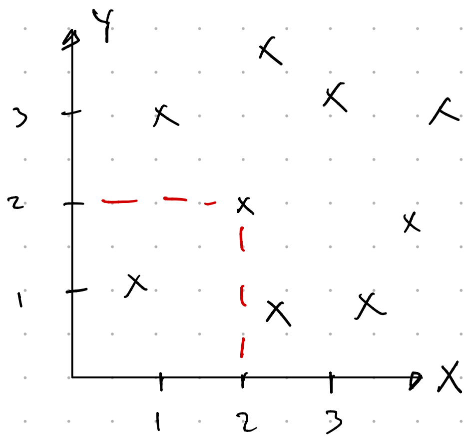
Multivariate RVs

Two Discrete RVs:

X, Y joint PMF:

$$P_{X,Y}(x_i, y_i) = f_{XY}(x_i, y_i) = P(X = x_i, Y = y_i)$$

$$R_{XY} = \{x_i, y_i \mid f_{XY}(x_i, y_i) > 0\}$$



$$f_{XY}(2,2) = 1/4$$

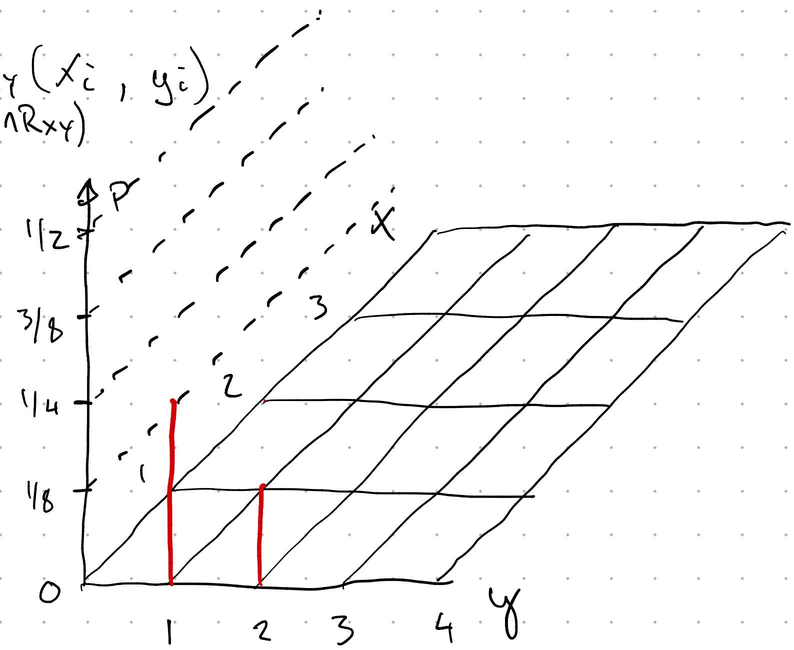
$$\sum_{(x_i, y_i) \in R_{XY}} f_{XY}(x_i, y_i) = 1$$

In general:

$$P((X,Y) \in A) = \sum_{(x_i, y_i) \in (A \cap R_{XY})} f_{XY}(x_i, y_i)$$

Example

	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6



a) Find $P(X=0, Y \leq 1) =$

b) Find Marginal PMFs of X and Y:

$f_X(0) =$, $f_Y(0) =$

$f_Y(1) =$

$f_X(1) =$, $f_Y(2) =$

In general Marginal PMF:

$$f_X(x) = \sum_{y_j} f_{XY}(x, y_j)$$

$$f_Y(y) = \sum_{x_i} f_{XY}(x_i, y)$$

Example cont'd:

c) Find $P(Y=1 \mid X=0) =$

d) Are X and Y independent?

