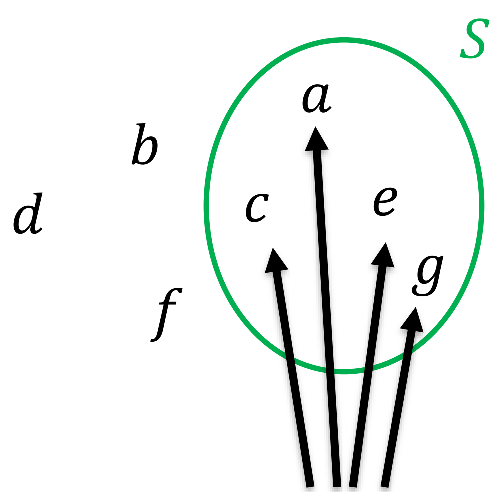
Gør tanke til handling
VIA University College

Set theory

Sets

Sets

Collections of definite and separate objects



The **elements** of the **set** S.

Specifying a set 1: Listing notation

One way of specifying a set is by simply listing its elements.

<u>Examples:</u>

$A = \{a, b, c, d, e\}$

$B = \{\text{elephant, rhino, hippo, giraffe, buffalo}\}$

$C = \{1,2,3\}$

$D = \{0.1, 1, \pi, 8.2939, \dots\}$

Note: order doesn't matter! $\{a, b, c\} = \{c, a, b\}$

Note: each element only appears once: $\cancel{\{a, a, a, b, b, c, a, b\}} \rightarrow \{a, b, c\}$.

Specifying a set 2: Set builder notation

Another way to specify a set is by *describing* it.

This is done using the symbol |, which means "such that".

<u>Examples:</u>

$A = \{l | l \text{ is a vowel in the English alphabet}\}$

$B = \{a | a \text{ is one of the five largest animals living in Africa}\}$

$C = \{n | n \text{ is an integer greater than 0 and less than 4}\}$

$D = \{x | x \in \mathbb{R}, 0 < x < 9\}$

An equivalent way to write C and D:

$C = \{n \text{ is an integer} | n \text{ is greater than 0 and less than 4}\}$

$D = \{x \in \mathbb{R} | 0 < x < 9\}$

A few important sets

The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

In the previous slide, we referred to the set $\mathbb{R}$ - the set of real numbers.

This is one of the fundamental number sets, which we will be referring to a lot duing the course. Below these sets are listed:

sometimes, 0 is included as well

- The set of natural numbers (positive integers): $\mathbb{N} = \{1,2,3,4, \ldots\}$
- The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1,0,1,2,3, \ldots\}$

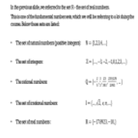
All the numbers that can be written as fractions

- The rational numbers: $\mathbb{Q} = \{-\frac{2}{5}, \frac{3}{3}, \frac{23}{393}, \frac{239{,}129}{2093}, \ldots\}$

