Test Catalog

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Mean with Known Variance

Statistical model:

- $X_1, X_2, \ldots, X_n$ are i.i.d. samples of a random variable X with mean μ and variance σ^2.
- Parameter Estimate:

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathrm{N}\left(\mu, \sigma^2/\mathrm{n}\right)$$

- Where the observation is $\bar{x} =$ 'the average of n samples drawn from X 's distribution'.
- NOTE: The statistical model is only true if n is sufficiently large ($n \geq 30$) or if the samples are drawn from a normal population with mean μ and variance σ^2.

Hypothesis test (two-tailed):

- $H_0 : \mu = \mu_0$ ($\mu \leq \mu_0$ if right-tailed test, and $\mu \geq \mu_0$ if left-tailed)
- $H_1 : \mu \neq \mu_0$ ($\mu > \mu_0$ if right-tailed test, and $\mu < \mu_0$ if left-tailed)
- Test statistic: $Z_0 = |\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}| \sim N(0, 1)$
- Critical value: $Z_{\text{crit}} = Z_{1-\frac{\alpha}{2}}$

  Python code: `stats.norm.ppf(`$1 - \frac{\alpha}{2}$`)`
- p-value: $2(1 - \Phi(Z_0))$

  Python code: `2*(1-stats.norm.cdf(`Z_0`))`
- Rejection Criteria
  a. Using Critical value and Test Statistic:

     Reject if $Z_{\text{crit}} < Z_0$
  b. Using p-value and significance level:

     Reject if p-value $< \alpha$
- If you need to make a one-tailed test, replace $\alpha/2$ above and find p-value as $1 - \Phi(Z_0)$

(1-α)-% confidence interval (two-sided): $\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$

(1-α)-% confidence interval (one-sided):

- Upper: $\mu \leq \bar{x} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$
- Lower: $\bar{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}} \leq \mu$

Mean with Unknown Variance

Statistical model:

- $X_1, X_2, \ldots, X_n$ are i.i.d. samples of a random variable X with mean μ and variance σ^2.
- Parameter Estimate:

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathrm{N}\left(\mu, \sigma^2/\mathrm{n}\right)$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- Where the observation is $\bar{x} =$ 'the average of n samples drawn from X 's distribution'.
- NOTE: The statistical model is only true if n is sufficiently large ($n \geq 30$) or if the samples are drawn from a normal population with mean μ and variance σ^2.

Hypothesis test (two-tailed):

- $H_0 : \mu = \mu_0$ ($\mu \leq \mu_0$ if right-tailed test, and $\mu \geq \mu_0$ if left-tailed)
- $H_1 : \mu \neq \mu_0$ ($\mu > \mu_0$ if right-tailed test, and $\mu < \mu_0$ if left-tailed)
- Test statistic: $T_0 = |\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}| \sim t(n-1)$
- Critical value: $T_{\text{crit}} = T_{1-\frac{\alpha}{2}}$

  Python code: `stats.t.ppf`$(1-\frac{\alpha}{2}, n-1)$
- p-value: $2(1 - t_{cdf}(T_0))$

  Python code: `2*(1-stats.t.cdf(`T_0`, n-1))`
- Rejection Criteria

  a. Using Critical value and Test Statistic:

     Reject if $T_{\text{crit}} < T_0$

  b. Using p-value and significance level:

     Reject if p-value $< \alpha$
- If you need to make a one-tailed test, replace $\alpha/2$ above and find p-value as $1 - t_{cdf}(T_0)$

(1-α)-% confidence interval (two-sided): $\bar{x} - t_{1-\alpha/2,n-1}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{1-\alpha/2,n-1}\frac{s}{\sqrt{n}}$

(1-α)-% confidence interval (one-sided):

- Upper: $\mu \leq \bar{x} + t_{1-\alpha,n-1}\frac{s}{\sqrt{n}}$
- Lower: $\bar{x} - t_{1-\alpha,n-1}, \frac{s}{\sqrt{n}} \leq \mu$

Proportion

Statistical model:

- $X \sim$ binomial (n, p)
- Parameter estimate: $\hat{p} = x/n$
- Where the observation is $x =$ 'number of successes out of n trials'
- Where the observation is $\bar{x} =$ 'the average of n samples drawn from X 's distribution'.
- NOTE: The statistical model is only true if $np \geq 5$ and $n(1-p) \geq 5$.

