \frac{a}{(1-r)} \quad \text{where: } a = \frac{1}{3},\ r = \frac{1}{3},\ |r| < 1$$

$$\Rightarrow \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} = \frac{1}{2} \quad \rightarrow \text{convergent}$$

$$\lim_{t\to\infty} \frac{1}{3^t} = \frac{\lim_{t\to\infty} 1}{\lim_{t\to\infty} 3^t} = \frac{1}{\infty} = 0$$

Problem 2

I wrote a program to compute the sequence $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(N)}$ and test it up to N = 20 where $f(x) = \frac{2x^3+5}{3x^2}$. I plotted the results of the sequence on the graph below and I also printed the output in the console as shown below.




```
x^(0) = 1.0000000000
x^(1) = 2.3333333333
x^(2) = 1.8616780045
x^(3) = 1.7220018801
x^(4) = 1.7100597366
x^(5) = 1.7099759508
x^(6) = 1.7099759467
x^(7) = 1.7099759467
x^(8) = 1.7099759467
x^(9) = 1.7099759467
x^(10) = 1.7099759467
x^(11) = 1.7099759467
x^(12) = 1.7099759467
x^(13) = 1.7099759467
x^(14) = 1.7099759467
x^(15) = 1.7099759467
x^(16) = 1.7099759467
x^(17) = 1.7099759467
x^(18) = 1.7099759467
x^(19) = 1.7099759467
x^(20) = 1.7099759467
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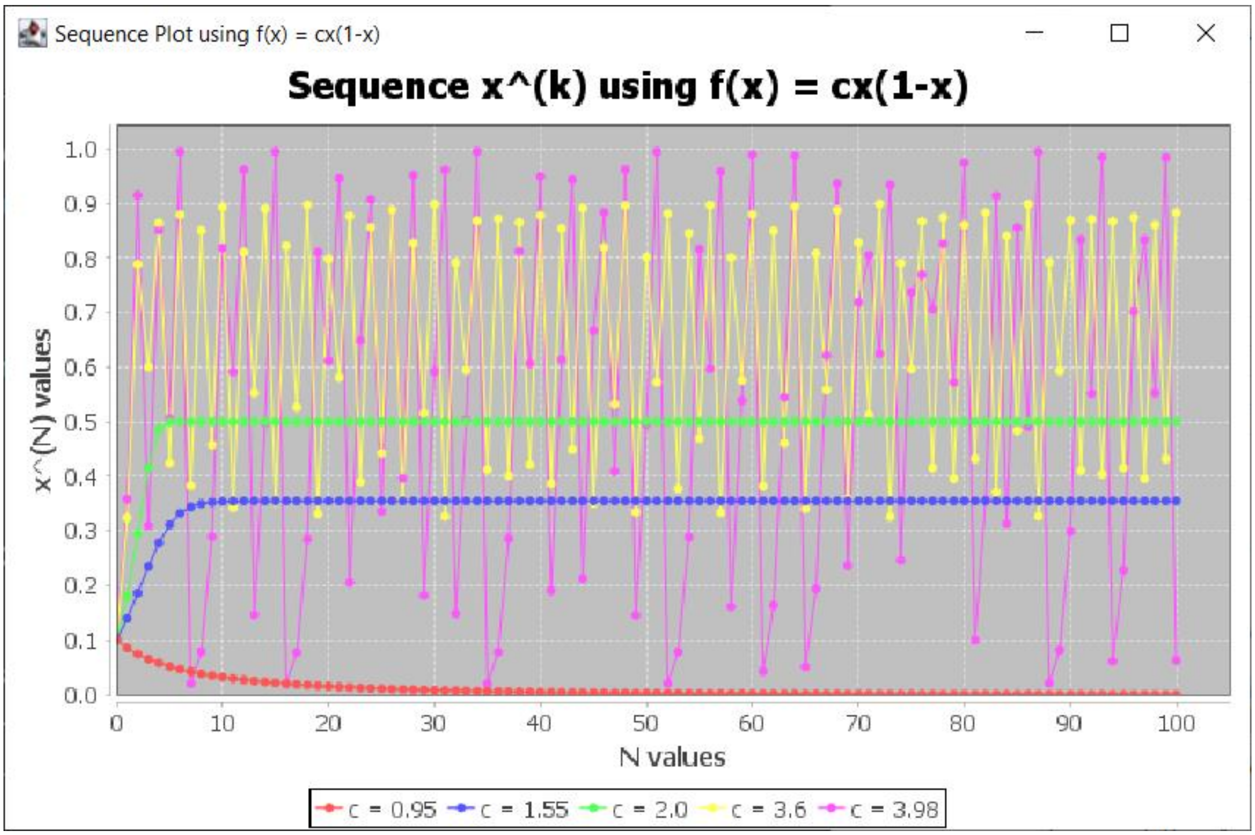
From the graph, the sequence can be seen to converge since it's approaching a limiting value. The limiting value from the printed results of my program is 1.7099759467. I verified this value by inserting this value into the function by x = 1.7099759467 and it satisfies the equation x = f(x). From my observation, the limiting value doesn't depend on $x^{(0)}$ necessarily for a large value of N that approaches infinity but it does affect at which term the sequence starts converging into this limiting value in this case. The term $x^{(0)}$ can influence the results where it might lead to a different limiting value or not finding a limiting value.

I also wrote a program to compute the sequence with f(x) = cx(1-x) for five different values of c. I plotted the sequence on the graph and I also printed the output in the console as shown below.



