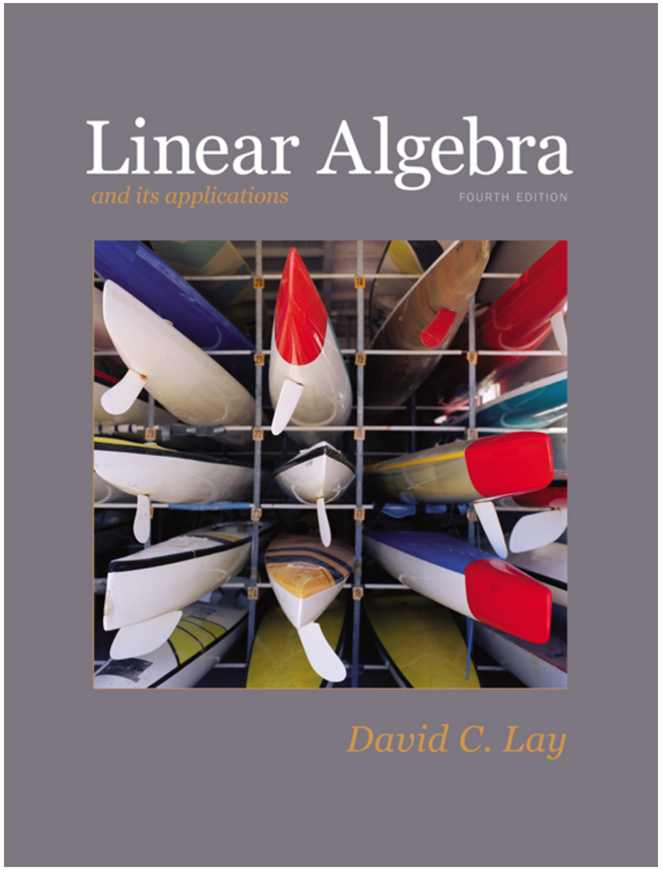
1 Linear Equations in Linear Algebra

1.1 SYSTEMS OF LINEAR EQUATIONS



LINEAR EQUATION

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers that are usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

LINEAR EQUATION

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

- The set of all possible solutions is called the **solution set** of the linear system.

- Two linear systems are called **equivalent** if they have the same solution set.

LINEAR EQUATION

- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be **inconsistent** if it has no solution.

MATRIX NOTATION

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- For the following system of equations,

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9,
\end{aligned}
$$

the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called the **coefficient matrix** of the system.

MATRIX NOTATION

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

- For the given system of equations,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the augmented matrix.

MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If m and n are positive numbers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)

- The basic strategy for solving a linear system is to replace one system with an equivalent system (*i.e.*, one with the same solution set) that is easier to solve.

SOLVING SYSTEM OF EQUATIONS

- **Example 1:** Solve the given system of equations.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \qquad \text{-----(1)}\\
2x_2 - 8x_3 &= 8 \qquad \text{-----(2)}\\
-4x_1 + 5x_2 + 9x_3 &= -9 \qquad \text{-----(3)}
\end{aligned}
$$

- **Solution:** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison.

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep x_1 in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ \underline{-4x_1 + 5x_2 + 9x_3 = -9} \\ -3x_2 + 13x_3 = -9 \end{array}$$

SOLVING SYSTEM OF EQUATIONS

- The result of this calculation is written in place of the original third equation.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Now, multiply equation 2 by $1/2$ in order to obtain 1 as the coefficient for x_2.

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline x_3 = 3 \end{array}$$

SOLVING SYSTEM OF EQUATIONS

- The new system has a triangular form.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ \hline x_3 = 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Now, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 term in equation 3 first to eliminate the $-4x_3$ and x_3 terms in equations 2 and 1.

SOLVING SYSTEM OF EQUATIONS

$$\begin{array}{r} 4x_3 = 12 \\ \underline{x_2 - 4x_3 = 4} \\ x_2 = 16 \end{array} \qquad \begin{array}{r} -x_3 = -3 \\ \underline{x_1 - 2x_2 + x_3 = 0} \\ x_1 - 2x_2 = -3 \end{array}$$

- Now, combine the results of these two operations.

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Move back to the x_2 in equation 2, and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Thus, the only solution of the original system is (29,16,3). To verify that (29,16,3) is a solution, substitute these values into the left side of the original system, and compute.