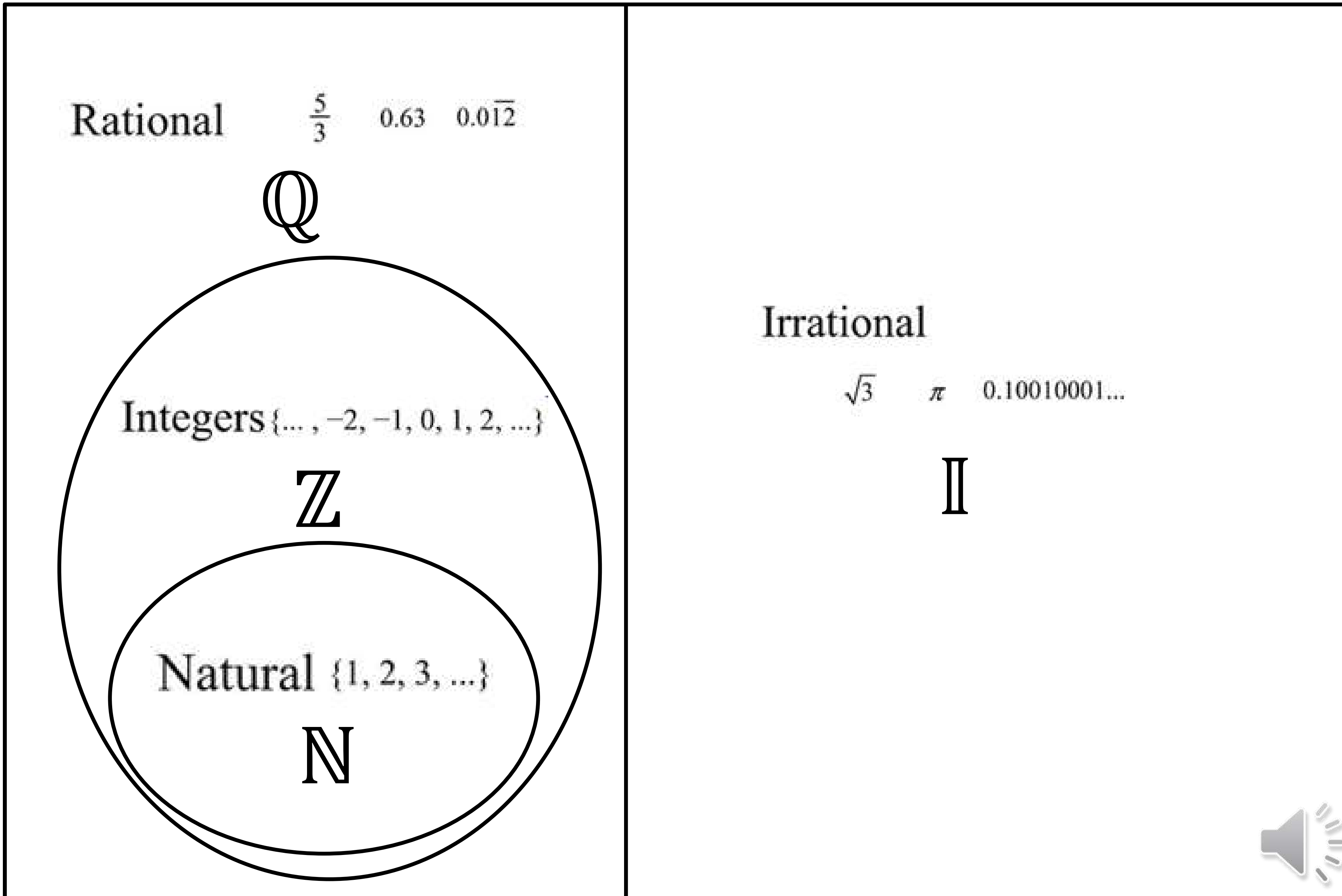
All the numbers that cannot be written as fractions

- The set of irrational numbers: $\mathbb{I} = \{\ldots, \sqrt{2},\ e, \pi, \ldots\}$
- The set of real numbers : $\mathbb{R} = \{-17.8923, -10, \}$

The continuum of all numbers from $-\infty$ to $+\infty$ - this consist of $\mathbb{Q}$ together with $\mathbb{I}$.



The real numbers: $\mathbb{R}$



The number line

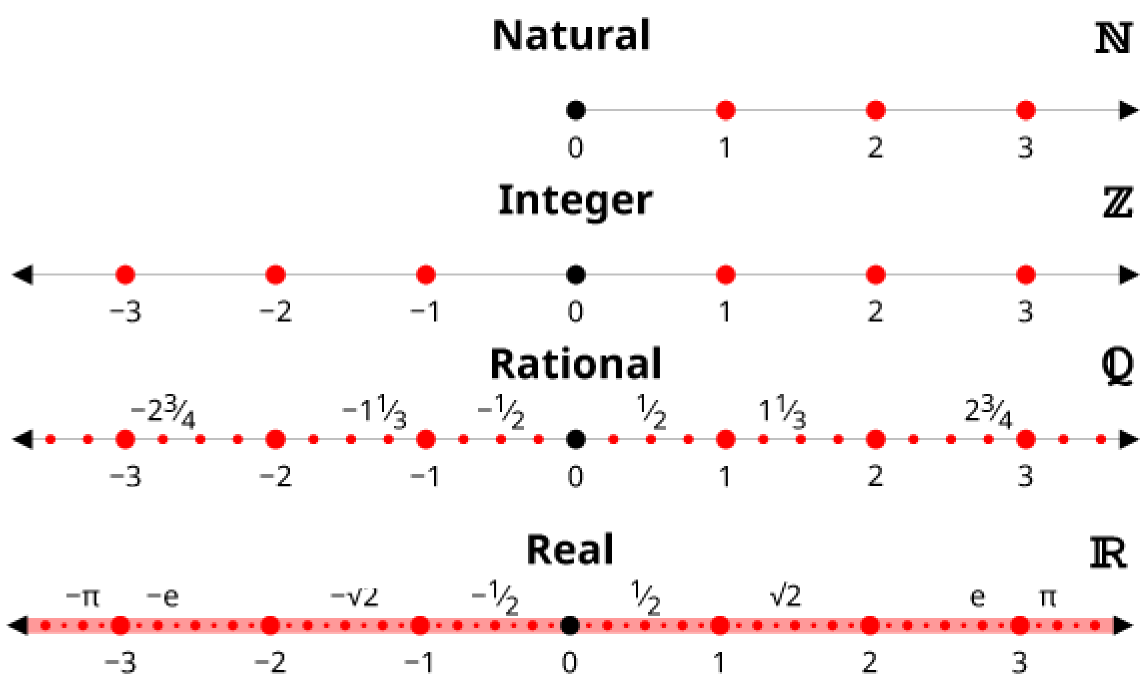
We can imagine that all numbers lie on a line (or axis) from $-\infty$ to ∞:

Start with the counting numbers (zero may be included):

Extend the line backward to include the negatives:

Insert all the fractions:

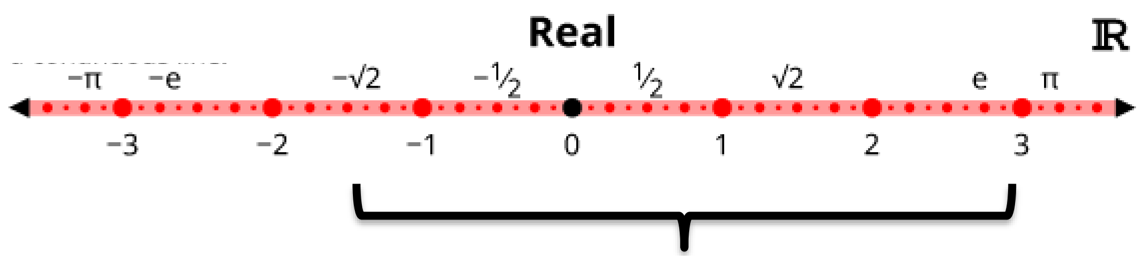
Fill in all the irrational numbers to make a continuous line:



source: https://thinkzone.wlonk.com/Numbers/NumberSets.htm

Interval notation

Consider an interval of the numbers on the real line – say from -$\sqrt{2}$ to 3:



We use the following notation to denote this interval:

$[-\sqrt{2}; 3]$ if the endpoints are included

$(-\sqrt{2}; 3)$ or $]-\sqrt{2}; 3[$ if the endpoints are not included

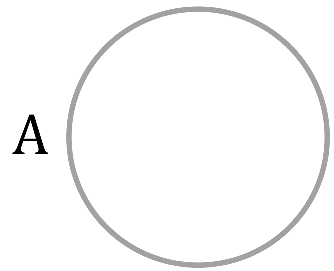
We may also consider intervals that extends all the way to $-\infty$ for example. For this, we use the same notation – however, note that ∞ or $-\infty$ are never included:

$(-\infty; 3]$ $\leftarrow$ interval from minus infinity to 3.

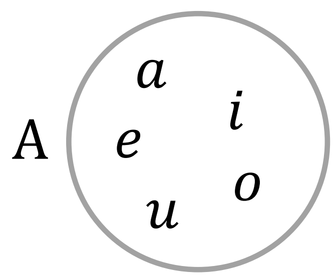
Venn diagrams and subsets

Venn diagrams

Graphical representations of sets as geometrical shapes (circles, ellipses, rectangles, …):



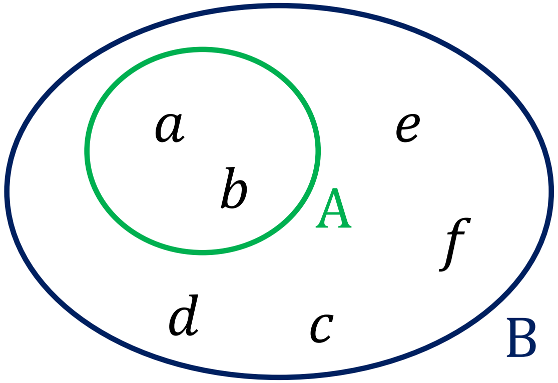
Sometimes the set's elements are also shown:



Subsets

A set A is a subset of another set B if all its elements are within it.
It is written $A \subseteq B$.
Note that this is fulfilled if A and B are equal!

