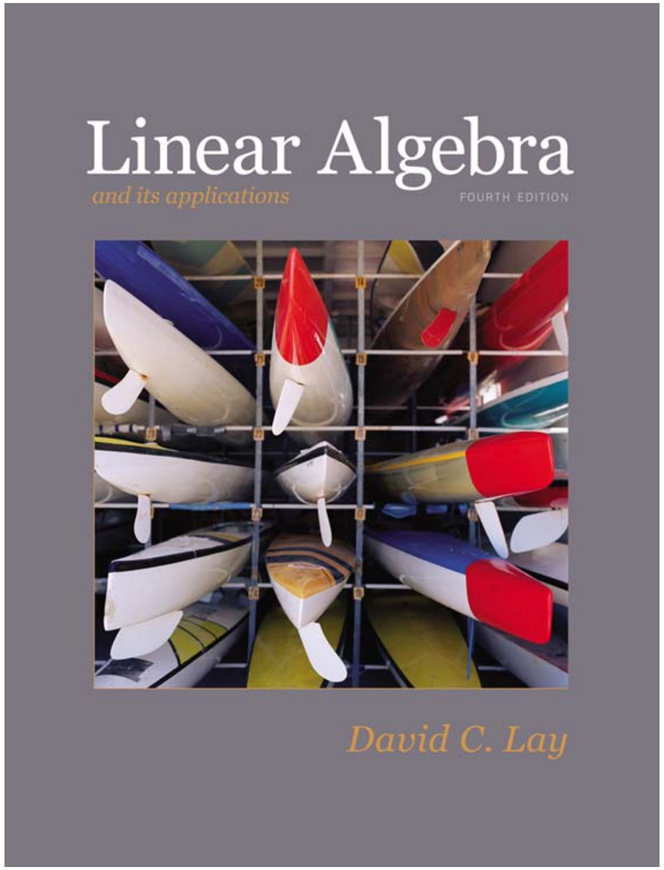
1 Linear Equations in Linear Algebra

1.7



- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathrm{v}_1 + x_2\mathrm{v}_2 + \ldots + x_p\mathrm{v}_p = 0$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0 \quad \text{----(1)}$$

- Equation (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.

- An indexed set is linearly dependent if and only if it is not linearly independent.

- **Example 1:** Let $\mathrm{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathrm{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

- **Solution:** We must determine if there is a nontrivial solution of the following equation.

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}+x_3\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

- Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.
- Choose any nonzero value for x_3—say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

- Substitute these values into equation (1) and obtain the equation below.

$$10\mathrm{v}_1 - 5\mathrm{v}_2 + 5\mathrm{v}_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

- Suppose that we begin with a matrix $A=\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ instead of a set of vectors.

- The matrix equation $A\mathbf{x}=\mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

- *Each linear dependence relation among the columns of A corresponds to a nontrivial solution of* $A\mathbf{x}=\mathbf{0}$.

- Thus, the columns of matrix A are linearly independent if and only if the equation $A\mathbf{x}=\mathbf{0}$ has only the trivial solution.

- A set containing only one vector – say, **v** – is linearly independent if and only if **v** is not the zero vector.
- This is because the vector equation $x_1\mathrm{v} = 0$ has only the trivial solution when $\mathrm{v} \neq 0$.
- The zero vector is linearly dependent because $x_1 0 = 0$ has many nontrivial solutions.

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

- **Theorem 7:** Characterization of Linearly Dependent Sets
- An indexed set $S = \{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
- In fact, if S is linearly dependent and $\mathrm{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathrm{v}_{j-1}$.

- **Proof:** If some $\mathbf{v}_j$ in S equals a linear combination of the other vectors, then $\mathbf{v}_j$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on $\mathbf{v}_j$.
- [For instance, if $\mathrm{v}_1 = c_2\mathrm{v}_2 + c_3\mathrm{v}_3$, then $0 = (-1)\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 + 0\mathrm{v}_4 + \ldots + 0\mathrm{v}_p$.]
- Thus S is linearly dependent.
- Conversely, suppose S is linearly dependent.
- If $\mathbf{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S.

- Otherwise, $\mathbf{v}_1 \neq 0$, and there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0.$$

- Let j be the largest subscript for which $c_j \neq 0$.

- If $j = 1$, then $c_1\mathbf{v}_1 = 0$, which is impossible because $\mathbf{v}_1 \neq 0$.

