**COMP 361: Elementary Numerical Methods**

Assignment 4

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COMP 361 - Section DD

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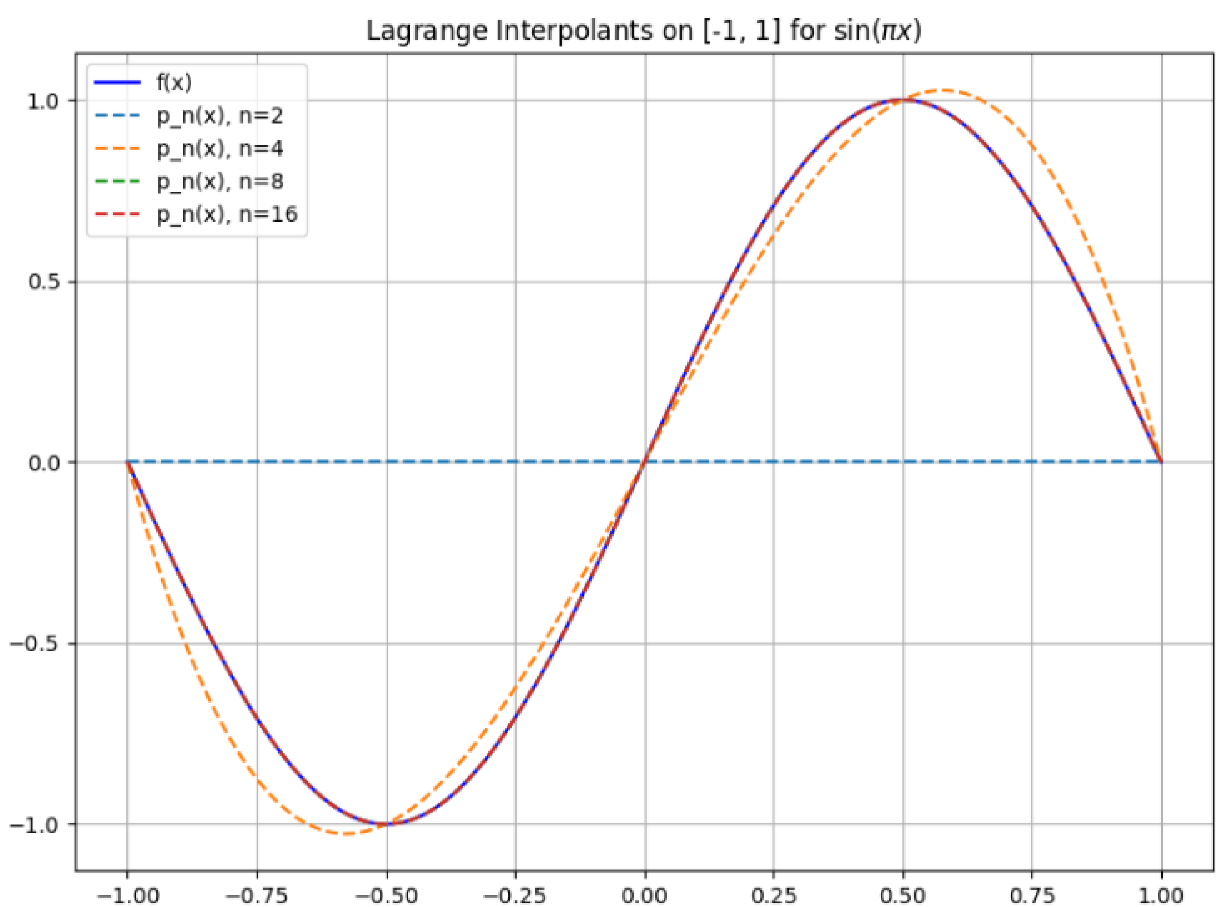
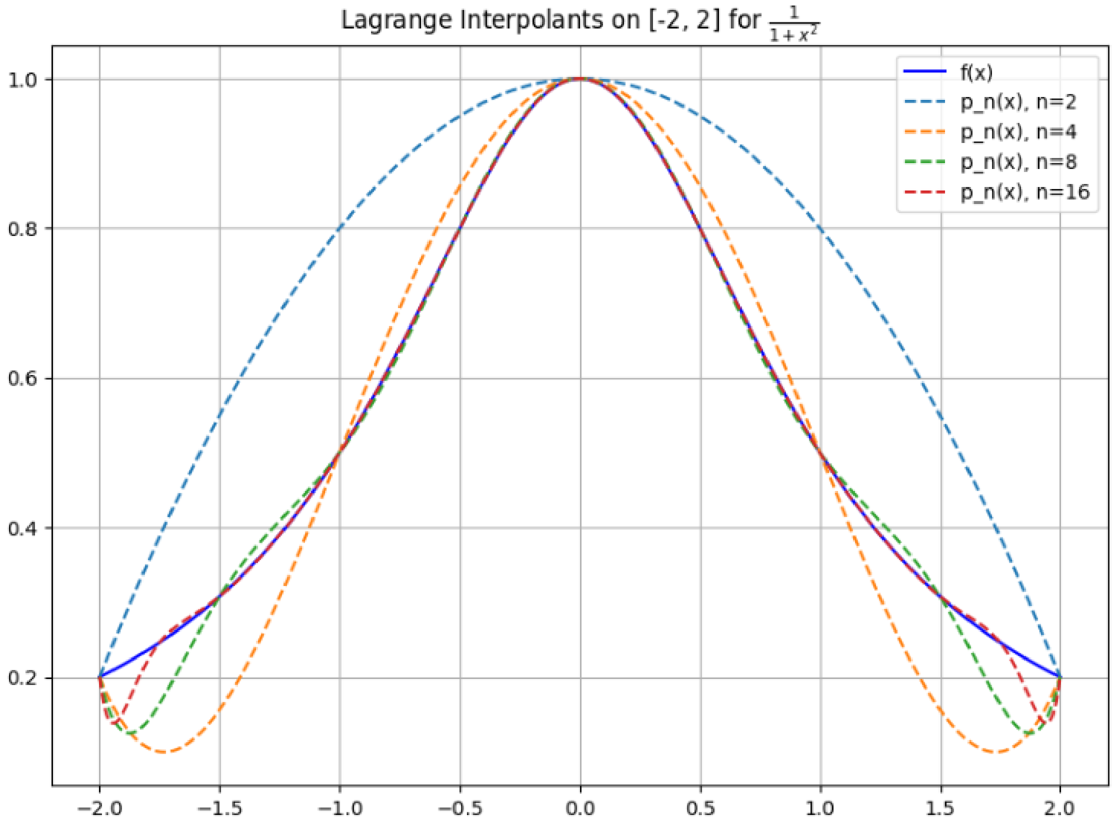
*“I certify that this submission is my original work and meets the Faculty’s Expectations of Originality.”*

