Exercise 4

Consider two random variables X and Y with joint PMF given by

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}, \quad \text{for } k,l = 1,2,3,\ldots$$

Find $P\left(X^2 + Y^2 \leq 10\right)$

$$R_X = \{1, 2, 3 \ldots\}$$

$$R_Y = \{1, 2, 3 \ldots\}$$

$$P_X(k) = \sum P(X=k, Y=l) = \sum_{l=1}^{\infty} \frac{1}{2^{k+l}}$$

$$= \frac{1}{2^k} \cdot \sum_{1}^{\infty} \frac{1}{2^l} = \frac{1}{2^k}$$

$$P_Y(l) = \sum_{k=1}^{\infty} \frac{1}{2^{k+l}} = \frac{1}{2^l} \sum_{1}^{\infty} \frac{1}{2^k} = \frac{1}{2^l}$$

$$P_X(k) \cdot P_Y(l) == P_{XY}(k,l) \rightarrow \text{TRUE}$$

$$P(X^2+Y^2 \leq 10) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$$
$$+ P(X=2, Y=1) + P(X=2, Y=2) + P(X=3, Y=1)$$
$$= \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} = \frac{11}{16}$$

Exercise 5

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2} & 0 \le x, \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

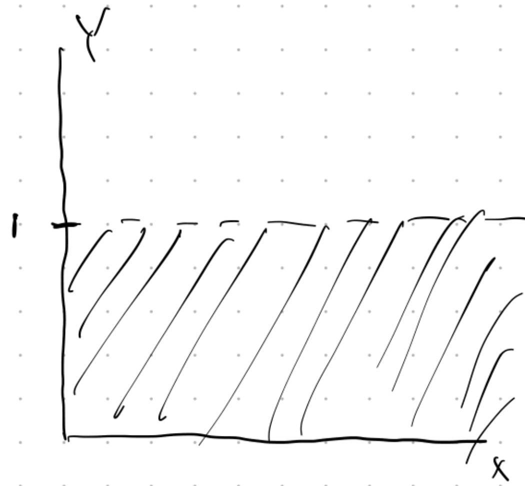
a. Find the constant c.

b. Find $P(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$.

c. Find $P(0 \le X \le 1)$.

x^{-2}

$\frac{1}{x^2}$



a)

$$\int_{y=0}^{1}\int_{x=0}^{\infty}\left(\frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2}\right)dx\,dy$$

$$= \int_0^1 \left(-\frac{1}{2}e^{-x} - \frac{cy}{(1+x)}\right)\Big|_0^{\infty} dy$$

$$= \int_0^1 \left(\frac{1}{2} + cy\right)dy = \frac{1}{2}y + \frac{1}{2}cy^2\Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{2}c = 1 \iff c = 1$$

$\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}$

b)

$$P(0 \le X \le 1, 0 \le Y \le \tfrac{1}{2}) =$$

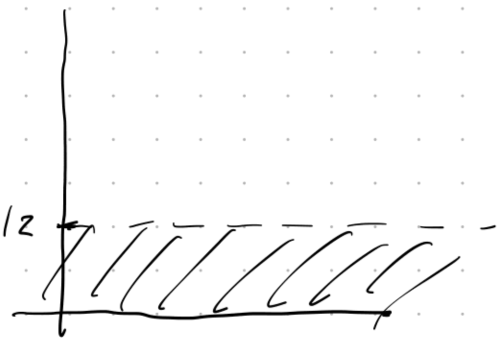
$$\int_0^{1/2}\int_0^1 \left(\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}\right)dx\,dy =$$

sympy.integrate$\left(\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}, (x, 0, 1), (y, 0, 1/2)\right)$

$$= 0.22 = \frac{5}{16} - \frac{1}{4e}$$

c)

$$P(0 \le X \le 1) = \int_0^1\int_0^1 \left(\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}\right)dy\,dx$$



$$= \underline{\underline{\frac{3}{4} - \frac{1}{2e} \approx 0.566}}$$

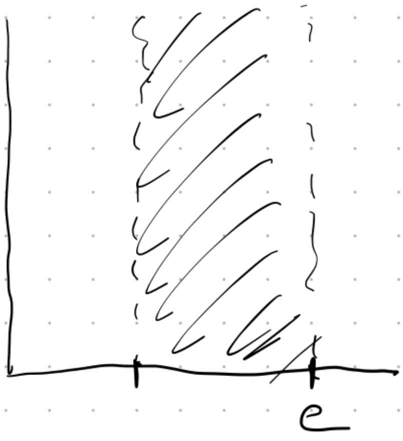
Exercise 6

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} e^{-xy} & 1 \le x \le e, \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.

b. Write an integral to compute $P(0 \le Y \le 1, 1 \le X \le \sqrt{e})$.



