4. Consider two random variables X and Y with joint PMF given by

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}, \qquad \text{for } k,l = 1,2,3,...$$

(a) Show that X and Y are independent and find the marginal PMFs of X and Y.

(b) Find $P(X^2 + Y^2 \leq 10)$.

Solution:

(a)

$$\begin{aligned} R_X &= \{1,2,3,\cdots\} \\ R_Y &= \{1,2,3,\cdots\} \end{aligned}$$

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}.$$

$$\begin{aligned} P_X(k) &= \sum_{l \in R_Y} P(X=k, Y=l) = \sum_{l=1}^{\infty} \frac{1}{2^{k+l}} \\ &= \frac{1}{2^k} \sum_{l=1}^{\infty} \frac{1}{2^l} = \frac{1}{2^k}. \end{aligned}$$

$$\begin{aligned} P_Y(l) &= \sum_{k \in R_X} P(X=k, Y=l) = \sum_{k=1}^{\infty} \frac{1}{2^{k+l}} \\ &= \frac{1}{2^l} \sum_{k=1}^{\infty} \frac{1}{2^l} = \frac{1}{2^l}. \end{aligned}$$

By calculating the marginal PMFs we observe that $P_{XY}(k,l) = P_X(k) \cdot P_Y(l)$ for all $k \in R_X$ and $l \in R_Y$ $(k,l = 1,2,3,\cdots)$. So, these two variables are independent.

(b)

There are different cases in which $X^2 + Y^2 \leq 10$:

$$\begin{aligned} P(X^2 + Y^2 \leq 10) &= P(X=1,Y=1) + P(X=1,Y=2) + P(X=2,Y=1) \\ &+ P(X=2,Y=2) + P(X=1,Y=3) + P(X=3,Y=1) \\ &= \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} = \frac{11}{16}. \end{aligned}$$

5. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2} & 0 \leq x, \ 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

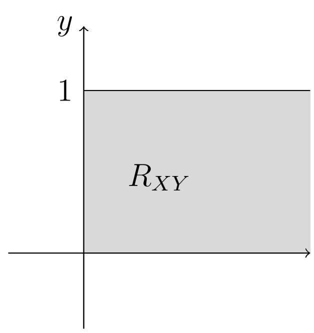
(a) Find the constant c.

(b) Find $P(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.

(c) Find $P(0 \leq X \leq 1)$.

Solution:

(a) We have:

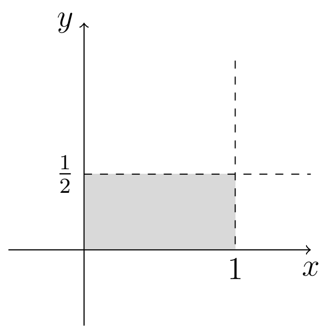


$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 \\ &= \int_{y=0}^{1}\int_{x=0}^{\infty}\left(\frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2}\right)dxdy \\ &= \int_{0}^{1}\left[-\frac{1}{2}e^{-x} - \frac{cy}{(1+x)}\right]_{0}^{\infty}dy \\ &= \int_{0}^{1}(\frac{1}{2} + cy)dy \\ &= \left[\frac{1}{2}y + \frac{1}{2}cy^2\right]_{0}^{1} \\ &= \frac{1}{2} + \frac{1}{2}c \end{aligned}$$

Thus, $c = 1$.

(b)

$$
\begin{aligned}
P(0 \le X \le 1, 0 \le Y \le \frac{1}{2}) \\
&= \int_{y=0}^{\frac{1}{2}} \int_{x=0}^{1} \frac{1}{2}e^{-x} + \frac{y}{(1+x)^2} dxdy \\
&= \int_{0}^{\frac{1}{2}} \left[-\frac{1}{2}e^{-x} - \frac{y}{1+x} \right]_0^1 dy \\
&= \int_{0}^{\frac{1}{2}} \left[(\frac{1}{2} + y) - (\frac{1}{2}e^{-1} + \frac{y}{2}) \right] dy \\
&= \frac{5}{16} - \frac{1}{4e}
\end{aligned}
$$



(c)

$$
\begin{aligned}
P(0 \le X \le 1) &= \int_{y=0}^{1} \int_{x=0}^{1} \left(\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2} \right) dxdy \\
&= \frac{3}{4} - \frac{1}{2e}
\end{aligned}
$$

6. Let X and Y be two jointly continuous random variables with joint PDF

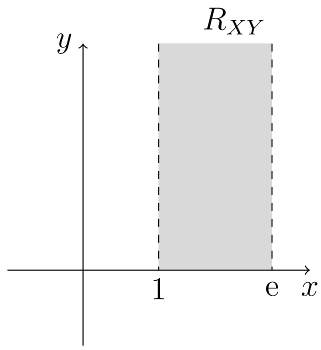
$$
f_{XY}(x,y) = \begin{cases} e^{-xy} & 1 \le x \le e, \;\; y > 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

(a) Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.

(b) Write an integral to compute $P(0 \le Y \le 1, 1 \le X \le \sqrt{e})$.

