Solutions for Homework 1

1. If $\lim\limits_{n\to\infty} a_n = a$, show that $\lim\limits_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$.

   Proof.
   $$\lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n/a_n}\right)^{n/a_n}\right)^{a_n} = e^a$$

   $\square$

2. Prove the Stirling Formula.
   $$\lim_{p\to\infty} \frac{\ln \Gamma(p)}{\frac{1}{2}\ln(2\pi) + (p - \frac{1}{2})\ln p - p} = 1$$
   $$\lim_{p\to\infty} \frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}} p^{p-\frac{1}{2}} e^{-p}} = 1$$

   Proof. The Gamma function is :
   $$\begin{aligned} \Gamma(p) &= \int_0^\infty x^p e^{-x} dx \\ &= \int_0^\infty e^{p\ln x - x} dx \\ &= \int_0^\infty e^{p(\ln x - \frac{x}{p})} dx \end{aligned}$$

   Let $f(x) = \ln x - \frac{x}{p}$. It's twice differentiable function on $[0, \infty)$. $f'(x) = \frac{1}{x} - \frac{1}{p}$, so $x = p$ is the unique maximum point and $f''(x) = -\frac{1}{x^2} < 0$. Then, according to Laplace's method [1], we have:

   $$\lim_{p\to+\infty} \frac{\int_0^\infty e^{pf(x)} dx}{\left(e^{pf(p)} \sqrt{\frac{2\pi}{p(-f''(p))}}\right)} = 1$$

   namely,

   $$\lim_{p\to+\infty} \frac{\Gamma(p)}{\left(e^{p(\ln p - \frac{p}{p})} \sqrt{\frac{2\pi}{p(\frac{1}{p^2})}}\right)} = 1$$
   $$\implies \lim_{p\to+\infty} \frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}} p^{p-\frac{1}{2}} e^{-p}} = 1$$

   It is easy to prove the other formula by L'Hospital's rule. $\square$

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   [1] http://en.wikipedia.org/wiki/Laplace%27s_method

3. $\sum\limits_{k=0}^{\infty} \text{Gamma}(x|k+\rho+1,\beta)\text{Poisson}(k|\lambda)$, ρ is a constant, $\rho > -1$

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \text{Gamma}(x|k+\rho+1,\beta)\text{Poisson}(k|\lambda) \\
=\ & \sum_{k=0}^{\infty} \frac{\beta^{k+\rho+1}}{\Gamma(k+\rho+1)} x^{k+\rho} e^{-\beta x} e^{-\lambda} \frac{\lambda^k}{k!} \\
=\ & \frac{\beta e^{-\lambda-\beta x}}{\lambda^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} (\lambda\beta x)^{k+\rho} \\
=\ & \beta e^{-\lambda-\beta x}\sqrt{(\beta x/\lambda)^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} \left(\sqrt{\lambda\beta x}\right)^{2k+\rho} \\
=\ & \beta e^{-\lambda-\beta x}\sqrt{(\beta x/\lambda)^\rho} I_\rho\left(2\sqrt{\lambda\beta x}\right)
\end{aligned}
$$

where I_ρ is the modified Bessel functions of the first kind, which is defined by

$$
I_\rho(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\rho+1)} \left(\frac{x}{2}\right)^{2k+\rho}
$$

4. Compute the following integrals:

   (a) $u_0 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)dx$

   (b) $u_1 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x\,dx$

   (c) $u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)(x-m_1)^2 dx$

   where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}dt$

   (a)

$$
\begin{aligned}
u_0 \ =\ & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
=\ & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}dy \\
& (\text{use } y=(x-\mu)/\sigma-\ \text{to replace x, then } Y\sim\mathcal{N}(0,1)) \\
=\ & \int_{-\infty}^{\infty} \Phi(\sigma y+\mu)\mathcal{N}(y|0,1)dy \\
=\ & \int_{-\infty}^{\infty} P(K\le\sigma Y+\mu|Y=y)\varphi(y)dy \\
& (K\sim\mathcal{N}(0,1) \text{ and is independent of } Y, \varphi(x) \text{ is the pdf of } \mathcal{N}(0,1)) \\
=\ & P(K\le\sigma Y+\mu) \\
& (\text{use the law of total probability}) \\
=\ & P(K-\sigma Y\le\mu)
\end{aligned}
$$

Let $Z = K - \sigma Y$, then $Z\sim\mathcal{N}(0,1+\sigma^2)$, so $\mu_0 = P(K-\sigma Y\le\mu) = P(Z\le\mu) = \Phi(\frac{\mu}{1+\sigma^2})$.

(b)

$$
\begin{aligned}
& \frac{\partial u_0}{\partial\mu} = \frac{\partial\Phi(\frac{\mu}{\sqrt{1+\sigma^2}})}{\partial\mu} \\
\implies & \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)\frac{x-\mu}{\sigma^2}dx = \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{1}{\sqrt{1+\sigma^2}} \\
\implies & \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)(x-\mu)dx = \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}} \\
\implies & u_1 = \mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}}
\end{aligned}
$$

(c)

$$u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 - 2m_1u_1 + m_1^2u_0$$

and

$$\frac{\partial u_1}{\partial \mu} = \frac{\partial(\mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}})}{\partial \mu}$$

$$\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x\frac{x-\mu}{\sigma^2}dx = -\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})$$

$$\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x(x-\mu)dx = \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right)$$

$$\implies \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 = \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right)$$

Hence,

$$u_2 = \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right) - 2m_1u_1 + m_1^2u_0$$

5. $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $0 < \theta < 1$.

   (a) Compute Jeffreys prior about θ.
   (b) If $\theta = \sin^2\alpha$, compute Jeffreys prior about α.

   (a)

$$p(\theta) \propto \sqrt{I(\theta)} = \sqrt{E\left[\left(\frac{d}{d\theta}\log f(x|\theta)\right)^2\right]} = \sqrt{E\left[\left(\frac{x}{\theta} - \frac{1-x}{1-\theta}\right)^2\right]}$$

$$= \sqrt{\theta\left(\frac{1}{\theta} - \frac{0}{1-\theta}\right)^2 + (1-\theta)\left(\frac{0}{\theta} - \frac{1}{1-\theta}\right)^2} = \frac{1}{\sqrt{\theta(1-\theta)}}.$$

   (b)

$$p(\alpha) \propto p(\theta)|\frac{d\theta}{d\alpha}|$$

$$\propto \frac{1}{|\sin\alpha\cos\alpha|}|2\sin\alpha\cos\alpha|$$

$$\propto 1$$