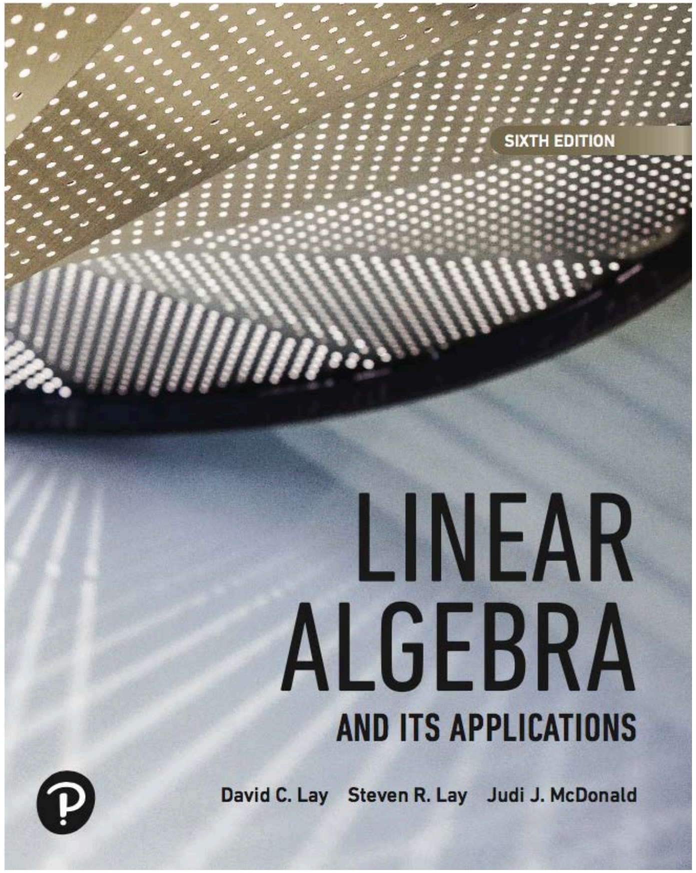
5

Eigenvalues and Eigenvectors

5.7

Applications To Differential Equations



DYNAMICAL SYSTEMS

- A system of differential equations is given by the equations:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}\, x_n \\
x_2' &= a_{21}x_1 + \cdots + a_{2n}\, x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + \cdots + a_{nn}\, x_n
\end{aligned}
$$

- where $x_1, \ldots, x_n$ are differentiable functions of t, with derivatives $x_1', \ldots, x_n'$, and the a_{ij} are constants.

DYNAMICAL SYSTEMS

- Written in matrix form, the system can be expressed as:

$$\mathbf{x}' = A\mathbf{x}$$

- This equation is *linear* because both differentiation of functions and multiplication of vectors by a matrix are linear transformations.

DYNAMICAL SYSTEMS

- A fundamental set of solutions is a *basis* for the set of all solutions, an n-dimensional vector space of functions.
- If a vector $\mathbf{x}_0$ is specified, then the **initial value problem** is to construct the (unique) function $\mathbf{x}$ such that $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x}_0$.

DYNAMICAL SYSTEMS

- Notice if $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ for some vector $\mathbf{v}$, then $\mathbf{x}'(t) = \lambda \mathbf{v}e^{\lambda t}$.
- For this choice of $\mathbf{x}(t)$, notice

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

if and only if

$$A\mathbf{v} = \lambda \mathbf{v}.$$

- Thus each eigenvalue–eigenvector pair of A provides a solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Dynamical Systems

- **Example:** Consider $A = \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix}$.

  Find all solutions to $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

- **Solution:** The eigenvalues of *A* are $\lambda_1 = -0.5$ and $\lambda_2 = -2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Dynamical Systems

- Set $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{-0.5t} + c_2\mathbf{v}_2 e^{-2t}$ and $\mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.
- Solving $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ results in $c_1 = 3$ and $c_2 = -2$.
- The solution to the initial boundary problem is

$$\mathbf{x}(t) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5t} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

Dynamical Systems

- The origin is called an **attractor**, or **sink**, of this dynamical system because all trajectories are drawn into the origin.
- The direction of greatest attraction is along the line through **0** and $\mathbf{v}_2$ corresponding to the more negative eigenvalue, $\lambda = -2$.