- So $j > 1$, and

$$c_1\mathrm{v}_1 + \ldots + c_j\mathrm{v}_j + 0\mathrm{v}_j + 0\mathrm{v}_{j+1} + \ldots + 0\mathrm{v}_p = 0$$

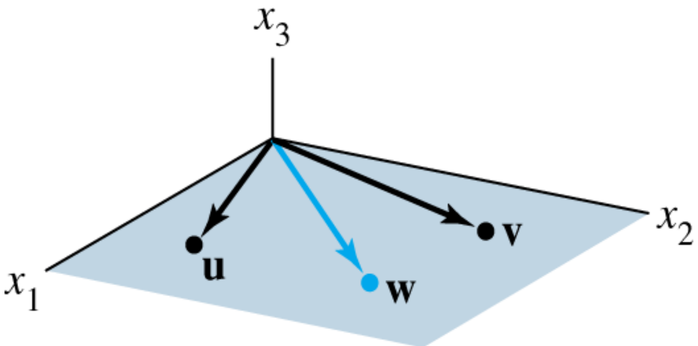
$$c_j\mathrm{v}_j = -c_1\mathrm{v}_1 - \ldots - c_{j-1}\mathrm{v}_{j-1}$$

$$\mathrm{v}_j = \left(-\frac{c_1}{c_j}\right)\mathrm{v}_1 + \ldots + \left(-\frac{c_{j-1}}{c_j}\right)\mathrm{v}_{j-1}.$$

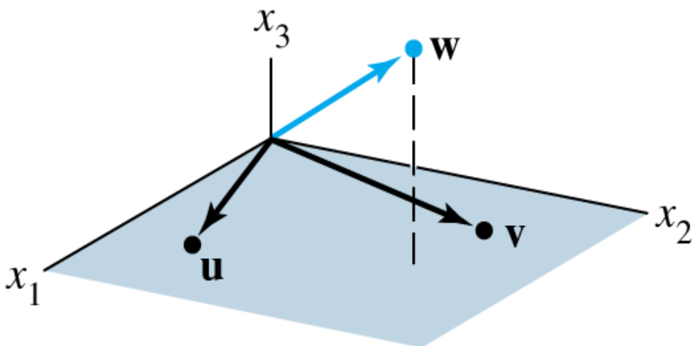
- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.
- **Example 2:** Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by **u** and **v**, and explain why a vector **w** is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

- **Solution:** The vectors **u** and **v** are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$.
- Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$).
- If **w** is a linear combination of **u** and **v**, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7.
- Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.
- By theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$).
- That vector must be **w**, since **v** is not a multiple of **u**.

- So **w** is in Span {**u**, **v**}. See the figures given below.



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

- Example 2 generalizes to any set {**u**, **v**, **w**} in $\mathbb{R}^3$ with **u** and **v** linearly independent.
- The set {**u**, **v**, **w**} will be linearly dependent if and only if **w** is in the plane spanned by **u** and **v**.

- **Theorem 8:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
- **Proof:** Let $A = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_p \end{bmatrix}$.
- Then A is $n \times p$, and the equation $A\mathrm{x} = 0$ corresponds to a system of n equations in p unknowns.
- If $p > n$, there are more variables than equations, so there must be a free variable.

- Hence $A\mathrm{x} = 0$ has a nontrivial solution, and the columns of A are linearly dependent.
- See the figure below for a matrix version of this theorem.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

- Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

- **Theorem 9:** If a set $S = \{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