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

- The results agree with the right side of the original system, so (29,16,3) is a solution of the system.

ELEMENTARY ROW OPERATIONS

- Elementary row operations include the following:
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Example 2:** Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \qquad \text{----(4)}$$

- **Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- To obtain an x_1 in in the first equation, interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

- To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \quad \text{----}(5)$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- Next, use the x_2 term in the second equation to eliminate the $-(1/2)x_2$ term from the third equation. Add $1/2$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \qquad \text{----(6)}$$

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \qquad \text{----(7)} \\ 0 &= 5/2 \end{aligned}$$

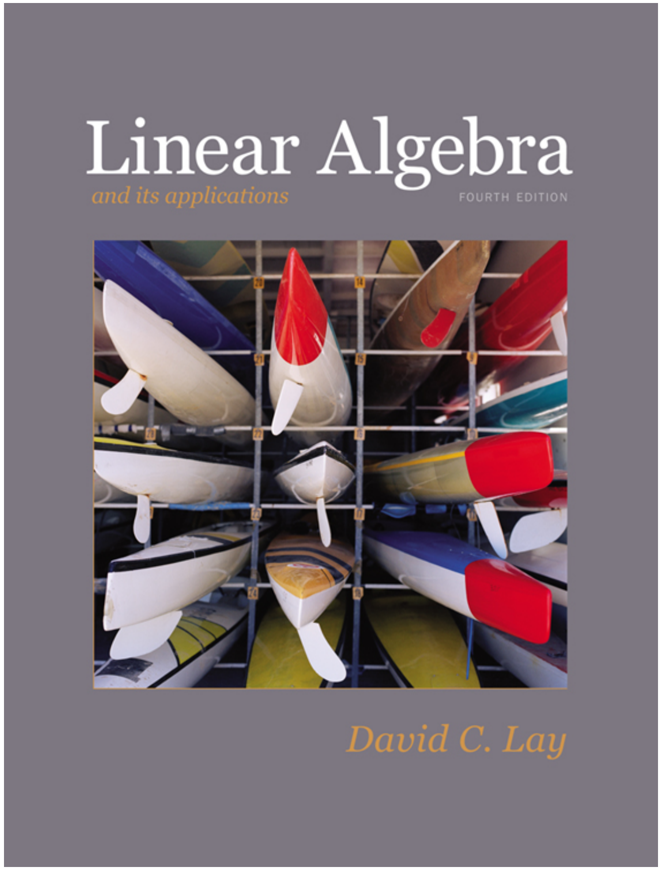
EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- The equation $0 = 5/2$ is a short form of $0x_1 + 0x_2 + 0x_3 = 5/2.$
- There are no values of x_1, x_2, x_3 that satisfy (7) because the equation $0 = 5/2$ is never true.
- Since (7) and (4) have the same solution set, the original system is inconsistent (*i.e.*, has no solution).

1 Linear Equations in Linear Algebra

1.2

Row Reduction and Echelon Forms



ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

    4. The leading entry in each nonzero row is 1.

    5. Each leading 1 is the only nonzero entry in its column.

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

ECHELON FORM

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

  Theorem 1: Uniqueness of the Reduced Echelon Form

  Each matrix is row equivalent to one and only one reduced echelon matrix.

PIVOT POSITION

- If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of** A; if U is in reduced echelon form, we call U **the reduced echelon form of** A.

- A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

PIVOT POSITION

- **Example 1:** Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

PIVOT POSITION

- Now, interchange rows 1 and 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

PIVOT POSITION

- Choose 2 in the second row as the next pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

- Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way a leading entry can be created in column 3. But, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

PIVOT POSITION

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

PIVOT POSITION

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot columns

- The pivots in the example are 1, 2 and –5.

ROW REDUCTION ALGORITHM

- **Example 2:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution:**
- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

ROW REDUCTION ALGORITHM

$$\begin{array}{c} \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

ROW REDUCTION ALGORITHM

- Interchange rows 1 and 3. (Rows 1 and 2 could have also been interchanged instead.)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.

ROW REDUCTION ALGORITHM

- We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add –1 times row 1 to row 2.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 4:** Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

ROW REDUCTION ALGORITHM

- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

Pivot

$$\begin{bmatrix} \boxed{3 \quad -9 \quad 12 \quad -9 \quad 6 \quad 15} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

- For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below.

