Discret Random Variables

A function that assigns a real number to each outcome:

$$X : S \longrightarrow \mathbb{R}$$

* Usually an R.V. is denoted by X, Y, Z (Upper case)
* The outcomes are denoted by x, y, z (lower case)

e.g. $P(X=x)$ if $x=2$

$P(X=2)$

$P(Y=y)$ if $y=3$

$P(Y=3)$

Random Variables are either Discrete or Continuous:

Discrete	Continuous
- The range is countable $\{1, 2, 3, \dots\}$	- The range is continuous $\hookrightarrow \mathbb{R}$
* Uniform	* Uniform
* Binomial	* Normal Distribution
* Geometric	* Chi-Squared
* Negative Binomial	* Log-Normal
* Hypergeometric	* Exponential
* Poisson	

Definition: Discrete Random Variable

Let X be a discrete R.V. with $R_X = \{x_1, x_2 \dots\}$

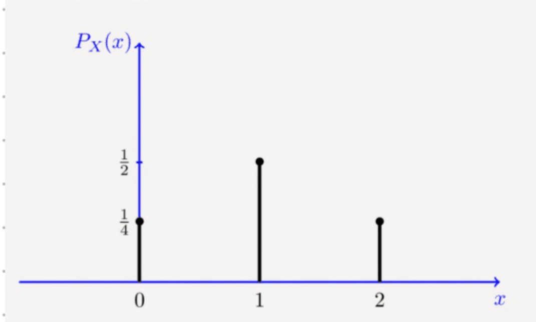
The function

$$P_X(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3 \dots$$

is called the probability mass function (PMF) of X.

Properties:

1) $f(x_k) = P_X(x_k)$
2) $0 \le f(x_k) \le 1$
3) $\sum_{k=1}^{n} f(x_k) = 1$



Independent Random variables:

Consider X and Y. We say X and Y are independent if:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

It follows:

$$P(Y = y \mid X = x) = P(Y = y)$$

Example:

I toss a coin twice and define X to be the number of heads I observe. Then, I toss the coin two more times and define Y to be the number of heads that I observe this time. Find $P\Big((X < 2) \text{ and } (Y > 1)\Big)$.

Bernoulli Distribution:

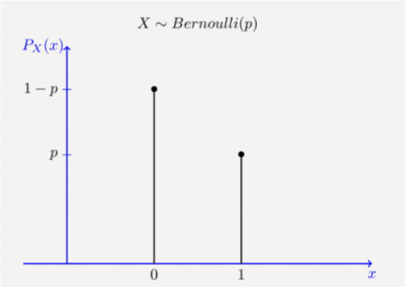
A Bernoulli R.V. can only take two values:

1 : success

0 : failure

Bernoulli PMF:

$$P_X(x) = \begin{cases} p & \text{for } x=1 \\ 1-p & \text{for } x=0 \\ 0 & \text{otherwise} \end{cases}$$



Binomial Distribution:

Given multiple independent Bernoulli experiment, the resulting R.V. has a Binomial PMF:

$$P_X(x) = f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

X = number of successes

p = probability of success

n = number of trials/Experiments

$\binom{n}{x}$ = Binomial Coefficient

↳ combination of subset:

In how many ways can I choose x elements from n when order does not matter:

We can summarize as

$$P(X=x) = \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}$$

* Flip a coin 12 times
  - ↳ each is a Bernoulli trial
  - ↳ together they form $B \sim (n,p)$
  - ↳ $n = 12$, $p = 1/2$

$P(X=2)$ means given 12 coin flips what is the probability of getting exactly 2 heads (if defined as succes)

* A footballer takes 4 penalties. Assume each trial has $p = 0.7$ (scores). Then $X \sim B(4, 0.7)$

Examples:

