Amsterdam University of Applied Sciences

Quantum Computing Introduction

Fundamentals: A Multi–Qubit World

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Multi–qubit systems

- 2–qubits system $\rightarrow$ 4 distinguishable states
- Normalized vector in a 4–D vector space
- Standard basis: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
  - Another notation: $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$
- Example:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

- When measuring the system:

$$\begin{aligned}
\mathbf{Pr}\{|00\rangle\} &= 0.5 \\
\mathbf{Pr}\{|01\rangle\} &= 0.5 \\
\mathbf{Pr}\{|10\rangle\} &= 0.0 \\
\mathbf{Pr}\{|11\rangle\} &= 0.0
\end{aligned}$$

Multi–qubit systems

- n–qubits system $\rightarrow 2^n$ distinguishable states
- Normalized vector in a 2^n–D vector space
- Standard basis consists of 2^n vectors
- $|0\rangle, |1\rangle, |2\rangle, \ldots, |2^n - 1\rangle$
  - $|0\rangle \rightarrow$ all qubits are in the 0 state
  - $|2^n - 1\rangle \rightarrow$ all qubits are in the 1 state
- As expected, basis vectors are orthonormal:
  - $\langle x|y\rangle = 0$, when $|x\rangle \neq |y\rangle$
  - $\langle x|x\rangle = 1$
- Uniform superposition:

$$|+\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

Multi–qubit systems

$$
\begin{aligned}
|\psi\rangle &= \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle \\
&= \alpha_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \alpha_i \in \mathbb{C} \wedge \sum_{i=0}^{2^n-1} \overline{\alpha_i}\alpha_i = 1
\end{aligned}
$$

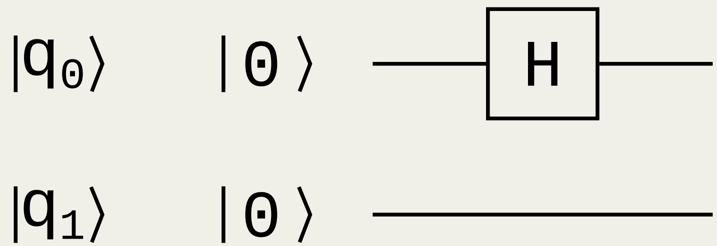
- WARNING: Bloch sphere does **NOT** hold anymore in a multi–qubit world!

Multi–qubit systems

- Also represented using Kronecker product:

$$
\begin{aligned}
|\psi\rangle &= |00\rangle \\
&= |0\rangle \otimes |0\rangle \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Quantum state evolution



$$
\begin{aligned}
H_0\,|00\rangle &= (I_1 \otimes H_0)\,|00\rangle \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) |00\rangle \\
&= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 0 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \right) |00\rangle \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} (|00\rangle + |01\rangle) \\
&= |0\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\
&= |0\rangle \otimes |+\rangle
\end{aligned}
$$

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Question 3 *What is the resulting transformation matrix* T_1 *when applied the following operation* $T_1 = Y \otimes S$

`Write down your solution here:`

$$T_1 =$$

Partial measurement

- Assume:

$$|\psi\rangle = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle$$

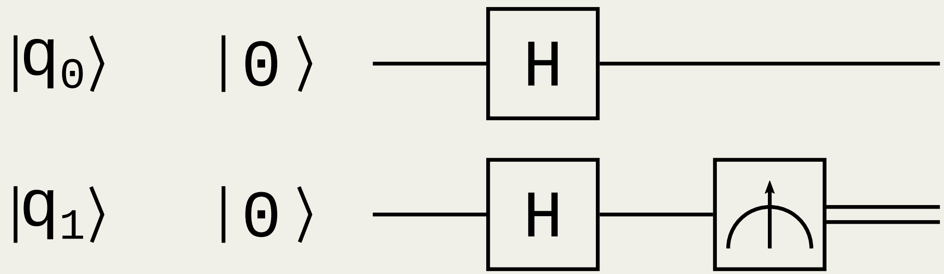
- When measuring only the first qubit (q_1):
  - What is $\mathbf{Pr}\{0\}$? or $\mathbf{Pr}\{1\}$?

$$\begin{aligned}\mathbf{Pr}\{0\} &= \mathbf{Pr}\{00\} + \mathbf{Pr}\{01\} \\ &= \overline{\alpha_0}\alpha_0 + \overline{\alpha_1}\alpha_1\end{aligned} \qquad \begin{aligned}\mathbf{Pr}\{1\} &= \mathbf{Pr}\{10\} + \mathbf{Pr}\{11\} \\ &= \overline{\alpha_2}\alpha_2 + \overline{\alpha_3}\alpha_3\end{aligned}$$

