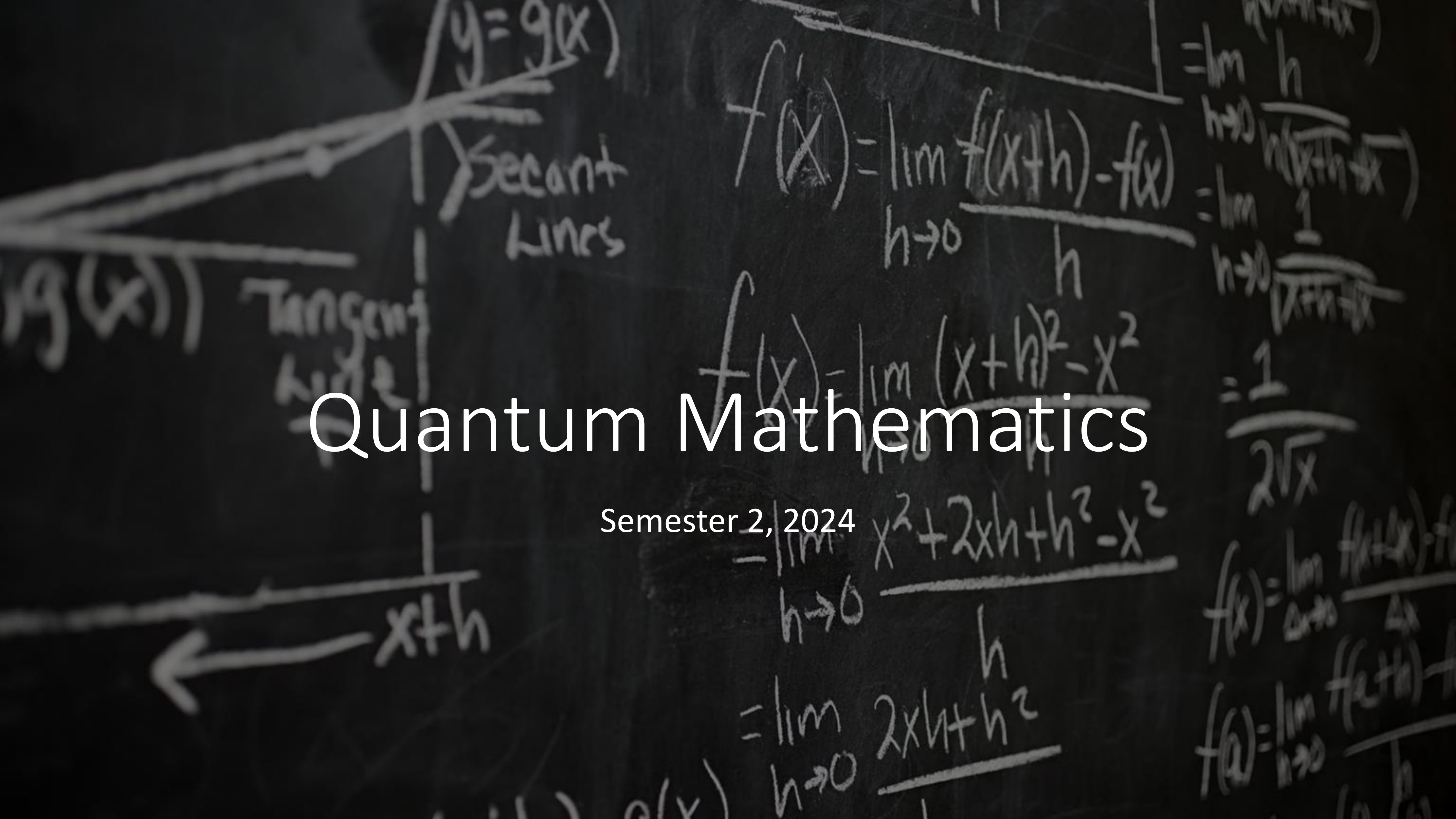


Quantum Mathematics

Semester 2, 2024

Mathematical tools

- Probability
- Complex vector space
  - Complex numbers
- All of quantum mechanics can be described by linear algebra
  - Vectors
  - Matrices
  - Lineair transformations
- Dirac notation

Probability

What is probability?

- number of times k that event A happens
- total number n of equally likely outcomes or events

$$P(A) = \frac{k}{n}$$

Probability

Example: what is the probability of throwing a 5 with a fair die?

- number of times favorable to throwing a 5 is $k = 1$
- total number of outcomes is $n = 6$

$$P(5) = \frac{k}{n} = \frac{1}{6}$$

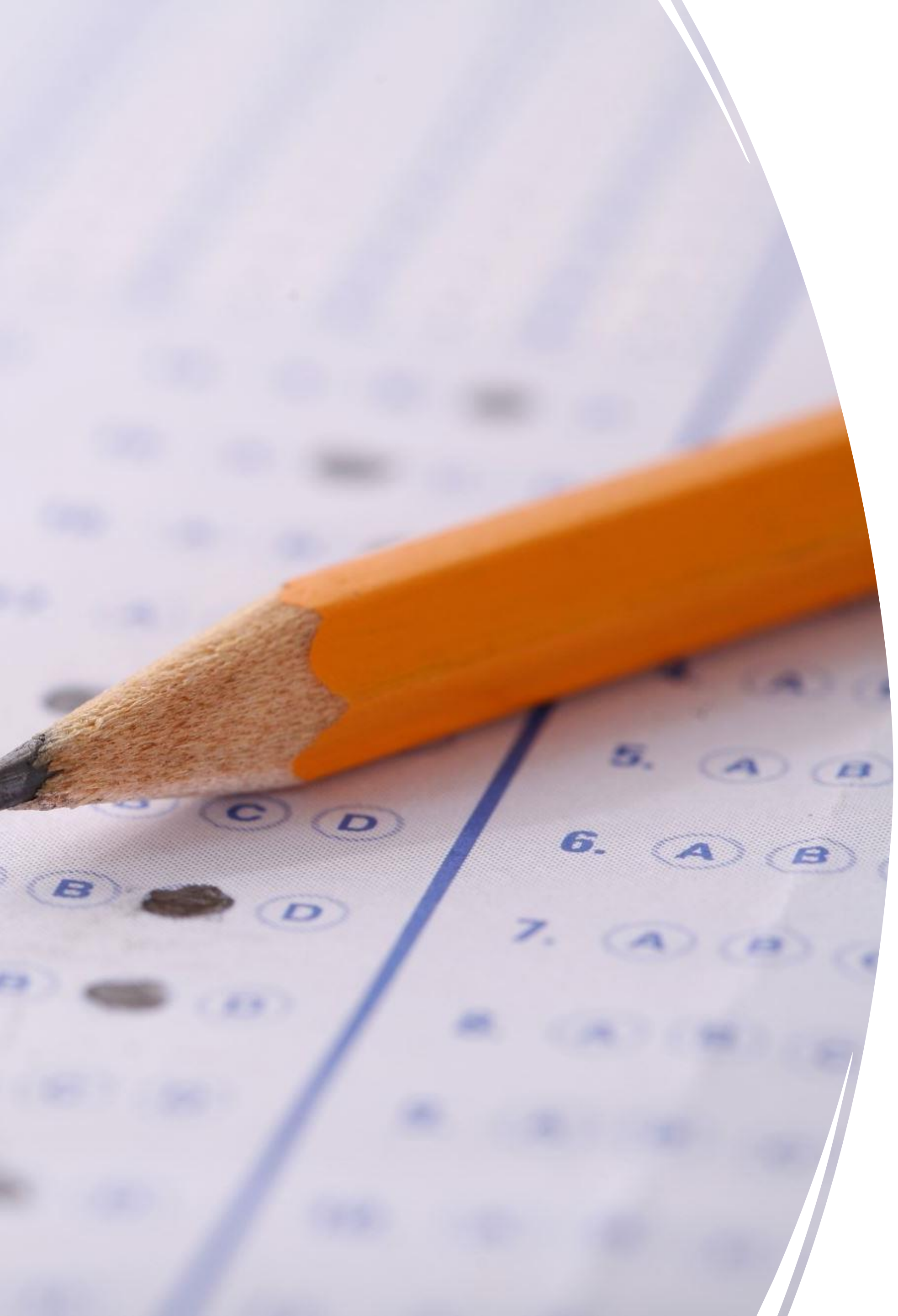
Probability

What about throwing with 2 dice?

Example: What is the probability of throwing two $5's$ with 2 fair dice.

- number of times favorable to throwing two $5's$ is $k = 1$
- total number of outcomes is $n = 36$

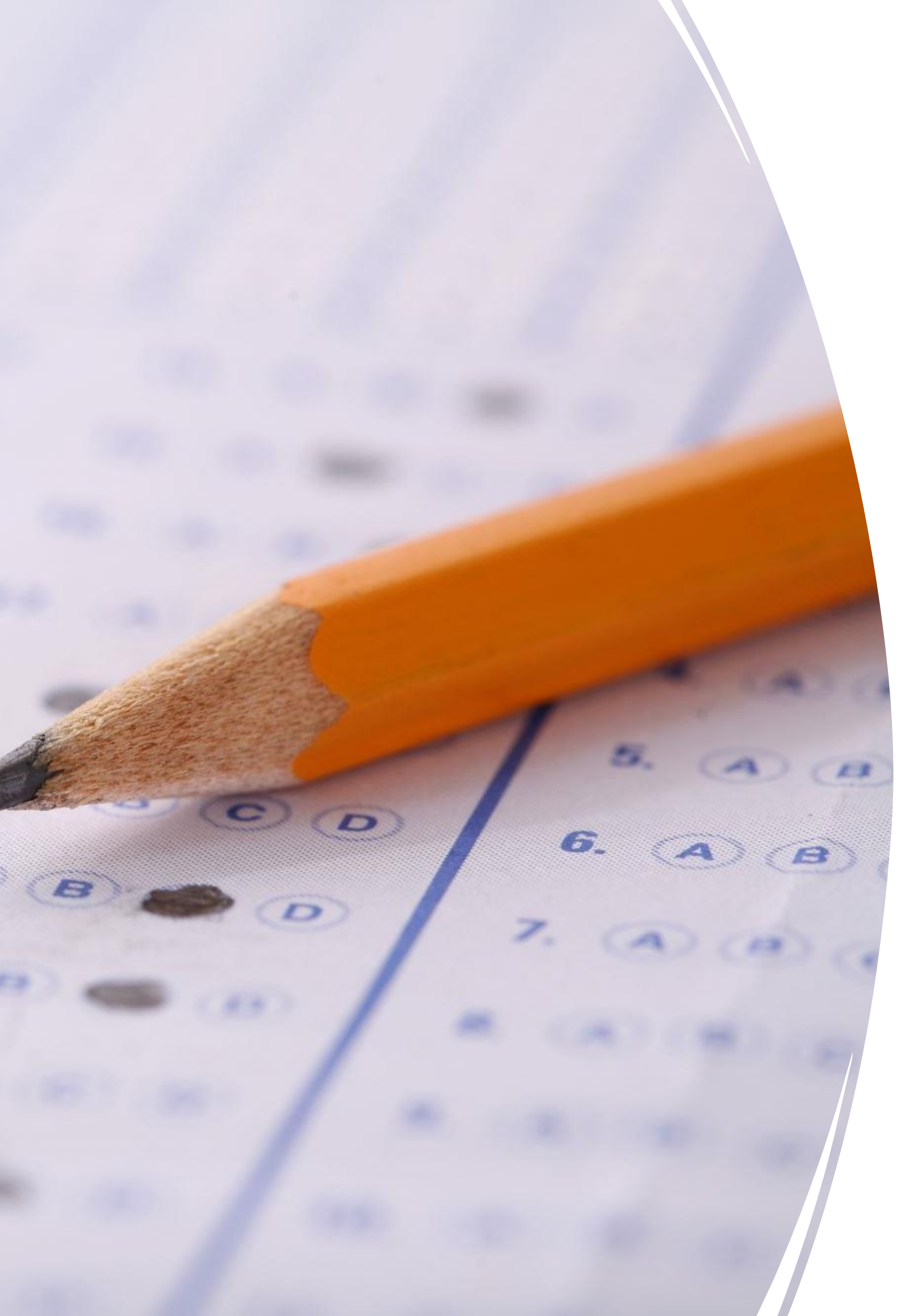
$$P(two\ 5's) = \frac{n}{k} = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$$



Quiz

Assume you throw with two fair dice and you add up the numbers.

What is the probability of throwing more than 7?



Quiz

Assume you throw with two fair dice and you add up the numbers.