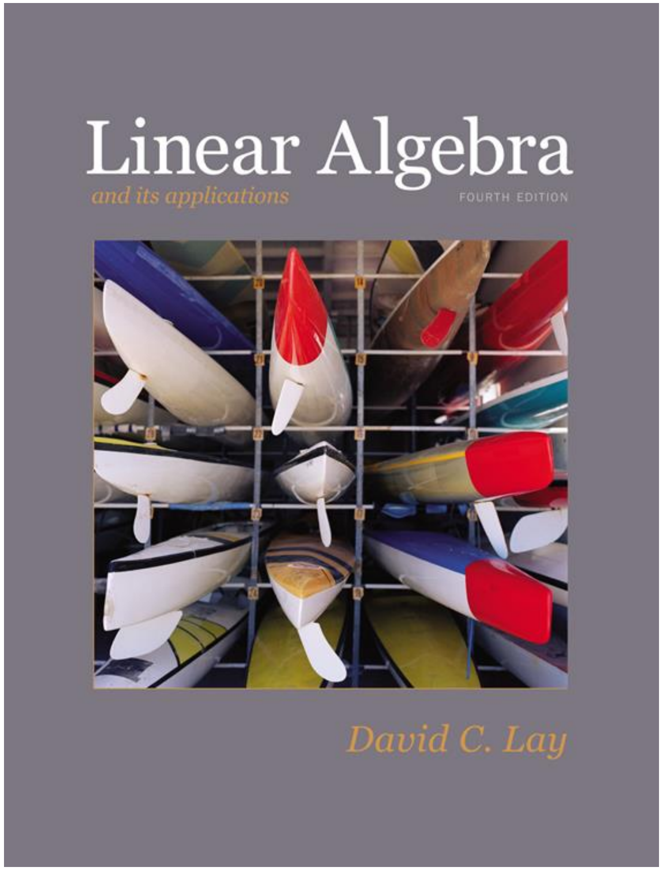
- **Proof:** By renumbering the vectors, we may suppose $\mathrm{v}_1 = 0$.

- Then the equation $1\mathrm{v}_1 + 0\mathrm{v}_2 + \ldots + 0\mathrm{v}_p = 0$ shows that S in linearly dependent.

1 Linear Equations in Linear Algebra

1.8

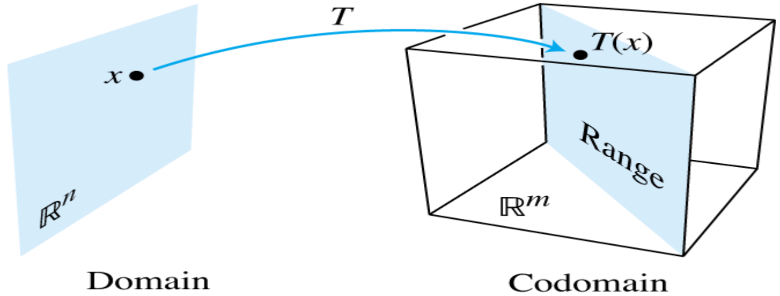
INTRODUCTION TO LINEAR TRANSFORMATIONS



LINEAR TRANSFORMATIONS

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.
- The notation $T:\mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the **range** of T. See the figure on the next slide.

MATRIX TRANSFORMATIONS



Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

- For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.
- For simplicity, we denote such a *matrix transformation* by $\mathbf{x} \mapsto A\mathbf{x}$.
- The domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

MATRIX TRANSFORMATIONS

- The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.
- **Example 1:** Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathrm{c} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.

  and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathrm{x}) = A\mathrm{x}$, so that

$$T(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

MATRIX TRANSFORMATIONS

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

MATRIX TRANSFORMATIONS

- **Solution:**

  a. Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

  b. Solve $T(\mathbf{x}) = \mathbf{b}$ for **x**. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}. \qquad \text{----(1)}$$

MATRIX TRANSFORMATIONS

- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

----(2)

- Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathrm{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$.

- The image of this **x** under *T* is the given vector **b**.

MATRIX TRANSFORMATIONS

c. Any **x** whose image under T is **b** must satisfy equation (1).
   - From (2), it is clear that equation (1) has a unique solution.
   - So there is exactly one **x** whose image is **b**.

d. The vector **c** is in the range of T if **c** is the image of some **x** in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some **x**.
   - This is another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent.

MATRIX TRANSFORMATIONS

- To find the answer, row reduce the augmented matrix.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

- The third equation, $0 = -35$, shows that the system is inconsistent.