<u>Example 1:</u>
$A = \{a, b\}; B = \{a, b, c, d, e, f\} \Rightarrow A \subseteq B$



<u>Example 2:</u>
$A = \{1,2,3,4\}; B = \{4,3,2,1\} \Rightarrow A \subseteq B$

Proper subsets

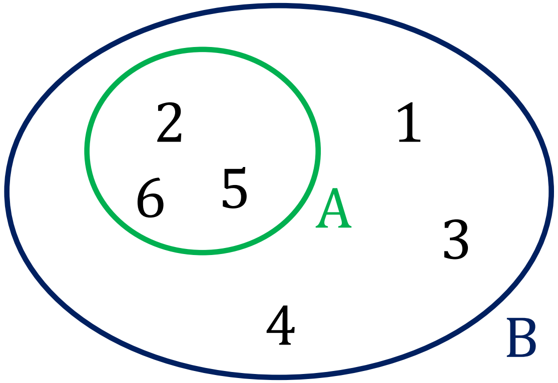
A set A is a ***proper*** subset of B if A is a subset of B but $A \neq B$.

This is written $A \subset B$.

<u>Example</u>

$A = \{2,5,6\}, B = \{1,2,3,4,5,6\} \Rightarrow A \subset B$

Note: in this case, it is also true that $A \subseteq B$.



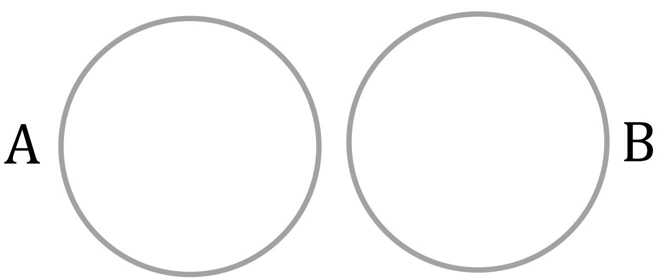
<u>Counterexample</u>

$S = \{1,2\}, T = \{1,2\} \Rightarrow S \subseteq T$, but $S \not\subset T$.

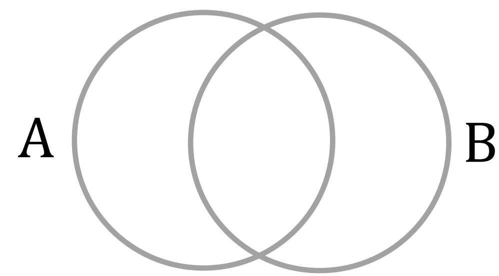
Disjoint sets

Two sets are ***disjoint*** if they do not have anything in common.

Disjoint:



Not disjoint:



Examples:

$\{1,2,3\}$ and $\{3,4,5\}$ are not disjoint.

$\{1,3,5\}$ and $\{2,4,6\}$ are disjoint.

$\{1\}$ and $\{a, b, \{1\}\}$ are disjoint.

Ø and any set (even Ø itself!) are disjoint.

The empty set

$$\emptyset = \{\}$$

The empty set is a subset of every set!

To illustrate this, imagine a set A and assume that $\emptyset$ is not a subset of A. That means that $\emptyset$ has at least one element which is not in A. But that's a contradiction, because $\emptyset$ has no elements!

The size – or order - of a set

The size (or order) of a set is the number of elements in the set.
It is written as $|A|$.

<u>Example</u>

$A = \{1,2,3,4\} \Rightarrow |A| = 4$

$B = \{6, 2, g, 9, \pi\} \Rightarrow |B| = 5$

$C = \{x \in \mathbb{N} | 0 < x < 3\} \Rightarrow |C| = 2$

$D = \{x \in \text{The Alphabet} | x \text{ is a vowel}\} \Rightarrow |D| = 5$

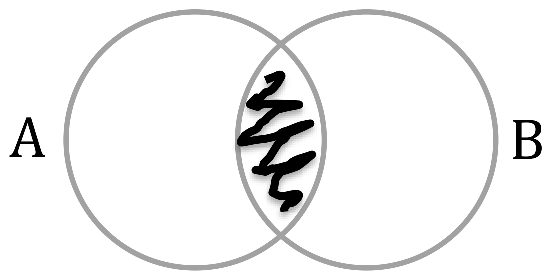
$|\emptyset| = 0$

Set operations

Intersection

The intersection between two sets is the set of elements they have in common.

