Continuous Random Variables

A continuous R.V. has only continuous values i.e. values that are uncountable and are related to real numbers, $\mathbb{R}$.

* Time it takes to complete SMP exam
  $R_x = [0; 180]$
* Age of fossil
  $R_x = [\text{min age}; \text{max age}]$
* Km/h of 1985 BMW 5-series
  $R_x = [0; \infty[$

Main difference to discrete:

DRV measured on exact values

CRV measured on intervals

↳ Makes no sense to find probability that exam took exactly 117 min., perhaps it took 117.0132149.... min but not exactly 117

↳ $P(X = 117) = 0$ → Note

↳ $P(116 \leq X \leq 118) = 0.71$

→ It makes perfect sense to find probability of 117 students attending exam: $P(X = 117)$

Uniform Distribution

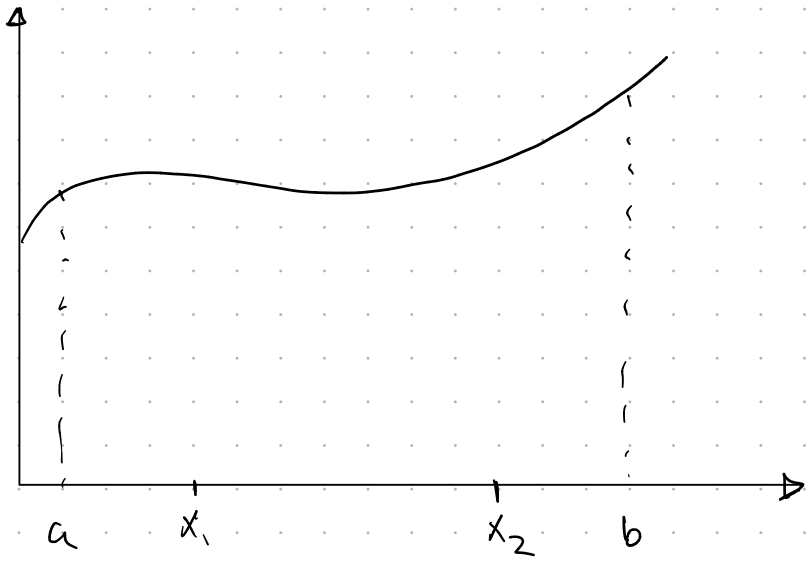
Choose a real number uniformly in $[a;b]$ and denote it X. By uniformly we mean that all intervals of same length in $[a;b]$ have same probability. Find CDF of X:

$$P(X=x)=0$$

Means "proportional to"

$$P(x_1 \le X \le x_2) \propto (x_2 - x_1) \quad , \; a \le x_1 \le x_2 \le b$$

$P(X \in [x_1;x_2])$



$$P(X \in [a,b]) = 1$$

$$P(X \in [x_1;x_2]) = \frac{x_2 - x_1}{b-a}$$

$$F(x) = P(X \le x)$$

$$F(x) = 0 \qquad x < a$$

$$F(x) = 1 \qquad x > b$$

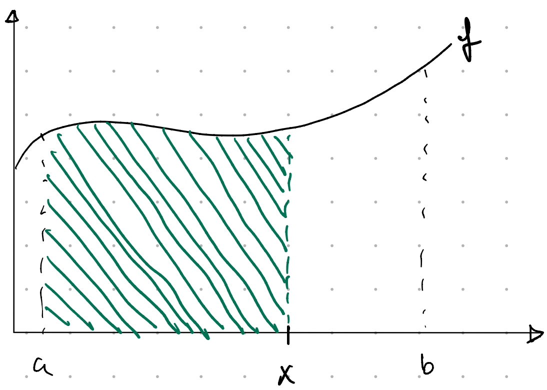
Including or not? Doesn't matter "$\le$" = "$<$"

$a \le X \le b$:

$$F(x) = P(X \le x)$$

$$= P(X \in [a;x])$$

$$= \frac{x-a}{b-a}$$



To summarise: The CDF of the Uniform CRV

$$F(x) = \begin{cases} 0 & \text{for } x<a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

More generally:

If X is a continuous R.V., then its CDF is a function s.t.:

$$F(x) = P(X \le x)$$

1. $0 \le F(x) \le 1$
2. $F(-\infty) = 0$, $F(\infty) = 1$

$-\infty$ = lower bound

∞ = Upper bound

3. $F(x)$ is non-decreasing as x increases
4. $P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$
5. $P(X \le x_1) = P(X < x_1)$
6. $P(X > x_1) = 1 - P(X \le x_1) = 1 - F(x_1)$

Probability Density Function (PDF):

Let X be a continuous R.V. The PDF is a function s.t.

$$f(x) = \lim_{h \to 0} \frac{F(x-h) - F(x)}{h} = \frac{dF(x)}{dx}$$

1. $f(x) \ge 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $\boxed{F(x) = \int_{-\infty}^{x} f(u)\,du}$ ← Important !!
4. $\int_{x_1}^{x_2} f(x)\,dx = P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$
5. $P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2)$
$= P(x_1 < X < x_2)$

Example:

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05$ for $0 \le x \le 20$. What is the probability that a current measurement is less than 10 milliamperes?

$$P(X<10) = \int_0^{10} 0.05\,dx = 0.05x\Big|_0^{10} = \underline{\underline{0.5}}$$

Let's find CDF:

$$F(x) = \int_0^x 0.05\,du = 0.05u\Big|_0^x = 0.05x$$

We get

$$F(x) = \begin{cases} 0 & x<0 \\ 0.05x & 0 \le x \le 20 \\ 1 & x > 20 \end{cases}$$

Example

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \ge 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped?

$$P(X>12.60) = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\,dx \,,$$

$$= -\frac{1}{20} \cdot 20e^{-20(x-12.5)}\Big|_{12.6}^{\infty} \quad \begin{matrix} F(\infty) \\ F(12.6) \end{matrix} \quad F(\infty)-F(12.6)$$

$$= 0 + \left(e^{-20(12.6-12.5)}\right)$$

$$= \underline{\underline{e^{-2}}} \approx \underline{\underline{0.135}}$$

OR

$$P(X>12.60) = 1 - \int_{12.5}^{12.6} 20e^{-20(x-12.5)}$$

$$= 1 - \left[-e^{-20(x-12.5)}\Big|_{12.5}^{12.6}\right.$$

$$= 1 + \left[e^{-2} - e^{0}\right] = \underline{\underline{e^{-2}}}$$

Let's Find CDF :

$$F(x) = \int_{12.5}^{x} 20 e^{-20(u-12.5)} du = -e^{-20(u-12.5)} \Big|_{12.5}^{x}$$

$$= -e^{-20(x-12.5)} - \left(-e^{-20(12.5-12.5)}\right)$$

$$= 1 - e^{-20(x-12.5)}$$

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & x \geq 12.5 \end{cases}$$

Example

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.01x} & 0 \leq x \end{cases}$$

Determine the probability density function of X. What proportion of reactions is complete within 200 milliseconds?

$$f(x) = F'(x) = \left(1 - e^{-0.01x}\right)'$$

$$= 0.01 e^{-0.01x}$$

$f(x) = x^n$

$f'(x) = n \cdot x^{n-1}$

$\left(f(g(x))\right)' = g'(x) \cdot f'(g(x))$

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.01 e^{-0.01x} & x \geq 0 \end{cases}$$

$$P(X \leq 200) = F(200) = 1 - e^{-0.01 \cdot 200}$$

$$= 1 - e^{-2} \approx \underline{\underline{0.865}}$$

Example:

Let
$$f(x) = \begin{cases} 4a \cdot e^{-2x} & , x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) Find a:
$$\int_0^\infty 4 \cdot a \cdot e^{-2x} \, dx = 1 \Rightarrow -\frac{4}{2} \cdot a \cdot e^{-2x} \Big|_0^\infty = 1 \Leftrightarrow 2a = 1$$
$$a = 1/2$$

b) Find $F(x)$:
$$\int_0^x 2 e^{-2u} \, du = -\frac{1}{2} \cdot 2 \cdot e^{-2u} \Big|_0^x = \left(-e^{-2x}\right) + 1$$
$$= 1 - e^{-2x}$$

c) Find $P(1 \leq X \leq 3)$
$$P(1 < X < 3) = F(3) - F(1)$$
$$= \left(1 - e^{-2 \cdot 3}\right) - \left(1 - e^{-2 \cdot 1}\right)$$
$$= e^{-2} - e^{-6} \rightarrow \int_1^3 2e^{-2x} \, dx$$

Expected Value:
$$E(X) = \boxed{\int_{-\infty}^{\infty} x f(x) \, dx}$$

Variance

General for all R.V.S

$$\mathrm{Var}(X) = E[(X - \mu_X)^2] = E[X^2] - (E[X])^2$$
$$\boxed{= \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx - (E[X])^2}$$

for C.R.V.

