Quantum Fourier Transformation

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Fourier Analysis

Fourier analysis studies how functions can be represented by sums of simpler trigonometric functions.

- Description of time domain signal in a unique combination of sinoids
- Sinoids describe frequency, amplitude and phase
- No information is lost, so the inverse is also possible

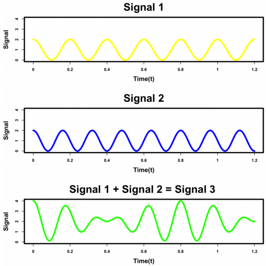


Figure: Fourier analysis. From: Elan Ness-Cohn

Fourier Transformation

Applications:

- Extracting frequencies from sound
- Uncertainty principle
- Partial Differential equations
- Vibration analysis
- Image processing.

Tutorials online:

- A visual introduction
- What is the Fourier Transform?

In classical computing it is being used as:

- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)

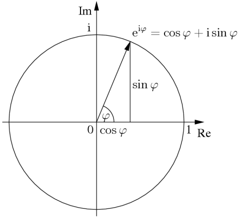
Discrete Fourier Transform

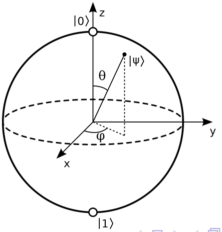
The discrete fourier transform (DFT) transforms a sequence of N complex numbers x_n to another sequence of complex number X_k:

$$X_k = \sum_{n=0}^{N-1} x_n . e^{-\frac{i2\pi}{N}kn} = \sum_{n=0}^{N-1} x_n . \left[\cos\left(\frac{2\pi}{N}kn\right) - i. \sin\left(\frac{2\pi}{N}kn\right)\right] \quad (1)$$

NB Eulers formula:

$$e^{ix} = \cos x + i \sin x \quad (2)$$





Quantum Fourier Transform

From computational base to Fourier base.

$$|X\rangle = \sum_{j=0}^{N-1} x_j |j\rangle \mapsto |Y\rangle = \sum_{k=0}^{N-1} y_k |k\rangle \tag{3}$$

and

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \omega_N^{jk} \tag{4}$$

where

$$\omega_N^{jk} = e^{2\pi i \frac{jk}{N}} \text{ and (Eulers formula) } e^{ix} = \cos x + i \sin x \tag{5}$$

One qubit

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow x_0 = \alpha, x_1 = \beta, \text{ and } N = 2 \tag{6}$$

Then

$$y_0 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i \frac{0\times 0}{2}} + \beta e^{2\pi i \frac{1\times 0}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha + \beta) \tag{7}$$

$$y_1 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i \frac{0\times 1}{2}} + \beta e^{2\pi i \frac{1\times 1}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha - \beta) \tag{8}$$

which gives the resulting state

$$U_{QFT}|\phi\rangle = \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle \tag{9}$$

This new state is the same state when a Hadamard gate is applied on the state

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{10}$$

The gates

The Hadamard gate:

$$H\,|x_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{2\pi i}{2} x_k}\,|1\rangle \right) \tag{11}$$

The Controlled Rotation gate:

$$C - ROT_k = \begin{bmatrix} I & 0 \\ 0 & U - ROT_k \end{bmatrix} \tag{12}$$

where:

$$U - ROT_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix} \tag{13}$$