Solutions for Homework 1

1. If $\lim_{n\to\infty} a_n = a$, show that $\lim_{n\to\infty} \left(1+\frac{a_n}{n}\right)^n = e^a$.

   Proof.
$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n/a_n}\right)^{n/a_n}\right)^{a_n} = e^a$$
□

2. Prove the Stirling Formula.
$$\begin{aligned}\lim_{p\to\infty}\frac{\ln\Gamma(p)}{\frac{1}{2}\ln(2\pi)+(p-\frac{1}{2})\ln p - p} &= 1\\ \lim_{p\to\infty}\frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}}p^{p-\frac{1}{2}}e^{-p}} &= 1\end{aligned}$$

   Proof. The Gamma function is :
$$\begin{aligned}\Gamma(p+1) &= \int_0^\infty x^p e^{-x}dx\\ &= \int_0^\infty e^{p\ln x - x}dx\\ &= \int_0^\infty e^{p(\ln x-\frac{x}{p})}dx\end{aligned}$$

   Let $f(x)=\ln x-\frac{x}{p}$. It's twice differentiable function on $[0,\infty)$. $f'(x)=\frac{1}{x}-\frac{1}{p}$, so $x=p$ is the unique maximum point and $f''(x)=-\frac{1}{x^2}<0$. Then, according to Laplace's method [1], we have:
$$\lim_{p\to+\infty}\frac{\int_0^\infty e^{pf(x)}\,dx}{\left(e^{pf(p)}\sqrt{\frac{2\pi}{p(-f''(p))}}\right)}=1$$

   namely,
$$\begin{aligned}&\lim_{p\to+\infty}\frac{\Gamma(p)}{\left(e^{p(\ln p-\frac{p}{p})}\sqrt{\frac{2\pi}{p(\frac{1}{p^2})}}\right)} &= 1\\ \Longrightarrow\quad &\lim_{p\to+\infty}\frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}}p^{p-\frac{1}{2}}e^{-p}} &= 1\end{aligned}$$

   It is easy to prove the other formula by L'Hospital's rule. □

[1] http://en.wikipedia.org/wiki/Laplace%27s_method

3. $\sum_{k=0}^{\infty}$ Gamma$(x|k+\rho+1,\beta)$Poisson$(k|\lambda)$, ρ is a constant, $\rho > -1$

$$\begin{aligned}
& \sum_{k=0}^{\infty} \text{Gamma}(x|k+\rho+1,\beta)\text{Poisson}(k|\lambda) \\
= & \sum_{k=0}^{\infty} \frac{\beta^{k+\rho+1}}{\Gamma(k+\rho+1)} x^{k+\rho} e^{-\beta x} e^{-\lambda} \frac{\lambda^k}{k!} \\
= & \frac{\beta e^{-\lambda-\beta x}}{\lambda^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} (\lambda\beta x)^{k+\rho} \\
= & \beta e^{-\lambda-\beta x} \sqrt{(\beta x/\lambda)^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} \left(\sqrt{\lambda\beta x}\right)^{2k+\rho} \\
= & \beta e^{-\lambda-\beta x} \sqrt{(\beta x/\lambda)^\rho} I_\rho\left(2\sqrt{\lambda\beta x}\right)
\end{aligned}$$

where I_ρ is the modified Bessel functions of the first kind, which is defined by

$$I_\rho(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\rho+1)} \left(\frac{x}{2}\right)^{2k+\rho}$$

4. Compute the following integrals:
   (a) $u_0 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)dx$
   (b) $u_1 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)xdx$
   (c) $u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)(x-m_1)^2dx$
   where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$

   (a)

$$\begin{aligned}
u_0 &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
& \quad \text{(use } y = (x-\mu)/\sigma - \text{ to replace x, then } Y \sim \mathcal{N}(0,1)) \\
&= \int_{-\infty}^{\infty} \Phi(\sigma y+\mu)\mathcal{N}(y|0,1)dy \\
&= \int_{-\infty}^{\infty} P(K \le \sigma Y+\mu|Y=y)\varphi(y)dy \\
& \quad (K \sim \mathcal{N}(0,1) \text{ and is independent of } Y, \varphi(x) \text{ is the pdf of } \mathcal{N}(0,1)) \\
&= P(K \le \sigma Y+\mu) \\
& \quad \text{(use the law of total probability)} \\
&= P(K - \sigma Y \le \mu)
\end{aligned}$$

   Let $Z = K - \sigma Y$, then $Z \sim \mathcal{N}(0, 1+\sigma^2)$, so $\mu_0 = P(K - \sigma Y \le \mu) = P(Z \le \mu) = \Phi(\frac{\mu}{1+\sigma^2})$.

   (b)

$$\begin{aligned}
& \frac{\partial u_0}{\partial \mu} = \frac{\partial \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})}{\partial \mu} \\
\Longrightarrow\ & \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)\frac{x-\mu}{\sigma^2}dx = \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{1}{\sqrt{1+\sigma^2}} \\
\Longrightarrow\ & \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)(x-\mu)dx = \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}} \\
\Longrightarrow\ & u_1 = \mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}}
\end{aligned}$$

(c)

$$u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 - 2m_1u_1 + m_1^2u_0$$

and

$$\begin{aligned}
\frac{\partial u_1}{\partial \mu} &= \frac{\partial(\mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\frac{\sigma^2}{\sqrt{1+\sigma^2}})}{\partial \mu} \\
\Longrightarrow \quad \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x\frac{x-\mu}{\sigma^2}dx &= -\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \\
\Longrightarrow \quad \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(\mu,\sigma^2)x(x-\mu)dx &= \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right) \\
\Longrightarrow \quad \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)x^2 &= \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
u_2 &= \mu u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})\right) - 2m_1u_1 + m_1^2u_0 \\
&= (m_1^2 - \mu^2)u_0 + 2(\mu - m_1)u_1 + \sigma^2\left(-\frac{\mu\sigma^2}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0\right) \\
&= (m_1^2 - \mu^2 + \sigma^2)u_0 + 2(\mu - m_1)u_1 - \frac{\mu\sigma^4}{\sqrt{(1+\sigma^2)^3}}\varphi(\frac{\mu}{\sqrt{1+\sigma^2}})
\end{aligned}$$

5. $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $0 < \theta < 1$.
   (a) Compute Jeffreys prior about θ.
   (b) If $\theta = \sin^2\alpha$, compute Jeffreys prior about α.

(a)

$$\begin{aligned}
p(\theta) \propto \sqrt{I(\theta)} &= \sqrt{\mathrm{E}\left[\left(\frac{d}{d\theta}\log f(x|\theta)\right)^2\right]} = \sqrt{\mathrm{E}\left[\left(\frac{x}{\theta} - \frac{1-x}{1-\theta}\right)^2\right]} \\
&= \sqrt{\theta\left(\frac{1}{\theta} - \frac{0}{1-\theta}\right)^2 + (1-\theta)\left(\frac{0}{\theta} - \frac{1}{1-\theta}\right)^2} = \frac{1}{\sqrt{\theta(1-\theta)}}.
\end{aligned}$$

(b)

$$\begin{aligned}
p(\alpha) \propto & p(\theta)|\frac{d\theta}{d\alpha}| \\
\propto & \frac{1}{|\sin\alpha\cos\alpha|}|2\sin\alpha\cos\alpha| \\
\propto & 1
\end{aligned}$$