Solutions for Homework 2

1. Compute the Laplace transforms of Gamma, Negative Binomial, Poisson distributions.

   Solution. (a) Gamma:

$$
\begin{aligned}
f(x;\alpha,\beta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x) \\
\Longrightarrow F(s) &= \int_0^\infty f(x)\exp(-sx)dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}\exp(-(s+\beta)x)dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(\beta+s)^\alpha} \\
&= \frac{\beta^\alpha}{(\beta+s)^\alpha}
\end{aligned}
$$

   (b) Negative Binomial

$$
\begin{aligned}
P(X=x;p,r) &= \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)}p^x(1-p)^r, \quad \text{for } x=0,1,2,\cdots \\
\Longrightarrow F(x) &= \sum_{x=0}^{\infty} P(X=x;p,r)\exp(-sx)dx \\
&= (1-p)^r\sum_{x=0}^{\infty}\frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)}(pe^{-s})^x dx \\
&= (1-p)^r(1-pe^{-s})^{-r}
\end{aligned}
$$

   (c) Poisson:

$$
\begin{aligned}
P(X=x;\lambda) &= \frac{\lambda^x}{x!}e^{-\lambda}, \quad \text{for } x=0,1,2,\cdots \\
\Longrightarrow F(x) &= \sum_{x=0}^{\infty} P(X=x;\lambda)\exp(-sx)dx \\
&= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{-s})^x}{x!} \\
&= e^{-\lambda(1-e^{-s})}
\end{aligned}
$$

2. Consider that

$$
\begin{aligned}
&w_1 = w\alpha &&, w_2 = w(1-\alpha), \\
&u_1 = u - \beta\sigma\sqrt{\frac{w_2}{w_1}} &&, u_2 = u + \beta\sigma\sqrt{\frac{w_1}{w_2}} \\
&\sigma_1^2 = r(1-\beta^2)\sigma^2 w/w_1, &&\sigma_2^2 = (1-r)(1-\beta^2)\sigma^2 w/w_2,
\end{aligned}
$$

   where $\alpha,\beta,r\in(0,1)$. Compute the Jacobian from $(w_1,w_2,u_1,u_2,\sigma_1^2,\sigma_2^2)$ to $(w,u,\sigma^2,\alpha,\beta,r)$

Solution.

$$\begin{aligned} dw_1 &= wd\alpha + \alpha dw \\ dw_2 &= -wd\alpha + (1-\alpha)dw \\ \Longrightarrow dw_1 \wedge dw_2 &= wd\alpha \wedge dw \end{aligned}$$

$$\begin{aligned} du_1 &= du - \sqrt{\frac{1-\alpha}{\alpha}} d(\beta\sigma) + \text{terms including } d\alpha \\ du_1 &= du + \sqrt{\frac{\alpha}{1-\alpha}} d(\beta\sigma) + \text{terms including } d\alpha \\ \Longrightarrow dw_1 \wedge dw_2 \wedge du_1 \wedge du_2 &= w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) d\alpha \wedge dw \wedge du \wedge d(\beta\sigma) \\ &= w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \frac{\beta}{2\sigma} d\alpha \wedge dw \wedge du \wedge d(\sigma^2) \\ &\quad + w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \sigma d\alpha \wedge dw \wedge du \wedge d\beta \end{aligned}$$

$$\begin{aligned} d\sigma_1^2 &= \frac{r(1-\beta^2)}{\alpha} d(\sigma^2) - \frac{2r\sigma^2\beta}{\alpha} d\beta + \frac{(1-\beta^2)\sigma^2}{\alpha} dr + C_1 d\alpha \\ d\sigma_2^2 &= \frac{(1-r)(1-\beta^2)}{1-\alpha} d(\sigma^2) - \frac{2(1-r)\sigma^2\beta}{1-\alpha} d\beta - \frac{(1-\beta^2)\sigma^2}{1-\alpha} dr + C_2 d\alpha \\ \Longrightarrow dw_1 \wedge dw_2 \wedge du_1 \wedge du_2 \wedge d\sigma_1^2 \wedge d\sigma_2^2 &= w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \left[\frac{\beta}{2\sigma}\left(\frac{2r\sigma^2\beta(1-\beta^2)\sigma^2}{\alpha(1-\alpha)} + \frac{2(1-\beta^2)\sigma^4\beta(1-r)}{\alpha(1-\alpha)}\right)\right. \\ &\quad d\alpha \wedge dw \wedge du \wedge d(\sigma^2) \wedge d\beta \wedge dr + \\ &\quad w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \left[\sigma\left(\frac{-r(1-\beta^2)^2\sigma^2}{\alpha(1-\alpha)} - \frac{(1-r)(1-\beta^2)^2\sigma^2}{\alpha(1-\alpha)}\right)\right] \\ &\quad d\alpha \wedge dw \wedge du \wedge d\beta \wedge d(\sigma^2) \wedge dr \\ &= w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \frac{(1-\beta)^2\sigma^3}{\alpha(1-\alpha)} d\alpha \wedge dw \wedge du \wedge d(\sigma^2) \wedge d\beta \wedge dr \\ &= -w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \frac{(1-\beta)^2\sigma^3}{\alpha(1-\alpha)} dw \wedge du \wedge d(\sigma^2) \wedge d\alpha \wedge d\beta \wedge dr \end{aligned}$$

