Uncertainty Quantification for Interpretable Machine Learning

– For Trustworthy Discoveries & Decision-making

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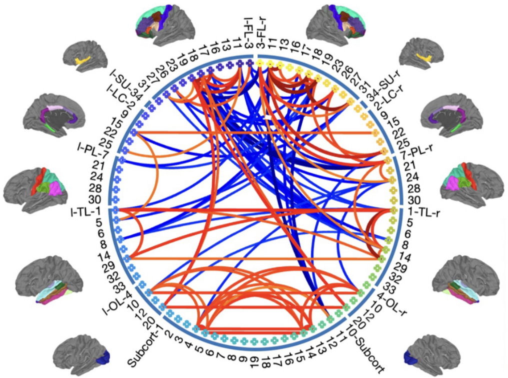
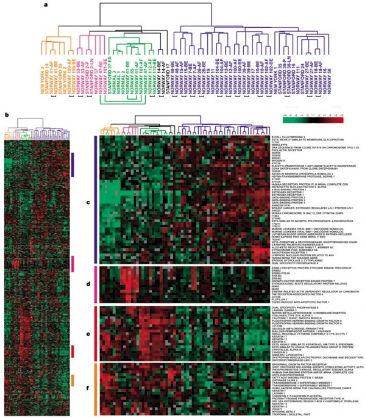
Background

Machine Learning Propels Discoveries



Association between brain regions from fMRI data



Hierarchical clustering for tumor data (Perou et al., 2000)

Machine Learning Propels Decision-making

Treatment in healthcare

Loan approval

Picture source: https://www.aamc.org/news/electronic-health-records-what-will-it-take-make-them-work;

https://auto.economictimes.indiatimes.com/news/auto-technology/us-lawmakers-raise-concerns-over-chinese-self-driving-testing-data-collection/105283633

Interpretable Machine Learning (IML)

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Generate human-understandable insights into the data, the ML model, or the model output

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Interpretable Machine Learning

Generate human-understandable insights into the data, the ML model, or the model output

- Insights into the data: functional association between brain regions; which treatment is more effective?
- Insights into the model: model diagnostics; safety check

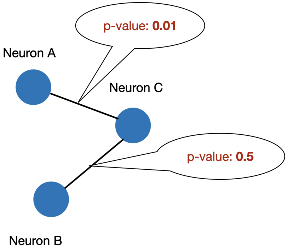
$\Rightarrow$ **scientific discoveries, decision-making**

Can we trust interpretable machine learning for discoveries and decision-making?

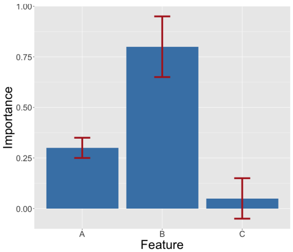
Trust in IML?

One Potential Solution

Provide uncertainty quantification (UQ) associated with machine learning interpretations!



p-values for detected association



Confidence intervals for feature importance

– draw conclusions/make decisions only based on *significant signals.*

Uncertainty Quantification: Challenges in the Modern Era

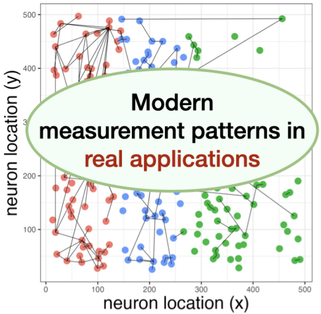
- Great tools in statistics & machine learning: selective inference, conformal inference, Bayesian inference...
- Numerous challenges from large-scale, complex data and models!

Rigorous uncertainty quantification in practical and complex scenarios?

Uncertainty Quantification: Challenges in the Modern Era

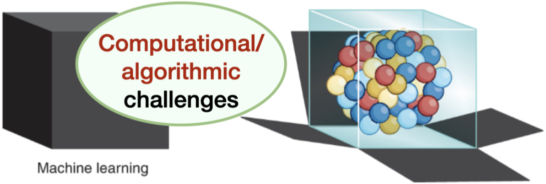
First part:

ML in science
UQ for graph learning



Second part:

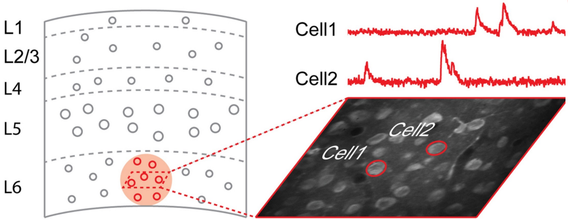
ML in the society
UQ for model-agnostic ML interpretations



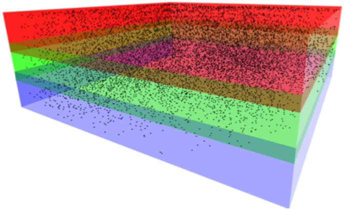
Uncertainty Quantification for Statistical Structure (Graph) Learning

Challenges from Data: Erose Measurements

Erose measurements: irregular, highly uneven measurements over a large system



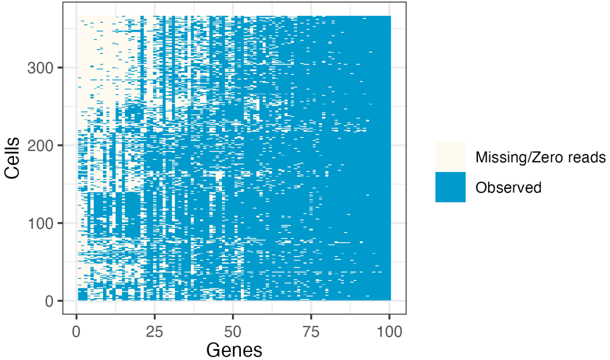
Calcium imaging data in neuroscience (Birkner et al., 2017)



Measurements in semi-overlapping cubes; the graph quilting problem (Vinci et al., 2019)

Challenges from Data: Erose Measurements

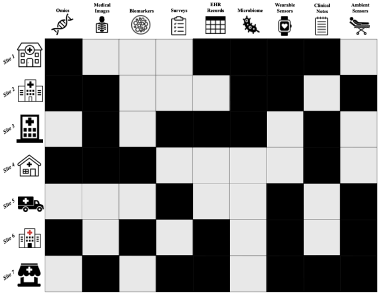
Erose measurements: irregular, highly uneven measurements over a large system



Single-cell RNA sequencing (Darmanis et al., 2015)

Challenges from Data: Erose Measurements

Erose measurements: irregular, highly uneven measurements over a large system



Patchwork learning in healthcare (Rajendran et al., 2023)

Table 1. Some examples of unequally spaced surveys.

Panel A: household surveys for monitoring poverty in developing countries[a]

Country	*Survey*	*Survey periods*
Bolivia	Encuesta Integrada de Hogares (EIH)	Mar 89, Nov 89, Sept 90, Nov 91, Nov 92, July–Dec 93, July–Dec 94, June 95
Brazil	Pesquisa Nacional por Amostra de Domicillios (PNAD)	Annual surveys since 1971, but surveys not taken in census years 1980 and 1991
Chile	Caracterización Socioeconómica Nacional (CASEN)	1985, 87, 90, 92, 94, 96
Ethiopia	Welfare monitoring survey	1995, 97, 98
Ghana	Ghana living standards survey	1987, 88, 91, 98
Kenya	Welfare monitoring survey	1992, 94, 97
Kyrgyz Republic	Poverty monitoring survey	1993, 96, 96, 97, 98
Mexico	Encuesta nacional de Ingreso-Gasto de los hogares (ENIGH)	1984, 89, 92, 94, 96
Nigeria	National consumer survey	1980, 85, 92, 96
Panama	Encuesta de Hogares–Mano de Obra (EMO)	1979, 89, 91, 95, 96
Peru	Encuesta Nacional de Hogares Sobre Medición de Niveles de Vida (ENNIV)	1985, 90, 91, 94
Senegal	Enquête Démographique et de Santé	1986, 92, 97
Thailand	Thailand Socio-Economic Survey (SES)	1975, 81, 86, 88, 90, 92, 94, 96, 98

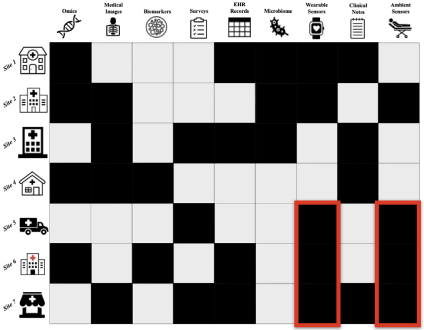
1985, 87, 90, 92, 94, 96
1995, 97, 98
1987, 88, 91, 98

Unevenly spaced time series in econometrics (Millimet and McDonough, 2017)

Structure Learning from Erose Measurements?

Common practices

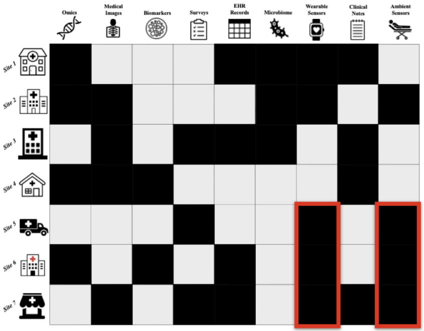
- Downsampling: focus on a complete block;
  - throw too much data away!



Structure Learning from Erose Measurements?

Common practices

- Downsampling: focus on a complete block;
  - throw too much data away!
- Ad-hoc imputation + downstream analysis;
  - low-rank completion methods?
    - provable mainly for random missingness
    - not low-rank?
    - extra uncertainty from imputation



Focus on graph learning from erose measurements in this talk

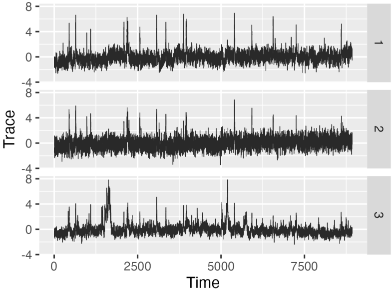
Why Graph Learning?

Graphical Model Structural Learning

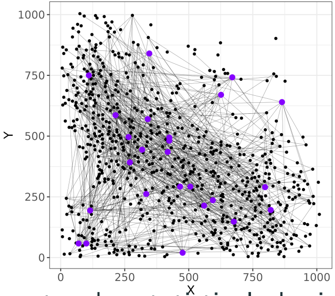
Extract **conditional dependency** relationships:

An edge between node j and k $\Longleftrightarrow$ Observations for j and k are conditionally dependent given all other nodes.

Functional Connectivity: a graph between neurons that reflect their co-firing patterns



$\Rightarrow$



Many applications: gene co-expression networks, sensor networks, statistical physics, ...

Gaussian Graphical Model Learning from Erose Measurements

- Focus on Gaussian graphical models in this talk
  - Nodes: $V = [p]$;
  - n samples of p-dimensional r.v.s: $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \sim \mathcal{N}(0, \Theta^{*-1})$;
  - $\Theta^*_{j,k} \neq 0 \Leftrightarrow j \not\perp\!\!\!\perp k |$all other nodes
  - Edges: $E = \{(j,k) : 1 \leq j, k \leq p, \Theta^*_{j,k} \neq 0\}$;
  - **Goal: identify non-zero entries in Θ^***

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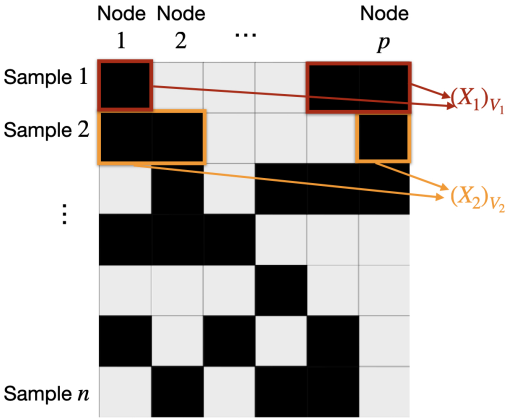
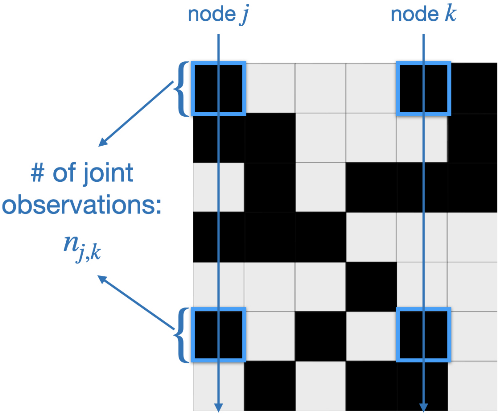


- **Erose measurements**
  Observe $(\boldsymbol{X}_i)_{V_i}$, $1 \leq i \leq n$; $V_i \subset [p]$ are irregular feature subsets, independent from $\boldsymbol{X}_i$.

