hmm

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Drawbacks of LR/Softmax

- Classification is non-linear
  - May not even represented as fixed-dimensional features
- Do not consider the relationship of labels within one data sample

The	lecture	is	really	boring
`determiner`	`?`	`verb`	`adverb`	`adjective`

Three	professors	lecture	IntroNLP
`CardinalNumber`	`Noun`	`?`	`ProperNoune`

https://www.merriam-webster.com/dictionary/lecture

lecture noun

lec·ture | \ ˈlek-chər , -shər\

Definition of *lecture* (Entry 1 of 2)

1 : a discourse given before an audience or c

2 : a formal reproof

lecture verb

lectured; **lecturing** \ ˈlek-chə-riŋ , ˈlek-shriŋ\

Definition of *lecture* (Entry 2 of 2)

intransitive verb

Motivation

- One data sample may have different labels, e.g.,
  - POS tagging
  - Parsing
  - Sentence generation
  - etc.

Markov Model

- Finite states $S = \{s_1, s_2, \cdots, s_n\}$
- You start from a state following the distribution $\boldsymbol{\pi} = [\pi_1, \pi_2, \cdots, \pi_n]$
- Transition only depends on the current state $\mathbb{P}[S^{(t)} = s_i \,|\, S^{(t-1)} = s_j]$

Example

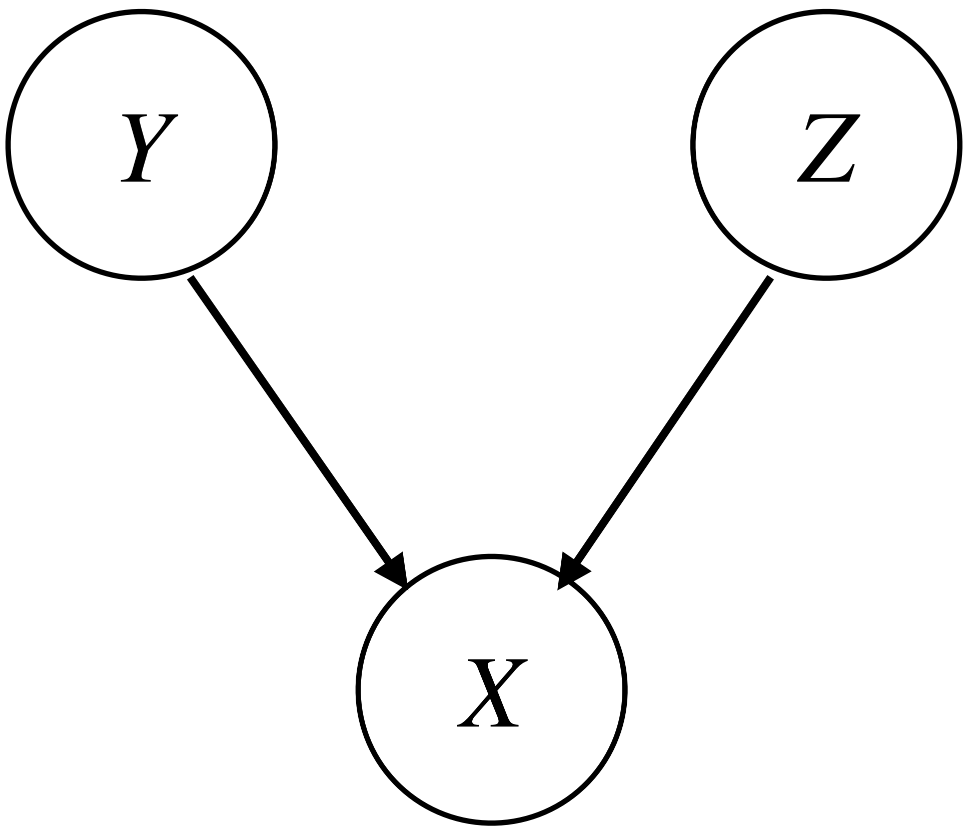
- Weather
- N-gram model

Bayesian Network in General

- Directed Acyclic Graph $G = \langle V, E \rangle$
  - Each node is a random variable
  - Each edge $a \to b$ represents that a is a direct "cause" of b
  - The joint probability can be represented as

$$p(x_1, \cdots, x_n) = \prod_{i=1}^{n} p(x_i \,|\, \underbrace{\mathrm{Par}(x_i)}_{\text{All parents}})$$

Is it correct?



$$p(X, Y, Z) = p(X \mid Y)p(X \mid Z)$$

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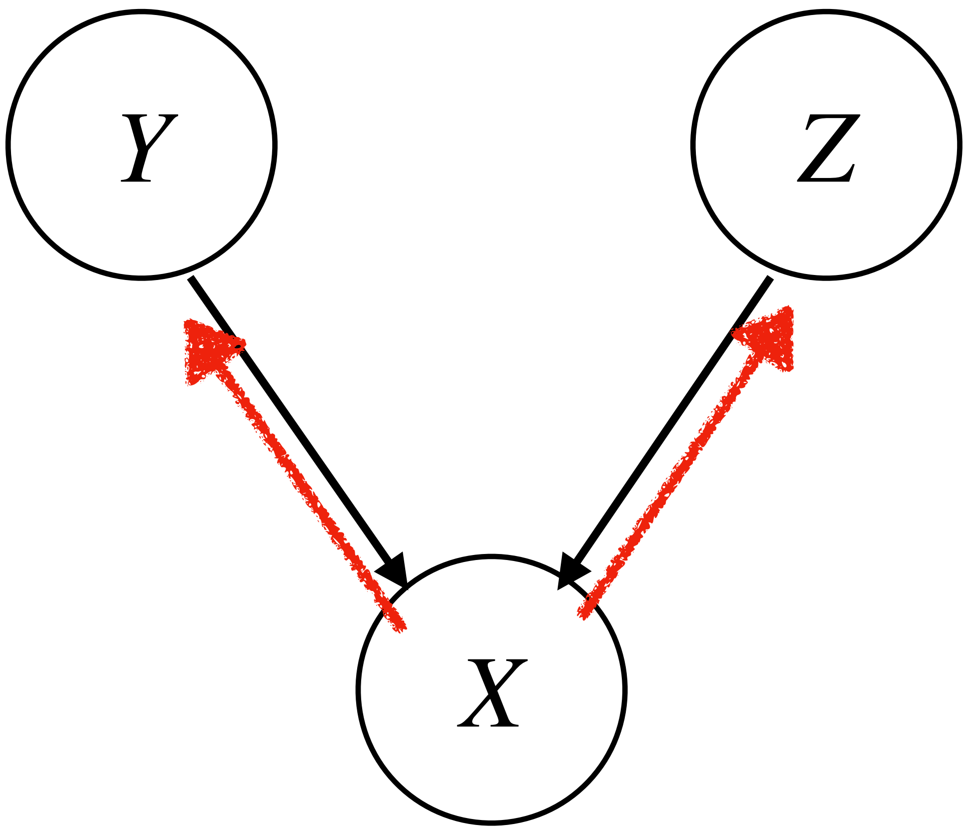
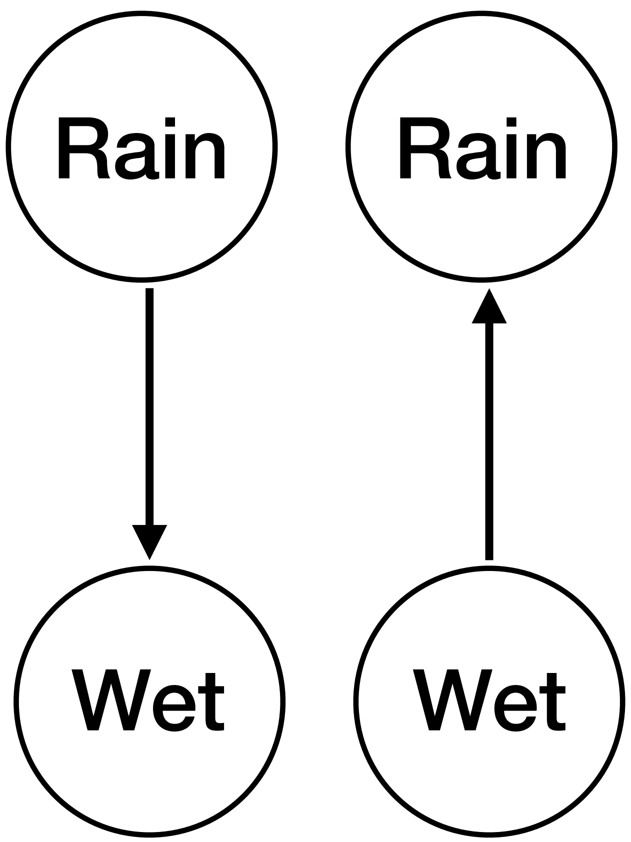
Can we reverse cause & effect?



Can we reverse cause & effect?



