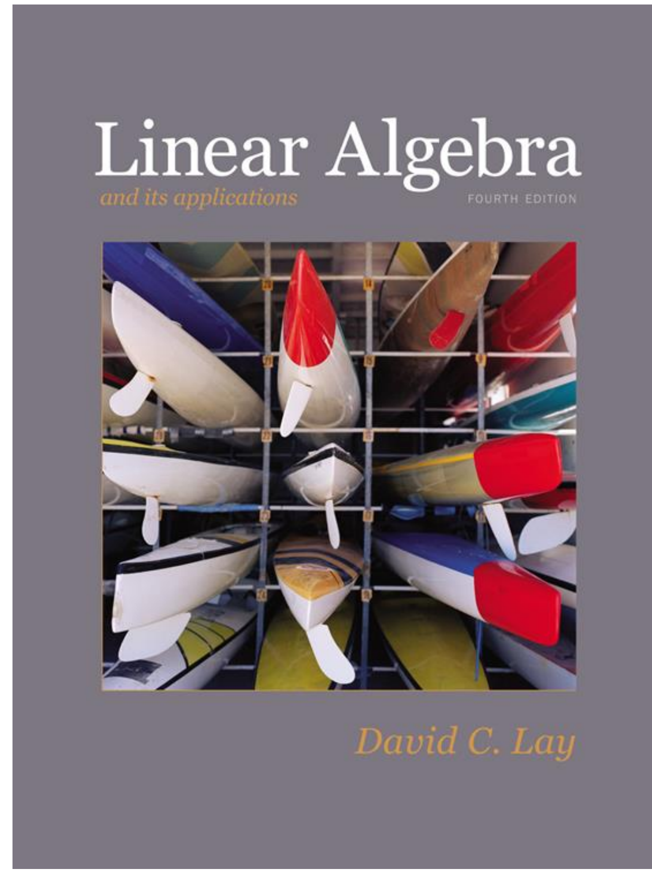
4 Vector Spaces

4.1

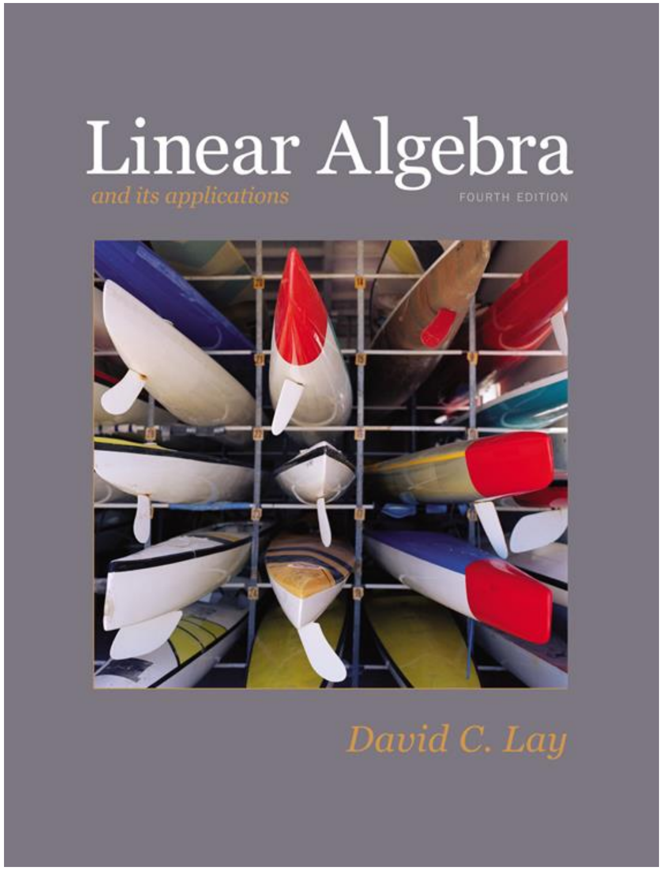
VECTOR SPACES AND SUBSPACES



4 Vector Spaces

4.2

NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS



PEARSON

NULL SPACE OF A MATRIX

- **Definition:** The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

- **Theorem 2:** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.
- **Proof:** Nul A is a subset of $\mathbb{R}^n$ because A has n columns.
- We need to show that Nul A satisfies the three properties of a subspace.

NULL SPACE OF A MATRIX

- **0** is in Null *A*.
- Next, let **u** and **v** represent any two vectors in Nul *A*.
- Then

$$A\mathrm{u} = 0 \text{ and } A\mathrm{v} = 0$$

- To show that $\mathrm{u} + \mathrm{v}$ is in Nul *A*, we must show that $A(\mathrm{u} + \mathrm{v}) = 0$.
- Using a property of matrix multiplication, compute

$$A(\mathrm{u} + \mathrm{v}) = A\mathrm{u} + A\mathrm{v} = 0 + 0 = 0$$

- Thus $\mathrm{u} + \mathrm{v}$ is in Nul *A*, and Nul *A* is closed under vector addition.

