Lecture Notes 2: Random Variables

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1 Random Variables

Definition 1.1. *A random variable X is a measure map $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each and come as "measurable" means that for every X, $\{\omega : X(\omega) \le x\} \in \mathcal{A}$.*

Example 1.1. *Flip a coin ten times. Let $X(\omega)$ be a number of heads in the sequence ω.*

Example 1.2. *Let $\Omega = \{(x, y)|x^2 + y^2 \le 1\}$. Consider drawing a point at random from Ω.*

Definition 1.2. *Let $A \subset \mathbb{R}$, $X^{-1} = \{\omega \in \Omega; X(\omega) \in A\} \in \mathcal{A}$. $P(X \in A) \triangleq P(X^{-1}(A)) = P(\{\omega \in \Omega|X(\omega) \in A\})$. $P(X = x) = P(X^{-1}(x)) = P(\{\omega \in \Omega|X(\omega) = x\})$*

Example 1.3. *Flip a coin twice and let X be the number of heads.*

ω	$P(\{\omega\})$	$X(\omega)$
TT	*1/4*	*0*
TH	*1/4*	*1*
HT	*1/4*	*1*
HH	*1/4*	*2*

X	*P(X)*
0	*1/4*
1	*1/2*
2	*1/4*

1.1 Distribution Function

Cumulative distribution function (or distribution function). CDF is the function $F_X : \mathbb{R} \to [0, 1]$.

$F_X(x) = P(X \le x)$.

Example 1.4. *From example 1.3, we can get*

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Theorem 1.1. *Let X have CDF F, Y have CDF G. If $F(x) = G(x)$ for all x, then $P(X \in A) = P(Y \in A)$ for all measurable A.*

Theorem 1.2. *A function F mapping $\mathbb{R} \to [0, 1]$ is a CDF for probability iff*

1. *F is non-deceasing, $x_1 < x_2 \implies F(x_1) \le F(x_2)$.*

2. F is normalized, i.e. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} = 1$.

3. F is right-continuous. $F(x) = F(x^+)$, *where* $F(x^+) = \lim_{y \to x, y > x} F(y)$.

Now we will get the proof of right-continuous.
Proof: Let $F(x_1) = P(X \leq x_1)$, $F(x_2) = P(X \leq x_2)$.
Let $X \in \mathbb{R}$, $y_1 > y_2 > \cdots$, and $\lim_{n \to +\infty} y_n = x$.
Let $A_i = (-\infty, y_i]$ and $A = (-\infty, x]$.
Note that $A = \cap_{i=1}^{\infty} A_i$ and $A_1 \supset A_2 \supset \cdots$
$\lim_{i \to \infty} P(A_i) = P(\cap_{i=1}^{\infty} A_i)$.
$F(x) = P(A) = P(\cap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} P(A_i) = \lim_{i \to \infty} F(y_i) = F(x^+)$.