## Problem 1

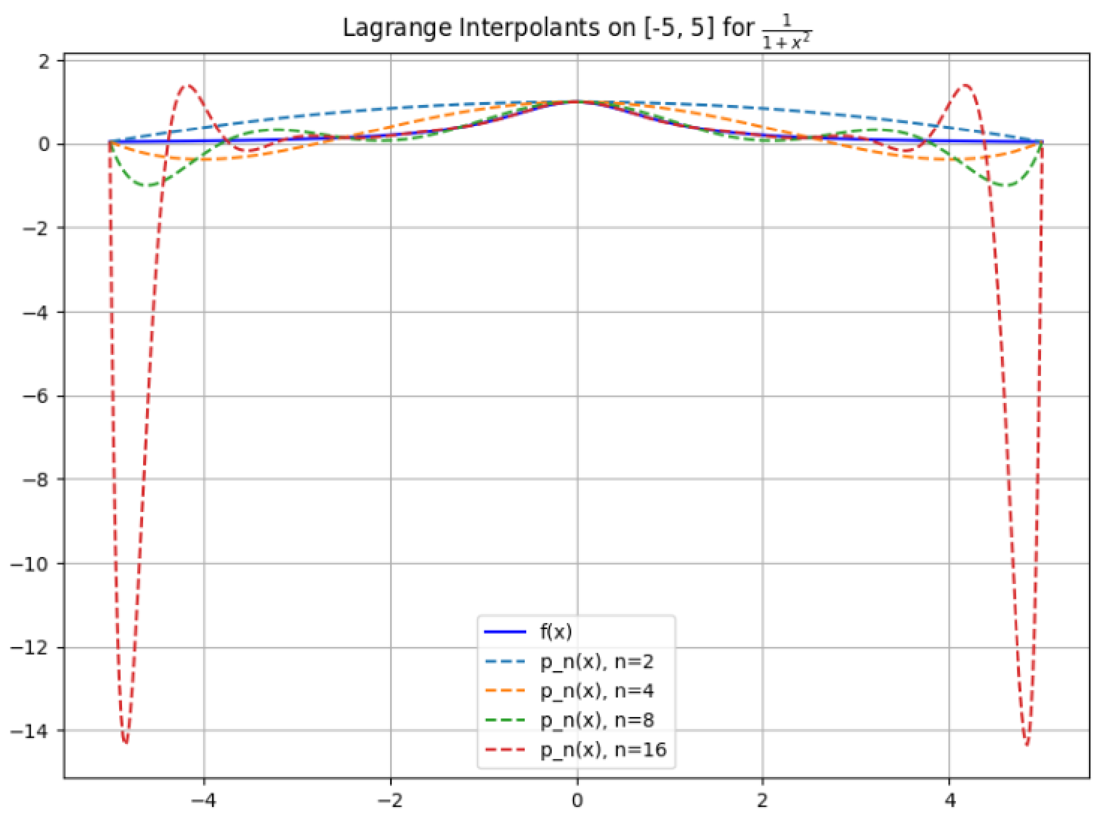
To estimate the maximum interpolation error, we compute the approximate maximum interpolation error for each of the twelve cases given in the problem. The calculations and plotting were processed by the written program in Python and the results are as shown below.

```
Function: sin(πx), Interval [-1, 1], n = 2, Maximum interpolation error: 1.000000
Function: sin(πx), Interval [-1, 1], n = 4, Maximum interpolation error: 0.180758
Function: sin(πx), Interval [-1, 1], n = 8, Maximum interpolation error: 0.001206
Function: sin(πx), Interval [-1, 1], n = 16, Maximum interpolation error: 0.000000
Function: 1/(1+x^2), Interval [-2, 2], n = 2, Maximum interpolation error: 0.305573
Function: 1/(1+x^2), Interval [-2, 2], n = 4, Maximum interpolation error: 0.161800
Function: 1/(1+x^2), Interval [-2, 2], n = 8, Maximum interpolation error: 0.099183
Function: 1/(1+x^2), Interval [-2, 2], n = 16, Maximum interpolation error: 0.071757
Function: 1/(1+x^2), Interval [-5, 5], n = 2, Maximum interpolation error: 0.646227
Function: 1/(1+x^2), Interval [-5, 5], n = 4, Maximum interpolation error: 0.438345
Function: 1/(1+x^2), Interval [-5, 5], n = 8, Maximum interpolation error: 1.045174
Function: 1/(1+x^2), Interval [-5, 5], n = 16, Maximum interpolation error: 14.386268
```