ROW REDUCTION ALGORITHM

- This produces the following matrix.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

ROW REDUCTION ALGORITHM

- Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.
- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \longleftarrow \text{Row } 1 + (-6) \times \text{row } 3 \\ \longleftarrow \text{Row } 2 + (-2) \times \text{row } 3 \\ \end{matrix}$$

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \longleftarrow \text{Row scaled by } \frac{1}{2}$$

ROW REDUCTION ALGORITHM

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1 + (9) \times \text{row } 2$$

- Finally, scale row 1, dividing by the pivot, 3.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

- This is the reduced echelon form of the original matrix.

- The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

SOLUTIONS OF LINEAR SYSTEMS

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

- There are 3 variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad \text{----(1)}$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

SOLUTIONS OF LINEAR SYSTEMS

- Whenever a system is consistent, as in (1), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

- In (1), solve the first and second equations for x_1 and x_2. (Ignore the third equation; it offers no restriction on the variables.)

SOLUTIONS OF LINEAR SYSTEMS

$$x_1 = 1 + 5x_3$$

$$x_2 = 4 - x_3 \qquad \text{----(2)}$$

$$x_3 \text{ is free}$$

- The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (2) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

- *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.*

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- The description in (2) is a *parametric description* of solutions sets in which the free variables act as parameters.

- *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- For instance, in system (1), add 5 times equation 2 to equation 1 and obtain the following equivalent system.

$$x_1 + 5x_2 = 21$$

$$x_2 + x_3 = 4$$

- We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.
- When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

EXISTENCE AND UNIQUENESS THEOREM

Theorem 2: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \dots \ 0 \quad b] \text{ with } b \text{ nonzero.}$$

- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

ROW REDUCTION TO SOLVE A LINEAR SYSTEM

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.

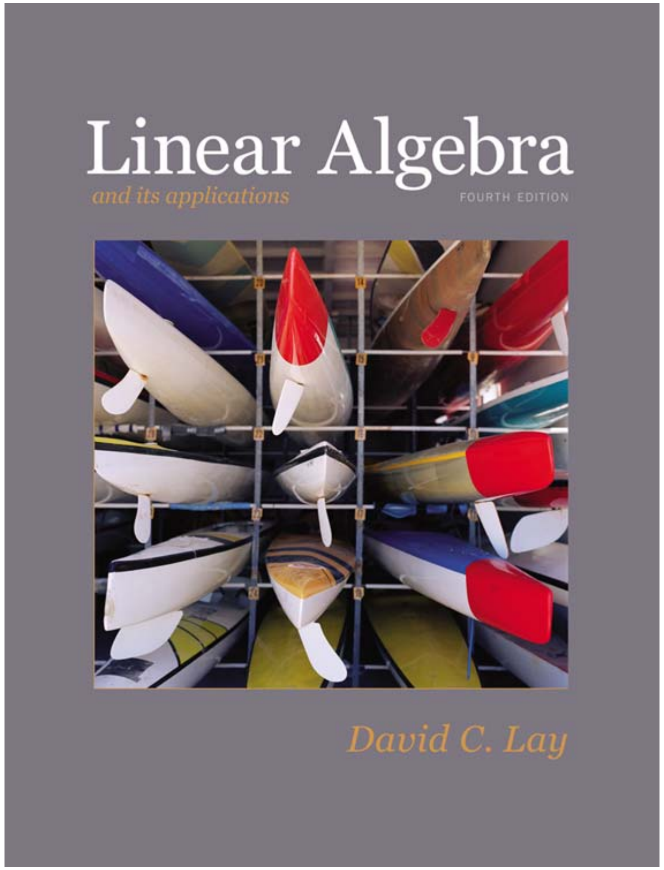
ROW REDUCTION TO SOLVE A LINEAR SYSTEM

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1 Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS



VECTOR EQUATIONS

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.

- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$ (read "r-two").

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains 2 entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number *c*, the **scalar multiple** of **u** by *c* is the vector *c***u** obtained by multiplying each entry in **u** by *c*.