$$p(R, W) = p(R)p(W \mid R)$$

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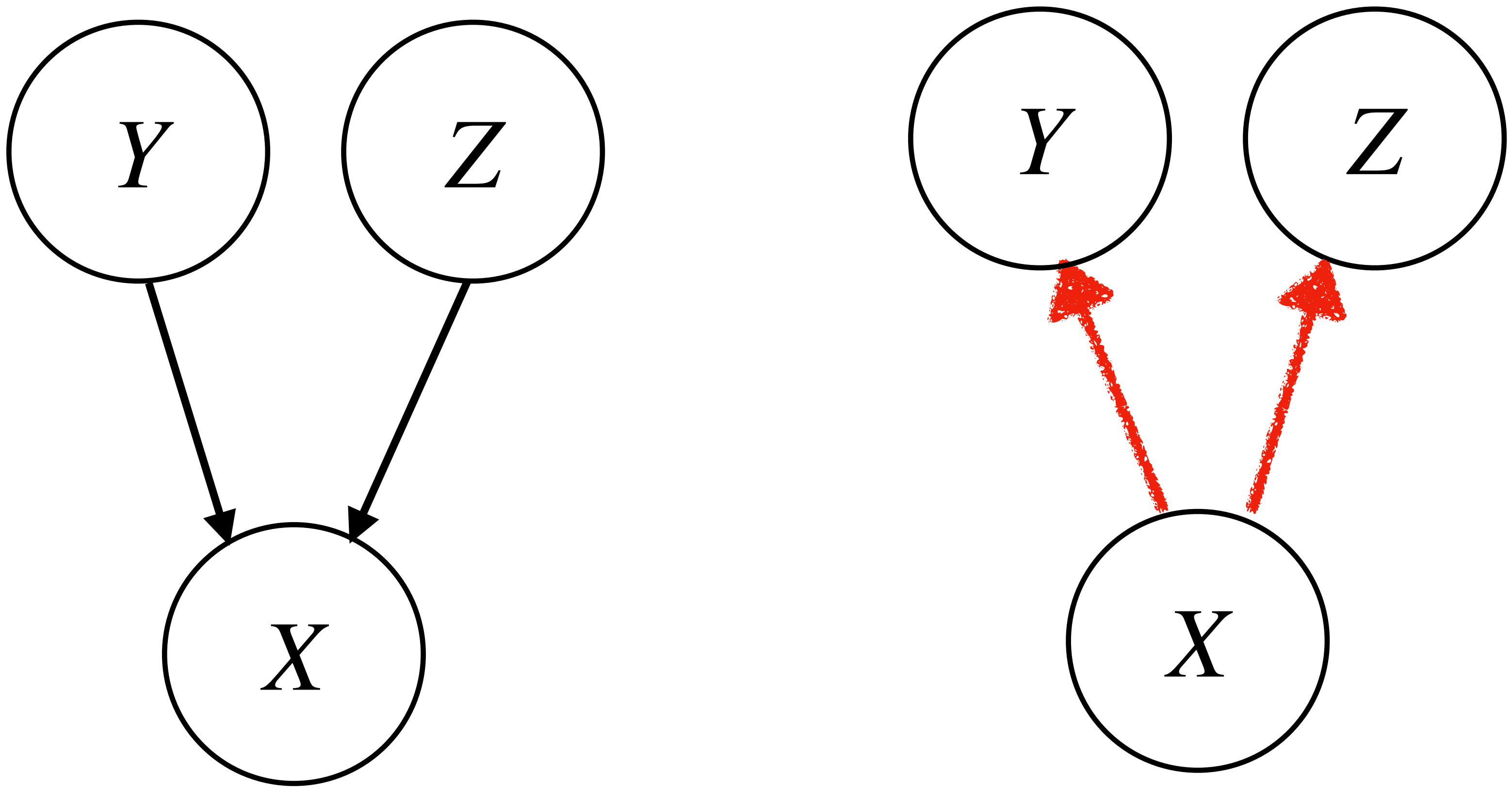
Can we reverse cause & effect?



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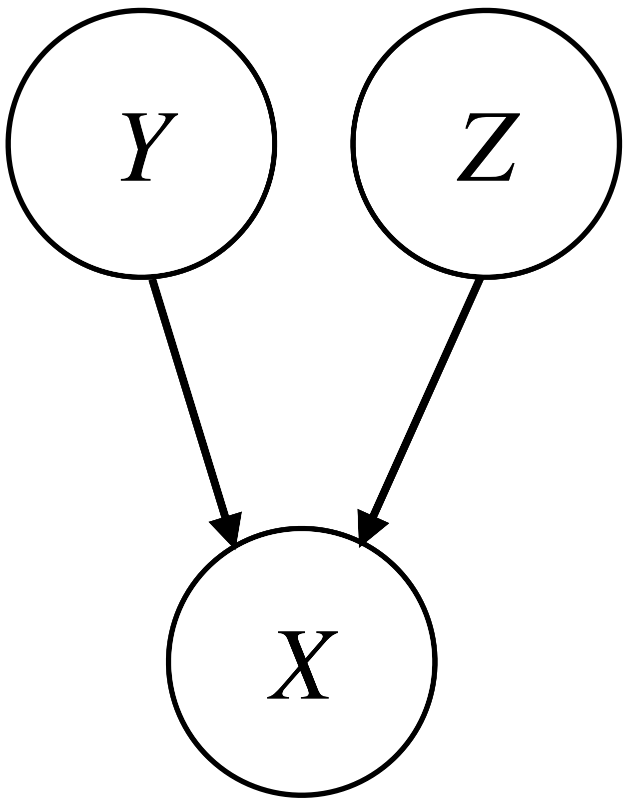
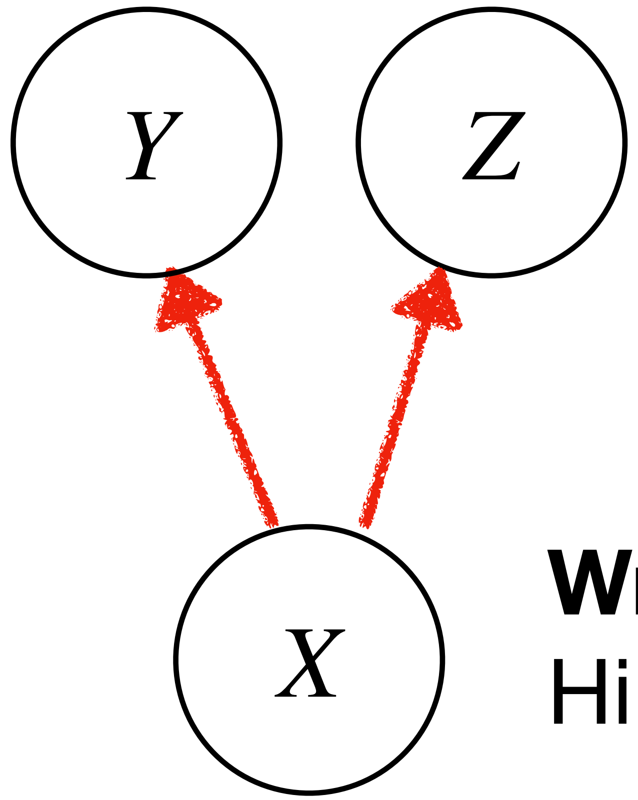




$Y \perp Z | X$

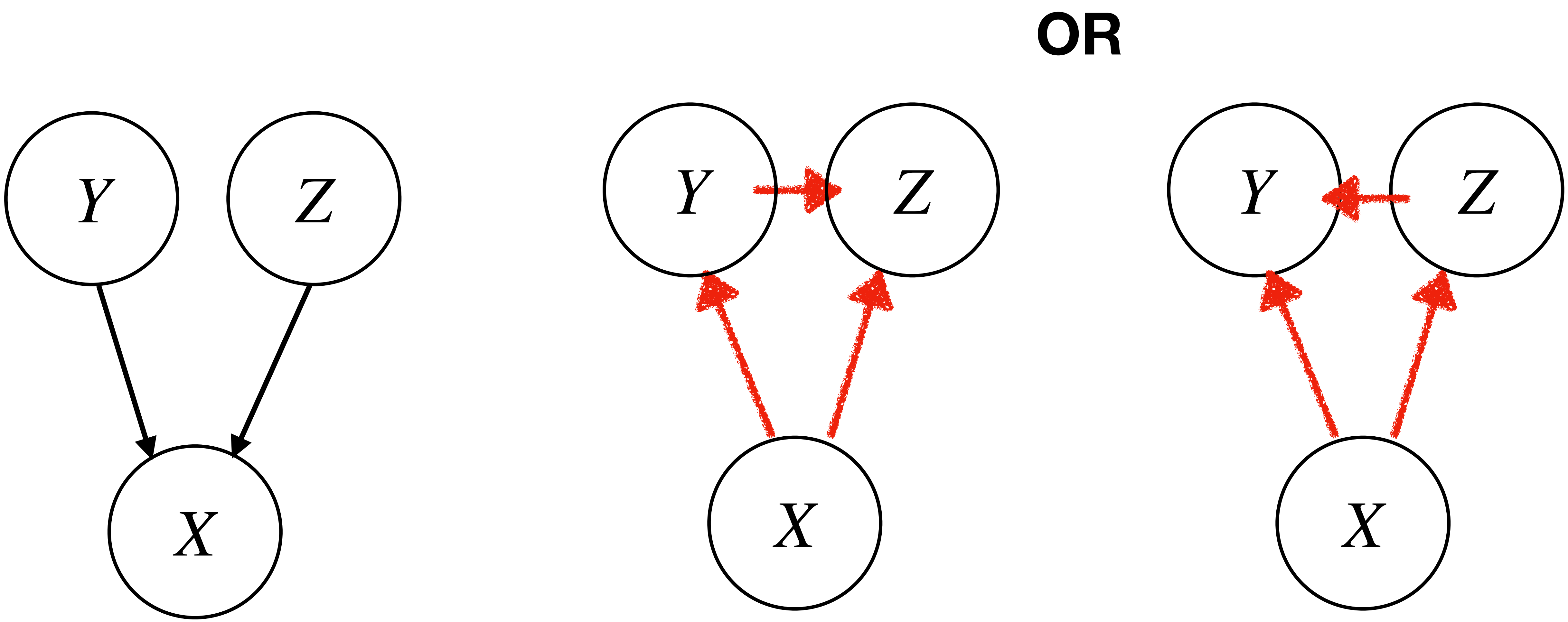
Written assignment: Prove.
Hint: By definition.

By the property of BNs,
$Y \perp Z | X. \implies 0$ mark

$Y \perp Z | X$

does not hold in general

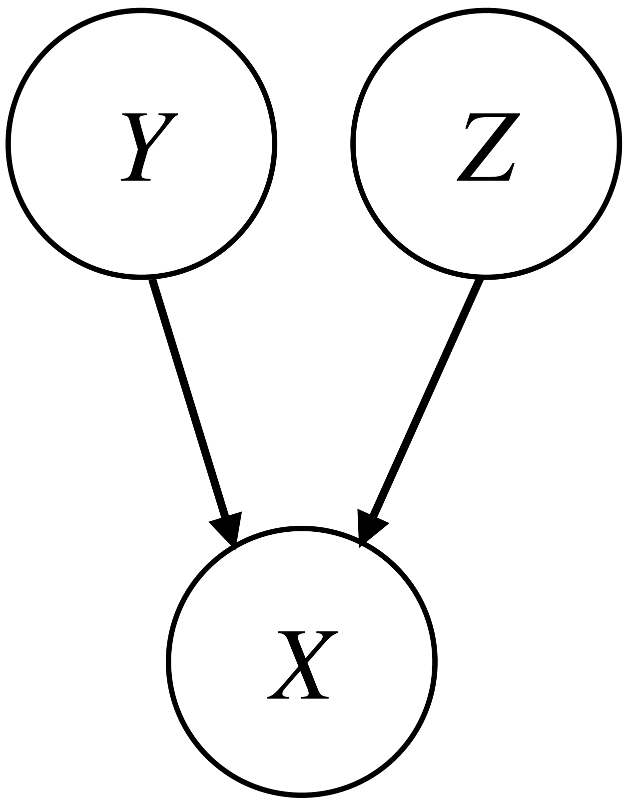
Can we reverse cause & effect?

