COMP 361: Elementary Numerical Methods

Assignment 3

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COMP 361 - Section DD

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"I certify that this submission is my original work and meets the Faculty's Expectations of Originality."

Problem 1

Using Newton's Method: $x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g\prime(x^{(k)})}$, where $x^{(0)} = 1$

$g(x) = x^3 - 5$

$g'(x) = 3x^2$

$$x^{(k+1)} = x^{(k)} - \frac{\left(x^{(k)}\right)^3 - 5}{3(x^{(k)})^2} = \frac{2\left(x^{(k)}\right)^3 + 5}{3(x^{(k)})^2}$$

Finding a fixed point x*: $x^* = x^* - \frac{g(x^*)}{g\prime(x^*)}$

$$x^* = x^* - \frac{(x^*)^3 - 5}{3(x^*)^2} \Rightarrow 0 = -\frac{(x^*)^3 - 5}{3(x^*)^2} \Rightarrow (x^*)^3 - 5 = 0 \Rightarrow x^* = \sqrt[3]{5} \approx 1.709975947$$

$$f(x) = x - \frac{x^3 - 5}{3x^2}$$

$$f'(x^*) = 1 - \frac{(x^*)^3 + 10}{3(x^*)^3}$$

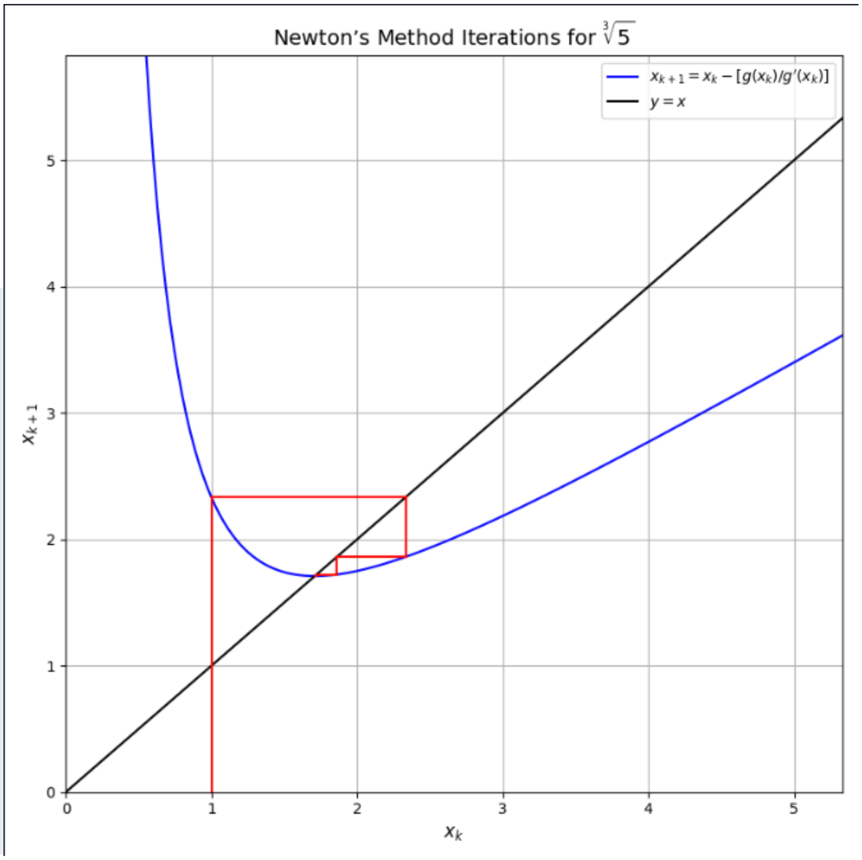
$$|f'(\sqrt[3]{5})| = \left|1 - \frac{\left(\sqrt[3]{5}\right)^3 + 10}{3\left(\sqrt[3]{5}\right)^3}\right| = 0$$

The fixed point is attracting since $|f'(x^*)| < 1$ and is quadratic since $|f'(x^*)| = 0$.

I wrote a code in Python to numerically carry out the first 10 iterations of Newton's method and the "$x^{(k+1)}$ versus $x^{(k)}$ diagram", using $x^{(0)} = 1$ and the iteration equation above as shown below.

From the graph, Newton's method converge for values of $x^{(0)} > 1$.

```
x(0) = 1.0000000000
x(1) = 2.3333333333
x(2) = 1.8616780045
x(3) = 1.7220018801
x(4) = 1.7100597366
x(5) = 1.7099759508
x(6) = 1.7099759467
x(7) = 1.7099759467
x(8) = 1.7099759467
x(9) = 1.7099759467
x(10) = 1.7099759467
```



The code source can be found in the same directory as this document and the name of the file containing the code is “cuberoot5N.py” (note that it cannot be executed directly by clicking the file, it must run in an environment that supports plotting for Python). I wrote the code and test it using this website: https://python-fiddle.com/.