  - What is the resulting vector state?

$$\begin{aligned} q_1 &\to |0\rangle \\ q_0 &\to \frac{\alpha_0 |0\rangle + \alpha_1 |1\rangle}{\sqrt{\overline{\alpha_0}\alpha_0 + \overline{\alpha_1}\alpha_1}} \end{aligned} \qquad \begin{aligned} q_1 &\to |1\rangle \\ q_0 &\to \frac{\alpha_2 |0\rangle + \alpha_3 |1\rangle}{\sqrt{\overline{\alpha_2}\alpha_2 + \overline{\alpha_3}\alpha_3}} \end{aligned}$$

Example



- Applying a Hadamard gate to each qubit

$$H_1 H_0 \left|00\right\rangle = \frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle)$$

- When measuring:

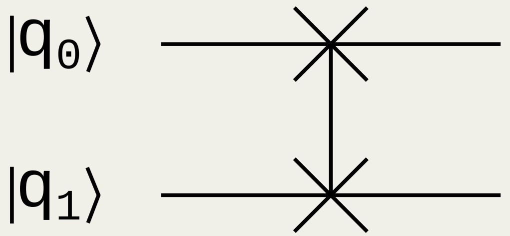
$$M(q_1) = 0 \rightarrow \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|01\right\rangle)$$

$$M(q_1) = 1 \rightarrow \frac{1}{\sqrt{2}}(\left|10\right\rangle + \left|11\right\rangle)$$

- Note the renormalization!

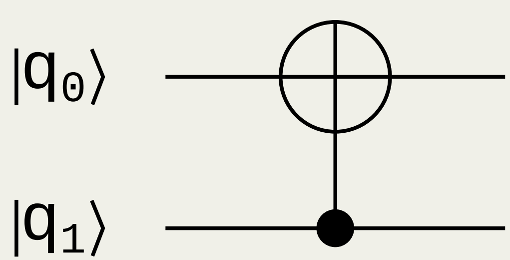
Swap gate

- Swaps the values of the two qubits:
  - $|00\rangle \rightarrow |00\rangle$
  - $|01\rangle \rightarrow |10\rangle$
  - $|10\rangle \rightarrow |01\rangle$
  - $|11\rangle \rightarrow |11\rangle$
- Represented by the matrix $SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Circuit notation:



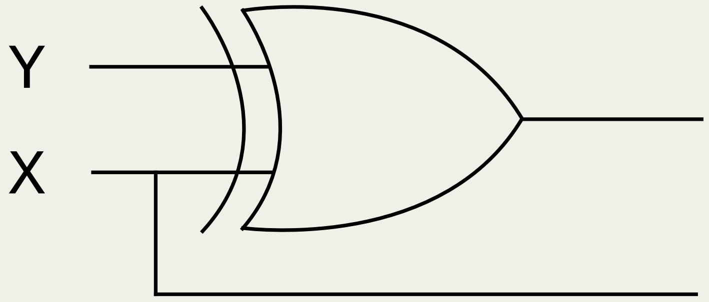
Controlled NOT gate

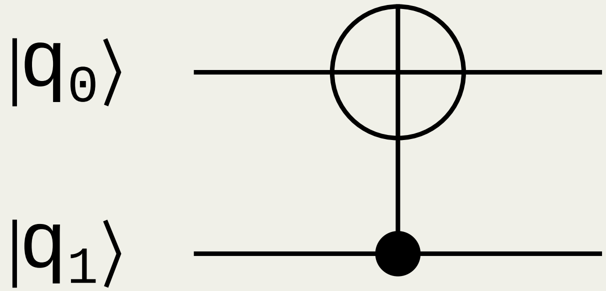
- Performs the NOT operation on the target qubit only when the control qubit is $|1\rangle$:
  - $|00\rangle \rightarrow |00\rangle$
  - $|01\rangle \rightarrow |01\rangle$
  - $|10\rangle \rightarrow |11\rangle$
  - $|11\rangle \rightarrow |10\rangle$
- Represented by the matrix $CNOT_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- Circuit notation:



Controlled NOT gate

- Similar to XOR operation, but reversible!





