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Quantum Search: Grover's algorithm

Quantum Capita Selecta

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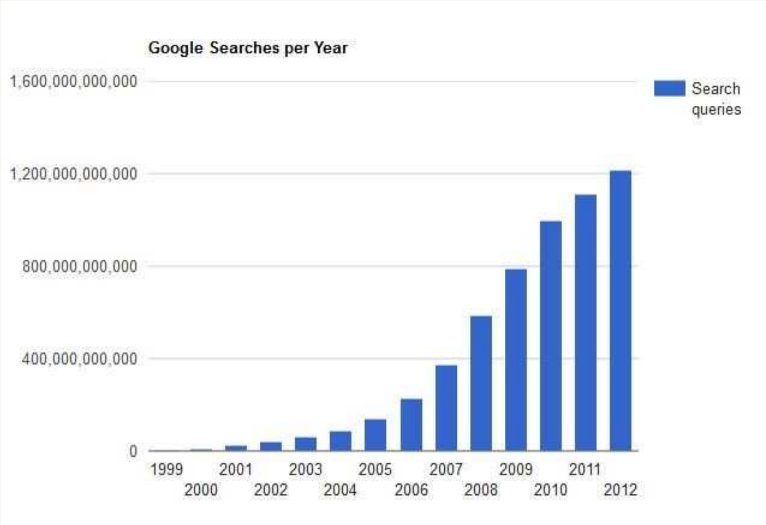
Searching



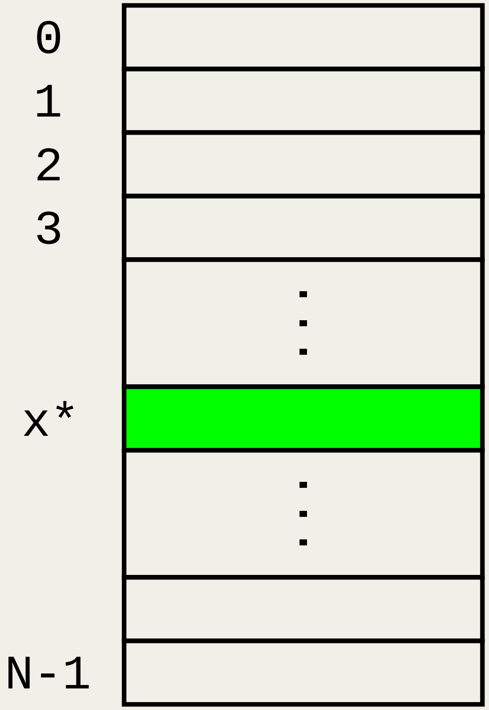
- We search a lot:
  - 82166 Google searches every second
  - 3.5 billion searches per day
- Structured search:
  - Based on the PageRank algorithm
  - Search data structures to improve performance

Searching

- Unstructured search:
  - Table with N entries
  - Special item: x^*
  - Find x^* (assuming 1 solution)
- Consider the n–to–1 bit function:

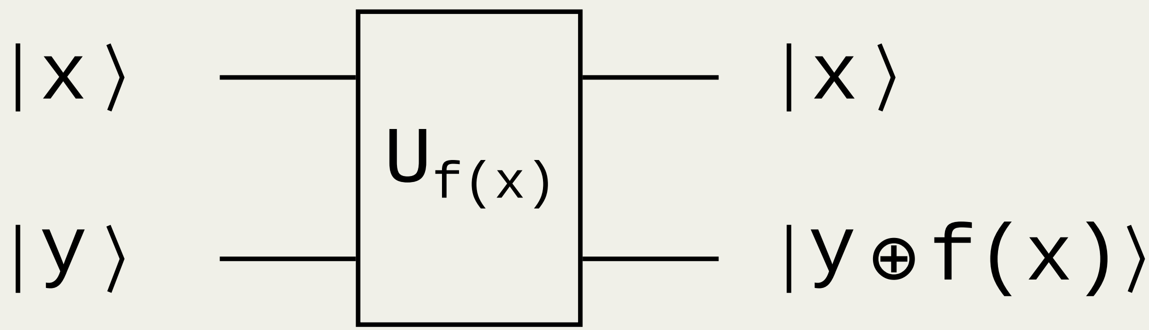
$$f(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{if } x \neq x^* \end{cases}$$