It is written $A \cap B$.



Note: This is analogous to the Boolean AND-operator.

$$A \cap B = \{x | x \in A \ \wedge x \in B\}$$

= "The set of elements which are in both *A* AND *B*"

<u>Example</u>:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

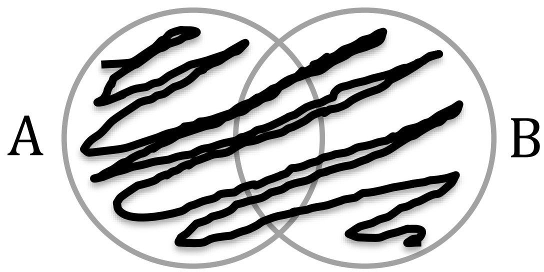
$$\Rightarrow A \cap B = \{\text{whale, sea lion}\}$$

Union

The union between two sets, A and B, is the set of all elements in A and all elements in B.

It is written $A \cup B$.

Note: This is analogous to the Boolean OR-operator.



$$A \cup B = \{x | x \in A \vee x \in B\}$$

= "The set of elements which are in either A OR B"

<u>Example</u>:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

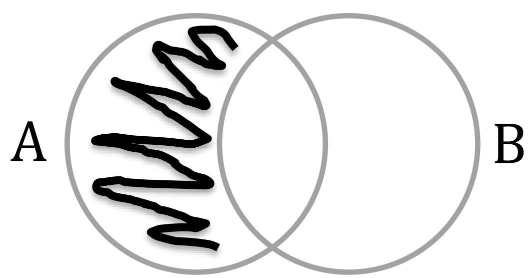
$$\Rightarrow A \cup B = \{\text{shark, whale, penguin,}$$
$$\text{sea lion, dog, cat, bear}\}$$

Set difference

The set difference between two sets A and B is the set of all the elements in A which are not elements in B.

This is also called the relative complement between A and B.

It is written $A - B$.



$$A - B = \{x | x \in A \wedge x \notin B\}$$

<u>Example:</u>

$A = \{\text{shark}, \text{whale}, \text{penguin}, \text{sea lion}\};$

$B = \{\text{dog}, \text{cat}, \text{whale}, \text{bear}, \text{sea lion}\}$

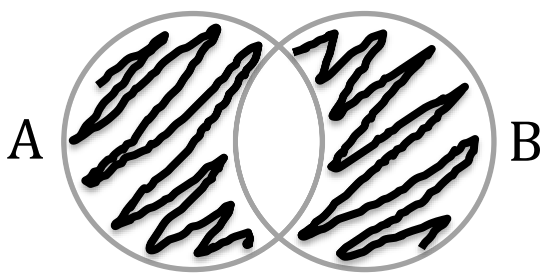
$\Rightarrow A - B = \{\text{shark}, \text{penguin}\}$

The symmetric difference

The symmetric difference between two sets A and B is the set of all the elements in A and all the elements in B except the part they have in common.

It is written $A \oplus B$.

Note: This is analogous to the Boolean XOR-operator.



$$A \oplus B = \{x | x \in (A \cup B) \land x \notin (A \cap B)\}$$

$=$ "The set of elements which are in either A or B, but not both"

<u>Example:</u>

$A = \{\text{shark, whale, penguin, sea lion}\}$;

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

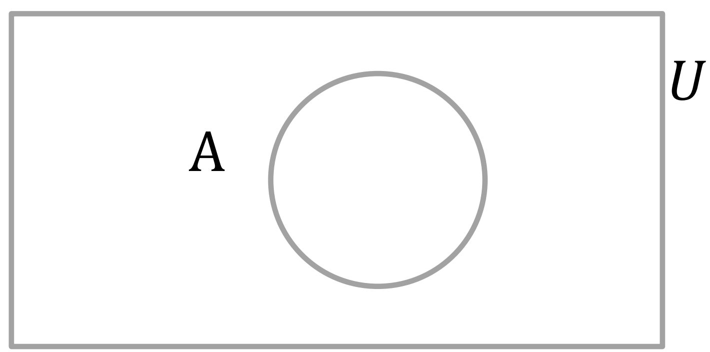
$\Rightarrow A \oplus B = \{\text{shark, penguin, dog, cat, bear}\}$

Universal set

The **universal set** – also called "the universe" – is the *set of all elements* from which any given set is drawn.