What is the probability of throwing more than 7?

$$P(x > 7) = \frac{5}{12}$$

Imaginary numbers

Imaginary unit: i

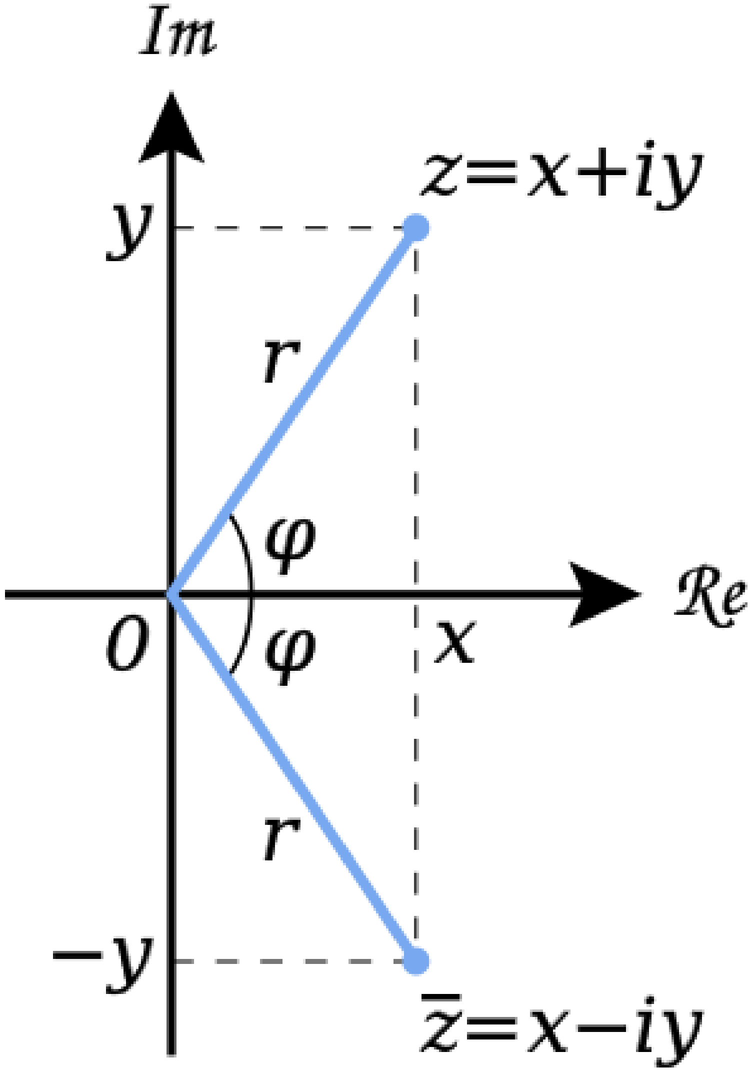
Defined as $i^2 = -1$ or $i = \sqrt{-1}$

Pure imaginary numbers: $2i, 7i, \pi i, \ldots$

Extends the real number system ($\mathbb{R}$) to the complex number system ($\mathbb{C}$)

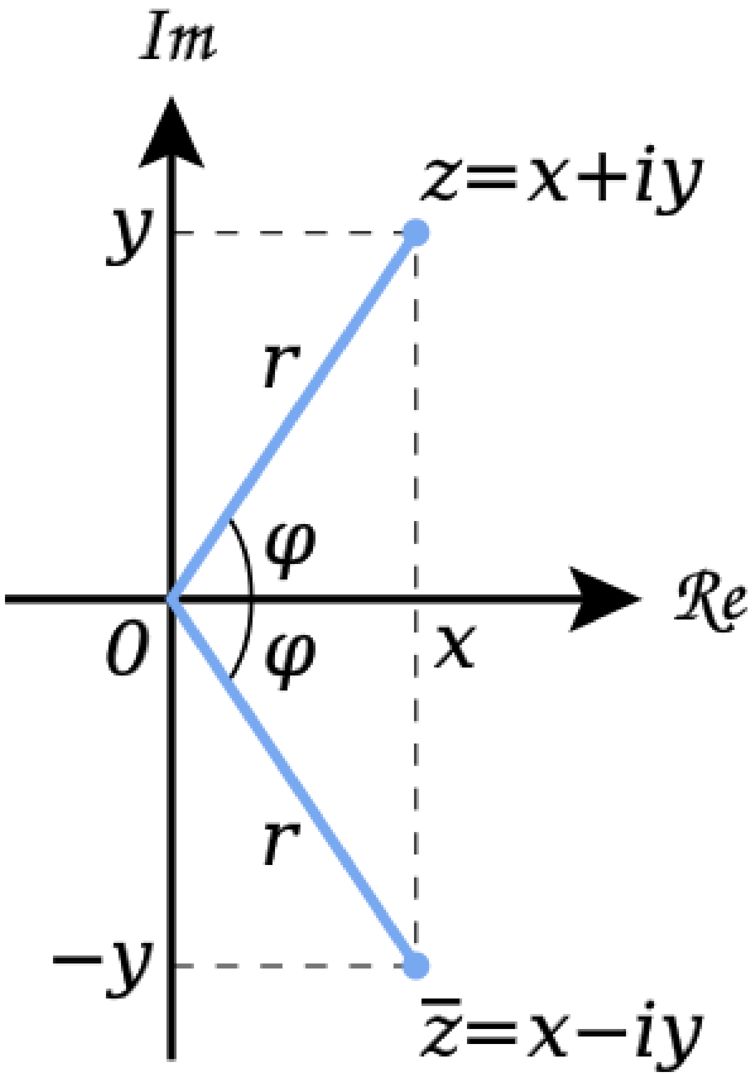
Complex numbers

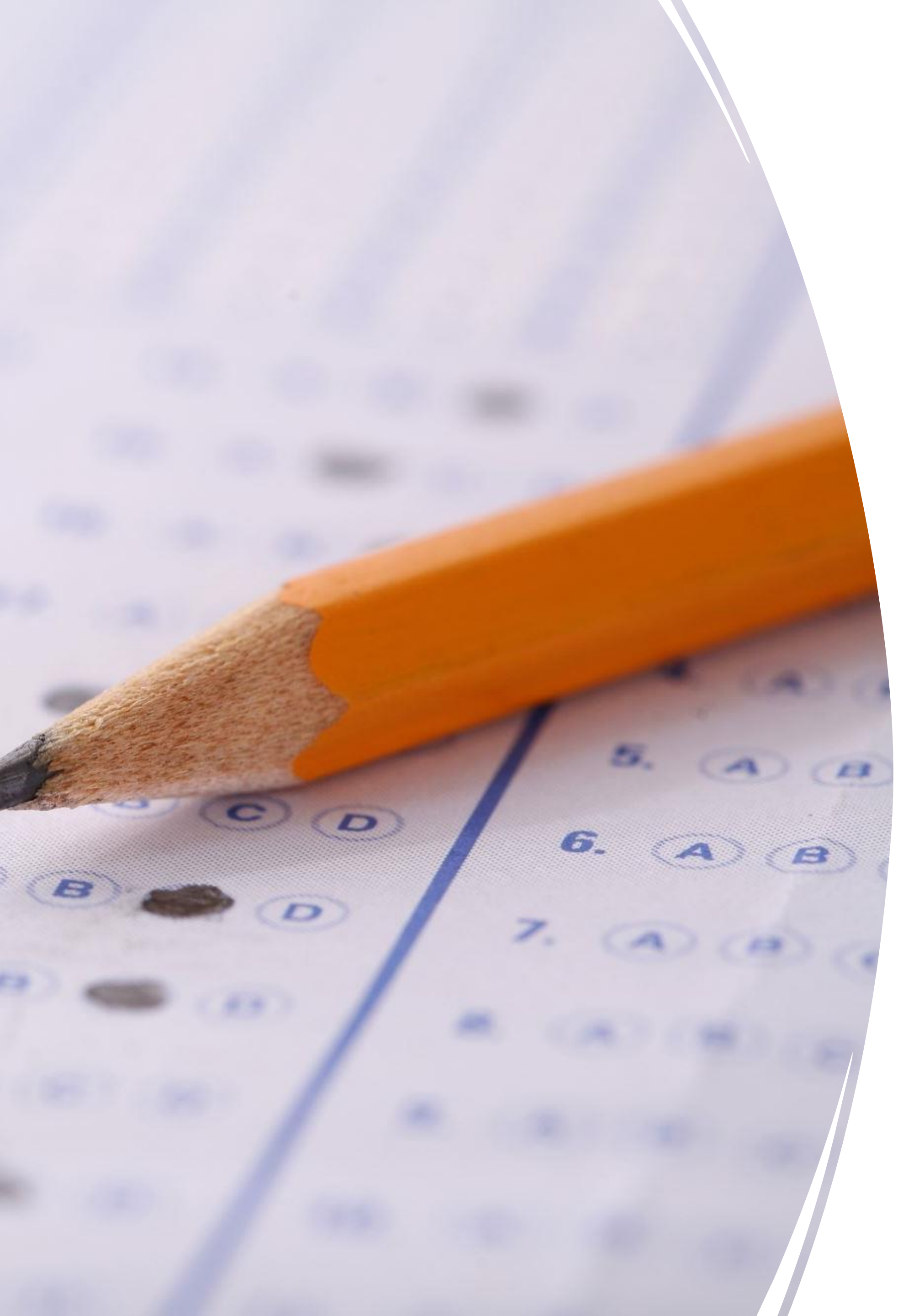
- Real and imaginary part
- Expressed in the form $z = x + yi$
  - Where $x, y \in \mathbb{R}$ and i is the imaginary unit
- Modules and argument:
  - $r = \sqrt{x^2 + y^2}$
  - $\varphi = \tan^{-1} \frac{y}{x}$



Complex numbers

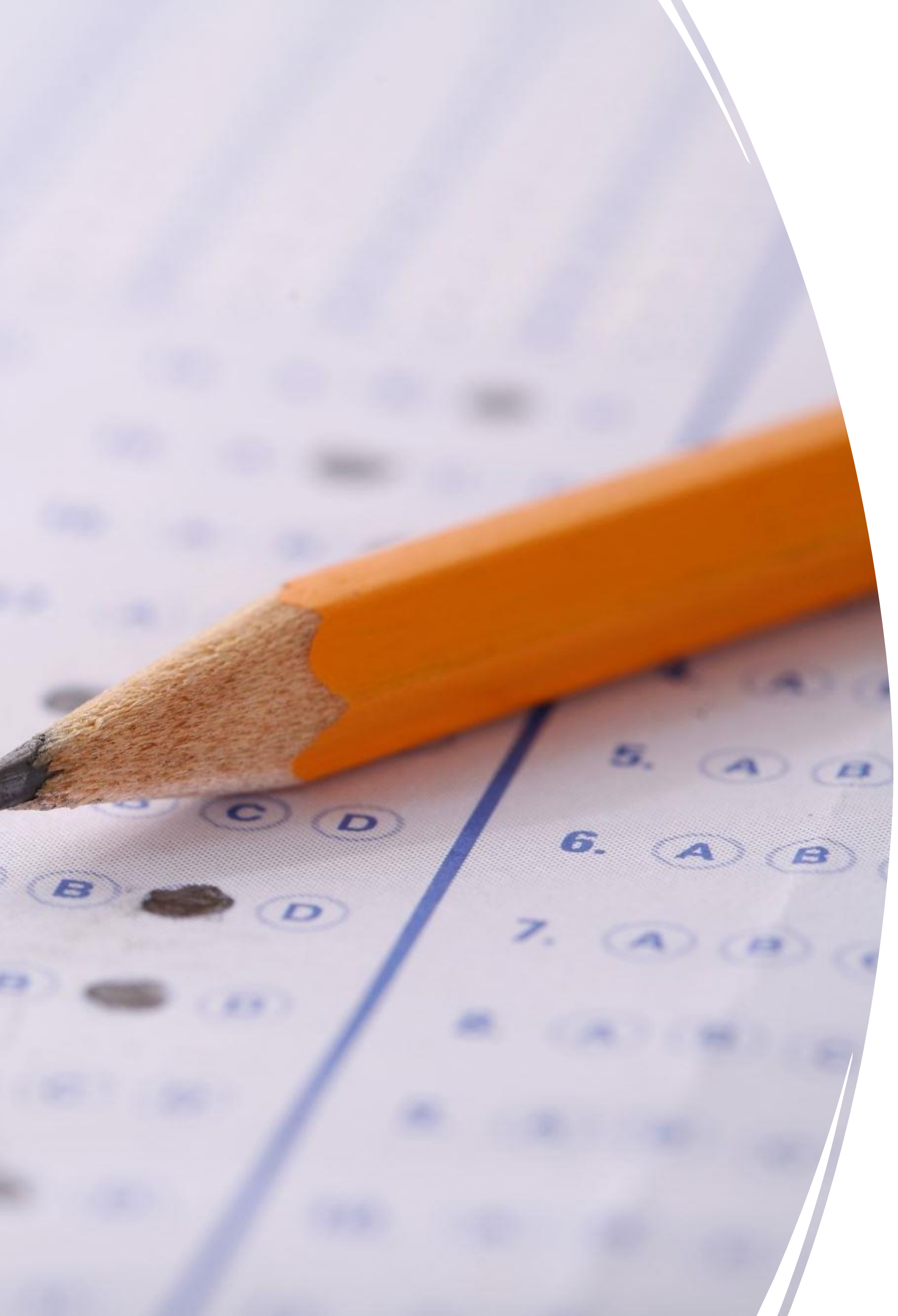
- Complex conjugate
  - The number with an equal real part and an imaginary part equal in magnitude but opposite in sign: $\bar{z} = x - yi$.





Quiz

What is the complex conjugate of the number $z = 4 + 3i$?



Quiz

What is the complex conjugate of the number $z = 4 + 3i$?

$$\bar{z} = 4 - 3i$$

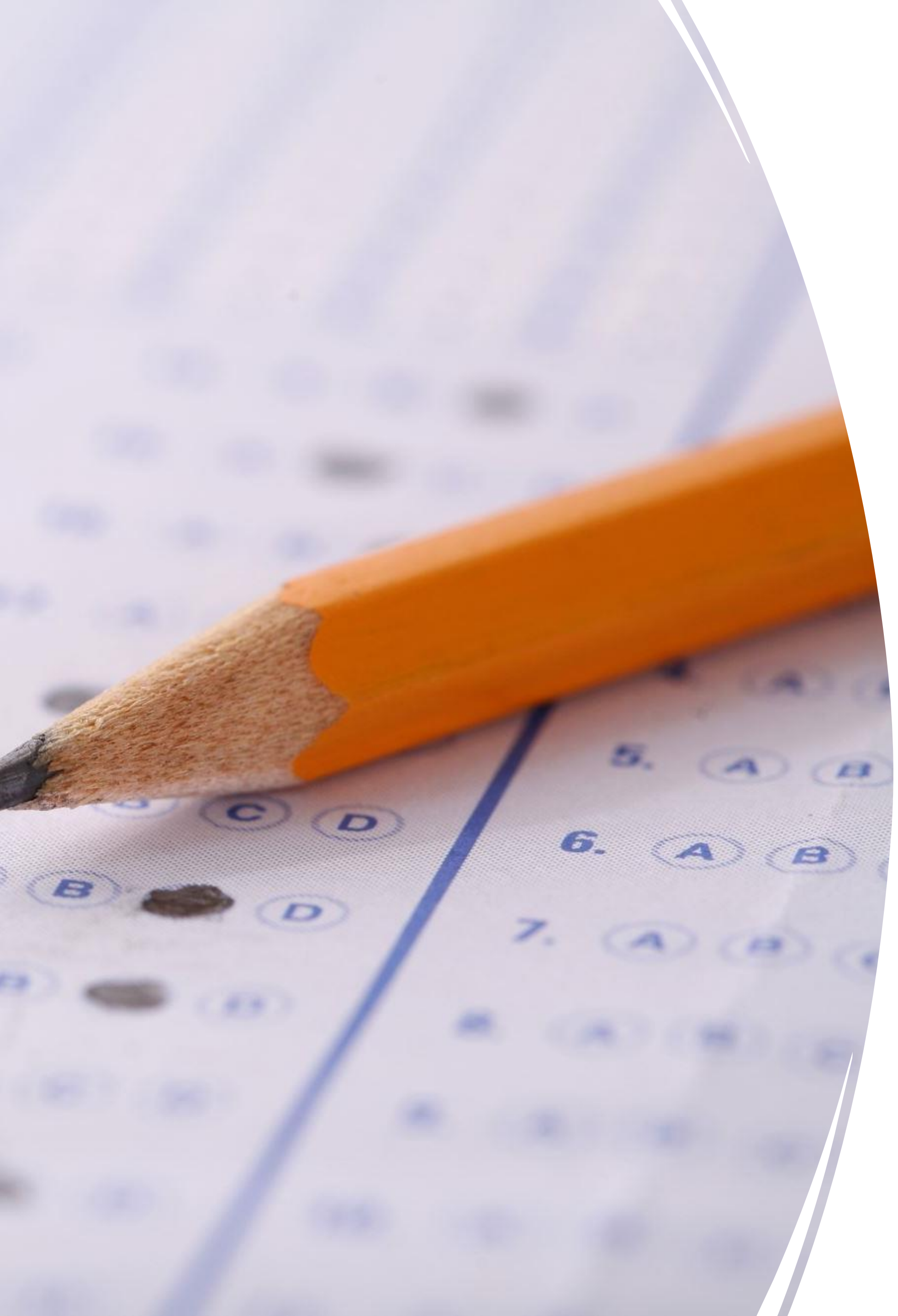
Complex numbers: addition

- Separately adding/substracting their real and imaginary parts

- Example: given $a = 7 - 5i$ and $b = 3 + 2i$;

