

# Introduction

Book: Not mandatory

- Uploaded relevant exercises
- Use another book or Google

Prerequisites: Recap today

Expect you to fill in the blanks

Exam: 3 hour in Wiseflow

Documentation must be uploaded

Must be .ipynb format

Tools: Python 3

Jupyter Notebook

↳ VSCode

↳ Jupyter lab

↳ DataSpell (Jetbrains)

↳ Google Colab

itslearning: I use in list mode

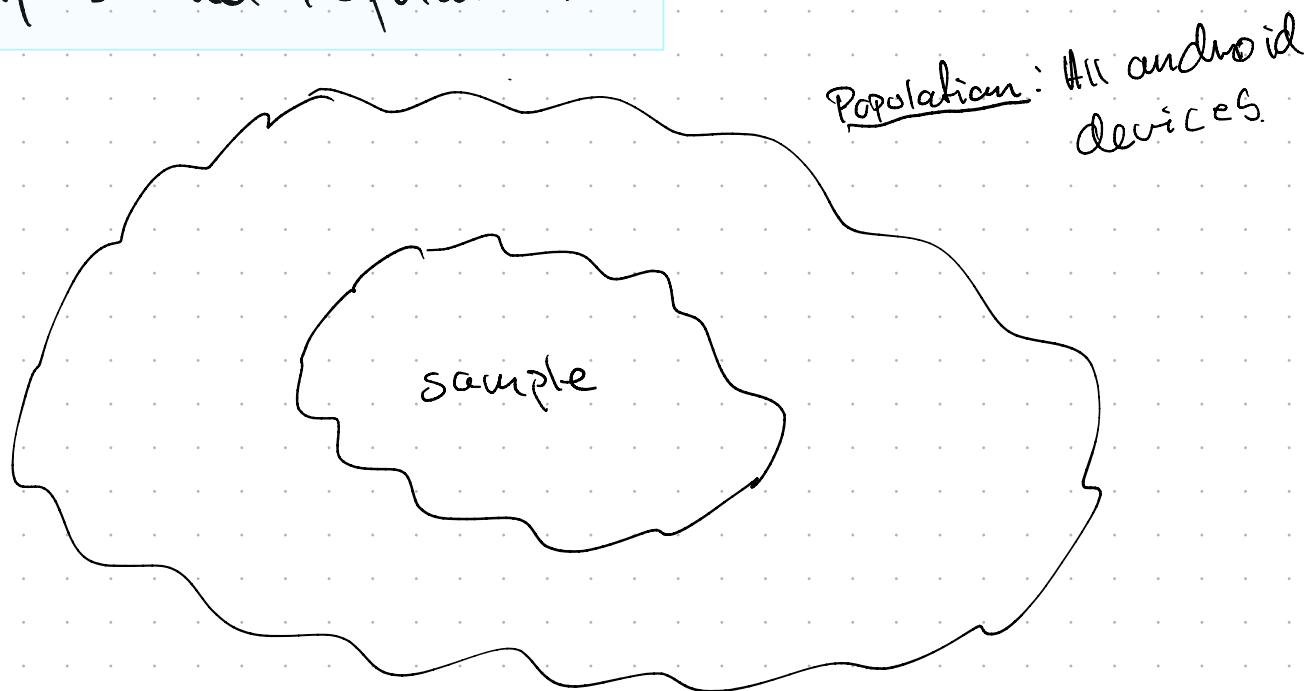
Session 0 will contain all material not associated with a specific session!

Wiseflow: You will receive multiple Wiseflow assignment

# Why Probability and statistics?

- Stochastic = Random
- Natural Processes  $\rightarrow$  Variability  $\rightarrow$  Uncertainty
  - $\hookrightarrow$  Statistics is the study of how to deal with uncertainty
  - $\hookrightarrow$  Probability is the unit in which we measure uncertainty

## Samples and Populations:



Sample = Representative subset of Population

Sample Size  $\rightarrow$  Big Data:

$$E = \frac{\sum x_i}{n}$$

$$\frac{2+2}{4} \rightarrow \frac{1}{2}$$

# Scales and Measurements:

**Nominal:** Values have no quantitative significance: 0 = male, 1 = female  
Is  $1 > 0$ ?

↳ Groups, classes, categories  
e.g. gender, color, jobs, etc.

**Ordinal:** Values are comparable, but difference is not known

↳ Order matters: Ranking.