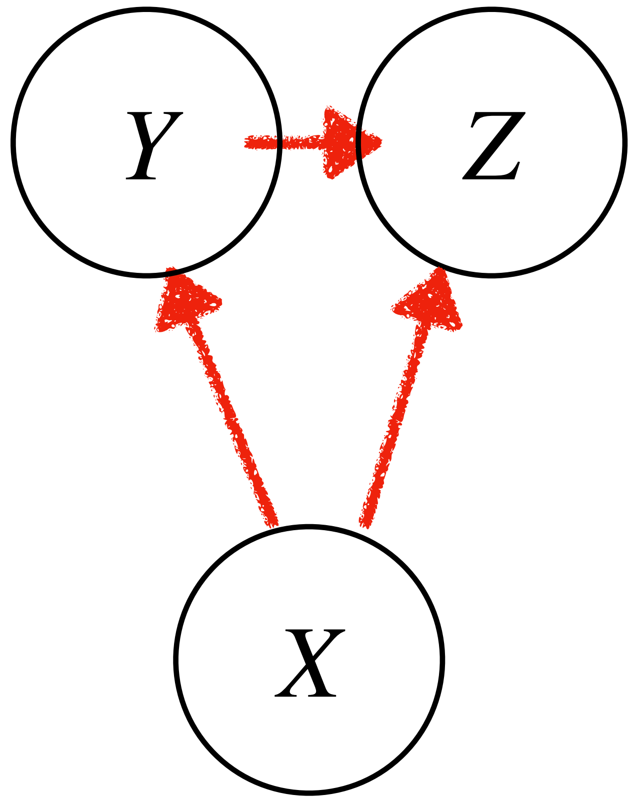


Cause and effect cannot be formally defined.

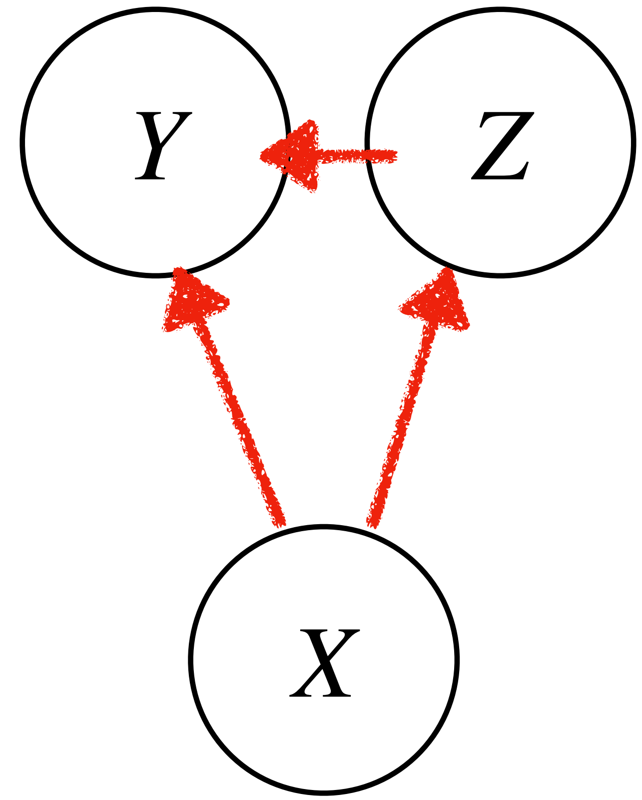
- In BN, "→" refers to conditional probability
- In logics, "→" refers to entailment

Can we reverse cause & effect?





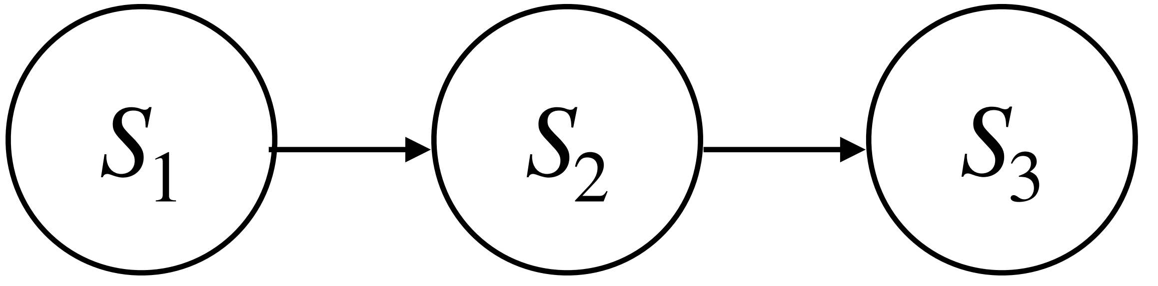
OR



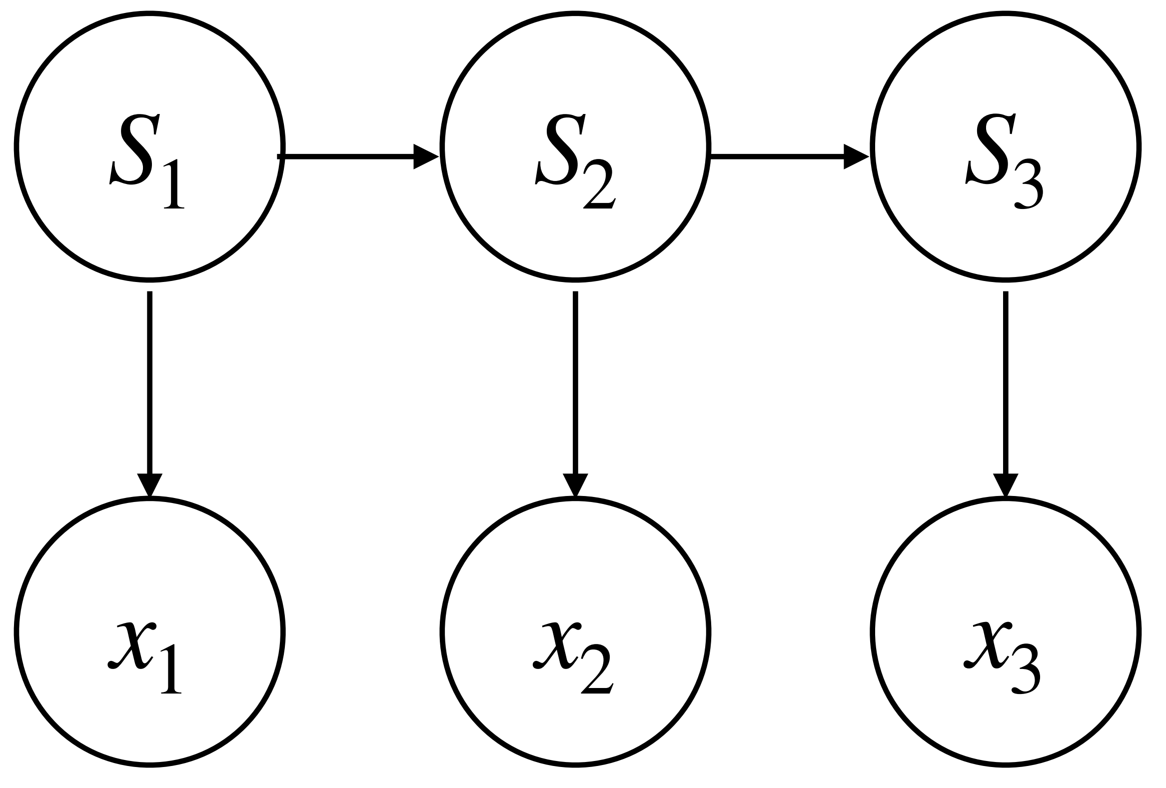
Cause and effect cannot be formally defined.

But with our intuition of cause and effect, we can simply our model.