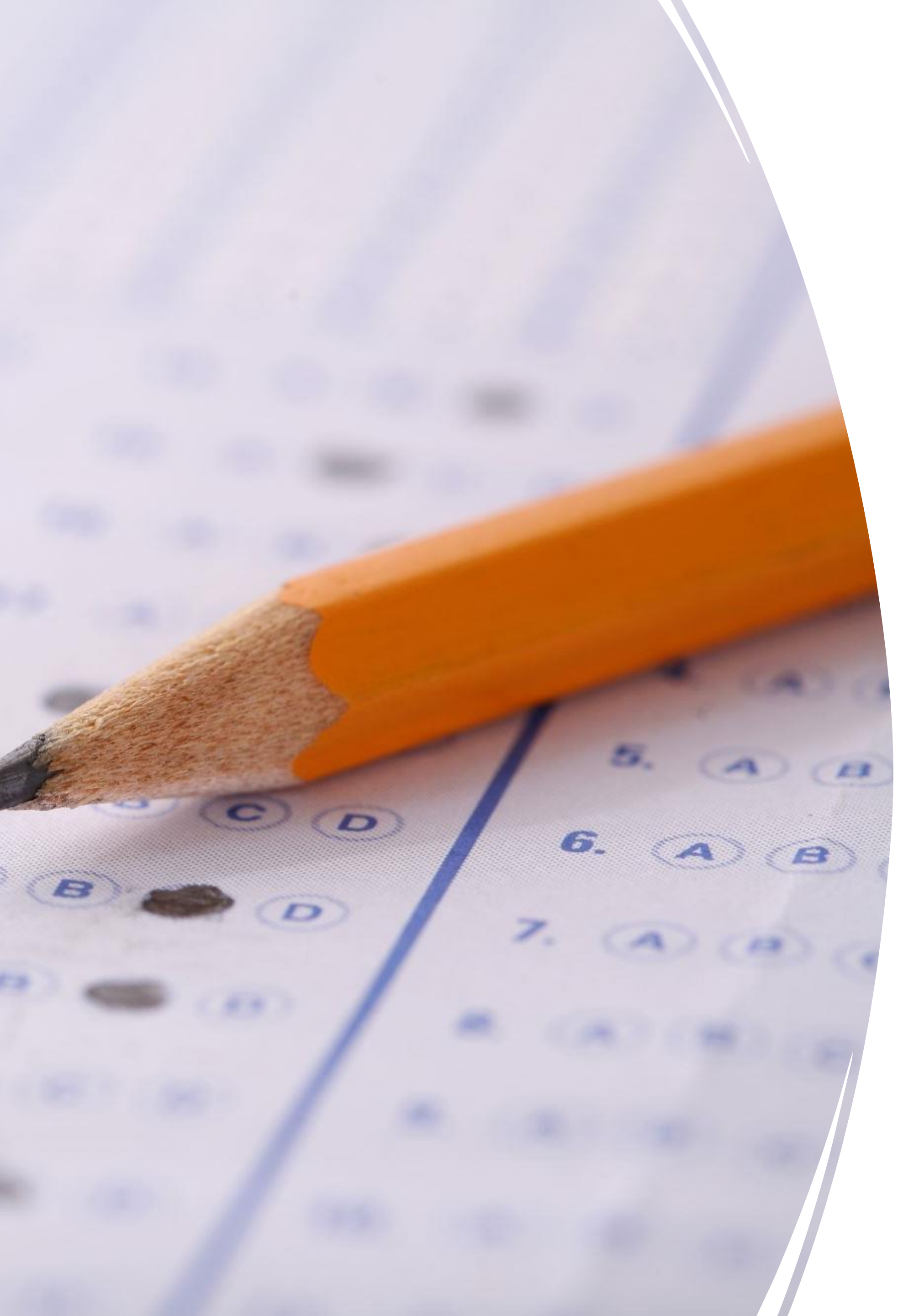
$$a + b = (7 - 5i) + (3 + 2i) = 7 - 5i + 3 + 2i = 10 - 3i$$

$$a - b = (7 - 5i) - (3 + 2i) = 7 - 5i - 3 - 2i = 4 - 7i$$



Quiz

Assuming that you have the following complex numbers: $a = 6 - i$ and $b = -2 + 5i$. What is the result of $a + b$?



Quiz

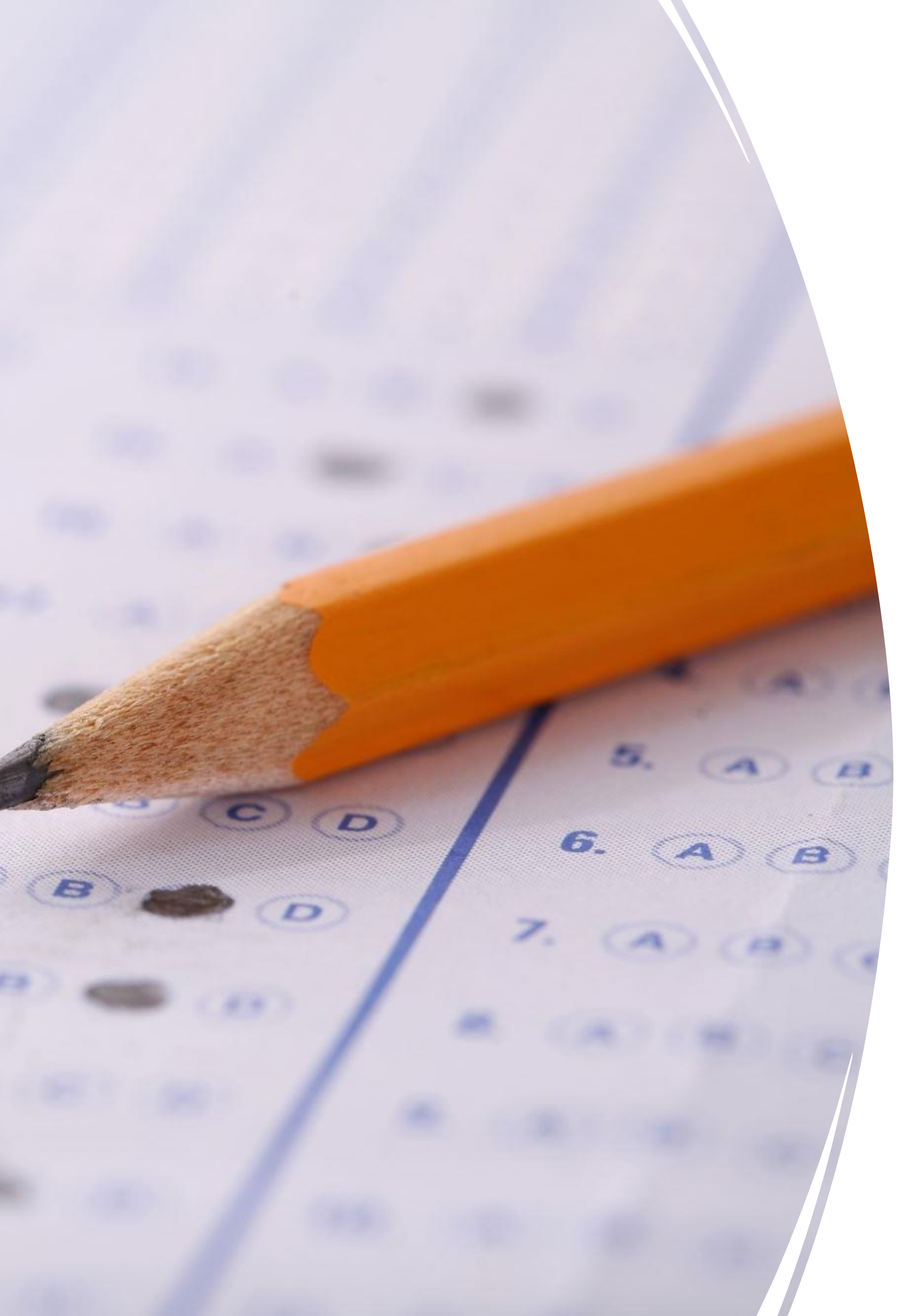
Assuming that you have the following complex numbers: $a = 6 - i$ and $b = -2 + 5i$. What is the result of $a + b$?

$$a + b = 4 + 4i$$

Complex numbers: multiplication

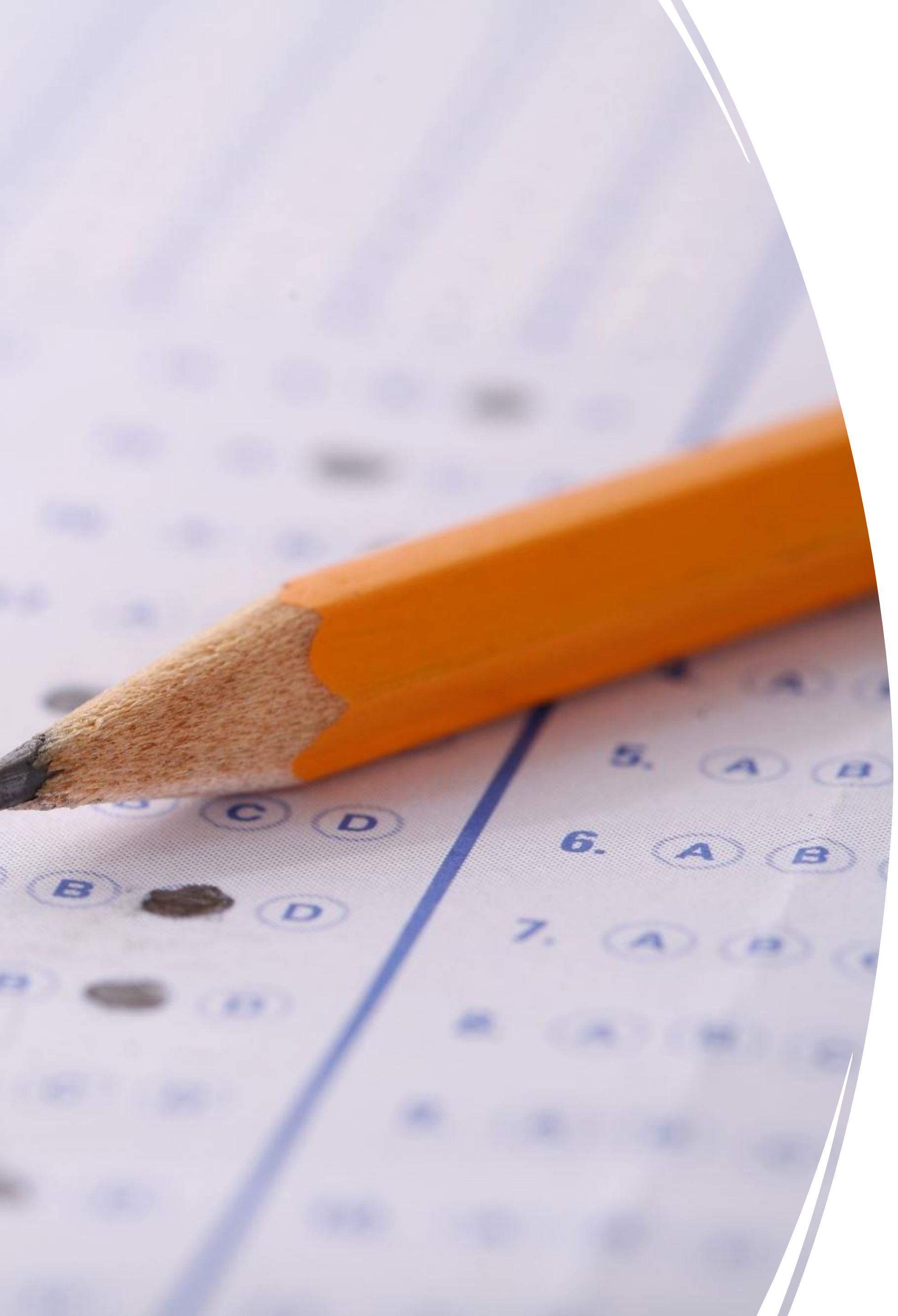
- Using the distributive and commutative properties and the definition $i^2 = -1$

- Example: given $a = 7 - 5i$ and $b = 3 + 2i$;
  $a \times b = (7 - 5i) \times (3 + 2i) \quad = 21 + 14i - 15i - 10i^2 = 21 - i + 10 = 31 - i$



