2 Matrix Algebra

INTRODUCTORY EXAMPLE

Computer Models in Aircraft Design

To design the next generation of commercial and military aircraft, engineers at Boeing's Phantom Works use 3D modeling and computational fluid dynamics (CFD). They study the airflow around a virtual airplane to answer important design questions before physical models are created. This has drastically reduced design cycle times and cost—and linear algebra plays a crucial role in the process.

The virtual airplane begins as a mathematical "wire-frame" model that exists only in computer memory and on graphics display terminals. (A model of a Boeing 777 is shown.) This mathematical model organizes and influences each step of the design and manufacture of the airplane—both the exterior and interior. The CFD analysis concerns the exterior surface.

Although the finished skin of a plane may seem smooth, the geometry of the surface is complicated. In addition to wings and a fuselage, an aircraft has nacelles, stabilizers, slats, flaps, and ailerons. The way air flows around these structures determines how the plane moves through the sky. Equations that describe the airflow are complicated, and they must account for engine intake, engine exhaust, and the wakes left by the wings of the plane. To study the airflow, engineers need a highly refined description of the plane's surface.

A computer creates a model of the surface by first superimposing a three-dimensional grid of "boxes" on the original wire-frame model. Boxes in this grid lie either completely inside or completely outside the plane, or they intersect the surface of the plane. The computer selects the boxes that intersect the surface and subdivides them, retaining only the smaller boxes that still intersect the surface. The subdividing process is repeated until the grid is extremely fine. A typical grid can include over 400,000 boxes.

The process for finding the airflow around the plane involves repeatedly solving a system of linear equations $A\mathbf{x} = \mathbf{b}$ that may involve up to 2 million equations and variables. The vector $\mathbf{b}$ changes each time, based on data from the grid and solutions of previous equations. Using the fastest computers available commercially, a Phantom Works team can spend from a few hours to several days setting up and solving a single airflow problem. After the team analyzes the solution, they may make small changes to the airplane surface and begin the whole process again. Thousands of CFD runs may be required.

This chapter presents two important concepts that assist in the solution of such massive systems of equations:

- *Partitioned matrices:* A typical CFD system of equations has a "sparse" coefficient matrix with mostly zero entries. Grouping the variables correctly leads to a partitioned matrix with many zero blocks. Section 2.4 introduces such matrices and describes some of their applications.

- *Matrix factorizations:* Even when written with partitioned matrices, the system of equations is complicated. To further simplify the computations, the CFD software at Boeing uses what is called an LU factorization of the coefficient matrix. Section 2.5 discusses LU and other useful matrix factorizations. Further details about factorizations appear at several points later in the text.

To analyze a solution of an airflow system, engineers want to visualize the airflow over the surface of the plane. They use computer graphics, and linear algebra provides the engine for the graphics. The wire-frame model of the plane's surface is stored as data in many matrices. Once the image has been rendered on a computer screen, engineers can change its scale, zoom in or out of small regions, and rotate the image to see parts that may be hidden from view. Each of these operations is accomplished by appropriate matrix multiplications. Section 2.7 explains the basic ideas.

Modern CFD has revolutionized wing design. The Boeing Blended Wing Body is in design for the year 2020 or sooner.

WEB

Our ability to analyze and solve equations will be greatly enhanced when we can perform algebraic operations with matrices. Furthermore, the definitions and theorems in this chapter provide some basic tools for handling the many applications of linear algebra that involve two or more matrices. For square matrices, the Invertible Matrix Theorem in Section 2.3 ties together most of the concepts treated earlier in the text. Sections 2.4 and 2.5 examine partitioned matrices and matrix factorizations, which appear in most modern uses of linear algebra. Sections 2.6 and 2.7 describe two interesting applications of matrix algebra, to economics and to computer graphics.

2.1 MATRIX OPERATIONS

If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A. See Fig. 1. For instance, the (3, 2)-entry is the number a_{32} in the third row, second column. Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$. Often, these columns are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and the matrix A is written as

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

Observe that the number a_{ij} is the ith entry (from the top) of the jth column vector $\mathbf{a}_j$.