- **Erose measurements**
  Even for assessing marginal dependency:
  joint sample sizes for node pairs $\{n_{j,k} : 1 \leq j, k \leq p\}$ are highly different



Gaussian Graphical Model Learning from Erose Measurements

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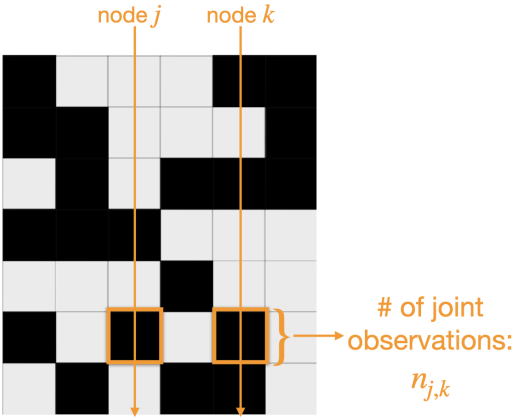


Prior Works on Graph Learning from Partial Observations

Estimation

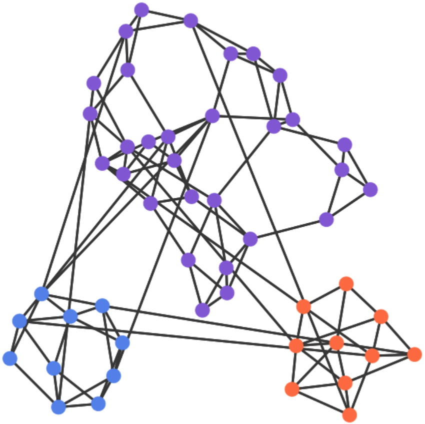
- Plug in covariance estimates into graphical Lasso (Kolar and Xing, 2012; Park et al., 2021)
- Most assume nodes missing with the same/similar probability!
- Existing characterization in minimum pairwise sample size $\min_{j,k} n_{j,k}$
- **Limited insights for our setting**

Inference

- Fully observed data
- Missing independently with same probability
- **Not applicable for our setting**

Toy Example: UQ Promotes Reliable Graph Learning

- Toy example: irregular patchwise observations
- $p = 30 + 10 + 10 = 50$ nodes in total



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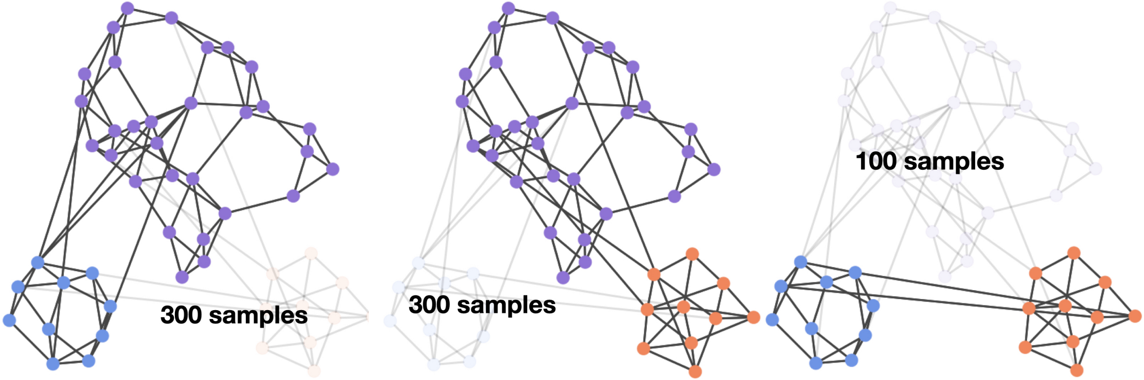
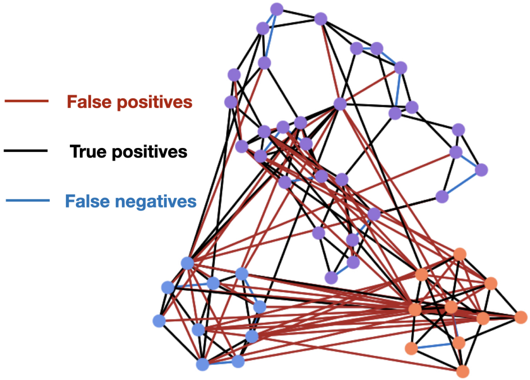


Measurement 1 Measurement 2 Measurement 3

Toy Example: UQ Promotes Reliable Graph Learning

- Plug-in estimate using graphical lasso



Toy Example: UQ Promotes Reliable Graph Learning

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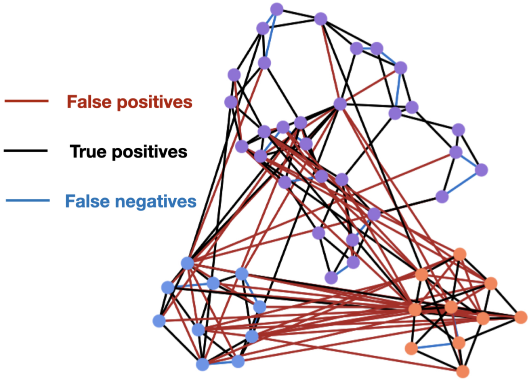
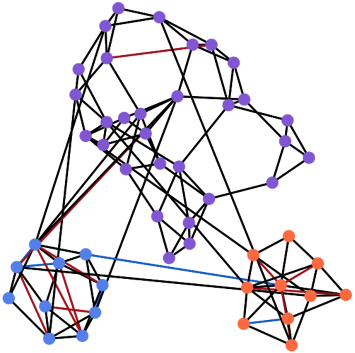


- We develop GI-JOE (**G**raph **I**nference when **J**oint **O**bservations are **E**rose) with FDR control



Problem Setup and Proposed Method

Recall: Model Setup

Gaussian graphical model:

- p-dimensional $\boldsymbol{X}_1, \ldots \boldsymbol{X}_n \sim \mathcal{N}(0, \Theta^{*-1})$;
- Nodes: $V = [p]$;
- Edges: $E = \{(j,k) : 1 \leq j,k \leq p, \Theta^*_{j,k} \neq 0\}$;

Observations

- $(\boldsymbol{X}_i)_{V_i}$, $1 \leq i \leq n$; $V_i \subset [p]$ are irregular feature subsets independent from $\boldsymbol{X}_i$
- Pairwise joint sample sizes $\{n_{j,k} : 1 \leq j,k \leq p\}$ are highly different

Recall: Model Setup

Observations

- $(\boldsymbol{X}_i)_{V_i}$, $1 \leq i \leq n$; $V_i \subset [p]$ are irregular feature subsets independent from $\boldsymbol{X}_i$
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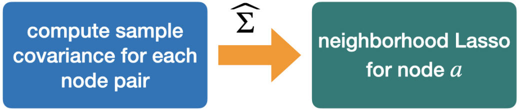
Edgewise-testing: $\mathcal{H}_0 : (a, b) \notin E$ **for** a, $b \in [p]$ **(Whole graph testing later)**

Edgewise Inference: Debiased Neighborhood Lasso

- Many existing methods are covariance-based.

- **Step 1**: Plug in $\widehat{\Sigma}$ into neighborhood Lasso (Meinshausen and Bühlmann, 2006) and debias it (Van de Geer et al., 2014):



$$\widehat{\theta}^{(a)} = \arg \min_{\theta \in \mathbb{R}^p, \theta_a = 0} \frac{1}{2} \theta^\top \widehat{\Sigma} \theta - \widehat{\Sigma}_{a,:} \theta + \sum_{j=1}^{p} \lambda_j |\theta_j|,$$

$|\widehat{\theta}_b^{(a)}|$ indicates edge strength of (a, b)

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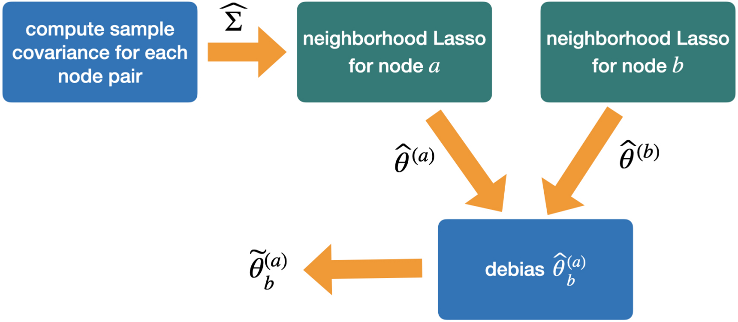

$|\widetilde{\theta}_b^{(a)}|$ also indicates edge strength of (a, b)

Edgewise Inference: Debiased Neighborhood Lasso

- **Step 2**: Normal approximation for $\widetilde{\theta}_b^{(a)}$ and variance estimation
  If fully observed with n samples: var. $\propto \frac{1}{n}$.
  Challenge: $\widehat{\Sigma}$ is computed from irregular data patches!

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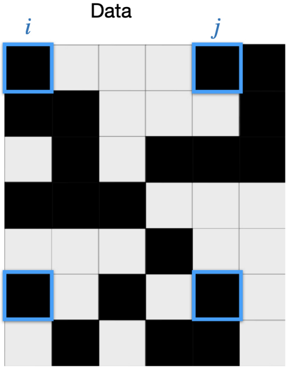

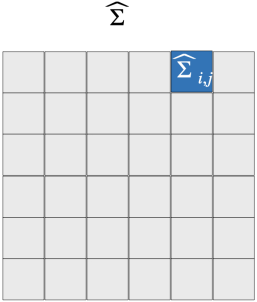

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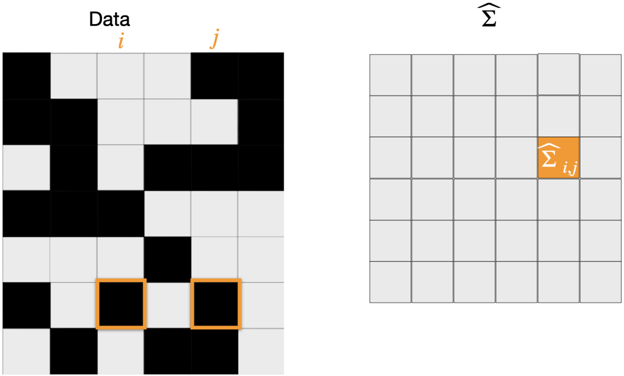

Edgewise Inference: Debiased Neighborhood Lasso

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  Challenge: $\widehat{\Sigma}$ is computed from irregular data patches!