Quiz

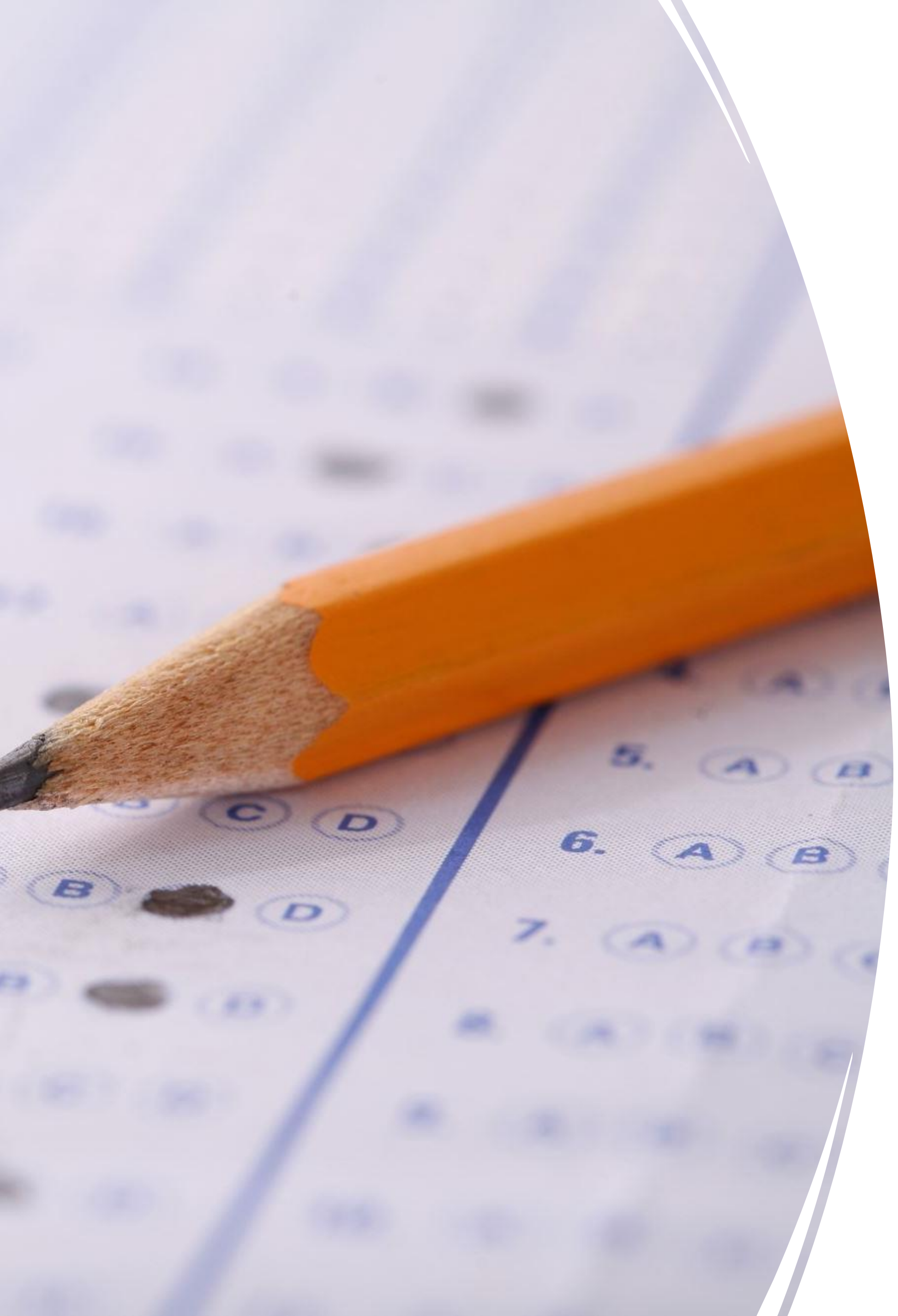
Assuming that you have the following complex numbers: $a = 3 + 4i$ and $b = 3 - 4i$. What is the result of $a \times b$?



Quiz

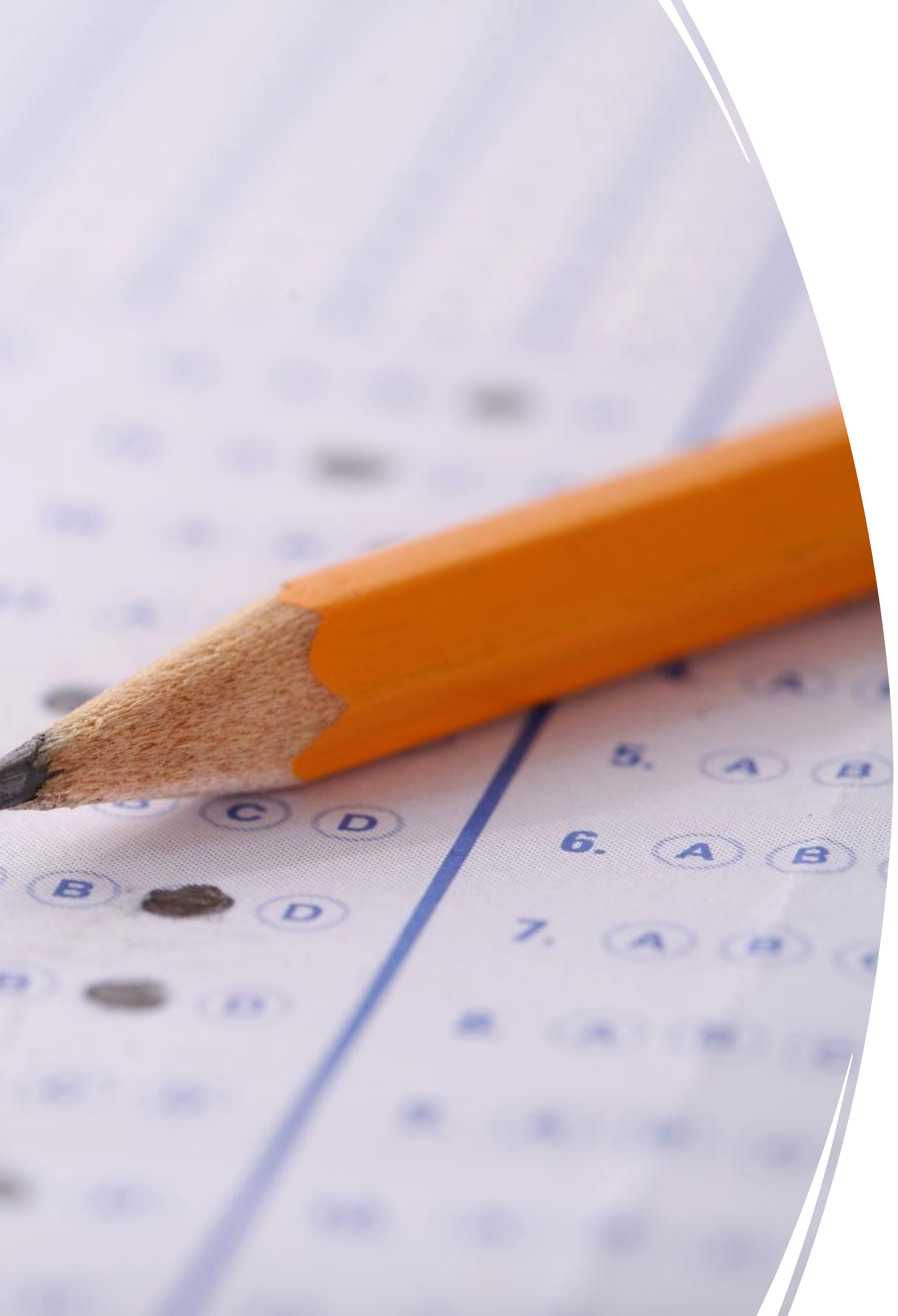
Assuming that you have the following complex numbers: $a = 3 + 4i$ and $b = 3 - 4i$. What is the result of $a \times b$?

$$a \times b = 25$$



Quiz

Assuming that you have the following complex numbers: $a = 2 + 5i, b = 3 - 4i$ and $-2 + 7i$. What is the result of $d = \left(b \times \bar{b}\right) + (a \times \bar{c})$?



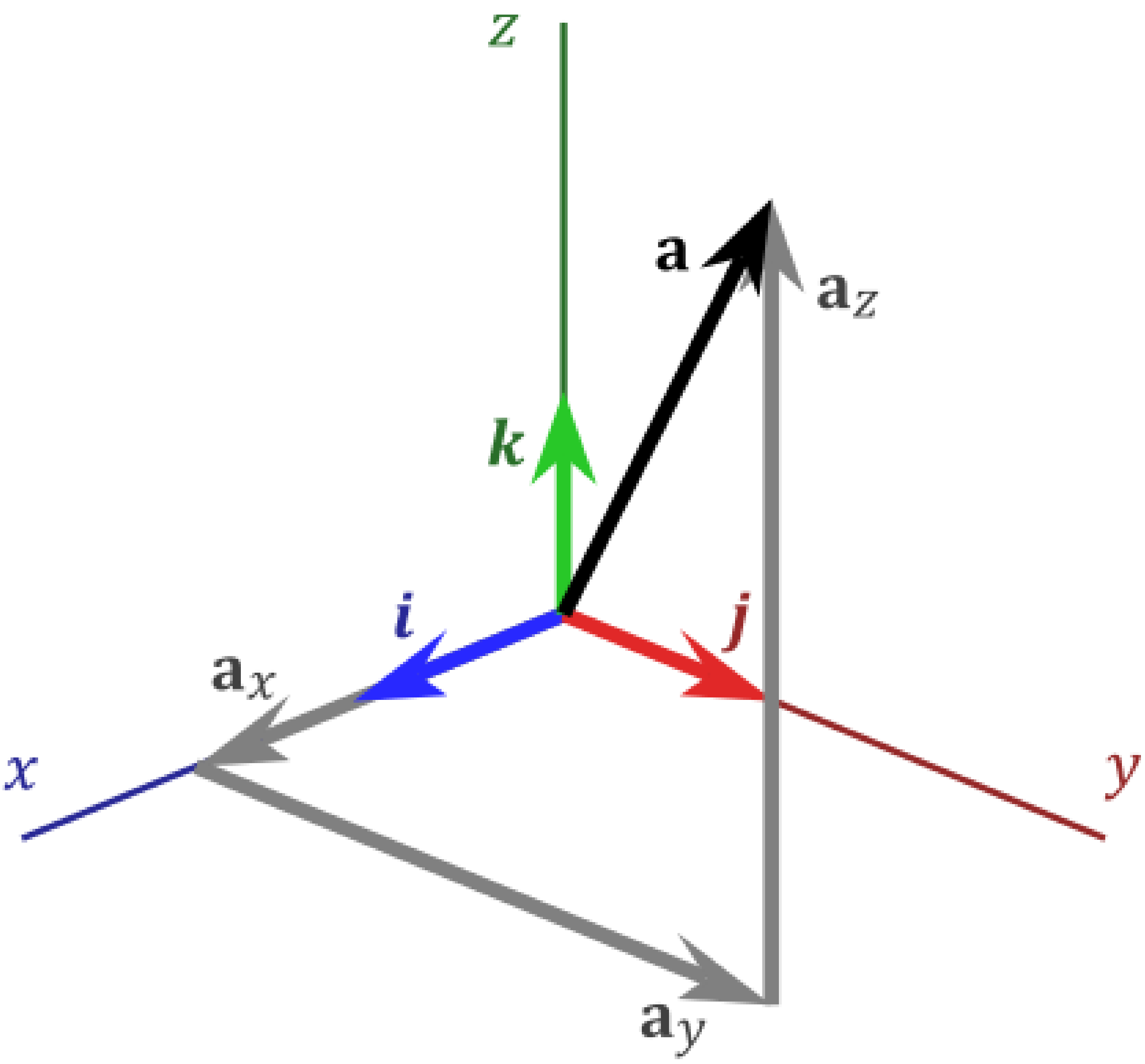
Quiz

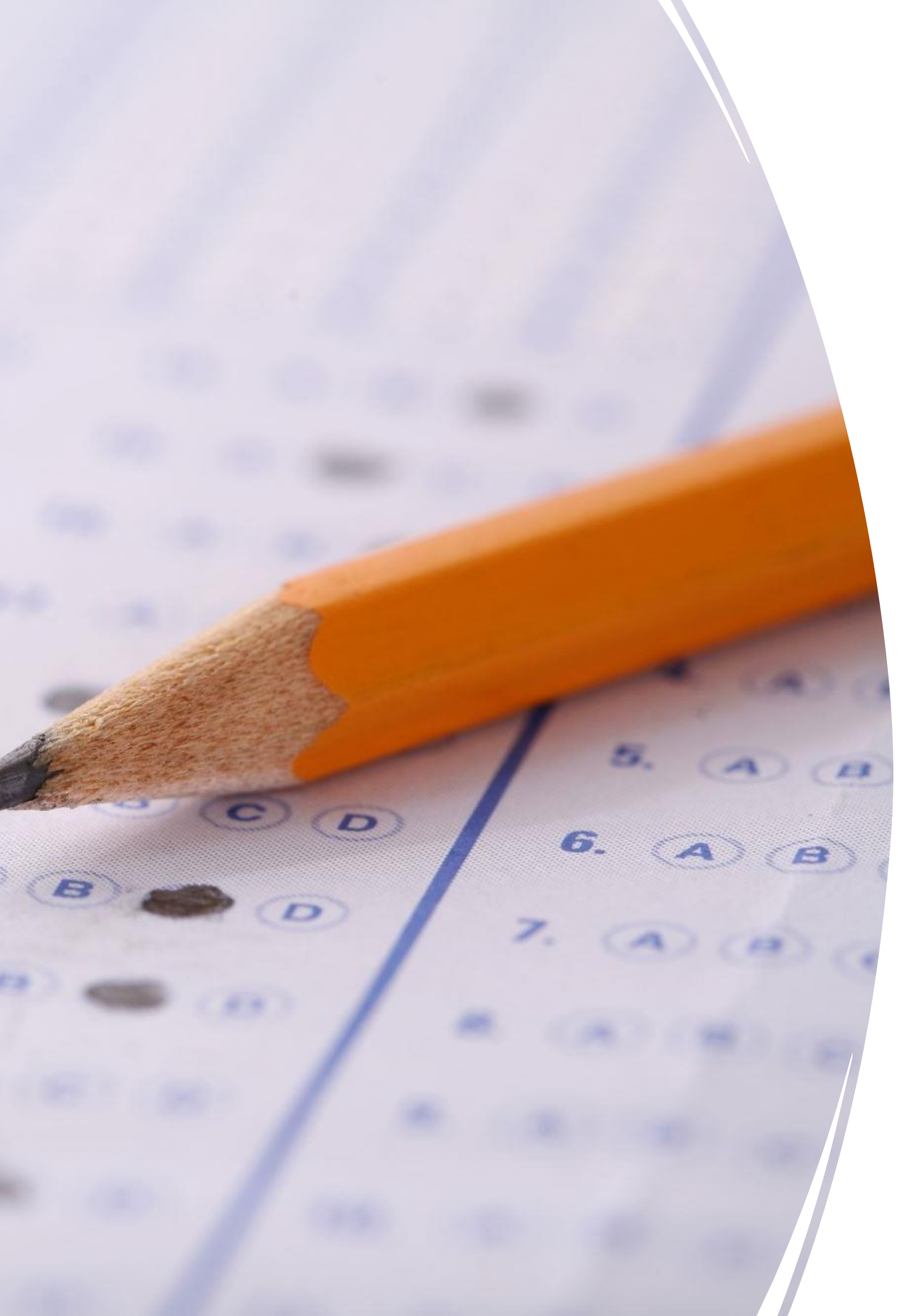
Assuming that you have the following complex numbers: $a = 2 + 5i, b = 3 - 4i$ and $-2 + 7i$. What is the result of $d = \left(b \times \bar{b}\right) + (a \times \bar{c})$?