In general:

X and Y are independent if

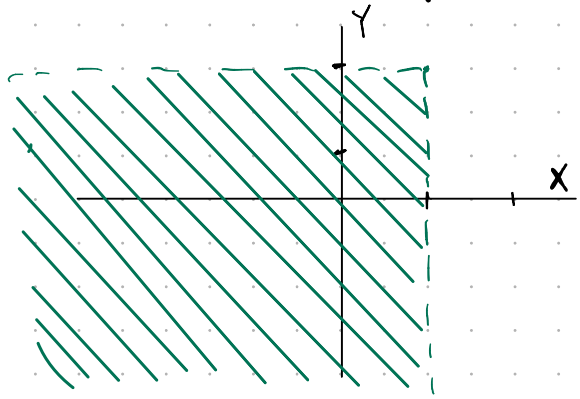
$$f_{XY}(x_i, y_j) = f_X(x_i) \cdot f_Y(y_j)$$

X, Y Joint CDF:

$$F_{XY}(x, y) = P(X \leq x \ , \ Y \leq y)$$

$$0 \leq F_{XY}(x, y) \leq 1$$

$$F_{XY}(1, 2) = P(X \leq 1 \ , \ Y \leq 2)$$



Marginal CDF:

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y)$$

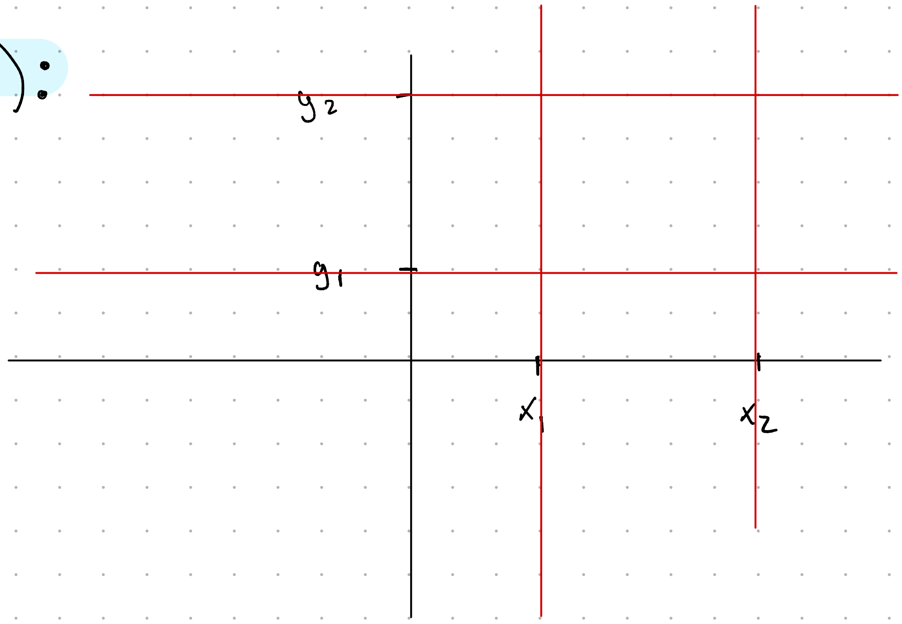
$$F_Y(x) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y)$$

Example:

Find $F_{XY}(0.5, 1) =$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$1/6$	$1/4$	$1/8$
$X=1$	$1/8$	$1/6$	$1/6$

$P(x_1 \le X \le x_2 \ , \ y_1 \le Y \le y_2)$:



$$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(X_1, X_2)$$

Conditioning and Independence

$$\boxed{f_{X|Y}(x_i \mid y_i)} = P(X = x_i \mid Y = y_i)$$

$$= \frac{P(X = x_i \ , \ Y = y_i)}{P(Y = y_i)} = \boxed{\frac{f_{XY}(x_i, y_i)}{f_Y(y_i)}}$$

Independence:

$$f_{X|Y}(x_i \mid y_i) = f_X(x_i)$$

$$\Updownarrow$$

$$\frac{f_{XY}(x_i, y_i)}{f_Y(y_i)} = f_X(x_i) \implies \boxed{f_{XY}(x_i, y_i) = f_X(x_i) \cdot f_Y(y_i)}$$

Conditional Expectation:

$$E(X \mid Y = y_i) = \sum_{x_i \in R_x} x_i \cdot f_{X|Y}(x_i \mid y_i)$$

Example:

Assume the following joint PMF:

$$f_{XY}(x,y) = \begin{cases} 1/3 & x=1 \wedge y=3 \\ 1/3 & x=2 \wedge y=0 \\ 1/3 & x=0 \wedge y=0 \\ 0 & \text{else} \end{cases}$$

a) Find $P_{X|Y=0}(X)$

$P_{X|Y=0}(x_i) =$

$P_{X|Y=0}(0) =$

$P_{X|Y=0}(1) =$

$P_{X|Y=0}(2) =$

$P_{X|Y=0}(x) = \begin{cases} \\ \end{cases}$

b) Find $E[X \mid Y=0]$:

$E[X \mid Y=0] =$

$=$

Example:

We throw a fair coin three times. Let X denote the number of heads on first toss and let Y denote the total number of heads:

X\Y	0	1	2	3
0	1/8	1/4	1/8	0
1	0	1/8	1/4	1/8

Find the expected number of heads given that the first toss resulted in a tail.

$$P_{Y|X=0}(y) = \frac{P_{XY}(y_i, 0)}{P_X(0)}$$

$P_X(0) = 1/2$, $P_X(1) = 1/2$

$P_{Y|X=0}(0) =$

$P_{Y|X=0}(1) =$

$P_{Y|X=0}(2) =$

$P_{Y|X=0}(3) =$

$E(Y|X=0) =$

$E(Y|X=1) =$

Expectation

Total Expectation:

1. $E[X] = \sum_i E[X|B_i] \cdot P[B_i]$, B_1, B_2 some Partition of S
2. $E[X] = \sum_{y_j \in R_Y} E[X|Y=y_j] P_Y(y_j)$
3. $E[X] = E[E[X|Y]]$

Two Continuous RVs

Joint PDF:

$$P_{XY}((X,Y) \in A) = \iint_A f_{XY}(x,y)\, dx\, dy$$

If $A = \mathbb{R}^2$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy$$

Joint CDF:

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial^2 x\, \partial^2 y} F_{XY}(x,y)$$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$.

a) $\int_0^1 \int_0^1 (x + cy^2)\, dx\, dy =$

$=$

b) $P(0 \leq X \leq 1/2, 0 \leq Y \leq 1/2) =$

$=$

$=$

The joint CDF satisfies:

1) $F_X(x) = F(x, \infty)$ (marginal of x)
2) $F_Y(y) = F(\infty, y)$ (marginal of y)
3) $F_{XY}(-\infty, \infty) = 1$
4) $P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) =$
   $F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_2)$

Marginal PDFs:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

c) Find Marginal PDFs:

$f_X(x) =$

$f_X(x) =$

$f_Y(y) =$

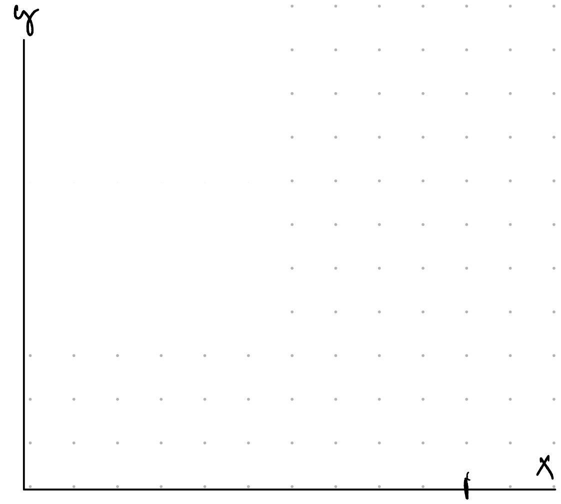
$f_Y(y) =$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find R_{XY} and show it in the $x-y$ plane.
b. Find the constant c.
c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.
d. Find $P(Y \le \frac{X}{2})$.
e. Find $P(Y \le \frac{X}{4} | Y \le \frac{X}{2})$.



a) $R_{XY} =$

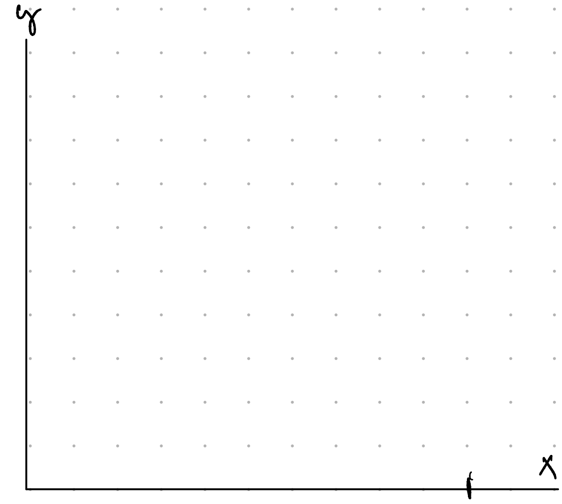
b)

