Recap - CRV Part 1

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx =$ area under $f(x)$ from a to b for any a and b (4.1)

If X is a *continuous random variable*, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \tag{4.2}$$

Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \tag{4.3}$$

for $-\infty < x < \infty$.

Probability Density Function from the Cumulative Distribution Function

Given $F(x)$,

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

Mean and Variance

Suppose that X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \tag{4.4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx \tag{4.5}$$

LOTUS

Assignment 1 (20%)

Let X denote a continuous stochastic variable with the following probability density function

$$f(x) = \begin{cases} cx^4 & for -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \quad \text{, where } c \text{ is a constant}$$

a) Show that the cumulative probability function of X is

$$F(x) = \begin{cases} 0 & for\ x < -1 \\ \frac{1}{5}c(x^5+1) & for -1 \leq x \leq 1 \\ 1 & for\ x > 1 \end{cases}$$

$$\int_{-1}^{x} c \cdot u^4 \, du = \frac{1}{5} c u^5 \Big|_{-1}^{x} = \frac{1}{5} \cdot c \cdot x^5 - \left(\frac{1}{5} \cdot c(-1)^5\right)$$
$$= \frac{1}{5} c x^5 + \frac{1}{5}c = \frac{1}{5} c(x^5+1)$$

b) Determine the constant c and restate both the probability density function and the cumulative probability function using the actual value of c

$$\int_{-1}^{1} f(x)\,dx = 1 \Leftrightarrow F(1) - F(-1) = 1$$
$$= \frac{2}{5} c = 1 \Rightarrow c = \frac{5}{2}$$

$$f(x) = \begin{cases} 5/2\, x^4 & x \in [-1;1] \\ 0 & \text{else} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x^5+1) & x \in [-1;1] \\ 1 & x > 1 \end{cases}$$

c) Compute $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$ and $P(X > 0)$

d) Find the expected value and variance of X

$$F(\tfrac{1}{2}) - F(-\tfrac{1}{2}) = \frac{1}{2}\left(\frac{1}{32}+1\right) - \frac{1}{2}\left(1 - \frac{1}{32}\right)$$
$$= \underline{\underline{\frac{1}{32}}}$$

$$E[X] = \int_{-1}^{1} x \cdot \frac{5}{2} x^4 \, dx = \frac{5}{12} x^6 \Big|_{-1}^{1} = 0$$

$$1 - F(0) = 1 - \frac{1}{2} = \underline{\underline{1/2}}$$

$$V[X] = Var(X) = \int_{-1}^{1} x^2 \cdot \frac{5}{2} x^4 \, dx - 0^2 = \frac{5}{14} x^7 \Big|_{-1}^{1} = 2 \cdot \frac{5}{14} = \frac{10}{14} = \underline{\underline{\frac{5}{7}}}$$

2)

Assignment 1 (15%)

Compute the expected value, $E(X)$, if X has a density function as follows:

a. $f(x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}} & x > 0 \\ 0 & otherwise \end{cases}$

$$\int_0^\infty x \frac{1}{4} x e^{-\frac{x}{2}} = \frac{-(x^2+4x+8)\cdot e^{-\frac{x}{2}}}{2}\Bigg|_0^\infty$$

$$= 0 - (-4) = \underline{\underline{4}}$$

b. $f(x) = \begin{cases} 5x^{-2} & x > 5 \\ 0 & otherwise \end{cases}$

$$\int_5^\infty x \cdot 5x^{-2} dx = \int_5^\infty 5x^{-1} dx$$

$$= 5 \cdot \log x \Big|_5^\infty = \infty$$

$\frac{1}{e^{\frac{1}{2}x}}$

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

c. If $E(X) = \frac{3}{5}$, find $\boldsymbol{a}$ and $\boldsymbol{b}$.

① $\int_0^1 (a + bx^2) dx = 1 \Rightarrow ax + \frac{1}{3}bx^3 \Big|_0^1 = 1$

$$\underline{a + \frac{1}{3}b = 1} \to a = -\frac{1}{3}b + 1$$

② $\int_0^1 x(a + bx^2) dx = \frac{3}{5} \Leftrightarrow \int_0^1 (ax + bx^3) dx = \frac{3}{5}$

$$= \frac{1}{2}ax^2 + \frac{1}{4}bx^4 \Big|_0^1 = \frac{3}{5}$$

$$= \underline{\frac{1}{2}a + \frac{1}{4}b = \frac{3}{5}}$$

$$\frac{1}{2}\left(-\frac{1}{3}b + 1\right) + \frac{1}{4}b = \frac{3}{5} \Leftrightarrow \underline{\underline{b = \frac{6}{5}}}$$

$$a + \frac{1}{3} \cdot \frac{6}{5} = 1 \Rightarrow \underline{\underline{a = \frac{3}{5}}}$$