The **diagonal entries** in an $m \times n$ matrix $A = [\,a_{ij}\,]$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the **main diagonal** of A. A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. An example is the $n \times n$ identity matrix, I_n. An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0. The size of a zero matrix is usually clear from the context.

$$\begin{array}{rc}
 & \text{Column } j \\
\text{Row } i & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\
 & \begin{array}{ccccc} \uparrow & \quad & \uparrow & \quad & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}$$

FIGURE 1 Matrix notation.

Sums and Scalar Multiples

The arithmetic for vectors described earlier has a natural extension to matrices. We say that two matrices are **equal** if they have the same size (i.e., the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal. If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B. Since vector addition of the columns is done entrywise, each entry in $A + B$ is the sum of the corresponding entries in A and B. The sum $A + B$ is defined only when A and B are the same size.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

but $A + C$ is not defined because A and C have different sizes. ■

If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A. As with vectors, $-A$ stands for $(-1)A$, and $A - B$ is the same as $A + (-1)B$.

EXAMPLE 2 If A and B are the matrices in Example 1, then

$$2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$ ■

It was unnecessary in Example 2 to compute $A - 2B$ as $A + (-1)2B$ because the usual rules of algebra apply to sums and scalar multiples of matrices, as the following theorem shows.

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Each equality in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal. Size is no problem because A, B, and C are equal in size. The equality of columns follows immediately from analogous properties of vectors. For instance, if the jth columns of A, B, and C are $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{c}_j$, respectively, then the jth columns of $(A+B)+C$ and $A+(B+C)$ are

$$(\mathbf{a}_j + \mathbf{b}_j) + \mathbf{c}_j \quad \text{and} \quad \mathbf{a}_j + (\mathbf{b}_j + \mathbf{c}_j)$$

respectively. Since these two vector sums are equal for each j, property (b) is verified.

Because of the associative property of addition, we can simply write $A+B+C$ for the sum, which can be computed either as $(A+B)+C$ or as $A+(B+C)$. The same applies to sums of four or more matrices.

Matrix Multiplication

When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. See Fig. 2.

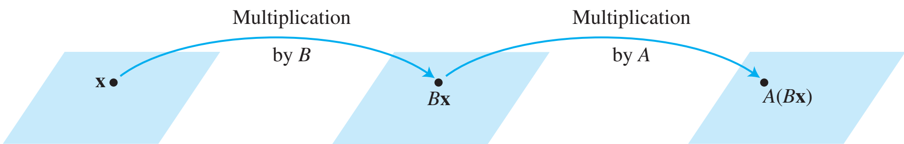

FIGURE 2 Multiplication by B and then A.

Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a *composition* of mappings—the linear transformations studied in Section 1.8. Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that

$$A(B\mathbf{x}) = (AB)\mathbf{x} \tag{1}$$

See Fig. 3.

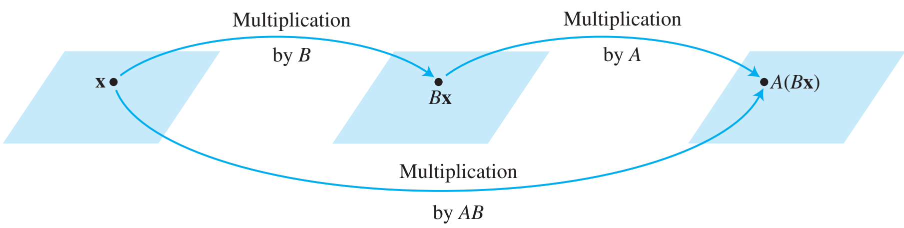

FIGURE 3 Multiplication by AB.

If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_p$ and the entries in $\mathbf{x}$ by $x_1, \dots, x_p$. Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_p\mathbf{b}_p$$

By the linearity of multiplication by A,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \cdots + A(x_p\mathbf{b}_p) \\ &= x_1A\mathbf{b}_1 + \cdots + x_pA\mathbf{b}_p \end{aligned}$$

The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \dots, A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights. In matrix notation, this linear combination is written as

$$A(B\mathbf{x}) = [\, A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p \,]\mathbf{x}$$

Thus multiplication by $[\, A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p \,]$ transforms $\mathbf{x}$ into $A(B\mathbf{x})$. We have found the matrix we sought!