```
Sequence for c = 0.95
x^(0) = 0.1000000000
x^(1) = 0.0855000000
x^(2) = 0.0742802625
x^(3) = 0.0653245698
x^(4) = 0.0580044069
x^(5) = 0.0519079009
x^(6) = 0.0467527972
x^(7) = 0.0423386245
x^(8) = 0.0385187621
x^(9) = 0.0351833137
x^(10) = 0.0322481757
x^(11) = 0.0296478194
x^(12) = 0.0273303849
x^(13) = 0.0252542632
x^(14) = 0.0233856611
x^(15) = 0.0216968334
x^(16) = 0.0201647767
x^(17) = 0.0187702506
x^(18) = 0.0174970319
x^(19) = 0.0163313415
x^(20) = 0.0152613973
x^(21) = 0.0142770627
x^(22) = 0.0133695668
x^(23) = 0.0125312804
x^(24) = 0.0117555350
x^(25) = 0.0110364753
x^(26) = 0.0103689379
x^(27) = 0.0097483519
x^(28) = 0.0091706555
x^(29) = 0.0086322268
x^(30) = 0.0081298259
x^(31) = 0.0076605452
x^(32) = 0.0072217682
x^(33) = 0.0068111336
x^(34) = 0.0064265049
x^(35) = 0.0060659447
x^(36) = 0.0057276916
x^(37) = 0.0054101409
x^(38) = 0.0051118277
x^(39) = 0.0048314121
x^(40) = 0.0045676660
x^(41) = 0.0043194623
x^(42) = 0.0040857644
x^(43) = 0.0038656173
x^(44) = 0.0036581406
x^(45) = 0.0034625207
x^(46) = 0.0032780051
x^(47) = 0.0031038968
x^(48) = 0.0029395495
x^(49) = 0.0027843631
x^(50) = 0.0026377799
x^(51) = 0.0024992809
x^(52) = 0.0023683828
x^(53) = 0.0022446349
x^(54) = 0.0021276167
x^(55) = 0.0020169354
x^(56) = 0.0019122240
x^(57) = 0.0018131390
x^(58) = 0.0017193590
x^(59) = 0.0016305827
x^(60) = 0.0015465277
x^(61) = 0.0014669291
x^(62) = 0.0013915384
x^(63) = 0.0013201219
x^(64) = 0.0012524602
x^(65) = 0.0011883470
x^(66) = 0.0011275881
x^(67) = 0.0010700008
x^(68) = 0.0010154131
x^(69) = 0.0009636629
x^(70) = 0.0009145976
x^(71) = 0.0008680730
x^(72) = 0.0008239535
x^(73) = 0.0007821109
x^(74) = 0.0007424242
x^(75) = 0.0007047794
x^(76) = 0.0006690685
x^(77) = 0.0006351898
x^(78) = 0.0006030470
x^(79) = 0.0005725492
x^(80) = 0.0005436103
x^(81) = 0.0005161491
x^(82) = 0.0004900885
x^(83) = 0.0004653559
x^(84) = 0.0004418824
x^(85) = 0.0004196028
x^(86) = 0.0003984554
x^(87) = 0.0003783818
x^(88) = 0.0003593267
x^(89) = 0.0003412377
x^(90) = 0.0003240652
x^(91) = 0.0003077622
x^(92) = 0.0002922841
x^(93) = 0.0002775887
x^(94) = 0.0002636361
x^(95) = 0.0002503882
x^(96) = 0.0002378093
x^(97) = 0.0002258651
x^(98) = 0.0002145234
x^(99) = 0.0002037535
x^(100) = 0.0001935264
Sequence for c = 1.55
x^(0) = 0.1000000000
x^(1) = 0.1395000000
x^(2) = 0.1860616125
x^(3) = 0.2347361677
x^(4) = 0.2784344039
x^(5) = 0.3114084643
x^(6) = 0.3323715106
x^(7) = 0.3439460688
x^(8) = 0.3497531144
x^(9) = 0.3525101037
x^(10) = 0.3537824323
x^(11) = 0.3543616555
x^(12) = 0.3546236825
x^(13) = 0.3547418758
x^(14) = 0.3547951199
x^(15) = 0.3548190913
x^(16) = 0.3548298808
x^(17) = 0.3548347366
x^(18) = 0.3548369218
x^(19) = 0.3548379051
x^(20) = 0.3548383476
x^(21) = 0.3548385468
x^(22) = 0.3548386364
x^(23) = 0.3548386767
x^(24) = 0.3548386948
x^(25) = 0.3548387030
x^(26) = 0.3548387067
x^(27) = 0.3548387083
x^(28) = 0.3548387091
x^(29) = 0.3548387094
x^(30) = 0.3548387096
x^(31) = 0.3548387096
x^(32) = 0.3548387097
x^(33) = 0.3548387097
x^(34) = 0.3548387097
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x^(36) = 0.3548387097
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x^(38) = 0.3548387097
x^(39) = 0.3548387097
x^(40) = 0.3548387097
x^(41) = 0.3548387097
x^(42) = 0.3548387097
x^(43) = 0.3548387097
x^(44) = 0.3548387097
x^(45) = 0.3548387097
x^(46) = 0.3548387097
x^(47) = 0.3548387097
x^(48) = 0.3548387097
x^(49) = 0.3548387097
x^(50) = 0.3548387097
x^(51) = 0.3548387097
x^(52) = 0.3548387097
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x^(56) = 0.3548387097
x^(57) = 0.3548387097
x^(58) = 0.3548387097
x^(59) = 0.3548387097
x^(60) = 0.3548387097
x^(61) = 0.3548387097
x^(62) = 0.3548387097
x^(63) = 0.3548387097
x^(64) = 0.3548387097
x^(65) = 0.3548387097
x^(66) = 0.3548387097
x^(67) = 0.3548387097
x^(68) = 0.3548387097
x^(69) = 0.3548387097
x^(70) = 0.3548387097
x^(71) = 0.3548387097
x^(72) = 0.3548387097
x^(73) = 0.3548387097
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x^(75) = 0.3548387097
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x^(81) = 0.3548387097
x^(82) = 0.3548387097
x^(83) = 0.3548387097
x^(84) = 0.3548387097
x^(85) = 0.3548387097
x^(86) = 0.3548387097
x^(87) = 0.3548387097
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x^(89) = 0.3548387097
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x^(97) = 0.3548387097
x^(98) = 0.3548387097
x^(99) = 0.3548387097
x^(100) = 0.3548387097
Sequence for c = 2.00
x^(0) = 0.1000000000
x^(1) = 0.1800000000
x^(2) = 0.2952000000