VECTOR EQUATIONS

- **Example 1:** Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find

$4\mathbf{u}, (-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

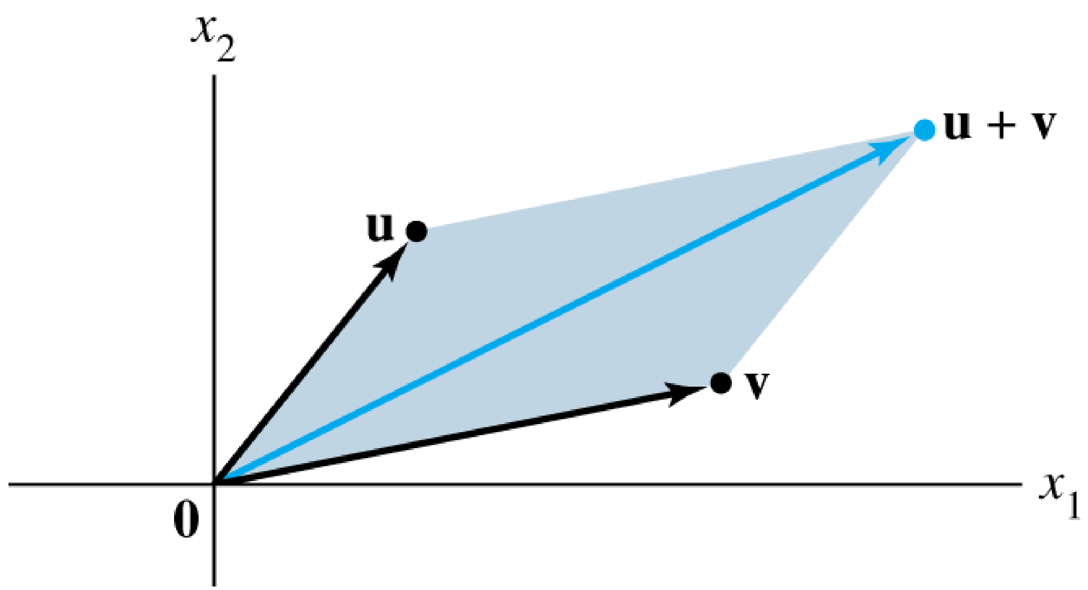
GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.

- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

PARALLELOGRAM RULE FOR ADDITION

- If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See the figure below.



VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n\times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$,

where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

LINEAR COMBINATIONS

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

  is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

LINEAR COMBINATIONS

- **Example 2:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad \text{----(1)}$$

If vector equation (1) has a solution, find it.

LINEAR COMBINATIONS

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}}_{\mathbf{a}_2} = \underbrace{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}_{\mathbf{a}_3},$$

which is same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

LINEAR COMBINATIONS

and $\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. ----(2)

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \quad \text{----(3)}$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

- Write this matrix in a way that identifies its columns.

$$\begin{bmatrix} a_1 & a_2 & b \end{bmatrix} \qquad \text{----(4)}$$

LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \qquad \text{----(5)}$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

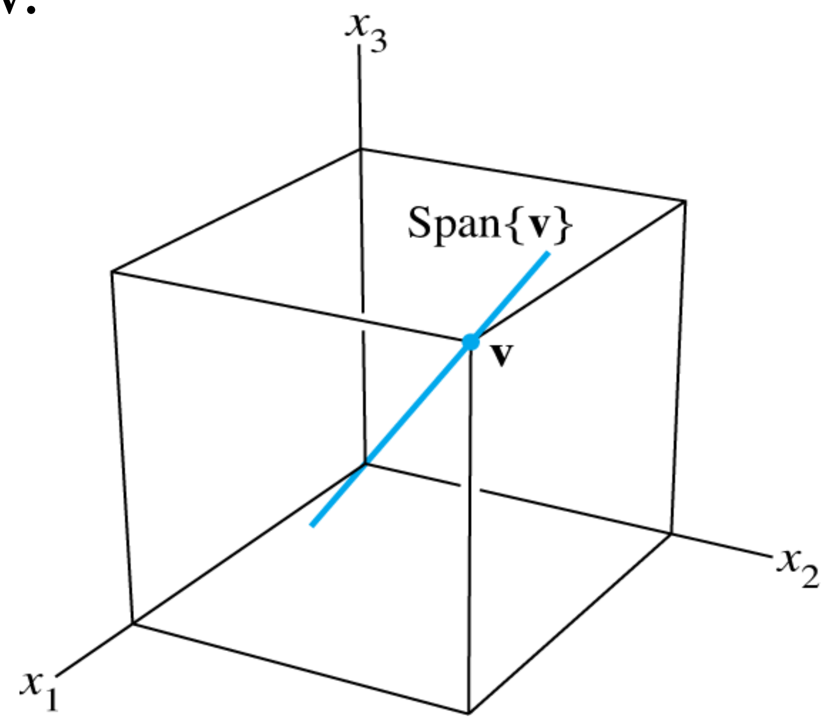
- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

