Gør tanke til handling

VIA University College

Combinatorics

- counting methods

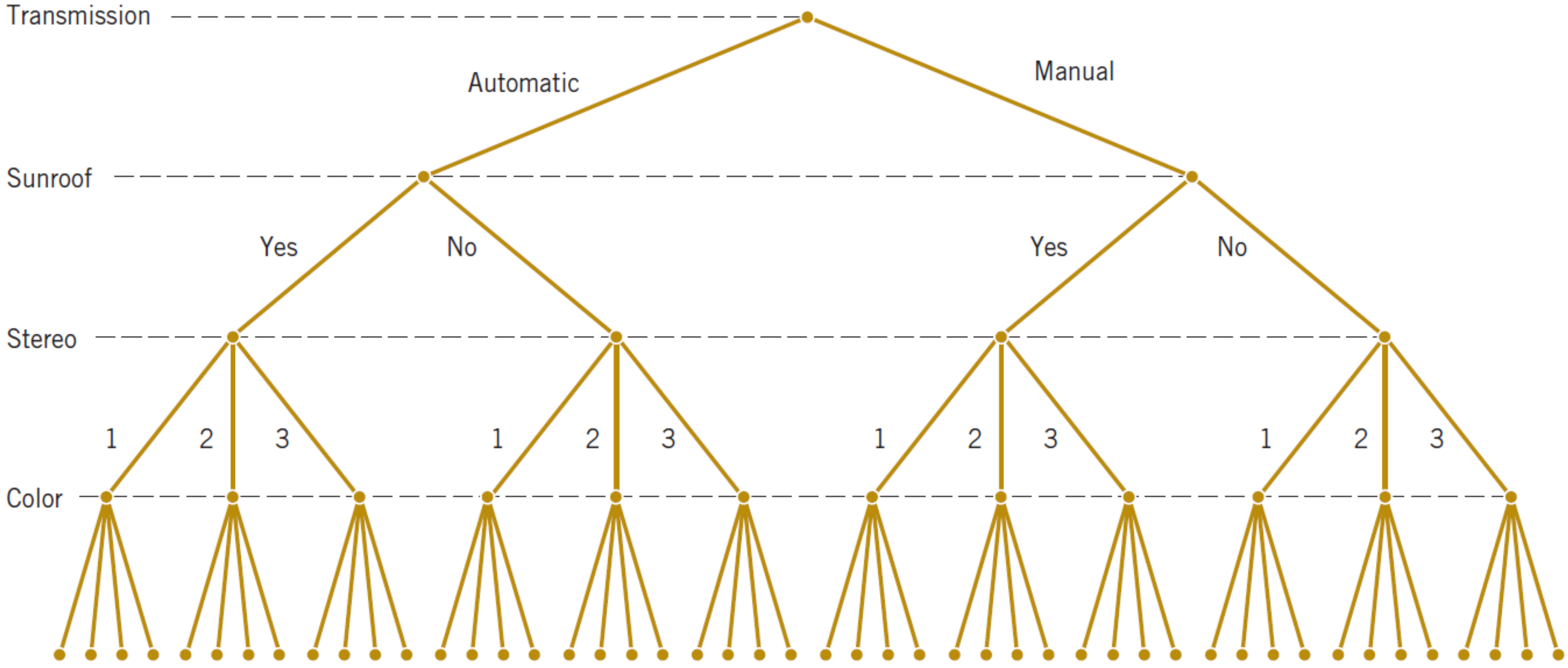
Example

You want to bye a car, and you have to make the following choices:

1) With manual or automatic transmission?
2) With or without a sunroof?
3) Which of three possible stereo systems?
4) Which one of four exterior colors?

Consider the set of all possible cars given the above choices. How do you calculate the number of elements in this set?

The multiplication rule



We calculate the total number of possible cars by multiplying the number of possibilities at each step:

First you have 2 possible choices of transmission, then 2 possible choices of sunroof, then 3 possible choices of stereo and then 4 possible number of choices of color. In total, that gives you

$2 \cdot 2 \cdot 3 \cdot 4 = 48$ possible cars.

Counting techniques

The "multiplication rule" is the fundamental technique we use to count the number of possibilities in many different scenarios.

In order to figure which numbers we have to multiply, we distiguish between whether we are dealing with

Permutations (order does matter) or **combinations** (order doesn't matter), and whether each step is **with** or **without replacement.**

In the next slide, we will see examples of each of the possible scenarios. We will then elaborate on these in the following slides.

Permutations vs. combinations and with vs. without replacement

	Permutations	Combinations
With replacement	1 The set of all 6-digit passwords with only lowercase letters (26 letters in the alphabet).	4 The set of all 4-scoup icecreams when there are 7 possible flavors.
Without replacement	2 The set of all different Scrum teams of 3 people that can be formed in a class with 30 students, when the first person becomes Product owner, the second person becomes Scrum Master and the third person becomes the Development Team.	3 The set of all different groups of 3 people that can be formed in a class with 30 students.