Problem 2

Using Chord Method: $x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(0)})}$, where $x^{(0)} = 1$

$g(x) = x^3 - 5$

$g'^{(x^{(0)})} = 3\left(x^{(0)}\right)^2 = 3(1)^2 = 3$

$$x^{(k+1)} = x^{(k)} - \frac{\left(x^{(k)}\right)^3 - 5}{3} = \frac{3x - \left(x^{(k)}\right)^3 + 5}{3}$$

Finding a fixed point x*: $x^* = x^* - \frac{g(x^*)}{g'(x^{(0)})}$

$$x^* = x^* - \frac{(x^*)^3 - 5}{3} \;=>\; 0 = -\frac{(x^*)^3 - 5}{3} \;=>\; (x^*)^3 - 5 = 0 \;=>\; x^* = \sqrt[3]{5} \approx 1.709975947$$

$$f(x) = x - \frac{x^3 - 5}{3}$$

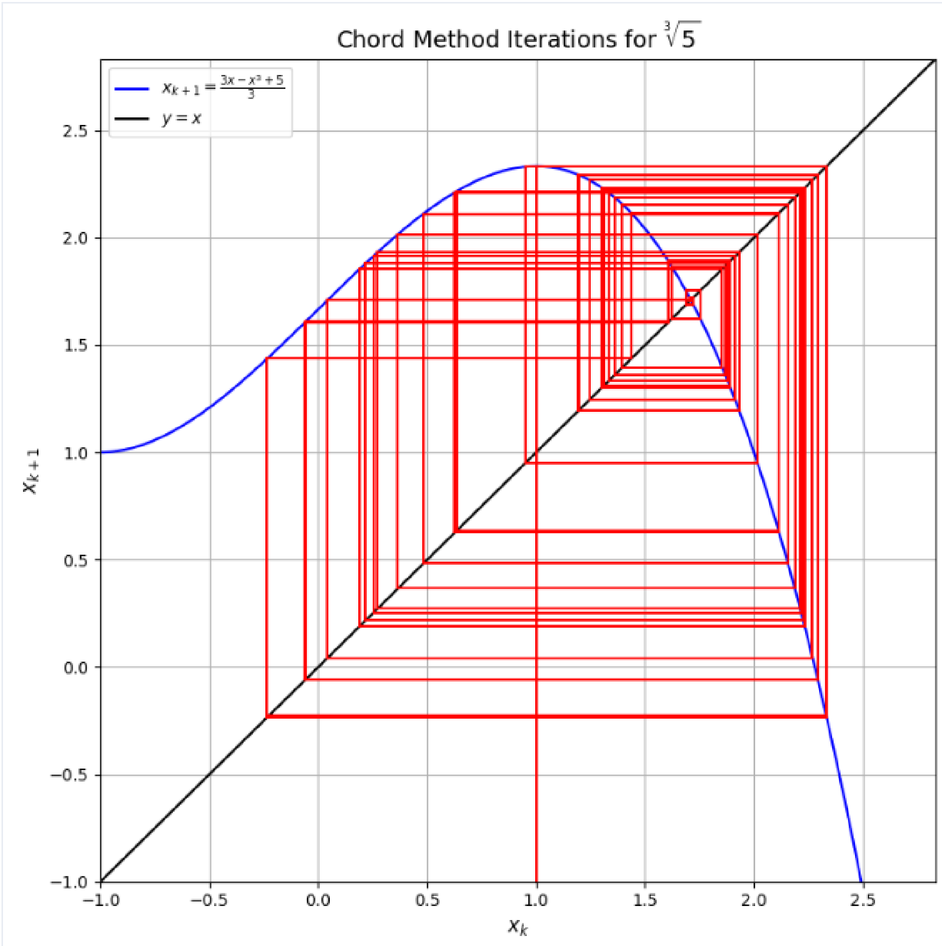
$f'(x^*) = 1 - (x^*)^2$

$|f'(\sqrt[3]{5})| = \left|1 - \sqrt[3]{5}^2\right| \approx 1.924017738$

The fixed point is repelling since $|f'(x^*)| > 1$ so it's not convergent.

I wrote a code in Python to numerically carry out the first 10 iterations of Chord method and the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" (showing around 60 iterations), using $x^{(0)} = 1$ and the iteration equation above as shown below.

```
x(0) = 1.0000000000
x(1) = 2.3333333333
x(2) = -0.2345679012
x(3) = 1.4364009050
x(4) = 2.1151840176
x(5) = 0.6274038354
x(6) = 2.2117476794
x(7) = 0.2719180864
x(8) = 1.9318829289
x(9) = 1.1951766955
x(10) = 2.2927610410
```



The code source can be found in the same directory as this document and the name of the file containing the code is “cuberoot5C.py” (note that it cannot be executed directly by clicking the file, it must run in an environment that supports plotting for Python). I wrote the code and test it using this website: https://python-fiddle.com/.