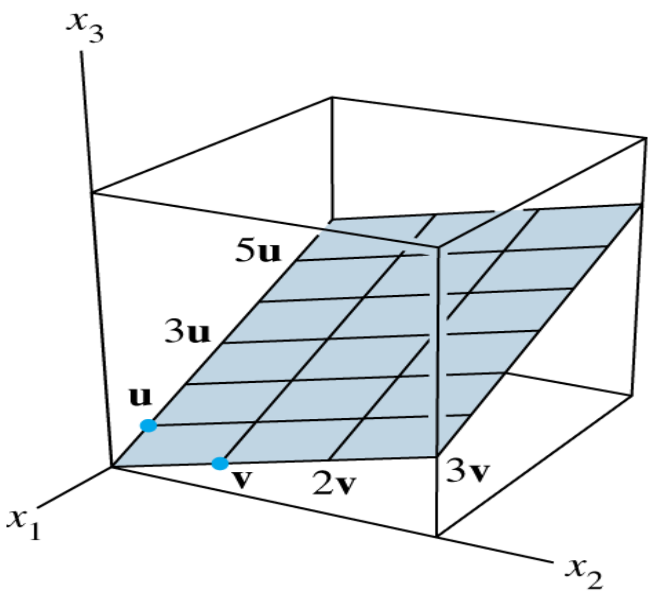
A GEOMETRIC DESCRIPTION OF SPAN {V}

- Let **v** be a nonzero vector in $\mathbb{R}^3$. Then Span {**v**} is the set of all scalar multiples of **v**, which is the set of points on the line in $\mathbb{R}^3$ through **v** and **0**. See the figure below.



A GEOMETRIC DESCRIPTION OF SPAN {U, V}

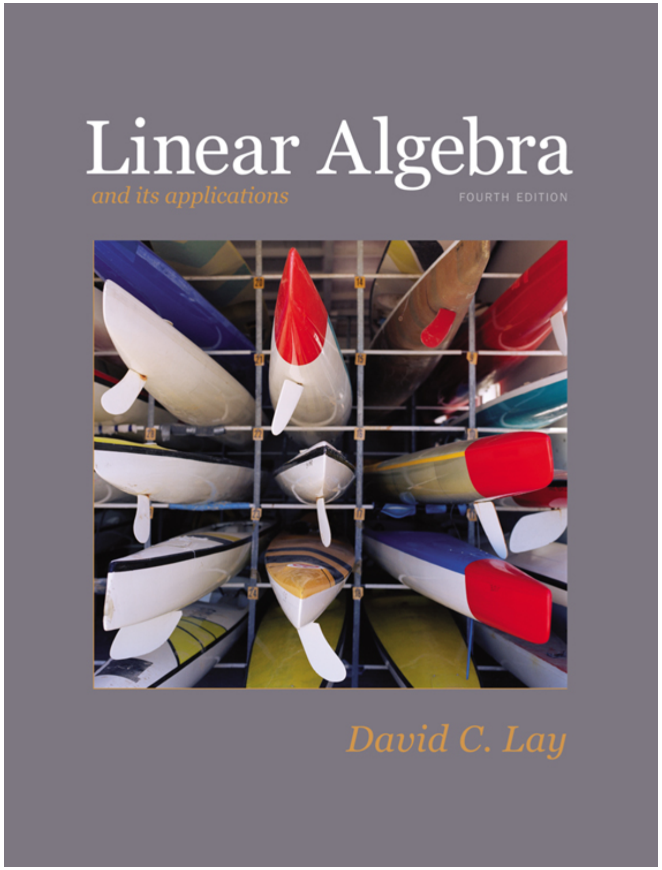
- If **u** and **v** are nonzero vectors in $\mathbb{R}^3$, with **v** not a multiple of **u**, then Span {**u, v**} is the plane in $\mathbb{R}^3$ that contains **u**, **v**, and **0**.
- In particular, Span {**u**, **v**} contains the line in $\mathbb{R}^3$ through **u** and **0** and the line through **v** and **0**. See the figure below.



