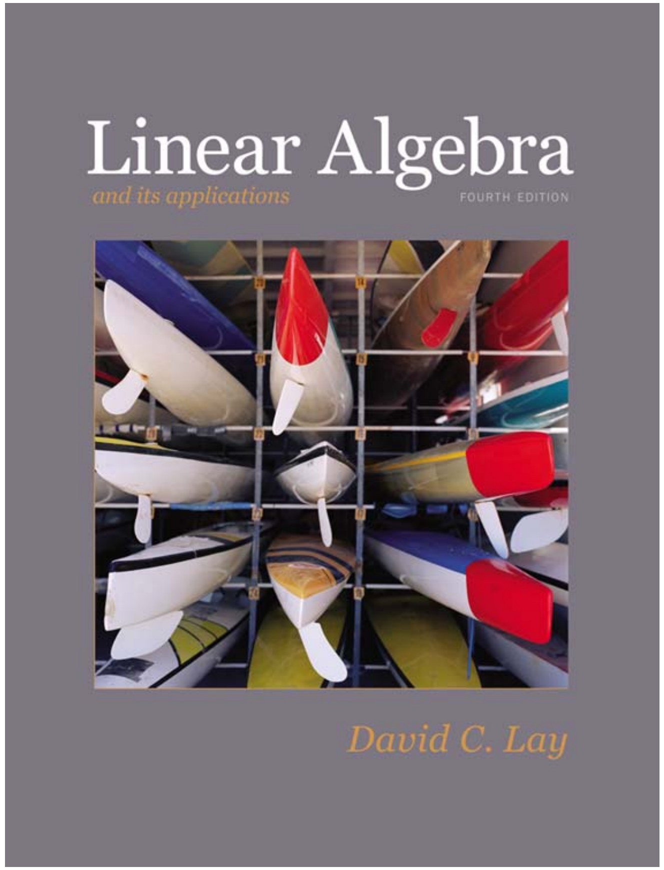
4 Vector Spaces

4.3

LINEARLY INDEPENDENT SETS; BASES



PEARSON

LINEAR INDEPENDENT SETS; BASES

- An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0 \qquad \text{----(1)}$$

  has *only* the trivial solution, $c_1 = 0, \ldots, c_p = 0$.

- The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \ldots, c_p$, *not all zero*, such that (1) holds.

- In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

LINEAR INDEPENDENT SETS; BASES

- **Theorem 4:** An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathrm{v}_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathrm{v}_1, \ldots, \mathrm{v}_{j-1}$.

- **Definition:** Let H be a subspace of a vector space V. An indexed set of vectors $\mathrm{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_p\}$ in V is a basis for H if
  (i) B is a linearly independent set, and
  (ii) The subspace spanned by B coincides with H; that is, $H = \mathrm{Span}\{\mathrm{b}_1, \ldots, \mathrm{b}_p\}$

LINEAR INDEPENDENT SETS; BASES

- The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.

- Thus a basis of V is a linearly independent set that spans V.

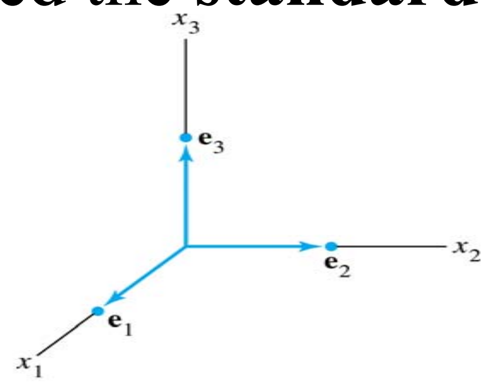
- When $H \neq V$, condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must belong to H, because Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

STANDARD BASIS

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See the following figure.



The standard basis for $\mathbb{R}^3$.

