Solutions for Homework 1

1. If $\lim_{n\to\infty} a_n = a$, show that $\lim_{n\to\infty} \left(1+\frac{a_n}{n}\right)^n = e^a$.

   Proof.
$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n/a_n}\right)^{n/a_n}\right)^{a_n} = e^a$$
□

2. If $nt > -1$, then $(1-t)^n \geq 1 - nt$.
3. If $-x < n < m$, then $(1+\frac{x}{n})^n \leq (1+\frac{x}{m})^m$.

   Proof.
$$(1+\frac{x}{n})^n \leq (1+\frac{x}{m})^m \Longleftrightarrow n\log(1+\frac{x}{n}) \leq m\log(1+\frac{x}{m})$$
   Consider function $f(t) = t\log(1+\frac{x}{t})$, $t > 0$.
$$\begin{aligned}\frac{df}{dt} &= \log(1+\frac{x}{t}) - \frac{x}{x+t}\\ \frac{d^2f}{dt^2} &= -\frac{x^2}{(t+x)^2t} \leq 0\end{aligned} \tag{1}$$
   Notice that $\lim_{t\to\infty}\frac{df}{dt} = 0$, so $\frac{df}{dt} \geq 0$. □

4. Compute the following integrals:
   (a) $u_0 = \int_{-\infty}^{\infty}\Phi(x)\mathcal{N}(x|\mu,\sigma^2)dx$
   (b) $u_1 = \int_{-\infty}^{\infty}\Phi(x)\mathcal{N}(x|\mu,\sigma^2)xdx$
   (c) $u_2 = \int_{-\infty}^{\infty}\Phi(x)\mathcal{N}(x|\mu,\sigma^2)(x-m_1)^2dx$
   where $\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$

   (a)
$$\begin{aligned}u_0 &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt\right)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\sigma y+\mu}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy\\ &\quad (\text{use } y=(x-\mu)/\sigma - \text{ to replace x, then } Y\sim\mathcal{N}(0,1))\\ &= \int_{-\infty}^{\infty}\Phi(\sigma y+\mu)\mathcal{N}(y|0,1)dy\\ &= \int_{-\infty}^{\infty}P(K\leq\sigma Y+\mu|Y=y)\varphi(y)dy\\ &\quad (K\sim\mathcal{N}(0,1) \text{ and is independent of } Y, \varphi(x) \text{ is the pdf of } \mathcal{N}(0,1))\\ &= P(K\leq\sigma Y+\mu)\\ &\quad (\text{use the law of total probability})\\ &= P(K-\sigma Y\leq\mu)\end{aligned}$$

   Let $Z = K - \sigma Y$, then $Z\sim\mathcal{N}(0,1+\sigma^2)$, so $\mu_0 = P(K-\sigma Y\leq\mu) = P(Z\leq\mu) = \Phi(\frac{\mu}{1+\sigma^2})$.

(b)

$$
\begin{aligned}
\frac{\partial u_0}{\partial \mu} &= \frac{\partial \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})}{\partial \mu} \\
\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)\frac{x-\mu}{\sigma^2}dx &= \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{1}{\sqrt{1+\sigma^2}} \\
\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)(x-\mu)dx &= \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}} \\
\implies u_1 &= \mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}}
\end{aligned}
$$

(c)

$$
u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 - 2m_1u_1 + m_1^2u_0
$$

and

$$
\begin{aligned}
\frac{\partial u_1}{\partial \mu} &= \frac{\partial(\mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}})}{\partial \mu} \\
\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x\frac{x-\mu}{\sigma^2}dx &= -\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \\
\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x(x-\mu)dx &= \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right) \\
\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 &= \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_2 &= \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right) - 2m_1u_1 + m_1^2u_0 \\
&= (m_1^2 - \mu^2)u_0 + 2(\mu - m_1)u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0\right) \\
&= (m_1^2 - \mu^2 + \sigma^2)u_0 + 2(\mu - m_1)u_1 - \frac{\mu\sigma^4}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})
\end{aligned}
$$

5. $X \sim Binom(n,\theta)$, $f(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{1-x}$, $0 < \theta < 1$. Compute Jeffreys prior about θ.

$$
\begin{aligned}
p(\theta) \propto \sqrt{I(\theta)} &= \sqrt{\mathrm{E}\left[\left(\frac{d}{d\theta}\log f(x|\theta)\right)^2\right]} \\
&= \sqrt{\mathrm{E}\left[\left(\frac{d}{d\theta}x\log\theta + \frac{d}{d\theta}(n-x)\log(1-\theta) + \frac{d}{d\theta}\log\binom{n}{x}\right)^2\right]} \\
&= \sqrt{\mathrm{E}\left[\left(\frac{x}{\theta} - \frac{n-x}{1-\theta}\right)^2\right]} \\
&= \sqrt{\mathrm{E}\left[\left(\frac{x-n\theta}{\theta(1-\theta)}\right)^2\right]} \\
&= \frac{1}{\theta(1-\theta)}\sqrt{Var(x)} \\
&= \sqrt{\frac{n}{\theta(1-\theta)}}.
\end{aligned}
$$

6. $\lim_{v\to\infty} \text{Gamma}(r|\frac{v}{2},\frac{v}{2}) = \delta_1(r)$

Proof. Applying Stirling's approximation, we have

$$\Gamma(\frac{v}{2}) = \sqrt{2\pi}(\frac{v}{2})^{\frac{v}{2}-\frac{1}{2}}e^{-\frac{v}{2}}$$

so

$$\begin{aligned}
\lim_{v\to\infty} \text{Gamma}(r|\frac{v}{2},\frac{v}{2}) &= \lim_{v\to\infty} \frac{(\frac{v}{2})^{\frac{v}{2}} r^{\frac{v}{2}-1} e^{-\frac{rv}{2}}}{\Gamma(\frac{v}{2})} \\
&= \lim_{v\to\infty} \frac{(\frac{v}{2})^{\frac{v}{2}} r^{\frac{v}{2}-1} e^{-\frac{rv}{2}}}{\sqrt{2\pi}(\frac{v}{2})^{\frac{v}{2}-\frac{1}{2}} e^{-\frac{v}{2}}} \\
&= \lim_{v\to\infty} \frac{\sqrt{v} r^{\frac{v}{2}-1} e^{-\frac{(r-1)v}{2}}}{2\sqrt{\pi}}
\end{aligned} \tag{2}$$

when $r=1$, $\lim_{v\to\infty} \text{Gamma}(r|\frac{v}{2},\frac{v}{2}) = \lim_{v\to\infty} \frac{1}{2}\sqrt{\frac{v}{\pi}} = \infty$.

when $r \neq 1$, $\frac{r}{e^{r-1}} < 1$, $\lim_{v\to\infty} \text{Gamma}(r|\frac{v}{2},\frac{v}{2}) = \lim_{v\to\infty} \frac{\sqrt{v}(\frac{r}{e^{r-1}})^{\frac{v}{2}}}{2\sqrt{\pi}r} = 0$. □