1 Linear Equations in Linear Algebra

1.4

THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$



MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

- $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Example 1:** For $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.
- **Solution:** Place $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$.
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$x_1 + 2x_2 - x_3 = 4 \qquad \text{----(1)}$$
$$-5x_2 + 3x_3 = 1$$

is equivalent to

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \text{----(2)}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the given example (1), the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \text{-----(3)}$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Theorem 3:** If A is an $m \times n$ matrix, with columns a1, …, an, and if b is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n a_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathrm{L} & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

EXISTENCE OF SOLUTIONS

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if **b** is a linear combination of the columns of A.
- **Theorem 4:** Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.
  a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.
  c. The columns of A span $\mathbb{R}^m$.
  d. A has a pivot position in every row.

PROOF OF THEOREM 4

- Statements (a), (b), and (c) are logically equivalent.
- So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or false.
- Let U be an echelon form of A.
- Given **b** in $\mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to an augmented matrix $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ for some **d** in $\mathbb{R}^m$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \ldots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

- If statement (d) is true, then each row of U contains a pivot position, and there can be no pivot in the augmented column.

PROOF OF THEOREM 4

- So $A\mathbf{x} = \mathbf{b}$ has a solution for any **b**, and (a) is true.
- If (d) is false, then the last row of U is all zeros.
- Let **d** be any vector with a 1 in its last entry.
- Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an *inconsistent* system.
- Since row operations are reversible, $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ can be transformed into the form $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.
- The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false.

COMPUTATION OF $A\mathbf{x}$

- **Example 2:** Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

- **Solution:** From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

COMPUTATION OF $A\mathbf{x}$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \text{---}(1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

- The first entry in the product $A\mathbf{x}$ is a sum of products (*a dot product*), using the first row of A and the entries in $\mathbf{x}$.

COMPUTATION OF $A\mathbf{x}$

- That is, $\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \end{bmatrix}.$

- Similarly, the second entry in $A\mathbf{x}$ can be calculated by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products.

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$.
- If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vertex $\mathbf{x}$.
- The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by I.
- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A*x

- **Theorem 5:** If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then
  a. $A(\mathrm{u}+\mathrm{v}) = A\mathrm{u} + A\mathrm{v}$;
  b. $A(c\mathrm{u}) = c(A\mathrm{u})$.
- **Proof:** For simplicity, take $n = 3$, $A = [\mathrm{a}_1 \quad \mathrm{a}_2 \quad \mathrm{a}_3]$, and **u**, **v** in $\mathbb{R}^3$.
- For $i = 1, 2, 3$, let u_i and v_i be the ith entries in **u** and **v**, respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A***x**

- To prove statement (a), compute $A(\mathrm{u}+\mathrm{v})$ as a linear combination of the columns of A using the entries in $\mathrm{u}+\mathrm{v}$ as weights.

$$A(\mathrm{u}+\mathrm{v}) = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

Entries in $\mathrm{u}+\mathrm{v}$

$$= (u_1+v_1)\mathrm{a}_1 + (u_2+v_2)\mathrm{a}_2 + (u_3+v_3)\mathrm{a}_3$$

Columns of A

$$= (u_1\mathrm{a}_1 + u_2\mathrm{a}_2 + u_3\mathrm{a}_3) + (v_1\mathrm{a}_1 + v_2\mathrm{a}_2 + v_3\mathrm{a}_3)$$

$$= A\mathrm{u} + A\mathrm{v}$$

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A*x

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$A(c\mathbf{u}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3$$