DEFINITION

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

This definition makes equation (1) true for all $\mathbf{x}$ in $\mathbb{R}^p$. Equation (1) proves that the composite mapping in Fig. 3 is a linear transformation and that its standard matrix is AB. *Multiplication of matrices corresponds to composition of linear transformations.*

EXAMPLE 3 Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

SOLUTION Write $B = [\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \,]$, and compute:

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

Then

$$AB = A[\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \,] = \begin{array}{c} \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{matrix} \end{array}$$

■

Notice that since the first column of AB is $A\mathbf{b}_1$, this column is a linear combination of the columns of A using the entries in $\mathbf{b}_1$ as weights. A similar statement is true for each column of AB.

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Obviously, the number of columns of A must match the number of rows in B in order for a linear combination such as $A\mathbf{b}_1$ to be defined. Also, the definition of AB shows that *AB has the same number of rows as A and the same number of columns as B.*

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$
\begin{array}{ccccc}
A & & B & & AB \\
\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} &
&
\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix} &
= &
\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \\
3 \times 5 & & 5 \times 2 & & 3 \times 2
\end{array}
$$

Match

Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A. ■

The definition of AB is important for theoretical work and applications, but the following rule provides a more efficient method for calculating the individual entries in AB when working small problems by hand.

ROW–COLUMN RULE FOR COMPUTING *AB*

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

To verify this rule, let $B = [\,\mathbf{b}_1 \ \cdots \ \mathbf{b}_p\,]$. Column j of AB is $A\mathbf{b}_j$, and we can compute $A\mathbf{b}_j$ by the row–vector rule for computing $A\mathbf{x}$ from Section 1.4. The ith entry in $A\mathbf{b}_j$ is the sum of the products of corresponding entries from row i of A and the vector $\mathbf{b}_j$, which is precisely the computation described in the rule for computing the (i, j)-entry of AB.

EXAMPLE 5 Use the row–column rule to compute two of the entries in AB for the matrices in Example 3. An inspection of the numbers involved will make it clear how the two methods for calculating AB produce the same matrix.

SOLUTION To find the entry in row 1 and column 3 of AB, consider row 1 of A and column 3 of B. Multiply corresponding entries and add the results, as shown below:

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6) + 3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

For the entry in row 2 and column 2 of AB, use row 2 of A and column 2 of B:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 1(3) + -5(-2) & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 13 & \square \end{bmatrix}$$

■

EXAMPLE 6 Find the entries in the second row of AB, where

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

SOLUTION By the row–column rule, the entries of the second row of AB come from row 2 of A (and the columns of B):

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square \\ -4 + 21 - 12 & 6 + 3 - 8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}$$ ■

Notice that since Example 6 requested only the second row of AB, we could have written just the second row of A to the left of B and computed

$$\begin{bmatrix} -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \end{bmatrix}$$

This observation about rows of AB is true in general and follows from the row–column rule. Let $\text{row}_i(A)$ denote the ith row of a matrix A. Then

$$\text{row}_i(AB) = \text{row}_i(A) \cdot B \tag{2}$$

Properties of Matrix Multiplication

The following theorem lists the standard properties of matrix multiplication. Recall that I_m represents the $m \times m$ identity matrix and $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

THEOREM 2

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

PROOF Properties (b)–(e) are considered in the exercises. Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known (or easy to check) that the composition of functions is associative. Here is another proof of (a) that rests on the "column definition" of the product of two matrices. Let

$$C = [\,\mathbf{c}_1 \ \ \cdots \ \ \mathbf{c}_p\,]$$

By the definition of matrix multiplication,

$$BC = [\, B\mathbf{c}_1 \quad \cdots \quad B\mathbf{c}_p \,]$$
$$A(BC) = [\, A(B\mathbf{c}_1) \quad \cdots \quad A(B\mathbf{c}_p) \,]$$