**Interval:** Values are comparable and difference and distance matter

↳ Zero is assigned arbitrarily

- \* Temperature
- \* Time of day
- \* Dates
- \* Likert scale

**Ratio:** Same as interval but with a "natural" zero:

- \* Height
- \* Weight
- \* Income
- \* Number of children

# Random Experiments

- Process where something uncertain is observed
- An outcome is the result of a random experiment
- A sample space  $S$  is the set of all possible outcomes.

Toss a coin:

$$S = \{H, T\}$$

Guess a bit string:

$$S = \{0, 1\}^n$$

Roll a die:

$$S = \{1, 2, 3, \dots, 6\}$$

Price of item:

$$S = \{0, \dots, \infty\}$$

Goals scored in football match:

$$S = \{0, \dots, \infty\}$$

Goals scored in Italian football:

$$S = \{0, 1\}^* \quad * \text{joke}$$

- An event  $A$  is a subset of the sample space.

# Probability

## Objective / classical

- Based on equally likely events
- long run relative frequency
- same for all observers

## Empirical

- Based on observation
- Relative frequency of large amount of observations

## Subjective

- Based on personal belief, experience, prejudice, etc.
- Different for all observers

## Axioms of Probability

1. For any event  $A$ ,  $P(A) \geq 0$ ,  $P(A) = P\{A\}$

2.  $P(S) = 1$

3. If  $A_1, A_2, A_3, \dots$  are disjoint

$$P(A_1 \cup A_2 \cup A_3 \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

Notation:  $P(A \cap B) = P(A \text{ and } B) = P(A, B)$

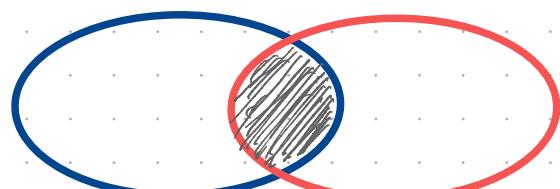
$P(A \cup B) = P(A \text{ or } B)$

$P(\bar{A}) = P(A^c) = P(A') = P(\text{not } A)$

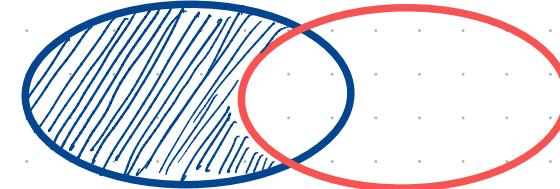
## Rules of Probability

a)  $P(A^c) = 1 - P(A)$

$A \cap B$



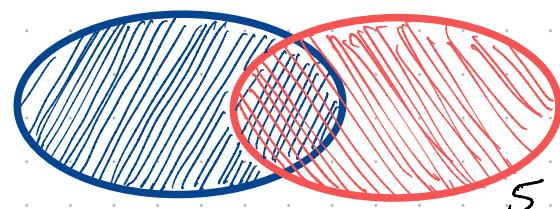
b)  $P(\emptyset) = 0$



c)  $P(A) \leq 1$

d)  $P(A) = P(A-B) + P(A \cap B)$   $A-B$

$A \cup B$



e)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

f)  $A \subset B \Rightarrow P(A) \leq P(B)$

↓  
Is a subset of

Example:

1. 60% Chance of rain today:  $P(A) = 0.6$
2. 50% Chance of rain tomorrow:  $P(B) = 0.5$
3. 30% Chance of no rain either day:  $P(A^c \cap B^c) = 0.3$

a. Probability of rain either day:

$$\begin{aligned} P(A \cup B) &= 1 - P(A \cap B)^c \\ &= 1 - P(A^c \cup B^c) \\ &= 1 - 0.3 = \underline{\underline{0.7}} \end{aligned}$$

b. Rain both days:

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.6 + 0.5 - 0.7 \\ &= \underline{\underline{0.4}} \end{aligned}$$

c. Rain today, not tomorrow:

$$\begin{aligned} P(A - B) &= P(A) - P(A \cap B) \\ &= 0.6 - 0.4 \\ &= \underline{\underline{0.2}} \end{aligned}$$

d. Rain today or tomorrow, not both

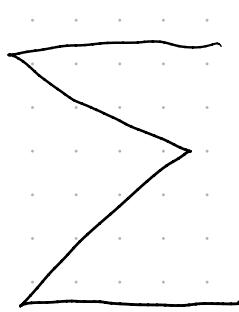
$$\begin{aligned} P((A \cup B) \setminus (A \cap B)) &= P(A \cup B) - P(A \cap B) \\ &= 0.7 - 0.4 = \underline{\underline{0.3}} \end{aligned}$$