Hypothesis test (two-tailed):

- $H_0 : p = \hat{p} \quad (p \leq \hat{p}$ if right-tailed test, and $\geq \hat{p}$ if left-tailed)
- $H_1 : p \neq \hat{p} \quad (p > \hat{p}$ if right-tailed test, and $p < \hat{p}$ if left-tailed)
- Test statistic: $Z_0 = |Z = \frac{X-np}{\sqrt{np(1-p)}}| = |\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}| \sim N(0,1)$
- Critical value: $Z_{\text{crit}} = Z_{1-\frac{\alpha}{2}}$

  Python code: `stats.norm.ppf(`$1 - \frac{\alpha}{2}$`)`
- p-value: $2(1 - \Phi(Z_0))$

  Python code: `2*(1-stats.norm.cdf(`Z_0`))`
- Rejection Criteria
  a. Using Critical value and Test Statistic:

     Reject if $Z_{\text{crit}} < Z_0$

  b. Using p-value and significance level:

     Reject if p-value $< \alpha$
- If you need to make a one-tailed test, replace $\alpha/2$ above and find p-value as $1 - \Phi(Z_0)$

(1-α)-% confidence interval (two-sided): $\hat{p} - z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

(1-α)-% confidence interval (one-sided):

- Lower: $\hat{p} - z_{1-\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p$
- Upper: $p \leq \hat{p} + z_{1-\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Comparing Means with Known Variance

Statistical model:

- $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim N\left(\mu_2, \sigma_2^2\right)$
- Parameter Estimate:

$$\hat{\delta} = \bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \sigma_1^2/n_1 + \sigma_2^2/n_2\right)$$

- Where the observation is $\bar{x}_1 - \bar{x}_2 =$ 'the difference between two sample means'

Hypothesis test (two-tailed):

- $H_0 : \mu_1 = \mu_2$ $\quad$ ($\mu_1 \leq \mu_2$ if right-tailed test, and $\mu_1 \geq \mu_2$ if left-tailed)
- $H_1 : \mu_1 \neq \mu_2$ $\quad$ ($\mu_1 > \mu_2$ if right-tailed test, and $\mu_< \mu_2$ if left-tailed)
- Test statistic: $Z_0 = | = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} | \sim N(0,1)$
- Critical value: $Z_{\text{crit}} = Z_{1-\frac{\alpha}{2}}$

  Python code: `stats.norm.ppf`$(1 - \frac{\alpha}{2})$
- p-value: $2(1 - \Phi(Z_0))$

  Python code: `2*(1-stats.norm.cdf(`Z_0`))`
- Rejection Criteria

  a. Using Critical value and Test Statistic:

     Reject if $Z_{\text{crit}} < Z_0$

  b. Using p-value and significance level:

     Reject if p-value $< \alpha$

- If you need to make a one-tailed test, replace $\alpha/2$ above and find p-value as $1 - \Phi(Z_0)$

Also, note this infobox from the book:

Tests on the Difference in Means, Variances Known

Null hypothesis: $\quad H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic:

$$Z_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad (10.2)$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\lvert z_0\rvert$ and probability below $-\lvert z_0\rvert$, $P = 2[1 - \Phi(\lvert z_0\rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Confidence Interval on the Difference in Means, Variances Known

If $\bar{x}_1$ and $\bar{x}_2$ are the means of independent random samples of sizes n_1 and n_2 from two independent normal populations with known variances σ_1^2 and σ_2^2, respectively, a $100(1-\alpha)\%$ **confidence interval for $\mu_1 - \mu_2$** is

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad (10.7)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

Comparing Means with Unknown Variance

Pooled Estimator of Variance

The **pooled estimator** of σ^2, denoted by S_p^2, is defined by

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} \tag{10.12}$$

Given the assumptions of this section, the quantity

$$T = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10.13}$$

has a t distribution with $n_1 + n_2 - 2$ degrees of freedom.