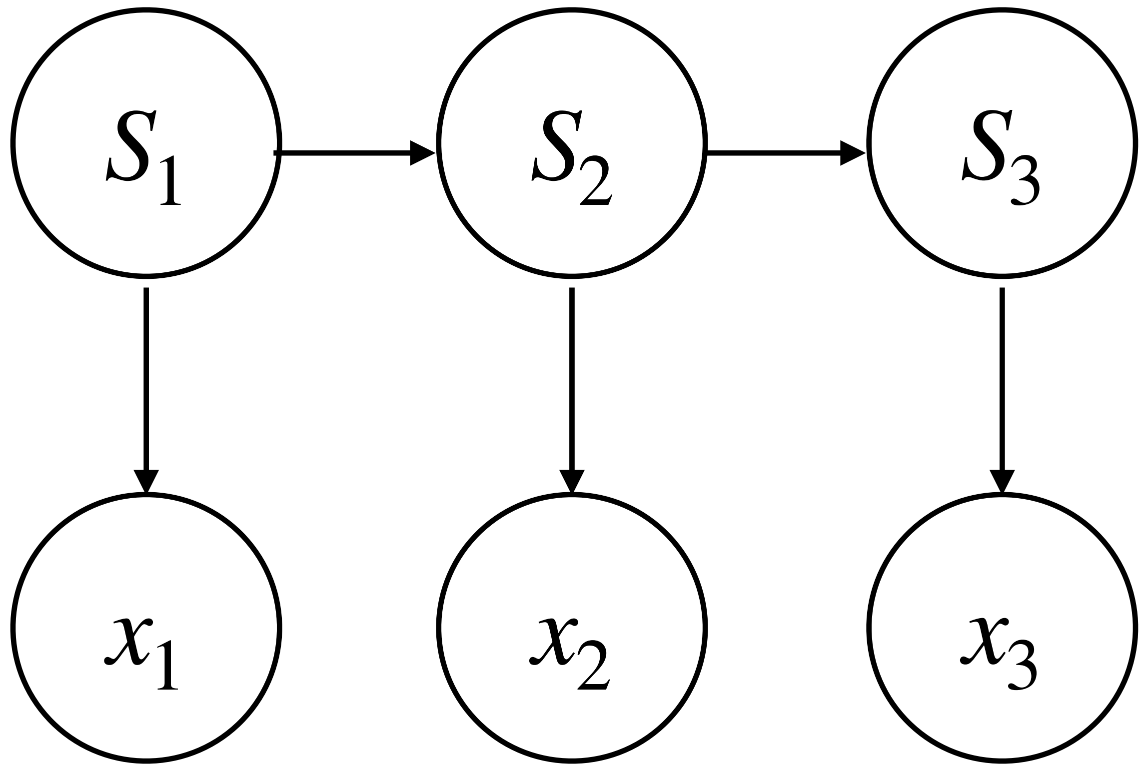
Markov Model



Hidden Markov Model



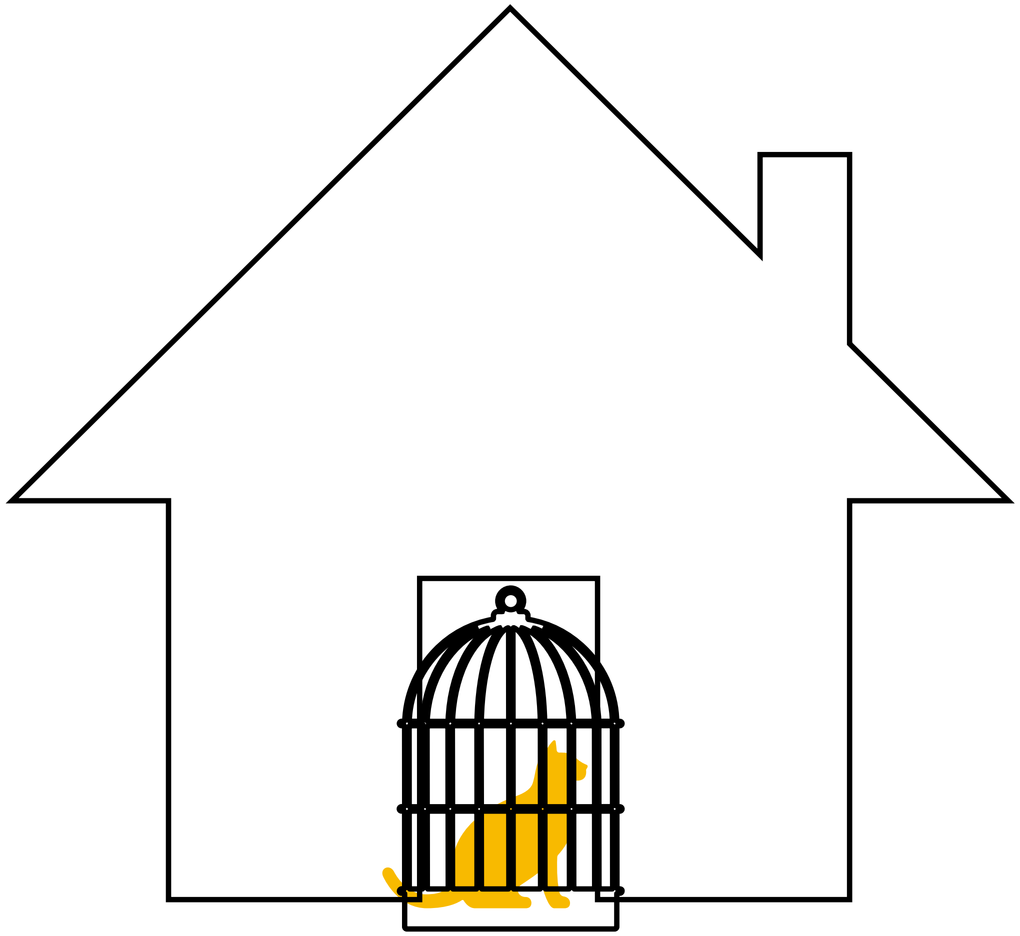
Hidden Markov Model

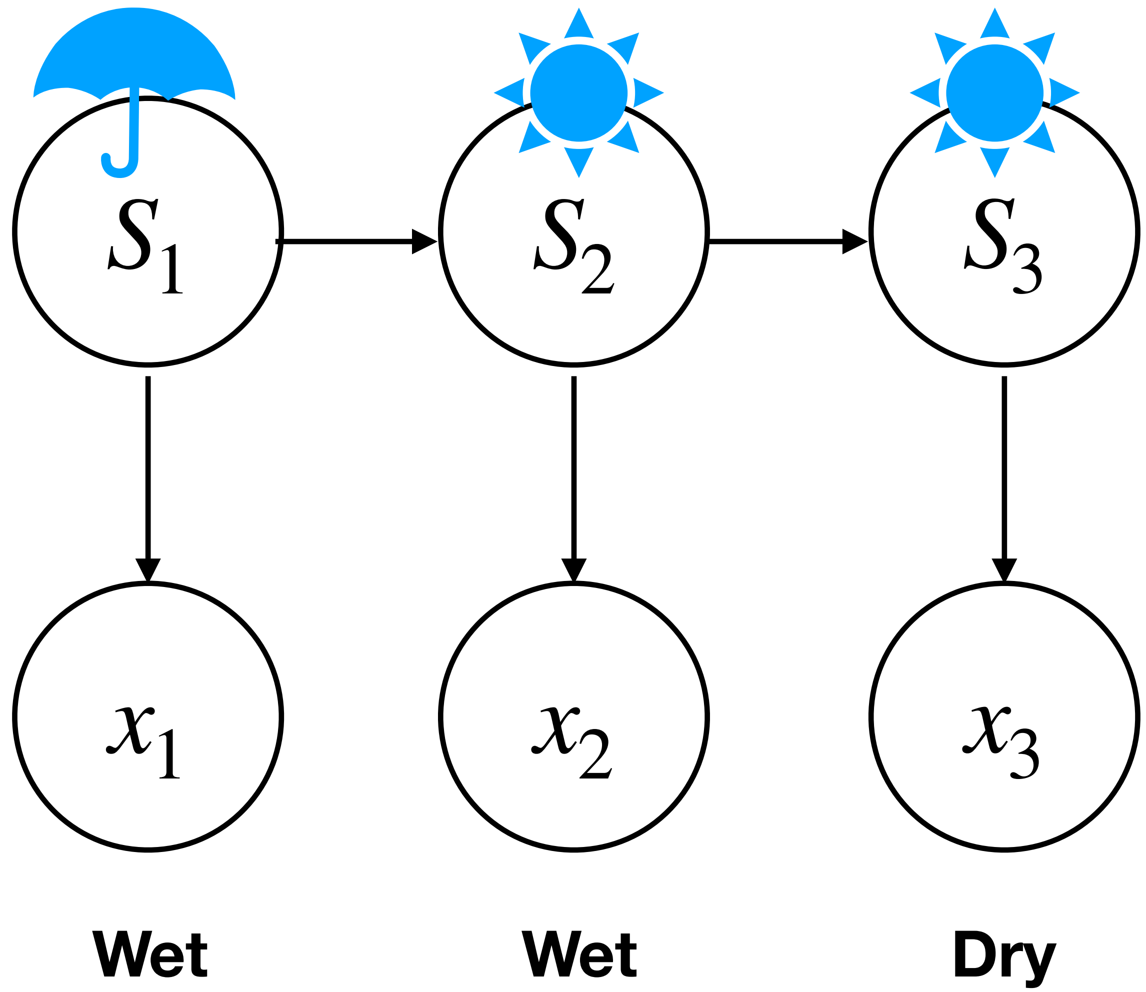


$$p(s_1, \cdots, s_T, x_1, \cdots x_T) = p(s_1) \prod_{t=2}^{T} p(s_t \mid s_{t-1}) \prod_{t=1}^{T} p(x_t \mid s_t)$$

Initial State Prob.	Transition Prob.	Emission Prob.
n	n^2	$v \cdot n$

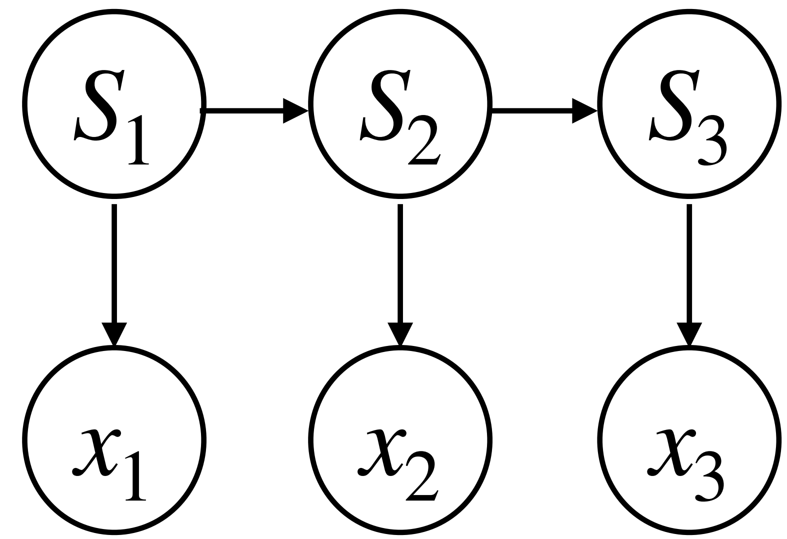
Example





Maximum Likelihood Estimation

- Training if fully observable
  - E.g., annotated by experts



$$p(s_1, \cdots, s_T, x_1, \cdots x_T) = p(s_1) \prod_{t=2}^{T} p(s_t \,|\, s_{t-1} \prod_{t=1}^{T} p(x_t \,|\, s_t)$$

$$\log p(\,\cdot\,) = \boxed{\log p(s_1)} + \boxed{\sum_{t=2}^{T} \log p(s_t \,|\, s_{t-1})} + \boxed{\sum_{t=1}^{T} \log p(x_t \,|\, s_t)}$$

Parameters factorize

MLE for Multinomial Distribution

- Counting
  - With one constraint $\pi_1 + \cdots \pi_n = 1$
  - You need to explicitly represent $\pi_n = 1 - \pi_1 - \cdots - \pi_{n-1}$
  - Or, you apply the Lagrangian multiplier method

$$\log p(\cdot) = \boxed{\log p(s_1)} + \sum_{t=2}^{T} \log p(s_t \mid s_{t-1}) + \sum_{t=1}^{T} \log p(x_t \mid s_t)$$

$$\pi_i = \frac{\sum_{i=1}^{M} \mathbb{I}\{S_1 = i\}}{M} = \frac{\text{\# of samples that start with stae } i}{\text{\# of all samples}}$$

