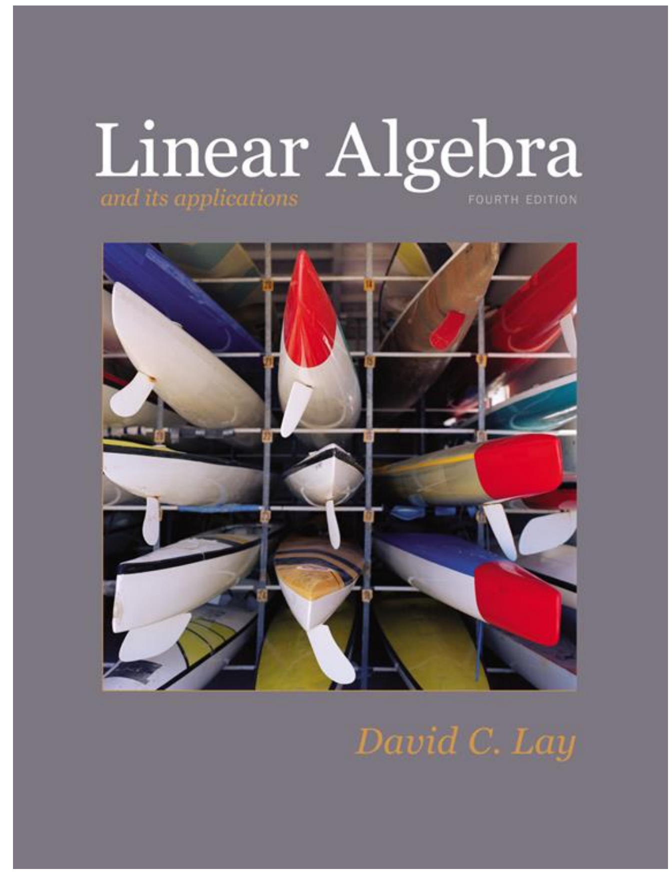
4 Vector Spaces

4.5

THE DIMENSION OF A VECTOR SPACE



PEARSON

DIMENSION OF A VECTOR SPACE

- **Theorem 9:** If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

- **Proof:** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with more than n vectors.

- The coordinate vectors $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector.

DIMENSION OF A VECTOR SPACE

- So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[\mathrm{u}_1]_{\mathrm{B}} + \ldots + c_p[\mathrm{u}_p]_{\mathrm{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ The zero vector in $\mathbb{R}^n$

- Since the coordinate mapping is a linear transformation,

$$[c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p]_{\mathrm{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

DIMENSION OF A VECTOR SPACE

- The zero vector on the right displays the n weights needed to build the vector $c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p$ from the basis vectors in B.
- That is, $c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p = 0 \cdot \mathrm{b}_1 + \ldots + 0 \cdot \mathrm{b}_n = 0$.
- Since the c_i are not all zero, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly dependent.
- Theorem 9 implies that if a vector space V has a basis $\mathrm{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$, then each linearly independent set in V has no more than n vectors.

DIMENSION OF A VECTOR SPACE

- **Theorem 10:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- **Proof:** Let B_1 be a basis of n vectors and B_2 be any other basis (of V).
- Since B_1 is a basis and B_2 is linearly independent, B_2 has no more than n vectors, by Theorem 9.
- Also, since B_2 is a basis and B_1 is linearly independent, B_2 has at least n vectors.
- Thus B_2 consists of exactly n vectors.

DIMENSION OF A VECTOR SPACE

- **Definition:** If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space **{0}** is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.
- **Example 1:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

DIMENSION OF A VECTOR SPACE

- H is the set of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly, $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, but $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$.
- By the Spanning Set Theorem, we may discard $\mathbf{v}_3$ and still have a set that spans H.

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- Finally, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and hence is a basis for H.
- Thus $\dim H = 3$.

