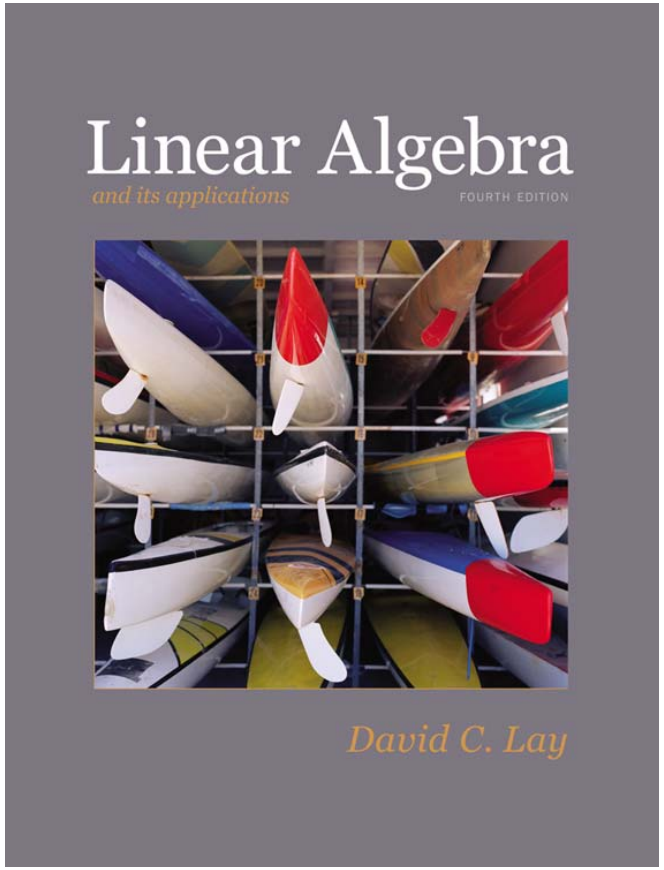
7 Symmetric Matrices and Quadratic Forms

7.1

DIAGONALIZATION OF SYMMETRIC MATRICES



SYMMETRIC MATRIX

- A **symmetric matrix** is a matrix A such that $A^T = A$.
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

SYMMETRIC MATRIX

- **Theorem 1:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
- **Proof:** Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that correspond to distinct eigenvalues, say, λ_1 and λ_2.
- To show that $\mathrm{v}_1 \cdot \mathrm{v}_2 = 0$, compute

$$\lambda_1 \mathrm{v}_1 \cdot \mathrm{v}_2 = (\lambda_1 \mathrm{v}_1)^T \mathrm{v}_2 = (A\mathrm{v}_1)^T \mathrm{v}_2$$ Since $\mathbf{v}_1$ is an eigenvector

$$= (\mathrm{v}_1^T A^T)\mathrm{v}_2 = \mathrm{v}_1^T (A\mathrm{v}_2)$$ Since $A^T = A$

$$= \mathrm{v}_1^T (\lambda_2 \mathrm{v}_2)$$ Since $\mathbf{v}_2$ is an eigenvector

$$= \lambda_2 \mathrm{v}_1^T \mathrm{v}_2 = \lambda_2 \mathrm{v}_1 \cdot \mathrm{v}_2$$

SYMMETRIC MATRIX

- Hence $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.
- But $\lambda_1 - \lambda_2 \neq 0$, so $v_1 \cdot v_2 = 0$.
- An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1} \qquad \text{----(1)}$$

- Such a diagonalization requires n linearly independent and orthonormal eigenvectors.
- When is this possible?
- If A is orthogonally diagonalizable as in (1), then

$$A^T = (PDP^T)^T = P^{TT}D^TP^T = PDP^T = A$$

SYMMETRIC MATRIX

- Thus A is symmetric.
- **Theorem 2:** An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

- **Example 1:** Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose characteristic equation is

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

SYMMETRIC MATRIX

- **Solution:** The usual calculations produce bases for the eigenspaces:

$$\lambda = 7 : \mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \lambda = -2 : \mathrm{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

- Although $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, they are not orthogonal.
- The projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is $\dfrac{\mathrm{v}_2 \cdot \mathrm{v}_1}{\mathrm{v}_1 \cdot \mathrm{v}_1}\mathrm{v}_1$.

SYMMETRIC MATRIX

- The component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ is

$$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

- Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$.

- (Note that $\mathbf{z}_2$ is linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so $\mathbf{z}_2$ is in the eigenspace).

SYMMETRIC MATRIX

- Since the eigenspace is two-dimensional (with basis $\mathbf{v}_1$, $\mathbf{v}_2$), the orthogonal set $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an *orthogonal basis* for the eigenspace, by the Basis Theorem.

- Normalize $\mathbf{v}_1$ and $\mathbf{z}_2$ to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$$\mathrm{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \mathrm{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

SYMMETRIC MATRIX

- An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathrm{u}_3 = \frac{1}{\left\|2\mathrm{v}_3\right\|} 2\mathrm{v}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

- By Theorem 1, $\mathbf{u}_3$ is orthogonal to the other eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

- Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

SYMMETRIC MATRIX

- Let

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

THE SPECTRAL THEOREM

- The set if eigenvalues of a matrix A is sometimes called the *spectrum* of A, and the following description of the eigenvalues is called a *spectral theorem*.

- **Theorem 3:** The Spectral Theorem for Symmetric Matrices
- An $n \times n$ symmetric matrix A has the following properties:
  a. A has n real eigenvalues, counting multiplicities.

THE SPECTRAL THEOREM

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

SPECTRAL DECOMPOSITION

- Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1,\ldots,\mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1,\ldots,\lambda_n$ are in the diagonal matrix D.
- Then, since $P^{-1} = P^T$,

$$A = PDP^T = \begin{bmatrix} \mathrm{u}_1 & \cdots & \mathrm{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathrm{u}_1^T \\ \vdots \\ \mathrm{u}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathrm{u}_1 & \cdots & \lambda_n \mathrm{u}_n \end{bmatrix} \begin{bmatrix} \mathrm{u}_1^T \\ \vdots \\ \mathrm{u}_n^T \end{bmatrix}$$

SPECTRAL DECOMPOSITION

- Using the column-row expansion of a product, we can write

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \quad \text{----(2)}$$

- This representation of A is called a **spectral decomposition** of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A.
- Each term in (2) is an $n \times n$ matrix of rank 1.
- For example, every column of $\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T$ is a multiple of $\mathbf{u}_1$.
- Each matrix $\mathbf{u}_j \mathbf{u}_j^T$ is a **projection matrix** in the sense that for each $\mathbf{x}$ in $\mathbb{R}^n$, the vector $(\mathbf{u}_j \mathbf{u}_j^T)\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto the subspace spanned by $\mathbf{u}_j$.

SPECTRAL DECOMPOSITION

- **Example 2:** Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

- **Solution:** Denote the columns of P by $\mathbf{u}_1$ and $\mathbf{u}_2$.

- Then $$A = 8\mathrm{u}_1\mathrm{u}_1^T + 3\mathrm{u}_2\mathrm{u}_2^T$$

SPECTRAL DECOMPOSITION

- To verify the decomposition of A, compute

$$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

and

$$8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A$$