Dynamical Systems

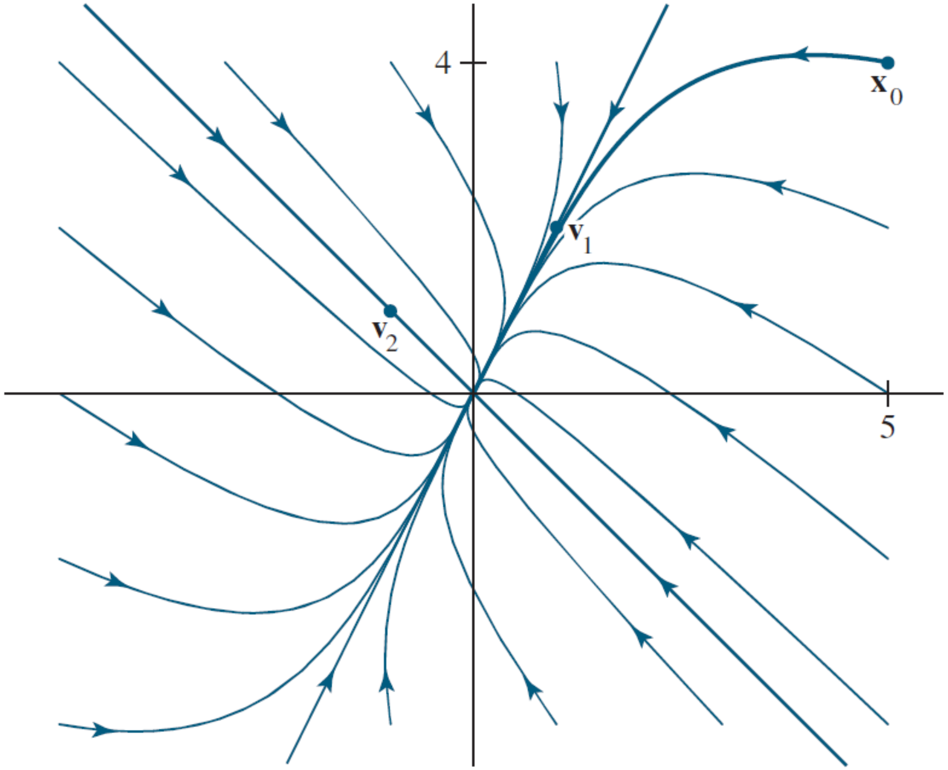


FIGURE 2 The origin as an attractor.

Dynamical Systems

- If the eigenvalues are positive instead of negative, the corresponding trajectories would be similar in shape, but the trajectories would be traversed **away** from the origin.
- In such a case, the origin is called a **repeller**, or **source**, of the dynamical system.
- The direction of greatest repulsion is the line containing the trajectory of the eigenfunction corresponding to the more positive eigenvalue.

Dynamical Systems

- A saddle point arises whenever the matrix *A* has both positive and negative eigenvalues.
- The origin is called a **saddle point** of the dynamical system because some trajectories approach the origin at first and then change direction and move away from the origin

Dynamical Systems

- The direction of greatest repulsion is the line through $\mathbf{v}_1$ and $\mathbf{0}$, corresponding to the positive eigenvalue.
- The direction of greatest attraction is the line through $\mathbf{v}_2$ and $\mathbf{0}$, corresponding to the negative eigenvalue.

