Exercise 6:

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = X^2$.

(a) Find CDF of Y.

(b) Find PDF of Y.

(c) Find EY.

Solution:

(a) First, we note that $R_Y = [0, 4]$. As usual, we start with the CDF. For $y \in [0, 4]$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(0 \leq X \leq \sqrt{y}) \quad \text{since } x \text{ is not negative} \\ &= \int_0^{\sqrt{y}} \frac{5}{32}x^4 dx \\ &= \frac{1}{32}(\sqrt{y})^5 \\ &= \frac{1}{32}y^2\sqrt{y} \end{aligned}$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{1}{32}y^2\sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 1 & \text{for } y > 4. \end{cases}$$

(b)

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{5}{64}y\sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(c)

To find the EY, we can directly apply LOTUS,

$$\begin{aligned} E[Y] = E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^2 x^2 \cdot \frac{5}{32}x^4 dx \\ &= \int_0^2 \frac{5}{32}x^6 dx \\ &= \frac{5}{32} \times \frac{1}{7} \times 2^7 = \frac{20}{7}. \end{aligned}$$

Exercise 7:

Consider two random variables X and Y with joint PMF given in Table 1.

Table 1: Joint PMF of X and Y in Problem prob:table-cov

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Find $\text{Cov}(X,Y)$ and $\rho(X,Y)$.

Solution:

First, we find PMFs of X and Y:

$R_X = \{0,1\}$ $\quad P_X(0) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{4+6+3}{24} = \frac{13}{24}$ $\quad P_X(1) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$

$R_Y = \{0,1,2\}$ $\quad P_Y(0) = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$

$P_Y(1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ $\quad P_Y(2) = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$

$$EX = 0 \cdot \frac{13}{24} + 1 \cdot \frac{11}{24} = \frac{11}{24}$$

$$EY = 0 \cdot \frac{7}{24} + 1 \cdot \frac{5}{12} + 2 \cdot \frac{7}{24} = 1$$

$$EXY = \sum ijP_{XY}(i,j) = 0 + 1 \cdot 0 \cdot \frac{1}{8} + 1 \cdot 1 \cdot \frac{1}{6} + 1 \cdot 2 \cdot \frac{1}{6} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Therefore:

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= EXY - EX \cdot EY = \frac{1}{2} - \frac{11}{24} \cdot 1 = \frac{1}{24} \\
\mathrm{Var}(X) &= EX^2 - (EX)^2 \\
EX^2 &= \frac{11}{24} \\
\mathrm{Var}(X) &= \frac{11}{24} \cdot \frac{13}{24} \\
\to \sigma_X &= \frac{\sqrt{11 \times 13}}{24} \approx 0.498 \\
EY^2 &= 0 \cdot \frac{7}{24} + 1 \cdot \frac{5}{12} + 4 \cdot \frac{7}{24} = \frac{19}{12} \\
\mathrm{Var}(Y) &= \frac{19}{12} - 1 = \frac{7}{12} \\
\to \sigma_Y &= \sqrt{\frac{7}{12}} \approx 0.76 \\
\to \rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \\
&= \frac{\frac{1}{24}}{\frac{\sqrt{11\times 13}}{24} \cdot \sqrt{\frac{7}{12}}} \approx 0.11
\end{aligned}
$$

Exercise 8:

Let X and Y be two independent $N(0,1)$ random variables and

$$
\begin{aligned}
&Z = 11 - X + X^2 Y \\
&W = 3 - Y.
\end{aligned}
$$

Find $\mathrm{Cov}(Z,W)$.

Solution:

$$
\begin{aligned}
\mathrm{Cov}(Z,W) &= \mathrm{Cov}(11 - X + X^2Y, 3 - Y) \\
&= \mathrm{Cov}(-X + X^2Y, -Y) = \mathrm{Cov}(X,Y) - \mathrm{Cov}(X^2Y, Y) \\
&= -\mathrm{Cov}(X^2Y, Y) \quad (\text{Since } X \text{ and } Y \text{are indep } \mathrm{Cov}(X,Y) = 0) \\
&= -E[X^2Y \cdot Y] + E[X^2Y] \cdot E[Y] \qquad (EY = 0) \\
&= -E[X^2Y^2] = -EX^2 \cdot EY^2 = -1
\end{aligned}
$$

Exercise 9:

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X^n)$, for $n = 1, 2, 3, \cdots$.

(b) Find variance of X.