x^(3) = 0.4161139200
x^(4) = 0.4859262512
x^(5) = 0.4996038592
x^(6) = 0.4999996861
x^(7) = 0.5000000000
x^(8) = 0.5000000000
x^(9) = 0.5000000000
x^(10) = 0.5000000000
x^(11) = 0.5000000000
x^(12) = 0.5000000000
x^(13) = 0.5000000000
x^(14) = 0.5000000000
x^(15) = 0.5000000000
x^(16) = 0.5000000000
x^(17) = 0.5000000000
x^(18) = 0.5000000000
x^(19) = 0.5000000000
x^(20) = 0.5000000000
x^(21) = 0.5000000000
x^(22) = 0.5000000000
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x^(24) = 0.5000000000
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x^(25) = 0.5000000000
x^(26) = 0.5000000000
x^(27) = 0.5000000000
x^(28) = 0.5000000000
x^(29) = 0.5000000000
x^(30) = 0.5000000000
x^(31) = 0.5000000000
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x^(65) = 0.5000000000
x^(66) = 0.5000000000
x^(67) = 0.5000000000
x^(68) = 0.5000000000
x^(69) = 0.5000000000
x^(70) = 0.5000000000
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x^(71) = 0.5000000000
x^(72) = 0.5000000000
x^(73) = 0.5000000000
x^(74) = 0.5000000000
x^(75) = 0.5000000000
x^(76) = 0.5000000000
x^(77) = 0.5000000000
x^(78) = 0.5000000000
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x^(82) = 0.5000000000
x^(83) = 0.5000000000
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x^(85) = 0.5000000000
x^(86) = 0.5000000000
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x^(88) = 0.5000000000
x^(89) = 0.5000000000
x^(90) = 0.5000000000
x^(91) = 0.5000000000
x^(92) = 0.5000000000
x^(93) = 0.5000000000
x^(94) = 0.5000000000
x^(95) = 0.5000000000
x^(96) = 0.5000000000
x^(97) = 0.5000000000
x^(98) = 0.5000000000
x^(99) = 0.5000000000
x^(100) = 0.5000000000
Sequence for c = 3.60
x^(0) = 0.1000000000
x^(1) = 0.3240000000
x^(2) = 0.7884864000
x^(3) = 0.6003921493
x^(4) = 0.8637170989
x^(5) = 0.4237555390
x^(6) = 0.8790724158
x^(7) = 0.3826947729
x^(8) = 0.8504621413
x^(9) = 0.4578346351
x^(10) = 0.8935994952
x^(11) = 0.3422859745
x^(12) = 0.8104546302
x^(13) = 0.5530245214
x^(14) = 0.8898782405
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x^(15) = 0.3527818474
x^(16) = 0.8219765359
x^(17) = 0.5267919972
x^(18) = 0.8974158800
x^(19) = 0.3314182260
x^(20) = 0.7976886677
x^(21) = 0.5809732457
x^(22) = 0.8763960005
x^(23) = 0.3899737828
x^(24) = 0.8564192335
x^(25) = 0.4426751879
x^(26) = 0.8881699173
x^(27) = 0.3575668151
x^(28) = 0.8269660362
x^(29) = 0.5151355603
x^(30) = 0.8991752933
x^(31) = 0.3263727067
x^(32) = 0.7914728269
x^(33) = 0.5941569283
x^(34) = 0.8680841022
x^(35) = 0.4122507372
x^(36) = 0.8722802408
x^(37) = 0.4010667204
x^(38) = 0.8647639422
x^(39) = 0.4210101592
x^(40) = 0.8775381782
x^(41) = 0.3868737265
x^(42) = 0.8539288065
x^(43) = 0.4490438398
x^(44) = 0.8906524910
x^(45) = 0.3506062725
x^(46) = 0.8196534510
x^(47) = 0.5321580165
x^(48) = 0.8962771031
x^(49) = 0.3346720472
x^(50) = 0.8016000049
x^(51) = 0.5725347734
x^(52) = 0.8810593439
x^(53) = 0.3772575951
x^(54) = 0.8457634873
x^(55) = 0.4696113991
x^(56) = 0.8966755186
x^(57) = 0.3335347187
x^(58) = 0.8002415165
x^(59) = 0.5754781145
x^(60) = 0.8794909953
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x^(61) = 0.3815517043
x^(62) = 0.8494920045
x^(63) = 0.4602792197
x^(64) = 0.8943201346
x^(65) = 0.3402418732
x^(66) = 0.8081184273
x^(67) = 0.5582269251
x^(68) = 0.8877946507
x^(69) = 0.3586151120
x^(70) = 0.8280371284
x^(71) = 0.5126099125
x^(72) = 0.8994275644
x^(73) = 0.3256474349
x^(74) = 0.7905642590
x^(75) = 0.5960606811
x^(76) = 0.8667804440
x^(77) = 0.4156995812
x^(78) = 0.8744163818
x^(79) = 0.3953245429
x^(80) = 0.8605549752
x^(81) = 0.4320003954
x^(82) = 0.8833537936
x^(83) = 0.3709435281
x^(84) = 0.8400399374
x^(85) = 0.4837422274
x^(86) = 0.8990484654
x^(87) = 0.3267371602
x^(88) = 0.7919279580
x^(89) = 0.5932010424
x^(90) = 0.8687288365
x^(91) = 0.4105405624
x^(92) = 0.8711892325
x^(93) = 0.4039867932
x^(94) = 0.8668132708
x^(95) = 0.4156128876
x^(96) = 0.8743637350
x^(97) = 0.3954664582
x^(98) = 0.8606618591
x^(99) = 0.4317228842
x^(100) = 0.8832176476
Sequence for c = 3.98
x^(0) = 0.1000000000
x^(1) = 0.3582000000
x^(2) = 0.9149731848
x^(3) = 0.3096330785
x^(4) = 0.8507665320
```

```
x^(5) = 0.5053121032
x^(6) = 0.9948876906
x^(7) = 0.0202429713
x^(8) = 0.0789361097
x^(9) = 0.2893666970
x^(10) = 0.8184217744
x^(11) = 0.5914581428
x^(12) = 0.9617089243
x^(13) = 0.1465629796
x^(14) = 0.4978274451
x^(15) = 0.9949812144
x^(16) = 0.0198745175
x^(17) = 0.0775284939
x^(18) = 0.2846409497
x^(19) = 0.8104095083
x^(20) = 0.6115108299
x^(21) = 0.9455100326
x^(22) = 0.2050528274