- **Theorem 11:** Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- **Proof:** If $H = \{0\}$, then certainly $\dim H = 0 \leq \dim V$.

- Otherwise, let $S = \{\mathrm{u}_1, \ldots, \mathrm{u}_k\}$ be any linearly independent set in H.

- If S spans H, then S is a basis for H.

- Otherwise, there is some u_{k+1} in H that is not in Span S.

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- But then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ will be linearly independent, because no vector in the set can be a linear combination of vectors that precede it (by Theorem 4).

- So long as the new set does not span H, we can continue this process of expanding S to a larger linearly independent set in H.

- But the number of vectors in a linearly independent expansion of S can never exceed the dimension of V, by Theorem 9.

THE BASIS THEOREM

- So eventually the expansion of S will span H and hence will be a basis for H, and $\dim H \le \dim V$.

- **Theorem 12:** Let V be a p-dimensional vector space, $p \ge 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

- **Proof:** By Theorem 11, a linearly independent set S of p elements can be extended to a basis for V.

THE BASIS THEOREM

- But that basis must contain exactly p elements, since $\dim V = p$.
- So S must already be a basis for V.
- Now suppose that S has p elements and spans V.
- Since V is nonzero, the Spanning Set Theorem implies that a subset S' of S is a basis of V.
- Since $\dim V = p$, S' must contain p vectors.
- Hence $S = S'$.

THE DIMENSIONS OF NUL *A* AND COL *A*

- Let *A* be an $m \times n$ matrix, and suppose the equation $A\mathbf{x} = \mathbf{0}$ has *k* free variables.

- A spanning set for Nul *A* will produce exactly *k* linearly independent vectors—say, $\mathbf{u}_1, \ldots, \mathbf{u}_k$—one for each free variable.

- So $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for Nul *A*, and the number of free variables determines the size of the basis.

DIMENSIONS OF NUL *A* AND COL *A*

- Thus, the dimension of Nul A is the number of free variables in the equation $A\mathrm{x} = 0$, and the dimension of Col A is the number of pivot columns in A.

- **Example 2:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

DIMENSIONS OF NUL *A* AND COL *A*

- **Solution:** Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable—x_2, x_4 and x_5.
- Hence the dimension of Nul A is 3.
- Also $\dim \operatorname{Col} A = 2$ because A has two pivot columns.

4 Vector Spaces

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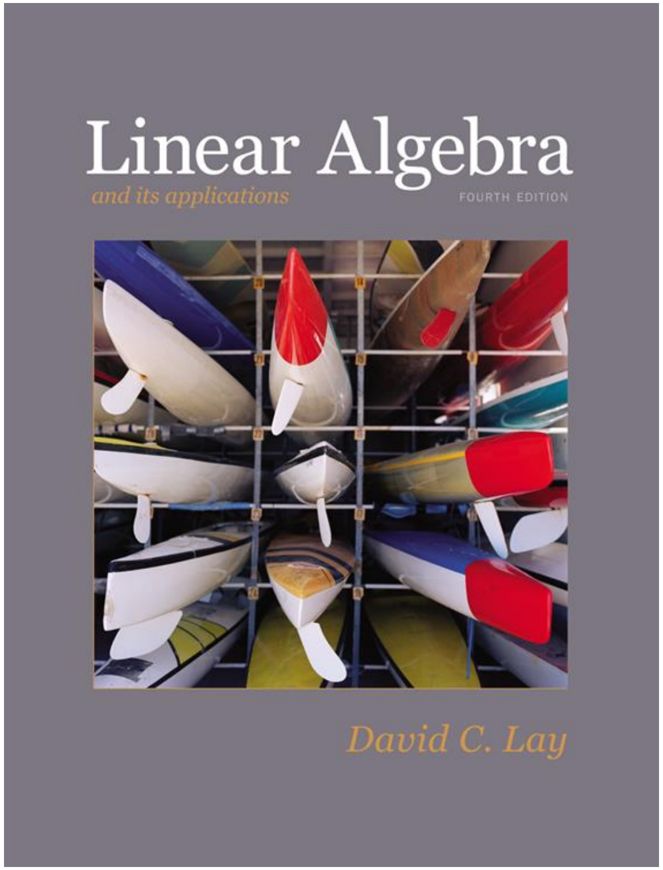
RANK



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THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A.
- Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

THE ROW SPACE

- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Proof:** If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A.