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal*

Null hypothesis: $H_0\colon \mu_1 - \mu_2 = \Delta_0$

Test statistic:
$$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10.14}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\lvert t_0\rvert$ and probability below $-\lvert t_0\rvert$	$t_0 > t_{\alpha/2,n_1+n_2-2}$ or $t_0 < -t_{\alpha/2,n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha,n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha,n_1+n_2-2}$

Paired Test

Statistical model:

- Let $(X_{11}, X_{21}), (X_{12}, X_{22}), \ldots, (X_{1n}, X_{2n})$ be a set of n paired observations for which we assume that the mean and variance of the population represented by X_1 are μ_1 and σ_1^2, and the mean and variance of the population represented by X_2 are μ_2 and σ_2^2.
- Define the difference for each pair of observations as $D_j = X_{1j} - X_{2j}, j = 1, 2, \ldots, n$. The D_j 's are assumed to be normally distributed with mean

$$\mu_D = E(X_1 - X_2) = E(X_1) - E(X_2) = \mu_1 - \mu_2$$

and variance σ_D^2, so testing hypotheses about the difference for μ_1 and μ_2 can be accomplished by performing a one-sample t -test on μ_D. Specifically, testing $H_0 : \mu_1 - \mu_2 = \Delta_0$ against $H_1 : \mu_1 - \mu_2 \neq \Delta_0$ is equivalent to testing

$$H_0 : \mu_D = \Delta_0$$
$$H_1 : \mu_D \neq \Delta_0$$

Paired *t*-Test

Null hypothesis: $H_0: \mu_D = \Delta_0$

Test statistic: $$T_0 = \frac{\overline{D} - \Delta_0}{S_D/\sqrt{n}} \qquad (10.24)$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_D \neq \Delta_0$	Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1: \mu_D > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1: \mu_D < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

Test for Independence

Statistical model:

- Let p_{ij} be the probability that a randomly selected element falls in the ijth cell given that the two classifications are independent.
- Then $p_{ij} = u_i v_j$, where u_i is the probability that a randomly selected element falls in row class i and v_j is the probability that a randomly selected element falls in column class j. Now by assuming independence, the estimators of u_i and v_j are

$$\hat{u}_i = \frac{1}{n}\sum_{j=1}^{c} O_{ij} \quad \hat{v}_j = \frac{1}{n}\sum_{i=1}^{r} O_{ij}$$

- Therefore, the expected frequency of each cell is

$$E_{ij} = n\hat{u}_i\hat{v}_j = \frac{1}{n}\sum_{j=1}^{c} O_{ij}\sum_{i=1}^{r} O_{ij}$$

Hypothesis test:

- H_0: The two categorical variables under examination are independent.
- H_1: The two categorical variables under examination are not independent.
- Test statistic: $\chi_0^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}}$
- Critical value: $\chi^2_{\text{crit}} = \chi^2_{1-\alpha,df}$, where $df = (r-1)(c-1)$ degree of freedom

  Python code: `stats.chi2.ppf(`$1-\alpha, df$`)`
- p-value: $1 - \chi^2_{\text{cdf}}(\chi_0^2, df)$

  Python code: `1-stats.chi2.cdf(`χ_0^2, df`)`
- Rejection Criteria
  a. Using Critical value and Test Statistic:

     Reject if $\chi^2_{\text{crit}} < \chi_0^2$

  b. Using p-value and significance level:

     Reject if p-value $< \alpha$