All entries of $\widehat{\Sigma}$ play a role: **from marginal to conditional dependency!**

Characterization of Debiased Neighborhood Lasso

A Closer Look into $\widetilde{\theta}_b^{(a)}$

With appropriately chosen tuning parameters in the neighborhood Lasso,

$$\widetilde{\theta}_b^{(a)} = -\frac{\Theta^*_{a,b}}{\Theta^*_{a,a}} + \text{mean-zero first order term} + \text{high-order residuals}$$

Characterization of Debiased Neighborhood Lasso

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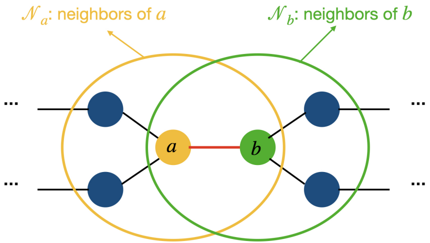
$$\widetilde{\theta}_b^{(a)} = -\frac{\Theta^*_{a,b}}{\Theta^*_{a,a}} + \text{mean-zero first order term} + \text{high-order residuals}$$

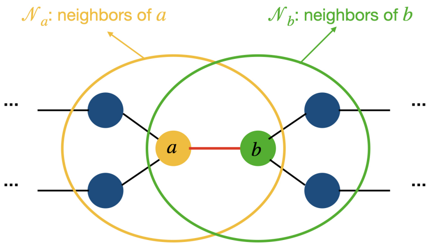
- mean-zero first-order term

$$\propto \sum_{j,k} (\widehat{\Sigma}_{j,k} - \Sigma^*_{j,k}) \underbrace{\Theta^*_{a,j}\Theta^*_{b,k}}_{\substack{\text{weight of} \\ \text{node pair } (j,k)}}$$

- only involve neighbors of a and b!



GI-JOE: Edge-wise Uncertainty Quantification



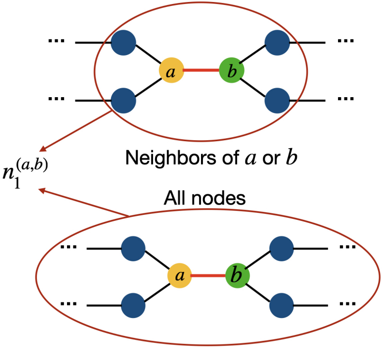
- **Step 2**: Estimate variance of first-order term
  - Variance contribution from each node pair (j, k): $\widehat{\theta}_j^{(a)}$, $\widehat{\theta}_k^{(b)}$, $1/n_{j,k}$
  - Plus some edge-edge correlations
  - Obtain $\widehat{\sigma}_n^2(a, b)$
- **Output**: Reject $\mathcal{H}_0 : (a, b) \notin E$ if $\frac{|\widetilde{\theta}_b^{(a)}|}{\widehat{\sigma}_n(a,b)} > z_{\alpha/2}$.

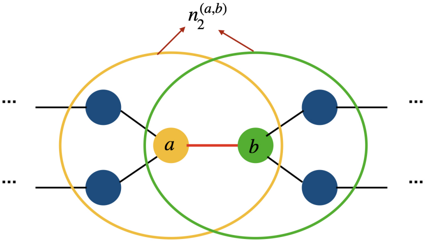
Edgewise Testing: Theoretical Guarantees

Assumption for Validity: Sufficient Local Sample Sizes

Minimum of the pairwise sample sizes $\{n_{j,k}\}$ for

$$\begin{cases} j \text{ is neighbor of } a \text{ or } b,\ k \text{ is any node:} & n_1^{(a,b)} \\ j \text{ is neighbor of } a,\ k \text{ is neighbor of } b\text{:} & n_2^{(a,b)} \end{cases}$$

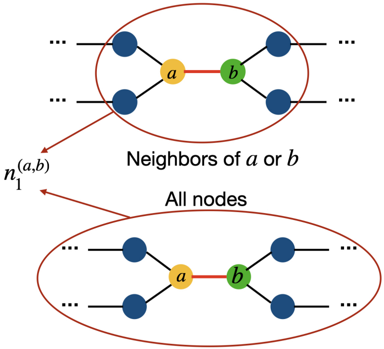


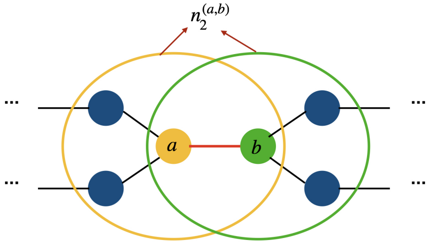


Assumption for Validity: Sufficient Local Sample Sizes

Main Assumption (Informal)

A1. The local sample size $n_1^{(a,b)}$ is sufficiently large, as a function of node degrees and graph size.





Statistical Validity of GI-JOE (Edge-wise Testing)

Main Theorem: Type I error and power (Informal)

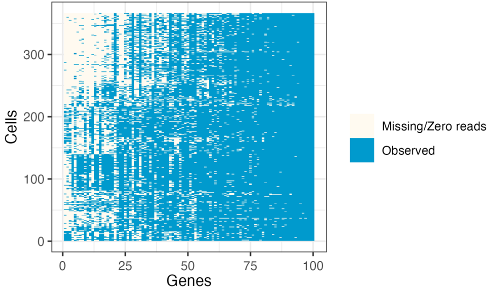
Suppose Assumption A1 holds. For testing $\mathcal{H}_0 : (a, b) \notin E$:

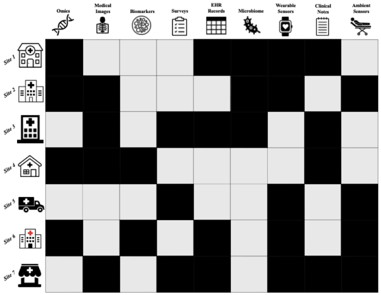
1. GI-JOE (edgewise testing) has asymptotically valid type I error control;
2. The asymptotic power is an increasing function of $|\Theta^*_{a,b}|\sqrt{n_2^{(a,b)}}$.

- Same sample size $n_{j,k} = n$ setting: reduces to prior requirements $n \gg d^2 \log^2 p$
- **First theory that allows for general erose measurements**.

Statistical Validity of GI-JOE (Edge-wise Testing)

- **First theory that allows for general erose measurements.**
  - arbitrary data-independent missing pattern! vs. nodes missing independently (Belloni et al., 2017).
  - localized sample size requirement! vs. global sample size $\min_{j,k\in[p]} n_{j,k}$ in existing estimation theory.





GI-JOE: FDR control

GI-JOE: FDR Control

Whole graph testing with FDR control?

- Want: 95% of the selected edges are true positives
- Take edgewise p-values; apply a variation of Benjamini-Hochberg's procedure
- **Valid for sparse graphs under mild sample size assumptions!**

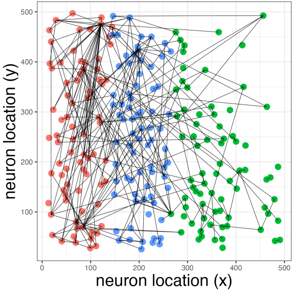
Empirical Studies

Application to Neuronal Functional Data

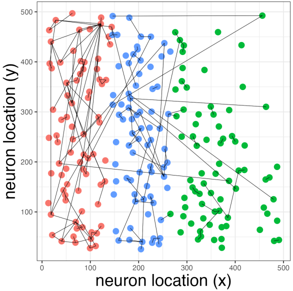
- Neuronal functional recordings of a mouse's visual cortex from Allen Brain Atlas
- Firing activities of $p = 227$ neurons, $n = 8931$ time points
- Goal: learn **functional connectivity amongst these neurons**
- Data is fully observed; we test how our method performs on manually masked data

Application to Neuronal Functional Data

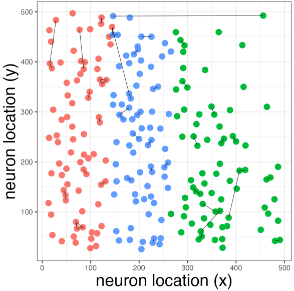
Manually mask functional data; three sets of neurons observed with high, medium, and low probabilities.



FDR-selected graph with **full data** (oracle)



GI-JOE (FDR), applied to **erose data**



DB-Glasso with minimum sample size, applied to **erose data**

Summary

- Erose measurements: challenge for reliable graph learning.

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- Edge-wise uncertainty hinges on neighbors; can be estimated by GI-JOE.

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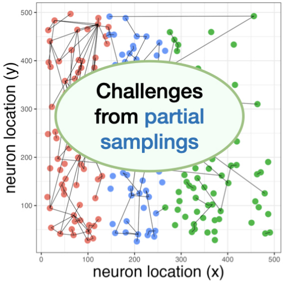
- Erose measurements: challenge for reliable graph learning.
- Edge-wise uncertainty hinges on neighbors; can be estimated by GI-JOE.
- Quantify different uncertainty levels over the graph with FDR control $\Rightarrow$ Better graph selection with erose data!
- Future directions: more reliable **feature selection / causal structural learning** from erosely measured data under dependency?
- **L. Zheng**, G. I. Allen, "Graphical Model Inference with Erosely Measured Data", *Journal of the American Statistical Association, Theory and Methods*, 2023.

From Complex Data Collection to Complex Machine Learning Systems

First part:

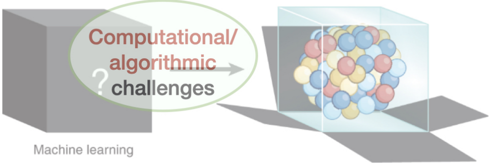
UQ for reliable scientific discoveries



Second part:

ML in the society

UQ for model-agnostic ML interpretations



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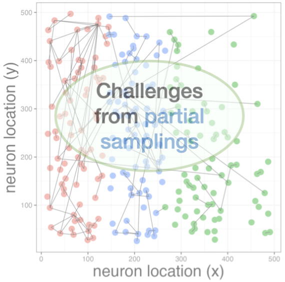

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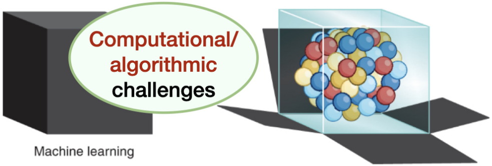

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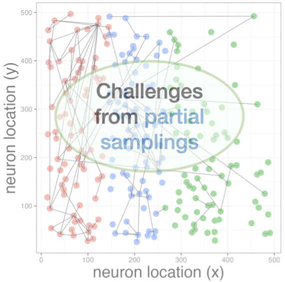

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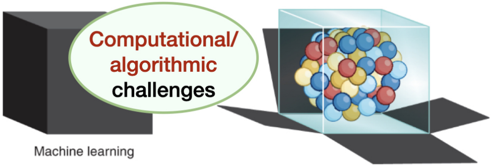


(Partial sampling can be a tool!)

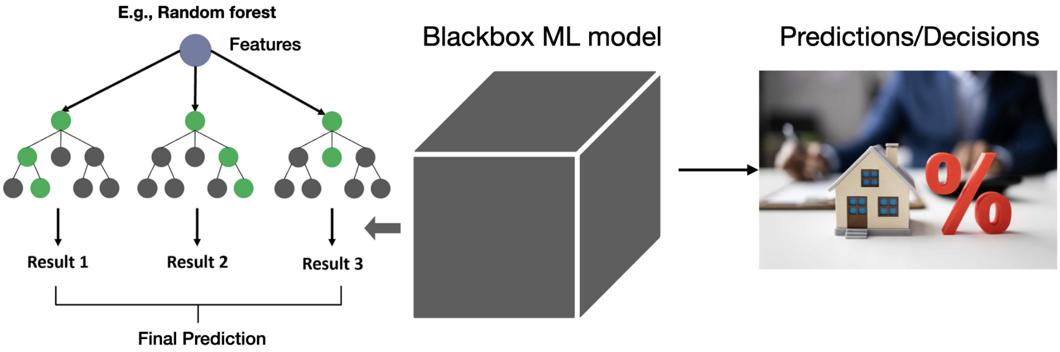
Uncertainty Quantification for Model-agnostic Machine Learning Interpretations

Interpreting Black-box Machine Learning Models

Machine learning is widely applied in **high-stakes applications**:

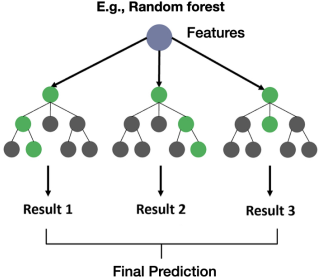
Can we trust machine learning? Make it interpretable!