Solution:

(a) We have:



for $1 < x < e$:

$$\begin{aligned} f_X(x) &= \int_0^\infty e^{-xy} dy \\ &= -\frac{1}{x} e^{-xy} \Big|_0^\infty \\ &= \frac{1}{x} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{1}{x} & 1 \le x \le e \\ 0 & \text{otherwise} \end{cases}$$

for $0 < y$

$$\begin{aligned} f_Y(y) &= \int_1^e e^{-xy} dx \\ &= \frac{1}{y}(e^{-y} - e^{-ey}) \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{y}(e^{-y} - e^{-ey}) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} P(0 \le Y \le 1, 1 \le X \le \sqrt{e}) &= \int_{x=1}^{\sqrt{e}} \int_{y=0}^{1} e^{-xy} dy dx \\ &= \frac{1}{2} - \int_1^{\sqrt{e}} \frac{1}{x} e^{-x} dx \end{aligned}$$

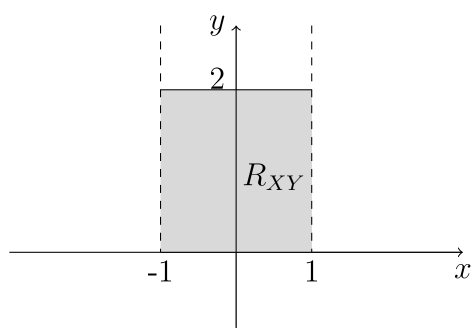
7. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}x^2 + \frac{1}{6}y & -1 \le x \le 1, \ \ 0 \le y \le 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.
(b) Find $P(X > 0, Y < 1)$.
(c) Find $P(X > 0 \text{ or } Y < 1)$.
(d) Find $P(X > 0 | Y < 1)$.
(e) Find $P(X + Y > 0)$.

Solution:

(a) for $-1 \le x \le 1$



$$\begin{aligned} f_X(x) &= \int_0^2 (\frac{1}{4}x^2 + \frac{1}{6}y)dy \\ &= \frac{1}{2}x^2 + \frac{1}{3} \end{aligned}$$

Thus,

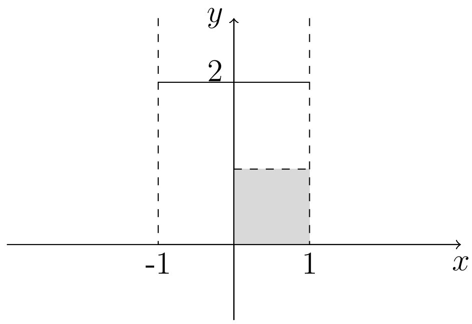
$$f_X(x) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{3} & -1 \le x \le 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le y \le 2$

$$\begin{aligned} f_Y(y) &= \int_{-1}^1 \left(\frac{1}{4}x^2 + \frac{1}{6}y\right) dx \\ &= \frac{1}{6} + \frac{1}{3}y \end{aligned}$$

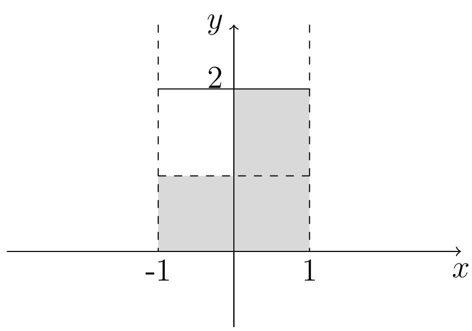
$$f_Y(y) = \begin{cases} \frac{1}{6} + \frac{1}{3}y & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) We have:



$$\begin{aligned} P(X > 0, Y < 1) &= \int_{x=0}^{1} \int_{y=0}^{1} \left(\frac{1}{4}x^2 + \frac{1}{6}y \right) dydx \\ &= \frac{1}{6} \end{aligned}$$

(c) We have:



$$\begin{aligned} P(X > 0 \quad \text{or} \quad Y < 1) &= 1 - P(X < 0 \quad \text{and} \quad Y > 1) \\ &= 1 - \int_{x=-1}^{0} \int_{y=1}^{2} \left(\frac{1}{4}x^2 + \frac{1}{6}y \right) dydx \\ &= \frac{2}{3} \end{aligned}$$

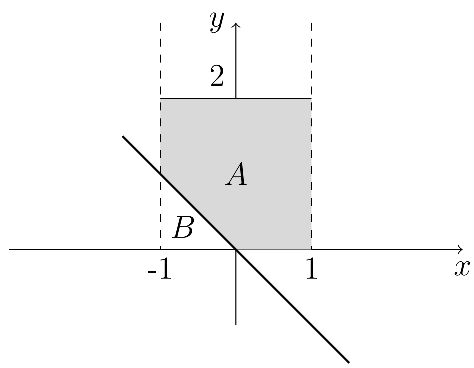
(d)

$$\begin{aligned}
P(X > 0 | Y < 1) &= \frac{P(X > 0 \quad \text{and} \quad Y < 1)}{P(Y < 1)} \\
&= \frac{1}{6P(Y < 1)}
\end{aligned}$$

$$\begin{aligned}
P(Y < 1) &= \int_0^1 (\frac{1}{3}y + \frac{1}{6})dy \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, $P(X > 0 | Y < 1) = \frac{1}{2}$.

(e) We have:



$$\begin{aligned}
P(X + Y > 0) &= P((x, y) \in A) \\
&= 1 - P(B) \\
&= 1 - \int_{x=-1}^{0} \int_{y=0}^{-x} \left(\frac{1}{4}x^2 + \frac{1}{6}y \right) dydx \\
&= \frac{131}{144}
\end{aligned}$$