Using Chord Method: $x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(0)})}$, where $x^{(0)} = 0.1$

$$g(x) = x^3 - 5$$

$$g'(x^{(0)}) = 3\left(x^{(0)}\right)^2 = 3(0.1)^2 = 0.03$$

$$x^{(k+1)} = x^{(k)} - \frac{\left(x^{(k)}\right)^3 - 5}{0.03} = \frac{0.03x - \left(x^{(k)}\right)^3 + 5}{0.03}$$

Finding a fixed point x*: $x^* = x^* - \frac{g(x^*)}{g'(x^{(0)})}$

$$x^* = x^* - \frac{(x^*)^3 - 5}{0.03} \;=>\; 0 = -\frac{(x^*)^3 - 5}{0.03} \;=>\; (x^*)^3 - 5 = 0 \;=>\; x^* = \sqrt[3]{5} \approx 1.709975947$$

$$f(x) = x - \frac{x^3 - 5}{0.03}$$

$$f'(x^*) = 1 - 100(x^*)^2$$

$$|f'(\sqrt[3]{5})| = \left|1 - 100\left(\sqrt[3]{5}\right)^2\right| \approx 291.4017738$$

The fixed point is repelling since $|f'(x^*)| > 1$ so it's not convergent.

I wrote a code in Java to numerically carry out the first 10 iterations of Chord method and the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" could not be drawn due to unreasonable big values produced by the iterations, using $x^{(0)} = 0.1$.

The code source can be found in the same directory as this document and the output for the first 10 iterations is in a text file named "chord_method_output.txt".

From the graph, it seems like all values of $x^{(0)} >= \sqrt[3]{5}$ for which the Chord method would converge to the cubic root of 5.

Problem 3

3.

$$x^{(k+1)} = c\,x^{(k)}(1-x^{(k)})$$

Finding fixed points : $x^* = cx^*(1-x^*)$

$$x^* = cx^*(1-x^*)$$
$$= cx^* - c(x^*)^2$$

$$\Rightarrow c(x^*)^2 - cx^* + x^* = 0$$
$$\Rightarrow x^*(cx^* - c + 1) = 0$$

$$x^* = 0$$
$$cx^* - c + 1 = 0 \Rightarrow x^* = \frac{c-1}{c} = 1 - \frac{1}{c}$$

$$f(x^*) = cx^*(1-x^*)$$
$$f'(x^*) = c(1-2x^*)$$

For $x^* = 0$: $c(1-2(0)) = c$

if $|c| < 1$, $x^* = 0 \rightarrow$ attracting

if $|c| > 1$, $x^* = 0 \rightarrow$ repelling

For $x^* = 1-\frac{1}{c}$: $c(1-2(1-\frac{1}{c})) = c(1-2+\frac{2}{c}) = c(\frac{2-c}{c}) = 2-c$

if $|2-c| < 1$, $x^* = 1-\frac{1}{c} \rightarrow$ attracting

if $|2-c| > 1$, $x^* = 1-\frac{1}{c} \rightarrow$ repelling

$c = 0.70$

$x^* = 1 - \frac{1}{(0.70)} \approx -0.428571 \rightarrow$ not in $[0,1]$

$x^* = 0 \rightarrow |f'(0)| = c = 0.70 < 1$

attracting , linear with rate 0.70

$c = 1.00$

$x^* = 1 - \frac{1}{(1.00)} = 0 \rightarrow |f'(0)| = c = 1$

inconclusive

$c = 1.80$

$x^* = 1 - \frac{1}{(1.80)} \approx 0.444 \rightarrow |f'(0.444)| = 2 - c = 2 - 1.80 \approx 0.2 < 1$

attracting, linear with rate 0.2

$x^* = 0 \rightarrow |f'(0)| = c = 1.80 > 1$

repelling

$c = 2.00$

$x^* = 1 - \frac{1}{(2.00)} = 0.5 \rightarrow |f'(0.5)| = 2 - c = 2 - 2.00 = 0$

attracting, quadratic

$x^* = 0 \rightarrow |f'(0)| = c = 2.00 > 1$

repelling

$c = 3.30$

$x^* = 1 - \frac{1}{(3.30)} \approx 0.69697 \rightarrow |f'(0.69697)| = |2 - 3.30| = 1.3 > 1$

repelling

$x^* = 0 \rightarrow |f'(0)| = c = 3.30 > 1$

repelling

$c = 3.50$

$x^* = 1 - \frac{1}{(3.50)} \approx 0.714286 \rightarrow |f'(0.714286)| = |2 - 3.50| = 1.5 > 1$

repelling

$x^* = 0 \rightarrow |f'(0)| = c = 3.50 > 1$

repelling

$c = 3.97$

$x^* = 1 - \frac{1}{(3.97)} \approx 0.7481 \rightarrow |f'(0.7481)| = |2 - 3.97| = -1.97 > 1$

repelling

$x^* = 0 \rightarrow |f'(0)| = c = 3.97 > 1$

repelling