$$d = 56 - 24i$$

Vectors

- Object that has magnitude and direction
- Algebraic operations which obey commutativity, associativity and distributivity laws
- Usually represented as $\boldsymbol{a}$ or $\vec{v}$
- Particular vector space: $\mathbb{R}^2$ or $\mathbb{R}^3$

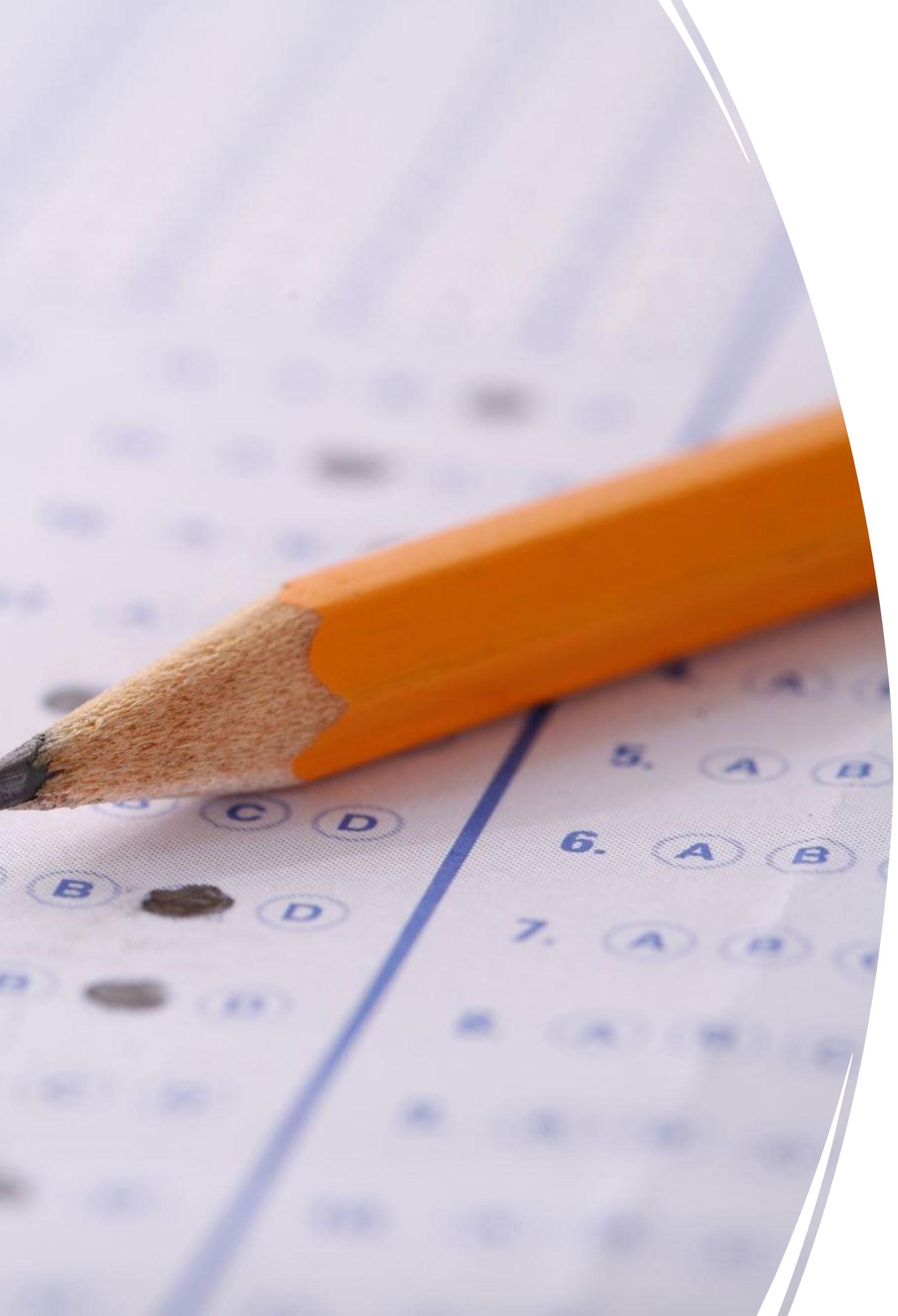




Quiz

Given the vectors $\overrightarrow{v_1} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

What is the result of $\vec{v} = \overrightarrow{v_1} + 3 \cdot \overrightarrow{v_2}$?



Quiz

Given the vectors $\overrightarrow{v_1} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

What is the result of $\vec{v} = \overrightarrow{v_1} + 3 \cdot \overrightarrow{v_2}$?

$$\vec{v} = \begin{pmatrix} 5 \\ 9 \\ -8 \end{pmatrix}$$

Vectors

- The **dot product** multiplies 2 vectors and produces a number
- Usually represented as: $\vec{a} \cdot \vec{b} = \lambda, \lambda \in \mathbb{R}$
- Given $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ then

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

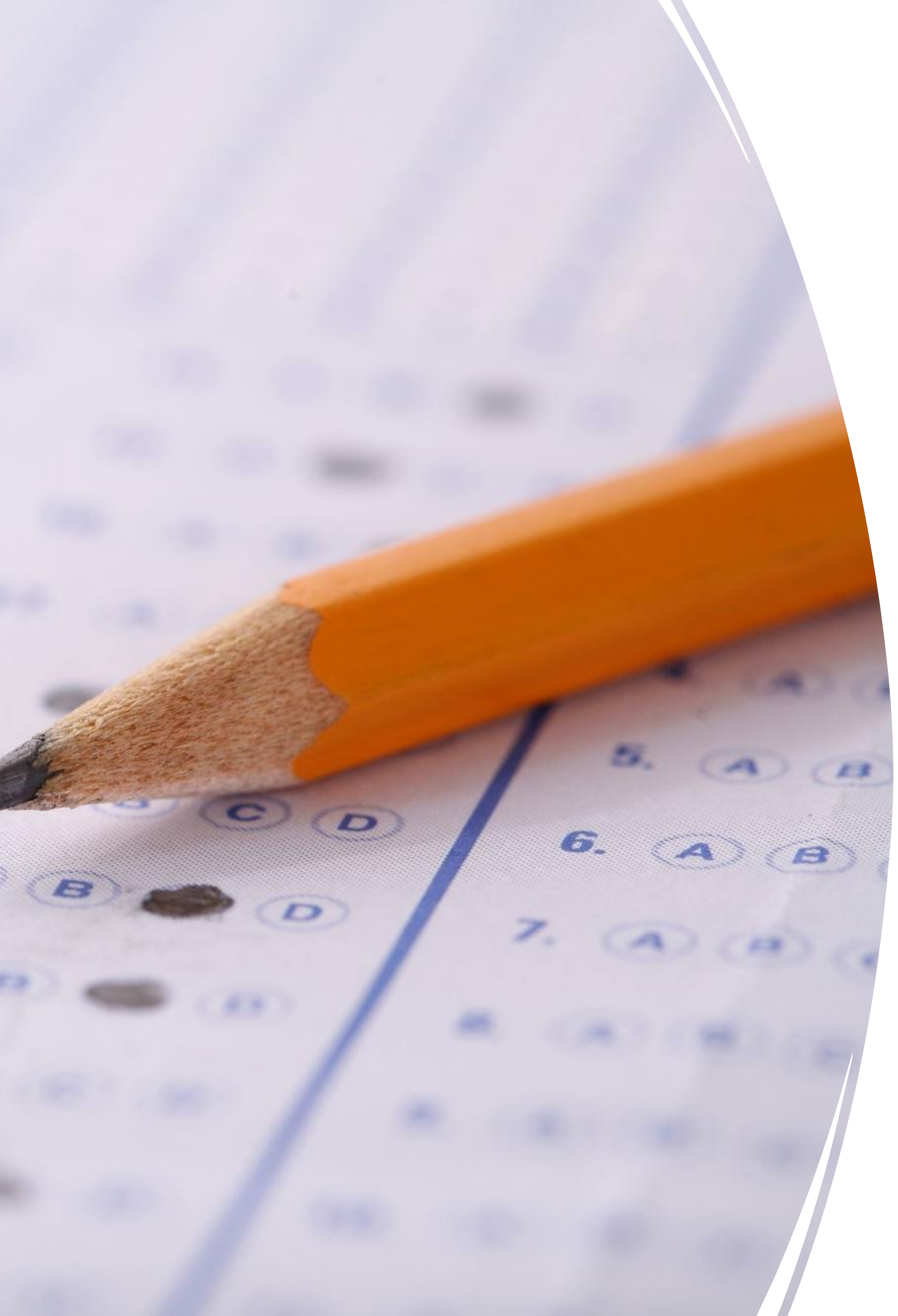
Vectors

- The **length** of a vector is the square root of the dot product of the vector with itself

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

- Must be a positive number: $\|\vec{a}\| \geq 0, \in \mathbb{R}$

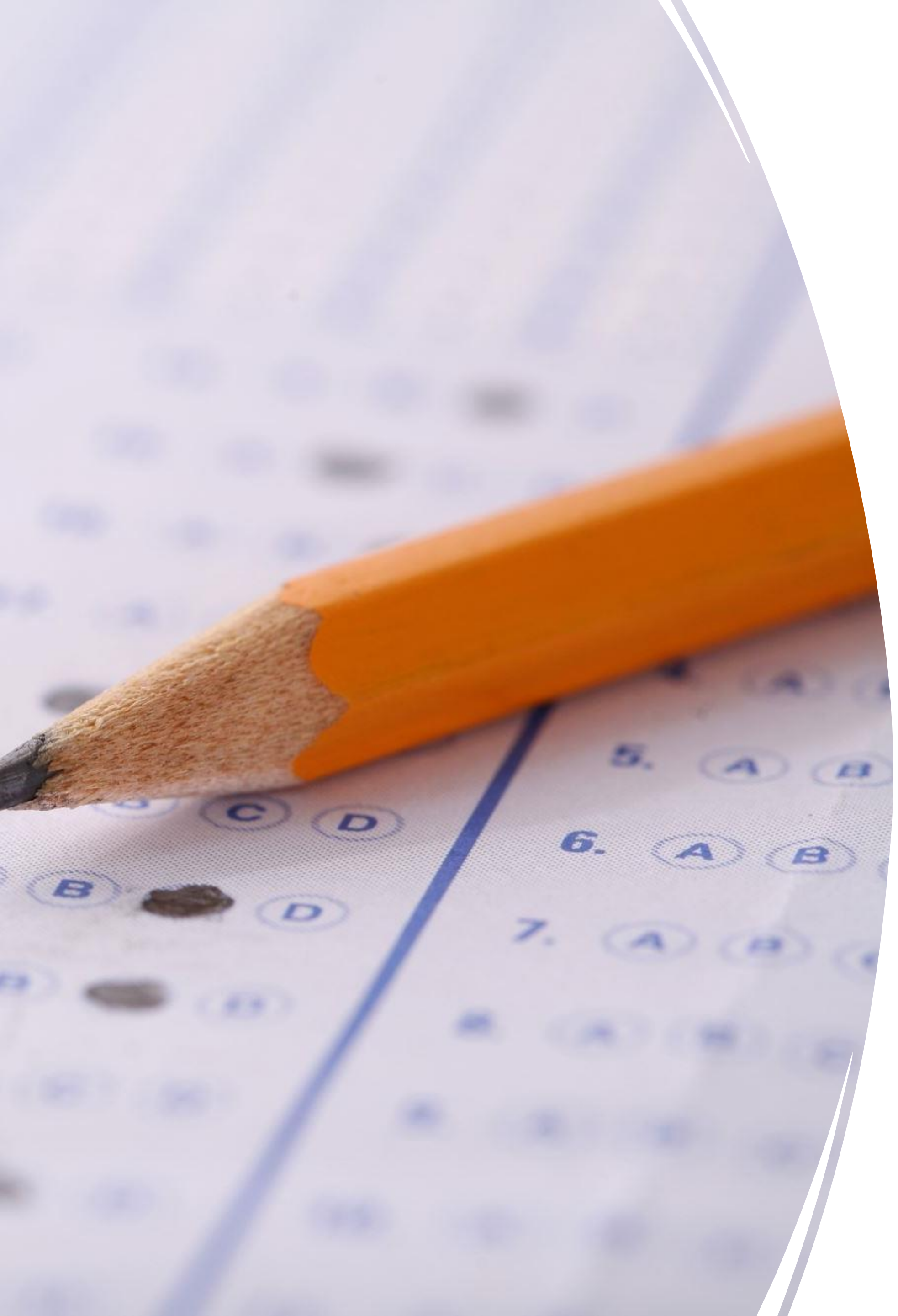
- A **unit vector** is a vector with length 1
- A vector of arbitrary length can be dividend by its length to create a unit vector. This is called normalizing a vector, written as $\hat{v}$



Quiz

Given the vector $\vec{v} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}$.

What is the lenght of vector $\vec{v}$?