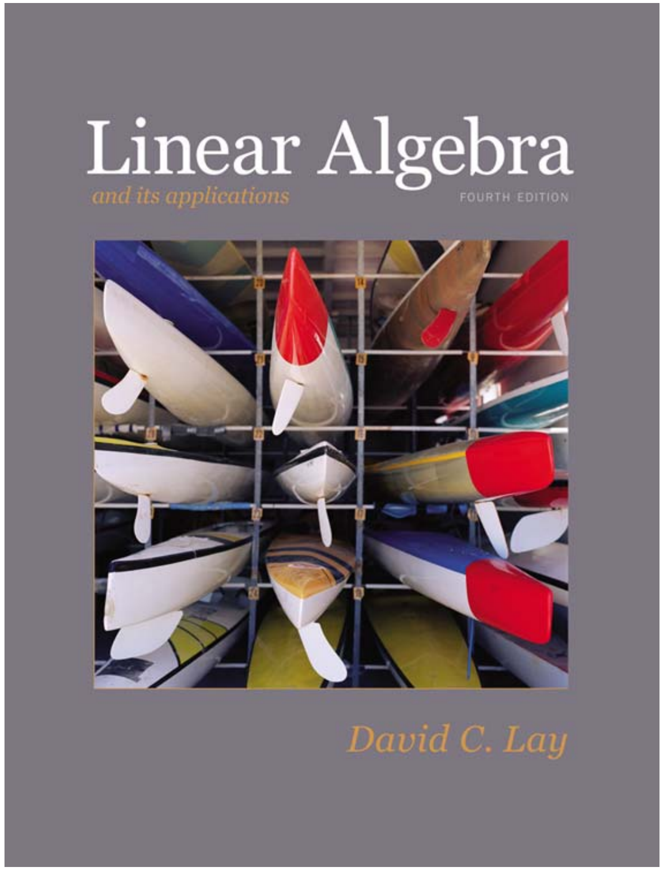
$$= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3)$$

$$= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3)$$

$$= c(A\mathbf{u})$$

1 Linear Equations in Linear Algebra

1.5



PEARSON

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
- This zero solution is usually called the **trivial solution**.
- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

- **Example 1:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let A be the matrix of coefficients of the system and row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \square \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \square \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

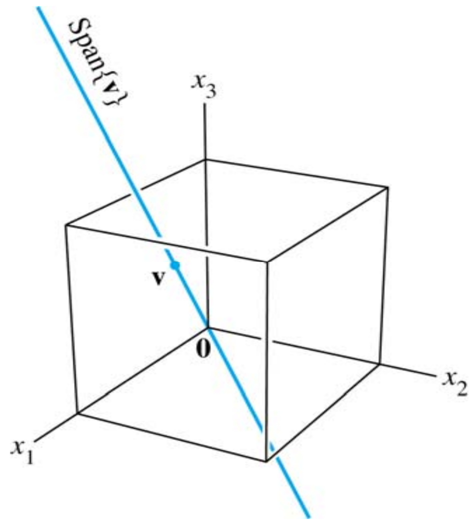
- Since x_3 is a free variable, $A\mathrm{x} = 0$ has nontrivial solutions (one for each choice of x_3.)
- Continue the row reduction of $\begin{bmatrix} A & 0 \end{bmatrix}$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

- Solve for the basic variables x_1 and x_2 to obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free.
- As a vector, the general solution of $A\mathrm{x} = 0$ has the form given below.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathrm{v}, \text{ where } \mathrm{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of **v**.
- The trivial solution is obtained by choosing $x_3 = 0$.
- Geometrically, the solution set is a line through **0** in $\mathbb{R}^3$. See the figure below.



- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (s, t in $\mathbb{R}$) is called a **parametric vector equation** of the plane.

- In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line.

- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in **parametric vector form**.

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

- **Example 2:** Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

- **Solution:** Row operations on $\begin{bmatrix} A & 0 \end{bmatrix}$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}.$$

- Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

- As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form given below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

- The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad \text{----(1)}$$

  describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
- The solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad \text{----(2)}$$

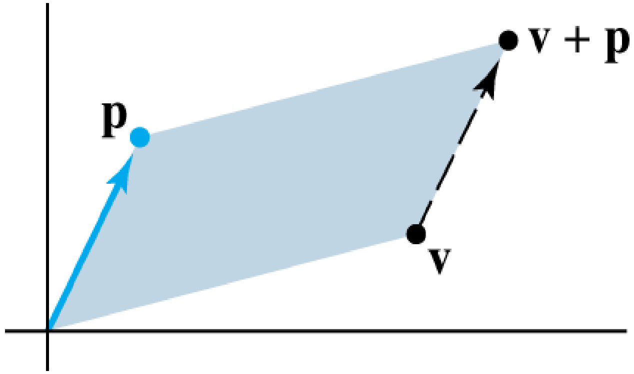
  [with the same **v** that appears in (1)].
- Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector **p** to the solutions of $A\mathbf{x} = \mathbf{0}$.

- The vector **p** itself is just one particular solution of $A\mathbf{x} = \mathbf{b}$ [corresponding to $t = 0$ in (1).]