1

The set of all 6-digit passwords with only lowercase letters.

$$\square\square\square\square\square\square$$

This is **permutation** because the order of the digits matter, and it is **with replacement** because you can use the same letter as many times as you want.

There are 26 possibilities for the first digit, 26 possibilities for the second digit, …., and 26 possibilities for the last digit. So, using the multiplication rule, we get a total of

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$$

possible passwords.

2

The set of all different Scrum teams of 3 people that can be formed in a class with 30 students, when the first person becomes Product owner, the second person becomes Scrum Master and the third person becomes the Development Team.

This is **permutation** because the order of the students matter, and it is **without replacement** because you cannot choose the same student more than once.

There are 30 possibilities for the first student (Product owner), 29 possibilities for the second student (Scrum Master), and 28 possibilities for the third student (Development Team). So, using the multiplication rule, we get a total of

$$30 \cdot 29 \cdot 28 = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot \ldots \cdot 1}{27 \cdot 26 \cdot \ldots \cdot 1} = \frac{30!}{27!}$$

Note: factorial

In combinatorics, we will make extensive use of the mathematical operation "factorial", which is defined as follows:

$$N! = N \cdot (N-1) \cdot (N-2) \cdot \ldots 3 \cdot 2 \cdot 1$$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

We often use the "trick" to use factorial to write e.g. $100 \cdot 99 \cdot 98 \cdot 97$ as $\frac{100!}{96!}$:

$$\frac{100!}{96!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot 95 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{96 \cdot 95 \cdot \ldots \cdot 3 \cdot 2 \cdot 1} = 100 \cdot 99 \cdot 98 \cdot 97$$

3

The set of all different groups of 3 people that can be formed in a class with 30 students.

This is **combination** because the order of the students doesn't matter, and it is **without replacement** because you cannot choose the same student more than once.

We know from the last slide that there are $\frac{30!}{27!}$ different permutations of three students. When dealing with permutations, we distinguish between the groups Alice-Bob-Eve and Bob-Alice-Eve. However, if the order doesn't matter, we would not count these as different groups. Therefore, we need to divide by the number of different permutations there is of a group of 3 students, which is $3!$. So we get

$$\frac{30!}{27! \cdot 3!}$$

4

The set of all 4-scoop icecreams when there are 7 possible flavors.

This is **combination** because the order of the scoops doesn't matter, and it is **with replacement** because you can choose the same flavor as many times as you want.

This is the toughest of the four cases, because we have to account for several different types of double counting. The result turns out to be

$$\frac{(4+7-1)!}{4!\,(7-1)!}$$

Proving the formula for this case is out of scope for the course.

Formulas for the different cases

In the next slide, we generalize the logic from the previous slides in a set of formulas that can be used in each of the 4 different cases.

Formulas

In all cases, n is the number to choose from and r is the number we choose.

	Permutations	Combinations
With replacement	1 n^r	4 $\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$
Without replacement	2 $P_r^n = \frac{n!}{(n-r)!}$	3 $C_r^n = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

This is called a "binomial coefficient" or "from n choose r"

Mixed cases

Note that you cannot always just apply one of the formulas from the previous slide! For instance, consider the question below:

> A hot dog can contain each of the 4 condiments mustard, ketchup, mayonnaise, and relish, and each of the 6 garnishes onions, sauerkraut, chili, cheese, coleslaw, and olives. The order of the condiments is irrelevant. So is the order of the garnishes, with the exception that onions (if chosen) has to be on top.
>
> How many different hot dogs can be made with
> 2 different condiments and 3 different garnishes?

It is important that you understand the logic behind the different methods of counting, in order to be able to solve such exercises! (the solution is in the next slide)

Mixed case - example

A hot dog can contain each of the 4 condiments mustard, ketchup, mayonnaise, and relish, and each of the 6 garnishes onions, sauerkraut, chili, cheese, coleslaw, and olives. The order of the condiments is irrelevant. So is the order of the garnishes, with the exception that onions (if chosen) has to be on top.

How many different hot dogs can be made with
2 different condiments and 3 different garnishes?

Solution: There are $C_2^4 = \frac{4!}{(4-2)!\cdot 2!} = 6$ possible combinations of condiments. We multiply this by the number of possible combinations of garnishes, where we distinguish between the cases with and without onions. There are $C_3^5 = \frac{5!}{(5-3)!\cdot 3!} = 10$ possibilities without onions and $C_2^5 = \frac{5!}{(5-2)!\cdot 2!} = 10$ possibilities with onions. In total, we get $6 \cdot (10 + 10) = 120$ possible hot dogs.