$=$

c) $f_X(x) =$

$f_Y(y) =$

$=$



d) $P(Y \le \frac{X}{2}) =$

$=$

e) $P(Y \leq \frac{X}{4} \mid Y \leq \frac{X}{2}) =$

$=$

$=$

$=$

Mean and Variance:

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y)\,dy\,dx$$

$$V[X] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \left(\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx\right)^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x,y)\,dy\,dx - \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y)\,dy\,dx\right)^2$$

$$= E[X^2] - (E[X])^2$$

Example

Given

$$f_{X,Y}(x,y) = \begin{cases} 2-x-y & , 0 \leq x,y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $Var(X)$:

$E[X] =$

$=$

$=$

$E[X^2] =$

$=$

$=$

$Var(X) =$

Conditioning

Same principles as in discrete case. Most importantly:

1. $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

2. $P(X \in A \mid Y = y) = \int_A f_{X|Y}(x|y)\, dy$

3. $F_{X|Y}(x|y) = P(X \leq x \mid Y = y) = \int_{-\infty}^{x} f_{X|Y}(x|y)\, dx$

4. $E[X \mid Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$

5. $V[X \mid Y = y] = E[X^2 \mid Y = y] - (E[X \mid Y = y])^2$

Independence:

Same as discrete!

Lotus still applies:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y)\, dy\, dx$$

Example:

$$f_{XY}(x,y) = \begin{cases} x + y & 0 \leq x, y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $E[XY^2]$

$E[XY^2] =$

$=$

Covariance and Correlation:

The **covariance** gives information about how X and Y are statistically related:

$$\begin{aligned} \operatorname{Cov}(X,Y) &= E[(X-EX)(Y-EY)] \\ &= E[XY] - (EX)\cdot(EY) \end{aligned}$$

Properties:

1. $\operatorname{Cov}(X,X) = \operatorname{Var}(X)$
2. If X and Y are independent
$\operatorname{Cov}(X,Y) = 0$
3. $\operatorname{Cov}(X,Y) = \operatorname{Cov}(Y,X)$
4. $\operatorname{Cov}(aX,Y) = a\cdot\operatorname{Cov}(X,Y)$, $a \in \mathbb{R}$
5. $\operatorname{Cov}(X+c,Y) = \operatorname{Cov}(X,Y)$
6. $\operatorname{Cov}(X+Y,Z) = \operatorname{Cov}(X,Z) + \operatorname{Cov}(Y,Z)$
7. $\operatorname{Cov}(X+Y,Z+W)$
$= \operatorname{Cov}(X,Z) + \operatorname{Cov}(X,W) + \operatorname{Cov}(Y,Z) + \operatorname{Cov}(Y,W)$
8. More generally:

$$\operatorname{Cov}\left(\sum_{i=1}^{m} a_i X_i \,,\, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \operatorname{Cov}(X_i, Y_j)$$

If $Z = X + Y$

9. $\operatorname{Var}(Z) = \operatorname{Cov}(Z,Z) = \operatorname{Cov}(X+Y, X+Y)$
$= \operatorname{Cov}(X,X) + \operatorname{Cov}(X,Y) + \operatorname{Cov}(Y,X) + \operatorname{Cov}(Y,Y)$
$= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\cdot\operatorname{Cov}(X,Y)$
10. More generally:
$\operatorname{Var}(aX+bY) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 2ab\operatorname{Cov}(X,Y)$

greek letter 'rho' = r

The correlation coefficient, ρ_{XY} or $\rho(X,Y)$ is obtained by normalising the covariance. More specifically we use the standardised versions of X and Y:

$$U = \frac{X - EX}{\sigma_X} \quad , \quad V = \frac{Y - EY}{\sigma_Y}$$

$$\rho_{XY} = \operatorname{Cov}(U,V) = \operatorname{cov}\left(\frac{X-EX}{\sigma_X}, \frac{Y-EY}{\sigma_Y}\right)$$

$$= \operatorname{Cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right) \qquad \text{(property 5)}$$

$$= \frac{\operatorname{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} \qquad \text{(property 4)}$$