INPUT		OUTPUT	
x	y	x	xor
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

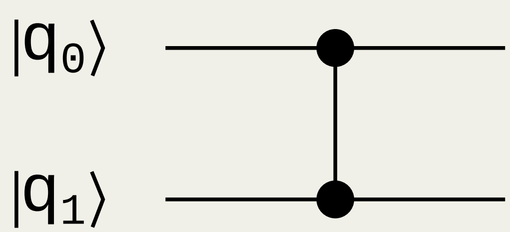
INPUT		OUTPUT	
$\lvert q_1\rangle$	$\lvert q_0\rangle$	$\lvert q_1\rangle$	$\lvert q_1\rangle \oplus \lvert q_0\rangle$
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

Controlled gates

- If U is a gate with matrix $U = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix}$
- Controlled–U operates on two qubits such that the first qubit serves as a control
  - $|00\rangle \to |00\rangle$
  - $|01\rangle \to |01\rangle$
  - $|10\rangle \to |1\rangle \otimes U\,|0\rangle$
  - $|11\rangle \to |1\rangle \otimes U\,|1\rangle$
- Represented by the matrix $CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{1,1} & u_{1,2} \\ 0 & 0 & u_{2,1} & u_{2,2} \end{bmatrix}$

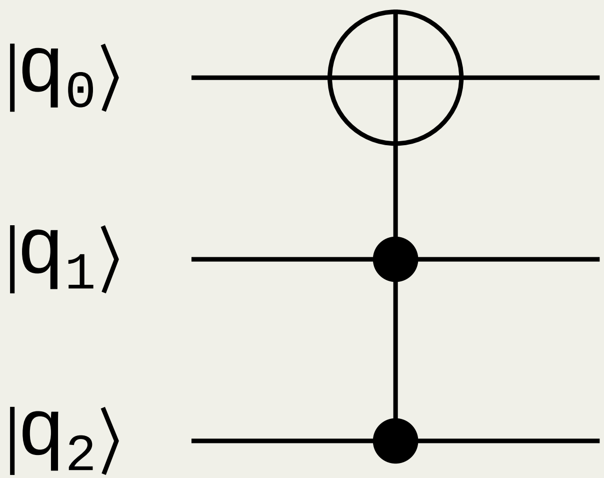
Controlled Z gate

- Performs the Z rotation on the target qubit only when the control qubit is $|1\rangle$:
  - $|00\rangle \rightarrow |00\rangle$
  - $|01\rangle \rightarrow |01\rangle$
  - $|10\rangle \rightarrow |10\rangle$
  - $|11\rangle \rightarrow -|11\rangle$
- Represented by the matrix $CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
- Circuit notation:



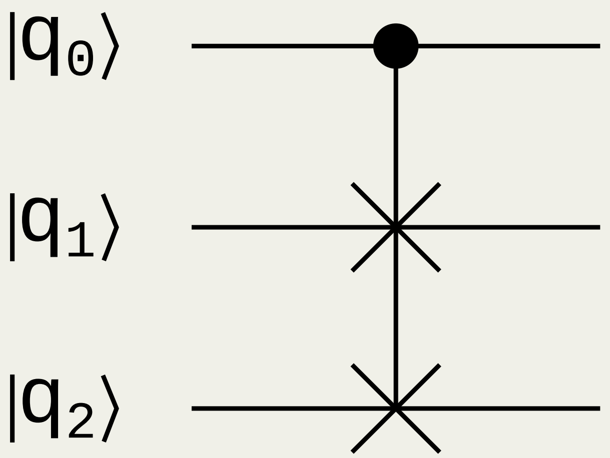
Toffoli gate (CCNOT)

- Operates on 3 qubits
- Performs the NOT operation on the target qubit only when the two control qubit are $|1\rangle$:



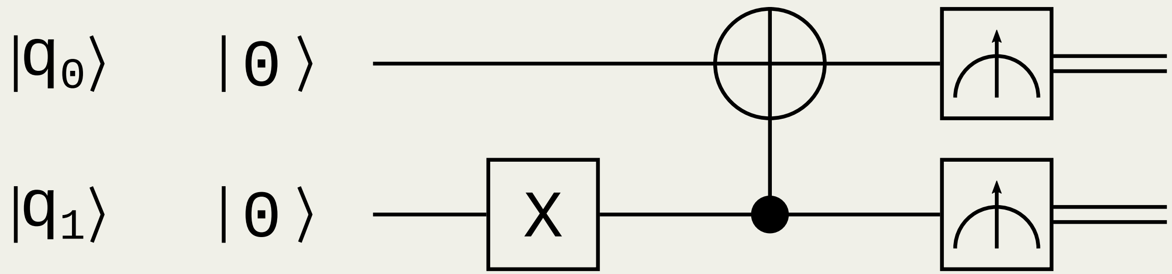
$$CCNOT_{1,\,2,\,0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Operates on 3 qubits:
  - First qubit: control qubit
  - Other two: target qubits
- Performs the SWAP operation between the target qubits only when the control qubit is $|1\rangle$:



$$CSWAP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example



$$
\begin{aligned}
\mathit{CNOT}_{1,\,0}(X_1\,|00\rangle) &= \mathit{CNOT}_{1,\,0}((X_1 \otimes I_0)\,|00\rangle) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) |00\rangle \right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= |11\rangle
\end{aligned}
$$

Question # 6

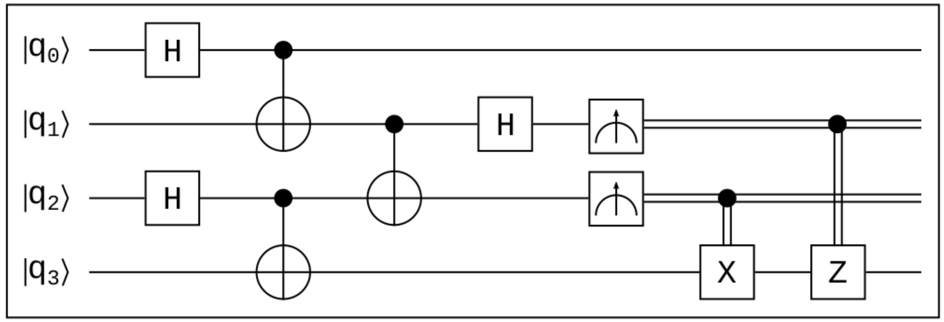


Figure 2: An arbitrary quantum circuit.

Question 6 *Consider the quantum circuit presented in Figure 2 and assume* $|q_0\rangle = |0\rangle$, $|q_1\rangle = |0\rangle$, $|q_2\rangle = |0\rangle$ *and* $|q_3\rangle = |0\rangle$*; hence,* $|\psi_{in}\rangle = |0000\rangle$*. Determine, by using the Dirac notation, what is the state vector of the quantum circuit just before the partial measurement?*

`Write down your solution here:`

$$\rightarrow |0000\rangle$$

Entanglement

- Two qubits are **entangled** iff the measurement of one qubit is correlated with the state of the other:
  - The state of those two qubits can **not be expressed as two individual states** (non–separable)
  - Are these qubits entangled?

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Entanglement

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- Two qubits are **entangled** iff the measurement of one qubit is correlated with the state of the other:
  - The state of those two qubits can **not be expressed as two individual states** (non–separable)
  - Are these qubits entangled?

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
\end{aligned}
$$

  - What about these?

$$
|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
$$

Concurrence

- Two qubits are **entangled** iff they have **nonzero concurrence**

$$C(|\psi\rangle) = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|$$

- Examples:

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ C(|\psi_1\rangle) &= 2\left|\frac{1}{2}\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{1}{2}\right| \\ &= 2\left|\frac{1}{4} - \frac{1}{4}\right| \\ &= 2\,|0| \\ &= 0 \end{aligned}$$

