

四. 计算题 (14 分)

解:

$$(1) \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 6 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 2 & 1 \end{pmatrix}$$

$$(2) A(E - C^{-1}B)^T C^T = E \Rightarrow A[C(E - C^{-1}B)]^T = E \Rightarrow A[C - B]^T = E$$

$$A = ([C - B]^T)^{-1} = \left[\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^T \right]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 2 & 1 \end{pmatrix}$$

五. 计算题 (16 分)

解 1:

$$(A, b) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & a & 1 \\ 3 & 2 & 4 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 4-a & -1 \\ 0 & 0 & a-2 & 1-b \end{pmatrix}$$

当 $a \neq 2$ 时, 方程组有唯一解;

当 $a = 2, b \neq 1$ 时, 方程组无解;

当 $a = 2, b = 1$ 时, $r(A) = r(A, b) = 2 < 3$, 方程组有无穷多组解,

$$\text{此时 } (A, b) \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{方程组通解为 } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad k \text{ 为任意常数。}$$

解 2:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & a \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & a-4 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 4-a \\ 1 & 2 \end{vmatrix} = a-2$$

当 $|A| \neq 0$ 即 $a \neq 2$ 时, 方程组有唯一解;

当 $|A| = 0$ 即 $a = 2$ 时, 方程组无解或有无穷多组解;

$$\text{此时, } (A, b) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 4 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1-b \end{pmatrix}$$

当 $a = 2, b \neq 1$ 时, 方程组无解;

当 $a = 2, b = 1$ 时, $r(A) = r(A, b) = 2 < 3$, 方程组有无穷多组解,

$$\text{此时 } (A, b) \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{方程组通解为 } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad k \text{ 为任意常数。}$$