MLE for Multinomial Distribution

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Written assignment

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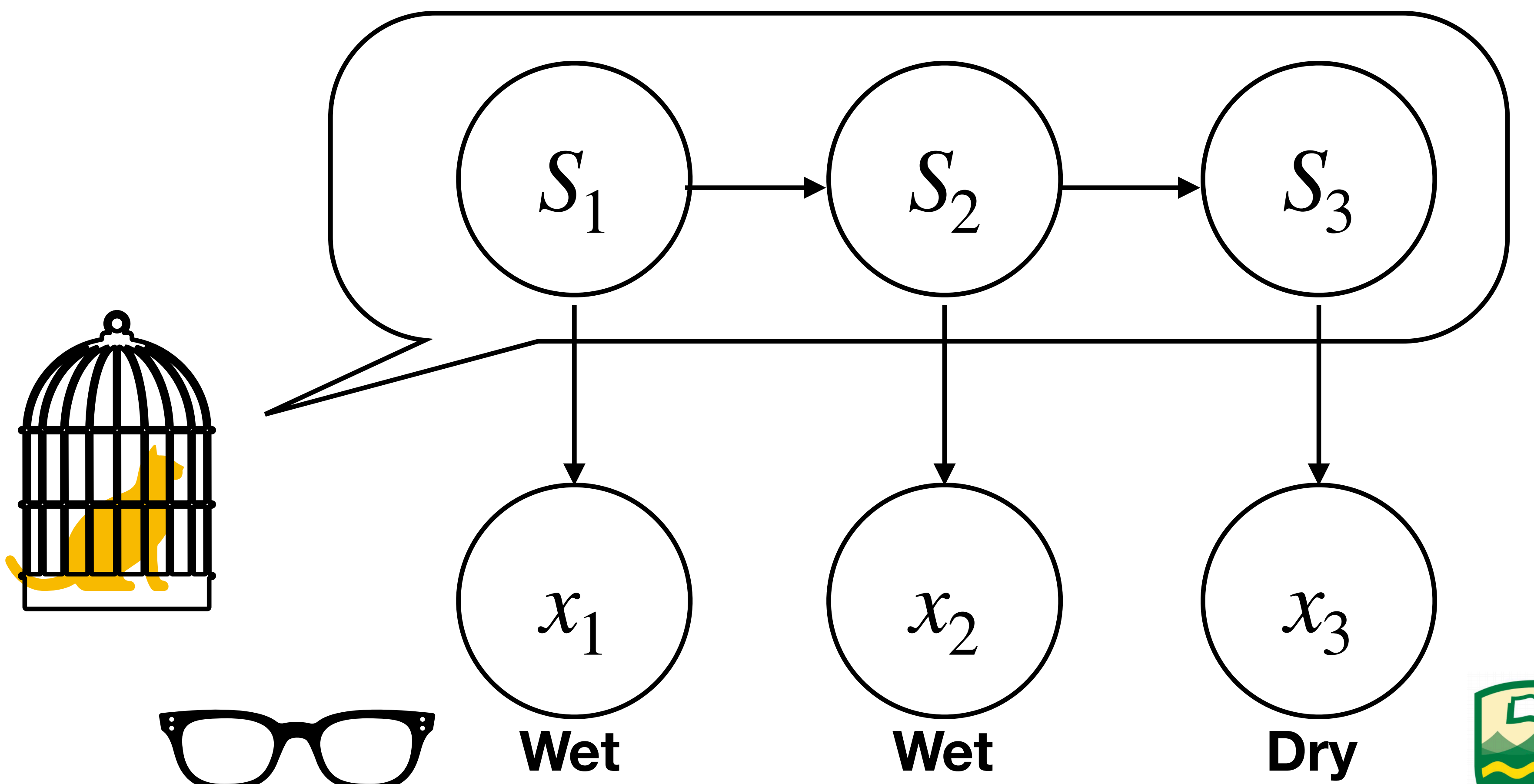
Inference

- Suppose the model is full trained
- During prediction, we observe $x_1, \cdots, x_T$
  - How can we know the states $s_1, \cdots, s_T$ that best explain $x_1, \cdots, x_T$?

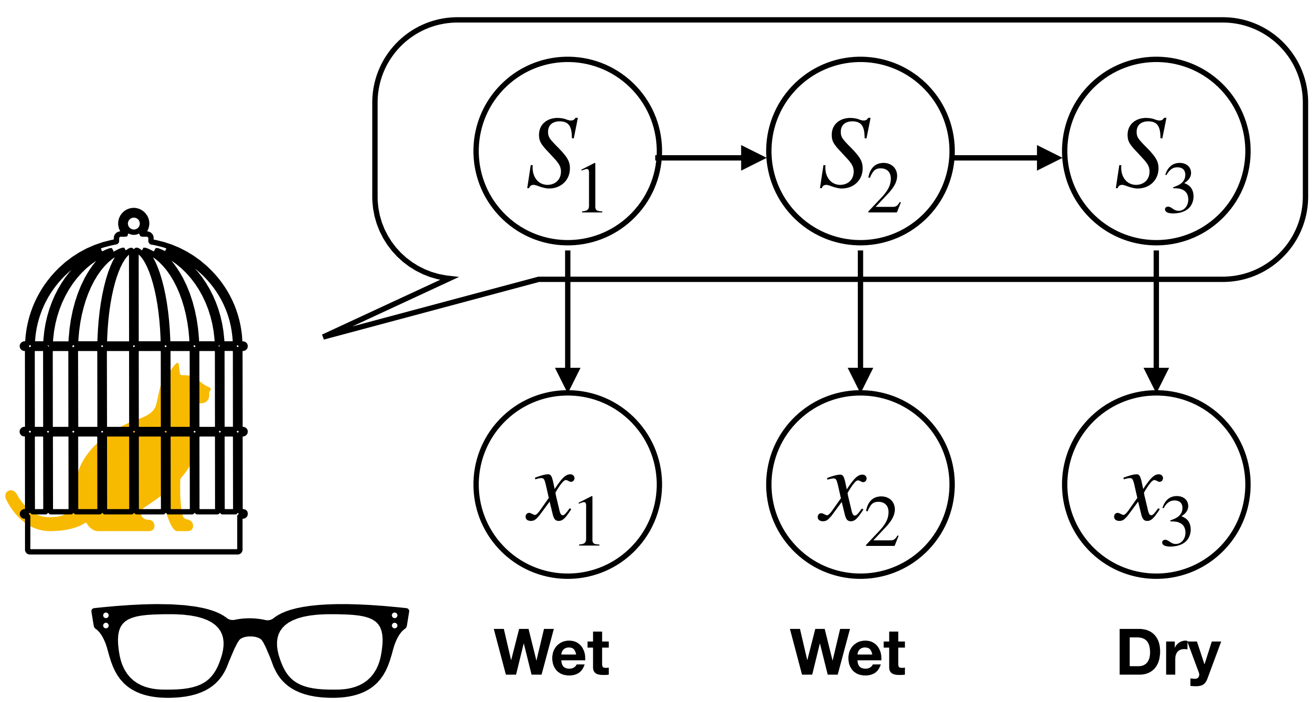


Inference Criteria

- We would like to predict the best (most probable) states
- Max *a posteriori* inference

$$s_1, \cdots, s_T = \underset{s_1,\cdots,s_T}{\operatorname{argmax}}\, p(s_1, \cdots, s_T \,|\, x_1, \cdots, x_T)$$

$$= \underset{s_1,\cdots,s_T}{\operatorname{argmax}}\, p(s_1, \cdots, s_T, x_1, \cdots, x_T)$$



Simplified notation may be used:

$$x_{1:t}, \quad x_1^t$$

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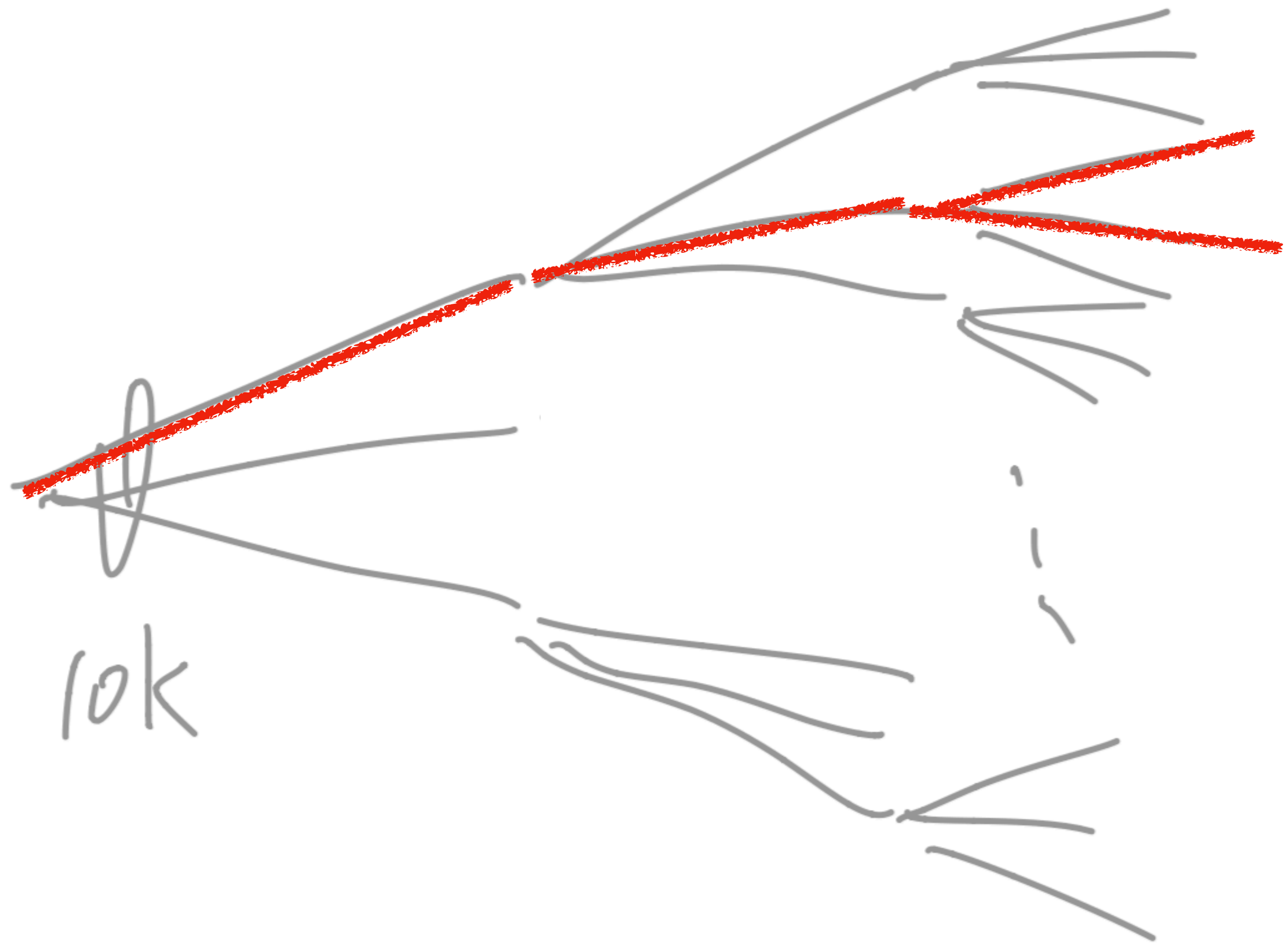
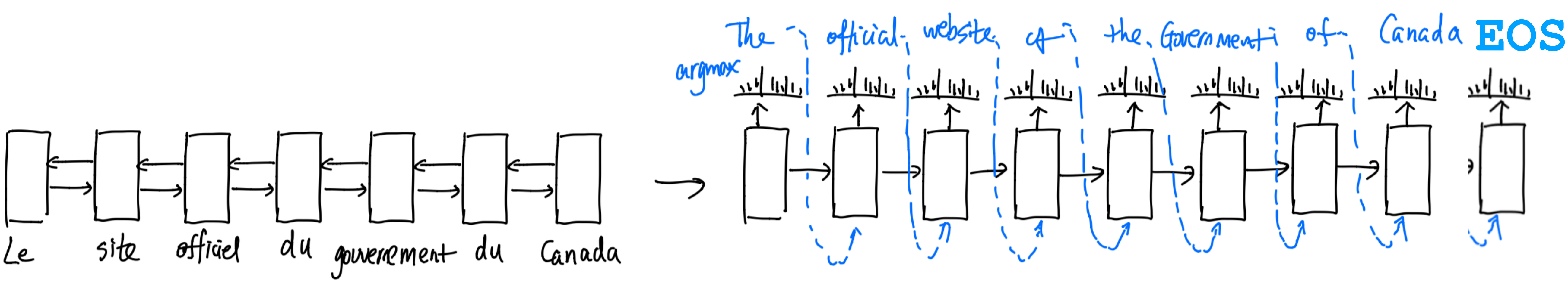
Recall Beam Search