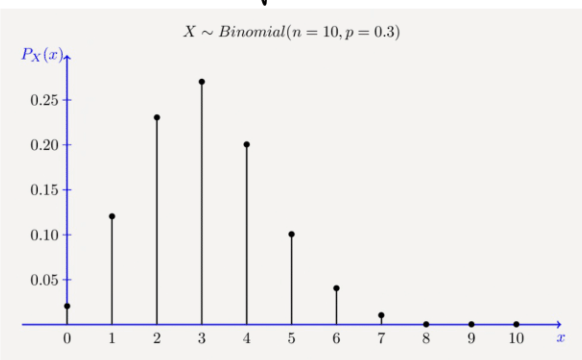
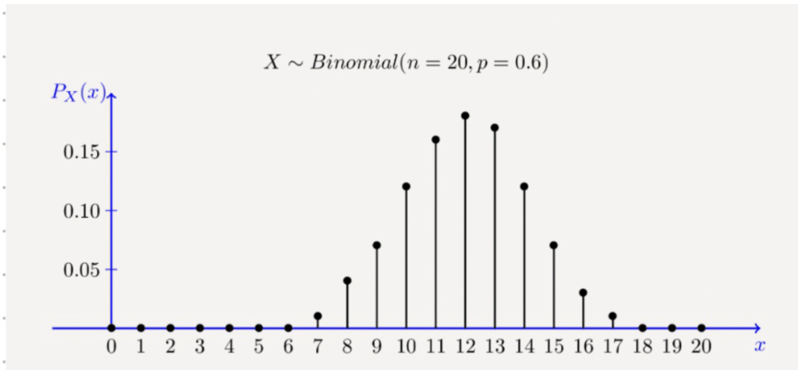


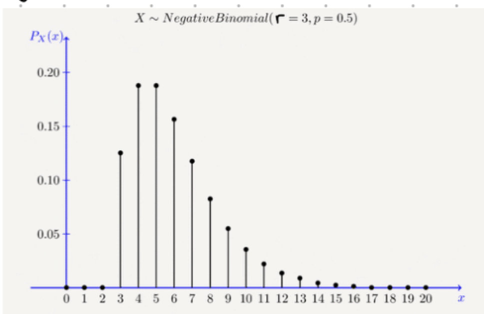
Fig.3.4 - PMF of a $Binomial(10, 0.3)$ random variable.



Negative Binomial Distribution: (Pascal)

Let X denote the number of trials until r successes. Then X has a negative binomial PMF:

$$f(x) = \binom{x-1}{r-1} \cdot (1-p)^{x-r} \cdot p^r$$



Geometric Distribution:

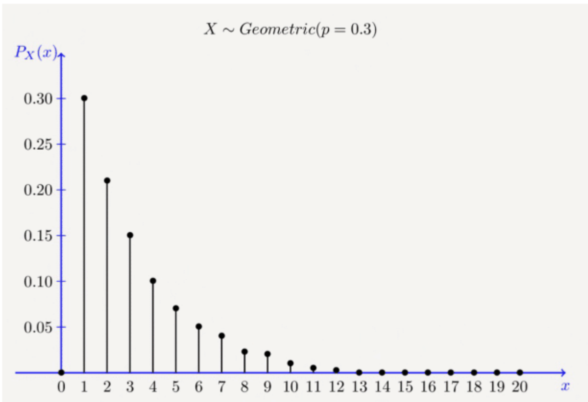
Let X denote the number of trials until the first success. Then X has a geometric PMF:

$$f(x) = (1-p)^{x-1} \cdot p$$

Relation to negative binomial:

Same but $r = 1$:

$$f(x) = \binom{x-1}{1-1} \cdot (1-p)^{x-1} \cdot p^1 = (1-p)^{x-1} \cdot p$$



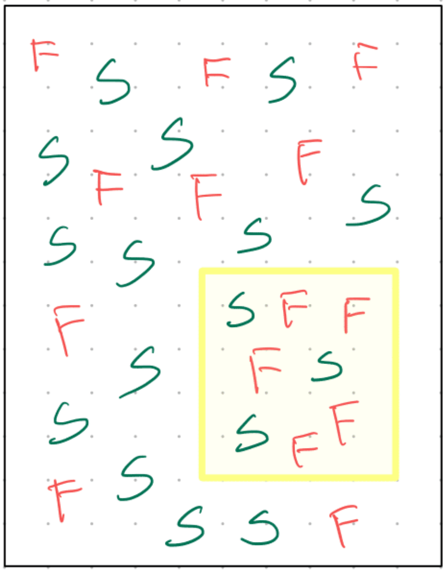
Hypergeometric Distribution

- Given a pool of size N
- With exactly r success
- and a random sample n

Now, let X denote the number of successes in n. Then X is a Hypergeometric R.V. with PMF:

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

N



$N = 30$

$r = 16$

$n = 8$

$x = 3$

Example: Lottery

Poisson Distribution:

Used to model the number of events occurring within a specific time/space interval and has PMF:

$$f(x) = P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

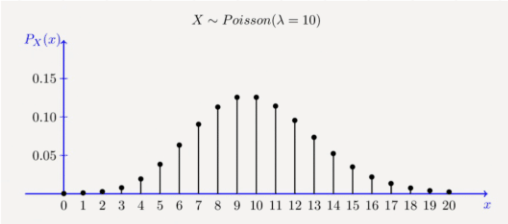


Fig.3.9 - PMF of a $Poisson(10)$ random variable.

