Frequent Direction Algorithms for Approximate Matrix Multiplication with Applications in CCA

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Outline

Introduction

Approximate matrix multiplication

we are given A, B each with a large number of rows n, and the goal is to compute some matrix X, s.t.

$$\|A^T B - X\|_X$$

is small, for some matrix norm $\|\cdot\|_X$.

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- Information retrieval
- Approximate leverage scores
- Large-scale k-means clustering

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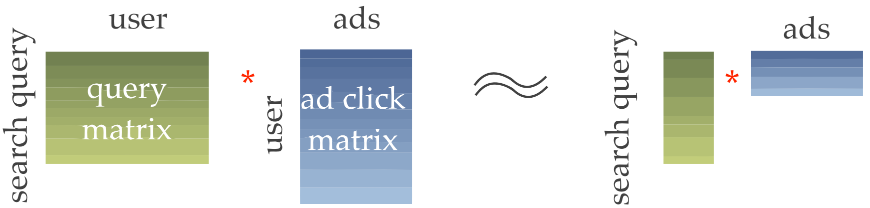
Approximate matrix multiplication



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Streaming Matrices

- Matrix too large to be stored.
- Streaming setting:
  - Use a small amount of memory.
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$$A^T B = \sum_i^n a_i^T b_i$$

Naive solution

Compute $A^T B$ in time $\mathcal{O}(nm_1 m_2)$.

Outline

Random Selection

- Random Selection. It randomly picks ℓ rows of A and B with probability

$$p_i = \|a_i\|\|b_i\| / \sum_{i=1}^{n} \|a_i\|\|b_i\|,$$

and with high probability,

$$\|A^T B - C^T D\|_2 \leq \frac{\mathcal{O}(1)}{\sqrt{\ell}} \|A\|_F \|B\|_F.$$

Random Selection

Get C and D in time $\mathcal{O}(n(m_1 + m_2)))$.

Outline

Random Projection

- Random Projection. Let $\Pi \in \mathbb{R}^{\ell \times n}$ be some subspace embedding, then

$$\|A^T B - (\Pi A)^T (\Pi B)\|_2 \leq \sqrt{\frac{\mathcal{O}(r)}{\ell}} \|A\|_2 \|B\|_2$$

holds with high probability, where $r = rank(A) + rank(B)$. For fast or sparse subspace embedding, ΠA and ΠB can be computed quickly.

Random Projection

For dense Π, get C and D in time $\mathcal{O}(n\ell(m_1 + m_2))$.
For sparse Π, get C and D in time $\mathcal{O}(nnz(A) + nnz(B))$.

Outline

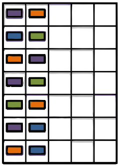
Frequent Directions

FD (Edo Liberty 2013)

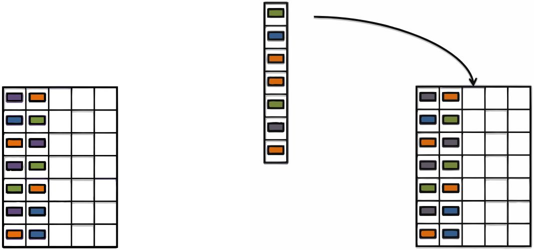
Give $A \in \mathbb{R}^{d \times n}$, FD efficiently maintains a matrix B with only ℓ columns s.t.

$$\|AA^T - BB^T\|_2 \leq 2\|A\|_F^2/\ell,$$

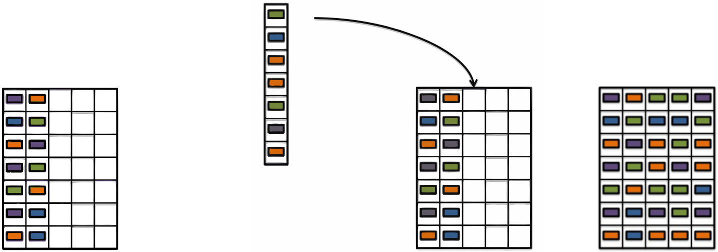
The FD algorithm runs in time $\mathcal{O}(nd\ell)$.