Recall from equation (1) that the definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x}$, so

$$A(BC) = [\, (AB)\mathbf{c}_1 \quad \cdots \quad (AB)\mathbf{c}_p \,] = (AB)C$$

The associative and distributive laws in Theorems 1 and 2 say essentially that pairs of parentheses in matrix expressions can be inserted and deleted in the same way as in the algebra of real numbers. In particular, we can write ABC for the product, which can be computed either as $A(BC)$ or as $(AB)C$.[1] Similarly, a product $ABCD$ of four matrices can be computed as $A(BCD)$ or $(ABC)D$ or $A(BC)D$, and so on. It does not matter how we group the matrices when computing the product, so long as the left-to-right order of the matrices is preserved.

The left-to-right order in products is critical because AB and BA are usually not the same. This is not surprising, because the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B. The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A. If $AB = BA$, we say that A and B **commute** with one another.

EXAMPLE 7 Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute. That is, verify that $AB \neq BA$.

SOLUTION

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$
$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

Example 7 illustrates the first of the following list of important differences between matrix algebra and the ordinary algebra of real numbers. See Exercises 9–12 for examples of these situations.

WARNINGS:

1. In general, $AB \neq BA$.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Powers of a Matrix

WEB

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k

[1] When B is square and C has fewer columns than A has rows, it is more efficient to compute $A(BC)$ than $(AB)C$.

copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times. If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself. Thus A^0 is interpreted as the identity matrix. Matrix powers are useful in both theory and applications (Sections 2.6, 4.9, and later in the text).

The Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE 8 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

■

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Proofs of (a)–(c) are straightforward and are omitted. For (d), see Exercise 33. Usually, $(AB)^T$ is not equal to $A^T B^T$, even when A and B have sizes such that the product $A^T B^T$ is defined.

The generalization of Theorem 3(d) to products of more than two factors can be stated in words as follows:

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

The exercises contain numerical examples that illustrate properties of transposes.

NUMERICAL NOTES

1. The fastest way to obtain AB on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate AB by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates AB by rows.)
2. The definition of AB lends itself well to parallel processing on a computer. The columns of B are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of AB.

PRACTICE PROBLEMS

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let
$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?
2. Let A be a 4×4 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.

2.1 EXERCISES

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC, CD
2. $A + 3B$, $2C - 3E$, DB, EC

In the rest of this exercise set and in those to follow, assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.
4. Compute $A - 5I_3$ and $(5I_3)A$, where
$$A = \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product AB in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing AB.

5. $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$
6. $A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$
7. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?
8. How many rows does B have if BC is a 5×4 matrix?
9. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?
10. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.
11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.
12. Let $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

13. Let $\mathbf{r}_1, \ldots, \mathbf{r}_p$ be vectors in $\mathbb{R}^n$, and let Q be an $m \times n$ matrix. Write the matrix $[\, Q\mathbf{r}_1 \quad \cdots \quad Q\mathbf{r}_p \,]$ as a *product* of two matrices (neither of which is an identity matrix).

14. Let U be the 3×2 cost matrix described in Example 6 in Section 1.8. The first column of U lists the costs per dollar of output for manufacturing product B, and the second column lists the costs per dollar of output for product C. (The costs are categorized as materials, labor, and overhead.) Let $\mathbf{q}_1$ be a vector in $\mathbb{R}^2$ that lists the output (measured in dollars) of products B and C manufactured during the first quarter of the year, and let $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{q}_4$ be the analogous vectors that list the amounts of products B and C manufactured in the second, third, and fourth quarters, respectively. Give an economic description of the data in the matrix UQ, where $Q = [\, \mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4 \,]$.

Exercises 15 and 16 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

15. a. If A and B are 2×2 matrices with columns $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}_1$, $\mathbf{b}_2$, respectively, then $AB = [\, \mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2 \,]$.

b. Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

c. $AB + AC = A(B + C)$

d. $A^T + B^T = (A + B)^T$

e. The transpose of a product of matrices equals the product of their transposes in the same order.