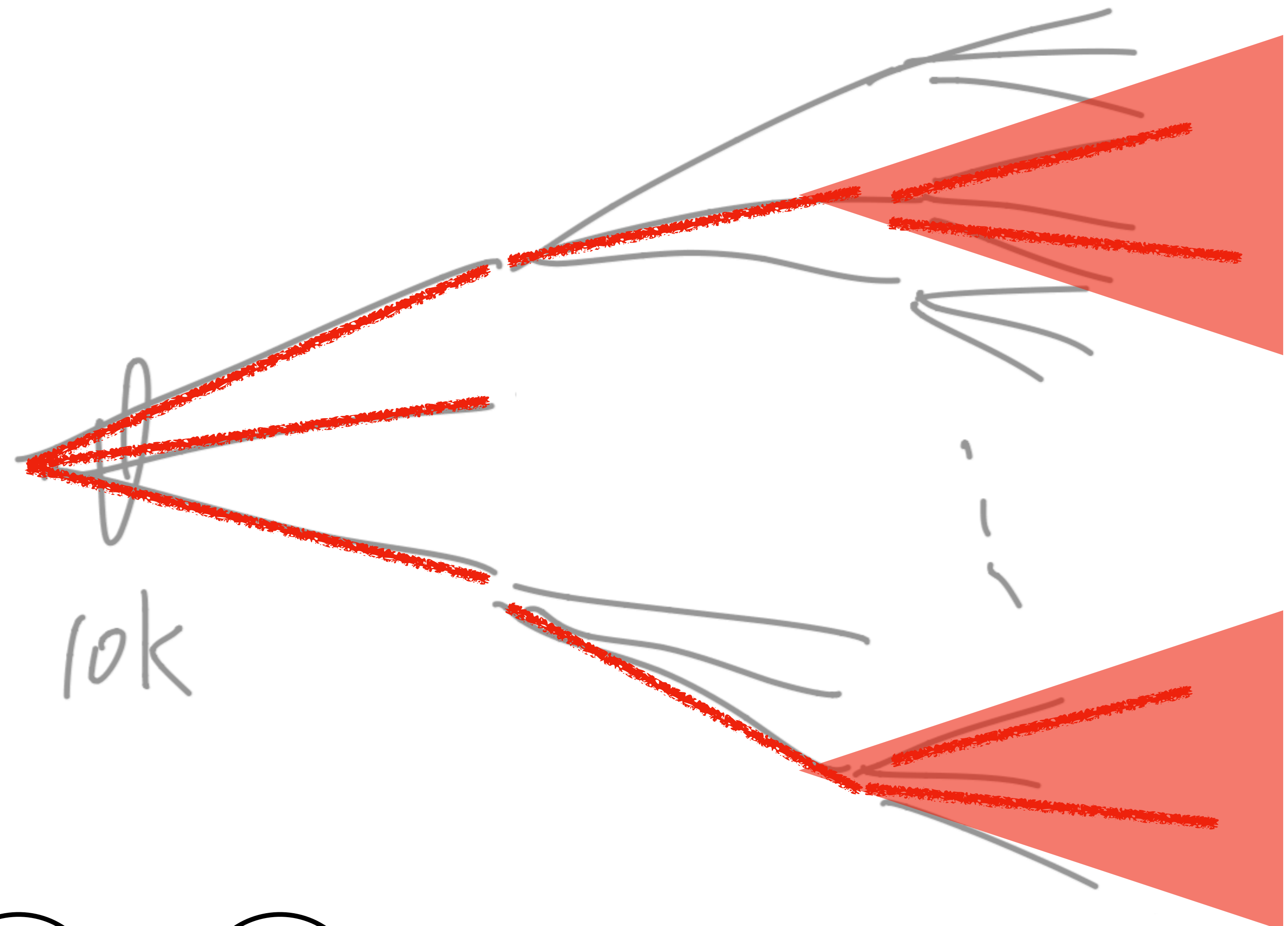
B=2

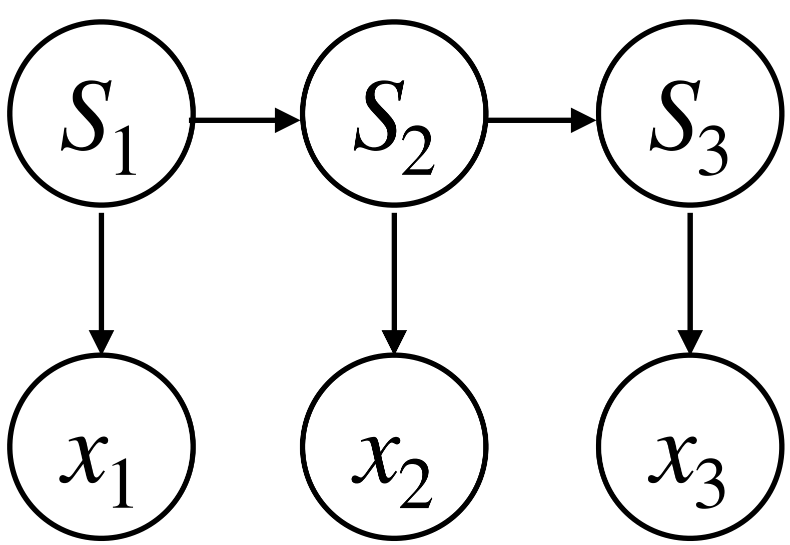




Search in HMM

Some sub-structures are shared in different paths





Markov Blanket

$$p(s_{1:T}, x_{1:T}) = \prod_{i=1}^{n} \left[\boxed{p(s_i \mid s_{i-1})} \boxed{p(x_i \mid s_i)} \right]$$

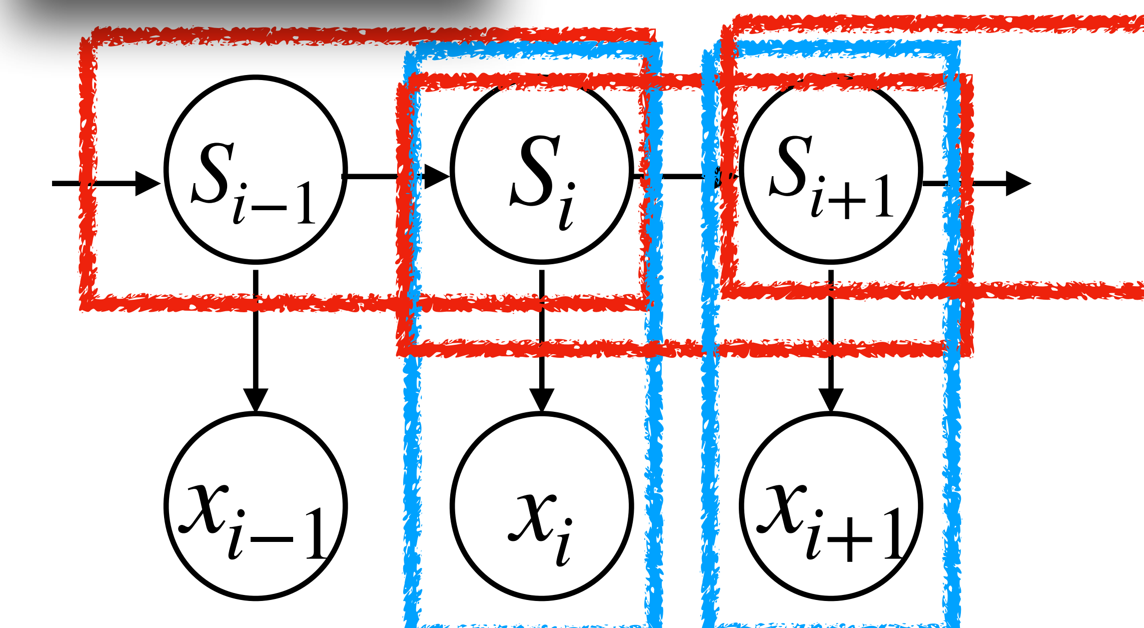
For simplicity, the first state's probability is denoted as

$$\mathbb{P}[s_1] \triangleq p(s_1 \mid s_0)$$

Key observation:

Factorized probability is local.