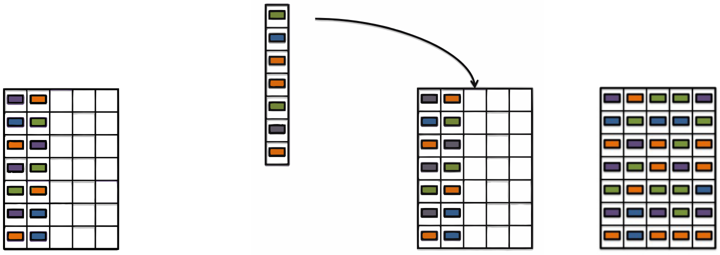
We keep a sketch of ℓ columns.

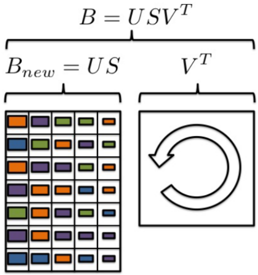


Input vectors are simply stored in empty columns.

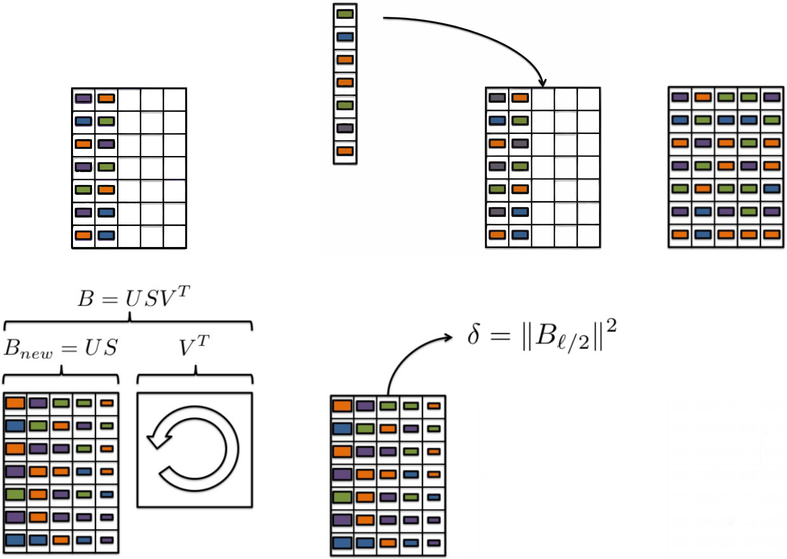


When the sketch is 'full' we need to zero out some columns.

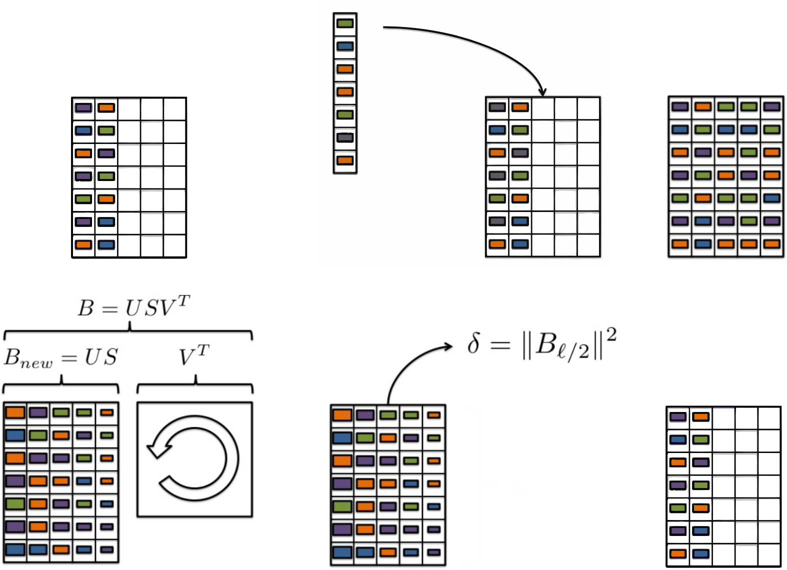




Using the SVD we compute $B = USV^T$ and set $B_{new} = US$.