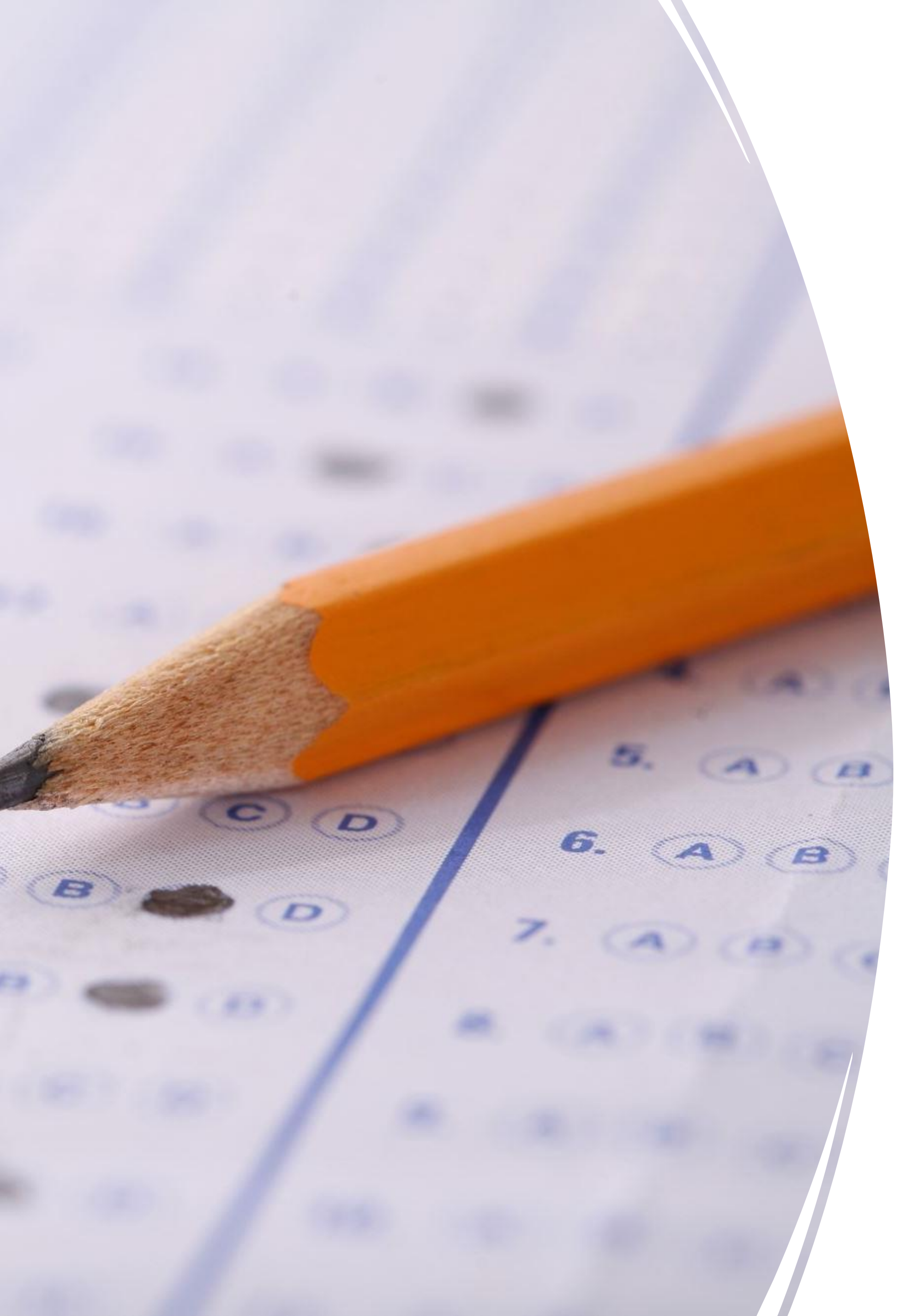


Quiz

Given the vector $\vec{v} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}$.

What is the lenght of vector $\vec{v}$?

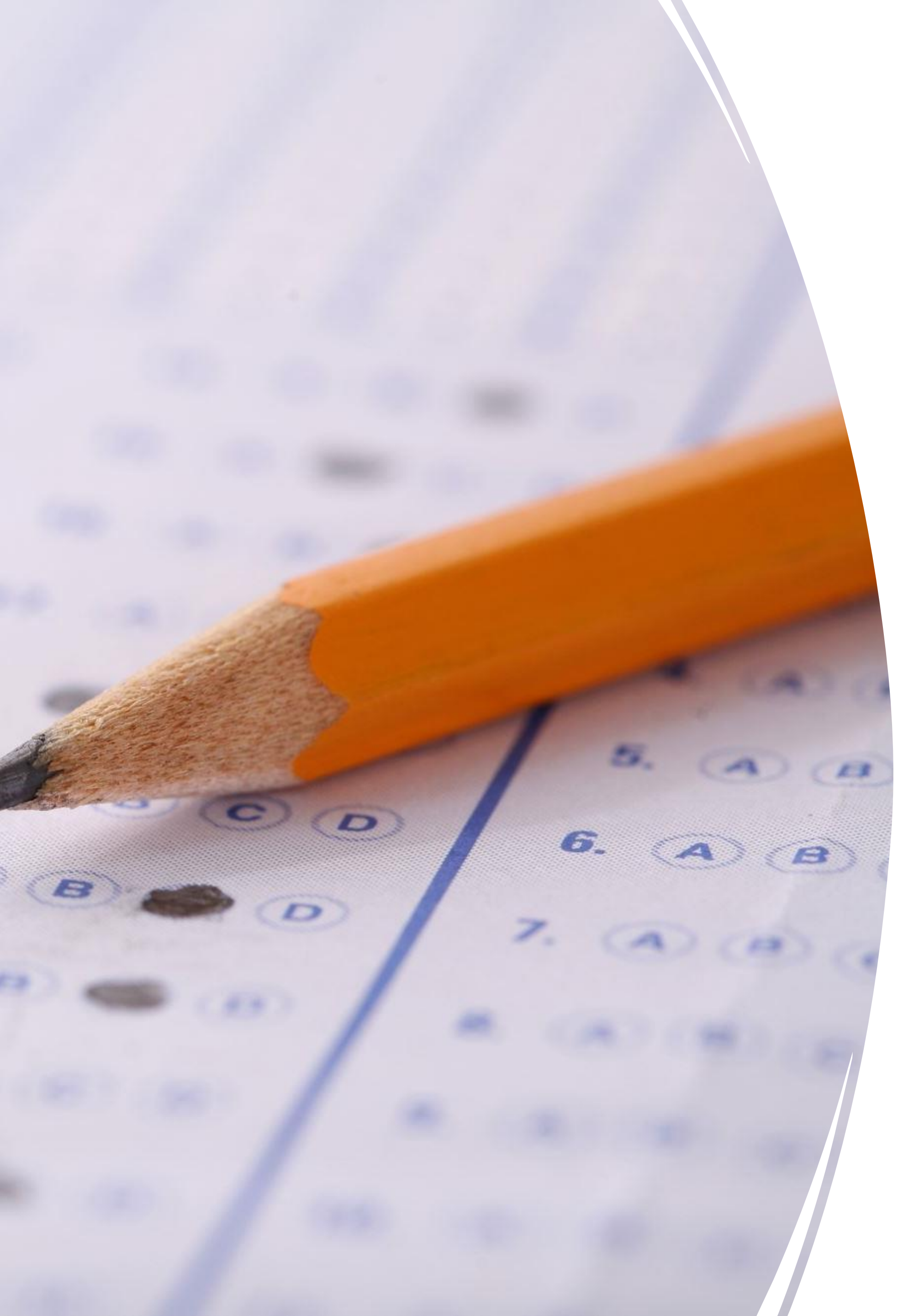
$$\|\vec{v}\| = 2\sqrt{5}$$



Quiz

Given the vector $\vec{v} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}$.

Confirm that this vector is not a unitary vector and determine its corresponding normalized vector.



Quiz

Given the vector $\vec{v} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}$.

$$\hat{v} = \begin{pmatrix} -\frac{3}{4\sqrt{2}} \\ \frac{9}{8\sqrt{2}} \\ \frac{3}{4\sqrt{2}} \\ -\frac{3}{8\sqrt{2}} \end{pmatrix}.$$

Matrices

- A rectangular arrangement of numbers, symbols or expressions into rows and columns
- Size of a matrix: number of rows and columns that it contains (dimensions)
- Possible to perform some arithmetic operations (addition and multiplication)

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} &
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\end{array}
$$

Matrices: Transpose

- Denoted as A^T
- Flips a matrix over its diagonal
- Write the rows of A as the columns of A^T
- Example:

$$A = \begin{pmatrix} 2 & -5 \\ 6 & 3 \\ -1 & 2\pi \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 6 & -1 \\ -5 & 3 & 2\pi \end{pmatrix}$$

Matrices: Trace

- Defined as: $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$
- Sum of the entries of the main diagonal
- The Trace is not defined for non-square matrices
- Example:

$$A = \begin{pmatrix} 1 & 3 \\ 8 & 4 \end{pmatrix}$$

$$\mathrm{Tr}(A) = 1 + 4 = 5$$

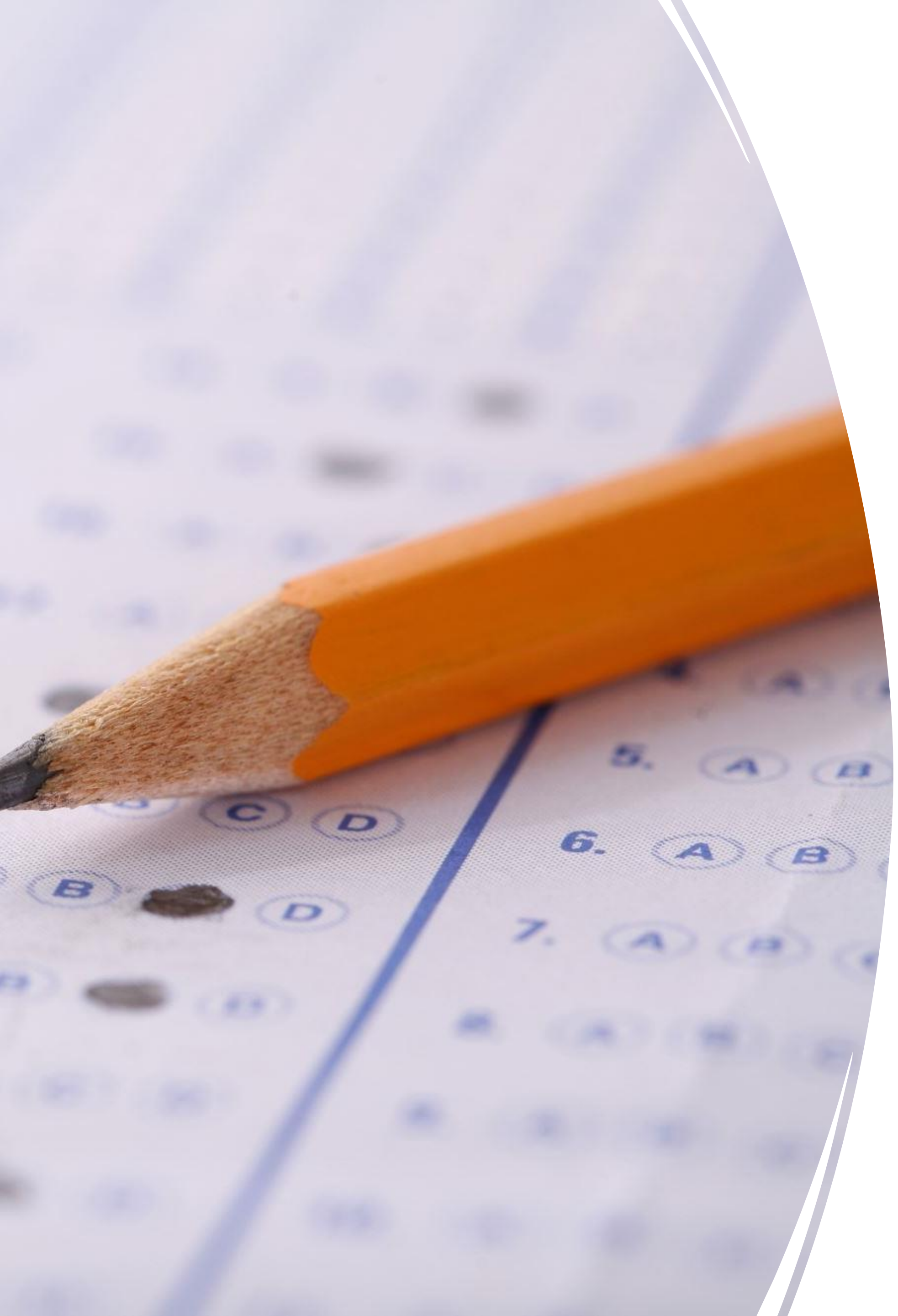
Matrices: Scalar multiplication

- Real numbers are known as **scalars**
- Scalar multiplication: each entry in the matrix is multiplied by the given scalar.
- Example:

$$A = \begin{pmatrix} 2 & -5 \\ 6 & 3 \\ -1 & 2\pi \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 2 & -5 \\ 6 & 3 \\ -1 & 2\pi \end{pmatrix}$$

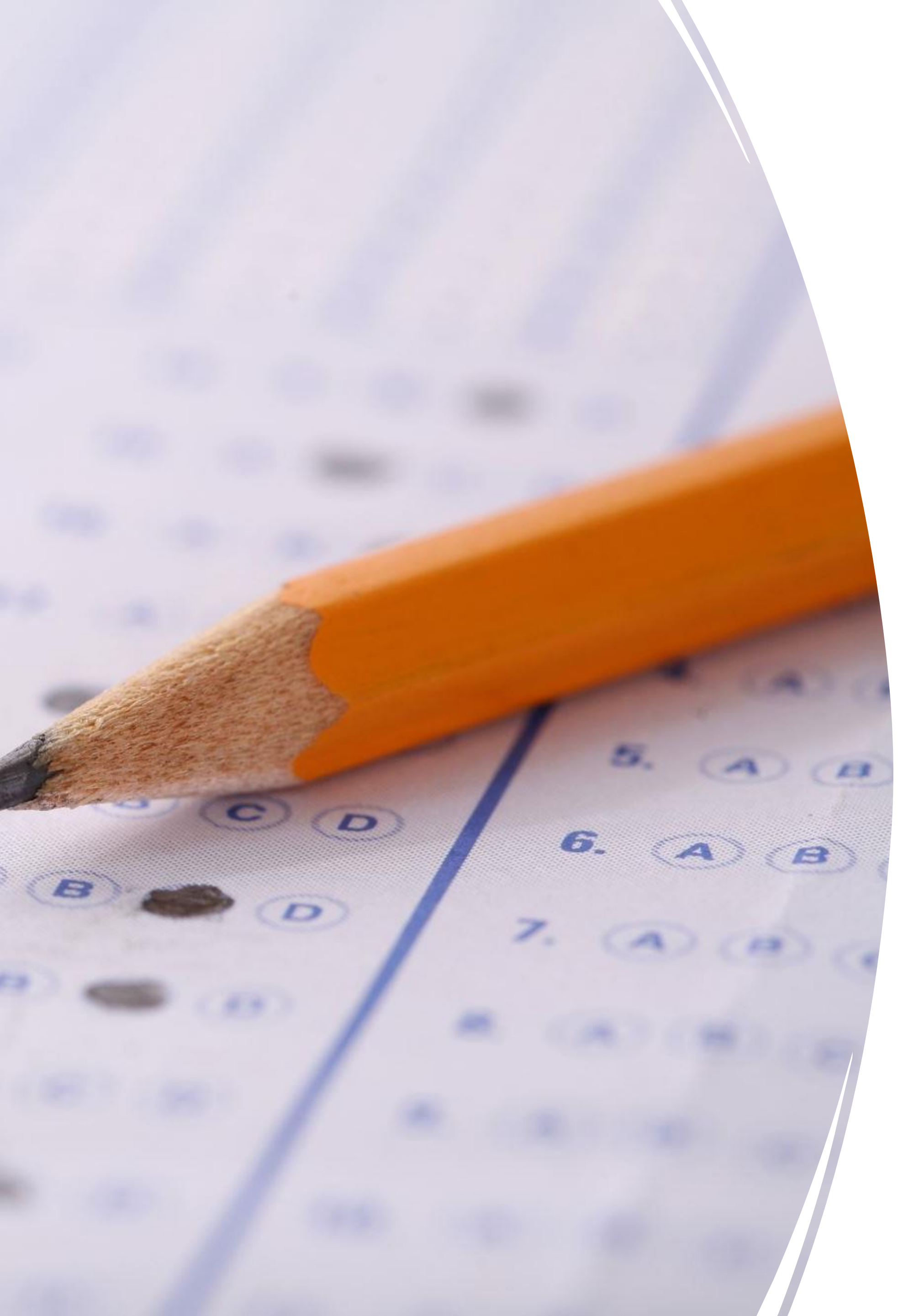
$$= \begin{pmatrix} 8 & -5 \\ 24 & 12 \\ -4 & 8\pi \end{pmatrix}$$



Quiz

Given the matrices $A = \begin{pmatrix} 0 & 4 & -2 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & -3 \end{pmatrix}$.