λ indicates the average number of events in a $\underbrace{\text{given interval}}_{\text{Important}}$

PMF Summary:

$f(x) = P(X=x)$ Allows us to find probality of exactly x success.

Assume we want to find $P(X \leq 22)$ where X is a Poisson R.V.:

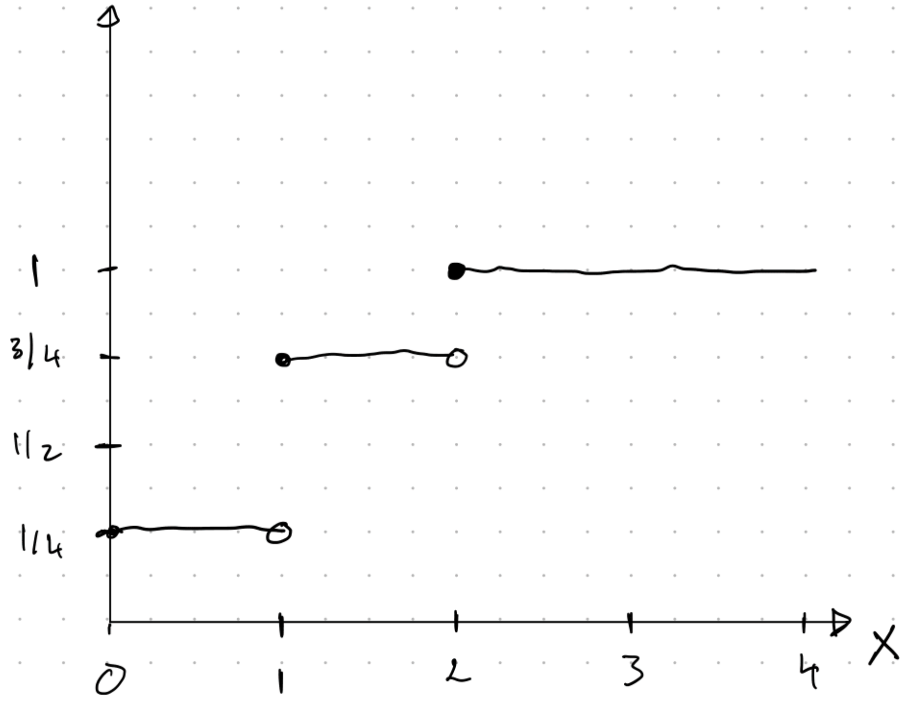
Cumulative Distribution Function:

$$F_X(x) = P(X \leq x) \quad , x \in \mathbb{R}$$

Example: Find CDF

Toss a coin twice. Let X denote heads

$$\boxed{X \sim \text{Binomial}(2, 1/2)}$$



We get:

$$\boxed{F_X(x) = \sum_{x_k \leq x} P_X(x_k)}$$

Intervals:

$P(3 \leq X \leq 7) =$

$P(3 \leq X < 7) =$

$P(3 < X \leq 7) =$

$P(3 < X < 7) =$

In general:

$$P(a < X \leq b) = F(b) - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a-1)$$

Example:

Let X be a discrete R.V with

$R_x = \{1, 2, 3 \ldots\}$ and $f(x) = \frac{1}{2^x}$

① Find CDF

②

$P(2 < X \leq 5) =$

③ $P(X > 4) =$

Expectation:

Expected value:

$$EX = E(X) = \sum_{x_k \in R_X} x_k \cdot P(X = x_k)$$

Recall Flipping two coins:

Variance:

$$\begin{aligned} Var(X) &= E[(X - \mu_X)^2] \\ &= \sum_{k=1}^{n} x_k^2 \cdot P(X = x_k) - \left(E(X)\right)^2 \end{aligned}$$