16. a. The first row of AB is the first row of A multiplied on the right by B.

b. If A and B are 3×3 matrices and $B = [\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \,]$, then $AB = [\, A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \,]$.

c. If A is an $n \times n$ matrix, then $(A^2)^T = (A^T)^2$

d. $(ABC)^T = C^T A^T B^T$

e. The transpose of a sum of matrices equals the sum of their transposes.

17. If $A = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -3 & -11 \\ 1 & 17 \end{bmatrix}$, determine the first and second columns of B.

18. Suppose the third column of B is all zeros. What can be said about the third column of AB?

19. Suppose the third column of B is the sum of the first two columns. What can be said about the third column of AB? Why?

20. Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can be said about the columns of AB? Why?

21. Suppose the last column of AB is entirely zeros but B itself has no column of zeros. What can be said about the columns of A?

22. Show that if the columns of B are linearly dependent, then so are the columns of AB.

23. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

24. Suppose A is a $3 \times n$ matrix whose columns span $\mathbb{R}^3$. Explain how to construct an $n \times 3$ matrix D such that $AD = I_3$.

25. Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint:* Think about the product CAD.]

26. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

In Exercises 27 and 28, view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, called the **scalar product**, or **inner product**, of $\mathbf{u}$ and $\mathbf{v}$. It is usually written as a single real number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix, called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$. The products $\mathbf{u}^T\mathbf{v}$ and $\mathbf{u}\mathbf{v}^T$ will appear later in the text.

27. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$.

28. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

29. Prove Theorem 2(b) and 2(c). Use the row–column rule. The (i, j)-entry in $A(B + C)$ can be written as

$$a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj})$$

or

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

30. Prove Theorem 2(d). [*Hint:* The (i, j)-entry in $(rA)B$ is $(ra_{i1})b_{1j} + \cdots + (ra_{in})b_{nj}$.]

31. Show that $I_m A = A$ where A is an $m \times n$ matrix. Assume $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

32. Show that $AI_n = A$ when A is an $m \times n$ matrix. [*Hint:* Use the (column) definition of AI_n.]

33. Prove Theorem 3(d). [*Hint:* Consider the jth row of $(AB)^T$.]

34. Give a formula for $(AB\mathbf{x})^T$, where $\mathbf{x}$ is a vector and A and B are matrices of appropriate sizes.

35. [**M**] Read the documentation for your matrix program, and write the commands that will produce the following matrices (without keying in each entry of the matrix).

a. A 4×5 matrix of zeros

b. A 5×3 matrix of ones

c. The 5×5 identity matrix

d. A 4×4 diagonal matrix, with diagonal entries 3, 4, 2, 5

A useful way to test new ideas in matrix algebra, or to make conjectures, is to make calculations with matrices selected at random. Checking a property for a few matrices does not prove that the property holds in general, but it makes the property more believable. Also, if the property is actually false, making a few calculations may help to discover this.

36. [**M**] Write the command(s) that will create a 5×6 matrix with random entries. In what range of numbers do the entries lie? Tell how to create a 4×4 matrix with random integer entries between -9 and 9. [*Hint:* If x is a random number such that $0 < x < 1$, then $-9.5 < 19(x - .5) < 9.5$.]

37. [**M**] Construct random 4×4 matrices A and B to test whether $AB = BA$. The best way to do this is to compute $AB - BA$ and check whether this difference is the zero matrix. Then test $AB - BA$ for three more pairs of random 4×4 matrices. Report your conclusions.

38. [**M**] Construct a random 5×5 matrix A and test whether $(A + I)(A - I) = A^2 - I$. The best way to do this is to compute $(A + I)(A - I) - (A^2 - I)$ and verify that this difference is the zero matrix. Do this for three random matrices. Then test $(A + B)(A - B) = A^2 - B^2$ the same way for three pairs of random 4×4 matrices. Report your conclusions.

39. [**M**] Use at least three pairs of random 4×4 matrices A and B to test the equalities $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$, as well as $(AB)^T = A^T B^T$. (See Exercise 37.) Report your conclusions. [*Note*: Most matrix programs use A' for A^T.]

