Quantum Error Correction

Quantum Capita Selecta

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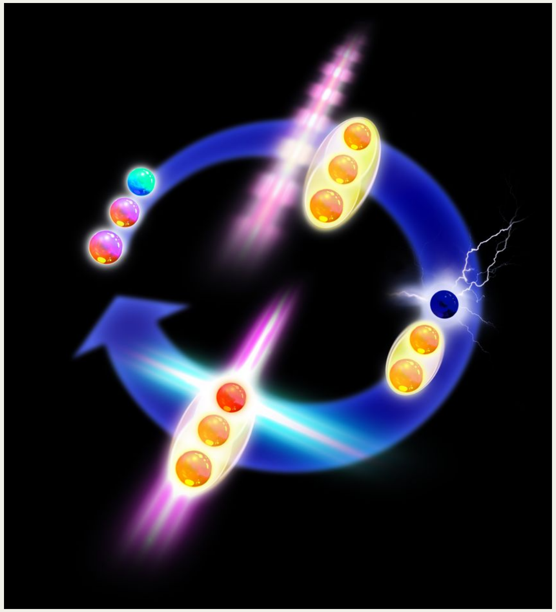
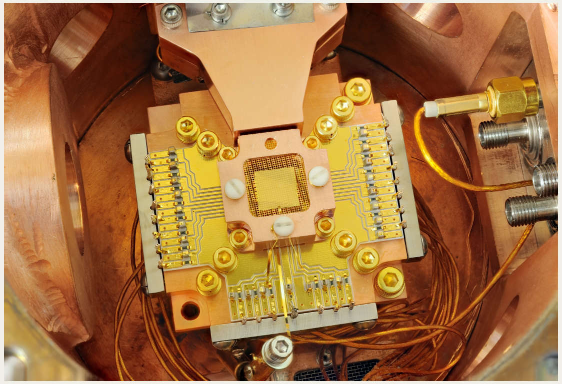
Why Quantum Error Correction (QEC)?

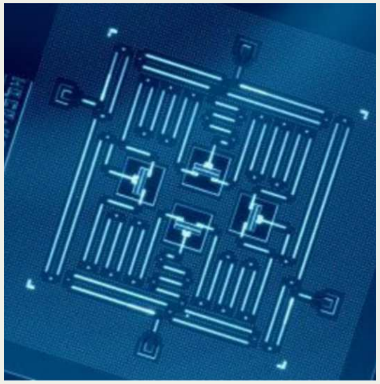
- Quantum Error Correction Codes (QECC) are algorithms too!
  - In general with Clifford gates only
- Sophisticated quantum–classic interaction
- Step towards quantum programming model



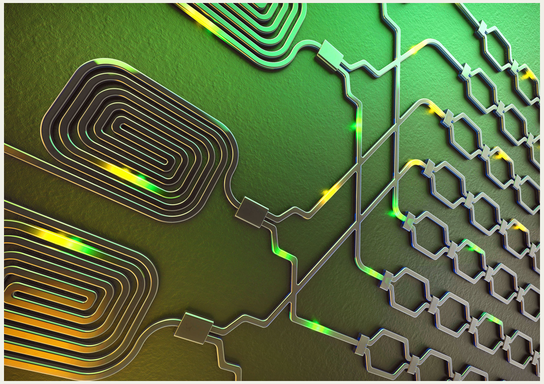
Quantum hardware



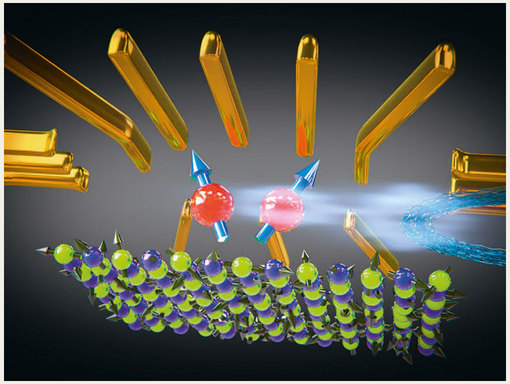
Ion traps



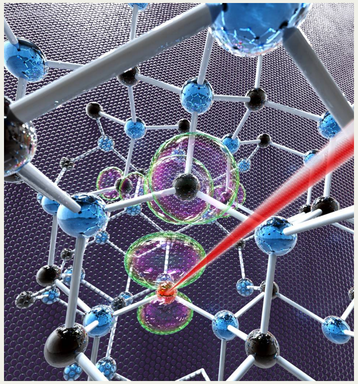
Superconductors



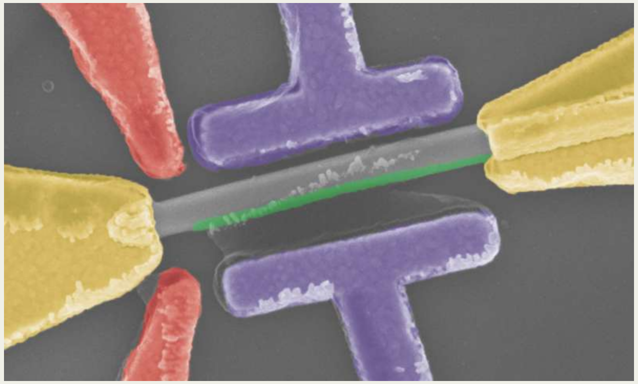
Photons



Quantum dots



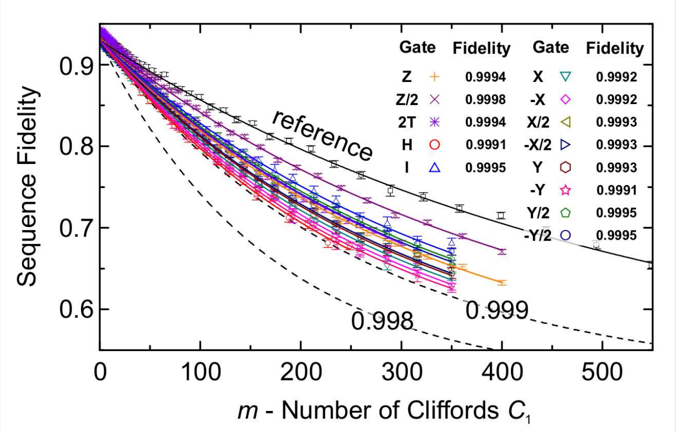
NV centers

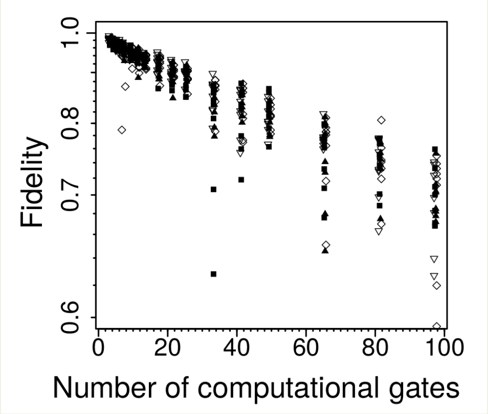


Majoranas

Qubits are fragile

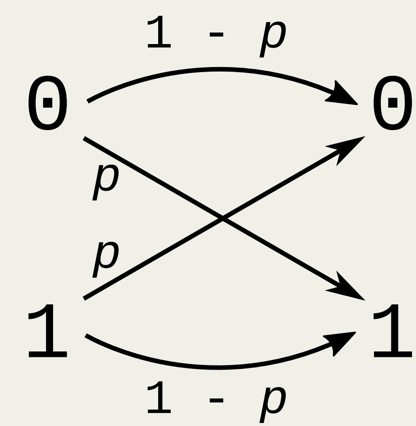
- Qubits lose their state in a short period
  - Tens of microseconds for superconducting qubits
- Quantum gates have high error rates
  - Around 0.1% for superconducting qubits
  - Factoring a 2000–bit number requires error rates around $4\text{x}10^{-13}$

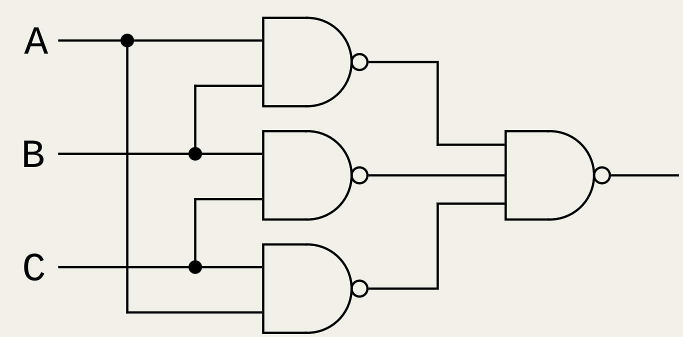




Classic Error Correction

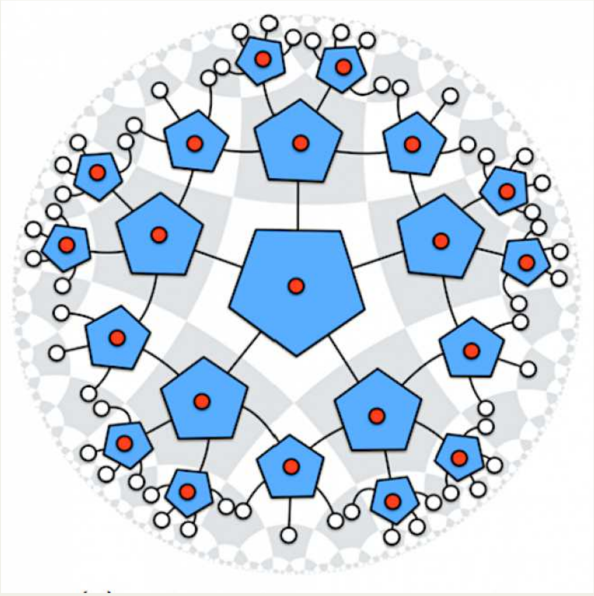
- Error/Noise:
  - Independent single–bit errors
  - Probability of error: p
- Parity bit/Checksum
  - Detection only
- Repetition code/majority voting
- More redundancy, larger code space, better error resistance





Quantum Error Correction

- Encode a quantum state redundantly into a larger Hilbert space (logical qubits)
- Costs:
  - Space
  - Time
- Benefits of logical qubits:
  - Longer coherence times
  - Lower error rates



Challenges of QEC

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- Measurements are destructive

$$\alpha|0\rangle + \beta|1\rangle = \begin{cases} M(|\psi\rangle) = |0\rangle \\ M(|\psi\rangle) = |1\rangle \end{cases}$$

- No cloning

$$\alpha|0\rangle + \beta|1\rangle \nRightarrow \begin{cases} \alpha|0\rangle + \beta|1\rangle \\ \alpha|0\rangle + \beta|1\rangle \\ \alpha|0\rangle + \beta|1\rangle \end{cases}$$

- Errors are continuous:

$$|\psi\rangle|e\rangle \rightarrow \sqrt{1-p}I|\psi\rangle|e\rangle + \sqrt{p}X|\psi\rangle|\hat{e}\rangle$$

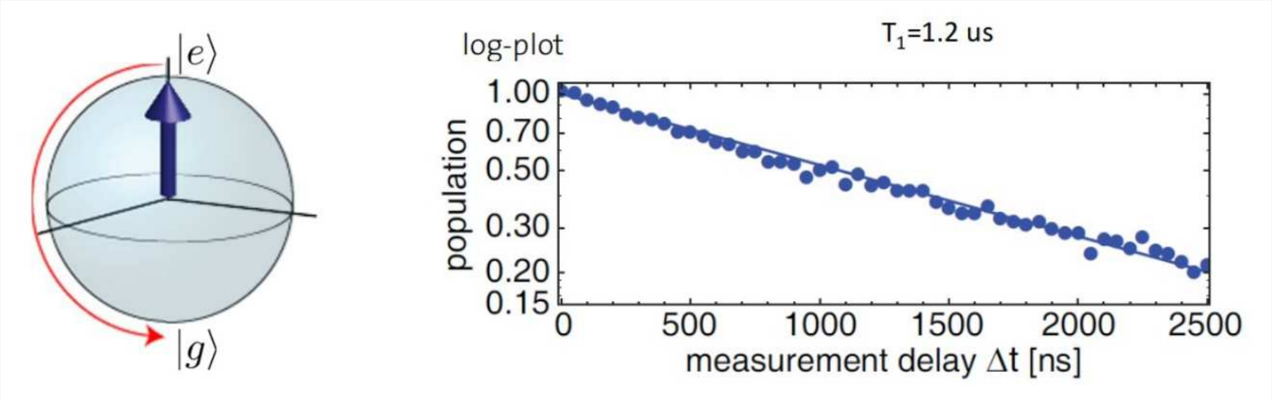
- Bit–flip channel (independent X errors)

$$|\psi\rangle\,|e\rangle \rightarrow X\,|\psi\rangle\,|\hat{e}\rangle$$

- Phase–flip channel (independent Z errors)

$$|\psi\rangle\,|e\rangle \rightarrow Z\,|\psi\rangle\,|\hat{e}\rangle$$

- Amplitude damping/relaxation



Quantum error correction

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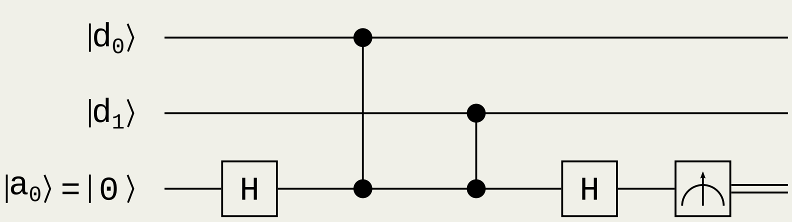
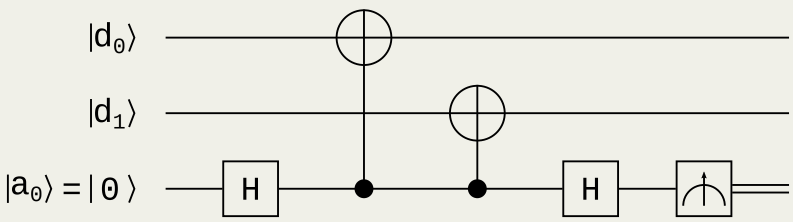
- Indirect measurements (stabilizers)
  - Learn about the state without fully collapsing it
  - Forces discrete errors
- All errors can be decomposed in X and Z errors
- Quantum memory

- Notation formalism, most important is that we can measure “stabilizers”

$Z_0 Z_1$ stabilizer



$X_0 X_1$ stabilizer



Bit–flip code

Specifications

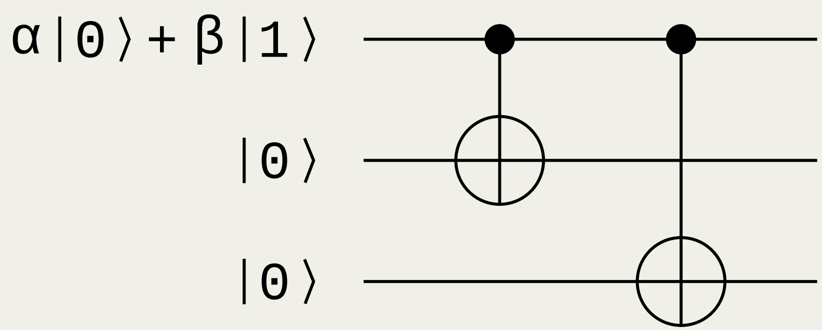
- 3 data qubits
- 2 ancilla qubits
- Detects up to two X errors
- Corrects up to one X error
- Not resistant to Z errors

Steps

- Encode state
- Extract error syndrome
- Decode error syndrome
- Correct
- Decode state

Encode state

- Encodes a single qubit state in a larger Hilbert space (redundancy)

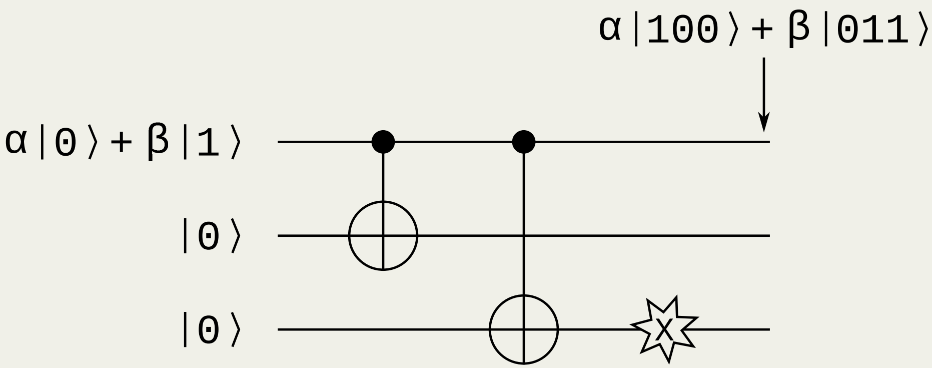


$$
\begin{aligned}
&\rightarrow |00\rangle \otimes |\psi\rangle \\
&\rightarrow |00\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) \\
&\rightarrow \alpha |000\rangle + \beta |001\rangle \\
\mathit{CNOT}_{0,\,1} &\rightarrow \alpha |000\rangle + \beta |011\rangle \\
\mathit{CNOT}_{0,\,2} &\rightarrow \alpha |000\rangle + \beta |111\rangle
\end{aligned}
$$

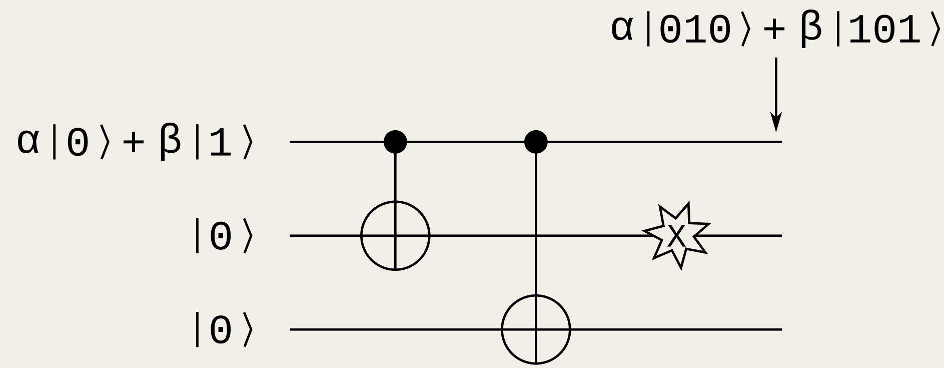
$$|\psi_L\rangle = \alpha |000\rangle + \beta |111\rangle$$

Errors

- Independent X errors can occur with probability p (bit–flip channel)
- Single error changes parity of the encoded state

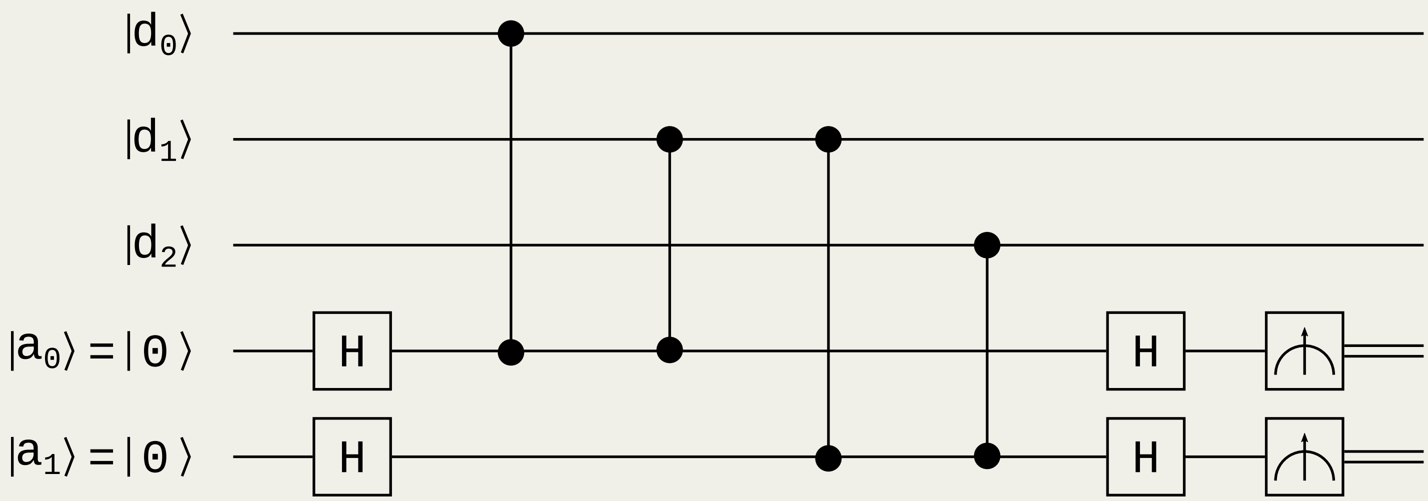


$$
\begin{aligned}
&\rightarrow \alpha\,|000\rangle + \beta\,|001\rangle \\
\textit{CNOT}_{0,\,1} &\rightarrow \alpha\,|000\rangle + \beta\,|011\rangle \\
\textit{CNOT}_{0,\,2} &\rightarrow \alpha\,|000\rangle + \beta\,|111\rangle \\
X_2 &\rightarrow \alpha\,|100\rangle + \beta\,|011\rangle
\end{aligned}
$$



$$
\begin{aligned}
&\rightarrow \alpha\,|000\rangle + \beta\,|001\rangle \\
\textit{CNOT}_{0,\,1} &\rightarrow \alpha\,|000\rangle + \beta\,|011\rangle \\
\textit{CNOT}_{0,\,2} &\rightarrow \alpha\,|000\rangle + \beta\,|111\rangle \\
X_1 &\rightarrow \alpha\,|010\rangle + \beta\,|101\rangle
\end{aligned}
$$

Extract error syndrome



- Measuring stabilizers: $Z_0 Z_1$, $Z_1 Z_2$
  - Error Syndrome Measurement
- For an encoded state in the code space:

$$M(a_0) = M(a_1) = 0$$
$$|\psi_L\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$$

Error syndrome measurement

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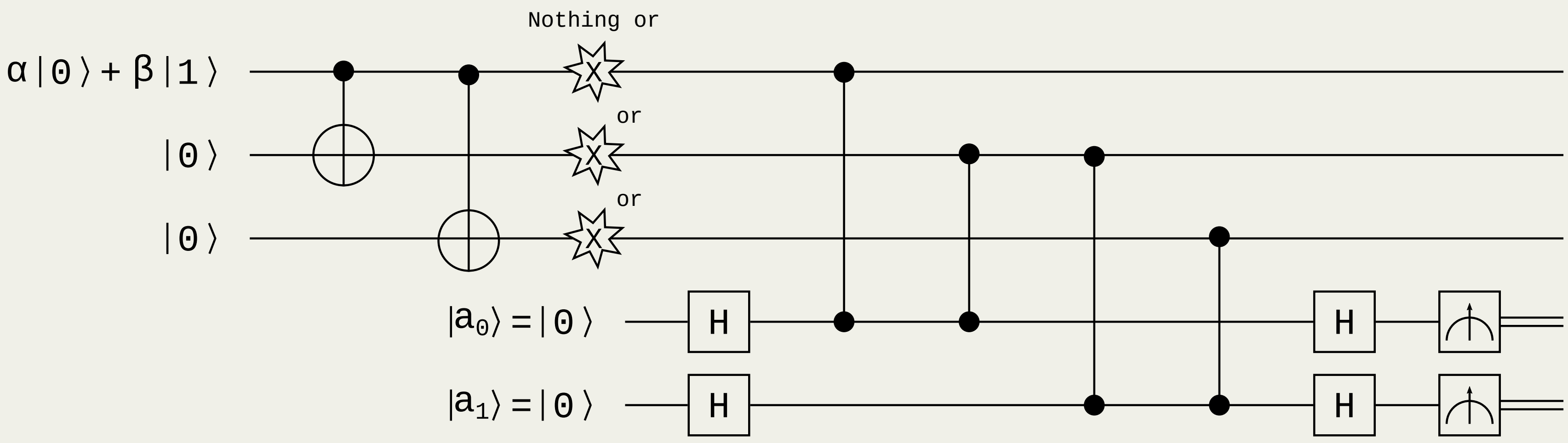

$$|\psi_L\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$$
$$X_{d_0} \rightarrow \alpha\,|001\rangle + \beta\,|110\rangle$$
$$X_{d_1} \rightarrow \alpha\,|010\rangle + \beta\,|101\rangle$$
$$X_{d_2} \rightarrow \alpha\,|100\rangle + \beta\,|011\rangle$$

Error	$M(a_1)$	$M(a_0)$
None	0	0
X_{d_0}	0	1
X_{d_1}	1	1
X_{d_2}	1	0

Decode error syndrome

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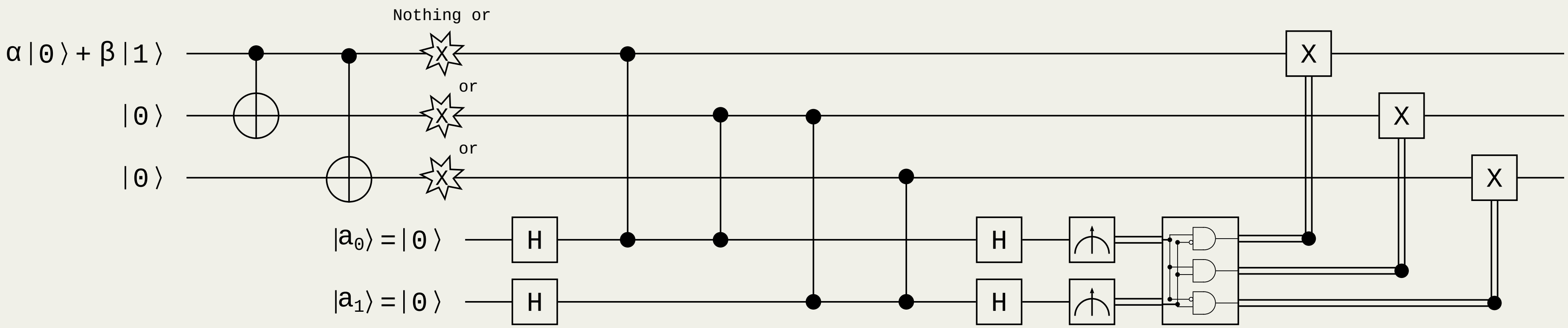
- Decoder reads error syndrome and outputs most likely error
  - Decoding is a fully classic calculation!
  - Find the least amount of errors that explains the error syndrome
  - Corrections can be wrong and generate an unrecoverable error

Error	$M(a_1)$	$M(a_0)$	$\mathbf{Pr}\{\}$
X_{d_0}	0	1	p
$X_{d_1 d_2}$	0	1	p^2

$$
\begin{aligned}
|\psi_L\rangle &= \alpha\,|000\rangle + \beta\,|111\rangle \\
X_{d_0} &\rightarrow \alpha\,|001\rangle + \beta\,|110\rangle \\
X_{d_1} X_{d_2} &\rightarrow \alpha\,|110\rangle + \beta\,|001\rangle
\end{aligned}
$$

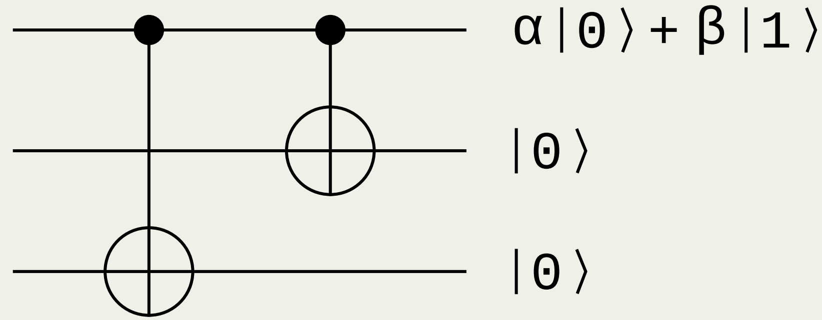
Correct

- Apply correction for most likely error



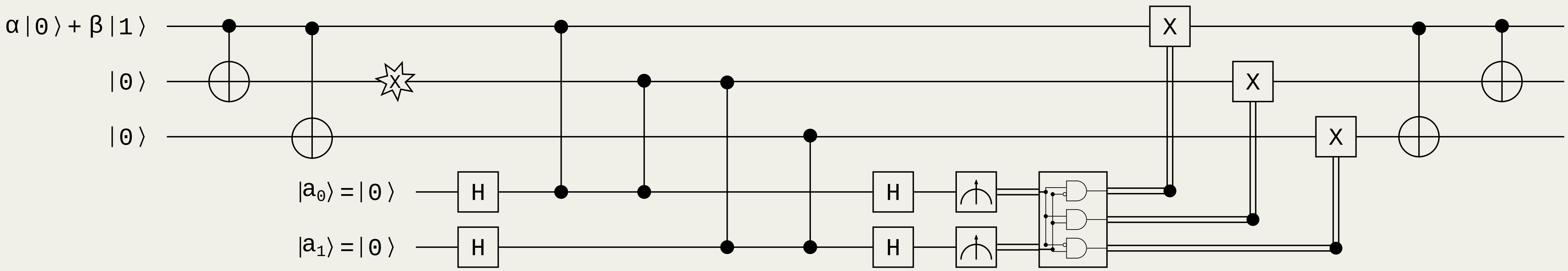
Decode state

- Reverse encoding circuit (uncompute)
- We can also measure any qubit directly in the encoded state



$$
\begin{aligned}
& \rightarrow \alpha\,|000\rangle + \beta\,|111\rangle \\
\mathit{CNOT}_{0,\,2} & \rightarrow \alpha\,|000\rangle + \beta\,|011\rangle \\
\mathit{CNOT}_{0,\,1} & \rightarrow \alpha\,|000\rangle + \beta\,|001\rangle \\
& \rightarrow |00\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle) \\
& \rightarrow |00\rangle \otimes |\psi\rangle
\end{aligned}
$$

Example



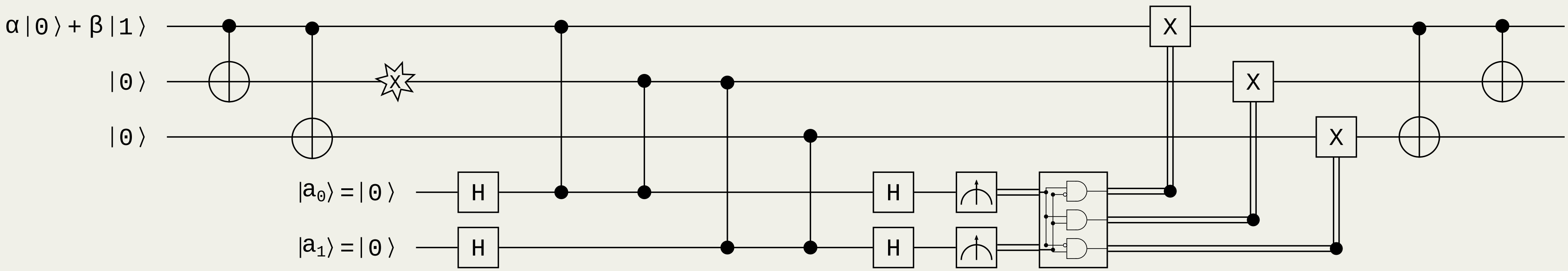
$$\rightarrow |a_1 a_0\rangle \otimes |00\rangle \otimes |\psi\rangle$$
$$\rightarrow |00\rangle \otimes |00\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)$$

$$\rightarrow \alpha\,|00000\rangle + \beta\,|00001\rangle$$
$$CNOT_{0,1} \rightarrow \alpha\,|00000\rangle + \beta\,|00011\rangle$$
$$CNOT_{0,2} \rightarrow \alpha\,|00000\rangle + \beta\,|00111\rangle$$

$$X_1 \rightarrow \alpha\,|00010\rangle + \beta\,|00101\rangle$$

$$H_{a_0} \rightarrow \frac{1}{\sqrt{2}}(\alpha\,|00010\rangle + \alpha\,|01010\rangle + \beta\,|00101\rangle + \beta\,|01101\rangle)$$
$$H_{a_1} \rightarrow \frac{1}{2}(\alpha\,|00010\rangle + \alpha\,|10010\rangle + \alpha\,|01010\rangle + \alpha\,|11010\rangle + \beta\,|00101\rangle + \beta\,|10101\rangle + \beta\,|01101\rangle + \beta\,|11101\rangle)$$

Example



$$\rightarrow \frac{1}{2}\left(\alpha\,|00010\rangle + \alpha\,|10010\rangle + \alpha\,|01010\rangle + \alpha\,|11010\rangle + \beta\,|00101\rangle + \beta\,|10101\rangle + \beta\,|01101\rangle + \beta\,|11101\rangle\right)$$

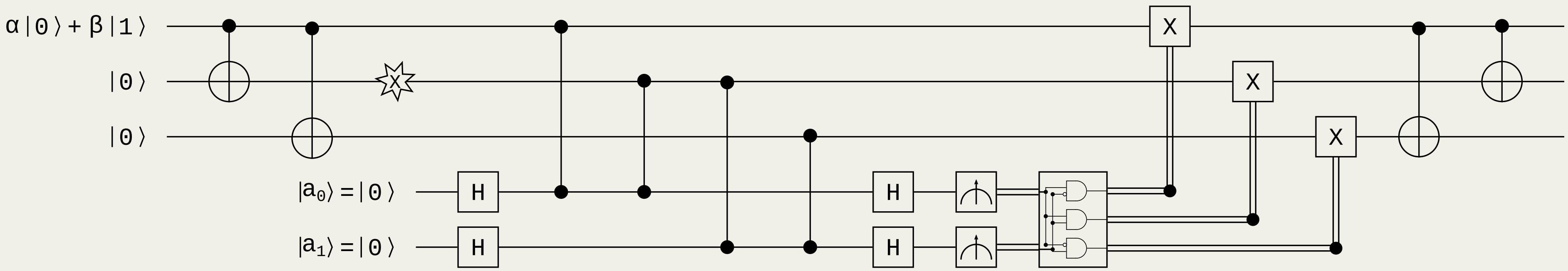
$$CZ_{0,\,a_0} \rightarrow \frac{1}{2}\left(\alpha\,|00010\rangle + \alpha\,|10010\rangle + \alpha\,|01010\rangle + \alpha\,|11010\rangle + \beta\,|00101\rangle + \beta\,|10101\rangle - \beta\,|01101\rangle - \beta\,|11101\rangle\right)$$

$$CZ_{1,\,a_0} \rightarrow \frac{1}{2}\left(\alpha\,|00010\rangle + \alpha\,|10010\rangle - \alpha\,|01010\rangle - \alpha\,|11010\rangle + \beta\,|00101\rangle + \beta\,|10101\rangle - \beta\,|01101\rangle - \beta\,|11101\rangle\right)$$

$$CZ_{1,\,a_1} \rightarrow \frac{1}{2}\left(\alpha\,|00010\rangle - \alpha\,|10010\rangle - \alpha\,|01010\rangle + \alpha\,|11010\rangle + \beta\,|00101\rangle + \beta\,|10101\rangle - \beta\,|01101\rangle - \beta\,|11101\rangle\right)$$

$$CZ_{2,\,a_1} \rightarrow \frac{1}{2}\left(\alpha\,|00010\rangle - \alpha\,|10010\rangle - \alpha\,|01010\rangle + \alpha\,|11010\rangle + \beta\,|00101\rangle - \beta\,|10101\rangle - \beta\,|01101\rangle + \beta\,|11101\rangle\right)$$

Example



$$\rightarrow \frac{1}{2}\left(\alpha\left|00010\right\rangle - \alpha\left|10010\right\rangle - \alpha\left|01010\right\rangle + \alpha\left|11010\right\rangle + \beta\left|00101\right\rangle - \beta\left|10101\right\rangle - \beta\left|01101\right\rangle + \beta\left|11101\right\rangle\right)$$

$$H_{a_0} \rightarrow \frac{1}{2\sqrt{2}}\left(\alpha\left|00010\right\rangle + \alpha\left|01010\right\rangle - \alpha\left|10010\right\rangle - \alpha\left|11010\right\rangle - \alpha\left|00010\right\rangle + \alpha\left|01010\right\rangle + \alpha\left|10010\right\rangle - \alpha\left|11010\right\rangle + \right.$$

$$\left. + \beta\left|00101\right\rangle + \beta\left|01101\right\rangle - \beta\left|10101\right\rangle - \beta\left|11101\right\rangle - \beta\left|00101\right\rangle + \beta\left|01101\right\rangle + \beta\left|10101\right\rangle - \beta\left|11101\right\rangle\right)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(2\alpha\left|01010\right\rangle - 2\alpha\left|11010\right\rangle + 2\beta\left|01101\right\rangle - 2\beta\left|11101\right\rangle\right)$$

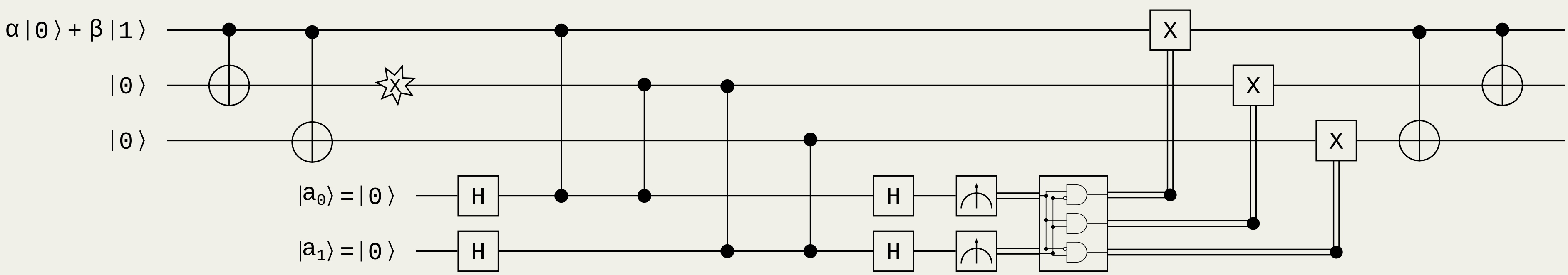
$$\rightarrow \frac{1}{\sqrt{2}}\left(\alpha\left|01010\right\rangle - \alpha\left|11010\right\rangle + \beta\left|01101\right\rangle - \beta\left|11101\right\rangle\right)$$

$$H_{a_1} \rightarrow \frac{1}{2}\left(\alpha\left|01010\right\rangle + \alpha\left|11010\right\rangle - \alpha\left|01010\right\rangle + \alpha\left|11010\right\rangle + \beta\left|01101\right\rangle + \beta\left|11101\right\rangle - \beta\left|01101\right\rangle + \beta\left|11101\right\rangle\right)$$

$$\rightarrow \frac{1}{2}\left(2\alpha\left|11010\right\rangle + 2\beta\left|11101\right\rangle\right)$$

$$\rightarrow \alpha\left|11010\right\rangle + \beta\left|11101\right\rangle$$

Example



$$\rightarrow \alpha\,|11010\rangle + \beta\,|11101\rangle$$

$$\rightarrow M(a_1) = M(a_0) = 1$$

$$\textit{CLAS} \rightarrow 010$$

$$X_1 \rightarrow \alpha\,|11000\rangle + \beta\,|11111\rangle$$

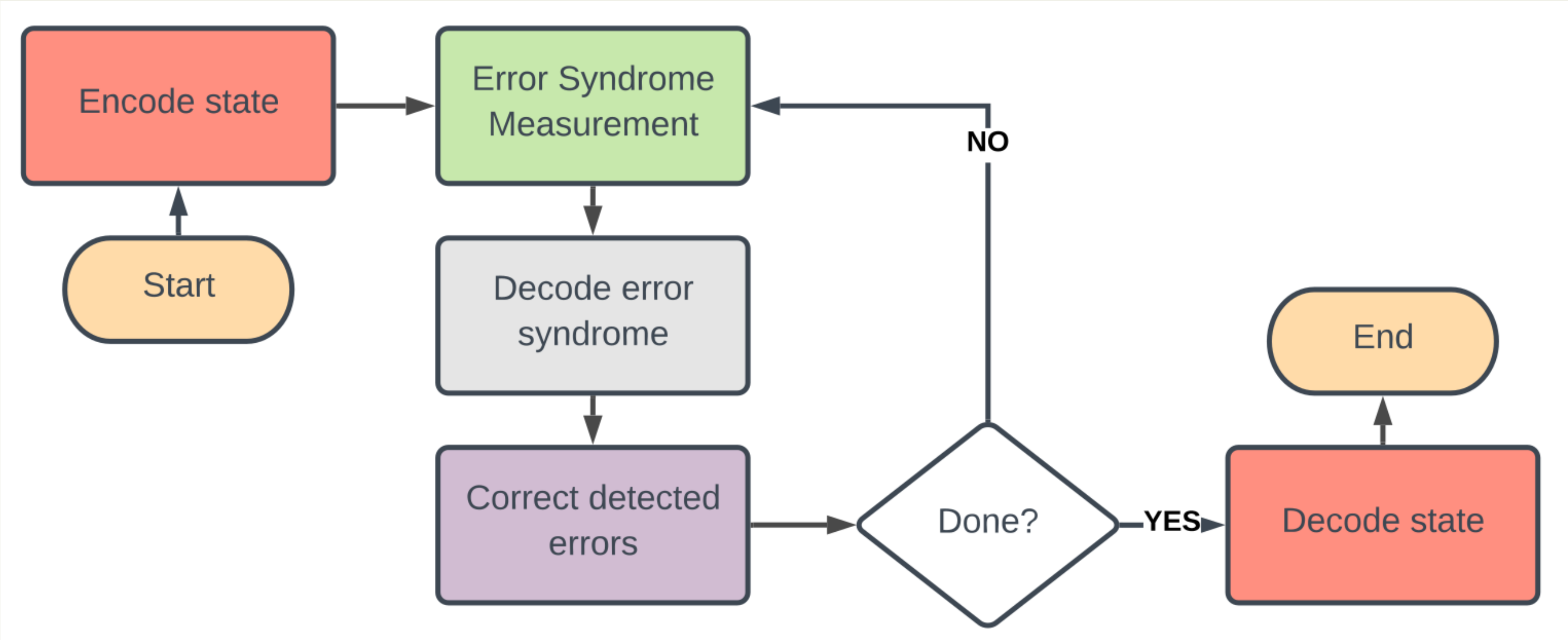
$$\textit{CNOT}_{0,\,2} \rightarrow \alpha\,|11000\rangle + \beta\,|11011\rangle$$

$$\textit{CNOT}_{0,\,1} \rightarrow \alpha\,|11000\rangle + \beta\,|11001\rangle$$

$$\rightarrow |11\rangle \otimes |00\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)$$

$$\rightarrow |a_1 a_0\rangle \otimes |00\rangle \otimes |\psi\rangle$$

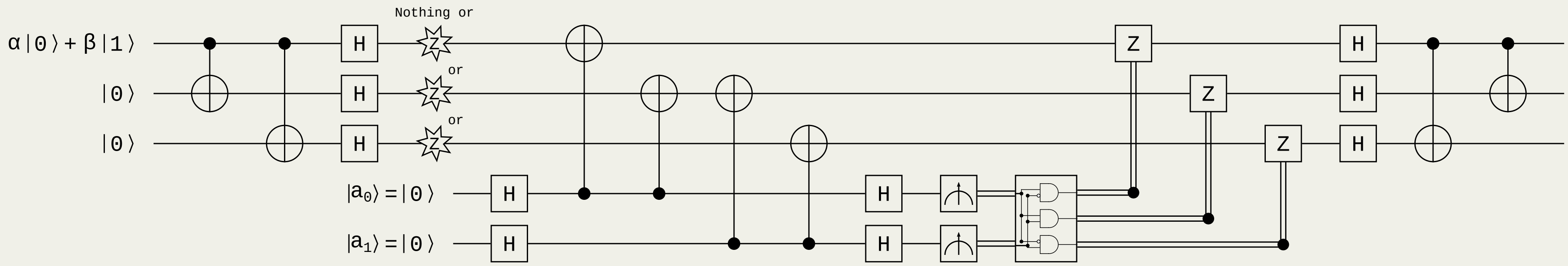
Bit–flip code summary



Phase–flip code

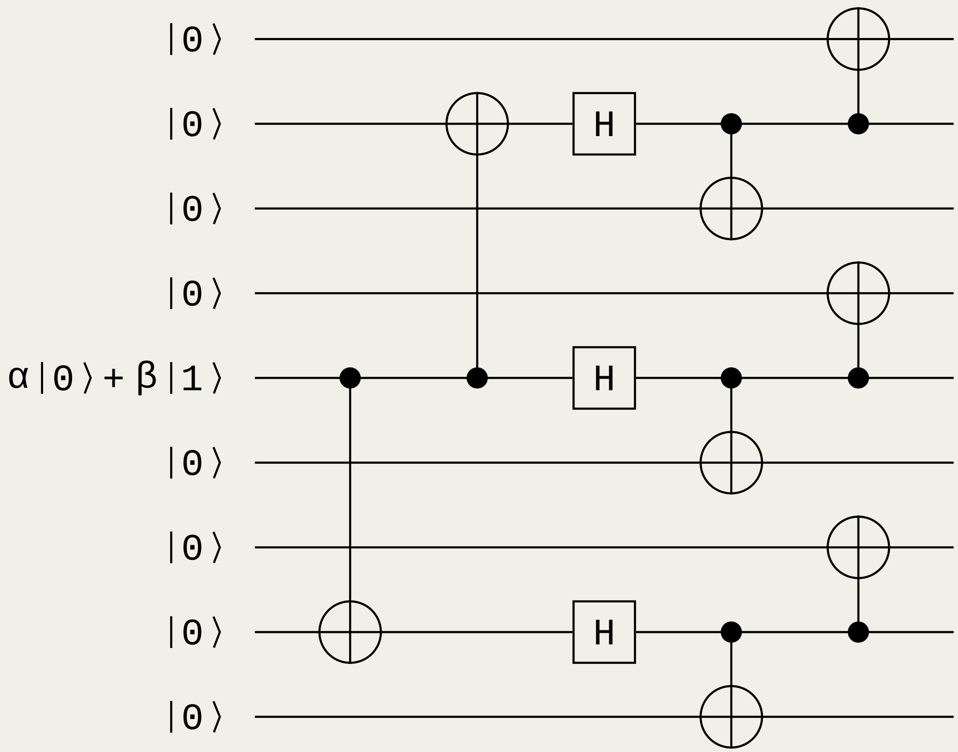
- Same principle in a different basis, now detecting Z errors

$$|\psi_L\rangle = \alpha\,|+++\rangle + \beta\,|---\rangle$$

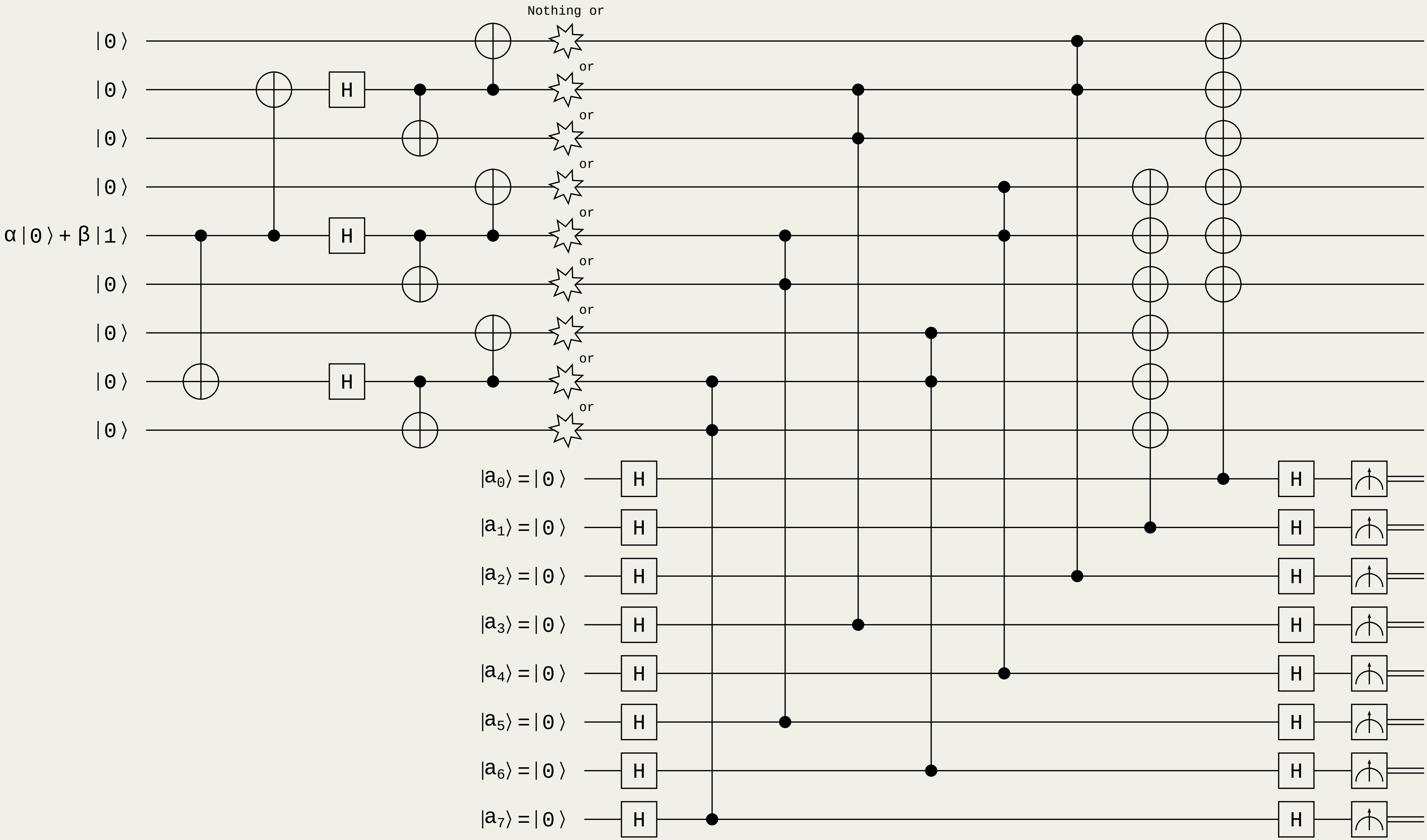


Shor code

- Concatenated code
  - Phase–flip code (detects Z errors)
  - Three times bit–flip code (detects X errors)
- First complete QECC

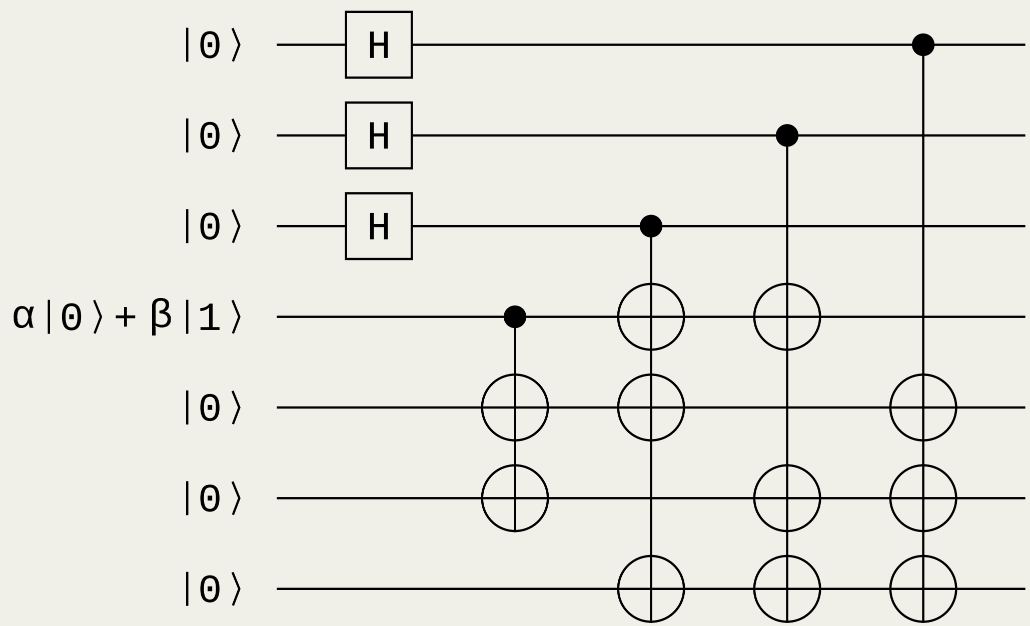


Shor code ESM



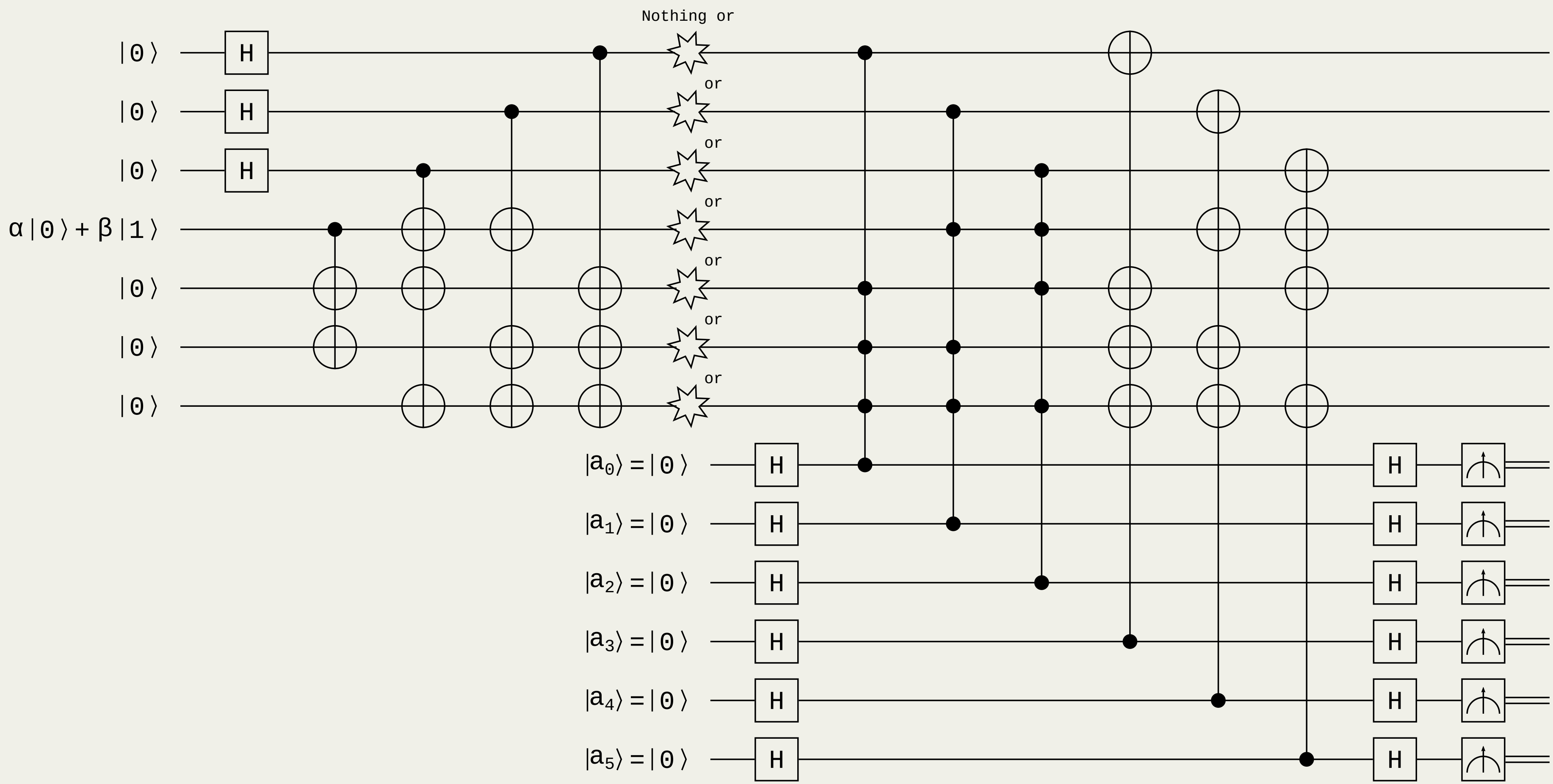
Steane code

- Mathematically a very convenient code
- Corrects up to one X and one Z error



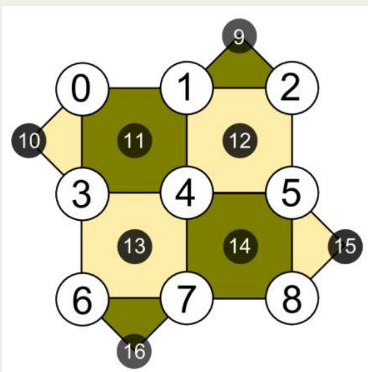
- $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \rightarrow |\Psi\rangle = \alpha\,|0_{Steane}\rangle + \beta\,|1_{Steane}\rangle$

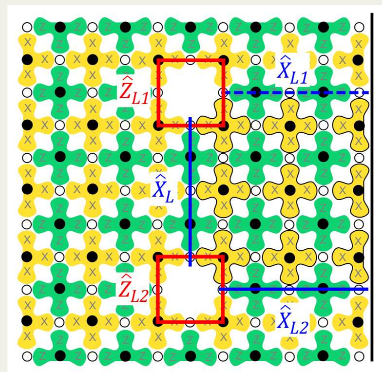
Steane code ESM

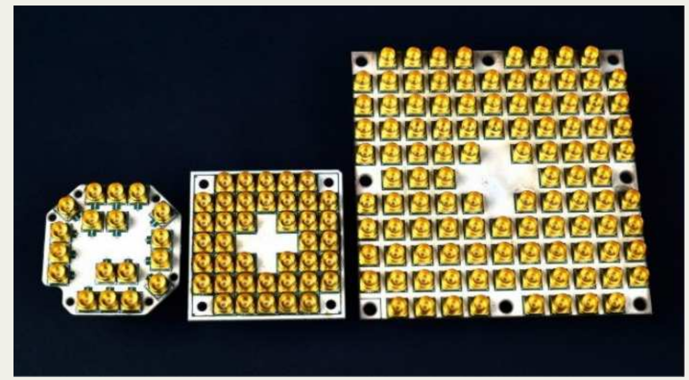


Other well–known codes

- Planar surface code (SC7, SC17, SC49) (target of QuTech)
- Double–defect surface code







Towards fault–tolerant Universal QC

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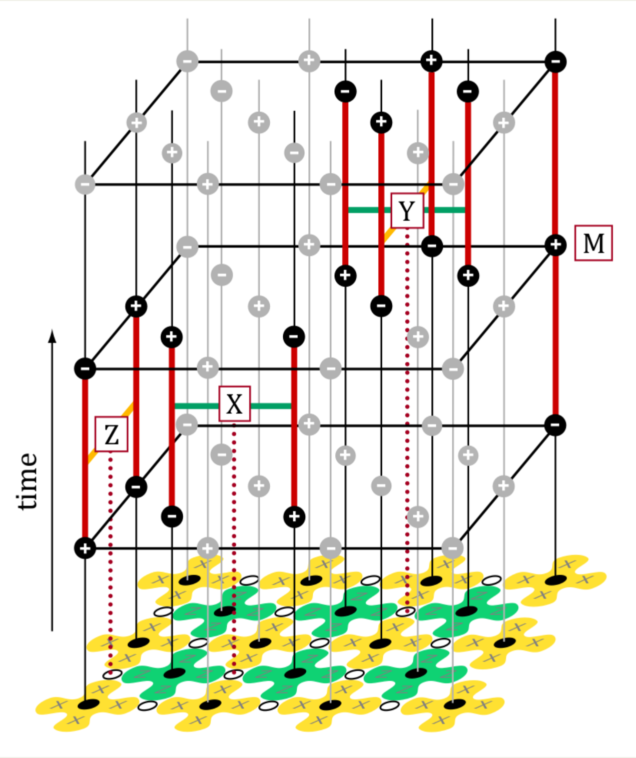
- Resistance to ancilla circuit errors
- Multiple logical qubits
- Logical operations (no decoding required)
  - Universal (e.g. X, Z, H, S, *CNOT*, T)




Questions?