

Discrete Random Variables

A function that assigns a real number to each outcome:

$$X: S \rightarrow \mathbb{R}$$

* Usually an R.V. is denoted

by X, Y, Z (Uppercase)

* The outcomes are denoted

by x, y, z (lower case)

e.g. $P(X=x)$ if $x=2$

$P(X=2)$

$P(Y=y)$ if $y=3$

$P(Y=3)$

Random Variables are either Discrete or Continuous:

Discrete

- The range is countable

$\{1, 2, 3, \dots\}$

* Uniform

* Binomial

* Geometric

* Negative Binomial

* Hypergeometric

* Poisson

Continuous

- The range is continuous

\mathbb{R}

* Uniform

* Normal Distribution

* Chi-Squared

* Log-Normal

* Exponential

Definition: Discrete Random Variable

Let X be a discrete R.V. with $R_X = \{x_1, x_2, \dots\}$

The function

$$P_X(x_k) = P(X=x_k), \text{ for } k=1, 2, 3, \dots$$

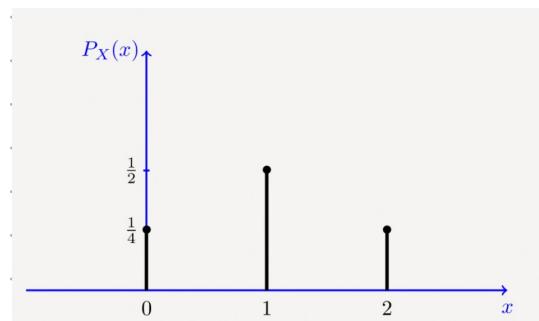
is called the probability mass function (PMF) of X .

Properties:

$$1) f(x_k) = P_X(x_k)$$

$$2) 0 \leq f(x_k) \leq 1$$

$$3) \sum_{k=1}^n f(x_k) = 1$$



Independent Random Variables:

Consider X and Y . We say X and Y are independent if:

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

It follows:

$$P(Y=y | X=x) = P(Y=y)$$

Example:

I toss a coin twice and define X to be the number of heads I observe. Then, I toss the coin two more times and define Y to be the number of heads that I observe this time. Find $P((X < 2) \text{ and } (Y > 1))$.

Bernoulli Distribution:

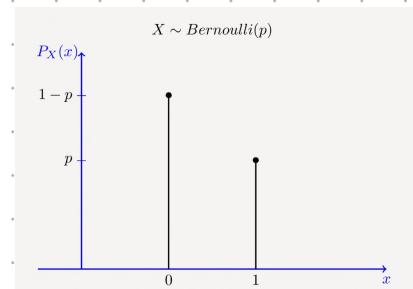
A Bernoulli R.V. can only take two values:

1 : success

0: failure

Bernoulli PMF:

$$P_x(x) = \begin{cases} p & \text{for } x=1 \\ 1-p & \text{for } x=0 \\ 0 & \text{otherwise} \end{cases}$$



Binomial Distribution:

Given multiple independent Bernoulli experiments, the resulting R.V. has a Binomial PMF:

$$P_x(x) = f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

x = number of successes

p = probability of success

n = number of trials/Experiments

$\binom{n}{x}$ = Binomial Coefficient

↳ Combinatorial of subset:

In how many ways can I choose x elements from n when order does not matter:

We can summarize as

$$P(X=x) = \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}$$

* Flip a coin 12 times

↳ each is a Bernoulli trial

↳ together they form $B \sim B(n, p)$

↳ $n = 12, p = 1/2$

$P(X=2)$ means given 12 coin flips

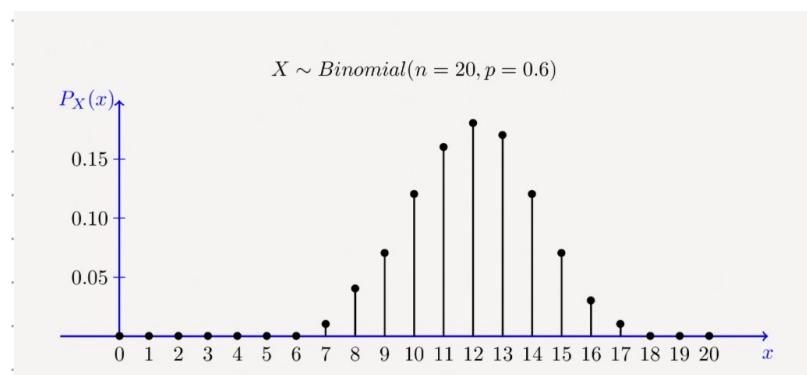
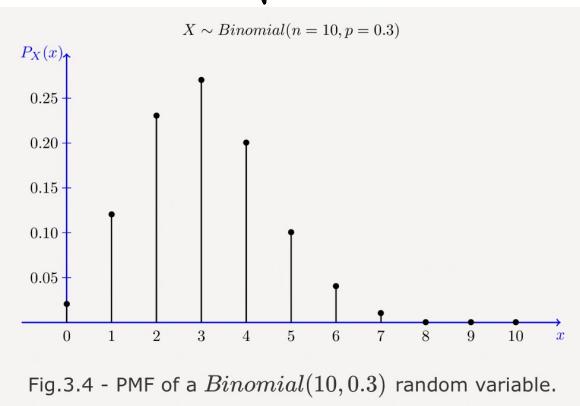
What is the probability of getting exactly 2 heads (if defined as success)

* A footballer takes 4 penalties.

Assume each trial has $p = 0.7$

(scores). Then $X \sim B(4, 0.7)$

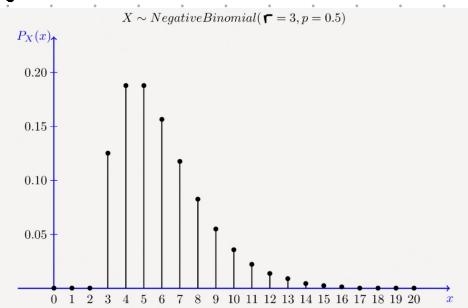
Examples:



Negative Binomial Distribution: (Pascal)

Let X denote the number of trials until r successes. Then X has a negative binomial PMF:

$$f(x) = \binom{x-1}{r-1} \cdot (1-p)^{x-r} \cdot p^r$$



Geometric Distribution:

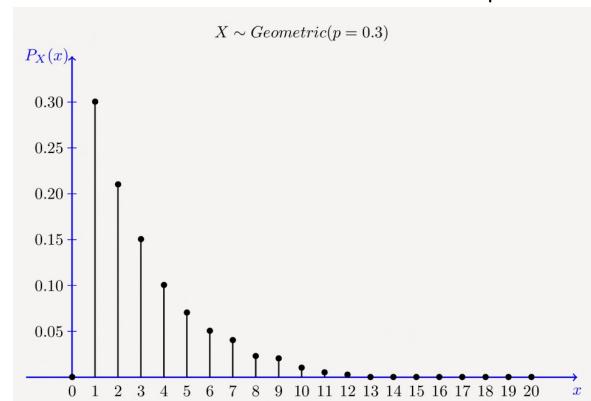
Let X denote the number of trials until the first success. Then X has a geometric PMF:

$$f(x) = (1-p)^{x-1} \cdot p$$

Relation to negative binomial:

Same but $r=1$:

$$f(x) = \binom{x-1}{1-1} \cdot (1-p)^{x-1} \cdot p^1 = (1-p)^{x-1} \cdot p$$



Hypergeometric Distribution

- Given a pool of size N

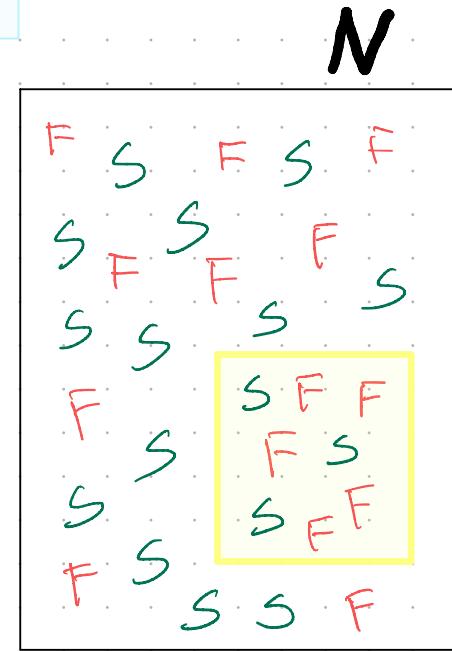
- With exactly r success

- and a random sample n

Now, let X denote the number of successes in n .

