Problems 1

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Please contact me if you find any mistakes in the solutions below.

Exercise 1

Heart failures are due to either natural occurrences (87%) or outside factors (13%). Outside factors are related to induced substances (73%) or foreign objects (27%). Natural occurrences are caused by arterial blockage (56%), disease (27%), and infection (e.g., staph infection) (17%).

a. Determine the probability that a failure is due to an induced substance. $0.13 \times 0.73 = \underline{\underline{0.0949}}$

b. Determine the probability that a failure is due to disease or infection. $0.87 \times (0.27 + 0.17) = \underline{\underline{0.3828}}$

Exercise 2

Computer keyboard failures are due to faulty electrical connects (12%) or mechanical defects (88%). Mechanical defects are related to loose keys (27%) or improper assembly (73%). Electrical connect defects are caused by defective wires (35%), improper connections (13%), or poorly welded wires (52%).

a. Find the probability that a failure is due to loose keys. $0.88 \times 0.27 = \underline{\underline{0.2376}}$

b. Find the probability that a failure is due to improperly connected or poorly welded wires. $0.12 \times (0.13 + 0.52) = \underline{\underline{0.078}}$

Exercise 3

Two teams A and B play a football match, and we are interested in the winner. The sample space can be defined as:

$$S = \{a, b, d\}$$

where a shows the outcome that A wins, b shows the outcome that B wins, and d shows the outcome that they draw. Suppose that we know that (1) the probability that A wins is $P(a) = P(\{a\}) = 0.5$ and (2) the probability of a draw is $P(d) = P(\{d\}) = 0.25$.

a. Find the probability that B wins. $P(b) = \underline{\underline{0.25}}$

b. Find the probability that B wins or a draw occurs. $P(\{b, d\}) = \underline{\underline{0.50}}$

Exercise 4

Let A and B be two events such that:

$$P(A) = 0.4, \quad P(B) = 0.7, \quad P(A \cup B) = 0.9$$

a. Find $P(A \cap B) = \underline{\underline{0.2}}$.

b. Find $P(A^c \cap B) = \underline{\underline{0.5}}$.

c. Find $P(A - B) = \underline{\underline{0.2}}$.

d. Find $P(A^c - B) = \underline{\underline{0.1}}$.

e. Find $P(A^c \cup B) = \underline{\underline{0.8}}$.

f. Find $P(A \cap (B \cup A^c)) = \underline{\underline{0.2}}$.

Exercise 5

Consider a random experiment with a sample space.

$$S = \{1, 2, 3, \cdots\}.$$

Suppose that we know:

$$P(k) = P(\{k\}) = \frac{c}{3^k} \quad \text{for} \quad k = 1, 2, \cdots$$

where c is a constant number.

a. Find $c = \underline{\underline{2}}$.

b. Find $P(\{2, 4, 6\}) \approx \underline{\underline{0.25}}$.

c. Find $P(\{3, 4, 5, \cdots\}) = \underline{\underline{\tfrac{1}{9}}}$.

Exercise 6

Let T be the time needed to complete a job at a certain factory. By using the historical data, we know that

$$P(T \le t) = \begin{cases} \frac{1}{16}t^2 & \text{for} \quad 0 \le t \le 4 \\ 1 & \text{for} \quad t > 4 \end{cases}$$

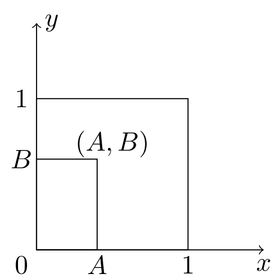
a. Find the probability that the job is completed in less than one hour, i.e., find $P(T \le 1) = \underline{\underline{1/16}}$.

b. Find the probability that the job needs more than 2 hours. $P(T > 2) = 1 - P(T < 2) = \underline{\underline{\tfrac{3}{4}}}$

c. Find the probability that $1 \le T \le 3$. $P(1 \le T \le 3) = P(T \le 3) - P(T < 1) = \underline{\underline{\tfrac{9}{16}}}$

Exercise 7

You choose a point (A, B) uniformly at random in the unit square $\{(x, y) : 0 \le x, y \le 1\}$.



What is the probability that the equation

$$AX^2 + X + B = 0$$

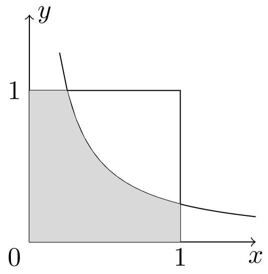
has real solutions?

Solution:

The equation has real roots if and only if:

$$1 - 4AB > 0 \quad \text{i.e.} \quad AB < \frac{1}{4}.$$

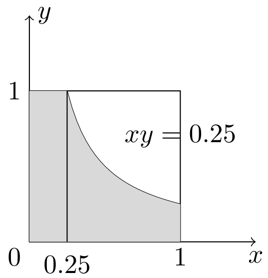
This area is shown here:



Since (A, B) is uniformly chosen in the square we can say that probability of having real roots is

$$\begin{aligned} P(R) &= \frac{\text{area of the shaded region}}{\text{area of the square}} \\ &= \frac{\text{area of the shaded region}}{1} \end{aligned}$$

To find the area of the shaded region we can set up the following integral:



$$\begin{aligned} Area &= \frac{1}{4} + \int_{\frac{1}{4}}^{1} \frac{1}{4x} dx \\ &= \frac{1}{4} + \frac{1}{4} \left[\ln(x)\right]_{\frac{1}{4}}^{1} \\ &= \frac{1}{4} + \frac{1}{4} \ln 4 \end{aligned}$$