x^(23) = 0.6487645381
x^(24) = 0.9069190666
x^(25) = 0.3359791556
x^(26) = 0.8879267072
x^(27) = 0.3960612221
x^(28) = 0.9520029872
x^(29) = 0.1818593323
x^(30) = 0.5921703319
x^(31) = 0.9611884271
x^(32) = 0.1484748343
x^(33) = 0.5031916303
x^(34) = 0.9949594577
x^(35) = 0.0199602382
x^(36) = 0.0778560717
x^(37) = 0.2857421252
x^(38) = 0.8122923811
x^(39) = 0.6068444055
x^(40) = 0.9495654066
x^(41) = 0.1906059618
x^(42) = 0.6140158099
x^(43) = 0.9432615725
x^(44) = 0.2130063298
x^(45) = 0.6671858405
x^(46) = 0.8837546010
x^(47) = 0.4088749766
x^(48) = 0.9619509958
x^(49) = 0.1456730843
x^(50) = 0.4953206985
```

```
x^(51) = 0.9949128545
x^(52) = 0.0201438406
x^(53) = 0.0785575038
x^(54) = 0.2880971652
x^(55) = 0.8162868107
x^(56) = 0.5968513604
x^(57) = 0.9576668597
x^(58) = 0.1613533613
x^(59) = 0.5385674474
x^(60) = 0.9890799570
x^(61) = 0.0429871669
x^(62) = 0.1637342961
x^(63) = 0.5449629981
x^(64) = 0.9869537486
x^(65) = 0.0512466658
x^(66) = 0.1935093712
x^(67) = 0.6211327079
x^(68) = 0.9366009310
x^(69) = 0.2363309157
x^(70) = 0.7183048837
x^(71) = 0.8053250514
x^(72) = 0.6239709197
x^(73) = 0.9338322200
x^(74) = 0.2459226274
x^(75) = 0.7380698612
x^(76) = 0.7694245100
x^(77) = 0.7060935251
x^(78) = 0.8259513265
x^(79) = 0.5721478163
x^(80) = 0.9742828766
x^(81) = 0.0997218969
x^(82) = 0.3573142119
x^(83) = 0.9139702482
x^(84) = 0.3129419618
x^(85) = 0.8557369756
x^(86) = 0.4913357927
x^(87) = 0.9947012274
x^(88) = 0.0209773684
x^(89) = 0.0817385275
x^(90) = 0.2987282156
x^(91) = 0.8337688818
x^(92) = 0.5516213676
x^(93) = 0.9843942330
x^(94) = 0.0611416638
x^(95) = 0.2284653756
x^(96) = 0.7015504121
```

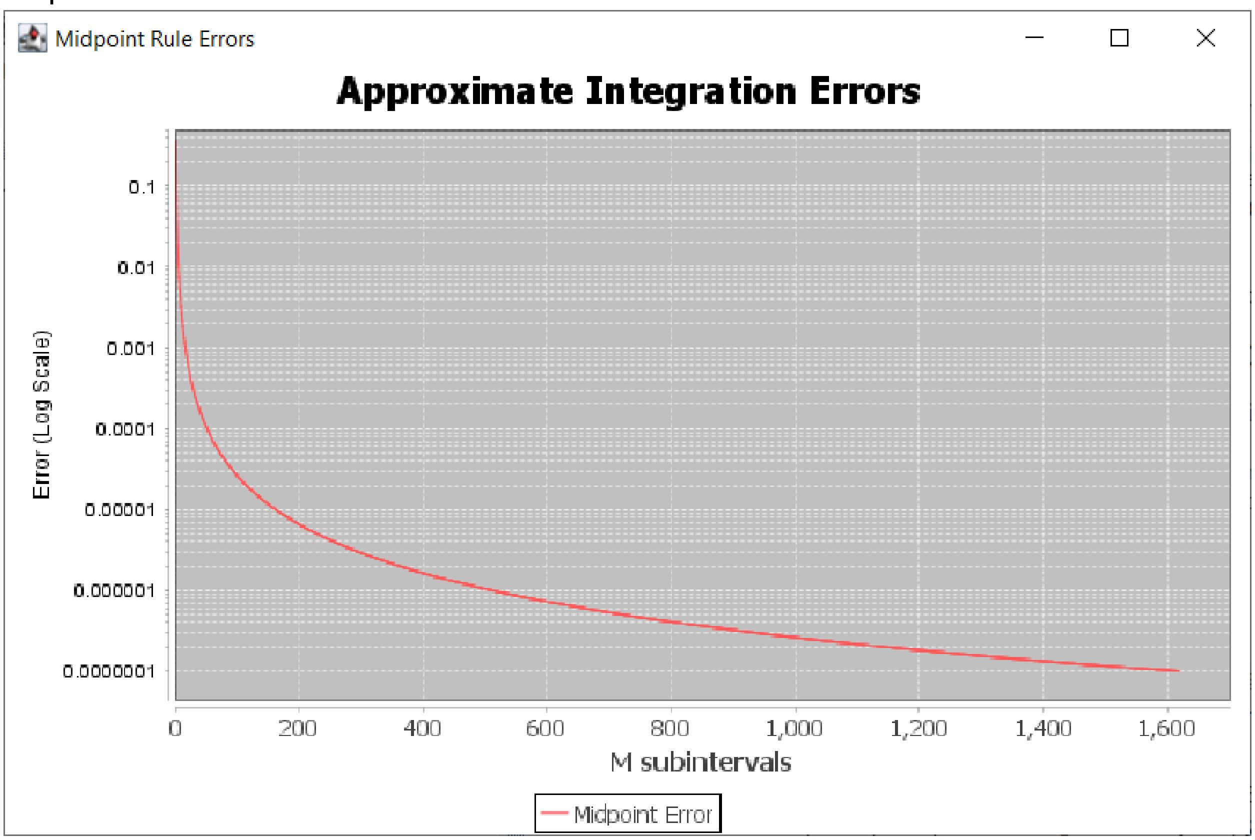
```
x^(97) = 0.8333221769
x^(98) = 0.5528073791
x^(99) = 0.9839012952
x^(100) = 0.0630413551
```

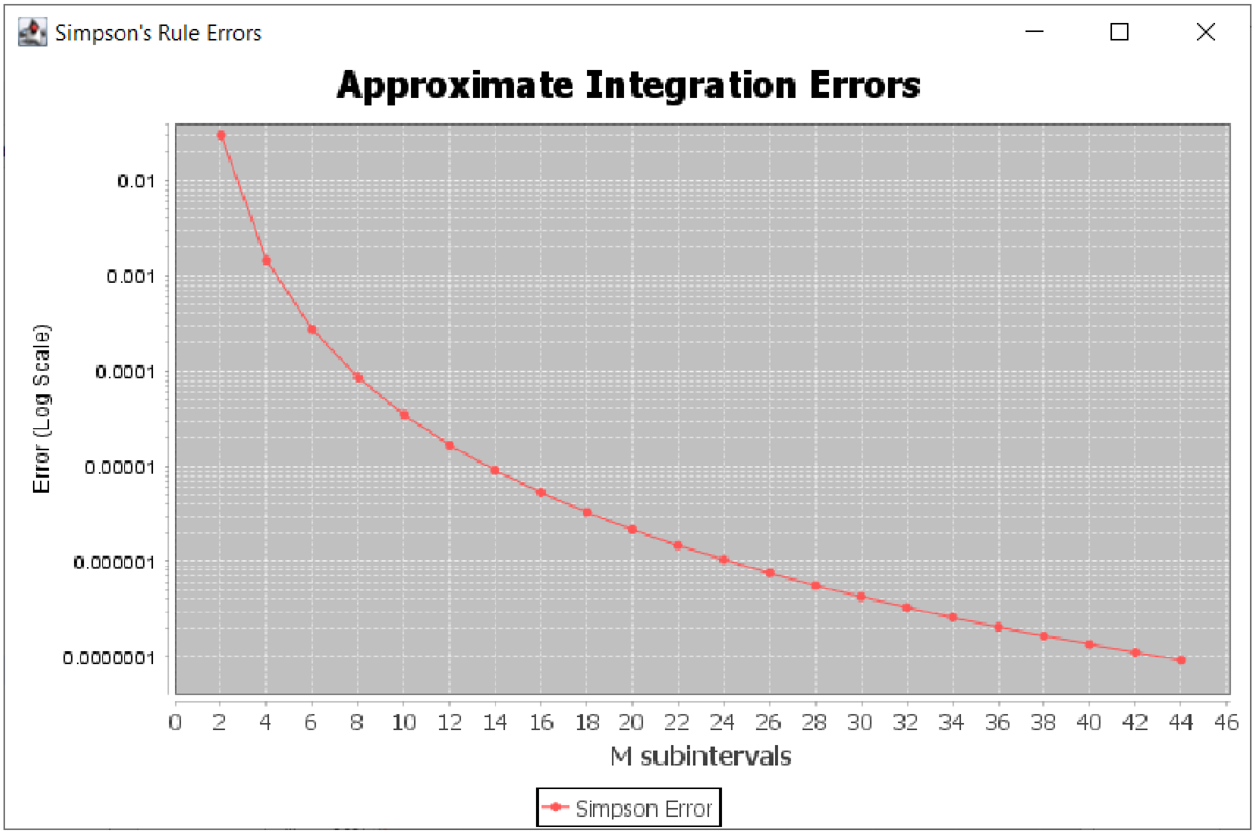
For c = 0.95, we can see from the graph (and the printed results from the program) that the sequence converges to 0 since c is less than 1. For c = 1.55 and c = 2.0, they are both values larger than one but smaller than 3 and the sequence of each of these c values converge respectively to a certain value such as 0.3548387097 (from the printed results in the program) for c = 1.55 and such as 0.5 for c = 2.0. For c = 3.6, we can see that the sequence seems to oscillate between some values. For c = 3.98, it doesn't seem that the sequence follows a certain pattern. This can be explained by the behavior of logistic map, $x_{k+1} = cx_k(1 - x_k)$, where c is assumed to lie in the interval of [0, 4] and x_k is bounded on [0, 1]. For 0 < c ≤ 1, the function converges to 0. For 1< c ≤ 3, the function converges to the stable fixed-point x* = (r-1)/r. For 3< c ≤ 3.57 (rounded value), the function converges between x* and x'* which depends on the c value. Then if c is larger than 3.57, it gets more complex and harder to predict the pattern.

Problem 3

a) I implemented the Midpoint Rule and the Simpson's Rule in a program in which each of them approximates the integral of f(x) = sin(πx) over the interval [0, 1] using values of M. The error is calculated by the absolute value of the difference between the approximation from the methods and the known exact value of the integral which is 2/π.

I plotted the results of the errors for each value of M on graphs and I also printed the output in the console as shown below.





Simpson's Rule Errors
Approximate Integration Errors
0.01
0.001
0.0001
0.00001
0.000001
0.0000001
Error (Log Scale)
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46
M subintervals
Simpson Error

M	Midpoint Error
1	0.3633802276324186
2	0.07048700881896615
3	0.030046894299085279999999999999992
4	0.016661710070606875
5	0.01059382313237658
6	0.0073307784917974809086446322915104
7	0.0053744001231235179002913754647525
8	0.00410908956779519375
9	0.003243614648377888235883520626287
10	0.002625549782384776
11	0.0021687897535340800242492575747911
12	0.0018216922607843552956747481987021
13	0.0015517515215535835892833038682972
14	0.0013376755167405570165348253099274
15	0.0011650431994603938892407783520881
16	0.001023804968564110000
17	0.00090678336532573102533695812403995
18	0.0008087412807439112571486351674625
19	0.0007257847608931909372993837462974
20	0.0006549697915373520
21	0.00059403713978782874212571533789056
22	0.00054122970896858084369752099505328
23	0.00049516431011043309058289587907808
24	0.00045474007735656672723120352396752
25	0.0004190720287648428
26	0.000387442179995041306923269506428096
27	0.000359263103690856299486382413357310
