Solutions for Homework 1

1. If $\lim\limits_{n\to\infty} a_n = a$, show that $\lim\limits_{n\to\infty} \left(1+\frac{a_n}{n}\right)^n = e^a$.

   Proof.
   $$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n/a_n}\right)^{n/a_n}\right)^{a_n} = e^a$$
   $\square$

2. Prove the Stirling Formula.
   $$\begin{aligned}\lim_{p\to\infty}\frac{\ln\Gamma(p)}{\frac{1}{2}\ln(2\pi)+(p-\frac{1}{2})\ln p - p} &= 1\\ \lim_{p\to\infty}\frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}}p^{p-\frac{1}{2}}e^{-p}} &= 1\end{aligned}$$

   Proof. The Gamma function is :
   $$\begin{aligned}\Gamma(p) = &\int_0^\infty x^p e^{-x}dx\\ = &\int_0^\infty e^{p\ln x - x}dx\\ = &\int_0^\infty e^{p(\ln x - \frac{x}{p})}dx\end{aligned}$$

   Let $f(x) = \ln x - \frac{x}{p}$. It's twice differentiable function on $[0,\infty)$. $f'(x) = \frac{1}{x} - \frac{1}{p}$, so $x = p$ is the unique maximum point and $f''(x) = -\frac{1}{x^2} < 0$. Then, according to Laplace's method [1], we have:
   $$\lim_{p\to+\infty}\frac{\int_0^\infty e^{pf(x)}\,dx}{\left(e^{pf(p)}\sqrt{\frac{2\pi}{p(-f''(p))}}\right)} = 1$$

   namely,
   $$\begin{aligned}\lim_{p\to+\infty}\frac{\Gamma(p)}{\left(e^{p(\ln p - \frac{p}{p})}\sqrt{\frac{2\pi}{p(\frac{1}{p^2})}}\right)} &= 1\\ \Longrightarrow \quad \lim_{p\to+\infty}\frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}}p^{p-\frac{1}{2}}e^{-p}} &= 1\end{aligned}$$

   It is easy to prove the other formula by L'Hospital's rule. $\square$

[1] http://en.wikipedia.org/wiki/Laplace%27s_method

3. $\sum\limits_{k=0}^{\infty}$ Gamma$(x|k+\rho+1,\beta)$Poisson$(k|\lambda)$, ρ is a constant, $\rho > -1$

$$\begin{aligned}
& \sum_{k=0}^{\infty} \text{Gamma}(x|k+\rho+1,\beta)\text{Poisson}(k|\lambda) \\
= & \sum_{k=0}^{\infty} \frac{\beta^{k+\rho+1}}{\Gamma(k+\rho+1)} x^{k+\rho} e^{-\beta x} e^{-\lambda} \frac{\lambda^k}{k!} \\
= & \frac{\beta e^{-\lambda-\beta x}}{\lambda^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} (\lambda\beta x)^{k+\rho} \\
= & \beta e^{-\lambda-\beta x} \sqrt{(\beta x/\lambda)^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\rho+1)k!} \left(\sqrt{\lambda\beta x}\right)^{2k+\rho} \\
= & \beta e^{-\lambda-\beta x} \sqrt{(\beta x/\lambda)^\rho} I_\rho\left(2\sqrt{\lambda\beta x}\right)
\end{aligned}$$

where I_ρ is the modified Bessel functions of the first kind, which is defined by

$$I_\rho(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\rho+1)} \left(\frac{x}{2}\right)^{2k+\rho}$$

4. Compute the following integrals:
   (a) $u_0 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)dx$
   (b) $u_1 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)xdx$
   (c) $u_2 = \int_{-\infty}^{\infty} \Phi(x)\mathcal{N}(x|\mu,\sigma^2)(x-m_1)^2dx$
   where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

   (a)

$$\begin{aligned}
u_0 &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
& \quad (\text{use } y = (x-\mu)/\sigma - \text{ to replace x, then } Y \sim \mathcal{N}(0,1)) \\
&= \int_{-\infty}^{\infty} \Phi(\sigma y+\mu)\mathcal{N}(y|0,1)dy \\
&= \int_{-\infty}^{\infty} P(K \le \sigma Y+\mu | Y=y) f(y) dy \\
& \quad (K \sim \mathcal{N}(0,1) \text{ and is independent of } Y, f(x) \text{ is the pdf of } \mathcal{N}(0,1)) \\
&= P(K \le \sigma Y+\mu) \\
& \quad (\text{use the law of total probability}) \\
&= P(K - \sigma Y \le \mu)
\end{aligned}$$

   Let $Z = K - \sigma Y$, then $Z \sim \mathcal{N}(0, 1+\sigma^2)$, so $\mu_0 = P(K - \sigma Y \le \mu) = P(Z \le \mu) = \Phi(\frac{\mu}{1+\sigma^2})$.

   (b)

   (c)