Solution:

(a)

Using LOTUS we have

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= \int_0^1 x^n (x^2 + \frac{2}{3}) dx \\ &= \int_0^1 (x^{n+2} + \frac{2}{3} x^n) dx \\ &= \left[\frac{1}{n+3} x^{n+3} + \frac{2}{3(n+1)} x^{n+1} \right]_0^1 \\ &= \frac{1}{n+3} + \frac{2}{3(n+1)} \\ &= \frac{5n+9}{3(n+1)(n+3)}. \quad \text{where } n = 1, 2, 3, \cdots \end{aligned}$$

(b)

We know that

$$\text{Var}(X) = EX^2 - (EX)^2.$$

So we need to find the values of EX and EX^2

$$E[X] = \frac{7}{12}$$

$$E[X^2] = \frac{19}{45}$$

Thus, we have

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{19}{45} - (\frac{7}{12})^2 = 0.0819.$$

Exercise 10:

We get

$$f_X(x) = \begin{cases} \frac{3}{8}x^2 & \text{hvis } 0 < x < 2 \\ 0 & \text{ellers.} \end{cases}$$

a

$$\begin{aligned} P(x > 3/4) &= 1 - \int_0^{3/4} \frac{3}{8}x^2 dx \\ &= 1 - \frac{1}{8}x^3 \Big|_0^{3/2} \\ &= 1 - \frac{1}{8}\left((3/4)^3\right) \\ &= 1 - \frac{1}{8} \cdot \frac{27}{64} = 1 - \frac{27}{512} \approx 0.9473 \end{aligned}$$

$$\begin{aligned} P(x < 3/4) &= 1 - P(x > 3/4) \\ &\approx 0.0527 \end{aligned}$$

b

$$\begin{aligned} E[(X+2)/3] &= \frac{1}{3}E(X) + \frac{2}{3} \\ &= \frac{1}{3}\int_0^2 x \cdot \frac{3}{8}x^2 dx + \frac{2}{3} \approx 1.667 \end{aligned}$$

$$E[(X^2] = \int_0^2 x^2 \cdot \frac{3}{8}x^2 dx \approx 2.4$$

c We first find the CDF og derive this to get the PDF. We note that $R_Z = (1, e^4)$, and for $y \in (1, e^4)$ we get

$$\begin{aligned} F_Z(z) &= P(Z < z) \\ &= P\left(e^{2X} < z\right) \\ &= P\left(X < \frac{\ln z}{2}\right) \\ &= \int_1^{\frac{\ln z}{2}} \frac{3}{8}x^2 dx = \frac{1}{8}\left(\frac{\ln z}{2}\right)^3 - = \frac{1}{64}(\ln z)^3 - \frac{1}{8} \end{aligned}$$

Now we get:

$$\begin{aligned} f_Z(z) &= F_Z'(z) \\ &= \frac{3\ln z}{64z} \end{aligned}$$

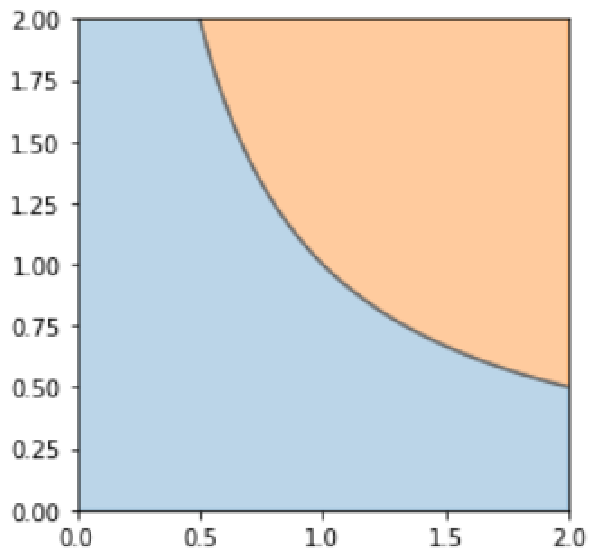
Thus,

$$f_Z(y) = \begin{cases} \frac{3}{64} \cdot z^{-1} \ln z & 1 < z < e^4 \\ 0 & \text{ellers} \end{cases}$$

d

$$\begin{aligned} \operatorname{Cov}(2X+3Y, X-4Y+9) &= 2\operatorname{Cov}(X, X-4Y) + 3\operatorname{Cov}(Y, X-4Y) \\ &= 2(\operatorname{Cov}(X,X) - 4\operatorname{Cov}(X,Y)) + 3(\operatorname{Cov}(Y,X) - 4\operatorname{Cov}(Y,Y)) \\ &= 2\operatorname{Var}(X) - 8\operatorname{Cov}(X,Y) + 3\operatorname{Cov}(X,Y) - 12\operatorname{Var}(Y) \\ &= 2\operatorname{Var}(X) - 12\operatorname{Var}(Y) \\ &= -3/2 \end{aligned}$$

e We choose to work on the orange area and then take the complement. We could have found the blue area directly.



$$
\begin{aligned}
P(XY < 1) &= P\left(Y < \frac{1}{X}\right) \\
&= 1 - \int_{1/2}^{2} \int_{1/x}^{2} \frac{9x^2y^2}{64} dydx \\
&= 1 - \frac{3}{64} \cdot \left(\frac{8}{3}x^3 - \ln x\right)\Bigg|_{1/2}^{2} \\
&\approx 0.08
\end{aligned}
$$