28	0.0003340504404568417798269773211161
29	0.000311401933514286125688956080728450
30	0.0002909812756652323089246356753304
31	0.00027250554623118396592764280730955
32	0.00025573535417217440625
33	0.000240467041059202288618570167407712
34	0.000226526466201998126633774138155029
35	0.00021376401735337598689460932468818
36	0.000202050578344959632292220349599264
37	0.0001912742494906819411070197698132
38	0.0001813376643252583166093783092350
39	0.00017215578189089311806646376469700
40	0.000163654060630023150
41	0.000155766940315867602315311965723975
42	0.00014843657403302996112034202238816
43	0.00014161176421910349946609137977472
44	0.000135247066079909694342440058979440
45	0.00012930202894800339732871498963945
46	0.000123740551850875189424977882514368
47	0.0001185303340532266737231627906075
48	0.00011364240490535195739478789417860
49	0.0001090507201810856333575559991048
50	0.00010473181437105572
51	0.0001006645002391407397210191002925
52	0.00009682960843928209601637261916720
53	0.000093209761202926333526448704264718
54	0.000089789175094774900927064611660488
55	0.00008655348864720969300377801349630
56	0.000083489611350562965276947634066048
57	0.000080585591027544580151168225240476
58	0.000077830497078156007453916394677808
59	0.00007521431746066086101123468815648
60	0.000072727867592685748966657259702218
61	0.00007036270962142101249122613422010
62	0.000068111080736003277464674348944604
63	0.000065965829382319521954025235589184
64	0.000063920358400928187500
65	0.00006196857424150391704352354557220
66	0.000060104841523593040422737358025792
67	0.000058323942309505758148442706303314
68	0.000056621039539311758708352148607160
69	0.000054991644148700971417710656974912
70	0.000053431585451674294911230919773265
71	0.000051936984423130946010953860128184
72	0.00005050422956088869754674222733536
73	0.000049129955046829483794699244020
74	0.00004781102095971846057354252801998
75	0.000046544495323117379982509811890944
76	0.000045327637795542319329578839829320
77	0.000044157884834082295549608274482068
78	0.00004303283618077539246215082387094
79	0.000041950242539237558833736794853358
80	0.0000409079943226314125
81	0.000039904111368076139585541071817688
82	0.000038936733524035580450410429053536
83	0.000038004112026452042170532308460708
84	0.000037104601589573492055112068293184
85	0.000036236653144268485453982336173653
86	0.00003539880716386780221652460334450
87	0.000034589687524050997429851045954160
88	0.00003380799584850298655093526611988
89	0.000033052506296501274690614108108250
90	0.00003232206075399337661199871127184
91	0.00003161556439250088104249629522834

```
1575        0.0000001055376842938487111093600713463
1576        0.00000010540379570139513357362520285262
1577        0.00000010527016172246628130422920775804
1578        0.00000010513678172600119564629515267820
1579        0.00000010500365505778402935562771384225
1580        0.00000010487078108773483018007125191870
1581        0.00000010473815916821786426056411590315
1582        0.00000010460578866681890603021386711356
1583        0.00000010447366894433542857209501148684
1584        0.00000010434179936861208947965522615186
1585        0.00000010421017931056780164280983028788
1586        0.00000010407880813966584271177177457583
1587        0.0000001039476852298859611921692462412
1588        0.00000010381680995417732067336930164601
1589        0.00000010368618169258987664121489592934
1590        0.00000010355579982748130920119492927432
1591        0.00000010342566372025656121418820047175
1592        0.0000001032957727740608598224759646920
1593        0.00000010316612636944039247840140653288
1594        0.00000010303672388971418746753526076408
1595        0.00000010290756472282827083605904155417
1596        0.00000010277864825871625193279616824152
1597        0.00000010264997389308913343841149458215
1598        0.00000010252154101657963416170999709862
1599        0.00000010239334902815144991449706299509
1600        0.0000001022653973253536875
1601        0.00000010213768531031948509263652463128
1602        0.00000010201021237565003755930951909776
1603        0.00000010188297793344994444700059497401
1604        0.00000010175598139086323910317305835666
1605        0.00000010162922215206472050310800656501
1606        0.00000010150269961207941342254151765414
1607        0.00000010137641320137083208987645793076
1608        0.00000010125036232954991597434948062896
1609        0.00000010112454640898749461218814268725
1610        0.00000010099896485147125073165935933116
1611        0.00000010087361708039013030513419503656
