

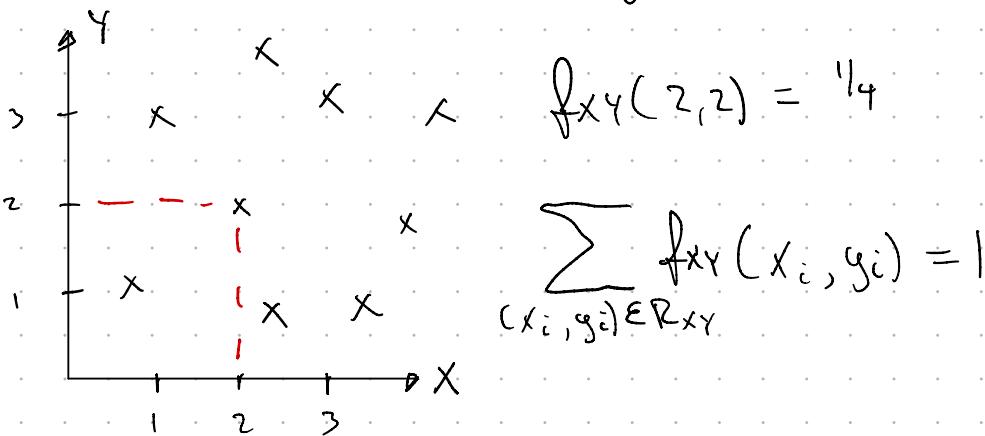
Multivariate RVs

Two Discrete RVs:

X, Y joint PMF:

$$P_{X,Y}(x_i, y_i) = f_{XY}(x_i, y_i) = P(X=x_i, Y=y_i)$$

$$R_{XY} = \{(x_i, y_i) \mid f_{XY}(x_i, y_i) > 0\}$$

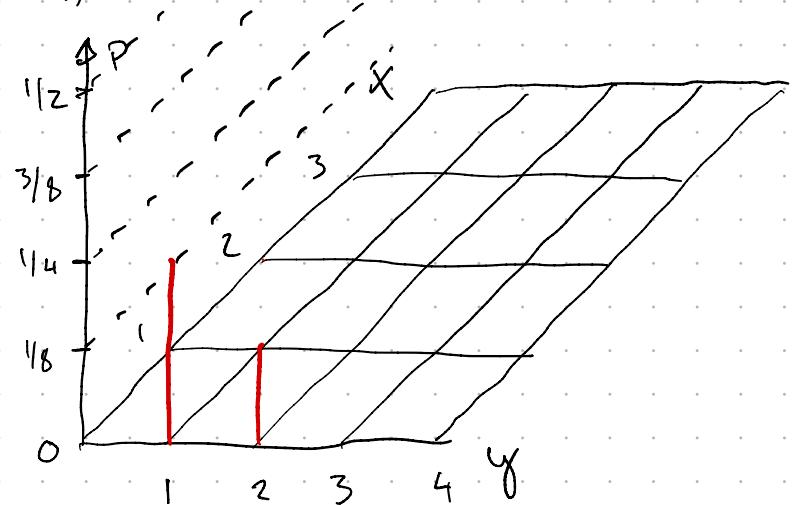


In general:

$$P((X, Y) \in A) = \sum_{(x_i, y_i) \in A \cap R_{XY}} f_{XY}(x_i, y_i)$$

Example

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$



a) Find $P(X=0, Y \leq 1) = f(0,0) + f(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

b) Find Marginal PMFs of X and Y:

$$f_X(0) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}, \quad f_Y(0) = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$$

$$f_X(1) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}, \quad f_Y(1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

In general Marginal PMF:

$$f_X(x) = \sum_{g_i} f_{xy}(x, g_i)$$

$$f_Y(y) = \sum_{x_i} f_{xy}(x_i, y)$$

Example cont'd:

c) Find $P(Y=1 | X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{1/4}{13/24} = \frac{24}{52} = \frac{6}{13}$

d) Are X and Y independent?

$$\begin{aligned} P(Y=1 | X=0) &= P(Y=1) \\ 6/13 &= 5/12 \end{aligned} \quad \left. \begin{array}{l} \text{No not} \\ \text{Independent} \end{array} \right\}$$

In general:

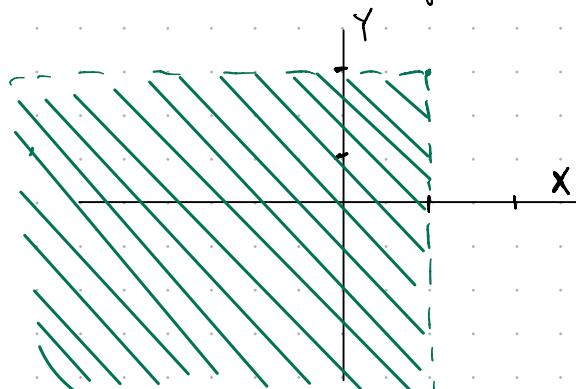
X and Y are independent if

$$f_{xy}(x_i, y_i) = f_x(x_i) \cdot f_y(y_i)$$

X, Y joint CDF:

$$F_{xy}(x, y) = P(X \leq x, Y \leq y)$$

$$0 \leq F_{xy}(x, y) \leq 1$$



$$F_{xy}(1, 2) = P(X \leq 1, Y \leq 2)$$

Marginal CDF:

$$F_x(x) = F_{xy}(x, \infty) = \lim_{y \rightarrow \infty} F_{xy}(x, y)$$

$$F_y(y) = F_{xy}(\infty, y) = \lim_{x \rightarrow \infty} F_{xy}(x, y)$$

Example:

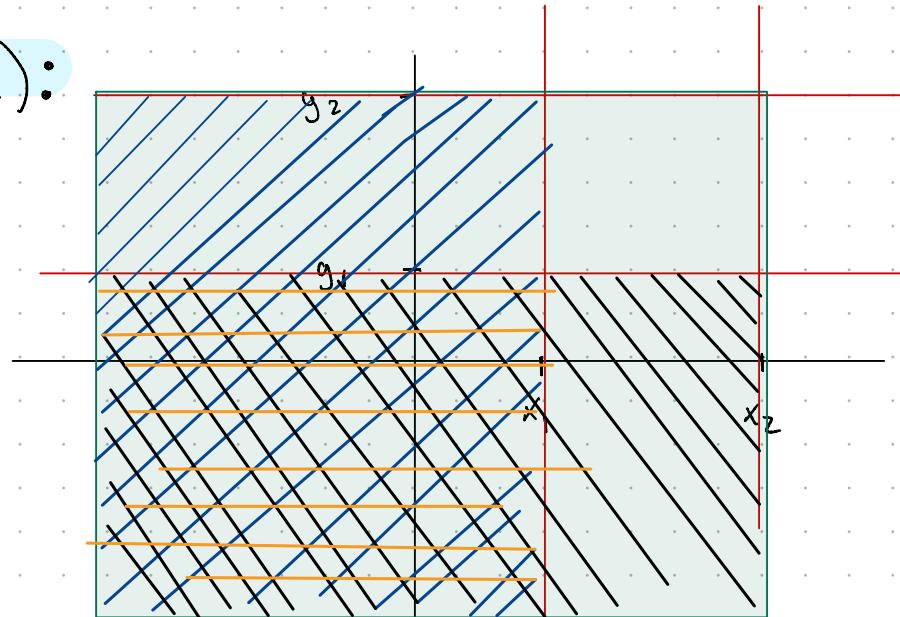
$$\text{Find } F_{XY}(0.5, 1) =$$

$$P(X \leq 0.5, Y \leq 1) =$$

$$f_{XY}(0,0) + f_{XY}(0,1) = \underline{\underline{\frac{5}{12}}}$$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	1/6	1/4	1/8
$X=1$	1/8	1/6	1/6

$P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2)$:



$$= F_{XY}(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

Conditioning and Independence

$$\begin{aligned} f_{X|Y}(x_i | y_i) &= P(X=x_i | Y=y_i) \\ &= \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)} = \frac{f_{XY}(x_i, y_i)}{f_Y(y_i)} \end{aligned}$$

Independence:

$$f_{X|Y}(x_i | y_i) = f_X(x_i)$$

$$\frac{f_{XY}(x_i, y_i)}{f_Y(y_i)} = f_X(x_i) \Rightarrow f_{XY}(x_i, y_i) = f_X(x_i) \cdot f_Y(y_i)$$

Conditional Expectation:

$$E(X|Y=y_i) = \sum_{x_i \in \mathbb{R}_X} x_i \cdot f_{X|Y}(x_i|y_i)$$

Example:

Assume the following joint PMF:

$$f_{XY}(x,y) = \begin{cases} 1/3 & x=1 \wedge y=3 \\ 1/3 & x=2 \wedge y=0 \\ 1/3 & x=0 \wedge y=0 \\ 0 & \text{else} \end{cases}$$

a) Find $P_{X|Y=0}(x)$

$$P_{X|Y=0}(x_i) = \frac{P_{XY}(x_i, 0)}{P_Y(0)}, \quad P_Y(0) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$P_{X|Y=0}(0) = \frac{1/3}{2/3} = 1/2$$

$$P_{X|Y=0}(1) = 0 = 0$$

$$P_{X|Y=0}(2) = \frac{1/3}{2/3} = 1/2$$

$$P_{X|Y=0}(x) = \begin{cases} 1/2 & x=0 \\ 0 & x=1 \\ 1/2 & x=2 \\ 0 & x \notin \mathbb{R}_X \end{cases}$$

b) Find $E[X|Y=0]$:

$$\begin{aligned} E[X|Y=0] &= 0 \cdot 1/2 + 1 \cdot 0 + 2 \cdot 1/2 \\ &= 1 \end{aligned}$$

Example:

We throw a fair coin three times. Let X denote the number of heads on first toss and let Y denote the total number of heads:

$X \setminus Y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

Find the expected number of heads given that the first toss resulted in a tail.

$$P_{Y|X=0}(y) = \frac{P_{XY}(y_i, 0)}{P_X(0)}$$

$$P_X(0) = \frac{1}{2}, P_X(1) = \frac{1}{2}$$

$$P_{Y|X=0}(0) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{Y|X=0}(1) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P_{Y|X=0}(2) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{Y|X=0}(3) = \frac{0}{\frac{1}{2}} = 0, \text{ so}$$

$$E(Y|X=0) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot 0 = 1$$

$$E(Y|X=1) = 0 \cdot 0 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$$

Expectation

Total Expectation:

- $E[X] = \sum_i E[X|B_i] \cdot P[B_i]$, B_1, B_2 some Partition of S
- $E[X] = \sum_{y_j \in R_Y} E[X|Y=y_j] P_Y(y_j)$
- $E[X] = E[E[X|Y]]$

Two Continuous RVs

Joint PDF:

$$P_{XY}((X, Y) \in A) = \iint_A f_{XY}(x, y) dx dy$$

If $A = \mathbb{R}^2$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy$$

Joint CDF:

$$\begin{aligned} F_{XY}(x, y) &= P(X \leq x, Y \leq y) \\ &= \int_{-\infty}^y \int_{-\infty}^x f_{XY}(x, y) dx dy \end{aligned}$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad 0 \leq x, y \leq 1$$

a. Find the constant c .

b. Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

$$a) \int_0^1 \int_0^1 (x + cy^2) dx dy = \int_0^1 \left(\frac{1}{2}x^2 + cxy^2 \Big|_0^1 \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} + cy^2 \right) dy = \frac{1}{2}y + \frac{1}{3}cy^3 \Big|_0^1 = \frac{1}{2} + \frac{1}{3}c = 1$$

$$c = \frac{3}{2}$$

b)

$$P(0 \leq X \leq 1/2, 0 \leq Y \leq 1/2) = \int_0^{1/2} \int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx dy$$

$$= \int_0^{1/2} \left(\frac{1}{2}x^2 + \frac{3}{2}xy^2\Big|_0^{1/2}\right) dy = \int_0^{1/2} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy =$$

$$= \left. \frac{1}{8}y + \frac{1}{4}y^3 \right|_0^{1/2} = \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$$

The Joint CDF satisfies:

1) $F_X(x) = F(x, \infty)$ (marginal of X)

2) $F_Y(y) = F(\infty, y)$ (marginal of y)

3) $F_{XY}(-\infty, \infty) = 1$

4) $P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) =$

$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

Marginal PDFs:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

c) Find Marginal PDFs:

$$f_X(x) = \int_0^1 \left(x + \frac{3}{2}y^2\right) dy = xy + \frac{1}{2}y^3\Big|_0^1 = x + \frac{1}{2}$$