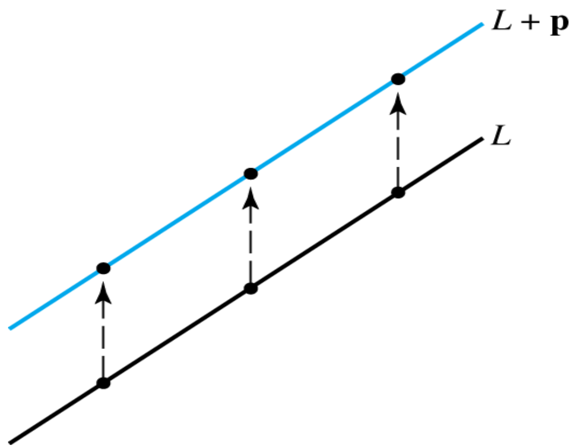
- Now, to describe the solution of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a *translation*.

- Given **v** and **p** in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding **p** to **v** is to *move* **v** in a direction parallel to the line through **p** and **0**.

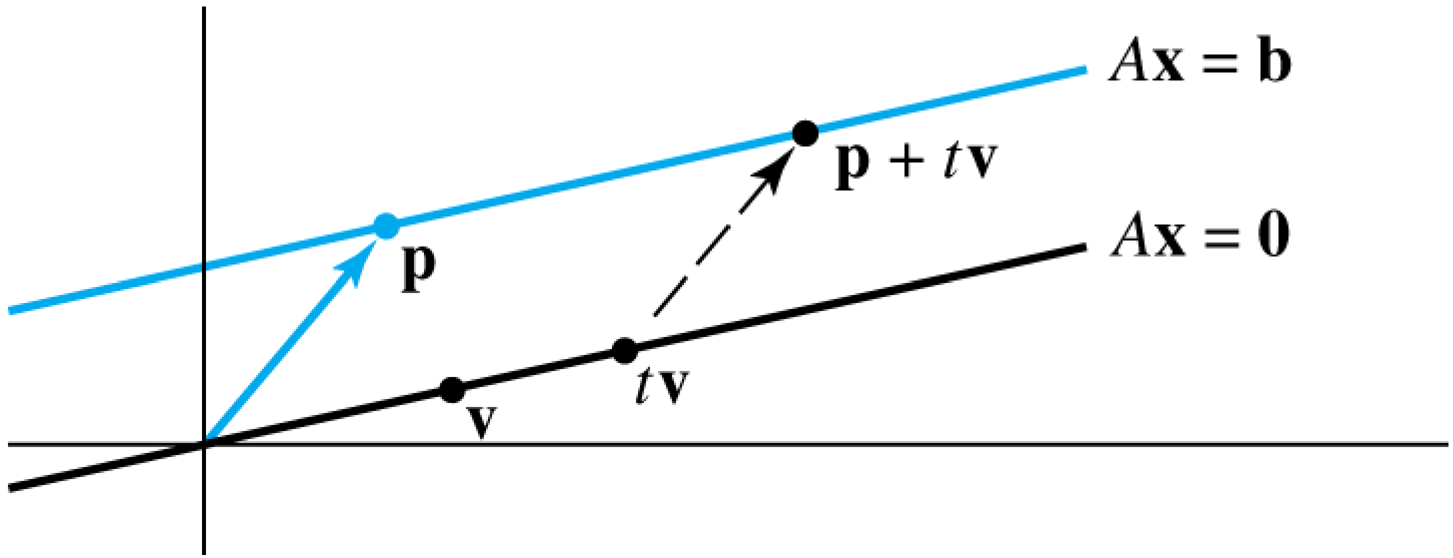
- We say that **v** is **translated by p** to $\mathbf{v}+\mathbf{p}$. See the following figure.



- If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector **p**, the result is a line parallel to L. See the following figure.



- Suppose L is the line through **0** and **v**, described by equation (2).
- Adding **p** to each point on L produces the translated line described by equation (1).
- We call (1) **the equation of the line through p parallel to v**.
- Thus the solution set of $A\mathrm{x} = \mathrm{b}$ *is a line through* **p** *parallel to the solution set of* $A\mathrm{x} = 0$. The figure on the next slide illustrates this case.



- The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = 0$ shown in the figure above generalizes to any consistent equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables.

- **Theorem 6:** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given **b**, and let **p** be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

- This theorem says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution **p** of $A\mathbf{x} = \mathbf{b}$ for the translation.

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.