Problems 5

Richard Brooks

Exercise 1

Recap ch. 4.
Let $X \sim N(3, 9)$.

a. Find $P(X > 0) = \Phi(1) \approx 0.8413$.

b. Find $P(-3 < X < 8) = \Phi\left(\frac{5}{3}\right) - \Phi(-2) \approx 0.9295$.

c. Find $P(X > 5 | X > 3) = 2 \times \left(1 - \Phi\left(\frac{2}{3}\right)\right) \approx 0.5050$.

Exercise 2

Recap ch. 4.
Let $X \sim N(3, 9)$ and $Y = 5 - X$.

a. Find $P(X > 2) = \Phi\left(\frac{1}{3}\right) \approx 0.6306$.

b. Find $P(-1 < Y < 3) = \Phi\left(\frac{1}{3}\right) - \Phi(-1) \approx 0.4719$.

c. Find $P(X > 4 | Y < 2) = 2\left(1 - \Phi\left(\frac{1}{3}\right)\right) \approx 0.7389$.

Exercise 3

Recap ch. 5.
Consider the set of points in the set C:

$$C = \{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}.$$

Suppose that we pick a point (X, Y) from this set completely at random.

a. What probability does each point have of being chosen? $\frac{1}{11}$

b. Find the joint and marginal PMFs of X and Y.

$$P_{XY}(x, y) = \begin{cases} \frac{1}{11} & (x, y) \in C \\ 0 & \text{otherwise} \end{cases}$$

$P_Y(-2) = P_{XY}(0, -2) = \frac{1}{11}$
$P_Y(-1) = P_{XY}(0, -1) + P_{XY}(-1, -1) + P_{XY}(1, -1) = \frac{3}{11}$
$P_Y(0) = P_{XY}(0, 0) + P_{XY}(1, 0) + P_{XY}(-1, 0) = \frac{3}{11}$
$P_Y(1) = P_{XY}(0, 1) + P_{XY}(-1, 1) + P_{XY}(1, 1) = \frac{3}{11}$
$P_Y(2) = P_{XY}(0, 2) = \frac{1}{11}$

$$P_X(i) = \begin{cases} \frac{3}{11} & \text{for } i = -1, 1 \\ \frac{5}{11} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

c. Find the conditional PMF of X given $Y = 1$.

$$P_{X|Y}(i \mid 1) = \frac{P_{XY}(i, 1)}{P_Y(1)}$$
$$= \frac{\frac{1}{11}}{\frac{3}{11}} = \frac{1}{3}, \quad \text{for } i = -1, 0, 1.$$

d. Are X and Y independent? No

e. Find $E[XY^2] = 0$.

f. Find $E[X|Y = 1] = 0$.

g. Find $\text{Var}(X|Y = 1) = 2/3$.

h. Find $E[X|\ |Y| \leq 1] = 0$.

i. Find $E[X^2|Y| \leq 1] = 2/3$.

Exercise 4

The number of accidents in a certain city is modeled by a Poisson random variable with average rate of 10 accidents per day. Suppose that the number of accidents in different days are independent. Use the central limit theorem to find the probability that there will be more than 3800 accidents in a certain year. Assume that there are 365 days in a year. $P(Y \geq 3800) \approx 0.0065$

$$\begin{aligned}
Y &= X_1 + X_2 + \cdots + X_n, \quad n = 365 \\
X_i &\sim \text{Poisson}(\lambda = 10). \quad \text{Thus:} \quad EX_i = 10 \\
\text{Var}(X_i) &= \lambda = 10 \\
EY &= 365 \times 10 = 3650 \\
\text{Var}(Y) &= 365 \times 10 = 3650 \\
\frac{Y - 3650}{\sqrt{3650}} & \text{ is approximately} \quad N(0,1) \quad \text{(by the CLT)} \\
P(Y \geq 3800) &= P\left(\frac{Y - 3650}{\sqrt{3650}} \geq \frac{3800 - 3650}{\sqrt{3650}}\right) \\
&= 1 - \Phi\left(\frac{3800 - 3650}{\sqrt{3650}}\right) \\
&\approx 1 - \Phi(2.48) \\
&\approx 0.0065
\end{aligned}$$

Exercise 5

Let X_1, X_2, X_3, ..., X_n be a random sample with unknown mean $EX_i = \mu$, and unknown variance $\text{Var}(X_i) = \sigma^2$. Suppose that we would like to estimate $\theta = \mu^2$. We define the estimator $\hat{\Theta}$ as

$$\hat{\Theta} = \left(\overline{X}\right)^2 = \left[\frac{1}{n}\sum_{k=1}^{n} X_k\right]^2$$

to estimate θ. Is $\hat{\Theta}$ an unbiased estimator of θ? (You need to do some calculations!). It is not. I actually showed this in the slide!!!!

We have

$$\begin{aligned}
E[\hat{\Theta}] &= E[\overline{X}^2] \\
&= (\overline{X}) + \left(E[\overline{X}]\right)^2 \\
&= \frac{\sigma^2}{n} + \mu^2.
\end{aligned}$$

Therefore, the bias of this is estimator is

$$\begin{aligned}
B(\hat{\Theta}) &= E[\hat{\Theta}] - \theta && (1) \\
&= \frac{\sigma^2}{n}. && (2)
\end{aligned}$$

Thus, $\hat{\Theta}_n$ is NOT an unbiased estimator of θ