1612        0.00000010074850251521218868304382459033
1613        0.00000010062362056774510204842021633923
1614        0.00000010049897068638522379228760785300
1615        0.00000010037455227505874928678638456100
1616        0.00000010025036476595462121801266110176
1617        0.00000010012640759160232330085556085617
1618        0.00000010000268017808093818662666940448
1619        0.00000009987918195791547721941472163649
Midpoint error < 10^(-7) at M = 1619: 0.00000009987918195791547721941472163649
```

```
M           Simpson error
2           0.030046894299085287077446652455886800000000000000004082155997157844
4           0.001451415090116976872056659561276641143819168358731591784400284215 6
6           0.000274761806738334661145294956949556309261472351214391784400284215 6
8           0.000085679455635380102694996447305050936436145857342408215599715784 4
10          0.000034860444074150748822663824510603001203385501115917844002842156
12          0.000016750848053267936335735672815673657849547281363208215599715784 4
14          0.000009022032642008901322765181165663038469554882754608215599715784 4
16          0.000005281094580006718014164890319064806658112720841591784400284215 6
18          0.000003293777047358605557828402586776694596508375188008215599715784 4
20          0.000002159554897496041077998578922127324386384495779208215599715784 4
22          0.0000014742517523719243750420419424739222690495686 4
24          0.0000010405159166254016107520496698778091734091241 6
26          0.000000755212316144621466071712610851885579682342808991784400284215 6
28          0.000000561340294125539663287373205208701525731220564791784400284215 6
30          0.000000425882059435032032174088524797261466761405 2
32          0.000000328931700211067340415778493120385099082436782208215599715784 4
34          0.0000002580673289218856388835633041706365623512693 2
36          0.0000002053009566324272906876833348214577681354834773008215599715784 4
38          0.0000001653583797941313265078907347859023581708458741391784400284215 6
40          0.0000001346747428533538723326227943078685370516403631991784400284215 6
42          0.0000001107896760210219201756715170501766705194995 2
44          0.0000000919729328887238408097656635569611345247843 2
Simpson error < 10^(-7) at M = 44: 0.0000000919729328887238408097656635569611345247843 2
```

From the printed results of the program, the smallest value of M for which the error is less than 10^{-7} is 1619 for the Midpoint Rule and 44 for the Simpson's Rule. The number of function evaluations required for the Midpoint Rule is equal to M since we calculate the midpoints M times. The number of functions evaluations required for the Simpson's Rule is equal to M/2 +1 since we compute the function once for each endpoint ($f(x_0)$ + $f(x_M)$) and M/2 times at the midpoints because we only compute the function for when M is even. From the graph, it seems as M gets bigger, the error will also decrease. From the problem, we know that h is defined as 1/M so as M increases, h will decrease which results in the errors also decreasing as the value of M gets larger. The problem states that the errors are approximately proportional to h^p where p is an integer that depends on the method (Midpoint or Simpson's rule). This implies that the error E = c * h^p for some constant c. By taking the logarithm of both sides of the equation we get :

log(E) = log(c * h^p) => log(E) = log(c) + log(h^p) => log(E) = log(c) + p* log(h)
We can see that this equation is linear with a slope of p so we can use the slope formula to find p. In our case, we can do the difference between the log of two error values divided by the difference of the log of their h values (h=1/M).

For Midpoint Rule: $p = \frac{\log(error\ of\ M{=}1500) - \log(error\ of\ M{=}1619)}{\log\left(\frac{1}{1500}\right) - \log(\frac{1}{1619})} \approx 2.000000238$

For Simpson's Rule: $p = \frac{\log(error\ of\ M{=}20) - \log(error\ of\ M{=}44)}{\log\left(\frac{1}{20}\right) - \log(\frac{1}{44})} \approx 4.002959762$

As M increases and gets very large, h will approach 0. For both methods, their approximate formula has a multiplication with h, meaning that their errors will eventually be 0 if the function is continuous on the interval which will result in the convergence to the exact value of the integral. As M increases, it might lead to computational problems such as the limitations of precision that result into errors not being able to be represented properly.

b)

$|e_n| \leq \frac{M(b-a)^3}{24n^2}$ $\quad f(x) = \int_0^1 \sin(\pi x)$

$f'(x) = \pi \cdot \cos(\pi x)$

$f''(x) = -\pi^2 \cdot \sin(\pi x)$

$|-\pi^2 \cdot \sin(\pi x)| = \pi^2 \cdot |\sin(\pi x)|$

max value on $[0,1] = 1$ at $x = 0.5$

$\Rightarrow M = \pi^2 \cdot |\sin(\pi \cdot \frac{1}{2})| = \pi^2$

$b = 1, a = 0$

$|E_M| \leq \frac{M(b-a)^3}{24n^2}$

$|E_M| \leq \frac{\pi^2(1-0)^3}{24n^2}$

$|E_M| \leq \frac{\pi^2}{24n^2} < 10^{-7}$

$\frac{\pi^2}{24 \cdot 10^{-7}} < n^2$

$\sqrt{\frac{\pi^2}{24 \cdot 10^{-7}}} < n$

$n \approx 2028$

$\frac{\pi^2}{24(2028)^2} \approx 0.9999 \times 10^{-7}$

$\approx 0.999 \times 10^{-7} < 10^{-7}$

c)

$$|e_n| \leqslant \frac{M(b-a)^5}{180n^4} \qquad f(x) = \int_0^1 \sin(\pi x)$$

$$f'(x) = \pi \cdot \cos(\pi x)$$
$$f''(x) = -\pi^2 \sin(\pi x)$$
$$f'''(x) = -\pi^3 \cos(\pi x)$$
$$f''''(x) = \pi^4 \sin(\pi x)$$

max on $[0,1] = 1$, $x = 0.5$

$$\Rightarrow M = \left|\pi^4 \sin\left(\pi \cdot \tfrac{1}{2}\right)\right| = \pi^4 \quad , b = 1, a = 0$$

$$|E_M| \leqslant \frac{\pi^4}{180n^4} < 10^{-7}$$

$$\frac{\pi^4}{180 \cdot 10^{-7}} < n^4$$

$$\sqrt[4]{\frac{\pi^4}{180 \cdot 10^{-7}}} < n$$

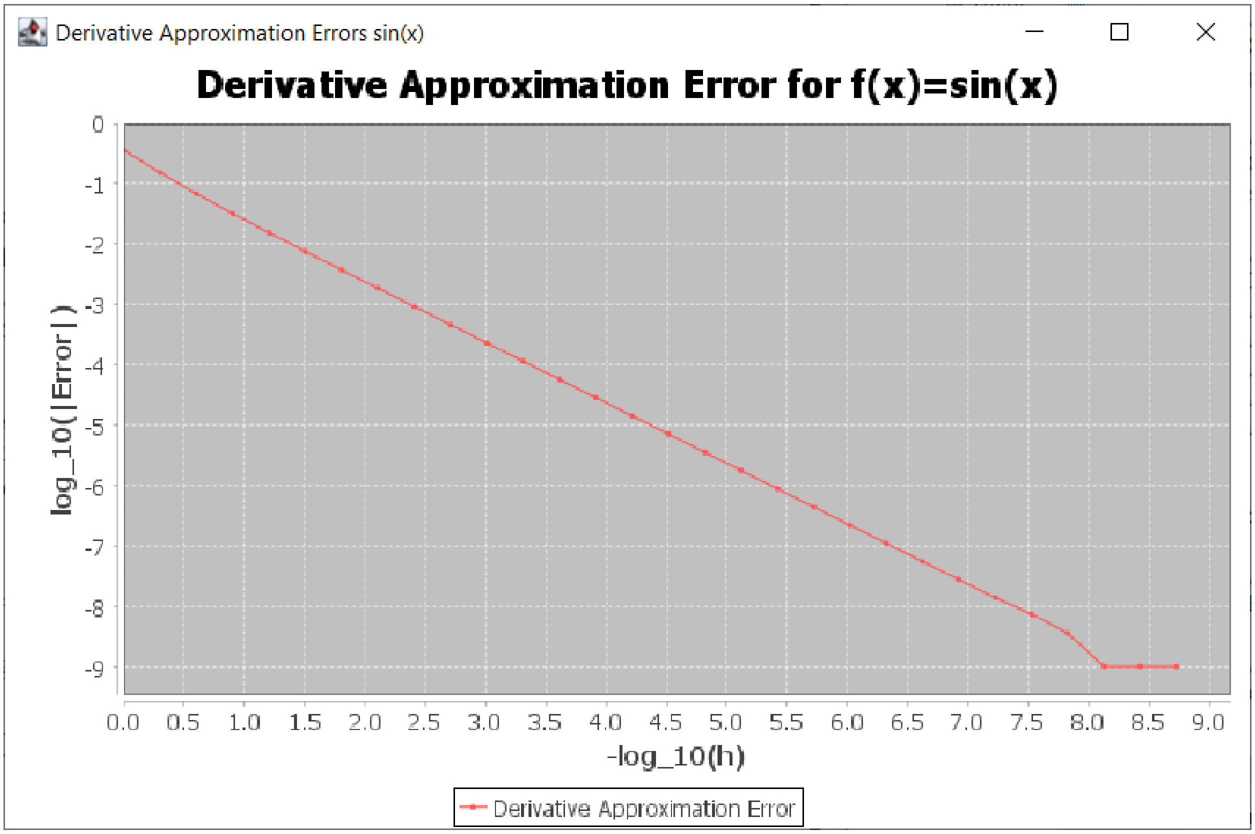
$$n \approx 48.23 \approx 49$$

$$\frac{\pi^4}{180 \cdot (49)^4} < 10^{-7}$$

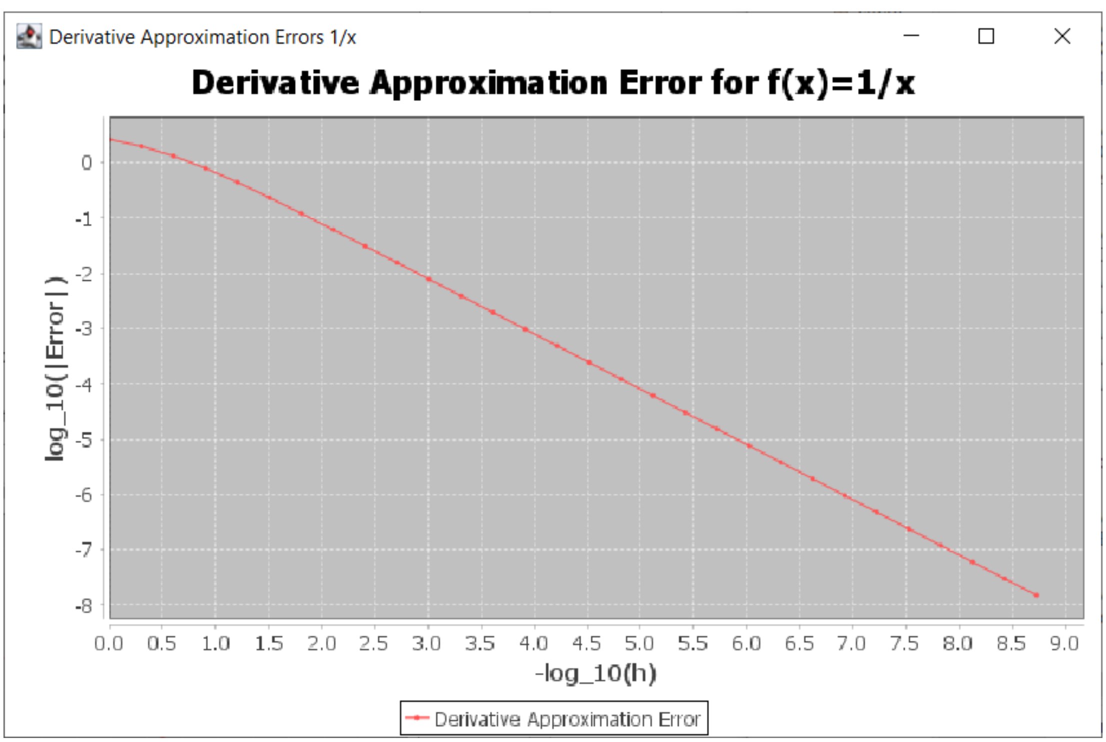
$$\approx 0.939 \times 10^{-7} < 10^{-7}$$

<u>Problem 4</u>

a) I wrote a program to approximate the derivative of the functions $f(x) = \sin(x)$ and $f(x) = \frac{1}{x}$. I plotted on the graph below the results of the computation for absolute errors from the approximation and I also printed the results in the console as shown below.