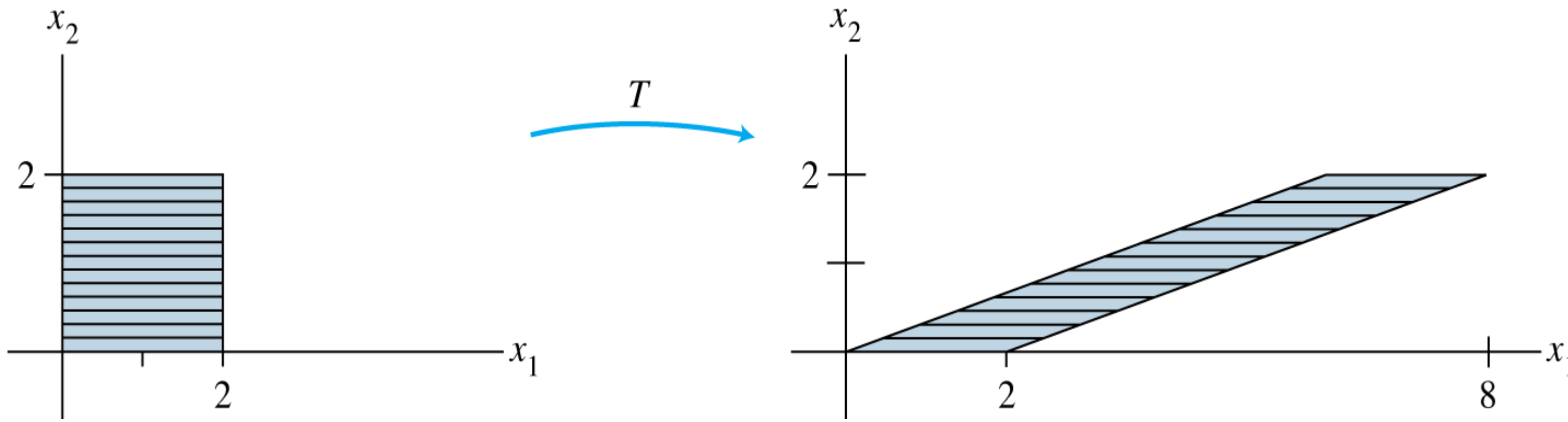
- So **c** is *not* in the range of *T*.

SHEAR TRANSFORMATION

- **Example 2:** Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathrm{x}) = A\mathrm{x}$ is called a **shear transformation**.

- It can be shown that if T acts on each point in the 2×2 square shown in the figure on the next slide, then the set of images forms the shaded parallelogram.

SHEAR TRANSFORMATION



- The key idea is to show that T maps line segments onto line segments and then to check that the corners of the square map onto the vertices of the parallelogram.
- For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

$$T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$

LINEAR TRANSFORMATIONS

and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

- *T* deforms the square as if the top of the square were pushed to the right while the base is held fixed.
- **Definition:** A transformation (or mapping) *T* is **linear** if:
  i. $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all **u**, **v** in the domain of *T*;
  ii. $T(c\mathrm{u}) = cT(\mathrm{u})$ for all scalars *c* and all **u** in the domain of *T*.

LINEAR TRANSFORMATIONS

- Linear transformations *preserve the operations of vector addition and scalar multiplication*.
- Property (i) says that the result $T(\mathrm{u}+\mathrm{v})$ of first adding **u** and **v** in $\mathbb{R}^n$ and then applying T is the same as first applying T to **u** and **v** and then adding $T(\mathbf{u})$ and $T(\mathbf{v})$ in $\mathbb{R}^m$.
- These two properties lead to the following useful facts.
- If T is a linear transformation, then

$$T(0) = 0 \qquad \text{----(3)}$$

LINEAR TRANSFORMATIONS

and $T(c\mathrm{u}+d\mathrm{v})=cT(\mathrm{u})+dT(\mathrm{v}).$ ----(4)

for all vectors **u**, **v** in the domain of *T* and all scalars *c*, *d*.

- Property (3) follows from condition (ii) in the definition, because $T(0)=T(0\mathrm{u})=0T(\mathrm{u})=0.$
- Property (4) requires both (i) and (ii):

$$T(c\mathrm{u}+d\mathrm{v})=T(c\mathrm{u})+T(d\mathrm{v})=cT(\mathrm{u})+dT(\mathrm{v})$$

- *If a transformation satisfies* (4) *for all* **u**, **v** and *c*, *d*, *it must be linear*.
- (Set $c=d=1$ for preservation of addition, and set for $d=0$ preservation of scalar multiplication.)

LINEAR TRANSFORMATIONS

- Repeated application of (4) produces a useful generalization:

$$T(c_1\mathrm{v}_1 + \ldots + c_p\mathrm{v}_p) = c_1T(\mathrm{v}_1) + \ldots + c_pT(\mathrm{v}_p) \quad \text{----(5)}$$

- In engineering and physics, (5) is referred to as a *superposition principle*.

- Think of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ as signals that go into a system and $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ as the responses of that system to the signals.

LINEAR TRANSFORMATIONS

- The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is the *same* linear combination of the responses to the individual signals.

- Given a scalar *r*, define $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathrm{x}) = r\mathrm{x}$.

- T is called a **contraction** when $0 \le r \le 1$ and a **dilation** when $r > 1$.