Let $\sigma = \|B_{\ell/2}\|^2$.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V^T
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FD-AMM

Algorithm 2 FD-AMM

Input: ℓ, $A \in R^{n\times m_1}$, $B \in \mathbb{R}^{n\times m_2}$

$G = [A \ B]$

$H = \mathrm{FD}(\ell, G)$

$C = H_{[\ell],1:m_1}$

$D = H_{[\ell],m_1+1:m_1+m_2}$

return C, D

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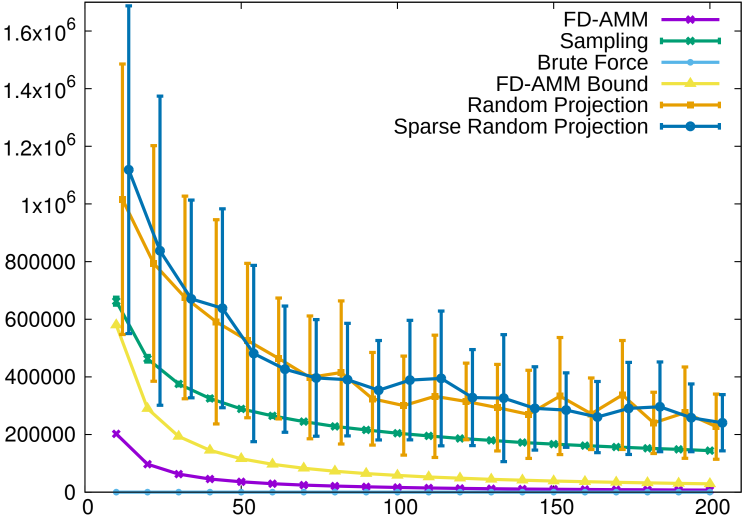
Theorem

If X is the result of applying the FD-AMM algorithm to matrices A, B, and sketch size ℓ, then

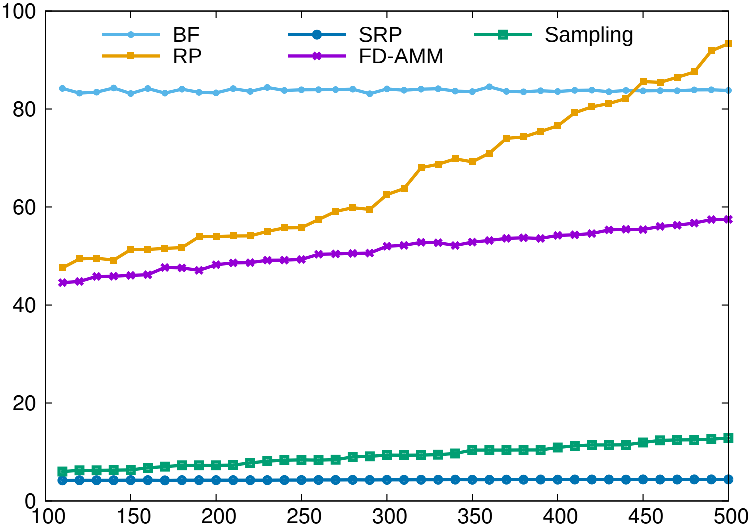
$$\|A^T B - C^T D\|_2 \leq (\|A\|_F^2 + \|B\|_F^2)/\ell.$$

Experiments

$\|A^T B - C^T D\|_2$ as a function of the sketch size ℓ.



Running time in second as a function of ℓ.



Thank You