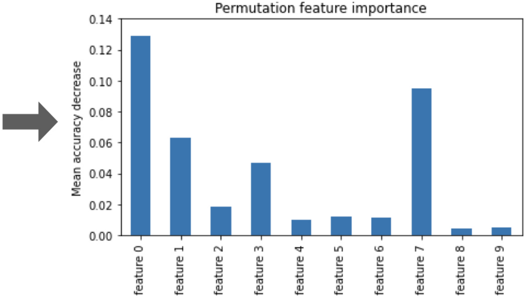
Interpreting Black-box Machine Learning Models



Why is this ML system rejecting my mortgage application?

Feature importance: How does my model's prediction rely on each feature?

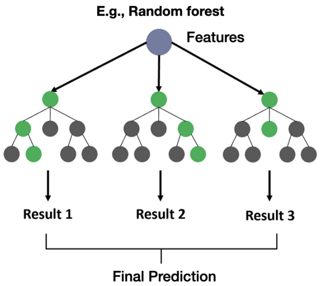


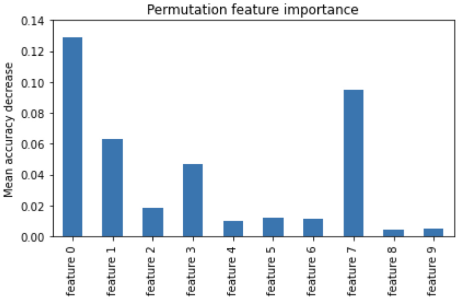


- Model-specific: defined for random forest, linear models, deep learning, etc.
- **Model-agnostic**: feature occlusion (Covert et al., 2021), permutation (König et al., 2021), Shapley values (Sundararajan and Najmi, 2020), etc.

Feature Importance for Interpretable Machine Learning

Feature importance: How does my model's prediction rely on each feature?





Can we trust feature importance? Uncertainty quantification?

Two Types of Feature Importance

Population feature importance

- Assume a data-generating model; infer about the population
- E.g., Conditional independence test, knockoff
- ML models are only tools
- Impossible without strong assumptions about the data or model (Shah and Peters, 2020)!

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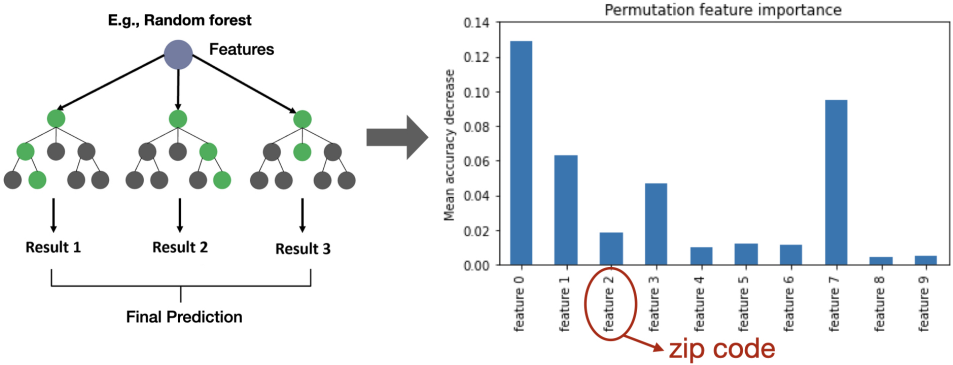
ML feature importance

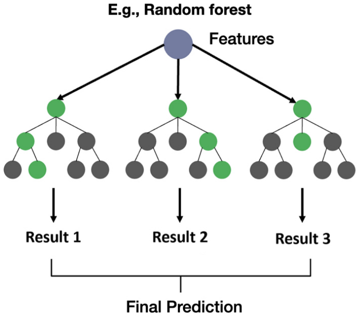
- Property of the model
- Which feature does my ML model rely on for decisions?
- Desired for model diagnostics, auditing, and deployment

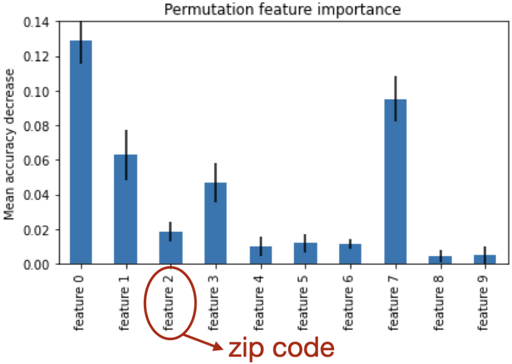
ML Feature Importance



- Want mortgage decision to rely less on sensitive features
- E.g., zip code is a proxy of race?
- Check feature importance

ML Feature Importance





UQ for ML feature importance:

- **has important societal consequences but is understudied!**

Feature Importance Inference

Population feature importance

- Inference for Lasso (Lee et al., 2016; Van de Geer et al., 2014)
- Conditional independence tests for random forest (Chi et al., 2022)
- Model-agnostic methods: Floodgate (Zhang and Janson, 2020), GCM (Shah and Peters, 2020), VIMP (Williamson et al., 2021)

ML feature importance

- Only a few works (Fisher et al., 2019; Lei et al., 2018; Rinaldo et al., 2019; Watson and Wright, 2021)
- Many are heuristic
- Most face computational challenges
- Efficient and rigorous UQ for ML feature importance?

Leave-One-Covariate-Out (LOCO) Inference
(Lei et al., 2018; Rinaldo et al., 2019):

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Inference target: Predictive power without feature j vs. with feature j.

Prior Work: LOCO Inference

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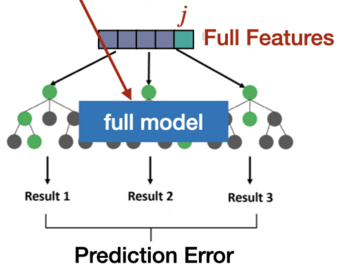
$$\Delta_j^*(\boldsymbol{X}, \boldsymbol{Y}) = \mathbb{E}\left[\mathrm{Error}\left(Y^{\mathrm{test}}, \mu_{-j}(X_{-j}^{\mathrm{test}}; \boldsymbol{X}_{-j}, \boldsymbol{Y})\right) - \mathrm{Error}\left(Y^{\mathrm{test}}, \mu(X^{\mathrm{test}}; \boldsymbol{X}, \boldsymbol{Y})\right) | \boldsymbol{X}, \boldsymbol{Y}\right]$$



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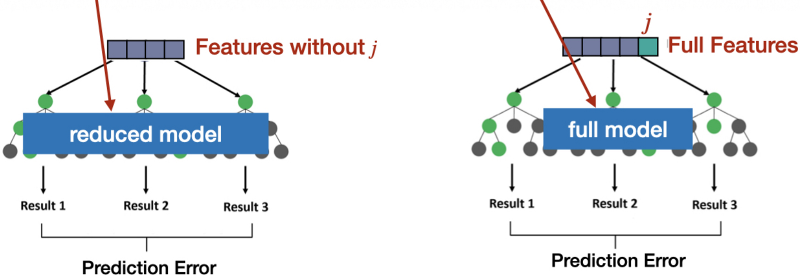

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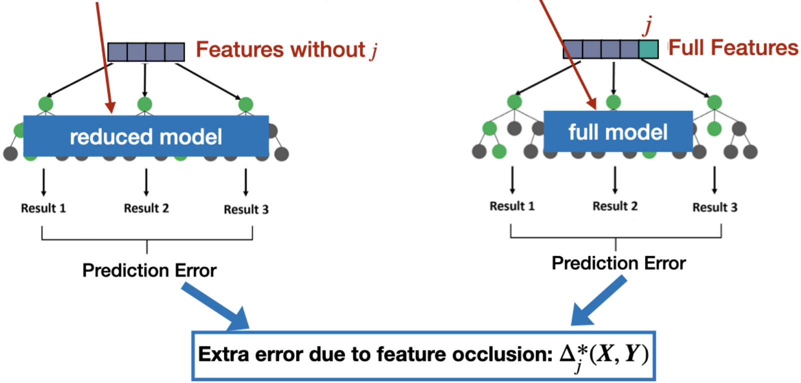

Prior Work: LOCO Inference

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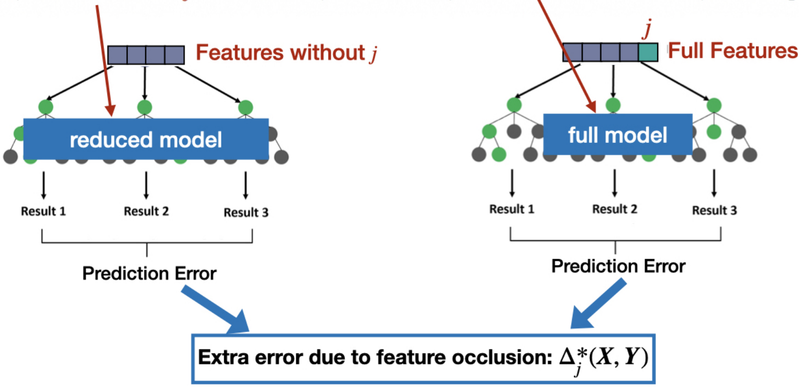

Property of the current models; model-agnostic

Prior Work: LOCO Inference

LOCO inference approach:

- Splits data; fits full and reduced models to training data
- Feature occlusion scores on test data $\Rightarrow$ confidence intervals

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Advantages

- Model-agnostic (applicability).
- Statistically valid without assuming data distribution/model choice.

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Advantages

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Challenges

- Data splitting loses statistical power;
- Interpretation is not for the full model & depends on random data splitting
- Model refitting for each feature: prohibitive computation after model training

Prior Work: LOCO Inference

LOCO inference approach:

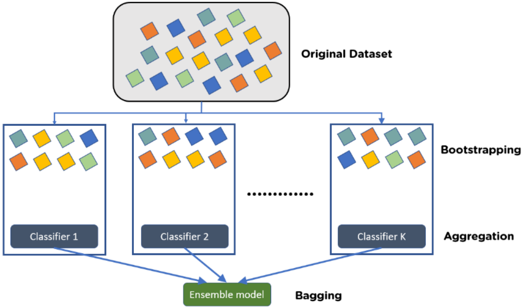
- Splits data; fits full and reduced models to training data
- Feature occlusion scores on test data $\Rightarrow$ confidence intervals

Our Goal

Can we utilize the general LOCO framework to perform ML feature importance inference, while avoiding data splitting and model refitting?

Our Approach: LOCO Inference for an Ensemble Framework

LOCO Inference for Ensemble Learning



Bagging

Picture source: https://www.simplilearn.com/tutorials/machine-learning-tutorial/bagging-in-machine-learning

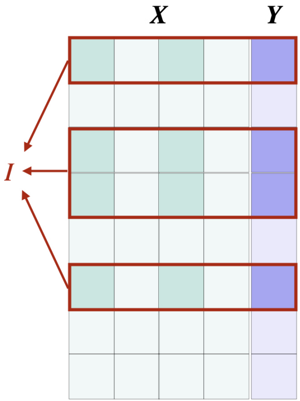
Inspiration: Jacknife+ After Bootstrap (Kim et al., 2020).