This depends on the context!

The universal set is usually denoted U and is drawn as a rectangle.



<u>Examples:</u>

For the set $\{-5, -3, 2, 3, 4\}$, $U = \mathbb{Z}$.

For the set $\{0.1, \pi, 5, 8.358\}$, $U = \mathbb{R}$.

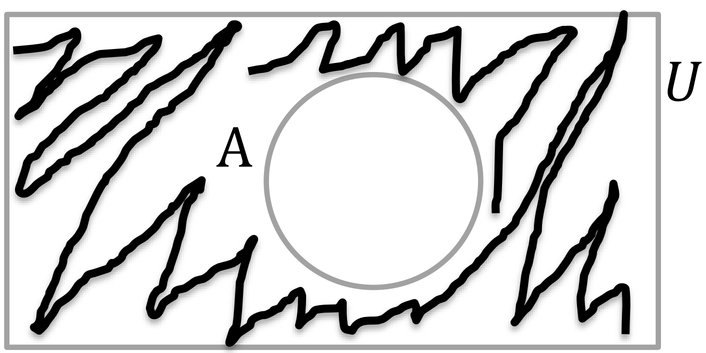
For the set $\{\text{Congo}, \text{Togo}, \text{Egypt}\}$, $U =$ countries

or maybe $U =$ African countries.

The complement

The complement to a set is everything in "the universe", or the universal set, that is not in the set.

It is written A^C (or sometimes $\bar{A}$ or A').



Note: This is analogous to the Boolean NOT-operator.

$$A^C = \{x \in U | x \notin A\}$$

$=$ "The set of elements which are NOT in A"

Examples:

$A = \{3,5\}$ and $U =$ primes below 10 $\Rightarrow A^C = \{2,7\}$

$B = \{\text{red}, \text{blue}\}$ and $U =$ primary colors $\Rightarrow A^C = \{\text{yellow}\}$

$C = \{x | 0 \leq x \leq 2\}$ and $U = \mathbb{R} \Rightarrow C^C = \{x | x < 0 \vee 2 < x\}$

Membership tables

Membership tables

Consider the following question

"An element is in the sets B and in the set C, but not in the set A. Is it in the set $(A \cup C) \cap B$?"

One way to answer any question of this type is to use a so-called **membership table.** A membership table for a combination of the sets A, B and C contains one row for all possible combinations of membership in each of these three sets, as you can see in the next slide.

Example of a membership table

This column represent the set we are interested in.

To answer whether an element is in $(A \cup C) \cap B$, it is helpful first to determine if it is in $A \cup C$.

A	B	C	$A \cup C$	$(A \cup C) \cap B$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

This row represent elements which are neither in A, B nor C.

This row represent elements which are both in B and C but not in A.

This row represent elements which are both in A, B and C.

If an element is neither in A nor C, it is not in $A \cup C$.

If an element is in A but not in C, it is in $A \cup C$.

If an element is in B but not in $A \cup C$, it is not in $(A \cup C) \cap B$.

If an element is in both $A \cup C$ and in B, it is in $(A \cup C) \cap B$.

Example of a membership table

This column represent the set we are interested in.

This row represent elements which are both in B and C but not in A.

A	B	C	$A \cup C$	$(A \cup C) \cap B$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

"An element is in the sets B and in the set C, but not in the set A. Is it in the set $(A \cup C) \cap B$?"

From the table, it is easy to see that the answer is <u>yes!</u>

Showing equalities with membership tables

We can use a membership-table to show that two sets are the same. This will be the case if the columns corresponding to the two sets are identical.

