

Functions of Continuous R.V.'s: Difficult

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

Let X be a Uniform(0,1) R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_Y(y) = \frac{y-a}{b-a} = \frac{y-0}{1-0} = y, f(x)=1$$

$$R_X = [0; 1], R_Y = [1; e]$$

$$F(Y) = P(Y \leq y)$$

$$= P(e^X \leq y)$$

$$= P(X \leq \ln y)$$

SG

$$F_Y = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

b) Find pdf of Y :

$$f_Y(y) = F'_Y(y) = \frac{1}{y} \text{ for } 1 \leq y \leq e, \text{ else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y} dy = y \Big|_1^e = \underline{\underline{e-1}}$$

Using hotus:

$$\begin{aligned} E[Y] &= E[e^X] = \int_0^1 e^x \cdot f_X(x) dx = e^x \Big|_0^1 \\ &= e^1 - e^0 = \underline{\underline{e-1}} \end{aligned}$$

Example

Let $f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

and let $Y = \frac{1}{X}$. Find PDF.

Method 1: like above we first find CDF

Note $R_Y = [1, \infty]$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P\left(\frac{1}{X} \leq y\right) \\ &= P(X \geq \frac{1}{y}) \\ &= F_X(y) = \int_0^{1/y} 4x^3 dx = x^4 \Big|_0^{1/y} = \frac{1}{y^4} \end{aligned}$$

$$f_Y(y) = F_Y'$$

Method 2: The Method of Transformation

Let X be a CRV and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonic and differentiable function

$$(x^{-1})' = -x^{-2}$$

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} & \text{where } g(x_1) = y \Rightarrow x_1 = g^{-1}(y) = \frac{1}{y} \\ 0 & \text{if } g(x) = y \text{ has no solution} \end{cases}$$

Exponential Distribution:

A CRV X is said to be exponentially distributed with $\lambda > 0$, shown as

$X \sim \text{Exponential}(\lambda)$, if its PDF is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & \end{cases}$$

Is often used to model time between events. let's find CDF:

λ is called the rate parameter.

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Example

Jobs are sent to printer at a rate of 3 jobs per hour (on average)

a) What is expected time between jobs:

b) What is probability that next job is sent within 5 minutes?

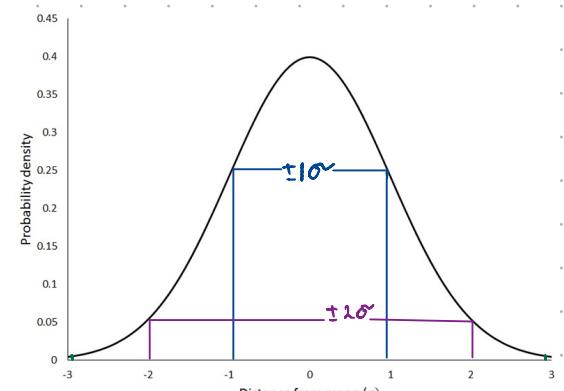
Normal Distribution

Characteristics:

Symmetric, bell shaped

Continuous for all intervals
s.t. any $P(X \in [a; b]) \neq 0$

$-\infty \leq X \leq \infty$



Two parameters, μ and σ^2

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Notation: $N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$

About $2/3$ of all cases lie in $[\mu-\sigma; \mu+\sigma]$

$$P(\mu-\sigma \leq X \leq \mu+\sigma) \approx 0.6826$$

About 95% of all cases lie in $[\mu-2\sigma; \mu+2\sigma]$

$$P(\mu-2\sigma \leq X \leq \mu+2\sigma) \approx 0.9544$$

Standard normal distribution

→ Special case of normal distribution with $\mu=0$ and $\sigma^2=1$, i.e. $N(0, 1)$

① Convert problem to a standardized normal variable, z-score

$$z = \frac{x-\mu}{\sigma} \sim N(0, 1)$$

② A table of z-scores exists

③ We can convert back:

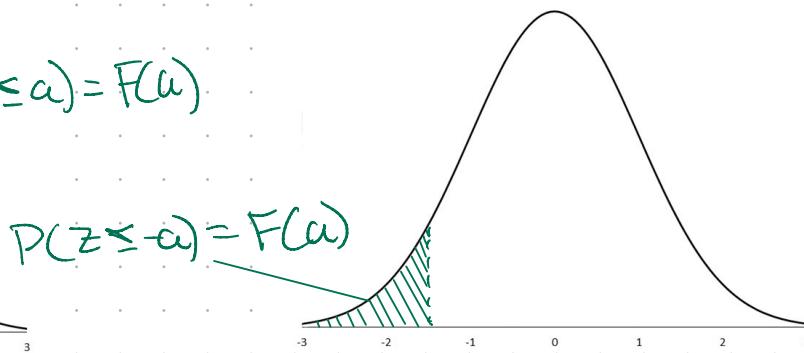
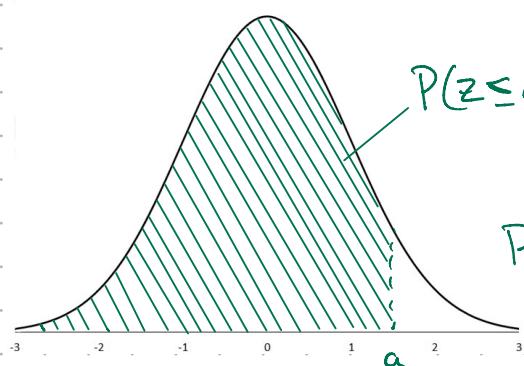
$$x = z \cdot \sigma + \mu$$

.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0006	.0005	.0005
-3.1	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0010	.0010
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0028	.0025	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0019
-2.7	.0035	.0031	.0028	.0025	.0023	.0020	.0018	.0016	.0014	.0013
-2.6	.0047	.0045	.0044	.0043	.0041	.0039	.0038	.0037	.0036	.0036
-2.5	.0062	.0060	.0058	.0057	.0055	.0052	.0051	.0049	.0048	.0048
-2.4	.0080	.0078	.0075	.0073	.0071	.0068	.0066	.0064	.0062	.0061
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0136	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0174	.0174	.0175	.0176	.0162	.0162	.0158	.0155	.0152	.0149
-2.0	.0219	.0219	.0219	.0212	.0202	.0202	.0192	.0192	.0189	.0183
-1.9	.0267	.0281	.0274	.0268	.0262	.0259	.0244	.0239	.0233	.0229
-1.8	.0356	.0351	.0344	.0340	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0468	.0463	.0457	.0451	.0446	.0440	.0434	.0428	.0422	.0417
-1.6	.0588	.0577	.0566	.0556	.0545	.0545	.0545	.0545	.0545	.0545
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0730	.0713	.0698	.0674	.0654	.0635	.0615	.0598	.0578	.0559
-1.3	.0778	.0759	.0738	.0717	.0695	.0674	.0653	.0632	.0610	.0589
-1.2	.1151	.1131	.1112	.1093	.1075	.1057	.1038	.1020	.1001	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1194	.1170
-1.0	.1557	.1535	.1513	.1491	.1469	.1449	.1429	.1407	.1387	.1367
-0.9	.1841	.1814	.1788	.1762	.1738	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2400	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2176	.2148
-0.6	.2678	.2660	.2632	.2604	.2575	.2546	.2517	.2487	.2458	.2429
-0.5	.3085	.3059	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3783	.3733	.3681	.3629	.3576	.3523	.3470	.3416	.3363	.3309
-0.2	.4077	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4403	.4364	.4325	.4286	.4247
-0.0	.4960	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	5000	5040	5080	5120	5160	5199	5239	5279	5319	5359
0.1	5479	5498	5479	5457	5437	5417	5397	5377	5357	5337
0.2	5793	5753	5713	5673	5633	5593	5553	5513	5473	5431
0.3	6179	6217	6255	6293	6331	6368	6406	6443	6480	6517
0.4	6554	6591	6628	6664	6700	6736	6772	6808	6844	6879
0.5	6930	6969	6995	7023	7051	7079	7106	7134	7161	7187
0.6	7257	7294	7324	7352	7380	7408	7434	7461	7487	7510
0.7	7580	7611	7642	7673	7704	7734	7764	7794	7823	7852
0.8	7881	7910	7939	7967	7995	8023	8051	8078	8106	8133
0.9	8181	8209	8237	8265	8293	8321	8349	8376	8403	8430
1.0	8413	8438	8461	8485	8509	8531	8554	8577	8599	8621
1.1	8643	8663	8686	8708	8729	8750	8770	8790	8810	8830
1.2	8849	8869	8888	8907	8925	8944	8962	8980	8997	9015
1.3	9039	9059	9078	9095	9112	9129	9146	9162	9178	9195
1.4	9192	9207	9222	9236	9251	9266	9279	9292	9306	9319
1.5	9332	9345	9357	9368	9379	9390	9406	9416	9429	9441
1.6	9452	9463	9474	9484	9495	9505	9515	9525	9535	9545
1.7	9553	9563	9573	9583	9593	9603	9613	9623	9633	9643
1.8	9641	9649	9655	9664	9671	9678	9685	9693	9699	9705
1.9	9713	9719	9725	9732	9738	9744	9750	9756	9761	9767
2.0	9777	9778	9783	9788	9793	9798	9803	9808	9813	9818
2.1	9831	9834	9839	9843	9848	9852	9856	9860	9864	9867
2.2	9881	9884	9886	9891	9895	9898	9904	9908	9914	9919
2.3	9929	9931	9933	9936	9939	9941	9944	9947	9951	9955
2.4	9969	9970	9972	9974	9976	9978	9980	9982	9984	9985
2.5	9998	9998	9998	9998	9998	9998	9998	9998	9998	9998
2.6	9995	9995	9995	9995	9995	9995	9995	9995	9995	9995
2.7	9997	9997	9997	9997	9997	9997	9997	9997	9997	9997
2.8	9994	9995	9995	9995	9995	9995	9995	9995	9995	9995
2.9	9981	9982	9982	9983	9984	9984	9985	9985	9986	9986
3.0	9987	9987	9987	9988	9988	9988	9989	9989	9990	9990
3.1	9990	9991	9991	9991	9992	9992	9992	9993	9993	9993
3.2	9992	9993	9994	9994	9994	9994	9994	9995	9995	9995
3.3	9995	9995	9995	9995	9995	9995	9995	9995	9995	9995
3.4	9997	9997	9997	9997	9997	9997	9997	9997	9997	9997