- In case $n = 2$:
  - $N = 2^n = 4$
  - Expected function calls: 2.25



Oracle

- Model for the interaction with the database
- Implements a function $f(x)$
  - $f(x)$ encodes information about the item x^*
- Query the oracle to fetch results from the function
- First possible implementation:
  - Add an extra (ancillary) qubit



Oracle

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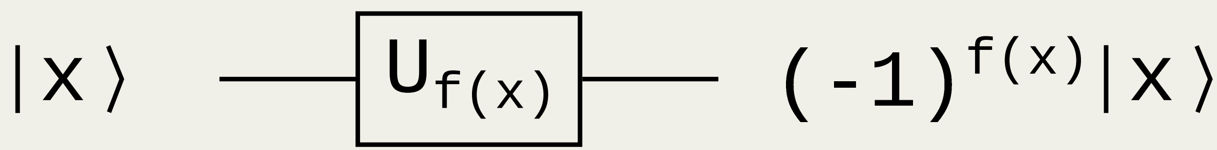
- Second possible implementation:
  - Flip the input phase if and only if $f(x) = 1$

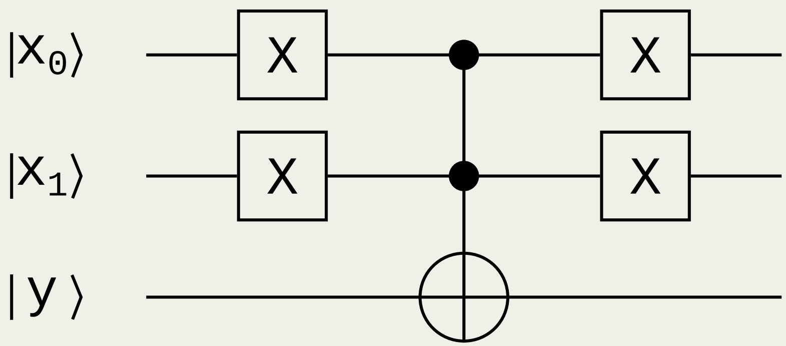
$$(-1)^{f(x)} |x\rangle = \begin{cases} |x\rangle & \text{if } f(x) = 0 \\ -|x\rangle & \text{if } f(x) = 1 \end{cases}$$

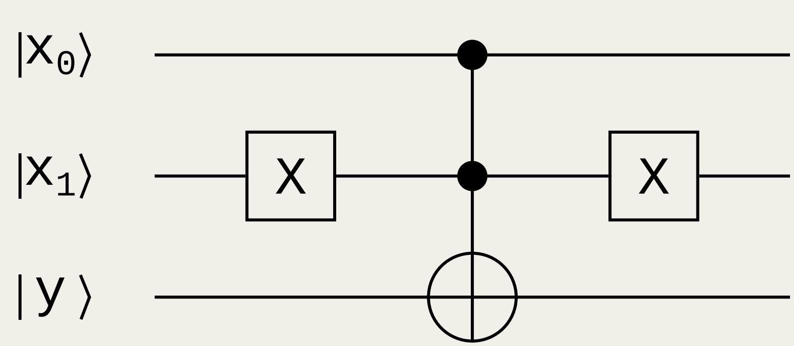


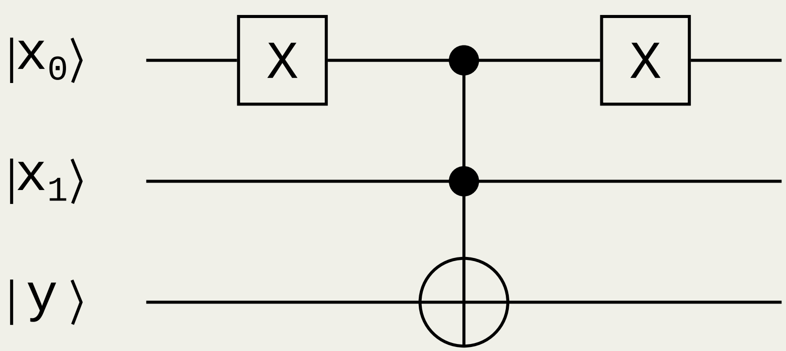
- Each implementation has its own particularities
- Solution is marked with a phase of -1

