1 The Model

We investigate the XXZ model in one-dimension (1D) with periodic boundary conditions (PBC),

$$H = J \sum_{\langle i,j \rangle} \alpha S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z, \tag{1}$$

where $\alpha \in [0, 1]$. In the limit of $\alpha = 1$ the above resembles exactly the Heisenberg model, while for $\alpha = 0$ it is the isotropic case of the XY model. Typically one puts the parameter α in front of the z component of the spin operators. However then the z axis is not a good quantization axis in $\alpha \to 0$ limit. Assuming natural alignment of the spin chain along the x-axis we decide to scale spin interaction along the chain while keeping the interactions in the perpendicular YZ plane untouched.

In particular we are interested in the $J < 0$ case for limiting and intermediate values of parameter α. The quantity of interest is the spectral function of the single spin excitation with momentum q,

$$S(q, \omega) = -\frac{1}{\pi} \lim_{\delta \to 0^+} \mathrm{Im} \langle \mathrm{GS} | S_q^+ G(\omega + i\delta) S_q^- | \mathrm{GS} \rangle, \tag{2}$$

where

$$G(\omega) = (\omega - H + E_{\mathrm{GS}})^{-1}. \tag{3}$$

We aim at identifying and understanding processes leading to change from single magnon branch in the $S(q, \omega)$ for $\alpha = 1$ to the two-spinon continuum in $\alpha = 0$ limit. In particular we would like to find out whether the magnon-magnon interactions play any (important) role in the above mentioned evolution of the spectral function.

2 Methodology

Let us firstly rewrite the XXZ Hamiltonian in somewhat more convenient form. We start by introducing the spin rising and lowering operators,

$$\begin{aligned} S_i^x &= \frac{1}{2} \left(S_i^+ + S_i^- \right), \\ S_i^y &= \frac{1}{2i} \left(S_i^+ - S_i^- \right). \end{aligned} \tag{4}$$

With the above on can split the model Hamiltonian into two parts, the Heisenberg Hamiltonian (denoted here as *parallel* $\|$) and *rotated* Heisenberg Hamiltonian (denoted as *perpendicular* $\perp$),

$$H = \frac{1 + \alpha}{2} H_{\|} + \frac{1 - \alpha}{2} H_{\perp}, \tag{5}$$

where

$$H_{\|} = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \frac{1}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right), \tag{6}$$

and

$$H_{\perp} = J \sum_{\langle i,j \rangle} S_i^z S_j^z - \frac{1}{2} \left(S_i^+ S_j^+ + S_i^- S_j^- \right). \tag{7}$$

2.1 Clarification of the notation

Let us comment on why we call $H_\perp$ the *rotated* Heisenberg Hamiltonian. Consider a transformation rotating spins at every second site, i.e. in the sublattice $\mathcal{B}$,

$$\mathrm{rot} : \forall j \in \mathcal{B} \; (S_j^\pm \to S_j^\mp) \wedge (S_j^z \to -S_j^\pm). \tag{8}$$

The stated-above transformation reveals the relation between $H_\parallel$ and $H_\perp$, namely,

$$\mathrm{rot}\, H_\parallel = -H_\perp. \tag{9}$$

Now let us comment on why we denoted these Hamiltonians as *parallel* and *perpendicular* accordingly. There are two reasons.

The first come from the magnetization conservation present in the Heisenberg model. Indeed, one can see that $S_i^+ S_j^- + S_i^- S_j^+$ operators allow the magnetic excitation (flipped spin) to spread only across states within the subspace of the Hilbert space corresponding to the given magnetization because number of spins up or/and down cannot be changed. On the other hand $S_i^+ S_j^+ + S_i^- S_j^-$ operators come with the opposing mechanism leading to changes in the magnetization of the system.

The second reason for the choice of the notation comes from the type of the *classical* ground state of the system realised by these models. For ferromagnetic coupling constant ($J < 0$) the ground state of $H_\parallel$ is the Ising ferromagnet (twofold degenerated due to $Z \to -Z$ symmetry of the system), i.e. neighbouring spins are aligned parallel in the *classical* ground state. Accordingly, for antiferromagnetic coupling constant ($J > 0$) the ground state of $H_\perp$ it the Ising antiferromagnet (also twofold degenerated due $Z \to -Z$ symmetry), i.e. neighbouring spins are aligned anti-parallel in the *classical* ground state.

2.2 Magnetization and Anti-magnetization

In order to understand how the classical antiferromagnet arises as the ground state of $H_\perp$ for $J > 0$ it is enough to look at the Eq. (9). Rotating the operators in the Heisenberg model and spins in the sublattice $\mathcal{B}$ one does not change the physics of the system. Therefore given the eigenstates $|\psi_n\rangle$ of $H_\parallel$ Hamiltonian with corresponding energies E_n one easily obtains eigenstates of the $H_\perp$ by reversing spins at sublattice $\mathcal{B}$. Then corresponding energies are $-E_n$. Incorporating the sign change into a coupling constant we see that Ising antiferromagnet is the ground state of $H_\perp$ for antiferromagnetic $J > 0$.

Above-described relation between eigenstates of these two Hamiltonians allow one to make even stronger statement, namely: In order to understand $H_\perp$ one just need to consider the Heisenberg model for opposing sign of the coupling constant J. Thus, any quantity of $H_\parallel$ should have its corresponding *anti-quantity* in $H_\perp$ related by the rotation of $\mathcal{B}$ sublattice. Therefore, for example, if $H_\parallel$ Hamiltonian conserves the magnetization,

$$M = \sum_{i=0,\ldots,N-1} S_i^z, \tag{10}$$

then $H_\perp$ Hamiltonian conserves the *anti-magnetization*

$$\bar{M} = \sum_{i=0,\ldots,N-1} (-1)^i S_i^z. \tag{11}$$

3 TODO

1. zrozumiec co z transformata fouriera