- $s_{i:T}, x_{i:T}$ only depends on s_{i-1}
- but not $s_{\le i-2}, x_{\le i-1}$



Recursion Variable

$$s_1, \cdots, s_T = \underset{s_1, \cdots, s_T}{\operatorname{argmax}} \, p(s_1, \cdots, s_T, x_1, \cdots, x_T)$$

$$p(s_{1:T}, x_{1:T}) = \prod_{i=1}^{n} \left[p(s_i \,|\, s_{i-1}) p(x_i \,|\, s_i) \right]$$

- Attempt#1: $\operatorname{argmax}_{s_{1:t}} p(x_1, \cdots, x_i, s_t)$ with the best s_t
  - But best choice for every step $\neq$ best choice globally

Recursion Variable

$$s_1, \cdots, s_T = \underset{s_1,\cdots,s_T}{\text{argmax}}\, p(s_1, \cdots, s_T, x_1, \cdots, x_T)$$

$$p(s_{1:T}, x_{1:T}) = \prod_{i=1}^{n} \left[p(s_i \mid s_{i-1}) p(x_i \mid s_i) \right]$$

- Attempt#1: $\max_{s_{1:t}} p(x_1, \cdots, x_i, s_t)$ with the best s_t
  - But best choice for every step $\neq$ best choice globally
- Attempt#2: $\max_{s_{1:t-1}} p(x_1, \cdots, x_t, s_t)$, for s_t being any state

$$M[t][j] \triangleq \max_{s_{1:t-1}} p(x_{1:t}, S_t = j)$$

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Dynamic Programming

$$M[t][j] \triangleq \max_{1:t-1} p(x_{1:t}, S_t = j)$$

Initialization

$$\begin{aligned} M[1][j] &= \max_{\varnothing} p(x_1, S_1 = j) && \text{[nothing to choose for "max"]} \\ &= p(x_1, S_1 = j) \\ &= p(S_1 = j) p(x_1 \mid S_1 = j) \\ &= \pi_j \cdot p(x_1 \mid s_1 = j) && \text{[both are model parameters]} \end{aligned}$$

Dynamic Programming

$$M[t][j] \triangleq \max_{1:t-1} p(x_{1:t}, S_t = j)$$

Recursion Step

- Assume $M[t-1][j] = \max_{s_{1:t-2}} p(x_{1:t-1}, S_{t-1} = j)$ known $(\forall j)$
- Goal: Figure out $M[t][j]$

$$M[t][j] = \max_{s_{1:t-1}} p(x_1, \cdots, x_t, S_t = j)$$

$$= \max_{s_{1:t-1}} p(x_1, \cdots, x_{t-1}, s_{t-1}) p(s_t = j \mid s_{t-1}) p(x_t \mid s_j)$$

$$= \max_{s_t} \boxed{\max_{s_{1:t-2}} p(x_1, \cdots, x_{t-1}, s_{t-1})} p(s_t = j \mid s_{t-1}) p(x_t \mid s_j)$$

Known by recursion assumption $M[t-1][s_t]$

Dynamic Programming

$$M[t][j] \triangleq \max_{1:t-1} p(x_{1:t}, S_t = j)$$

Recursion Step

- Assume $M[t-1][j] = \max_{s_{1:t-2}} p(x_{1:t-1}, S_{t-1} = j)$ known
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$$M[t][j] = \max_{s_{1:t-1}} p(x_1, \cdots, x_t, S_t = j)$$

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Known by recursion assumption $M[t-1][s_t]$

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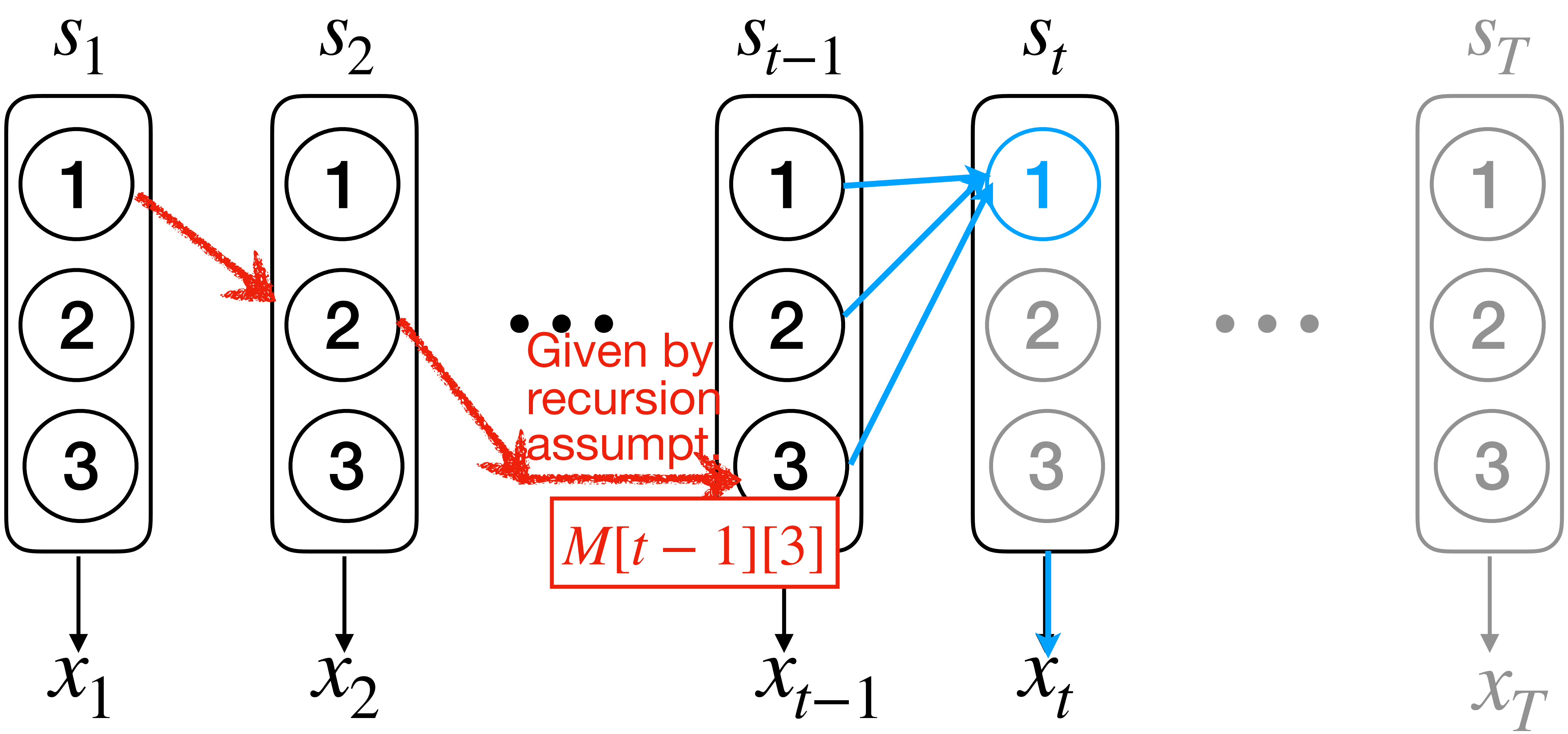
Illustration

$$M[t][j] \triangleq \max_{1:t-1} p(x_{1:t}, S_t = j)$$

Recursion Step

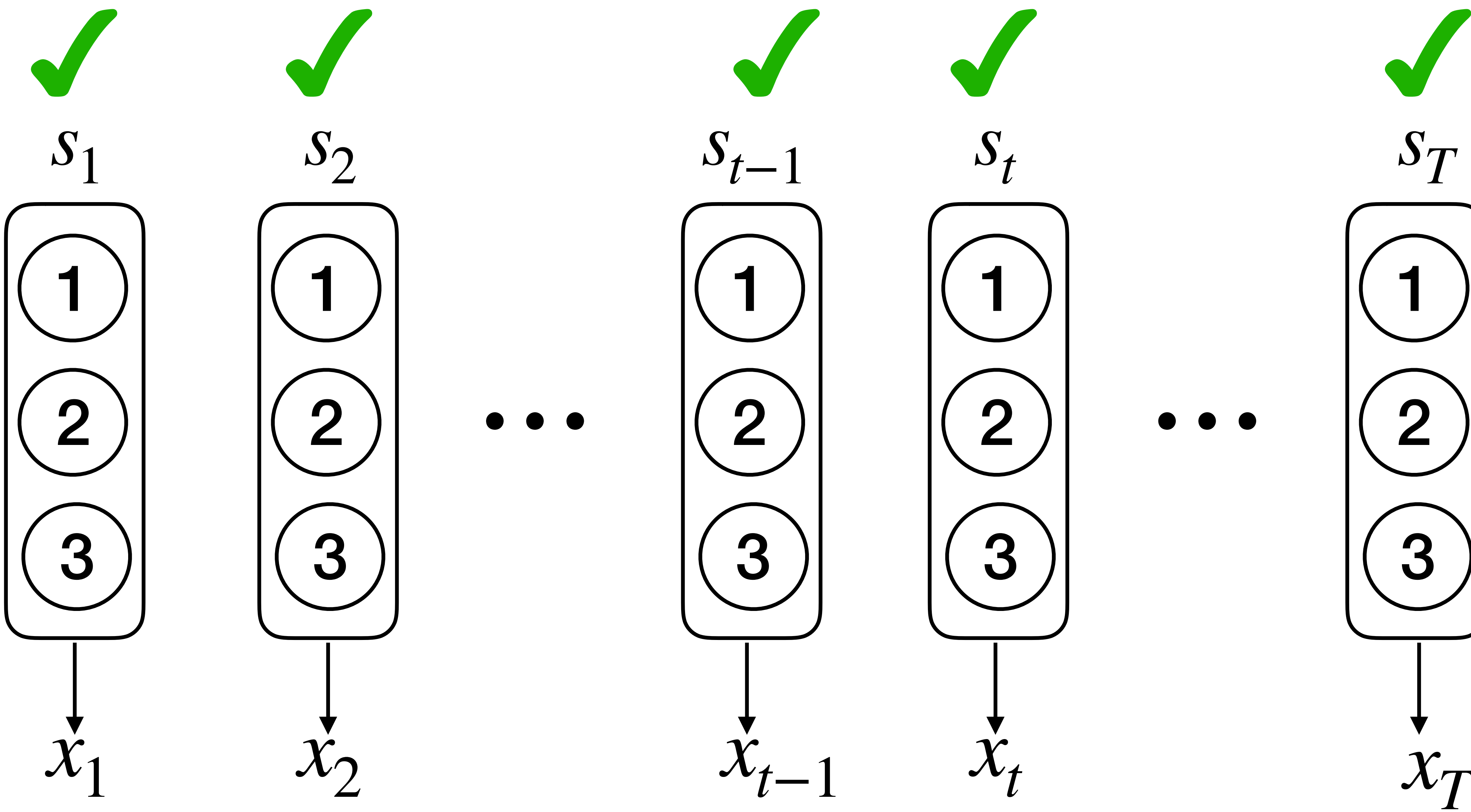
- Assume $M[t-1][j] = \max_{s_{1:t-2}} p(x_{1:t-1}, S_{t-1} = j)$ known
- Goal: Figure out $M[t][j]$ $(\forall j)$



$M[t][j] =$ **max**$_{s_{t-1}}$ **{** $\rightarrow \rightarrow \rightarrow \quad \nearrow \downarrow$ **}**

Dynamic Programming

Termination: $M[T][j]$ is done $(\forall j)$



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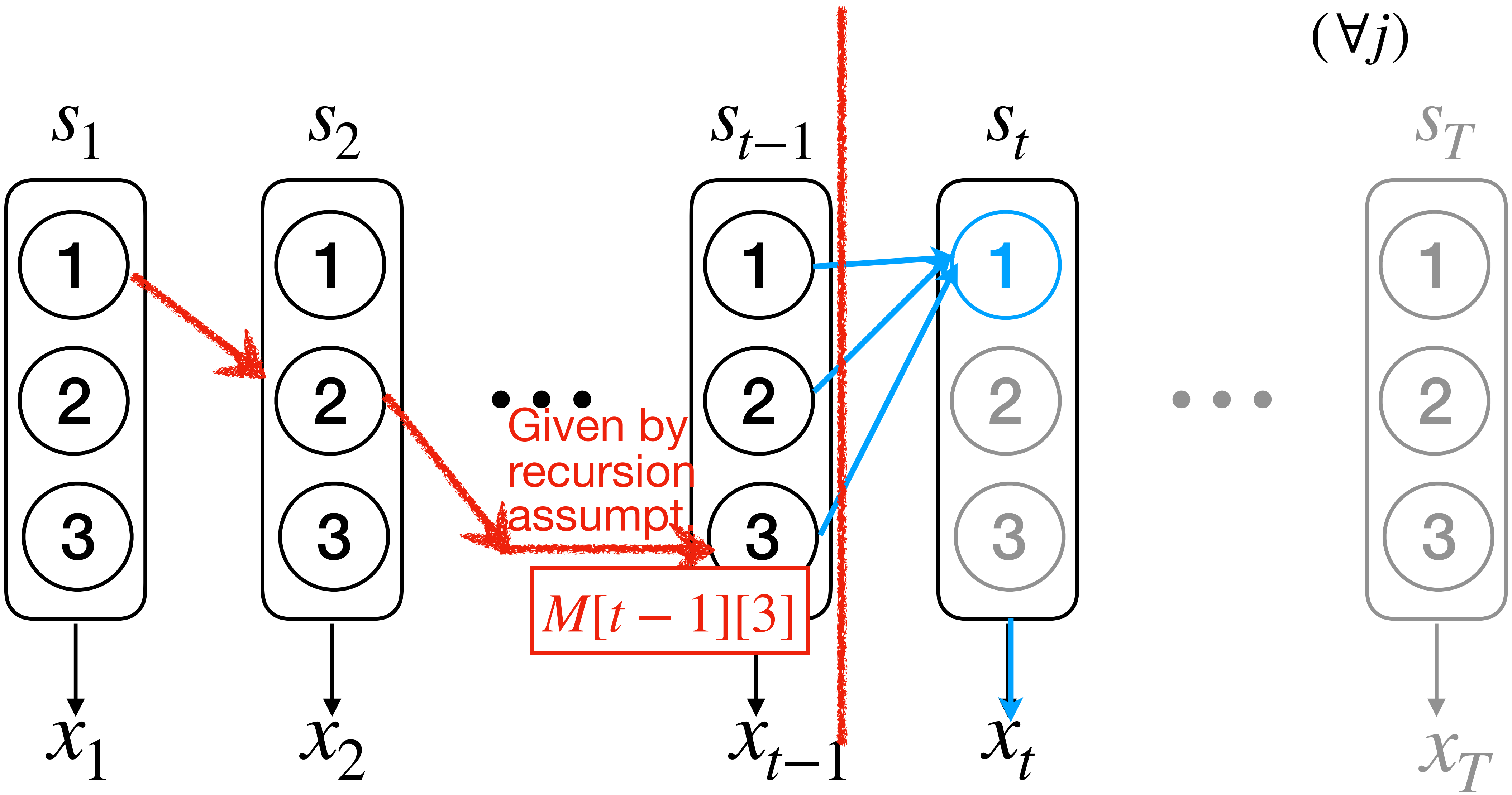
Backtracking the States

$$M[t][j] \triangleq \max_{1:t-1} p(x_{1:t}, S_t = j)$$

$$B[t][j] = \operatorname{argmax}_i \{M[t-1][i] \cdot P(S_t = j \mid S_{t-1} = i) \cdot P(x_t \mid S_j)\}$$

$(\forall j)$



$M[t][j] = \max_{s_{t-1}} \{ \rightarrow \rightarrow \rightarrow \nearrow \downarrow \}$

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Written Assignment

- Suppose an HMM is given
  - States $S = \{1, \cdots, n\}$
  - Parameters π_j, $P(S_{t-1} = j \mid S_t = i), P(x_t \mid S_t = j)$ known
- Suppose an HMM is given
  - We would like to find the state and output sequences of length T that have the highest jointly probability

$$s_{1:T}, x_{1:T} = \underset{s_{1:T}, x_{1:T}}{\operatorname{argmax}}\, p(s_{1:T}, x_{1:T})$$

  - Think of the problem $x_{1:T} = \operatorname{argmax}_{x_{1:T}} p(x_{1:T})$

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  - States $S = \{1, \cdots, n\}$
  - Parameters π_j, $P(S_{t-1} = j \mid S_t = i)$, $P(x_t \mid S_t = j)$ known
- Goal
  - To find the state and output sequences of length T that have the highest jointly probability

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  - Think of the problem $x_{1:T} = \operatorname{argmax}_{x_{1:T}} p(x_{1:T})$ [optional]

Written Assignment

- Requirements
  - Design a DP algorithm, stating the initialization, recursion, and termination of the algorithm

    (don't forget backpointers)
  - For any recursion variable, a clear definition is needed
  - The recursion step should be supported by derivation
  - Given pseudo code that generates $s_{1:T}, x_{1:T}$

Written Assignments

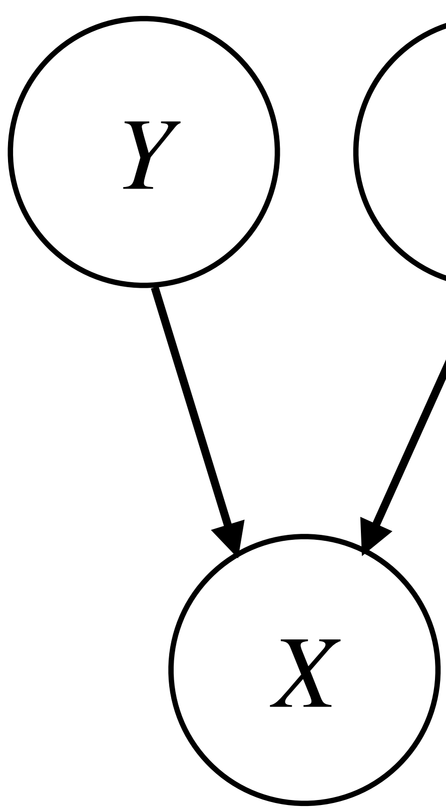
- Every week, we solve problems that have been mentioned in Monday's and Wednesday's lectures.
- Every assignment is due on next Monday
- Automatically extended to next Wednesday [**before class**]
- Further extensions require good reasons (self-approved extension may result in 0 mark).

Problem 1

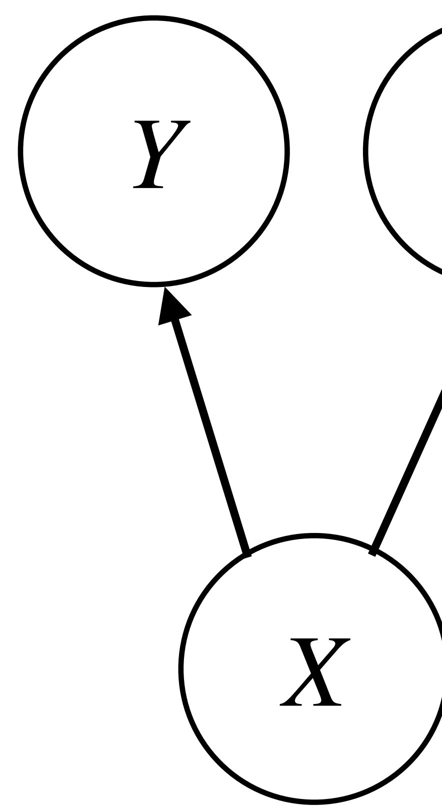
Show that $Y \perp Z | X$ does not hold in general for BN (1), but $Y \perp Z | X$ must be true for BN (2).

Note: If your solution involves showing some example, please provide your own example.

(1)



(2)



Problem 2

Give the MLE estimation for HMM transition and emission probabilities

- Figure out what are the parameters
- Give the formula to estimate these parameters (either by indicator functions or natural language expressions)

It's strongly recommended to derive MLE for multinomial distributions, but is optional for this assignment.

$$\log p(\cdot) = \log p(s_1) + \boxed{\sum_{t=2}^{T} \log p(s_t \mid s_{t-1})} + \boxed{\sum_{t=1}^{T} \log p(x_t \mid s_t)}$$

$$\pi_i = \frac{\sum_{i=1}^{M} \mathbb{I}\{S_1 = i\}}{M} = \frac{\text{\# of samples that start with state } i}{\text{\# of all samples}}$$

Problem 3

- Suppose an HMM is given
  - States $S = \{1, \cdots, n\}$
  - Parameters π_j, $P(S_{t-1} = j \mid S_t = i)$, $P(x_t \mid S_t = j)$ known
- Goal
  - To find the state and output sequences of length T that have the highest jointly probability

$$s_{1:T}, x_{1:T} = \operatorname*{argmax}_{s_{1:T}, x_{1:T}} p(s_{1:T}, x_{1:T})$$

  - Think of the problem $x_{1:T} = \operatorname{argmax}_{x_{1:T}} p(x_{1:T})$ [optional]

Problem 3

- Requirements
  - Design a DP algorithm, stating the initialization, recursion, and termination of the algorithm

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Thank you!

Q&A