<u>Example: Showing that $A \cap B = B - (B - A)$</u>

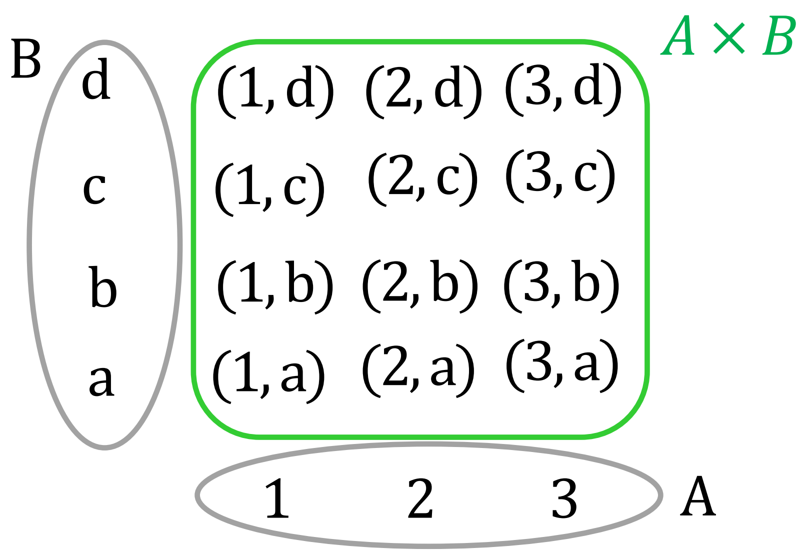
A	B	$A \cap B$	$B - A$	$B - (B - A)$
1	1	1	0	1
1	0	0	0	0
0	1	0	1	0
0	0	0	0	0

For all combinations of membership in A and B, an element is in $B - (B - A)$ if and only if it is in $A \cap B$. This proves that $A \cap B = B - (B - A)$.

Cartesian products and tuples

Cartesian product 1

The Cartesian product between two sets A and B is the combination of every element in A with every element in B.



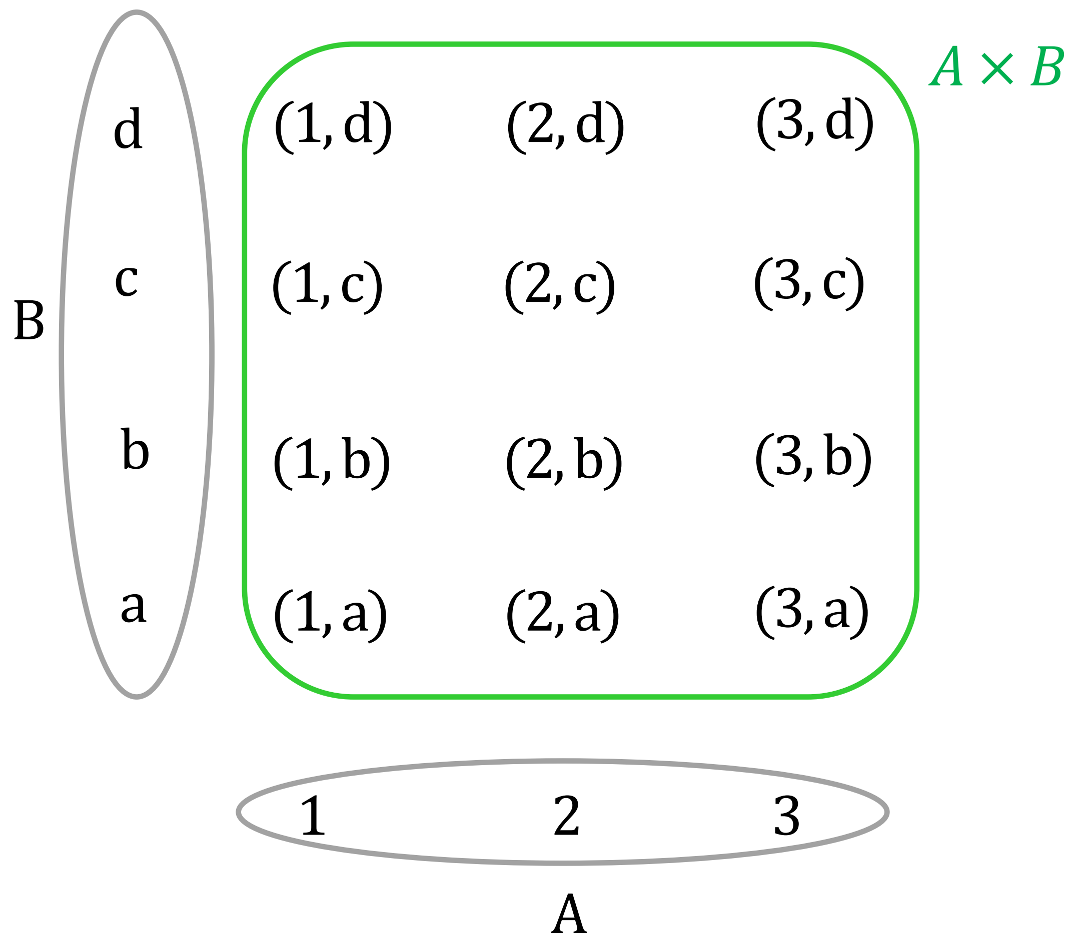
$$A \times B = \{(x, y) | x \in A \land y \in B\}$$

