Lecture Notes 2: Random Variables

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2 Random Variables

Definition 2.1. *A random variable X is a measure map $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each out come Ω and "measurable" means that for every X, $\{\omega : X(\omega) \leq x\} \in \mathcal{A}$.*

Example 2.1. *Flip a coin ten times. Let $X(\omega)$ be a number of heads in the sequence ω. If $w = HHHTTTHHTT$, $X(\omega)$=5.*

Example 2.2. *Let $\Omega = \{(x, y)|x^2 + y^2 \leq 1\}$. Consider drawing a point at random from Ω. $\omega = (x, y) \in \Omega$, $X(\omega) = x, X(\omega) = y, X(\omega) = x + y$ are possible random variables.*

Definition 2.2. *Let $A \subset \mathbb{R}$, $X^{-1} = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{A}$. $P(X \in A) \triangleq P(X^{-1}(A)) = P(\{\omega \in \Omega | X(\omega) \in A\})$. $P(X = x) = P(X^{-1}(x)) = P(\{\omega \in \Omega | X(\omega) = x\})$*

Note for simplicity, we will use $\{X > 0\}$ to denote $\{\omega \in \Omega : X(\omega) > 0\}$, $P(X > 0)$ to denote $P(\{X > 0\})$.

Example 2.3. *Flip a coin twice and let X be the number of heads.*

ω	$P(\{\omega\})$	$X(\omega)$
TT	1/4	0
TH	1/4	1
HT	1/4	1
HH	1/4	2

X	$P(X)$
0	1/4
1	1/2
2	1/4

2.1 Distribution Function

Cumulative distribution function (or distribution function). CDF is the function $F_X : \mathbb{R} \to [0, 1]$.

$F_X(x) = P(X \leq x)$.

Example 2.4. *From example 1.3, we can get*

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Theorem 2.1. *Let X have CDF F, Y have CDF G. If $F(x) = G(x)$ for all x, then $P(X \in A) = P(Y \in A)$ for all measurable A.*

Theorem 2.2. *A function F mapping $\mathbb{R} \to [0,1]$ is a CDF for probability iff*

1. *F is non-deceasing, $x_1 < x_2 \implies F(x_1) \leq F(x_2)$.*
2. *F is normalized, i.e. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$.*
3. *F is right-continuous. $F(x) = F(x^+)$, where $F(x^+) = \lim_{y \to x, y > x} F(y)$.*

Now we will get the proof of right-continuous.
Proof: Let $y_1 > y_2 > \cdots$, and $\lim_{n \to +\infty} y_n = x$.
Then $F(y_1) = P(Y \leq y_1)$, $F(y_2) = P(Y \leq y_2)$, ...
Let $A_i = (-\infty, y_i]$ and $A = (-\infty, x]$.
Note that $A = \cap_{i=1}^{\infty} A_i$ and $A_1 \supset A_2 \supset \cdots$
$\lim_{i \to \infty} P(A_i) = P(\cap_{i=1}^{\infty} A_i)$.
$F(x) = P(A) = P(\cap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} P(A_i) = \lim_{i \to \infty} F(y_i) = F(x^+)$.