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \int_0^1 \left(x + \frac{3}{2}y^2\right) dx = \frac{1}{2}x^2 + \frac{3}{2}xy^2\Big|_0^1 = \frac{3}{2}y^2 + \frac{1}{2}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x, y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- a. Find R_{XY} and show it in the $x - y$ plane.
- b. Find the constant c .
- c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.
- d. Find $P(Y \leq \frac{X}{2})$.
- e. Find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$.

a) $R_{XY} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\}$

b) $\int_0^1 \int_0^x c x^2 y \, dy \, dx = \int_0^1 \left[\frac{1}{2} c x^2 y^2 \right]_0^x \, dx$

$$= \int_0^1 \frac{1}{2} c x^4 \, dx = \frac{1}{10} c x^5 \Big|_0^1 = \frac{1}{10} c = 1 \Rightarrow c = 10$$

c) $f_X(x) = \int_0^x 10 x^2 y \, dy = 10 x^2 \cdot \frac{1}{2} y^2 \Big|_0^x = 5x^4$

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

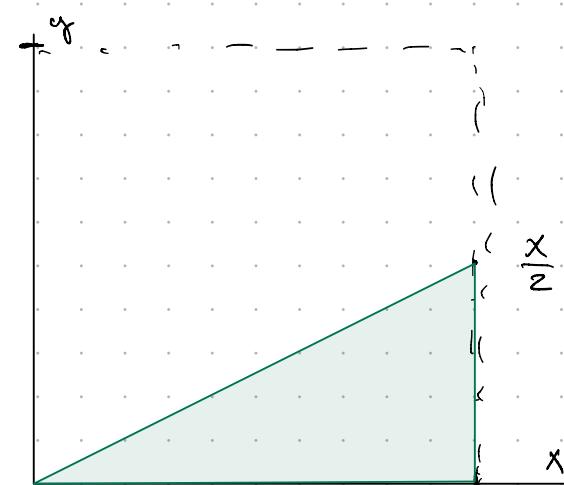
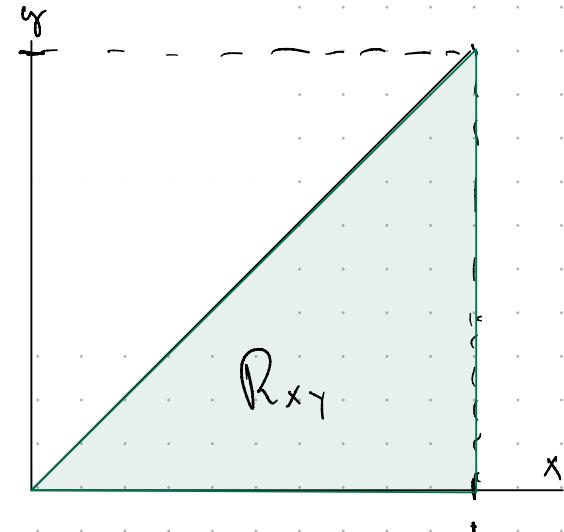
$$\begin{aligned} f_Y(y) &= \int_y^1 10 x^2 y \, dx = \frac{10}{3} x^3 y \Big|_y^1 = \frac{10}{3} y - \frac{10}{3} y^4 \\ &= \frac{10}{3} y(1 - y^3) \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{10}{3} y(1 - y^3) & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

d) $P(Y \leq \frac{X}{2}) = \int_0^1 \int_0^{x/2} 10 x^2 y \, dy \, dx$

$$= \int_0^1 5x^2 y^2 \Big|_0^{x/2} \, dx = \int_0^1 \frac{5}{4} x^4 \, dx$$

$$= \frac{5}{4} x^5 \Big|_0^1 = \underline{\underline{\frac{5}{4}}}$$



$$\begin{aligned}
 e) P(Y \leq \frac{x}{4} | Y \leq \frac{x}{2}) &= \frac{P(Y \leq \frac{x}{4}, Y \leq \frac{x}{2})}{P(Y \leq \frac{x}{2})} = \frac{P(Y \leq \frac{x}{4})}{P(Y \leq \frac{x}{2})} \\
 &= 4 \int_0^{\frac{x}{4}} 10x^2 \cdot y dy dx = 4 \int_0^{\frac{x}{4}} 5x^2 y^2 \Big|_0^{x/4} \\
 &= 4 \int_0^{\frac{x}{4}} \frac{5}{16} x^4 dx = 4 \left[\frac{1}{16} x^5 \right]_0^{\frac{x}{4}} = \underline{\underline{\frac{1}{4}}}
 \end{aligned}$$