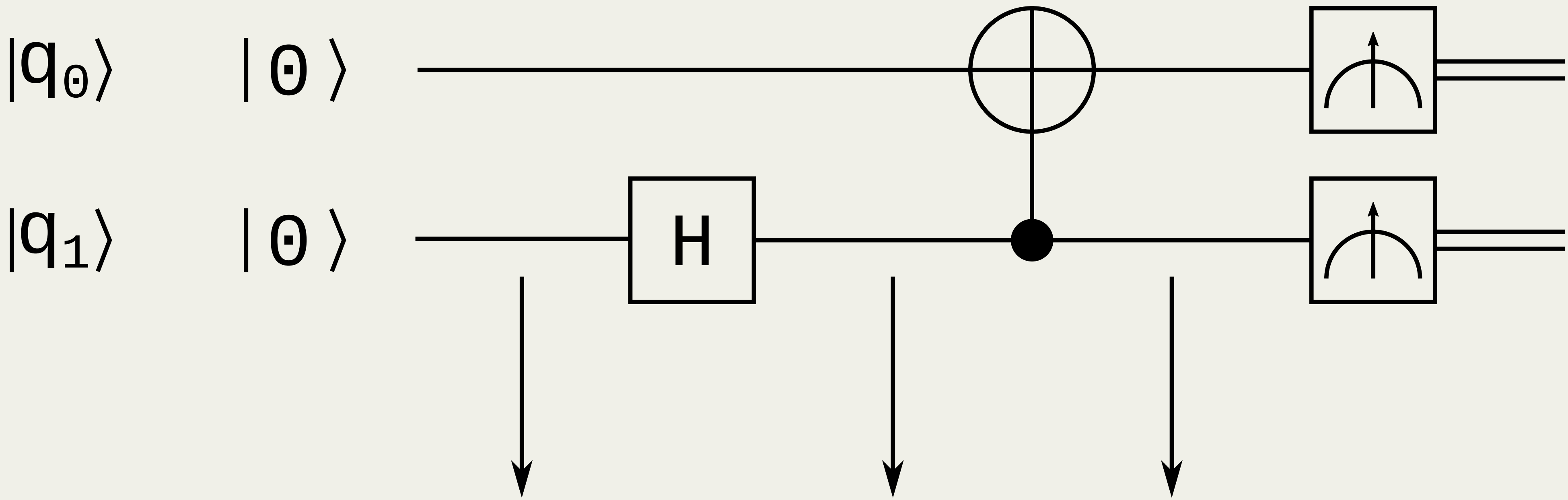
$$\begin{aligned} |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ C(|\psi_2\rangle) &= 2\left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) - 0\right| \\ &= 2\left|\frac{1}{2}\right| \\ &= 1 \end{aligned}$$

Bell states

- Maximal entangled states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell states



$|00\rangle$

$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

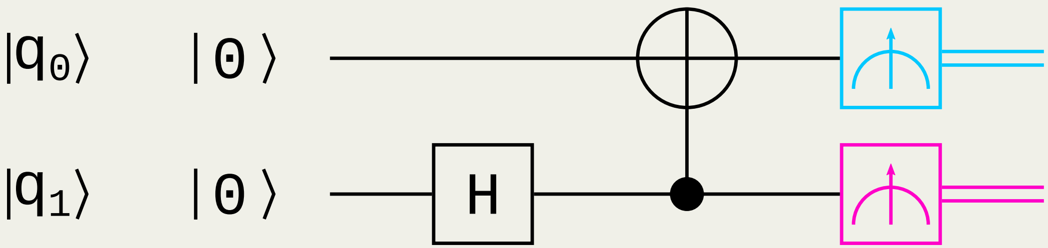
$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$

- Requires superposition **and** a two–qubit gate

Bell states

$$
\begin{aligned}
\mathit{CNOT}_{1,0}(H_1 |00\rangle) &= \mathit{CNOT}_{1,0}((H_1 \otimes I_0) |00\rangle) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) |00\rangle \right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\end{aligned}
$$

Measurement correlations



$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

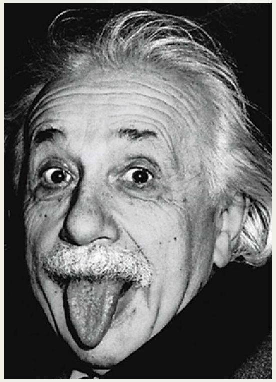
$$\mathbf{Pr}\left\{|00\rangle\right\} = \overline{\alpha_0}\alpha_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = 0.5$$

$$M(q_1) = 0 \rightarrow |00\rangle \rightarrow M(q_0) = 0$$
$$M(q_1) = 1 \rightarrow |11\rangle \rightarrow M(q_0) = 1$$

$$\mathbf{Pr}\left\{|11\rangle\right\} = \overline{\alpha_3}\alpha_3 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = 0.5$$

Measurement result of two qubits always the same!

“Spooky action at a distance”
– Albert Einstein



Question # 10

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Question 10 *Considering the state vector obtained in Question 6. Assume that the measuring process returned the following values:* $M(|q_2\rangle) = 1$ *and* $M(|q_1\rangle) = 1$. *What is the final state vector of the quantum circuit after applying the corrections? (Note: Remember to renormalize the vector).*

`Write down your solution here:`

What is next?

- Assignment is already available (check DLO)
- ESK Wiskunde lectures and workshops
- Q & A, Discussion
  - Questions about the assignment or the material
  - Feel free to propose any discussion
  - You can always write me an email


Questions?