Law of the unconscious statistician (LOTUS):
$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, dx}$$

PDF

Example

Let

$$f(x)=\begin{cases}2x & 0\le x\le 1\\ 0 & \text{else}\end{cases}$$

Find $E(X)$: $\int_0^1 x\cdot 2x\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \underline{\underline{2/3}}$

Example

Let

$$f(x)=\begin{cases}x^2(2x+\frac{3}{2}) & 0\le x\le 1\\ 0 & \text{else}\end{cases}$$

If $Y=\frac{2}{X}+3$, Find $Var(Y)$

$2\cdot\frac{1}{X}+3$

$a\;X+b$

$$Var(aX+b) = a^2\cdot Var(X)$$

$$Var(Y) = 2^2\cdot Var\left(\frac{1}{X}\right)$$

$$V\left(\frac{1}{X}\right) = \underbrace{E\left[\frac{1}{X^2}\right]} - \left(E\left[\frac{1}{X}\right]\right)^2$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{X^2}\cdot x^2\left(2X+\frac{3}{2}\right)dx$$

$$= \int_0^1\left(2x+\frac{3}{2}\right)dx = x^2+\frac{3}{2}x\Big|_0^1$$

$$= 5/2$$

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x}\cdot x^2\left(2x+\frac{3}{2}\right)dx = \int_0^1\left(2x^2+\frac{3}{2}x\right)dx$$

$$= \frac{2}{3}x^3+\frac{3}{4}x^2\Big|_0^1 = \frac{2}{3}+\frac{3}{4}$$

$$= \underline{\underline{\frac{17}{12}}}$$

$$Var(Y) = 4\left(\frac{5}{2}-\left(\frac{17}{12}\right)^2\right) = 4\left(\frac{5}{2}-\frac{289}{144}\right)$$

$$= 4\cdot\frac{360-289}{144} = 4\cdot\frac{71}{144} = \underline{\underline{\frac{71}{36}}}$$

Uniform R.V.

Recall $F(x) = \dfrac{x-a}{b-a}$, so

$$f(x) = F'(x) = \left(\frac{x}{b-a}\right)' - \left(\frac{a}{b-a}\right)'$$

$$= \frac{1}{b-a}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}$$

Expected value:

$$E(X) = \int_a^b x \cdot f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{2}x^2 \cdot \frac{1}{b-a}\Big|_a^b = \frac{x^2}{2(b-a)}\Big|_a^b$$

$$= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}} = \frac{b+a}{2}$$

$$= \frac{a+b}{2}$$

Variance

$$Var(X) = \int_a^b x^2 \cdot \frac{1}{b-a} = \frac{1}{3} \cdot x^3 \cdot \frac{1}{b-a}\Big|_a^b$$

$$\vdots$$

$$= \frac{(b-a)^2}{12}$$

Functions of Continuous R.V.'s: Difficult

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

Let X be a Uniform$(0,1)$ R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_x(X) = \frac{x-a}{b-a} = \frac{x-0}{1-0} = x \quad , f_x(x) = 1$$

$$R_x = [0;1] \quad , R_Y = [1;e]$$

$$F(Y) = P(Y \leq y)$$
$$= P(e^X \leq y)$$
$$= P(X \leq \ln y) \quad , \text{so}$$

$$F(Y) = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y \leq e \\ 1 & y > e \end{cases}$$

b) Find pdf of Y:

$$f(y) = F'(y) = \frac{1}{y} \qquad 1 \leq y \leq e \text{, else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y}\, dy = y\Big|_1^e = e-1$$

Using LOTUS:

$$E[Y] = \int_0^1 e^x \cdot 1\, dx = e^x\Big|_0^1 = e-1$$

$$=$$