Lagrange Interpolants on [-2, 2] for $\frac{1}{1+x^2}$
f(x)
p_n(x), n=2
p_n(x), n=4
p_n(x), n=8
p_n(x), n=16
1.0
0.8
0.6
0.4
0.2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Lagrange Interpolants on [-5, 5] for $\frac{1}{1+x^2}$
f(x)
p_n(x), n=2
p_n(x), n=4
p_n(x), n=8
p_n(x), n=16
2
0
−2
−4
−6
−8
−10
−12
−14
−4
−2
0
2
4

The code source can be found in the same directory as this document and the name of the file containing the code is “LagrangeInterpolation.py” (note that it cannot be executed directly by clicking the file, it must run in an environment that supports plotting for Python). I wrote the code and test it using this website: [https://python-fiddle.com/](https://python-fiddle.com/).

Using the Lagrange Interpolation Theorem and Table, we derived the tight upper bound on the maximum interpolation error for n=2, 4, 8, 16 as shown below.

$$f(x) = \sin(\pi x) \quad \text{on} \quad [-1, 1]$$

$$f^{(1)}(x) = \pi \cos(\pi x)$$
$$f^{(2)}(x) = -\pi^2 \sin(\pi x)$$
$$f^{(3)}(x) = -\pi^3 \cos(\pi x)$$
$$f^{(4)}(x) = \pi^4 \sin(\pi x)$$
$$\vdots$$
$$f^{(n+1)}(x) = \pm \pi^{n+1} \sin(\pi x) \quad \text{or} \quad f^{(n+1)}(x) = \pm \pi^{n+1} \cos(\pi x)$$

$$\max_{x \in [-1,1]} |\sin(\pi x)| = 1 \quad , \quad \max_{x \in [-1,1]} |\cos(\pi x)| = 1$$

$$\max_{x \in [-1,1]} \left| f^{(n+1)}(x) \right| = \left| \pm \pi^{n+1} \cdot 1 \right| = \pi^{n+1}$$

$$\max_{x \in [-1,1]} |\sin(\pi x) - p(x)| \leq \max_{x \in [-1,1]} \left| f^{(n+1)}(x) \right| \cdot \frac{\max |\omega_{n+1}(x)|}{(n+1)!}$$
$$\leq \pi^3 \cdot \frac{0.38490}{3!} \approx 1.989052649 \text{ when } n=2$$
$$\leq \pi^5 \cdot \frac{0.11340}{5!} \approx 0.2891886021 \text{ when } n=4$$
$$\leq \pi^9 \cdot \frac{0.01877}{9!} \approx 1.5418783 \times 10^{-3} \text{ when } n=8$$
$$\leq \pi^{17} \cdot \frac{0.00095}{17!} \approx 7.5545513 \times 10^{-10} \text{ when } n=16$$

From my findings, it seems that the upper bound found using the theorem is slightly larger than the actual (approximate) maximum interpolation error found by computing it above with the written program. The approximate maximum interpolation error found by computing is within the upper bound.