THE SPANNING SET THEOREM

- **Theorem 5:** Let $S = \{v_1, \ldots, v_p\}$ be a set in V, and let $H = \text{Span}\{v_1, \ldots, v_p\}$.
  a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
  b. If $H \neq \{0\}$, some subset of S is a basis for H.
- **Proof:**
  a. By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $v_1, \ldots, v_{p-1}$—say,

THE SPANNING SET THEOREM

$$v_p = a_1 v_1 + \ldots + a_{p-1} v_{p-1} \qquad \text{----(2)}$$

- Given any **x** in H, we may write

  $$x = c_1 v_1 + \ldots + c_{p-1} v_{p-1} + c_p v_p \qquad \text{----(3)}$$

  for suitable scalars $c_1, \ldots, c_p$.

- Substituting the expression for $\mathbf{v}_p$ from (2) into (3), it is easy to see that **x** is a linear combination of $v_1, \ldots v_{p-1}$.

- Thus $\{v_1, \ldots, v_{p-1}\}$ spans H, because **x** was an arbitrary element of H.

THE SPANNING SET THEOREM

b. If the original spanning set S is linearly independent, then it is already a basis for H.

- Otherwise, one of the vectors in S depends on the others and can be deleted, by part (a).
- So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for H.
- If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because $H \neq \{0\}$.

THE SPANNING SET THEOREM

- **Example 1:** Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

  and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
  Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find a basis for the subspace H.

- **Solution:** Every vector in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$$

THE SPANNING SET THEOREM

- Now let **x** be any vector in H—say,

$$\mathrm{x} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 .$$

- Since $\mathrm{v}_3 = 5\mathrm{v}_1 + 3\mathrm{v}_2$, we may substitute

$$\mathrm{x} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3(5\mathrm{v}_1 + 3\mathrm{v}_2)$$

$$= (c_1 + 5c_3)\mathrm{v}_1 + (c_2 + 3c_3)\mathrm{v}_2$$

- Thus **x** is in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in H already belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- We conclude that H and Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the set of vectors.
- It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

BASIS FOR COL *B*

- **Example 2:** Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution:** Each nonpivot column of B is a linear combination of the pivot columns.
- In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.
- By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B.

BASIS FOR COL B

- Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4).

- Thus S is a basis for Col B.

BASES FOR NUL A AND COL A

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.

- **Proof:** Let B be the reduced echelon form of A.
- The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
- Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.

BASES FOR NUL *A* AND COL *A*

- For this reason, every nonpivot column of A is a linear combination of the pivot columns of A.
- Thus the nonpivot columns of a may be discarded from the spanning set for Col A, by the Spanning Set Theorem.
- This leaves the pivot columns of A as a basis for Col A.

- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form.
- But, be careful to use the pivot columns of A itself for the basis of Col A.

BASES FOR NUL *A* AND COL *A*

- Row operations can change the column space of a matrix.
- The columns of an echelon form B of A are often not in the column space of A.

- **Two Views of a Basis**
- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.
- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.