So,

$$\det(J) = -w\left(\sqrt{\frac{1-\alpha}{\alpha}} + \sqrt{\frac{\alpha}{1-\alpha}}\right) \frac{(1-\beta)^2\sigma^3}{\alpha(1-\alpha)}$$

3. Show the conditional distribution of multinomial distribution in Theorem 5.7.

Proof. Using definition of conditional distribution,

$$\begin{aligned} f(x^{(m)}|x_{m+1}, \cdots, x_k) &= \frac{n!}{\prod_{i=1}^k x_i!(n - \sum_{i=1}^k x_i)!} \prod_{i=1}^k \theta_i^{x_i} (1 - \sum_{t=1}^k \theta_t)^{n - \sum_{j=1}^k x_j} / \\ &\quad \frac{n!}{\prod_{i=m+1}^k x_i!(n - \sum_{i=m+1}^k x_i)!} \prod_{i=m+1}^k \theta_i^{x_i} (1 - \sum_{t=m+1}^k \theta_t)^{n - \sum_{j=m+1}^k x_j} \\ &= M_{m-1}(x^{(m)}|(\theta_1', \ldots, \theta_m'), n-s) \end{aligned}$$

where $\theta_i' = \frac{\theta_i}{\sum_{j=1}^m \theta_j}, (1 \le i \le m)$ and $s = \sum_{i=m+1}^k x_i$. ☐

4.

$$\mathbb{P}(\boldsymbol{X}|\boldsymbol{\theta}, n) \sim \textit{Multinomial Distribution},$$
$$\mathbb{P}(\boldsymbol{\theta}|\boldsymbol{\alpha}) \sim \textit{Dirichlet Distribution}.$$

Compute $\mathbb{P}(\theta|\boldsymbol{X})$.

Solution.
$$\mathbb{P}(\boldsymbol{X}|\boldsymbol{\theta}) = \frac{n!}{\prod_{i=1}^{k} x_i!}\prod_{i=1}^{k}\theta_i^{x_i}$$
where $x_1 + \cdots + x_k = n$, $\theta_1 + \cdots + \theta_k = 1$.
$$\mathbb{P}(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^{k}\alpha_i)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\theta_1^{\alpha_1-1}\cdots\theta_k^{\alpha_k-1}$$
So,
$$\begin{aligned}
\mathbb{P}(\theta|\boldsymbol{X}) &\propto \mathbb{P}(\boldsymbol{X}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta}|\boldsymbol{\alpha}) \\
&\propto \frac{n!}{\prod_{i=1}^{k} x_i!}\prod_{i=1}^{k}\theta_i^{x_i}\frac{\Gamma(\sum_{i=1}^{k}\alpha_i)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\theta_1^{\alpha_1-1}\cdots\theta_k^{\alpha_k-1} \\
&\propto \frac{n!}{\prod_{i=1}^{k} x_i!}\frac{\Gamma(\sum_{i=1}^{k}\alpha_i)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\theta_1^{x_1+\alpha_1-1}\cdots\theta_k^{x_k+\alpha_k-1}
\end{aligned}$$
Ignoring the constant part, we can see it's also a Dirichlet distribution.
$$\mathbb{P}(\boldsymbol{\theta}|\boldsymbol{X}) \sim Dir(\boldsymbol{\theta}|\boldsymbol{x}+\boldsymbol{\alpha})$$

5. If $vec(\boldsymbol{X}^T) \sim N_{np}(vec(\boldsymbol{\mu}^T), \mathbf{B}\otimes\mathbf{A})$, show the p.d.f of $\boldsymbol{X}$ is
$$\frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}\mathrm{tr}(\boldsymbol{A}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})^T\boldsymbol{B}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})))$$

