1 The Model

The t-J model Hamiltonian reads,

$$H = -t \sum_{\langle i,j \rangle} \left(\tilde{c}^\dagger_{i\sigma} \tilde{c}_{j\sigma} + H.c \right) + J \sum_{\langle i,j \rangle} \left(\frac{1}{2} \left[S^+_i S^-_j + S^-_i S^+_j \right] + S^z_i S^z_j - \frac{1}{4} \tilde{n}_i \tilde{n}_j \right), \tag{1}$$

where $\tilde{c}^\dagger_{i\sigma} = c^\dagger_{i\sigma}(1 - n_{i\bar{\sigma}})$ can create electrons only on unoccupied sites. In order to introduce holes and magnons we start with rotation of spins in one sublattice of the system. This results in

$$H_{\text{rot}} = -t \sum_{\langle i,j \rangle} \left(\tilde{c}^\dagger_{i\sigma} \tilde{c}_{j\bar{\sigma}} + H.c \right) + J \sum_{\langle i,j \rangle} \left(\frac{1}{2} \left[S^+_i S^+_j + S^-_i S^-_j \right] - S^z_i S^z_j - \frac{1}{4} \tilde{n}_i \tilde{n}_j \right). \tag{2}$$

This allows for introduction of holes and magnons accoridng to the following transformations

$$\begin{aligned} \tilde{c}^\dagger_{i\uparrow} &= h_i, & \tilde{c}_{i\uparrow} &= h^\dagger_i (1 - a^\dagger_i a_i), \\ \tilde{c}^\dagger_{i\downarrow} &= h_i a^\dagger_i, & \tilde{c}_{i\downarrow} &= h^\dagger_i a_i, \end{aligned} \tag{3}$$

$$\begin{aligned} S^+_i &= h_i h^\dagger_i (1 - a^\dagger_i a_i) a_i, & S^z_i &= (\frac{1}{2} - a^\dagger_i a_i) h_i h^\dagger_i, \\ S^-_i &= a^\dagger_i (1 - a^\dagger_i a_i) h_i h^\dagger_i, & \tilde{n}_i &= 1 - h^\dagger_i h_i = h_i h^\dagger_i. \end{aligned} \tag{4}$$

Here magnons can be understood as deviations from state that after the rotation has all the spins pointing up. In the end the model (up to the shift by a constant) reads,

$$\mathcal{H} = \mathcal{H}_t + \mathcal{H}_J, \tag{5}$$

where,

$$\mathcal{H}_t = t \sum_{\langle i,j \rangle} \left[h^\dagger_i h_j \left(a_i + a^\dagger_j (1 - a^\dagger_i a_i) \right) + h^\dagger_j h_i \left(a_j + a^\dagger_i (1 - a^\dagger_j a_j) \right) \right], \tag{6}$$

$$\begin{aligned} \mathcal{H}_J = {} & \frac{J}{2} \sum_{\langle i,j \rangle} h_i h^\dagger_i \left[(1 - a^\dagger_i a_i)(1 - a^\dagger_j a_j) a_i a_j + a^\dagger_i a^\dagger_j (1 - a^\dagger_i a_i)(1 - a^\dagger_j a_j) \right] h_j h^\dagger_j \\ & + \frac{J}{2} \sum_{\langle i,j \rangle} h_i h^\dagger_i \left[a^\dagger_i a_i + a^\dagger_j a_j - 2 a^\dagger_i a_i a^\dagger_j a_j - 1 \right] h_j h^\dagger_j. \end{aligned} \tag{7}$$

Now let consider a staggered magnetic field,

$$\mathcal{H}_B = \frac{B}{2} \sum_{\langle i,j \rangle} \left[(-1)^i S^z_i + (-1)^j S^z_j \right]. \tag{8}$$

Performing the same set of transformations as to the Hamiltonian we obtain (up to the constant energy shift),

$$\mathcal{H}_B = \frac{B}{2} \sum_{\langle i,j \rangle} \left[a^\dagger_i a_i h_i h^\dagger_i + a^\dagger_j a_j h_j h^\dagger_j \right] \approx \frac{B}{2} \sum_{\langle i,j \rangle} h_i h^\dagger_i \left[a^\dagger_i a_i + a^\dagger_j a_j \right] h_j h^\dagger_j, \tag{9}$$

where ommited terms on the righthandside of the approximation sign are modifying the magnetic field only around the hole and they are equal to $\frac{B}{2}\left[a_i^\dagger a_i h_j h_j^\dagger + a_j^\dagger a_j h_i h_i^\dagger\right]$. In the end,

$$\begin{aligned}\mathcal{H}_{J+B} = \mathcal{H}_J + \mathcal{H}_B \approx& \\ &= \frac{J}{2}\sum_{\langle i,j\rangle} h_i h_i^\dagger \left[(1-a_i^\dagger a_i)(1-a_j^\dagger a_j)a_i a_j + a_i^\dagger a_j^\dagger(1-a_i^\dagger a_i)(1-a_j^\dagger a_j)\right] h_j h_j^\dagger \\ &+ \frac{J}{2}\sum_{\langle i,j\rangle} h_i h_i^\dagger \left[\left(1+\frac{B}{J}\right)\left(a_i^\dagger a_i + a_j^\dagger a_j\right) - 2a_i^\dagger a_i a_j^\dagger a_j - 1\right] h_j h_j^\dagger.\end{aligned} \tag{10}$$

Let introduce new coupling constant $J_s = J + B$ and parameter $\lambda = J/J_s$. Then, in the limit of a single hole, we can write,

$$\begin{aligned}\mathcal{H}_{J+B} \approx& \frac{J_s\lambda}{2}\sum_{\langle i,j\rangle} h_i h_i^\dagger \left[(1-a_i^\dagger a_i)(1-a_j^\dagger a_j)a_i a_j + a_i^\dagger a_j^\dagger(1-a_i^\dagger a_i)(1-a_j^\dagger a_j)\right] h_j h_j^\dagger \\ &+ \frac{J_s}{2}\sum_{\langle i,j\rangle} h_i h_i^\dagger \left[a_i^\dagger a_i + a_j^\dagger a_j - 2\lambda a_i^\dagger a_i a_j^\dagger a_j\right] h_j h_j^\dagger.\end{aligned} \tag{11}$$

For $\lambda \neq 1$ coupling constants in xy plane and in z direction are different. In the end the resulting model can be undestood as XXZ model with rescaled magnon-magnon interaction.