What is the resulting matrix C of the following operations $C = 3A + B^T$?



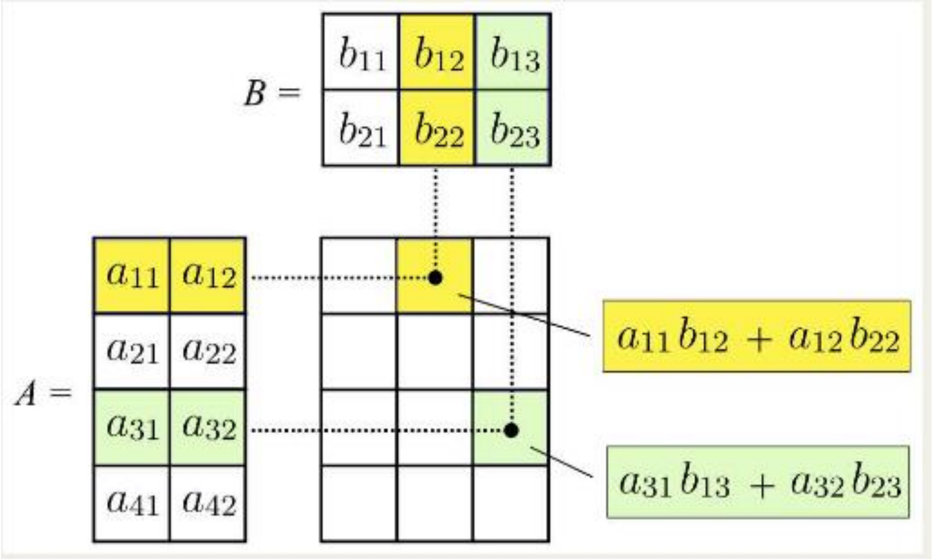
Quiz

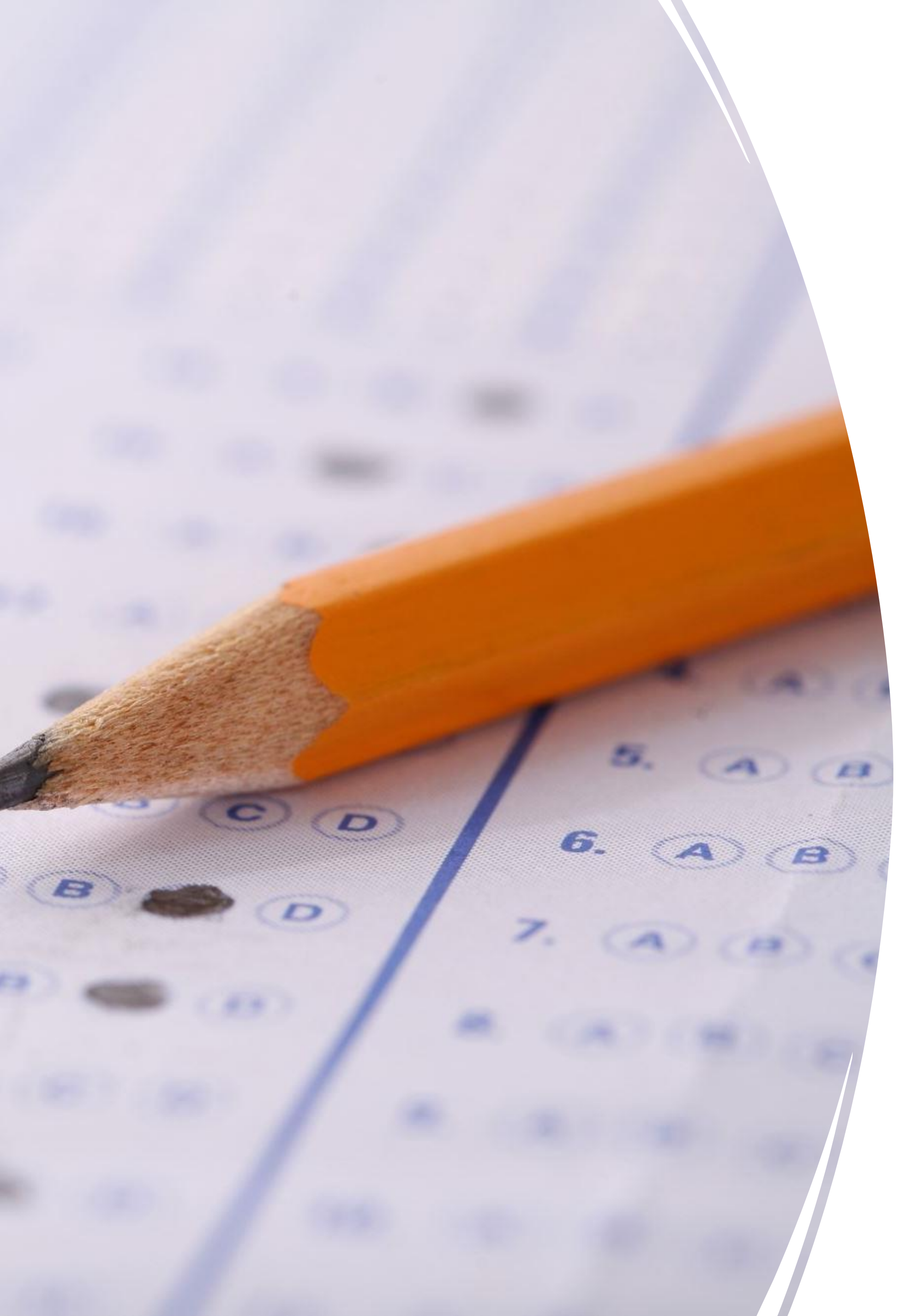
What is the resulting matrix C of the following operations $C = 3A + B^T$?

$$C = \begin{pmatrix} 0 & 11 & -5 \\ -5 & 3 & 5 \\ 3 & 0 & -3 \end{pmatrix}$$

Matrices: Multiplication

- Refers to the product of two matrices (A and B)
- **If and only if** the number of columns of the left matrix A is the same as the number of rows of the right matrix B
- Each entry in the resulting matrix is the **dot product** of a row in the matrix A and a column in the matrix B

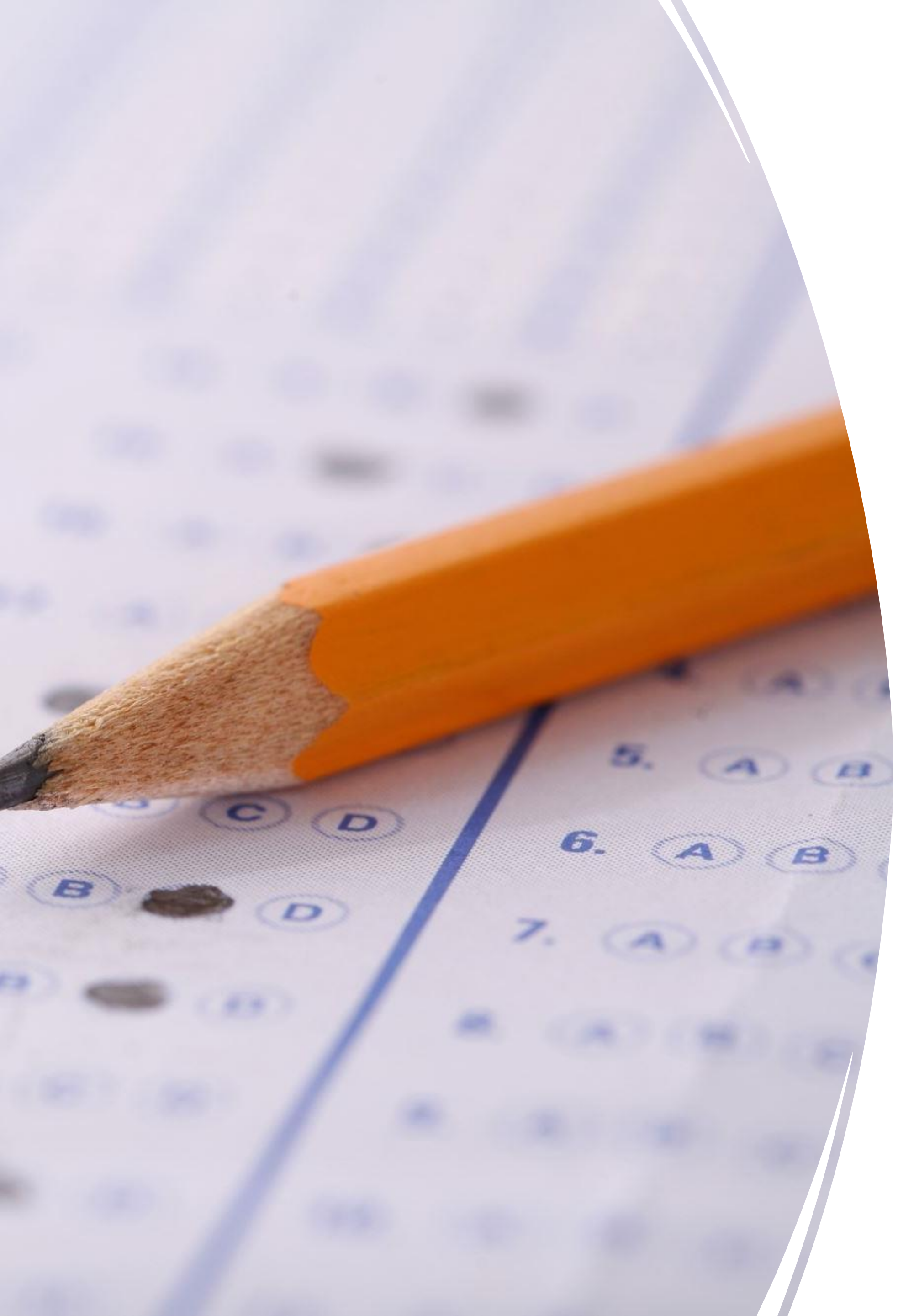




Quiz

Given the matrices $A = \begin{pmatrix} 3 & 0 & 4 & 1 \\ 5 & 2 & 6 & 7 \\ 1 & 0 & 2 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 6 \\ 1 & 5 & 2 \\ 0 & 3 & 3 \end{pmatrix}$.

What is the resulting matrix C of the following operation $C = A \times B$?



Quiz

What is the resulting matrix C of the following operation $C = A \times B$?

$$C = \begin{pmatrix} 4 & 26 & 11 \\ 10 & 64 & 45 \\ 2 & 35 & 28 \end{pmatrix}$$

Matrices: Matrix-vector multiplication

- Consider a vector as a 1-column matrix
- Number of columns in the matrix **must be equal** to the number of rows in the vector
- For instance: $A\vec{v}$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{1,1}v_1 + a_{1,2}v_2 + \cdots + a_{1,n}v_n \\ a_{2,1}v_1 + a_{2,2}v_2 + \cdots + a_{2,n}v_n \\ \vdots \\ a_{m,1}v_1 + a_{m,2}v_2 + \cdots + a_{m,n}v_n \end{pmatrix}$$

- Wait a second!! Remember the **dot product**?

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

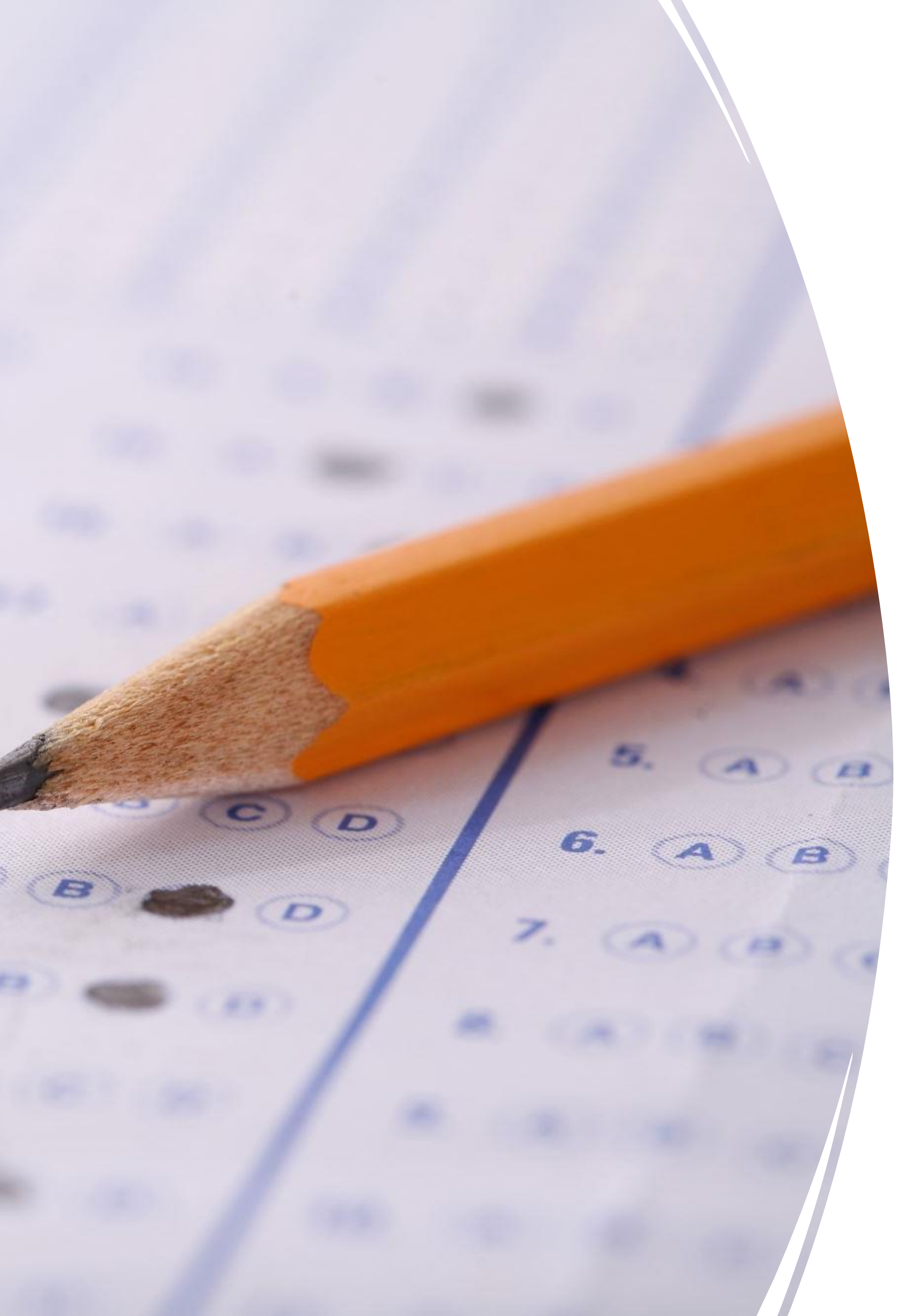
Matrices: Matrix-vector multiplication

- Updated definition for dot product: $\vec{a} \cdot \vec{b} = \vec{a}^T \cdot \vec{b}$
- Given $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$$\vec{a} \cdot \vec{b} = \vec{a}^T \cdot \vec{b}$$