a) $f_X(x) = \int_0^\infty e^{-xy}\,dy = -\frac{1}{x} e^{-xy}\Big|_0^\infty$

$= \frac{1}{x}$

$f_Y(y) = \int_1^e e^{-xy}\,dx = -\frac{1}{y} \cdot e^{-xy}\Big|_1^e$

$= -\frac{1}{y} \cdot e^{-e \cdot y} - \left(-\frac{1}{y} \cdot e^{-y}\right)$

$= \frac{1}{y}\left(e^{-y} - e^{-ey}\right)$, $y > 0$

b) $\int_1^{\sqrt{e}} \int_0^1 e^{-xy}\,dy\,dx = \int_1^{\sqrt{e}} -\frac{1}{x} e^{-xy}\Big|_0^1 dx = \int_1^{\sqrt{e}} \left[-\frac{1}{x} e^{-x} + \frac{1}{x}\right] dx$

Exercise 7

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}x^2 + \frac{1}{6}y & -1 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

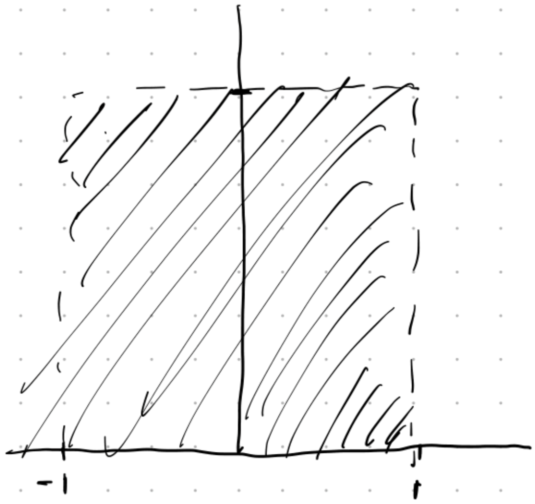
a. Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.

b. Find $P(X > 0, Y < 1)$.

c. Find $P(X > 0 \text{ or } Y < 1)$.

d. Find $P(X > 0 | Y < 1)$.

e. Find $P(X + Y > 0)$.

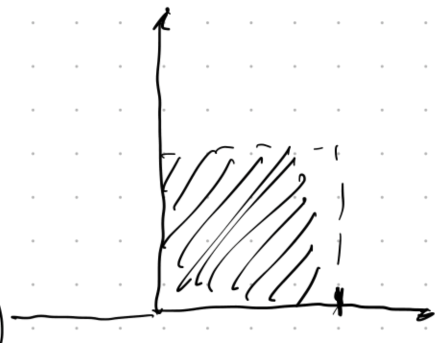


a) $f_X(x) = \int_0^2 \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dy$

$= \frac{1}{2}x^2 + \frac{1}{3}$

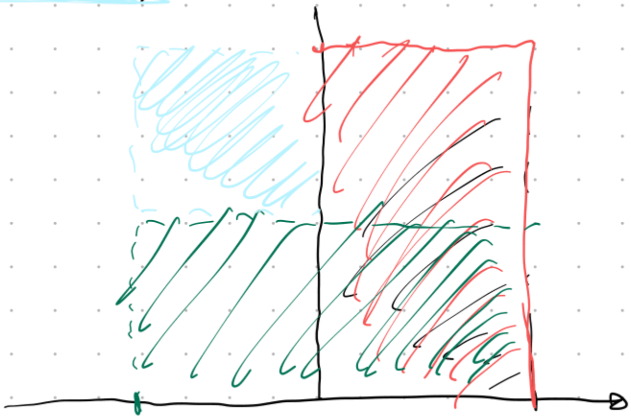
$f_X =$

$f_Y(y) = \int_{-1}^{1} \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dx = \underline{\frac{1}{3}y + \frac{1}{6}}$



b) $\int_0^1 \int_0^1 \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dy\, dx = \frac{1}{6}$

c) $P(X > 0 \text{ or } Y < 1) = 1 - P(X < 0 \text{ and } Y > 1)$



$1 - \underbrace{\int_{-1}^{0} \int_{1}^{2} \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dy\, dx}$

$= 2/3$

d) $P(X \ge 0 | Y < 1) = \frac{P(X > 0, Y < 1)}{P(Y < 1)}$

$= \frac{1}{6 \cdot P(Y < 1)}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(Y < 1) = \int_0^1 \left(\frac{1}{3}y + \frac{1}{6}\right) dy = \frac{1}{3}$

$P(X \ge 0 | Y < 1) = \frac{1}{6 \cdot 1/3} = \frac{1}{2}$

$$P(X+Y>0) = P(Y>-x)$$

$$P(X+Y \geq 0) = 1 - P(B)$$

$$= 1 - \int_{-1}^{0} \int_{0}^{-x} \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dy\, dx$$

$$= \frac{131}{144} \approx 0.909$$