n	h_n	Error
1	1	0.3595131138905214
2	0.5	0.1534916694829858
3	0.25	0.0687296762138484
4	0.125	0.0322086872002992
5	0.0625	0.0155484029812992
6	0.03125	0.0076332431686320
7	0.015625	0.0037811443082896
8	0.0078125	0.00188167368908512
9	0.00390625	0.00093860862057952
10	0.001953125	0.00046874680555744
11	0.0009765625	0.0002342339706544
12	0.00048828125	0.0001170821202608
13	0.000244140625	0.00005853234293472
14	0.0001220703125	0.0000292639916720
15	0.00006103515625	0.00001463145126624
16	0.000030517578125	0.0000073155895984
17	0.0000152587890625	0.0000036577616560
18	0.00000762939453125	0.0000018288747184
19	0.000003814697265625	0.0000009144378032
20	0.0000019073486328125	0.00000045721672416
21	0.00000095367431640625	0.0000002285904560
22	0.000000476837158203125	0.0000001142956720
23	0.0000002384185791015625	0.0000000572531376
24	0.00000011920928955078125	0.0000000287318704
25	0.000000059604644775390625	0.0000000144712368
26	0.0000000298023223876953125	0.0000000077603504
27	0.00000001490116119384765625	0.00000000373381856
28	0.000000007450580596923828125	0.0000000010494640
29	0.0000000037252902984619140625	0.0000000010494640
30	0.00000000186264514923095703125	0.0000000010494640



n	h_n	Error
1	1	2.666666666666666666666666666666667
2	0.5	2
3	0.25	1.333333333333333333333333333333332
4	0.125	0.8
5	0.0625	0.444444444444444444444444444444448
6	0.03125	0.235294117647058823529411764705888
7	0.015625	0.121212121212121212121212121212096
8	0.0078125	0.061538461538461538461538461538432
9	0.00390625	0.031007751937984496124031007751936
10	0.001953125	0.01556420233463035019455252918272
11	0.0009765625	0.007797270955165692007797270955008
12	0.00048828125	0.003902439024390243902439024390144
13	0.000244140625	0.001952171791117618350414836506624
14	0.0001220703125	0.000976324139614351964852330971136
15	0.00006103515625	0.000488221652630294153545709748224
16	0.000030517578125	0.000244125724748245346353371971584
17	0.0000152587890625	0.000122066587323384906466477473792
18	0.00000762939453125	0.00006103422494163602239956058112
19	0.000003814697265625	0.000030517345296132689417347530752
20	0.0000019073486328125	0.000015258730855061130290487885824
21	0.00000095367431640625	0.000007629379979362527155824099328
22	0.000000476837158203125	0.00000381469362764966235192950784
23	0.0000002384185791015625	0.000001907347723318231907734388736
24	0.00000011920928955078125	0.000000953674089032628766865620992
25	0.000000059604644775390625	0.000000476837101359712915448397824
26	0.0000000298023223876953125	0.00000023841856489070863l8272512
27	0.00000001490116119384765625	0.000000119209285998067677078224896
28	0.000000007450580596923828125	0.000000059604643887212218487406592
29	0.0000000037252902984619140625	0.000000029802322165650709322661888
30	0.00000000186264514923095703125	0.000000014901161138336505050169344

From observing the graphs and the printed results, we notice that the errors from approximating the derivative in both cases decrease consistently as h decreases and the slope of the line in both graph is almost linear on a logarithmic scale for both axes. In the graph for the derivative approximation error for $f(x) = \sin(x)$, as h becomes very small, the line seemed to have become flat which is probably due to numerical instability.

b)

$$\lim_{x\to 0} f(x) = \frac{\sqrt{1+x^2}-1}{x^2} - \frac{x^2 \cdot \sin(x)}{x-\tan(x)}$$

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

$$\sqrt{1+x^2}-1 = \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

$$\frac{\sqrt{1+x^2}-1}{x^2} = \frac{1}{2} - \frac{x^2}{8} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$x^2 \cdot \sin(x) = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$x - \tan(x) = -\frac{x^3}{3} - \frac{2x^5}{15} - \cdots$$

$$\frac{x^2 \cdot \sin(x)}{x-\tan(x)} = \frac{x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots}{-\frac{x^3}{3} - \frac{2x^5}{15} - \cdots} = \frac{x^3\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{x^3\left(-\frac{1}{3} - \frac{2x^2}{15} - \cdots\right)} = \frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{\left(-\frac{1}{3} - \frac{2x^2}{15} - \cdots\right)}$$

$$\lim_{x\to 0}\left(\left(\frac{1}{2} - \frac{x^2}{8} + \cdots\right) - \left(\frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{\left(-\frac{1}{3} - \frac{2x^2}{15} - \cdots\right)}\right)\right)$$

$$= \lim_{x\to 0}\left(\left(\frac{1}{2} - \frac{0^2}{8} + \cdots\right) - \left(\frac{\left(1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \cdots\right)}{\left(-\frac{1}{3} - \frac{2\cdot 0^2}{15} - \cdots\right)}\right)\right)$$

$$= \left(\frac{1}{2} - \left(\frac{1}{-\frac{1}{3}}\right)\right) = \frac{1}{2} - (-3) = \frac{1}{2} + 3 = \frac{7}{2}$$

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