**Problem 2**

2. $x_0 = -2h$ , $x_1 = -h$ , $x_2 = 0$ , $x_3 = h$ , $x_4 = 2h$ , $X = X_2$

$$f''(x_2) \cong f_0 l_0''(x_2) + f_1 l_1''(x_2) + f_2 l_2''(x_2) + f_3 l_3''(x_2) + f_4 l_4''(x_2)$$

$$l_0 = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$\Rightarrow \frac{(x+h)(x)(x-h)(x-2h)}{(-2h+h)(-2h)(-2h-h)(-2h-2h)} = \frac{(x+h)(x)(x-h)(x-2h)}{24h^4}$$

$$l_0'' = \frac{6x^2-6xh-h^2}{12h^4} = \frac{-h^2}{12h^4} = -\frac{1}{12h^2} \quad \text{where } x = x_2 = 0$$

$$l_1 = \frac{(x+2h)(x)(x-h)(x-2h)}{(-h+2h)(-h)(-h-h)(-h-2h)} = \frac{(x+2h)(x)(x-h)(x-2h)}{-6h^4}$$

$$l_1'' = -\frac{6x^2-3xh-4h^2}{3h^4} = \frac{4h^2}{3h^4} = \frac{16}{12h^2} \quad \text{where } x = x_2 = 0$$

$$l_2 = \frac{(x+2h)(x+h)(x-h)(x-2h)}{(2h)(h)(-h)(-2h)} = \frac{(x+2h)(x+h)(x-h)(x-2h)}{4h^4}$$

$$l_2'' = \frac{6x^2-5h^2}{2h^4} = -\frac{5h^2}{2h^4} = -\frac{30}{12h^2} \quad \text{where } x = x_2 = 0$$

$$l_3 = \frac{(x+2h)(x+h)(x)(x-2h)}{(h+2h)(h+h)(h)(h-2h)} = \frac{(x+2h)(x+h)(x)(x-2h)}{-6h^4}$$

$$l_3'' = -\frac{6x^2+3xh-4h^2}{3h^4} = \frac{4h^2}{3h^4} = \frac{16}{12h^2} \quad \text{where } x = x_2 = 0$$

$$l_4 = \frac{(x+2h)(x+h)(x)(x-h)}{(2h+2h)(2h+h)(2h)(2h-h)} = \frac{(x+2h)(x+h)(x)(x-h)}{24h^4}$$

$$l_4'' = \frac{6x^2+6xh-h^2}{12h^4} = -\frac{h^2}{12h^4} = -\frac{1}{12h^2}$$

$$f''(x_2) \cong \frac{-f_0+16f_1-30f_2+16f_3-f_4}{12h^2}$$

2. Taylor expansions

$$f(-2h) = f(0) - 2hf'(0) + \frac{(-2h)^2}{2} f''(0) - \frac{(-2h)^3}{2\cdot 3} f^{(3)}(0) + \frac{(-2h)^4}{2\cdot 3\cdot 4} f^{(4)}(0) - \frac{(-2h)^5}{2\cdot 3\cdot 4\cdot 5} f^{(5)}(0) + \frac{(-2h)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} f^{(6)}(0) - \cdots$$

$$f(-h) = f(0) - hf'(0) + \frac{(-h)^2}{2} f''(0) - \frac{(-h)^3}{2\cdot 3} f^{(3)}(0) + \frac{(-h)^4}{2\cdot 3\cdot 4} f^{(4)}(0) - \frac{(-h)^5}{2\cdot 3\cdot 4\cdot 5} f^{(5)}(0) + \frac{(-h)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} f^{(6)}(0) - \cdots$$

$$f(0) = f(0)$$

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \frac{h^3}{2\cdot 3} f^{(3)}(0) + \frac{h^4}{2\cdot 3\cdot 4} f^{(4)}(0) + \frac{h^5}{2\cdot 3\cdot 4\cdot 5} f^{(5)}(0) + \frac{h^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} f^{(6)}(0) + \cdots$$

$$f(2h) = f(0) + 2hf'(0) + \frac{(2h)^2}{2} f''(0) + \frac{(2h)^3}{2\cdot 3} f^{(3)}(0) + \frac{(2h)^4}{2\cdot 3\cdot 4} f^{(4)}(0) + \frac{(2h)^5}{2\cdot 3\cdot 4\cdot 5} f^{(5)}(0) + \frac{(2h)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} f^{(6)}(0) + \cdots$$

$$\frac{-f_0 + 16f_1 - 30f_2 + 16f_3 - f_4}{12h^2} - f_2''$$

$$\Rightarrow \frac{h^4 f^{(6)}(x_2)}{90}$$