1 The Model

We investigate the XXZ model in one-dimension (1D) with periodic boundary conditions (PBC),

$$H = J \sum_{\langle i,j \rangle} \alpha S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z, \tag{1}$$

where $\alpha \in [0, 1]$. In the limit of $\alpha = 1$ the above resembles exactly the Heisenberg model, while for $\alpha = 0$ it is the isotropic case of the XY model. Typically one puts the parameter α in front of the z component of the spin operators. However then the z axis is not a good quantization axis in $\alpha \to 0$ limit. Assuming natural alignment of the spin chain along the x-axis we decide to scale spin interaction along the chain while keeping the interactions in the perpendicular YZ plane untouched.

In particular we are interested in the $J < 0$ case for limiting and intermediate values of parameter α. The quantity of interest is the spectral function of the single spin excitation with momentum q,

$$S(q, \omega) = -\frac{1}{\pi} \lim_{\delta \to 0^+} \mathrm{Im} \left\langle \mathrm{GS} \right| S_q^+ G(\omega + i\delta) S_q^- \left| \mathrm{GS} \right\rangle, \tag{2}$$

where

$$G(\omega) = (\omega - H + E_{\mathrm{GS}})^{-1}. \tag{3}$$

We aim at identifying and understanding processes leading to change from single magnon branch in the $S(q, \omega)$ for $\alpha = 1$ to the two-spinon continuum in $\alpha = 0$ limit. In particular we would like to find out whether the magnon-magnon interactions play any (important) role in the above mentioned evolution of the spectral function.

2 Methodology

Let us firstly rewrite the XXZ Hamiltonian in somewhat more convenient form. We start by introducing the spin rising and lowering operators,

$$\begin{aligned} S_i^x &= \frac{1}{2} \left(S_i^+ + S_i^- \right), \\ S_i^y &= \frac{1}{2i} \left(S_i^+ - S_i^- \right). \end{aligned} \tag{4}$$

With the above on can split the model Hamiltonian into two parts, the Heisenberg Hamiltonian (denoted here as *parallel* $\|$) and *rotated* Heisenberg Hamiltonian (denoted as *perpendicular* $\perp$),

$$H = \frac{1+\alpha}{2} H_{\|} + \frac{1-\alpha}{2} H_{\perp}, \tag{5}$$

where

$$H_{\|} = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \frac{1}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right), \tag{6}$$

and

$$H_{\perp} = J \sum_{\langle i,j \rangle} S_i^z S_j^z - \frac{1}{2} \left(S_i^+ S_j^+ + S_i^- S_j^- \right). \tag{7}$$

2.1 Clarification of the notation

Let us comment on why we call $H_\perp$ the *rotated* Heisenberg Hamiltonian. Consider a transformation rotating spins at every second site, e.g. on the sublattice $\mathcal{B}$,

$$\mathrm{rot} : \forall j \in \mathcal{B}\ (S_j^{\pm} \to S_j^{\mp}) \wedge (S_j^z \to -S_j^z). \tag{8}$$

The stated-above transformation reveals the relation between $H_\parallel$ and $H_\perp$, namely,

$$\mathrm{rot}\, H_\parallel = -H_\perp. \tag{9}$$

Now let us comment on why we denoted these Hamiltonians as *parallel* and *perpendicular* accordingly. There are two reasons.

The first come from the magnetization conservation present in the Heisenberg model. Indeed, one can see that $S_i^+ S_j^- + S_i^- S_j^+$ operators allow the magnetic excitation (flipped spin) to spread only across states within the subspace of the Hilbert space corresponding to the given magnetization because number of spins up or/and down cannot be changed. On the other hand $S_i^+ S_j^+ + S_i^- S_j^-$ operators come with the opposing mechanism leading to changes in the magnetization of the system.

The second reason for the choice of the notation comes from the type of the *classical* ground state of the system realised by these models. For ferromagnetic coupling constant ($J < 0$) the ground state of $H_\parallel$ is the Ising ferromagnet (twofold degenerated due to $Z \to -Z$ symmetry of the system), i.e. neighbouring spins are aligned parallel in the *classical* ground state. Accordingly, for antiferromagnetic coupling constant ($J > 0$) the ground state of $H_\perp$ it the Ising antiferromagnet (also twofold degenerated due $Z \to -Z$ symmetry), i.e. neighbouring spins are aligned anti-parallel in the *classical* ground state.

2.2 Magnetization and Staggered Magnetization

In order to understand how the classical antiferromagnet arises as the ground state of $H_\perp$ for $J > 0$ it is enough to look at the Eq. (9). Rotating the operators in the Heisenberg model and spins in the sublattice $\mathcal{B}$ one does not change the physics of the system. Therefore given the eigenstates $|\psi_n\rangle$ of $H_\parallel$ Hamiltonian with corresponding energies E_n one easily obtains eigenstates of the $H_\perp$ by reversing spins at sublattice $\mathcal{B}$. Then corresponding energies are $-E_n$. Incorporating the sign change into a coupling constant we see that Ising antiferromagnet is the ground state of $H_\perp$ for antiferromagnetic $J > 0$.

Above-described relation between eigenstates of these two Hamiltonians allow one to make even stronger statement, namely: in order to understand $H_\perp$ one just need to consider the Heisenberg model for opposing sign of the coupling constant J. Thus, any quantity of $H_\parallel$ should have its corresponding *anti*-quantity in $H_\perp$ related by the rotation of $\mathcal{B}$ sublattice. Therefore, for example, if $H_\parallel$ Hamiltonian conserves the magnetization,

$$M = \sum_{i=0,\ldots,N-1} S_i^z, \tag{10}$$

then $H_\perp$ Hamiltonian conserves the staggered magnetization

$$M_s = \sum_{i=0,\ldots,N-1} (-1)^i S_i^z. \tag{11}$$

2.3 Interplay of the two parts of the Hamiltonian

The ground state of the Heisenberg model $H_{\parallel}$ for antiferromagnetic coupling constant is given by a certain linear combination of the S^z spins configurations. It belongs to the subspace where absolute value of the magnetization is smallest possible, i.e. $M = 0$ for even number of spins or $|M| = 1$ for odd number of spins. Accordingly the ground state of rotated Heisenberg $H_{\perp}$ for ferromangetic coupling constant is a linear combination of states belonging to the subspace with staggered magnetization $M_s = 0$ (assuming even number of spins). Thus the full model tries to maximize magnetization and minimize staggered magnetization. It is important to notice that this is not the same and the two parts of the Hamiltonian play towards rather orthogonal than totally parallel or antiparallel goal. While pure ferromagenetic Heisenberg model prefers simple ferromagnetic ground state the contribution from rotated Heisenberg will lead to changes in magnetization via double spin flips processes, i.e. $S_i^+ S_j^+ + S_i^- S_j^-$. This will further unlock the $S_i^+ S_j^- + S_i^- S_j^+$ terms leading to the break of the staggered magnetization conservation of the pure rotated Heisenberg. In the end the ground state will be a complicated linear combination of states belonging to various magnetization and staggered magnetization subspaces. Shortly speaking,

$$[H_{\parallel}, H_{\perp}] \neq 0. \tag{12}$$

3 TODO

1. zrozumiec co z transformata fouriera