Rules:

$$1. P(Z \leq a) = F(a)$$



Example:

$$\text{Find } P(Z \leq a) \text{ for } a = \{-1.65, -1.00, 1.00, 1.65\}$$

$$P(Z \leq -1.65) =$$

$$P(Z \leq -1.00) =$$

$$P(Z \leq 1.00) =$$

$$P(Z \leq 1.65) =$$

$$\text{Find } a \text{ for } \{0.6026, 0.9750, 0.3446\}$$

$$P(Z \leq a) =$$

$$a =$$

$$P(Z \leq a) =$$

$$a =$$

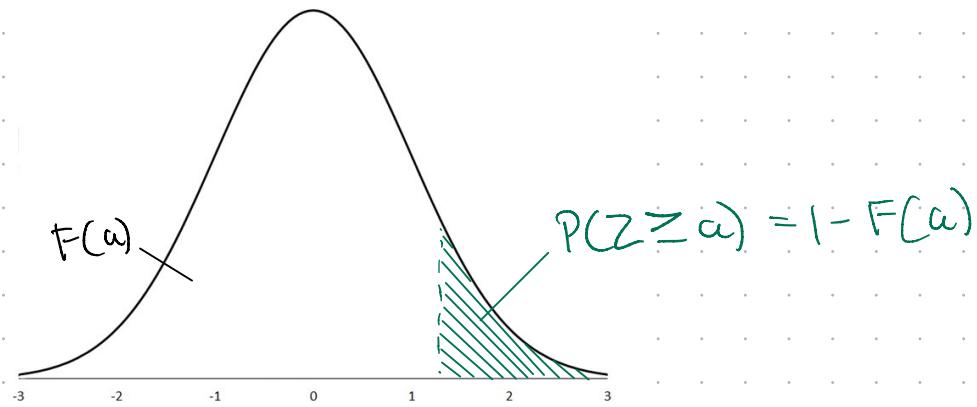
$$P(Z \leq a) =$$

$$a =$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0008	.0007
-3.0	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0165	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1036	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1738	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5949	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8103	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9812	.9817	
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985		
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989			
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992			
3.2	.9993	.9993	.9994	.9994	.9994	.9994				
3.3	.9995	.9995	.9995	.9996	.9996	.9996				
3.4	.9997	.9997	.9997	.9997	.9997					

$$2. P(Z \geq a) = 1 - F(a) = F(-a) = P(Z \leq -a)$$



Example:

$$P(Z \geq 1.5) =$$

$$P(Z \geq 2) =$$

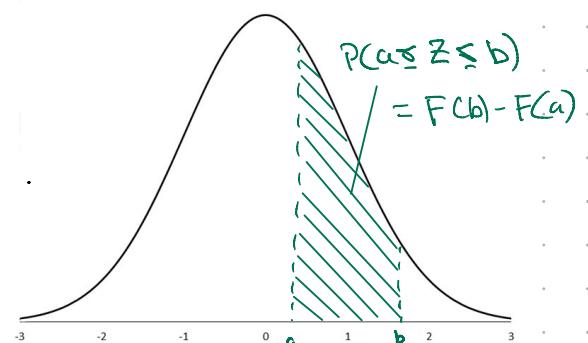
$$3. P(a \leq Z \leq b) = F(b) - F(a), \quad b \geq a$$

Example:

$$P(-1 \leq Z \leq 1.5) =$$

$$=$$

$$=$$



Example:

Family income $\sim N(\$25000, \$10.000^2)$

If poverty level is \$10,000, what percentage of population live in Poverty:

$X = \text{Family income}$: