COMP 361/5611 - Elementary Numerical Methods
Assignment 3 - Due Sunday, November 12, 2023

Problem 1. (20%) Show how to use Newton's method to compute the cubic root of 5. Numerically carry out the first 10 iterations of Newton's method, using $x^{(0)} = 1$. Analytically determine the fixed points of the Newton iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram (i.e., trajectories of fixed-point iterations - such graphs are shown in the Notes pp. 118-125) (red curves))", again taking $x^{(0)} = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. For which values of $x^{(0)}$ will Newton's method converge?

Problem 2. (20%) Also use the Chord method to compute the cubic root of 5. Numerically carry out the first 10 iterations of the Chord method, using $x^{(0)} = 1$. Analytically determine the fixed points of the Chord iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. If the convergence is linear then determine analytically the rate of convergence. Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram", as in the Lecture Notes, again taking $x^{(0)} = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. (If done by hand then make sure that your diagram is sufficiently accurate, for otherwise the graphical results may be misleading.)

Do the same computations and analysis for the Chord Method when $x^{(0)} = 0.1$.

More generally, analytically determine all values of $x^{(0)}$ for which the Chord method will converge to the cubic root of 5.

Problem 3. (20%) Consider the *discrete logistic equation*, discussed in the Lecture Notes, and given by

$$x^{(k+1)} = cx^{(k)}(1 - x^{(k)}), \qquad k = 0, 1, 2, 3, \cdots .$$

For each of the following values of c, determine analytically the fixed points and whether they are attracting or repelling: $c = 0.70$, $c = 1.00$, $c = 1.80$, $c = 2.00$, $c = 3.30$, $c = 3.50$, $c = 3.97$. (You need only consider "physically meaningful" fixed points, namely those that lie in the interval $[0, 1]$.) If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. If the convergence is linear then analytically determine the rate of convergence. For each case include a statement that describes the behavior of the iterations, as also shown in the Lecture Notes.

Problem 4. (20%) Consider the example of solving a system of nonlinear equations by Newton's method, as given on Pages 95-97 of the Lecture notes. Write a program to carry out this iteration, using Gauss elimination to solve the 2 by 2 linear systems that arise. Use each of the following 16 initial data sets for the Newton iteration:

$$(x_1^{(0)} \ , \ x_2^{(0)}) \ = \ (i, j) \ , \qquad i = 0, 1, 2, 3 \ , \quad j = 0, 1, 2, 3 \ .$$

Present and discuss your numerical results in a concise manner.

Problem 5. (20%)

- Derive Newton's method for finding $\sqrt{R}$.
- Perform three iterations of Newton's method for computing $\sqrt{2}$, starting with $x_0 = 1$.
- Perform three iterations of the bisection method for computing $\sqrt{2}$, starting with interval [1,2].
- Find theoretically the minimum number of iterations in both schemes to achieve 10^{-6} accuracy.
- Find numerically the minimum number of iterations in both schemes to achieve 10^{-6} accuracy and compare your results with the theoretical estimates.