- Many ensemble methods are good predictors
- Conformal inference (Jackknife+) for bagging is computationally free with no data-splitting!

Idea: Minipatch Ensembles.

Minipatch Ensemble Learning

Minipatch ensembles: like bagging, but double-subsampling for both observations and features (Yao and Allen, 2020).



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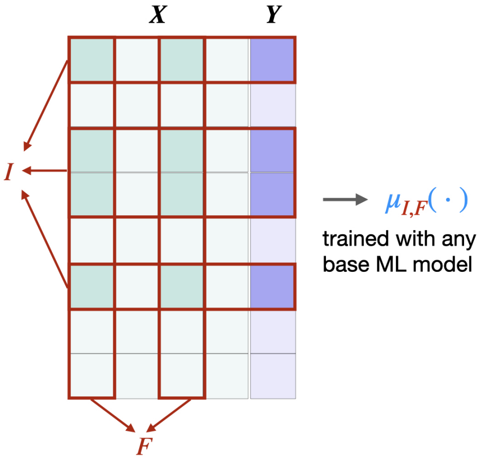

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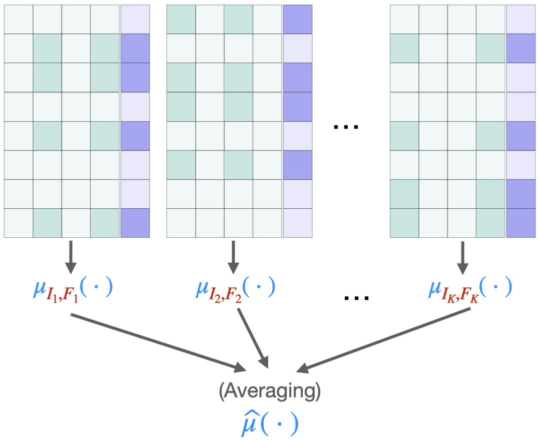
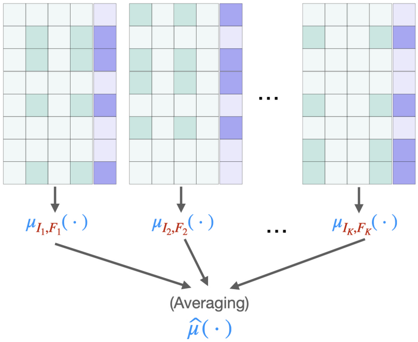


Minipatch Ensemble Learning

Inspiration: Bagging; Random Forests (Louppe and Geurts, 2012); Stochastic Optimization & Dropout.

Advantages:

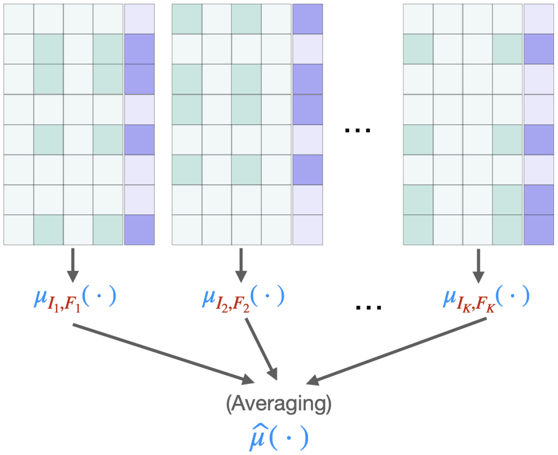
- Fast and easily parallelizable
- Ensemble diversity; **implicit regularization** (LeJeune et al., 2020; Yao et al., 2021)



LOCO Inference for Minipatch Ensembles?

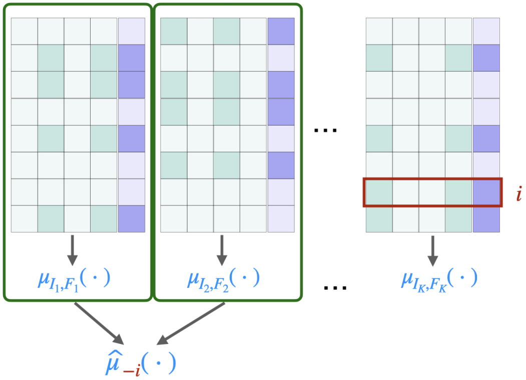
Algorithm: LOCO for Minipatch

- **Step 1**. Fit minipatch learning predictor: $\hat{\mu}$.



Algorithm: LOCO for Minipatch

- **Step 2. LOO** (leave-one-observation-out) predictor: $\hat{\mu}_{-i}(X_i)$.
  - Ensemble minipatches without observation i.
  - Compute test error on sample i.



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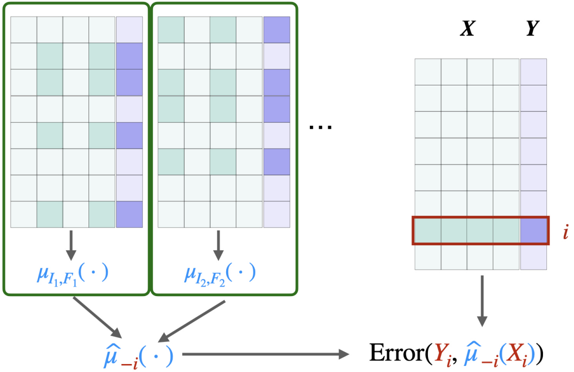

No data-splitting!

Simple model averaging;

Free computationally!

Algorithm: LOCO for Minipatch

- **Step 3. LOCO-LOO** predictor: $\hat{\mu}_{-i}^{-j}(X_i)$.
    - Ensemble minipatches without observation i and without feature j.
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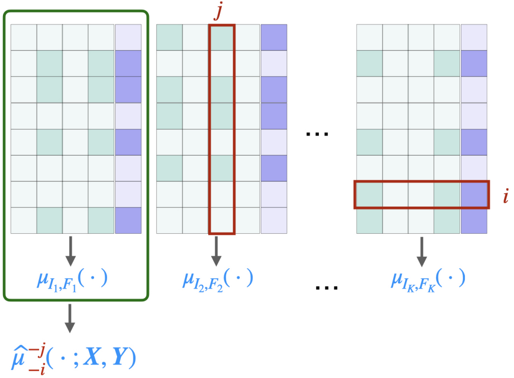

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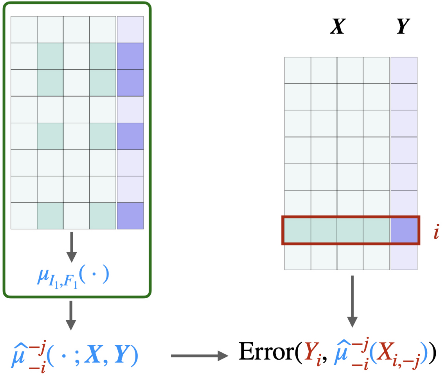


Simple model averaging;
Free computationally!

Algorithm: LOCO for Minipatch

- **Step 4**. Compute feature occlusion scores for observations $1 \leq i \leq N$:

$$\widehat{\Delta}_j(X_i, Y_i) = \mathrm{Error}(Y_i, \hat{\mu}_{-i}^{-j}(X_i)) - \mathrm{Error}(Y_i, \hat{\mu}_{-i}(X_i)).$$

Importance of feature j for predicting sample i.

- **Step 5**. Construct asymptotically normal interval from $\{\widehat{\Delta}_j(X_i, Y_i)\}_{i=1}^N$:

$$\hat{\mathbb{C}}_j = \left[\bar{\Delta}_j - \frac{z_{\alpha/2}\hat{\sigma}_j}{\sqrt{N}}, \bar{\Delta}_j + \frac{z_{\alpha/2}\hat{\sigma}_j}{\sqrt{N}}\right],$$

$\bar{\Delta}_j$: mean occlusion score, $\hat{\sigma}_j$: standard deviation.

Algorithm: LOCO for Minipatch

Full Algorithm

- **Step 1**. Fit minipatch learning predictor.
- **Step 2&3**. For each sample i, compute **LOO** and **LOCO-LOO** predictor by simple model averaging.
- **Step 4&5**. Construct the normal confidence interval.

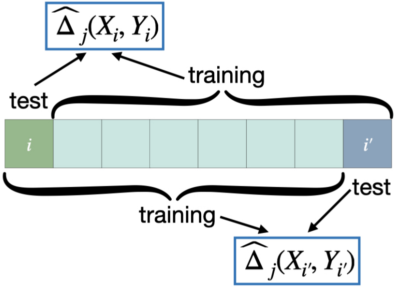
Algorithmic advantages

- **No model-refitting** $\Rightarrow$ once predictive model is trained, confidence intervals are computationally free!
- **No data-splitting** $\Rightarrow$ powerful; feature importance inference for the current model at hand!

Theoretical Guarantees

Does LOCO-MP confidence interval have valid coverage?

- **Leave-one-observation-out instead of data-splitting** $\Rightarrow$ dependency amongst $\{\widehat{\Delta}_j(X_i, Y_i)\}_{i=1}^N$!
- $\widehat{\Delta}_j(X_i, Y_i)$ and $\widehat{\Delta}_j(X_{i'}, Y_{i'})$ switches i and i' for training and testing; share $N-2$ training samples.
- Central limit theorem not applicable!



Theoretical Guarantees

A1. Smoothness of $\mathrm{Error}()$.

A2. Minipatch predictors have bounded difference (automatically hold for classification).

A3. Small MP: $n = o(\sqrt{N})$

A4. Large number of MPs: $K \gg \frac{N}{\sigma_j^2}$

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Theorem
Suppose samples $(\boldsymbol{X}_i, Y_i)$ are i.i.d., and assumptions A1-A4 hold. Then

$$\lim_{N \to \infty} \mathbb{P}(\Delta_j^* \in \hat{\mathbb{C}}_j) = 1 - \alpha.$$

Valid asymptotic coverage under mild assumptions; applicable to any data distributions and base ML models.

Theoretical Guarantees

- **Algorithmic stability**: prediction is stable against change in one training sample.
- Stability facilitates statistical inference under dependency (Bayle et al., 2020)!

Theoretical Guarantees

- **Algorithmic stability**: prediction is stable against change in one training sample.
- Stability facilitates statistical inference under dependency (Bayle et al., 2020)!
- Minipatch ensembles are stable with any base model and any data distribution!
- Independent interest: stability also helps with conformal inference, selective inference.

Simultaneous Predictive Inference

Predictive inference is also free after training!

