

Solutions for Homework 1

1. If $\lim_{n \rightarrow \infty} a_n = a$, show that $\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$.

Proof.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{n/a_n}\right)^{n/a_n} \right)^{a_n} = e^a$$

□

2. Prove the Stirling Formula.

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{\ln \Gamma(p)}{\frac{1}{2} \ln(2\pi) + (p - \frac{1}{2}) \ln p - p} &= 1 \\ \lim_{p \rightarrow \infty} \frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}} p^{p-\frac{1}{2}} e^{-p}} &= 1 \end{aligned}$$

Proof. The Gamma function is :

$$\begin{aligned} \Gamma(p) &= \int_0^\infty x^p e^{-x} dx \\ &= \int_0^\infty e^{p \ln x - x} dx \\ &= \int_0^\infty e^{p(\ln x - \frac{x}{p})} dx \end{aligned}$$

Let $f(x) = \ln x - \frac{x}{p}$. It's twice differentiable function on $[0, \infty)$. $f'(x) = \frac{1}{x} - \frac{1}{p}$, so $x = p$ is the unique maximum point and $f''(x) = -\frac{1}{x^2} < 0$. Then, according to Laplace's method ¹, we have:

$$\lim_{p \rightarrow +\infty} \frac{\int_0^\infty e^{pf(x)} dx}{\left(e^{pf(p)} \sqrt{\frac{2\pi}{p(-f''(p))}} \right)} = 1$$

namely,

$$\begin{aligned} \lim_{p \rightarrow +\infty} \frac{\Gamma(p)}{\left(e^{p(\ln p - \frac{p}{p})} \sqrt{\frac{2\pi}{p(\frac{1}{p^2})}} \right)} &= 1 \\ \implies \lim_{p \rightarrow +\infty} \frac{\Gamma(p)}{(2\pi)^{\frac{1}{2}} p^{p-\frac{1}{2}} e^{-p}} &= 1 \end{aligned}$$

It is easy to prove the other formula by L'Hospital's rule.

□

¹http://en.wikipedia.org/wiki/Laplace%27s_method

3. $\sum_{k=0}^{\infty} \text{Gamma}(x|k + \rho + 1, \beta) \text{Poisson}(k|\lambda)$, ρ is a constant, $\rho > -1$

$$\begin{aligned}
& \sum_{k=0}^{\infty} \text{Gamma}(x|k + \rho + 1, \beta) \text{Poisson}(k|\lambda) \\
&= \sum_{k=0}^{\infty} \frac{\beta^{k+\rho+1}}{\Gamma(k + \rho + 1)} x^{k+\rho} e^{-\beta x} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \frac{\beta e^{-\lambda - \beta x}}{\lambda^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \rho + 1) k!} (\lambda \beta x)^{k+\rho} \\
&= \beta e^{-\lambda - \beta x} \sqrt{(\beta x / \lambda)^\rho} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \rho + 1) k!} \left(\sqrt{\lambda \beta x} \right)^{2k+\rho} \\
&= \beta e^{-\lambda - \beta x} \sqrt{(\beta x / \lambda)^\rho} I_\rho \left(2\sqrt{\lambda \beta x} \right)
\end{aligned}$$

where I_ρ is the modified Bessel functions of the first kind, which is defined by

$$I_\rho(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \rho + 1)} \left(\frac{x}{2} \right)^{2k+\rho}$$

4. Compute the following integrals:

- (a) $u_0 = \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(x|\mu, \sigma^2) dx$
(b) $u_1 = \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(x|\mu, \sigma^2) x dx$
(c) $u_2 = \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(x|\mu, \sigma^2) (x - m_1)^2 dx$

where $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

- (a)

$$\begin{aligned}
u_0 &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\sigma y + \mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&\quad (\text{use } y = (x - \mu)/\sigma \text{ to replace } x, \text{ then } Y \sim \mathcal{N}(0, 1)) \\
&= \int_{-\infty}^{\infty} \Phi(\sigma y + \mu) \mathcal{N}(y|0, 1) dy \\
&= \int_{-\infty}^{\infty} P(K \leq \sigma Y + \mu | Y = y) \varphi(y) dy \\
&\quad (K \sim \mathcal{N}(0, 1) \text{ and is independent of } Y, \varphi(x) \text{ is the pdf of } \mathcal{N}(0, 1)) \\
&= P(K \leq \sigma Y + \mu) \\
&\quad (\text{use the law of total probability}) \\
&= P(K - \sigma Y \leq \mu)
\end{aligned}$$

Let $Z = K - \sigma Y$, then $Z \sim \mathcal{N}(0, 1 + \sigma^2)$, so $\mu_0 = P(K - \sigma Y \leq \mu) = P(Z \leq \mu) = \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})$.