<u>Examples:</u>

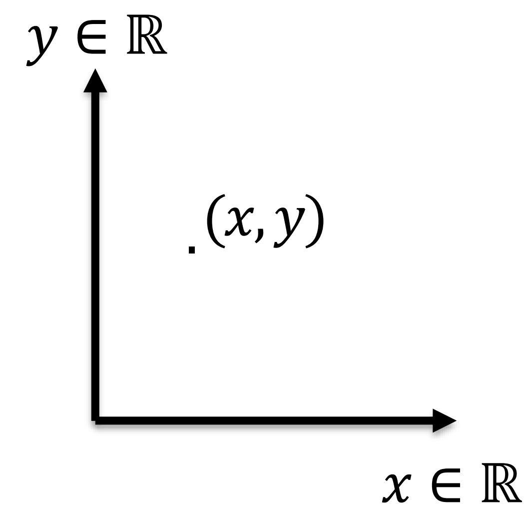
$C = \{0,1\}, D = \{x, y\} \Rightarrow C \times D = \{(0, x), (0, y), (1, x), (1, y)\}$

$S = \{u, v, z\}, T = \{u, v\} \Rightarrow S \times T = \{(u, u), (u, v), (v, u), (v, v), (z, u), (z, v)\}$

Cartesian product : $A \times B$

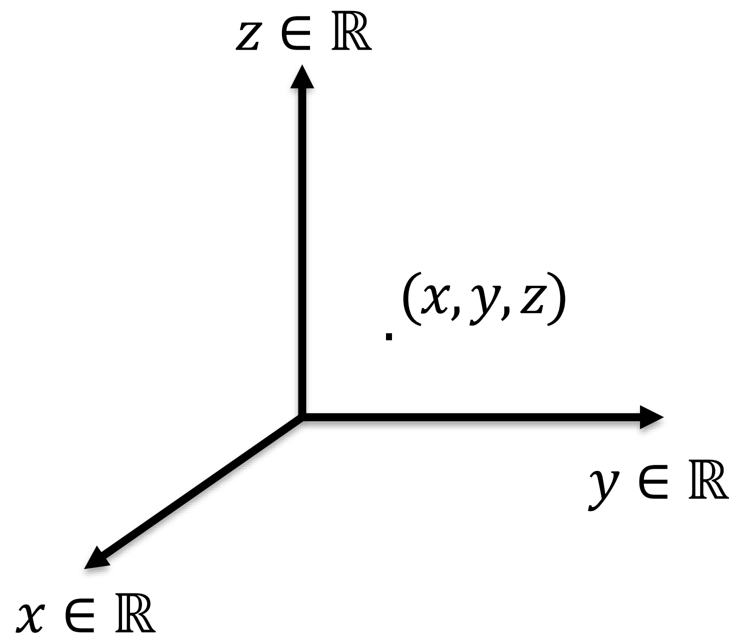


Cartesian coordinate system



Cartesian product between more than two sets

$$A \times B \times C = \{(x, y, z) | x \in A \wedge y \in B \wedge z \in C\}$$



$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) | a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}$$

Tuples

The elements of a Cartesian product are called **tuples.**

An n-tuple contains n elements.

Examples:

2-tuple: $(3,1)$, 3-tuple: $(\text{Denmark}, \text{Norway}, \text{Sweden})$, 5-tuple: (a, b, c, d, e)

A 2-tuple is also called an **ordered pair.**

Unlike sets, tuples are *ordered*, i.e. it matters which element comes first:

For sets: $\{1,2,3\} = \{2,3,1\}$

For tuples: $(1,2,3) \neq (2,3,1)$