Then X is a Hypergeometric R.V. with PMF:

$$f(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}$$



$$N = 30$$

$$r = 16$$

$$n = 8$$

$$x = 3$$

Example: lottery

Poisson Distribution:

Used to model the number of events occurring within a specific time/space interval and has PMF:

$$f(x) = P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

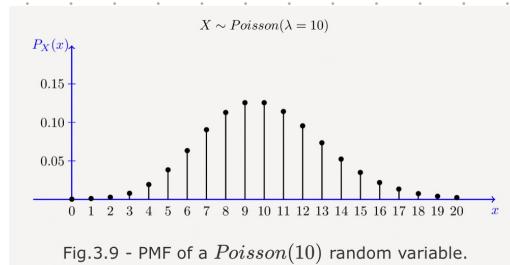


Fig.3.9 - PMF of a $\text{Poisson}(10)$ random variable.

λ indicates the average number of events in a given interval

Important

PMF Summary:

$f(x) = P(X=x)$ Allows us to find probability of exactly x success.

Assume we want to find $P(X \leq 22)$ where X is a Poisson R.V.:

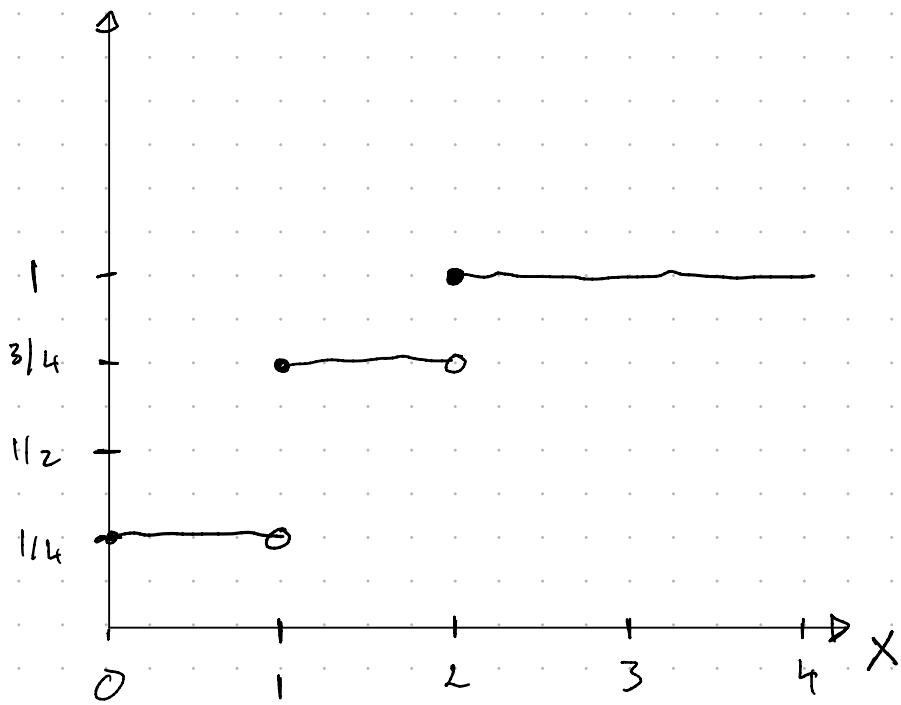
Cumulative Distribution Function:

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}$$

Example: Find CDF

Toss a coin twice. Let X denote heads

$$X \sim \text{Binomial}(2, \frac{1}{2})$$



We get:

$$F_X(x) = \sum_{X_k \leq x} P_X(X_k)$$

Intervals:

$$P(3 \leq X \leq 7) =$$

$$P(3 \leq X < 7) =$$

$$P(3 < X \leq 7) =$$

$$P(3 < X < 7) =$$

In general:

$$P(a < X \leq b) = F(b) - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a-1)$$

Example:

Let X be a discrete R.V with
 $R_X = \{1, 2, 3, \dots\}$ and $f(x) = \frac{1}{2^x}$

① Find CDF

②

$$P(2 < X \leq 5) =$$

③

$$P(X > 4) =$$

Expectation:

Expected value:

$$EX = E(x) = \sum_{x_k \in R_x} x_k \cdot P(X=x_k)$$

Recall Flipping two coins:

Variance:

$$\begin{aligned} \text{Var}(x) &= E[(x - \mu_x)^2] \\ &= \sum_{k=1}^n x_k^2 \cdot P(X=x_k) - (E(x))^2 \end{aligned}$$