- (b)

$$\begin{aligned}
& \frac{\partial u_0}{\partial \mu} = \frac{\partial \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})}{\partial \mu} \\
\Rightarrow & \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(\mu, \sigma^2) \frac{x - \mu}{\sigma^2} dx = \varphi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) \frac{1}{\sqrt{1+\sigma^2}} \\
\Rightarrow & \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(\mu, \sigma^2) (x - \mu) dx = \varphi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) \frac{\sigma^2}{\sqrt{1+\sigma^2}} \\
\Rightarrow & u_1 = \mu u_0 + \varphi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) \frac{\sigma^2}{\sqrt{1+\sigma^2}}
\end{aligned}$$

(c)

$$u_2 = \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(x|\mu, \sigma^2) x^2 - 2m_1 u_1 + m_1^2 u_0$$

and

$$\begin{aligned} \frac{\partial u_1}{\partial \mu} &= \frac{\partial(\mu u_0 + \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \frac{\sigma^2}{\sqrt{1+\sigma^2}})}{\partial \mu} \\ \Rightarrow \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(\mu, \sigma^2) x \frac{x - \mu}{\sigma^2} dx &= -\frac{\mu \sigma^2}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \\ \Rightarrow \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(\mu, \sigma^2) x(x - \mu) dx &= \sigma^2 \left(-\frac{\mu \sigma^2}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \right) \\ \Rightarrow \int_{-\infty}^{\infty} \Phi(x) \mathcal{N}(x|\mu, \sigma^2) x^2 &= \mu u_1 + \sigma^2 \left(-\frac{\mu \sigma^2}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \right) \end{aligned}$$

Hence,

$$\begin{aligned} u_2 &= \mu u_1 + \sigma^2 \left(-\frac{\mu \sigma^2}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 + \frac{\mu}{\sqrt{1+\sigma^2}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \right) - 2m_1 u_1 + m_1^2 u_0 \\ &= (m_1^2 - \mu^2) u_0 + 2(\mu - m_1) u_1 + \sigma^2 \left(-\frac{\mu \sigma^2}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) + u_0 \right) \\ &= (m_1^2 - \mu^2 + \sigma^2) u_0 + 2(\mu - m_1) u_1 - \frac{\mu \sigma^4}{\sqrt{(1+\sigma^2)^3}} \varphi(\frac{\mu}{\sqrt{1+\sigma^2}}) \end{aligned}$$

5. $f(x; \theta) = \theta^x (1 - \theta)^{1-x}$, $0 < \theta < 1$.

(a) Compute Jeffreys prior about θ .

(b) If $\theta = \sin^2 \alpha$, compute Jeffreys prior about α .

(a)

$$\begin{aligned} p(\theta) &\propto \sqrt{I(\theta)} = \sqrt{\mathbb{E} \left[\left(\frac{d}{d\theta} \log f(x|\theta) \right)^2 \right]} = \sqrt{\mathbb{E} \left[\left(\frac{x}{\theta} - \frac{1-x}{1-\theta} \right)^2 \right]} \\ &= \sqrt{\theta \left(\frac{1}{\theta} - \frac{0}{1-\theta} \right)^2 + (1-\theta) \left(\frac{0}{\theta} - \frac{1}{1-\theta} \right)^2} = \frac{1}{\sqrt{\theta(1-\theta)}}. \end{aligned}$$

(b)

$$\begin{aligned} p(\alpha) &\propto p(\theta) \left| \frac{d\theta}{d\alpha} \right| \\ &\propto \frac{1}{|\sin \alpha \cos \alpha|} |2 \sin \alpha \cos \alpha| \\ &\propto 1 \end{aligned}$$