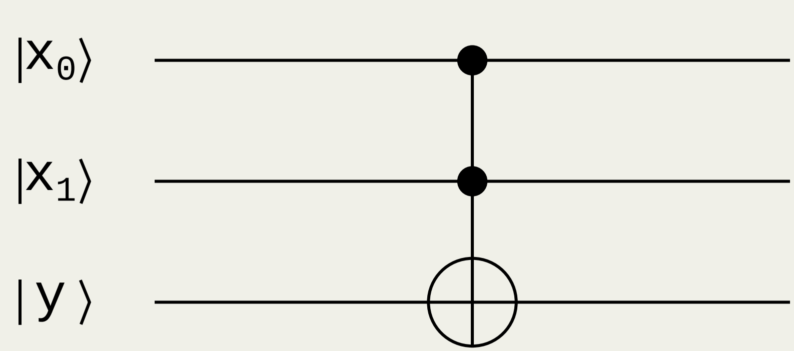
Oracle ($n = 2$)

- Using ancillary qubits





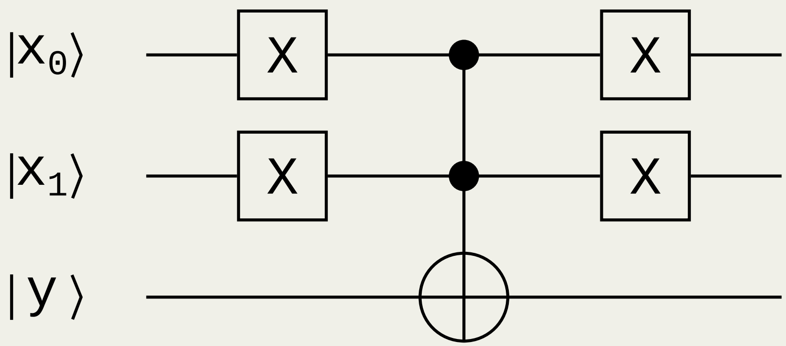




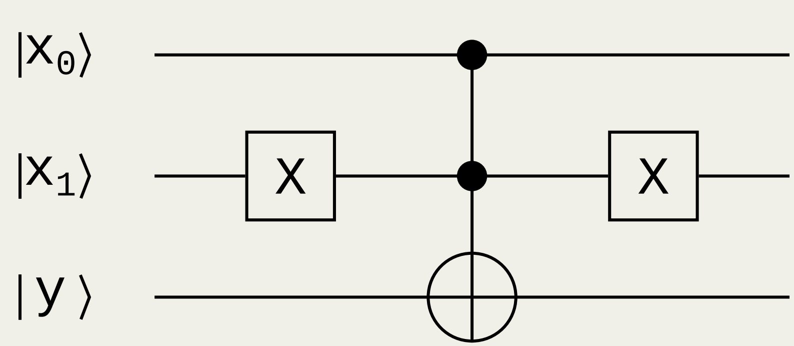
Oracle ($n = 2$)