NULL SPACE OF A MATRIX

- Finally, if c is any scalar, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$$

which shows that $c\mathbf{u}$ is in Nul A.

- Thus Nul A is a subspace of $\mathbb{R}^n$.

- **An Explicit Description of Nul *A***
- There is no obvious relation between vectors in Nul A and the entries in A.
- We say that Nul A is defined *implicitly*, because it is defined by a condition that must be checked.

NULL SPACE OF A MATRIX

- No explicit list or description of the elements in Nul A is given.
- *Solving* the equation $A\mathbf{x} = 0$ amounts to producing an *explicit* description of Nul A.

- **Example 1:** Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

NULL SPACE OF A MATRIX

- **Solution:** The first step is to find the general solution of $A\mathrm{x} = 0$ in terms of free variables.

- Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.
- Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\uparrow \atop \mathrm{u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\uparrow \atop \mathrm{v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\uparrow \atop \mathrm{w}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

NULL SPACE OF A MATRIX

$$= x_2\mathrm{u} + x_4\mathrm{v} + x_5\mathrm{w}. \qquad \text{----(1)}$$

- Every linear combination of **u**, **v**, and **w** is an element of Nul *A*.
- Thus {**u**, **v**, **w**} is a spanning set for Nul *A*.

1. The spanning set produced by the method in Example (1) is automatically linearly independent because the free variables are the weights on the spanning vectors.
2. When Nul *A* contains nonzero vectors, the number of vectors in the spanning set for Nul *A* equals the number of free variables in the equation $A\mathrm{x} = 0$.

COLUMN SPACE OF A MATRIX

- **Definition:** The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

- **Theorem 3:** The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.
- A typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

COLUMN SPACE OF A MATRIX

- The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the range of the linear transformation $\mathrm{x} \mapsto A\mathrm{x}$.
- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathrm{x} = \mathrm{b}$ has a solution for each **b** in $\mathbb{R}^m$.
- **Example 2:** Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

COLUMN SPACE OF A MATRIX

a. Determine if **u** is in Nul A. Could **u** be in Col A?

b. Determine if **v** is in Col A. Could **v** be in Nul A?

- **Solution:**

a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

COLUMN SPACE OF A MATRIX

- **u** is *not* a solution of $A\mathrm{x} = 0$, so **u** is not in Nul A.
- Also, with four entries, **u** could not possibly be in Col A, since Col A is a subspace of $\mathbb{R}^3$.

b. Reduce $[A \quad \mathrm{v}]$ to an echelon form.

$$[A \quad \mathrm{v}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

- The equation $A\mathrm{x} = \mathrm{v}$ is consistent, so **v** is in Col A.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- With only three entries, **v** could not possibly be in Nul A, since Nul A is a subspace of $\mathbb{R}^4$.

- Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a linear transformation instead of a matrix.
- **Definition:** A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector **x** in V a unique vector $T(\mathbf{x})$ in W, such that
  i. $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all **u**, **v** in V, and
  ii. $T(c\mathbf{u})=cT(\mathbf{u})$ for all **u** in V and all scalars c.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathrm{u}) = 0$ (the zero vector in W).
- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.
- The kernel of T is a subspace of V.
- The range of T is a subspace of W.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

Nul *A*	Col *A*
1. Nul *A* is a subspace of $\mathbb{R}^n$.	1. Col *A* is a subspace of $\mathbb{R}^m$.
2. Nul *A* is implicitly defined; *i.e.*, you are given only a condition $(A\mathrm{x} = 0)$ that vectors in Nul *A* must satisfy.	2. Col *A* is explicitly defined; *i.e.*, you are told how to build vectors in Col *A*.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

3. It takes time to find vectors in Nul *A*. Row operations on $\begin{bmatrix} A & 0 \end{bmatrix}$ are required.	3. It is easy to find vectors in Col *A*. The columns of a are displayed; others are formed from them.
4. There is no obvious relation between Nul *A* and the entries in *A*.	4. There is an obvious relation between Col *A* and the entries in *A*, since each column of *A* is in Col *A*.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

5. A typical vector **v** in Nul *A* has the property that $A\mathbf{v} = 0$.	5. A typical vector **v** in Col *A* has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector **v**, it is easy to tell if **v** is in Nul *A*. Just compare *A***v**.	6. Given a specific vector **v**, it may take time to tell if **v** is in Col *A*. Row operations on $[A \quad \mathbf{v}]$ are required.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

7. Nul $A = \{0\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every **b** in $\mathbb{R}^m$.
8. Nul $A = \{0\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.
   1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
   2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
   3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
   4. There is a zero vector 0 in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

VECTOR SPACES AND SUBSPACES

5. For each **u** in *V*, there is a vector $-\mathbf{u}$ in *V* such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by *c*, denoted by *c***u**, is in *V*.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+c\mathbf{v}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

- Using these axioms, we can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the **negative** of **u**, in Axiom 5 is unique for each **u** in *V*.