$$= \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

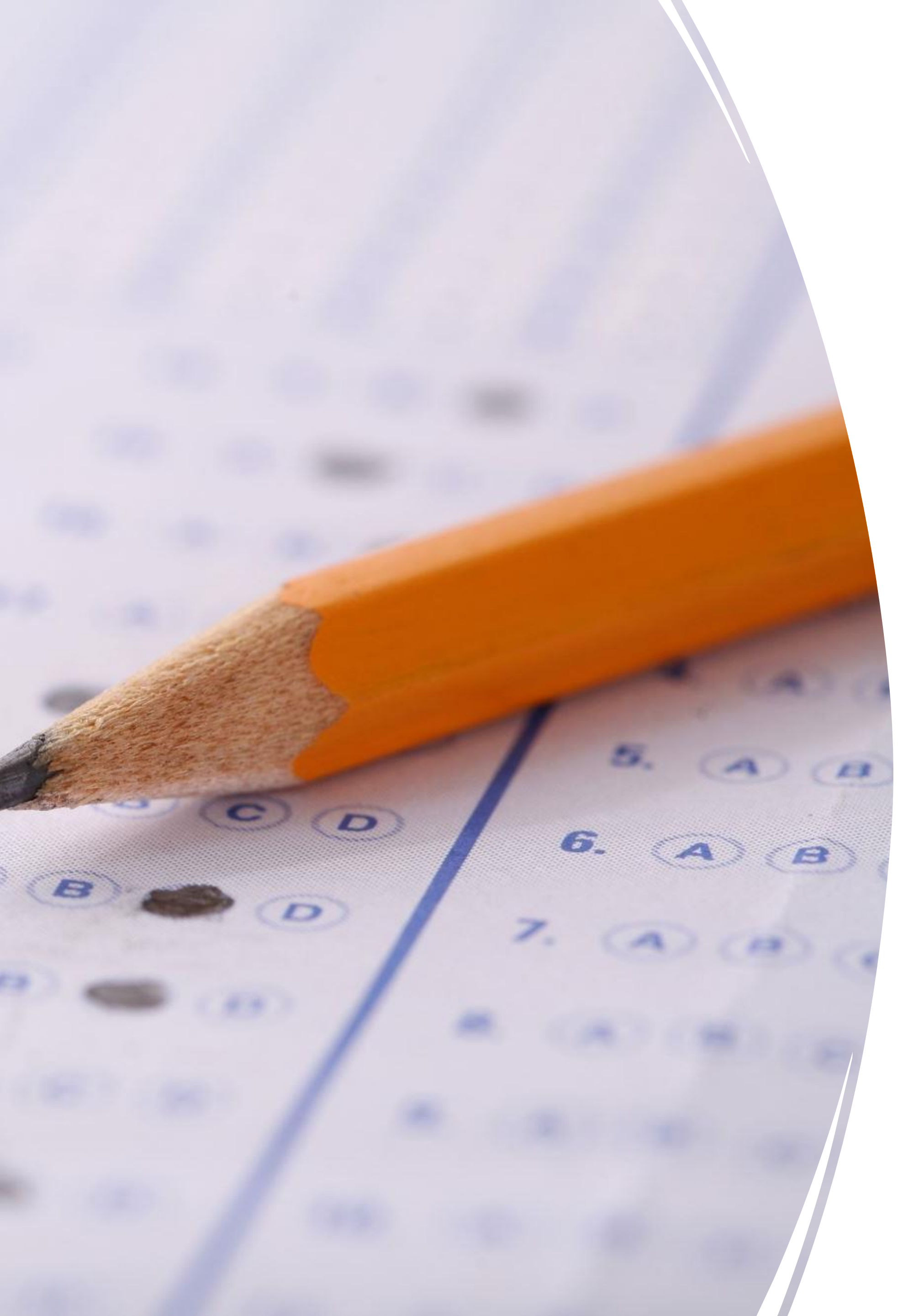
$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



Quiz

What is the resulting vector $\vec{v}$ of the following matrix-vector multiplication?

$$\vec{v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



Quiz

What is the resulting vector $\vec{v}$ of the following matrix-vector multiplication?

$$\vec{v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrices: Tensor product

- Operation on two matrices of arbitrary size resulting in a block matrix
- $A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$
- $A \otimes (B + C) = A \otimes B + A \otimes C$
- $(A + B) \otimes C = A \otimes C + B \otimes C$
- $(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

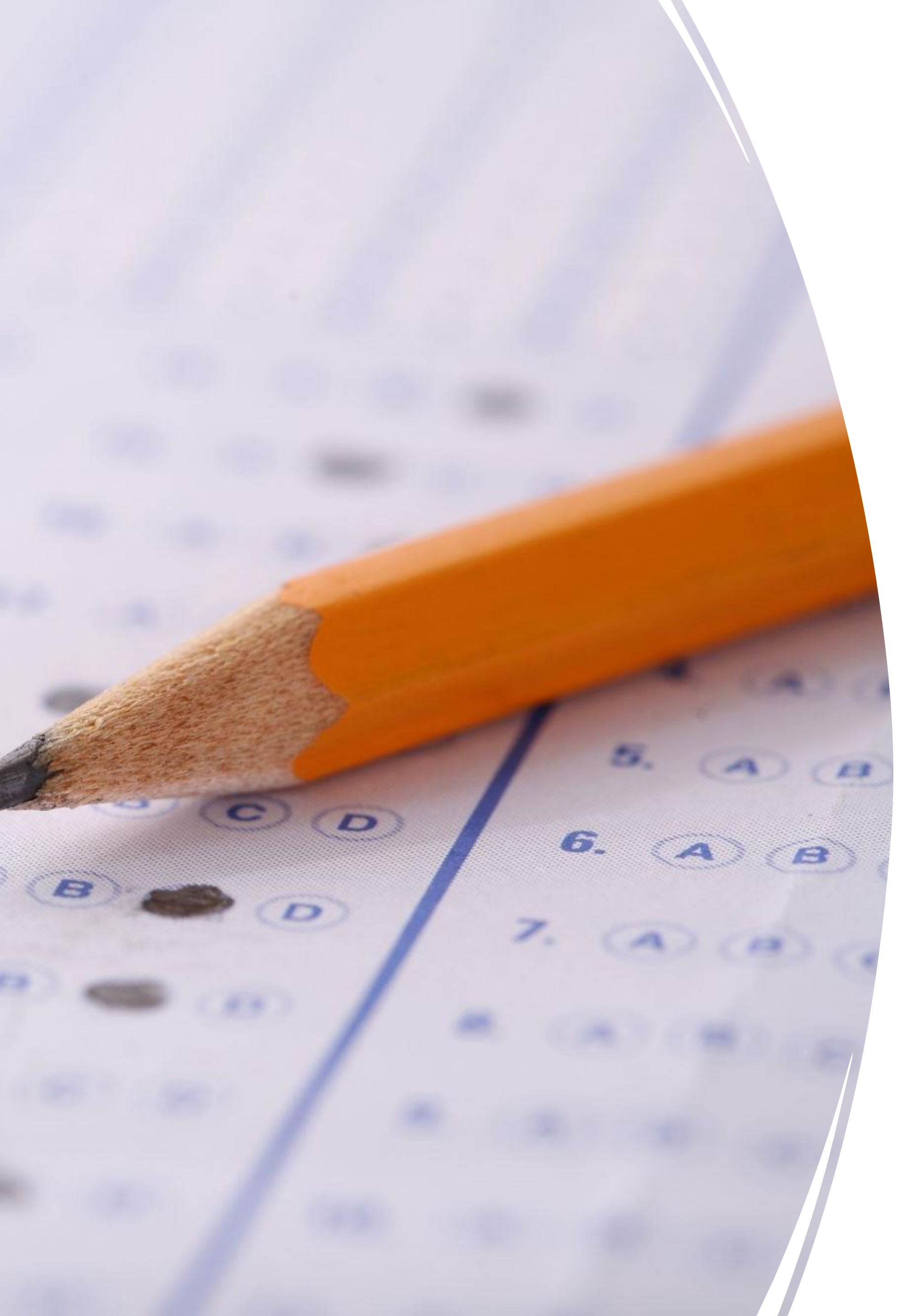
Matrices: Tensor product

- Example:

$A = \begin{pmatrix} 2 & -5 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$

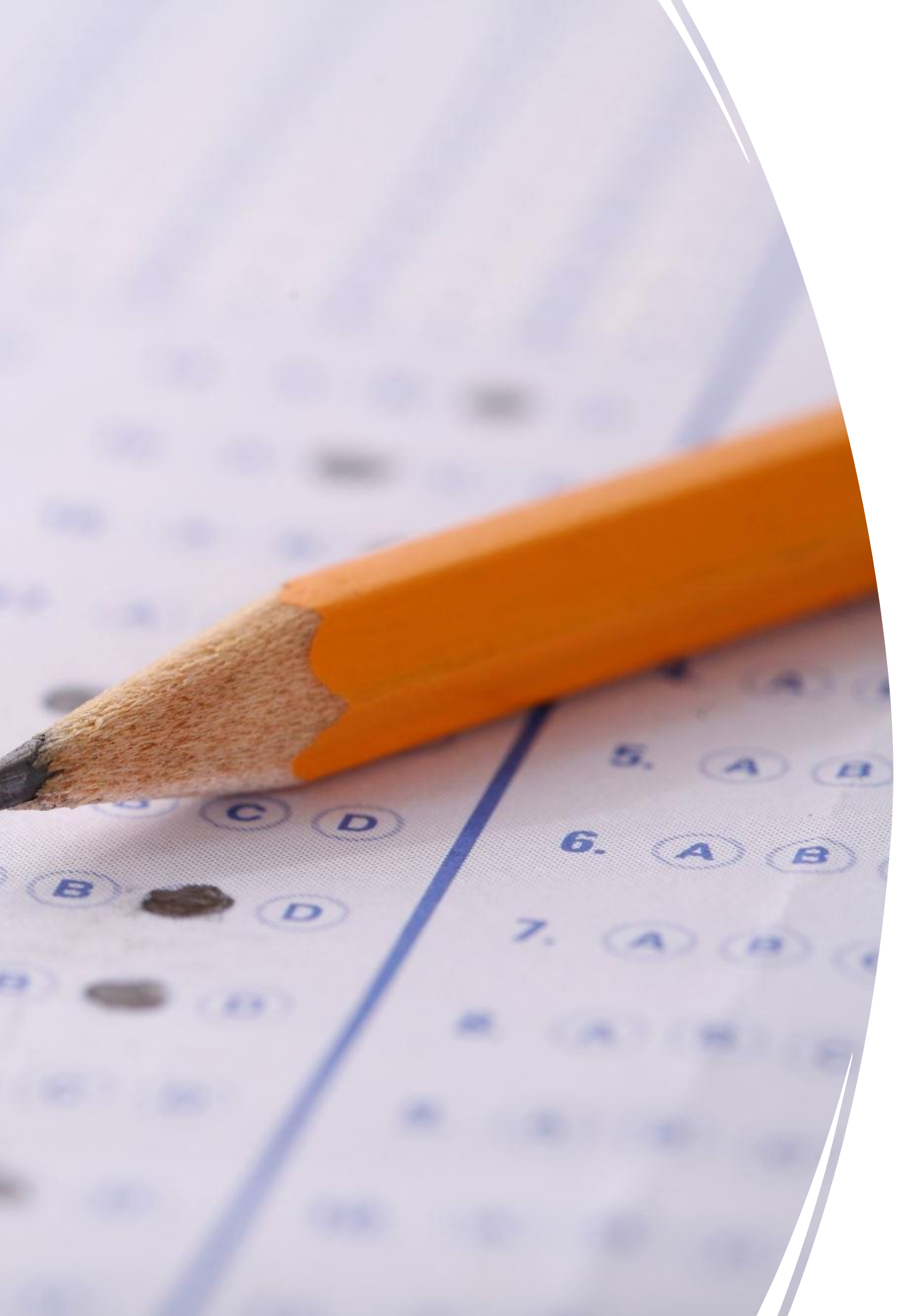
$$A \otimes B = \begin{pmatrix} 2\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} & -5\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} & 3\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 2 & 8 & -5 & -20 \\ -2 & 4 & 5 & -10 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & -3 & 6 \end{pmatrix}$$



Quiz

Given the matrices $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. What is the resulting matrix T of the following operation $T = Y \otimes S$?



Quiz

Given the matrices $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. What is the resulting matrix T of the following operation $T = Y \otimes S$?

$$T = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Final Quiz of this section!

Given the matrices $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. What is the resulting matrix T of the following operation $T = Y \otimes S$?

What is the resulting vector $\vec{x}$ of the following matrix-vector operations?

$$\vec{x} = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Final Quiz of this section!

Given the matrices $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. What is the resulting matrix T of the following operation $T = Y \otimes S$?

What is the resulting vector $\vec{x}$ of the following matrix-vector operations?

$$\vec{x} = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$


ANY
QUESTIONS?