- Using ancillary qubits

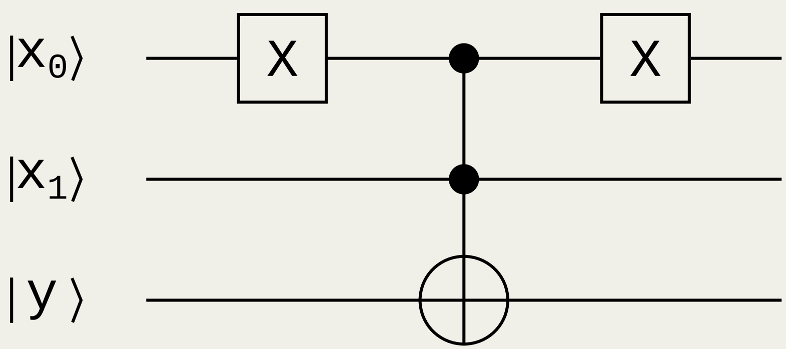
$x^* = 00$



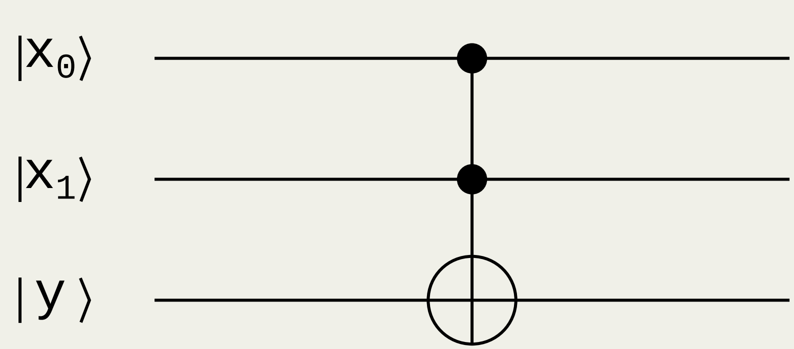
$x^* = 01$



$x^* = 10$



$x^* = 11$



Oracle ($n = 2$)

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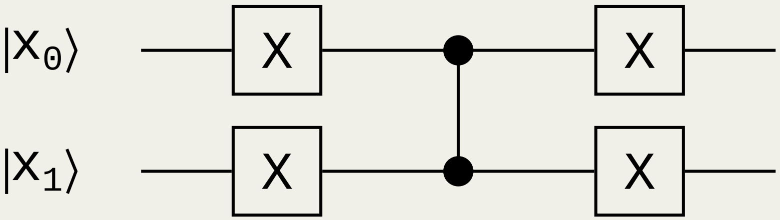
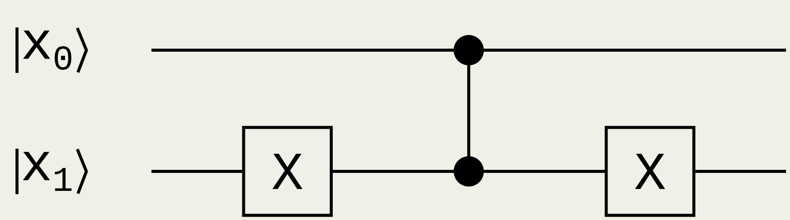
- Flipping phase

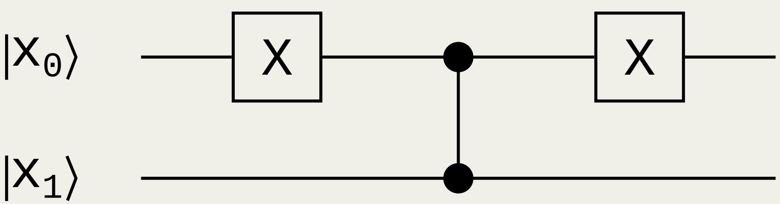
$x^* = 00$



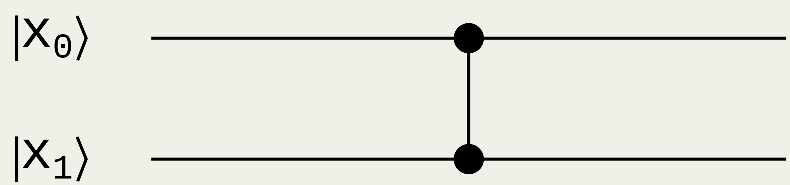
$x^* = 01$



$x^* = 10$

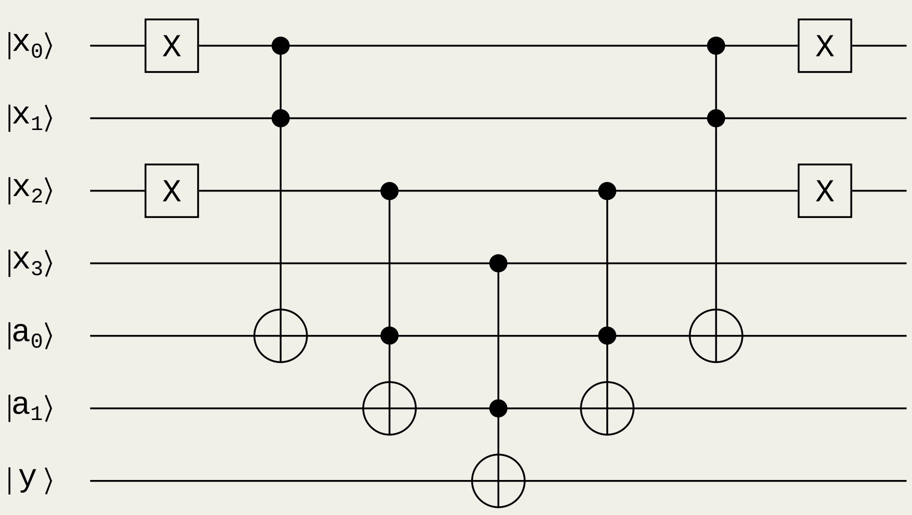


$x^* = 11$



Oracle ($n > 2$)