40. [**M**] Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute S^k for $k = 2, \ldots, 6$.

41. [**M**] Describe in words what happens when A^5, A^{10}, A^{20}, and A^{30} are computed for

$$A = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/3 & 1/6 \\ 1/4 & 1/6 & 7/12 \end{bmatrix}$$

SOLUTIONS TO PRACTICE PROBLEMS

1. $A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. So $(A\mathbf{x})^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$. Also,

$$\mathbf{x}^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}.$$

The quantities $(A\mathbf{x})^T$ and $\mathbf{x}^T A^T$ are equal, by Theorem 3(d). Next,

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 25 + 9 \end{bmatrix} = 34$$

A 1×1 matrix such as $\mathbf{x}^T\mathbf{x}$ is usually written without the brackets. Finally, $A^T\mathbf{x}^T$ is not defined, because $\mathbf{x}^T$ does not have two rows to match the two columns of A^T.

2. The fastest way to compute $A^2\mathbf{x}$ is to compute $A(A\mathbf{x})$. The product $A\mathbf{x}$ requires 16 multiplications, 4 for each entry, and $A(A\mathbf{x})$ requires 16 more. In contrast, the product A^2 requires 64 multiplications, 4 for each of the 16 entries in A^2. After that, $A^2\mathbf{x}$ takes 16 more multiplications, for a total of 80.

2.2 THE INVERSE OF A MATRIX

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This section investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

Recall that the multiplicative inverse of a number such as 5 is 1/5 or 5^{-1}. This inverse satisfies the equations

$$5^{-1} \cdot 5 = 1 \quad \text{and} \quad 5 \cdot 5^{-1} = 1$$

The matrix generalization requires *both* equations and avoids the slanted-line notation (for division) because matrix multiplication is not commutative. Furthermore, a full generalization is possible only if the matrices involved are square.[1]

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

EXAMPLE 1 If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$. ■

Here is a simple formula for the inverse of a 2×2 matrix, along with a test to tell if the inverse exists.

THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The simple proof of Theorem 4 is outlined in Exercises 25 and 26. The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

[1] One could say that an $m \times n$ matrix A is invertible if there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. However, these equations imply that A is square and $C = D$. Thus A is invertible as defined above. See Exercises 23–25 in Section 2.1.

EXAMPLE 2 Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

SOLUTION Since $\det A = 3(6) - 4(5) = -2 \neq 0$, A is invertible, and

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Invertible matrices are indispensable in linear algebra—mainly for algebraic calculations and formula derivations, as in the next theorem. There are also occasions when an inverse matrix provides insight into a mathematical model of a real-life situation, as in Example 3, below.

THEOREM 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF Take any $\mathbf{b}$ in $\mathbb{R}^n$. A solution exists because if $A^{-1}\mathbf{b}$ is substituted for $\mathbf{x}$, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$. So $A^{-1}\mathbf{b}$ is a solution. To prove that the solution is unique, show that if $\mathbf{u}$ is any solution, then $\mathbf{u}$, in fact, must be $A^{-1}\mathbf{b}$. Indeed, if $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain

$$A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}, \quad I\mathbf{u} = A^{-1}\mathbf{b}, \quad \text{and} \quad \mathbf{u} = A^{-1}\mathbf{b}$$

EXAMPLE 3 A horizontal elastic beam is supported at each end and is subjected to forces at points 1, 2, 3, as shown in Fig. 1. Let $\mathbf{f}$ in $\mathbb{R}^3$ list the forces at these points, and let $\mathbf{y}$ in $\mathbb{R}^3$ list the amounts of deflection (that is, movement) of the beam at the three points. Using Hooke's law from physics, it can be shown that

$$\mathbf{y} = D\mathbf{f}$$

where D is a *flexibility matrix*. Its inverse is called the *stiffness matrix*. Describe the physical significance of the columns of D and D^{-1}.