Solution. Since any matrix $\boldsymbol{X}$ may be considered in the vector form $vec(\boldsymbol{X})$. The way of ordering the elements can have no effect on the distribution. So,
$$\begin{aligned}
\mathbb{P}(\boldsymbol{X}) &= \mathbb{P}(vec(\boldsymbol{X}^T)) \\
&= \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{B}\otimes\boldsymbol{A}|^{\frac{1}{2}}}\exp(-\frac{1}{2}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T))^T(\boldsymbol{B}\otimes\boldsymbol{A})^{-1}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T)))
\end{aligned}$$
Since $|\boldsymbol{A}\otimes\boldsymbol{B}| = |\boldsymbol{A}|^n|\boldsymbol{B}|^p$ ($\boldsymbol{A}$ is $p\times p$, $\boldsymbol{B}$ is $n\times n$),
$$\mathbb{P}(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T))^T(\boldsymbol{B}\otimes\boldsymbol{A})^{-1}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T)))$$
Since $(\boldsymbol{B}\otimes\boldsymbol{A})^{-1} = \boldsymbol{B}^{-1}\otimes\boldsymbol{A}^{-1}$,
$$\mathbb{P}(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T))^T(\boldsymbol{B}^{-1}\otimes\boldsymbol{A}^{-1})(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T)))$$
Since $(\mathbf{B}^T\otimes\mathbf{A})\ \mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{AXB})$,
$$\mathbb{P}(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}(vec(\boldsymbol{X}^T)-vec(\boldsymbol{\mu}^T))^T vec(\boldsymbol{A}^{-1}(\boldsymbol{X}^T-\boldsymbol{\mu}^T)(\boldsymbol{B}^{-1})^T))$$
Since $vec(\boldsymbol{X})^T vec(\boldsymbol{Y}) = \mathrm{tr}(\boldsymbol{X}^T\boldsymbol{Y})$,
$$\begin{aligned}
\mathbb{P}(\boldsymbol{X}) &= \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}\mathrm{tr}((X-\boldsymbol{\mu})\boldsymbol{A}^{-1}(\boldsymbol{X}^T-\boldsymbol{\mu}^T)(\boldsymbol{B}^{-1})^T)) \\
&= \frac{1}{(2\pi)^{\frac{np}{2}}|\boldsymbol{A}|^{\frac{n}{2}}|\boldsymbol{B}|^{\frac{n}{2}}}\exp(-\frac{1}{2}\mathrm{tr}(\boldsymbol{A}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})^T\boldsymbol{B}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})))
\end{aligned}$$

6. Prove theorem 6.6.

Proof. The joint distribution of A,B is

$$p(A,B)=\frac{etr(\frac{1}{2}\Sigma^{-1}(A+B))|A|^{\frac{r_1-p-1}{2}}|B|^{\frac{r_2-p-1}{2}}}{2^{\frac{p(r_1+r_2)}{2}}|\Sigma|^{\frac{r_1+r_2}{2}}\Gamma_p(\frac{1}{2}r_1)\Gamma_p(\frac{1}{2}r_2)}$$

We have $A=T^TUT$, $B=T^TT-T^TUT$, So,

$$\begin{aligned}
p(U,T^TT) &= \frac{etr(\frac{1}{2}\Sigma^{-1}(T^TT))|T^TT|^{\frac{r_1-p-1}{2}}|U|^{\frac{r_1-p-1}{2}}|T^TT|^{\frac{r_2-p-1}{2}}|I-U|^{\frac{r_2-p-1}{2}}|T^TT|^{\frac{p+1}{2}}}{2^{\frac{p(r_1+r_2)}{2}}|\Sigma|^{\frac{r_1+r_2}{2}}\Gamma_p(\frac{1}{2}r_1)\Gamma_p(\frac{1}{2}r_2)} \\
&= \frac{etr(\frac{1}{2}\Sigma^{-1}(T^TT))|T^TT|^{\frac{r_1+r_2-p-1}{2}}}{2^{\frac{p(r_1+r_2)}{2}}|\Sigma|^{\frac{r_1+r_2}{2}}\Gamma_p(\frac{r_1+r_2}{2})}\frac{\Gamma_p(\frac{r_1+r_2}{2})}{\Gamma_p(\frac{1}{2}r_1)\Gamma_p(\frac{1}{2}r_2)}|U|^{\frac{r_1-p-1}{2}}|I-U|^{\frac{r_2-p-1}{2}}
\end{aligned}$$

So, A+B and U are independent ,$A+B\sim W_p(\Sigma,r1+r2)$ and $p(U)\propto|U|^{\frac{r_1-p-1}{2}}|I-U|^{\frac{r_2-p-1}{2}}$. $\square$