Gør tanke til handling

VIA University College

Sequences and summation

Sequences

A sequence is simply a progression of numbers, for example

$$\{1,3,5,7,9,\ldots\}$$

where the ... denotes that the progression continues forever. The sequence above is infinite; however, sequences can also be finite, such as

$$\left\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\right\}.$$

Note that, despite the notation, **sequences are not sets!**

We rely on the context to distinguish between sequences and sets.

The **terms** of a sequence

The individual numbers in a sequence are called the sequence's **terms**. These are typically enumerated by either $1, 2, 3, 4, \ldots$ or $0, 1, 2, 3, \ldots$

The sequence $\{a_1, a_2, a_3, a_4, a_5\}$ can be written shortly as

$$\{a_n\}_{n=1}^{5} = \{a_1, a_2, a_3, a_4, a_5\}.$$

<u>Example</u>

Consider the sequence $\{a_n\}_{n=0}^{3} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right\}$. The term corresponding to $n = 2$ in this sequence is $a_2 = \frac{1}{4}$.

Specifying a sequence

In the previous slides, you saw two examples of sequences, both specified by listing the terms they contain. We have two additional ways to specify a sequence:

1) Using an **explicit formula**, where each term a_n is expressed as a function of n.

2) Using a **recursive formula** (also called a **recurrence relation**), where each term a_n is given by some formula involving the previous term(s).

A few examples are given in the following slides.

Explicit formula for a sequence

In this case, we give a formula which can be used to find a_n for any value of n.

Examples:

- The sequence $\{\mathrm{a_n}\}_{\mathrm{n=1}}^{\infty} = \{1,3,5,7,9,\dots\}$ can be expressed as $\{\mathrm{a_n}\}_{\mathrm{n=1}}^{\infty}$ with $a_n = 2n - 1$
- The sequence $\{\mathrm{a_n}\}_{\mathrm{n=0}}^{3} = \left\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\right\}$ can be expressed as $\{\mathrm{a_n}\}_{\mathrm{n=0}}^{3}$ with $a_n = \frac{1}{2^n}$.

Reccursive formula for a sequence

In this case, we state how it is possible to find each term by doing something to the previous term(s).

- The sequence $\{a_n\}_{n=1}^{\infty} = \{1,3,5,7,9,\ldots\}$ can be expressed as $\{a_n\}_{n=1}^{\infty}$ with $a_n = a_{n-1} + 2$ and $a_1 = 1$.
- The sequence $\{a_n\}_{n=0}^{3} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right\}$ can be expressed as $\{a_n\}_{n=0}^{3}$ with $a_n = \frac{a_{n-1}}{2}$ and $a_0 = 1$

Note: When a sequence is defined by a reccurence relation, you always need some "initial value" – i.e. value for the first tem(s) of the sequence – to get started.

Specifying a sequence

Note that it is not always possible to find an explicit or a recursive formula for a sequence!

For example, think of a sequence of random numbers.

Example: Arithmetic sequence

If the difference between adjacent terms is always the same (i.e. $a_n - a_{n-1} = \text{constant}$), the sequence is called **arithmetic.** The explicit formula for an arithmetic sequence is

$\{s_n\}_{n=0}^{\infty}$ with $s_n = a + n \cdot d$

The first few terms of this sequence are

$$\{s_n\}_{n=0}^{\infty} = \{a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots\}$$

Example: Geometric sequence

If the ratio between adjacent terms is always the same (i.e. $\frac{a_n}{a_{n-1}} = \text{constant}$), the sequence is called **geometric.** A geometric sequence can be described by the explicit formula

$$\{g_n\}_{n=0}^{\infty} \text{ with } g_n = a \cdot r^n$$

The first few terms of this sequence are

$$\{g_n\}_{n=0}^{\infty} = \{a, ar, ar^2, ar^3, \dots, ar^n, \dots\}$$

Example: The harmonic sequence*

The sequence given by

$$\{h_n\}_{n=1}^{\infty} \text{ with } h_n = \frac{1}{n}$$

is called **the harmonic sequence**. The first few terms of this sequence are:

$$\{h_n\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$$

* ***The* harmonic sequence** is actually just the most famous of a whole class of harmonic sequences given as $\{h_n\}_{n=1}^{\infty}$ with $h_n = \frac{1}{a \cdot n + d}$, i.e. the reciprocals of an arithmetic sequence.

Converting between recurrence relations and explicit formulae

Converting from explicit formula to recurrence relation

Exercise:

Find a recurrence relation for the sequence $\{a_n\}_{n=1}$ defined by $a_n = n!$

Solution:

$a_n = a_{n-1} \cdot n$ with $a_1 = 1$.

Converting from a recurrence relation to an explicit formula

It is often more useful to have an explicit formula for a sequence than a recurrence relation.

Therefore, we say that we have "solved" a recurrence relation when we have found an explicit formula for the sequence.

Example:
Solving a reccurence relations

<u>Example:</u>

A solution to the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 3$ is given by the closed formula $a_n = 3n$.