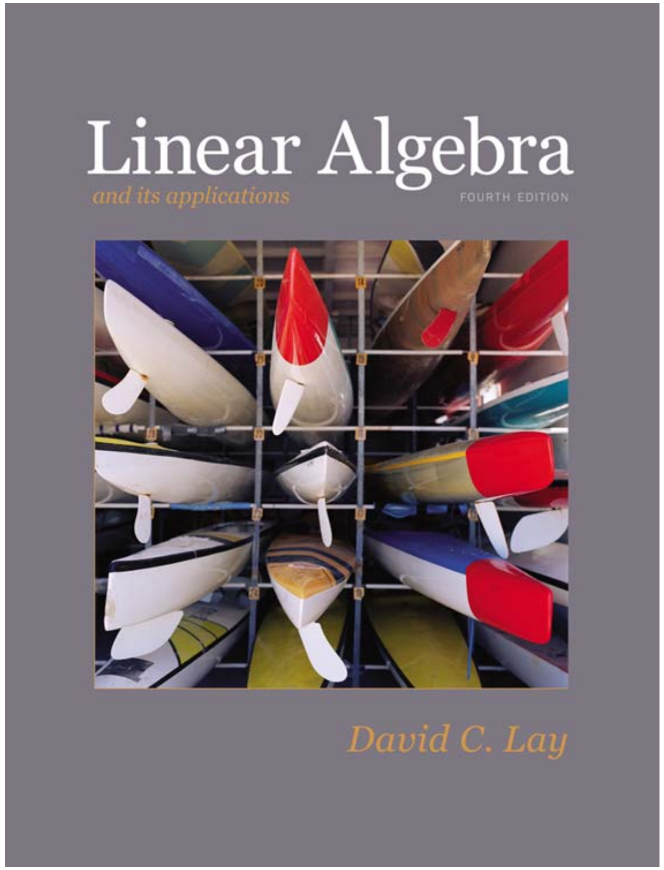
TWO VIEWS OF A BASIS

- Thus a basis is a spanning set that is as small as possible.
- A basis is also a linearly independent set that is as large as possible.
- If S is a basis for V, and if S is enlarged by one vector—say, $\mathbf{w}$—from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.

4 Vector Spaces

4.4

COORDINATE SYSTEMS



THE UNIQUE REPRESENTATION THEOREM

- **Theorem 7:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Then for each **x** in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \qquad \text{----(1)}$$

- **Proof:** Since $\mathcal{B}$ spans V, there exist scalars such that (1) holds.
- Suppose **x** also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

for scalars $d_1, \ldots, d_n$.

THE UNIQUE REPRESENTATION THEOREM

- Then, subtracting, we have
$$0 = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \ldots + (c_n - d_n)\mathbf{b}_n \quad \text{----(2)}$$

- Since B is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le n$.

- **Definition:** Suppose $\mathrm{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and **x** is in V. **The coordinates of x relative to the basis B** (or the **B-coordinate of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$.

THE UNIQUE REPRESENTATION THEOREM

- If $c_1, \ldots, c_n$ are the **B**-coordinates of **x**, then the vector in $\mathbb{R}^n$

$$[\mathrm{x}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to B)**, or the **B-coordinate vector of x**.

- The mapping $\mathrm{x} \mapsto [\mathrm{x}]_{\mathrm{B}}$ is the **coordinate mapping (determined by B)**.

COORDINATES IN $\mathbb{R}^n$

- When a basis B for $\mathbb{R}^n$ is fixed, the B-coordinate vector of a specified **x** is easily found, as in the example below.
- **Example 1:** Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and

  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to B.
- **Solution:** The B-coordinate c_1, c_2 of **x** satisfy

$$\underbrace{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{b}_1} + \underbrace{c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\mathbf{x}}$$

COORDINATES IN $\mathbb{R}^n$

or

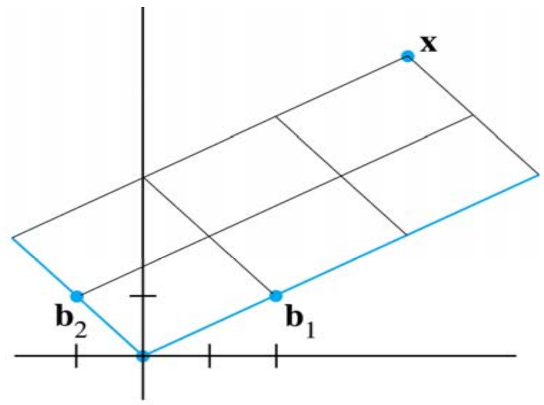
$$\begin{array}{cc} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{b}_1 \quad \mathbf{b}_2 \end{array} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ \mathbf{x} \end{array} \qquad \text{----(3)}$$

- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3, c_2 = 2$.
- Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$ and

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

COORDINATES IN $\mathbb{R}^n$

- See the following figure.



The $\mathcal{B}$-coordinate vector of **x** is (3, 2).

- The matrix in (3) changes the B-coordinates of a vector **x** into the standard coordinates for **x**.
- An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{b_1, \ldots, b_n\}$.
- Let P_B

$$= \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

COORDINATES IN $\mathbb{R}^n$

- Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \qquad \text{----(4)}$$

- $P_{\mathcal{B}}$ is called the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

- Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

- Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

COORDINATES IN $\mathbb{R}^n$

- Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts **x** into its B-coordinate vector:

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

- The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping.

- Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

THE COORDINATE MAPPING

- **Theorem 8:** Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space *V*. Then the coordinate mapping $x \mapsto [x]_B$ is a one-to-one linear transformation from *V* onto $\mathbb{R}^n$.

- **Proof:** Take two typical vectors in *V*, say,

$$u = c_1 b_1 + \ldots + c_n b_n$$

$$w = d_1 b_1 + \ldots + d_n b_n$$

- Then, using vector operations,

$$u + v = (c_1 + d_1)b_1 + \ldots + (c_n + d_n)b_n$$

THE COORDINATE MAPPING

- It follows that

$$[\mathbf{u}+\mathbf{w}]_B = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_B + [\mathbf{w}]_B$$

- So the coordinate mapping preserves addition.

- If r is any scalar, then

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n$$

THE COORDINATE MAPPING

- So

$$[r\mathbf{u}]_B = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_B$$

- Thus the coordinate mapping also preserves scalar multiplication and hence is a linear transformation.
- The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, cp$ are scalars, then

$$[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p]_B = c_1[\mathbf{u}_1]_B + \ldots + c_p[\mathbf{u}_p]_B \text{ ----(5)}$$

THE COORDINATE MAPPING

- In words, (5) says that the B-coordinate vector of a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.
- The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto $\mathbb{R}^n$.
- In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W.
- The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

THE COORDINATE MAPPING

- *Every vector space calculation in V is accurately reproduced in W, and vice versa.*
- In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$.
- **Example 2:** Let $\mathrm{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathrm{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$

  and $\mathrm{B} = \{\mathrm{v}_1, \mathrm{v}_2\}$. Then B is a basis for $H = \mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2\}$. Determine if **x** is in H, and if it is, find the coordinate vector of **x** relative to B.

THE COORDINATE MAPPING

- **Solution:** If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

- The scalars c_1 and c_2, if they exist, are the B-coordinates of $\mathbf{x}$.

THE COORDINATE MAPPING

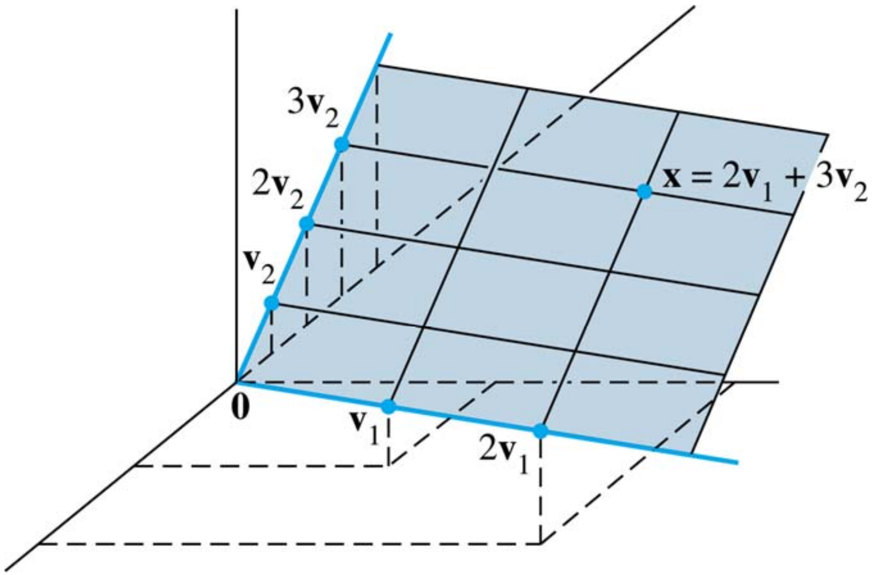
- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

THE COORDINATE MAPPING

- The coordinate system on H determined by B is shown in the following figure.



A coordinate system on a plane H in $\mathbb{R}^3$.