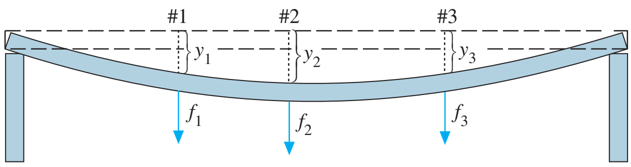

FIGURE 1 Deflection of an elastic beam.

SOLUTION Write $I_3 = [\,\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3\,]$ and observe that

$$D = DI_3 = [\,D\mathbf{e}_1 \;\; D\mathbf{e}_2 \;\; D\mathbf{e}_3\,]$$

Interpret the vector $\mathbf{e}_1 = (1, 0, 0)$ as a unit force applied downward at point 1 on the beam (with zero force at the other two points). Then $D\mathbf{e}_1$, the first column of D, lists the beam deflections due to a unit force at point 1. Similar descriptions apply to the second and third columns of D.

To study the stiffness matrix D^{-1}, observe that the equation $\mathbf{f} = D^{-1}\mathbf{y}$ computes a force vector $\mathbf{f}$ when a deflection vector $\mathbf{y}$ is given. Write

$$D^{-1} = D^{-1}I_3 = [\,D^{-1}\mathbf{e}_1 \;\; D^{-1}\mathbf{e}_2 \;\; D^{-1}\mathbf{e}_3\,]$$

Now interpret $\mathbf{e}_1$ as a deflection vector. Then $D^{-1}\mathbf{e}_1$ lists the forces that create the deflection. That is, the first column of D^{-1} lists the forces that must be applied at the three points to produce a unit deflection at point 1 and zero deflections at the other points. Similarly, columns 2 and 3 of D^{-1} list the forces required to produce unit deflections at points 2 and 3, respectively. In each column, one or two of the forces must be negative (point upward) to produce a unit deflection at the desired point and zero deflections at the other two points. If the flexibility is measured, for example, in inches of deflection per pound of load, then the stiffness matrix entries are given in pounds of load per inch of deflection. ■

The formula in Theorem 5 is seldom used to solve an equation $A\mathbf{x} = \mathbf{b}$ numerically because row reduction of $[\,A \quad \mathbf{b}\,]$ is nearly always faster. (Row reduction is usually more accurate, too, when computations involve rounding off numbers.) One possible exception is the 2×2 case. In this case, mental computations to solve $A\mathbf{x} = \mathbf{b}$ are sometimes easier using the formula for A^{-1}, as in the next example.

EXAMPLE 4 Use the inverse of the matrix A in Example 2 to solve the system

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

SOLUTION This system is equivalent to $A\mathbf{x} = \mathbf{b}$, so

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$ ■

The next theorem provides three useful facts about invertible matrices.

THEOREM 6

a. If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,
$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,
$$(A^T)^{-1} = (A^{-1})^T$$

PROOF To verify statement (a), find a matrix C such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I$$

In fact, these equations are satisfied with A in place of C. Hence A^{-1} is invertible, and A is its inverse. Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$. For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$. Similarly, $A^T(A^{-1})^T = I^T = I$. Hence A^T is invertible, and its inverse is $(A^{-1})^T$. ■

The following generalization of Theorem 6(b) is needed later.

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

There is an important connection between invertible matrices and row operations that leads to a method for computing inverses. As we shall see, an invertible matrix A is row equivalent to an identity matrix, and we can find A^{-1} by *watching the row reduction of A to I*.

Elementary Matrices

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. The next example illustrates the three kinds of elementary matrices.

EXAMPLE 5 Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

SOLUTION Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Addition of -4 times row 1 of A to row 3 produces E_1A. (This is a row replacement operation.) An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A. ∎

Left-multiplication (that is, multiplication on the left) by E_1 in Example 5 has the same effect on any $3 \times n$ matrix. It adds -4 times row 1 to row 3. In particular, since $E_1 \cdot I = E_1$, we see that E_1 *itself* is produced by this same row operation on the identity. Thus Example 5 illustrates the following general fact about elementary matrices. See Exercises 27 and 28.