- Leave-one-observation-out residuals are free to compute
- Use quantiles of LOO residuals to construct distribution-free predictive intervals (**conformal inference**)
- Similar to Jacknife+ after bootstrap (Kim et al., 2020)

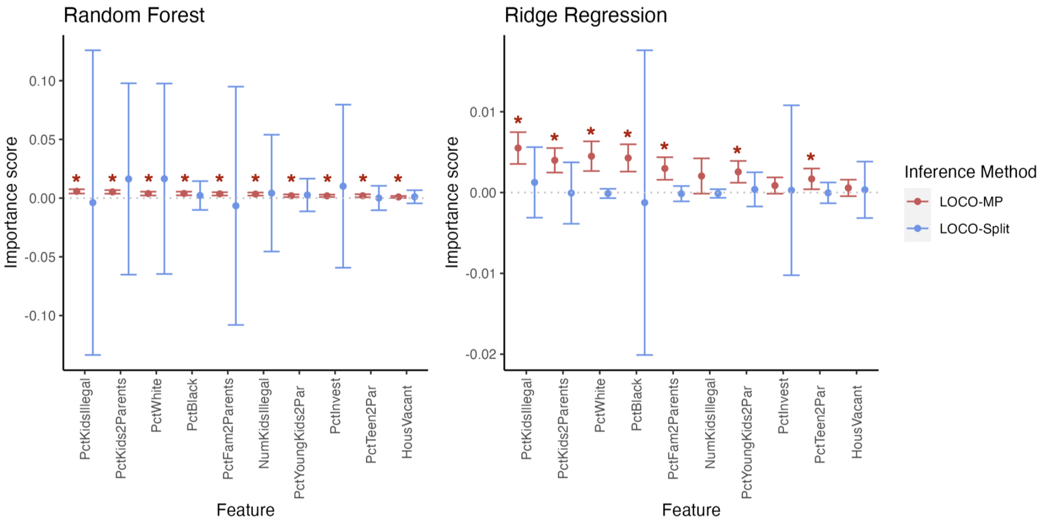
> Simultaneous, immediate inference for both feature importance & prediction $\Rightarrow$ convenient safety check for ML systems

Empirical Studies

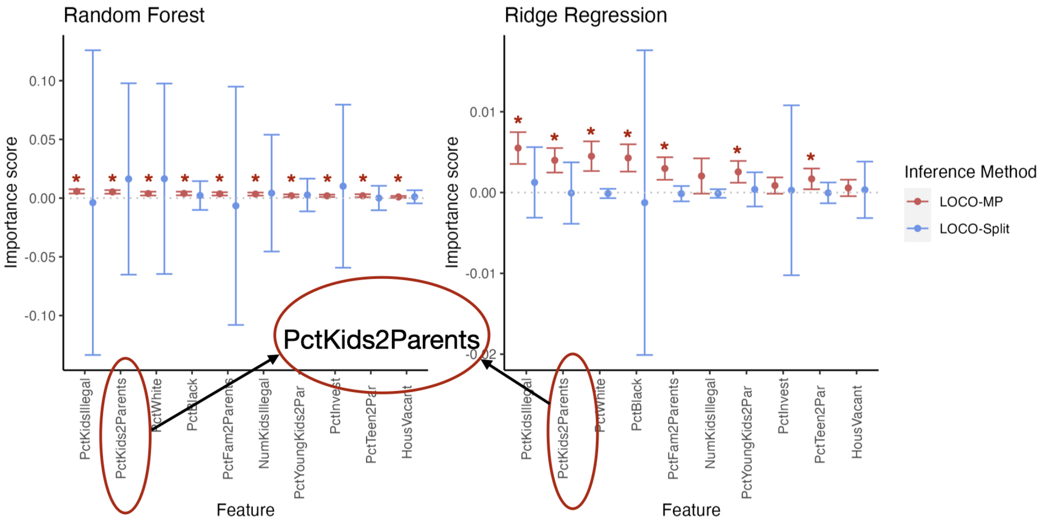
Real Data Example

- Communities and Crimes data (Redmond, 2009).
- 1994 observations, 122 features.
- Predict the per capita violent crime rate based on community features.

Real Data Example



Real Data Example



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  - Relationship to population feature importance?
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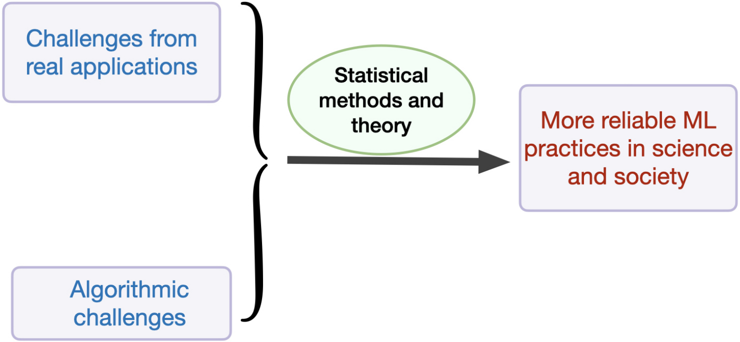
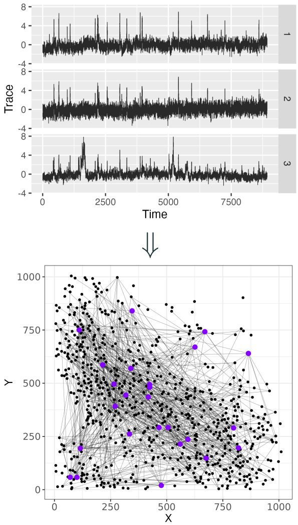
- L. Gan*, **L. Zheng***, G. I. Allen ($*$: equal contribution), "Model-Agnostic Confidence Intervals for Feature Importance: A Fast and Powerful Approach Using Minipatch Ensembles", https://arxiv.org/abs/2206.02088.

Other Works and Future Directions

Research Theme

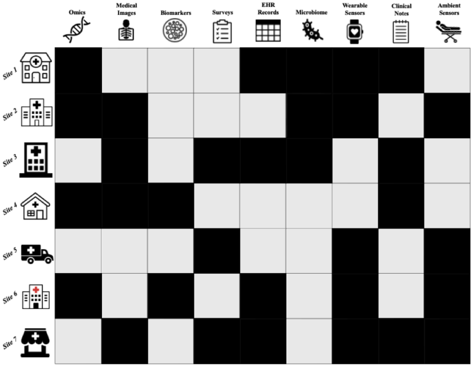


Learning Functional Connectivity in Neuroscience



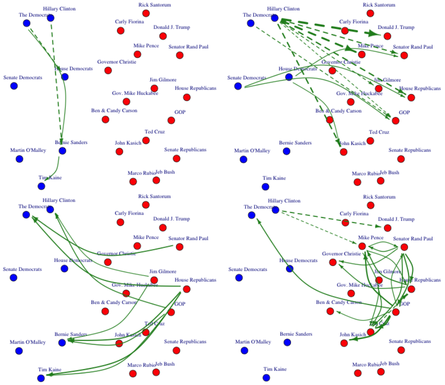
- Uncertainty quantification: GI-JOE
- Low-rank covariance completion for graph quilting
  A. Chang, **L. Zheng**, G. I. Allen,
  under revision at JASA, Applications and Case Studies
- Nonparanormal graph quilting
  STAT, 2023.

Clustering for Patchwise Multi-modal Healthcare Data



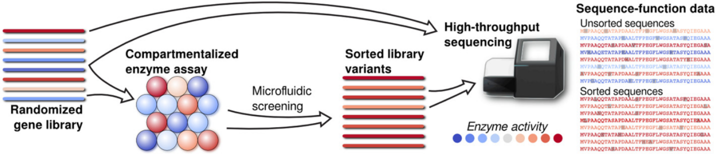
- Provable spectral clustering for patchwork learning
- PCA for patchwork learning?

Granger Causal Network on Social Media and Stock Market



- Hypothesis testing for Granger causal edges in linear AR(p) models
  L. Zheng, G. Raskutti,
  Electronic Journal of Statistics, 2019
- Context-dependent Granger causal network learning for mixed data types **L. Zheng**, G. Raskutti, R. Willett, B. Mark
  Journal of Machine Learning Research, 2020

Protein Engineering from Label-contaminated Data

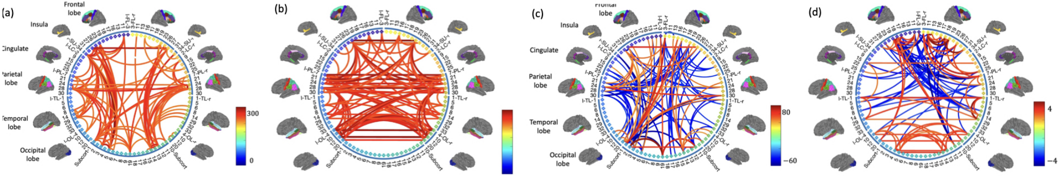


High-dimensional classification with positive-unlabeled data

L. Zheng, G. Raskutti,

under revision at Electronic Journal of Statistics

Joint Analysis of Functional & Structural Brain Connectivity in Neuroimaging



Joint Tensor PCA for Multi-modal Populations of Networks

J. Liu, **L. Zheng**, Z. Zhang, G. I. Allen.

Addressing Algorithmic Challenges for Large-scale Machine Learning

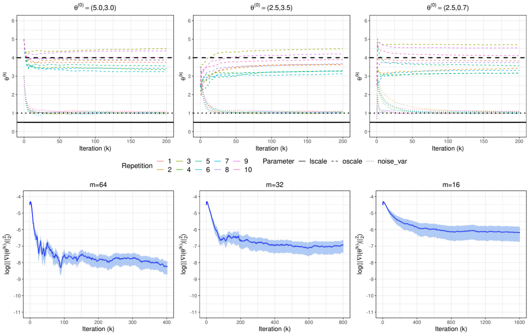
Subsampling helps both computationally and statistically

- LOCO-MP for free inference of ML interpretation

- **Provable Convergence: Stochastic Gradient Descent can Speed up Gaussian Processes!**

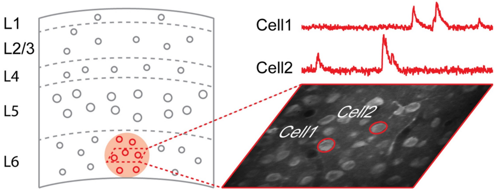
  H. Chen, **L. Zheng**, R. Al Kontar, G. Raskutti

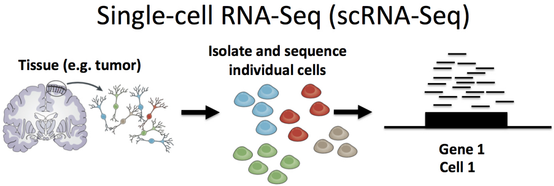
  Journal of Machine Learning Research, 2022



Future Directions

Reliable statistical learning from messy biomedical data?



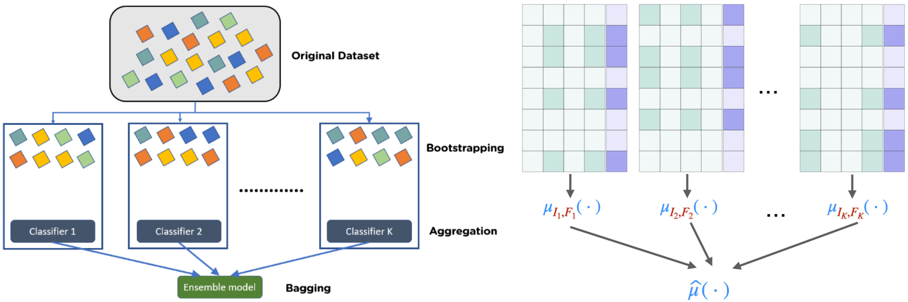


Picture source: https://speakerdeck.com/stephaniehicks/welcome-to-the-world-of-single-cell-rna-sequencing?slide=3

latent neurons; imputation for frequent dropouts

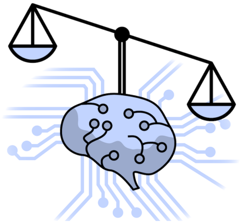
Future Directions

Exploit ensemble learning for statistical & computational advantages, and convenient uncertainty quantification?



Future Directions

Fairness in machine learning interpretations?



ML interpretations can

- inherit bias from data
- only depict the majority group

Picture source:

https://sitn.hms.harvard.edu/uncategorized/2020/fairness-machine-learning/

Acknowledgments

Coauthors

Luqin Gan

Genevera I. Allen

Thank you!