OR

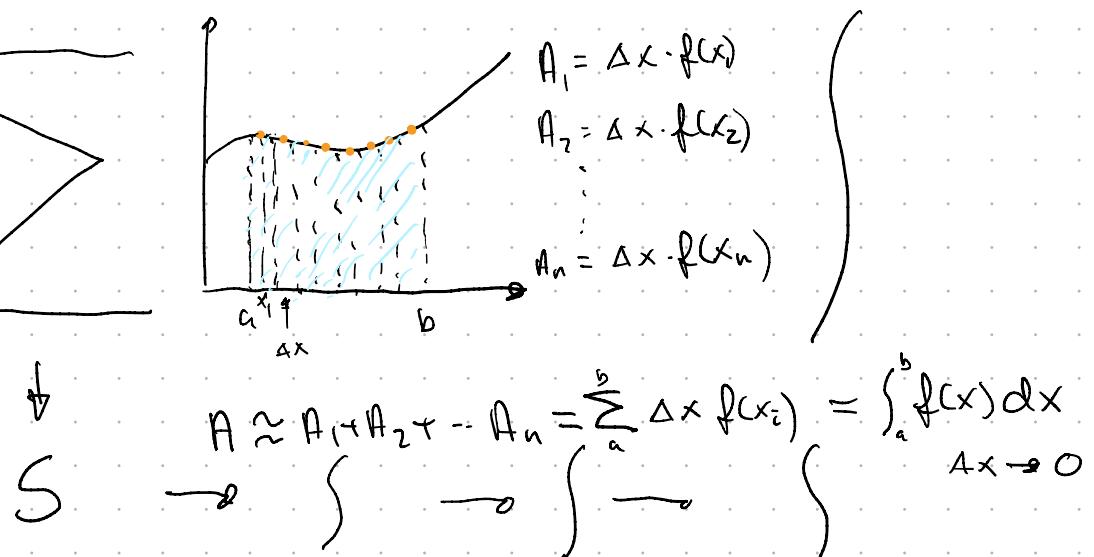
$$\begin{aligned} &= P(A - B) + P(B - A) \\ &= 0.2 + (0.5 - 0.4) = \underline{\underline{0.3}} \end{aligned}$$

# Discrete Probability Models

## Discrete



## Continuous



If a sample space is countable, we use a discrete probability model

If  $A \subset S$ :

$$P(A) = P\left(\bigcup_{S_j \in A} \{S_j\}\right) = \sum_{S_j \in A} P(S_j)$$

Example:

Assume you win  $K-2$  Kroner with  $P = \frac{1}{2^K}$ ,  $K \in \mathbb{N}$

a. loose 1:  $K-2 = -1 \Rightarrow K=1$  so  $P = 1/2$

b. loose 0:  $K-2 = 0 \Rightarrow K=2$  so  $P = 1/4$

c. Win 1:  $K-2 = 1 \Rightarrow K=3$  so  $P = 1/8$

d. Win 2:  $K-2 = 2 \Rightarrow K=4$  so  $P = 1/16$

e. Win 3:  $K-2 = 3 \Rightarrow K=5$  so  $P = 1/32$

f. Win [1; 3]:  $P_1 + P_2 + P_3 = 7/32$

g. More than 2:  $P_3 + P_4 + P_5 + \dots$

$$= 1/32 + 1/64 + 1/128 + \dots = \underline{\underline{1/16}} \quad 7$$

# Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) > 0$$

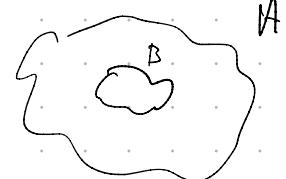
Some important consequences:

$$\rightarrow P(A^c|B) = 1 - P(A|B)$$

$$\rightarrow P(\emptyset|B) = 0$$

$$\rightarrow \text{If } B \subset A, P(A|B) = 1 = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$\rightarrow \text{If } A \subset B, P(A|B) = \frac{P(A)}{P(B)}$$



## Chain Rule

$$P(A_1 \cap A_2 \cap A_3 \dots A_k) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1, A_2) \cdot \dots \cdot P(A_k|A_{k-1}, A_{k-2}, \dots)$$