We can confirm this by checking the first few terms:

$a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 3$ → $a_0 = 0, a_1 = 3, a_2 = 6, a_3 = 9, \dots$

$a_n = 3n$ → $a_0 = 0, a_1 = 3, a_2 = 6, a_3 = 9, \dots$

Iteration

One way to solve a recurrence relation is by using **iteration**.

"repeated procedure applied to the result of a previous calculation"

This is exactly what a recurrence relation allows us to do!

In the following slides we will see two different iterative approaches to solve the recurrence relation given by $a_n = a_{n-1} + 3$ with $a_1 = 2$.

Iteration: Forward substitution

Solve the recurrence relation given by $a_n = a_{n-1} + 3$ with $a_1 = 2$.

Solution:

$$\begin{aligned}
a_2 &= 2 + 3 \\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&\vdots \\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1).
\end{aligned}$$

So we see that the recurrence relation is solved by the closed formula $a_n = 2 + 3(n - 1)$.

Iteration: Backward substitution

Solve the recurrence relation given by $a_n = a_{n-1} + 3$ with $a_1 = 2$.

Solution:

$$\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1).
\end{aligned}$$

Again we see that the recurrence relation is solved by the explicit formula $a_n = 2 + 3(n-1)$.

Example

Suppose that a person deposits $\$10{,}000$ in a savings account at a bank yielding 11% of compound interest per year.

Find an explicit formula for the amount of money in the account after n years.

<u>Solution:</u>

Let P_n be the amount of money after n years. We can then write a recurrence relation for P_n:
$P_n = P_{n-1} \cdot 1.11$ with $P_0 = 10{,}000$.

Use forward substitution to find an explicit formula for P_n.:

$$P_1 = 10{,}000 \cdot 1.11$$
$$P_2 = 10{,}000 \cdot 1.11 \cdot 1.11 = 10{,}000 \cdot 1.11^2$$
$$P_3 = 10{,}000 \cdot 1.11 \cdot 1.11 \cdot 1.11 = 10{,}000 \cdot 1.11^3$$
$$\vdots$$
$$\boldsymbol{P_n = 10{,}000 \cdot 1.11^n}$$

Series

A **series** is simply the sum of a sequence.

We will introduce some notation for series in the following slides.

Series example and sigma notation

<u>Example:</u>

When we sum all the terms of the sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n = \frac{1}{2^n}$ we get the series

$$\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots = \sum_{n=0} \frac{1}{2^n}$$

If we only want to add the terms up to a certain value of n, we write

$$\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots \frac{1}{2^8} = \sum_{n=0}^{8} \frac{1}{2^n}$$

upper limit

index of summation

lower limit

If we do not specify any limits, all terms of the sequence are added.

Example

What is the value of the series $\sum a_n$ where the terms are given by the sequence $\{a_n\}_{n=0}^{4} = \{-1,3,8,7,1.3\}$?

Answer: $\sum a_n = -1 + 3 + 8 + 7 + 1.3 = 18.3$

Example: The harmonic series

The sum of the harmonic sequence is known as the **harmonic series**.

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

What do you think the value of this sum is?

Anwer:

$$\sum_{n=1}^{\infty} h_n = \infty$$

When the sum is not a finite number, the series is said to be **divergent**.

Example: geometric series

The sum of a geometric sequence is known as a **geometric series**. Fx:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

What do you think the value of this sum is?

Anwer:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

When the sum is a finite number, the series is said to **convergent.**

Closed form formula for a series

Sometimes, it is possible to express a series as a simple, analytical expression. This is e.g. the case for the geometric series if the absolute value of r is smaller than 1:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1.$$

Closed form formulae for some important series

The table below (from Rosen) gives the closed form formulae for some important sums

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Changing limits

Often, it is useful to change the lower/upper limits, which can be done in a straightforward manner:

$$\sum_{j=1}^{n} a_j = \sum_{i=0}^{n-1} a_{i+1}$$

<u>Example:</u>

Given that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, find the value of $\sum_{k=-3}^{10} (k+4)^3$.

Solution: Let $i = k + 4$. Then

$\sum_{k=-3}^{10} (k+4)^3 = \sum_{i=1}^{14} i^3 = \frac{14^2 \cdot 15^2}{4} = 11025.$

Exercises

Use the table and the method of changing limits to find the value of the sums below

a) $\sum_{i=0}^{15}(i+1)$

b) $\sum_{i=1}^{11} 2 \cdot 3^{i-1}$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

<u>Solutions:</u>

a) Let $k = i + 1$. Then

$$\sum_{i=0}^{15}(i+1) = \sum_{k=1}^{16} k = \frac{16(16+1)}{2} = 136$$

b) Let $k = i - 1$. Then

$$\sum_{i=1}^{11} 2 \cdot 3^{i-1} = \sum_{k=0}^{10} 2 \cdot 3^k = \frac{2 \cdot 3^{11} - 2}{3-1} = 177{,}146$$