Supplementary Details for GI-JOE

GI-JOE: Procedure Details

- In neighborhood Lasso,

$$\widehat{\theta}^{(a)} = \arg\min_{\theta\in\mathbb{R}^p, \theta_a=0} \frac{1}{2}\theta^\top\widehat{\Sigma}\theta - \widehat{\Sigma}_{a,:}\theta + \sum_{j=1}^{p}\lambda_j|\theta_j|, \quad \lambda_j = C\sqrt{\frac{\log p}{\min_k n_{j,k}}}.$$

- Variance for node pair (a, b) is estimated by:

$$\widehat{\sigma}_n^2(a,b) = \sum_{j,k,j',k'} \frac{n_{j,k,j',k'}}{n_{j,k}n_{j',k'}} \widehat{\Theta}^{(a)}_{j,b}\widehat{\theta}^{(a)}_{k}\widehat{\Theta}^{(a)}_{j',b}\widehat{\theta}^{(a)}_{k'}(\widehat{\Sigma}_{j,j'}\widehat{\Sigma}_{k,k'} + \widehat{\Sigma}_{j,k'}\widehat{\Sigma}_{k,j'}).$$

$n_{j,k,j',k'}$: number of joint measurements of quadraple (j,k,j',k');

$n_{j,k}$: number of joint measurements of pair (j,k);

$n_{j',k'}$: number of joint measurements of pair (j',k'),

$$\widehat{\sigma}_n^2(a,b) \propto (n_2^{(a,b)})^{-1}.$$

GI-JOE: Assumptions for Valid Edgewise Testing

Assumption

The local sample sizes $n_1^{(a,b)}$, $n_2^{(a,b)}$, degrees of node a, b (d_a, d_b), graph size p satisfy

$$n_1^{(a,b)} \gg (d_a + d_b)^2 (\log p)^2 \frac{n_2^{(a,b)}}{n_1^{(a,b)}},$$

$$n_1^{(a,b)} \gg (d_a + d_b)^2 \log p \left(\frac{n_2^{(a,b)}}{n_1^{(a,b)}} \right)^2 .$$

GI-JOE: Valid FDR Control (Informal)

Theorem: Valid FDR control (Informal)

Assume

1. $n_1^{(a,b)}$ is sufficiently large for all (a, b) (holds even if $\max n_{j,k} \gg \min n_{j,k}$);
2. Most edge pairs (a, b), (a', b') are only weakly correlated (satisfied by most sparse graphs).

The edge set selected by GI-JOE (FDR) has **asymptotically valid FDR control**.

GI-JOE: Detailed Assumptions for Valid FDR Control

Assumption (Sample Size Condition)
For all node pairs $(a, b) \in [p] \times [p]$,

$$n_1^{(a,b)} \gg C(d+1)^2(\log p)^5 \log\log p \left(\frac{n_2^{(a,b)}}{n_1^{(a,b)}}\right)^2, \quad n_2^{(a,b)} \geq C(d+1)^6(\log p)^6.$$

- Weaker assumption than $p < n^C$ for $C > 0$ in prior literature (Liu, 2013);
- Let $g(d,p) = C(d+1)^2(\log p)^5 \log\log p$, then this is implied by

$$n_{\min} \gg g(d,p), \quad \frac{n_{\min}}{g(d,p)} \gg \left(\frac{n_{\max}}{g(d,p)}\right)^{2/3}$$

GI-JOE: Detailed Assumptions for Valid FDR Control

Assumption (Edge-edge correlations)
Total number of edge pairs: p^4.

- $\mathcal{A}_1$: *set of strongly correlated edge pairs;* $|\mathcal{A}_1| \leq Cp^2$
- $\mathcal{A}_2$: *set of moderately correlated edge pairs;* $|\mathcal{A}_2| \ll p^{4-\varepsilon}$ *for a small constant* $\varepsilon > 0$.

- In full observational setting, this is implied by (i) each node only has constant number of strongly connected neighbors; (ii) $d \ll p^{1-c}$;
- Empirical evidence supports this assumption for general graph and measurement patterns.

Proof Sketch for GI-JOE: Edgewise Testing

$$\widetilde{\theta}_b^{(a)} = -\frac{\Theta^*_{a,b}}{\Theta^*_{a,a}} + \text{mean-zero first-order term} + \text{high-order residuals}$$

- Mean-zero first-order term $\asymp \frac{1}{\sqrt{n_2^{(a,b)}}}$
- High-order residuals carefully controlled: $\lesssim \frac{\log p}{n_1^{(a,b)}}$ (collects errors from neighborhood Lasso)
- Variance estimates depend on (i) neighborhood Lasso; (ii) $\widehat{\Sigma}_{j,k} - \Sigma^*_{j,k}$ mainly for the $j \in \mathcal{N}_a$, $k \in \mathcal{N}_b$.

Proof Highlights for GI-JOE: Edge-wise Testing

- Want to control **high-order terms** and **variance estimation errors**

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- Want to control **high-order terms** and **variance estimation errors**
- Neighborhood Lasso errors reweighted by sample size:
  $\varepsilon^{(a)} = \widehat{\theta}^{(a)} - \theta^{(a)*}$, $\varepsilon^{(b)} = \widehat{\theta}^{(b)} - \theta^{(b)*}$, need to control

$$\left\langle \frac{1}{\sqrt{\boldsymbol{N}}}, \varepsilon^{(a)} \varepsilon^{(b)\top} \right\rangle;$$

$$\left\langle \frac{1}{\boldsymbol{\mathcal{N}}}, \varepsilon^{(a)} \otimes \varepsilon^{(b)} \otimes \varepsilon^{(a)} \otimes \varepsilon^{(b)} \right\rangle.$$

- Under-measured nodes are weighted more.

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- Under-measured nodes are weighted more.

Node-wise ℓ_1 penalty: $\lambda_j \asymp \sqrt{\frac{\log p}{\min_k n_{j,k}}} \Rightarrow$ weighted error bounds.

Key Proof Tool for GI-JOE: FDR Control

Key technical tool

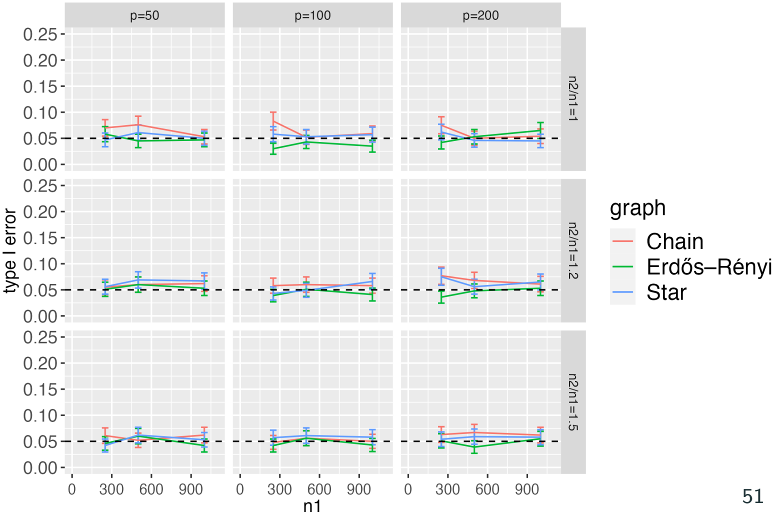
For each node pair, test statistics $(\xi^{(a,b)}, \xi^{(a',b')})$ converge to a two-variate independent Gaussian distribution:

Let $(Z_1, Z_2) \sim \mathcal{N}(0, I_2)$, we have

$$
\begin{aligned}
&\mathbb{P}(|\xi^{(a,b)}| > t_1, |\xi^{(a',b')}| > t_2) \\
\leq &\mathbb{P}(|Z_1| > t_1 - \varepsilon, |Z_2| > t_2 - \varepsilon) + C\exp\{-c\varepsilon\sqrt{n_2^{(a,b)}}\}.
\end{aligned}
$$

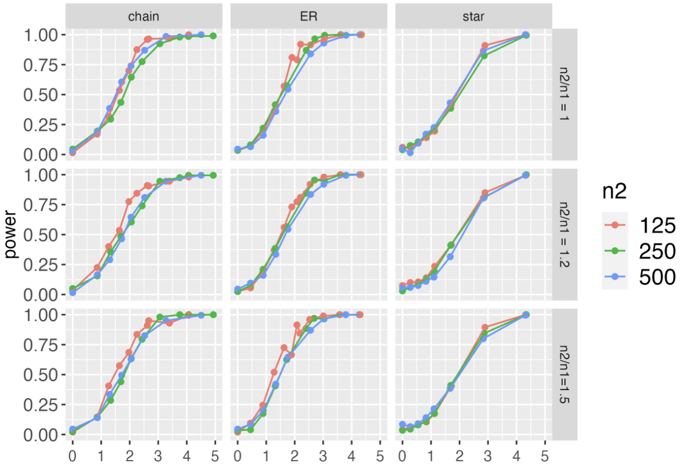
GI-JOE Simulation: Edge-wise Testing

Valid type I error with differing pairwise sample sizes!



GI-JOE Simulation: Edge-wise Testing

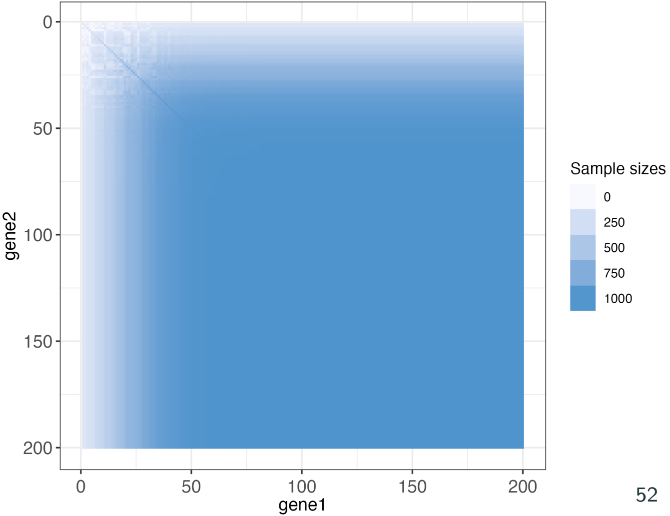
Power depends on signal strength & **localized sample size** $n_2^{(a,b)}$



$n_2^{(a,b)} \leftarrow \dfrac{\tilde{\theta}_b^{(a)*}}{\sigma_n(a,b)}$

Simulation: Graph Selection Comparison

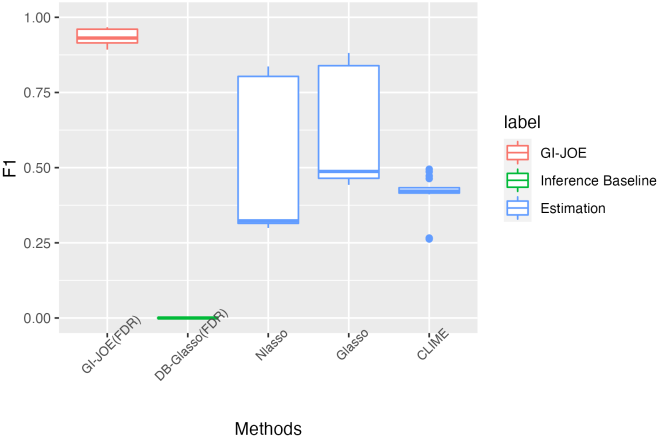
- Simulate data from a **scale-free graph** with 200 nodes
- **Real measurement pattern** in a real single-cell RNA sequencing data set (Chu et al., 2016)



Simulation: Graph Selection Comparison

F1-score: $2/(\mathrm{TPR}^{-1}+\mathrm{TDR}^{-1})$;
the higher the better

- Our method GI-JOE with FDR control
- Baseline inference methods: Plug-in method with debiased graphical lasso, minimum sample size
- Estimation methods: graphical lasso, neighborhood lasso, CLIME



Assumptions for Validity of LOCO-MP

A1. $\mathrm{Error}()$ is Lipschitz continuous.

A2. Bounded difference in MP predictions $||\hat{\mu}_{I,F}(X) - \hat{\mu}_{I',F'}(X)|| \leq B$.

Assumptions for Validity of LOCO-MP

A1. $\mathrm{Error}()$ is Lipschitz continuous.