## Independence

$$P(A|B) = P(A) \text{ iff. } A \text{ and } B \text{ are independent}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

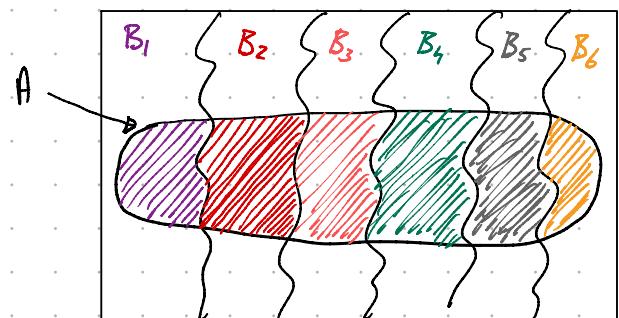
$$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B)$$

## Law of total probability

$$P(A) = \sum P(A \cap B_i)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots$$

$$P(A) = \sum P(A|B_i) \cdot P(B_i)$$



## Bages' Rule

$$P(A|B) \cdot P(B) = P(A \cap B) = P(B|A) \cdot P(A)$$

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$P(B_j|A) = \frac{P(A|B_j) \cdot P(B_j)}{\sum_i P(A|B_i) \cdot P(B_i)}$$

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})}$$

## Conditional Independence (not recap)

Two events A and B are said to be  
conditionally independent given an event C if

$$P(A \cap B|C) = P(A|C) \cdot P(B|C), P(C) > 0$$

Understanding Conditional independence:

$$P(A|B,C) = A \text{ given } B \text{ and } C$$

If A and B are conditionally independent,  
then  $P(A|B,C)$  does not depend on  
B and we get  $P(A|B,C) = P(A|C)$

$$\text{and also } P(A,B|C) = P(B|C), \text{i.e.}$$

A and B are independent given C.

## Example

A box contains two coins: a regular coin and one fake two-headed coin ( $P(H) = 1$ ). I choose a coin at random and toss it twice. Define the following events.

- A= First coin toss results in an  $H$ .
- B= Second coin toss results in an  $H$ .
- C= Coin 1 (regular) has been selected.

Find  $P(A|C)$ ,  $P(B|C)$ ,  $P(A \cap B|C)$ ,  $P(A)$ ,  $P(B)$ , and  $P(A \cap B)$ . Note that  $A$  and  $B$  are NOT independent, but they are *conditionally* independent given  $C$ .

Regular Coin:  $P(H) = 1/2$ ,  $P(T) = 1/2$

Fake Coin:  $P(H) = 1$ ,  $P(T) = 0$

$$P(A|C) = 1/2$$

$$P(B|C) = 1/2$$

$$P(A \cap B|C) = 1/2 \cdot 1/2 = 1/4$$

$$\begin{aligned} P(A) &= P(A|C) \cdot P(C) + P(A|\bar{C}) \cdot P(\bar{C}) \\ &= 1/2 \cdot 1/2 + 1 \cdot 1/2 \\ &= 3/4 \end{aligned}$$

$$\begin{aligned} P(B) &= \text{Same as } P(A) \\ &= 3/4 \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(A \cap B|C) \cdot P(C) + P(A \cap B|\bar{C}) \cdot P(\bar{C}) \\ &= P(A|C) \cdot P(B|C) \cdot P(C) + P(A|\bar{C}) \cdot P(B|\bar{C}) \cdot P(\bar{C}) \\ &= 1/2 \cdot 1/2 \cdot 1/2 + 1 \cdot 1 \cdot 1/2 \\ &= \frac{1}{8} + \frac{1}{2} = 5/8 \end{aligned}$$

# Random Variables

A random variable  $X$  is a function from the sample space to  $\mathbb{R}$ :

$$X: S \rightarrow \mathbb{R}$$

The range  $R_X$  is the set of all possible values of  $X$

Ex:

- Toss a coin 100 times. Let  $X$  be the number of heads:  $R_X = \{0, 1, \dots, 100\}$
- Toss a coin until the first head. Let  $Y$  be the number of tosses until first head:  
$$R_Y = \mathbb{N}_+$$
- Let  $Z$  denote the number of times you have to stop at a red light on your way to school:

$$P(Z=0)$$

$$P(Z=1)$$

$$P(Z \leq 2)$$

:

etc.