Dynamical Systems (12 of 12)

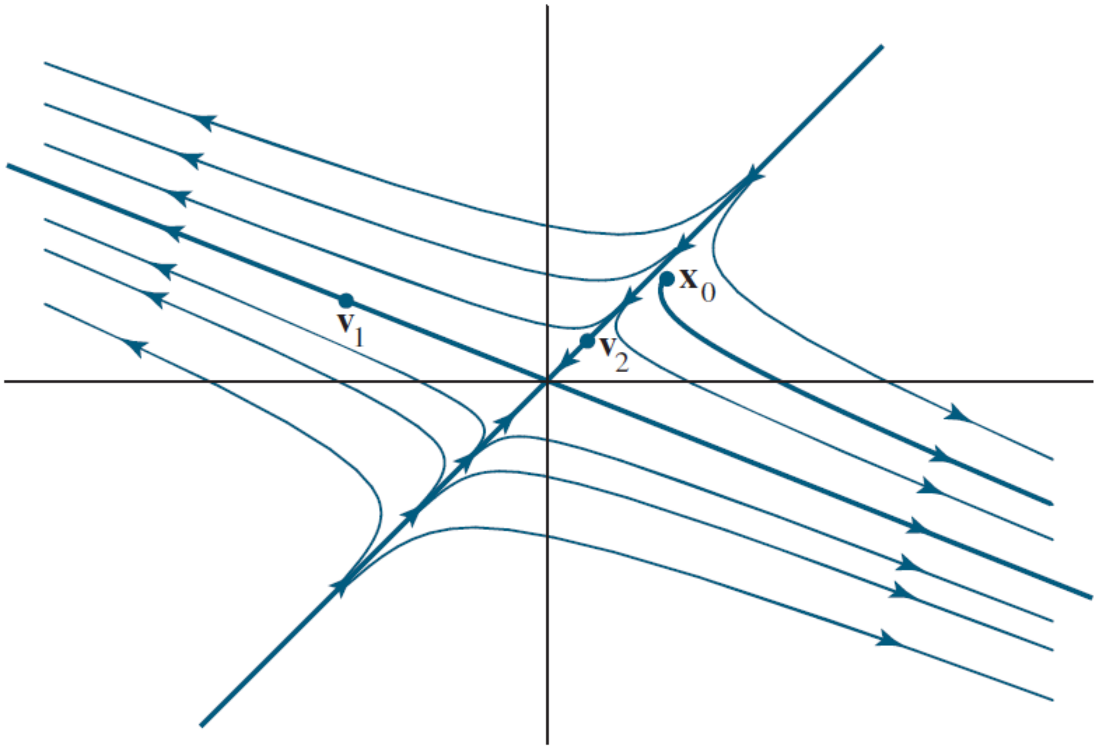


FIGURE 3 The origin as a saddle point.

Decoupling a Dynamical System

- Suppose the eigenfunctions for *A* are

  $\mathbf{v}_1 e^{\lambda_1 t}, \mathbf{v}_2 e^{\lambda_2 t}, \ldots, \mathbf{v}_n e^{\lambda_n t}$ with $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ linearly independent.

  eigenvectors.

- Let $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$, and let *D* be the diagonal matrix with

  entries $\lambda_1 \ldots \lambda_n$, so that $A = PDP^{-1}$.

- Now make a **change of variable**, defining a new function

  $\mathbf{y}$ by $\mathbf{y}(t) = P^{-1}\mathbf{x}(\mathrm{t})$, or, equivalently, $\mathbf{x}(t) = P\mathbf{y}(t)$

DECOUPLING A DYNAMICAL SYSTEM

- Substituting $\mathbf{x}(t)=P\mathbf{y}(t)$ into $\mathbf{x}'(t) = A\mathbf{x}(t)$ results in $P\mathbf{y}'(t) = PDP^{-1}\mathbf{x}(t) = PD\mathbf{y}(t)$.
- Multiplying both sides of the equation by P^{-1} creates $\mathbf{y}'(t) = D\mathbf{y}(t)$.
- The change of variable from **x** to **y** has *decoupled* the system of differential equations, because the derivative of each scalar function y_k depends only on y_k.

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