- It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A.

THE ROW SPACE

- Thus the row space of B is contained in the row space of A.
- Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B.
- So the two row spaces are the same.

THE ROW SPACE

- If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. (Apply Theorem 4 to the nonzero rows of B in reverse order, with the first row last).

- Thus the nonzero rows of B form a basis of the (common) row space of B and A.

THE ROW SPACE

- **Example 1:** Find bases for the row space, the column space, and the null space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

THE ROW SPACE

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B).
- Thus

  Basis for Row A: $\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$

THE ROW SPACE

- For the column space, observe from B that the pivots are in columns 1, 2, and 4.
- Hence columns 1, 2, and 4 of A (not B) form a basis for Col A:

$$\text{Basis for Col } A: \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A.

THE ROW SPACE

- However, for Nul A, we need the *reduced echelon form*.

- Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THE ROW SPACE

- The equation $A\mathrm{x}=0$ is equivalent to $C\mathrm{x}=0$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.

THE ROW SPACE

- The calculations show that

$$\text{Basis for Nul } A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.

THE RANK THEOREM

- **Definition:** The **rank** of A is the dimension of the column space of A.
- Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.
- The dimension of the null space is sometimes called the **nullity** of A.
- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}\, A + \dim \text{Nul}\, A = n$$

THE RANK THEOREM

- **Proof:** By Theorem 6, rank A is the number of pivot columns in A.
- Equivalently, rank A is the number of pivot positions in an echelon form B of A.
- Since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A, the rank of A is also the dimension of the row space.
- The dimension of Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
- Expressed another way, the dimension of Nul A is the number of columns of A that are *not* pivot columns.

THE RANK THEOREM

- (It is the number of these columns, not the columns themselves, that is related to Nul A).

- Obviously,

$$\begin{Bmatrix} \text{number of} \\ \text{pivot columns} \end{Bmatrix} + \begin{Bmatrix} \text{number of} \\ \text{nonpivot columns} \end{Bmatrix} = \begin{Bmatrix} \text{number of} \\ \text{columns} \end{Bmatrix}$$

- This proves the theorem.

THE RANK THEOREM

- **Example 2:**
  a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
  b. Could a 6×9 matrix have a two-dimensional null space?
- **Solution:**
  a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank $A = 7$.
  b. No. If a 6×9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem.

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

  - But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6.

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. Dim Col $A = n$

  p. rank $A = n$

RANK AND THE INVERTIBLE MATRIX THEOREM

q. $\text{Nul } A = \{0\}$

r. $\text{Dim Nul } A = 0$

- **Proof:** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning.

- The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

RANK AND THE INVERTIBLE MATRIX THEOREM

- Statement (g), which says that the equation $A\mathrm{x} = \mathrm{b}$ has at least one solution for each **b** in $\mathbb{R}^n$, implies (n), because Col A is precisely the set of all **b** such that the equation $A\mathrm{x} = \mathrm{b}$ is consistent.
- The implications $(\mathrm{n}) \Rightarrow (\mathrm{o}) \Rightarrow (\mathrm{p})$ follow from the definitions of dimension and rank.
- If the rank of A is n, the number of columns of A, then dim Nul $A = 0$, by the Rank Theorem, and so Nul $A = \{0\}$.

RANK AND THE INVERTIBLE MATRIX THEOREM

- Thus $(p) \Rightarrow (r) \Rightarrow (q)$.
- Also, (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.