Mean and Variance:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{XY}(x,y) dy dx$$

$$V[X] = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \left(\int_{-\infty}^{\infty} x \cdot f_X(x) dx \right)^2$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_{XY}(x,y) dy dx - \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{XY}(x,y) dy dx \right)^2 \\
 &= E[X^2] - (E[X])^2
 \end{aligned}$$

Example

Given

$$f_{X,Y}(x,y) = \begin{cases} 2-x-y & , 0 \leq x, y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $\text{Var}(X)$:

$$\begin{aligned}
 E[X] &= \int_0^1 \int_0^1 x (2-x-y) dy dx = \int_0^1 \int_0^1 (2x - x^2 - xy) dy dx \\
 &= \int_0^1 (2xy - x^2y - \frac{1}{2}y^2) \Big|_0^1 dx = \int_0^1 (2x - x^2 - \frac{1}{2}x) dx \\
 &= x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^2 \Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \underline{\underline{\frac{5}{12}}}
 \end{aligned}$$

$$E[X^2] = \int_0^1 \int_0^1 x^2 (2-x-y) dy dx$$

`sympy.integrate(x**2 * (2-x-y), (y, 0, 1), (x, 0, 1))`

$$= \underline{\underline{\frac{1}{4}}}$$

$$\text{Var}(X) = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}$$

Conditioning

Same principles as in discrete case. Most importantly:

$$1. f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$2. P(X \in A | Y=y) = \int_A f_{X|Y}(x|y) dy$$

$$3. F_{X|Y}(x|y) = P(X \leq x | Y=y) = \int_{-\infty}^x f_{X|Y}(x|y) dx$$

$$4. E[X | Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y) dx$$

$$5. V[X | Y=y] = E[X^2 | Y=y] - (E[X | Y=y])^2$$

Independence:

Same as discrete!

Lotus still applies:

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y) dy dx$$

Example:

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \leq x, y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $E[XY^2]$

$$E[XY^2] = \text{sympy.integrate}((x+y) \cdot xy^2, (y, 0, 1), (x, 0, 1))$$

$$= \frac{17}{72}$$

Covariance and Correlation:

The Covariance gives information about how X and Y are statistically related:

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - EX)(Y - EY)] \\ &= E[XY] - (EX) \cdot (EY)\end{aligned}$$

Properties:

1. $\text{Cov}(X, X) = \text{Var}(X)$

2. If X and Y are independent

$$\text{Cov}(X, Y) = 0$$

3. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

4. $\text{Cov}(aX, Y) = a \cdot \text{Cov}(X, Y)$,

5. $\text{Cov}(X+c, Y) = \text{Cov}(X, Y)$

6. $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

7. $\text{Cov}(X+Y, Z+W)$

$$= \text{Cov}(X, Z) + \text{Cov}(X, W) + \text{Cov}(Y, Z) + \text{Cov}(Y, W)$$

8. More generally:

$$\text{Cov}\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \text{Cov}(X_i, Y_j)$$

If $Z = X+Y$

$$\begin{aligned}9. \text{Var}(Z) &= \text{Cov}(Z, Z) = \text{Cov}(X+Y, X+Y) \\ &= \text{Cov}(X, X) + \text{Cov}(X, Y) + \text{Cov}(Y, X) + \text{Cov}(Y, Y) \\ &= \text{Var}(X) + \text{Var}(Y) + 2 \cdot \text{Cov}(X, Y)\end{aligned}$$

10. More generally:

$$\text{Var}(aX+bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y)$$

The **correlation coefficient**, ρ_{xy} or $\rho(X, Y)$ is obtained by normalising the covariance. More specifically we use the standardised versions of X and Y :

$$U = \frac{X - E(X)}{\sigma_x}, \quad V = \frac{Y - E(Y)}{\sigma_y}$$

$$\rho_{xy} = \text{Cov}(U, V) = \text{cov}\left(\frac{X - E(X)}{\sigma_x}, \frac{Y - E(Y)}{\sigma_y}\right)$$

$$= \text{cov}\left(\frac{X}{\sigma_x}, \frac{Y}{\sigma_y}\right) \quad (\text{Property 5})$$

$$= \frac{\text{Cov}(X, Y)}{\sigma_x \cdot \sigma_y} \quad (\text{Property 4})$$