VECTOR SPACES AND SUBSPACES

- For each **u** in *V* and scalar *c*,

$$0\mathbf{u} = \mathbf{0} \qquad \text{----(1)}$$

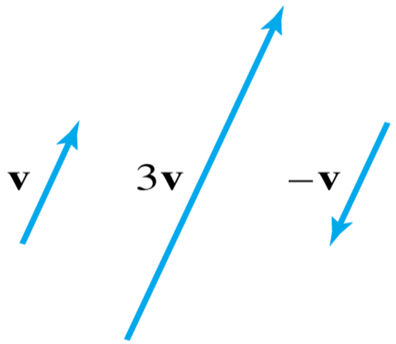
$$c\mathbf{0} = \mathbf{0} \qquad \text{----(2)}$$

$$-\mathbf{u} = (-1)\mathbf{u} \qquad \text{----(3)}$$

- **Example 1:** Let *V* be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule, and for each **v** in *V*, define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of **v**, pointing in the same direction as **v** if $c \geq 0$ and otherwise pointing in the opposite direction.

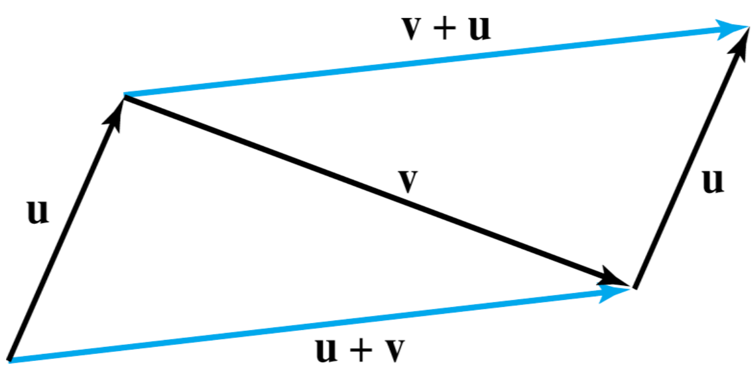
VECTOR SPACES AND SUBSPACES

- See the following figure below. Show that V is a vector space.

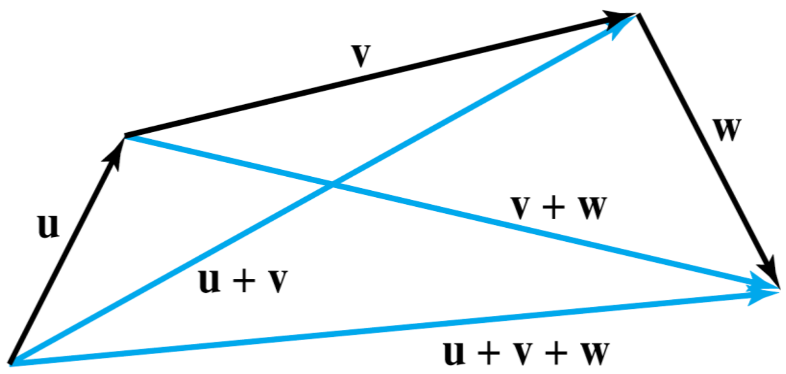


- **Solution:** The definition of V is geometric, using concepts of length and direction.
- No *xyz*-coordinate system is involved.
- An arrow of zero length is a single point and represents the zero vector.
- The negative of $\mathbf{v}$ is $(-1)\mathbf{v}$.
- So Axioms 1, 4, 5, 6, and 10 are evident. See the figures on the next slide.

SUBSPACES



$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}.$



$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$

- **Definition:** A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition. That is, for each **u** and **v** in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

SUBSPACES

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V.
- Every subspace is a vector space.
- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

- As the term **linear combination** refers to any sum of scalar multiples of vectors, and Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

A SUBSPACE SPANNED BY A SET

- **Example 2:** Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.
- **Solution:** The zero vector is in H, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
- To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \text{ and } \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

- By Axioms 2, 3, and 8 for the vector space V,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

A SUBSPACE SPANNED BY A SET

- So $\mathbf{u}+\mathbf{w}$ is in H.

- Furthermore, if c is any scalar, then by Axioms 7 and 9,

$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

  which shows that c**u** is in H and H is closed under scalar multiplication.

- Thus H is a subspace of V.

A SUBSPACE SPANNED BY A SET

- **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

- We call Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in H such that

$$H = \text{Span}\{\text{v}_1,\ldots\text{v}_p\}.$$