A2. Bounded difference in MP predictions $||\hat{\mu}_{I,F}(X) - \hat{\mu}_{I',F'}(X)|| \leq B$.
(automatically hold for classification)

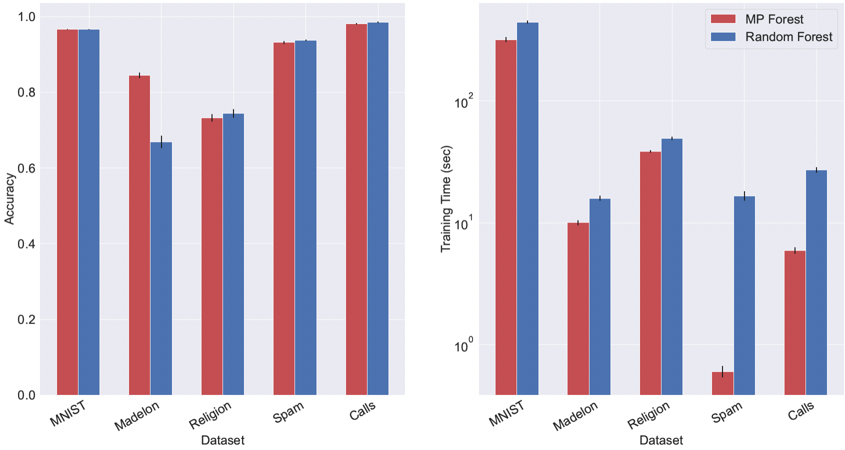
A3. MP size: $n = o(\frac{\sigma_j}{LB}\sqrt{N})$.

A4. $K \gg (\frac{L^2B^2N}{\sigma_j^2} + \frac{LB\sqrt{N}}{\sigma_j} + 1)\log(N)$.

Supplementary Details for LOCO-MP

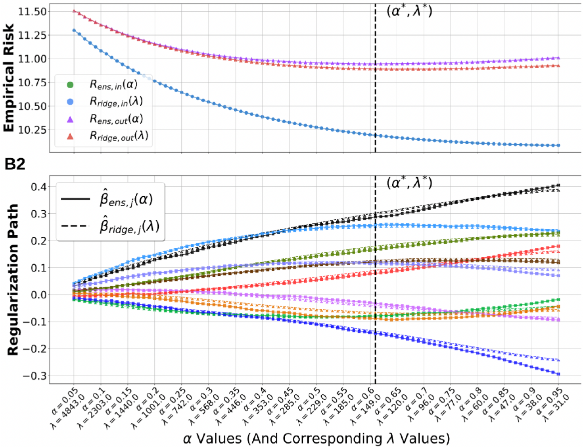
What is the Minipatch Learning Predictor?

When base models are trees, Minipatch predictor is similar to random forest



What is the Minipatch Learning Predictor?

When base models are linear regression, Minipatch predictor is equivalent to ridge regression (LeJeune et al., 2020; Yao et al., 2021)



Minipatch Feature Importance vs. Population Feature Importance?

Special Case: Linear Model. For independent features,

- Δ_j^* concentrates around $\tilde{\Delta}_j^*$: $\tilde{\Delta}_j^* \asymp 2\gamma \left(\beta_j^{*2} - \frac{\|\beta_{\setminus j}^*\|_2^2}{M-1} \right)$ (with $\gamma = m/M$).
- Under assumptions on the minipatch size and number; valid coverage for $\tilde{\Delta}_j^*$.

Minipatch Feature Importance vs. Population Feature Importance?

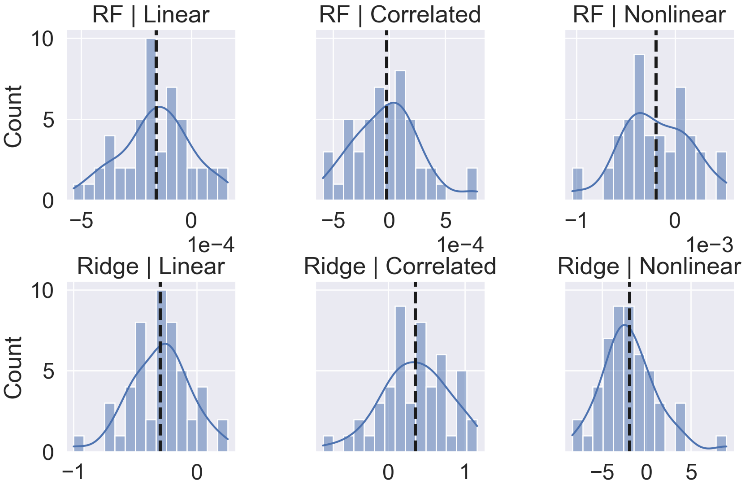
Special Case: Linear Model. For correlated features:

- When x_1 and x_2 have correlation $\rho \to 1$, we prove that $\Delta_1^* \to \Delta_2^*$ and are a function of $(\beta_1^* + \beta_2^*)^2$ for LOCO-MP.

As a comparison: original LOCO inference tends to miss correlated features.

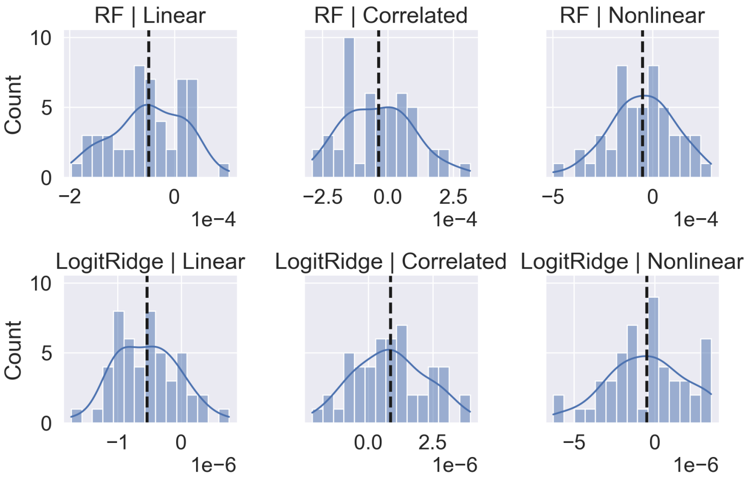
Minipatch Feature Importance vs. Population Feature Importance?

Histograms of the inference target for a noise feature in the regression setting



Minipatch Feature Importance vs. Population Feature Importance?

Histograms of the inference target for a noise feature in the classification setting



LOCO-MP: Detailed Assumption for Valid Coverage

A1. $\mathrm{Error}(Y, \widehat{Y})$ is Lipschitz-L w.r.t. the prediction $\widehat{Y}$.

A2. Bounded difference in MP predictions $||\hat{\mu}_{I,F}(X) - \hat{\mu}_{I',F'}(X)|| \leq B$.
(automatically hold for classification)

A3. Minipatch size: $n = o(\frac{\sigma_j}{LB}\sqrt{N})$.

A4. Minipatch number: $K \gg (\frac{L^2B^2N}{\sigma_j^2} + \frac{LB\sqrt{N}}{\sigma_j} + 1)\log(N)$, $K \gg \frac{M}{m}\log M$.

LOCO-MP Simulations: Validate Coverage

Simulation Set-up:

- Vary N, $M = 200$ (unless otherwise specified) & 10 true features.
- 3 Scenarios:
  1. Sparse Linear Regression (or Logistic Regression); iid features.
  2. Sparse Linear Regression (or Logistic Regression); correlated features.
     - Adjacent features have correlation 0.5.
  3. Sparse Non-linear Regression (or Logistic Regression); iid features.
     - Polynomial and MARS spline non-linearity.
- Minipatch LOCO (LOCO-MP) run with $m = \sqrt{M}$ and $n = \sqrt{N}$ and $K = 10,000$.

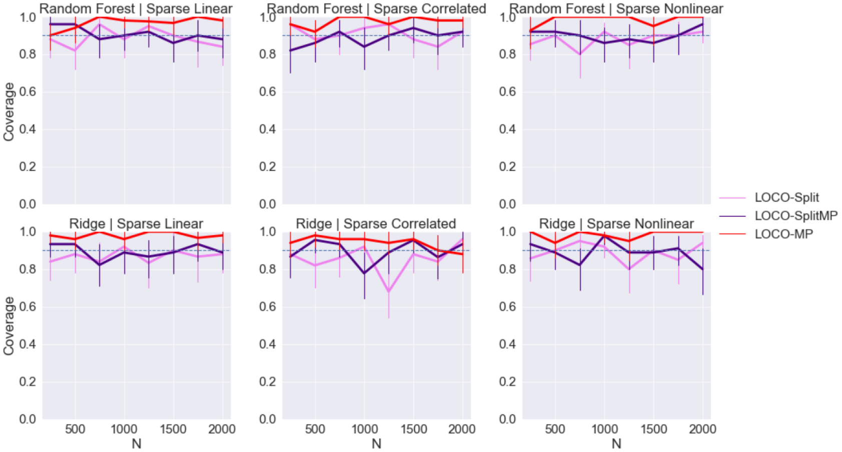
LOCO-MP Simulations: Validate Coverage

Simulation Set-up:

- $M = 200$, varying N, 10 signal features
- Sparse linear & nonlinear (logistic) regression; i.i.d. or correlated features
- LOCO-MP with $m = \sqrt{M}$, $n = \sqrt{N}$, $K = 10,000$

LOCO-MP Simulations: Comparative Results

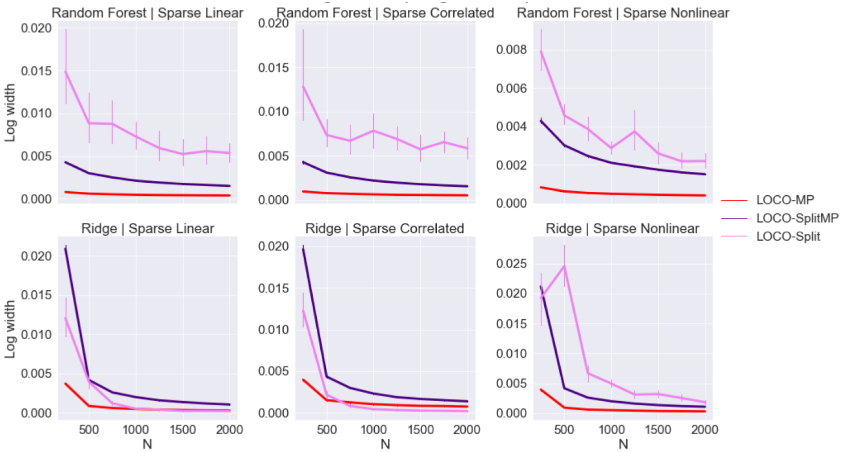
Theory Validation: Coverage.



Coverage for regression simulations for a null feature.

LOCO-MP Simulations: Comparative Results

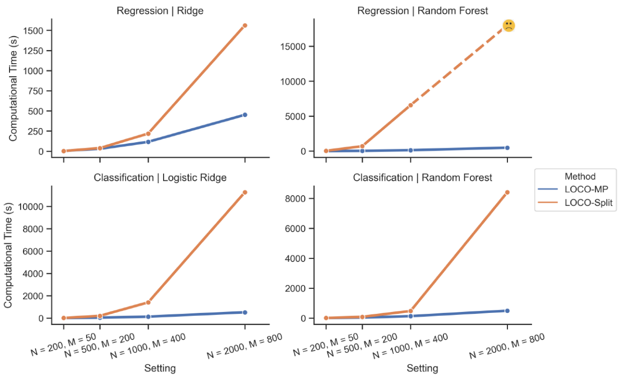
Interval Width:



Log interval width for regression simulations for a null feature.

LOCO-MP Simulations: Comparative Results

Computational Time:



Computational time for inference on all features in sparse linear regression and classification.

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