> If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Since row operations are reversible, as shown in Section 1.1, elementary matrices are invertible, for if E is produced by a row operation on I, then there is another row operation of the same type that changes E back into I. Hence there is an elementary matrix F such that $FE = I$. Since E and F correspond to reverse operations, $EF = I$, too.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

EXAMPLE 6 Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

SOLUTION To transform E_1 into I, add $+4$ times row 1 to row 3. The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$$

The following theorem provides the best way to "visualize" an invertible matrix, and the theorem leads immediately to a method for finding the inverse of a matrix.

THEOREM 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

PROOF Suppose that A is invertible. Then, since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ (Theorem 5), A has a pivot position in every row (Theorem 4 in Section 1.4). Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

Now suppose, conversely, that $A \sim I_n$. Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \dots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim \cdots \sim E_p(E_{p-1} \cdots E_1 A) = I_n$$

That is,

$$E_p \cdots E_1 A = I_n \tag{1}$$

Since the product $E_p \cdots E_1$ of invertible matrices is invertible, (1) leads to

$$\begin{aligned} (E_p \cdots E_1)^{-1}(E_p \cdots E_1)A &= (E_p \cdots E_1)^{-1} I_n \\ A &= (E_p \cdots E_1)^{-1} \end{aligned}$$

Thus A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = [\,(E_p \cdots E_1)^{-1}\,]^{-1} = E_p \cdots E_1$$

Then $A^{-1} = E_p \cdots E_1 \cdot I_n$, which says that A^{-1} results from applying $E_1, \dots, E_p$ successively to I_n. This is the same sequence in (1) that reduced A to I_n.

An Algorithm for Finding A^{-1}

If we place A and I side-by-side to form an augmented matrix $[\,A \quad I\,]$, then row operations on this matrix produce identical operations on A and on I. By Theorem 7, either there are row operations that transform A to I_n and I_n to A^{-1} or else A is not invertible.

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

SOLUTION

$$\begin{aligned}
[\,A \quad I\,] &= \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}
\end{aligned}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

It is a good idea to check the final answer:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is not necessary to check that $A^{-1}A = I$ since A is invertible. ■

Another View of Matrix Inversion

Denote the columns of I_n by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then row reduction of $[\,A \quad I\,]$ to $[\,I \quad A^{-1}\,]$ can be viewed as the simultaneous solution of the n systems

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad \ldots, \quad A\mathbf{x} = \mathbf{e}_n \tag{2}$$

where the "augmented columns" of these systems have all been placed next to A to form $[\,A \quad \mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n\,] = [\,A \quad I\,]$. The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of the systems in (2). This observation is useful because some applied problems may require finding only one or two columns of A^{-1}. In this case, only the corresponding systems in (2) need be solved.

WEB

NUMERICAL NOTE

In practical work, A^{-1} is seldom computed, unless the entries of A^{-1} are needed. Computing both A^{-1} and $A^{-1}\mathbf{b}$ takes about three times as many arithmetic operations as solving $A\mathbf{x} = \mathbf{b}$ by row reduction, and row reduction may be more accurate.

PRACTICE PROBLEMS

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

2.2 EXERCISES

Find the inverses of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$

5. Use the inverse found in Exercise 1 to solve the system

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

6. Use the inverse found in Exercise 3 to solve the system

$$\begin{aligned} 7x_1 + 3x_2 &= -9 \\ -6x_1 - 3x_2 &= 4 \end{aligned}$$

7. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

a. Find A^{-1}, and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4$$

b. The four equations in part (a) can be solved by the *same* set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $[\, A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4 \,]$.

8. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. In order for a matrix B to be the inverse of A, the equations $AB = I$ and $BA = I$ must both be true.

b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

d. If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$.

e. Each elementary matrix is invertible.

10. a. If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

b. If A is invertible, then the inverse of A^{-1} is A itself.

c. A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

d. If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{e}_j$ is consistent for every $j \in \{1, 2, \ldots, n\}$, then A is invertible. Note: $\mathbf{e}_1, \ldots, \mathbf{e}_n$ represent the columns of the identity matrix.

e. If A can be row reduced to the identity matrix, then A must be invertible.

11. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

12. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

13. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?