- What about $n > 2$?
- Is there a n–qubit Toffoli gate?
  - Of course, because everything holds up on paper...
  - ... but, in Qiskit?
- Chaining some Toffoli gates
  - Using ancilla qubits to store intermediate results



Grover's algorithm

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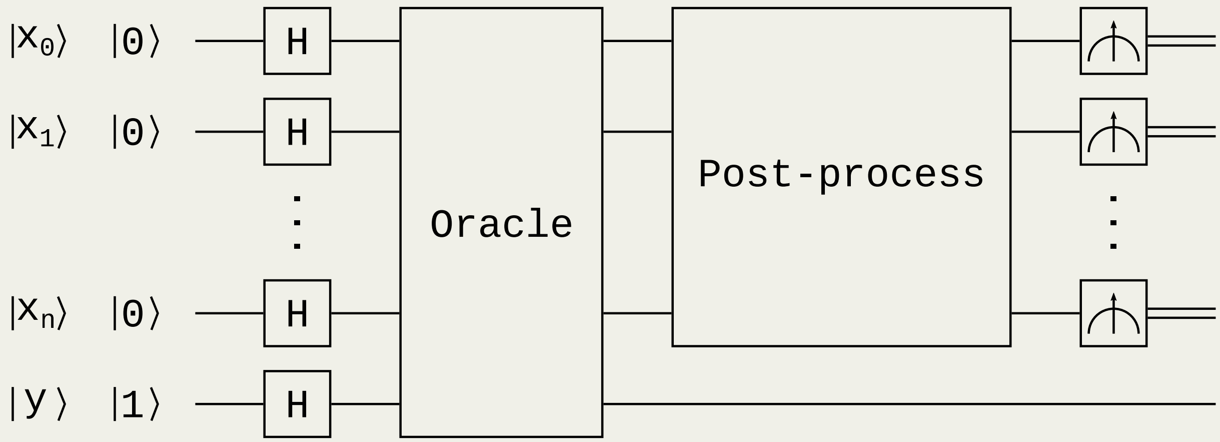
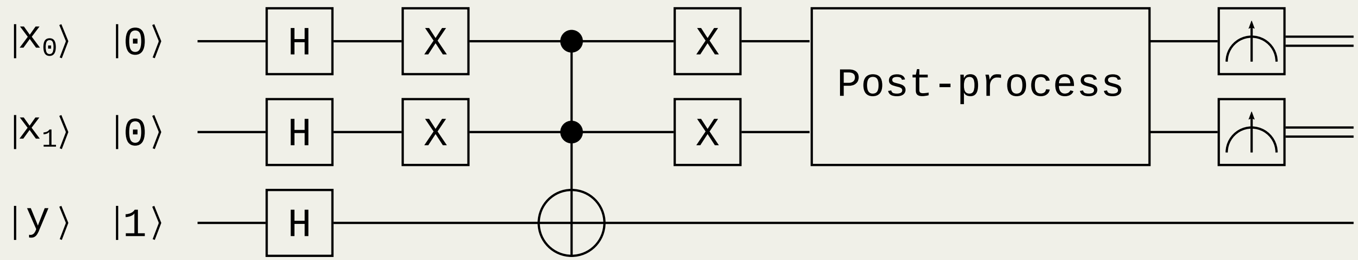


1. Prepare input state (equal superposition state)
2. Apply oracle (phase–kickback)
3. Post–process (inversion around mean)
4. Measure

Grover's algorithm



$$\rightarrow |1\rangle \otimes |0\rangle \otimes |0\rangle$$

$$\rightarrow |1\rangle \otimes |00\rangle$$

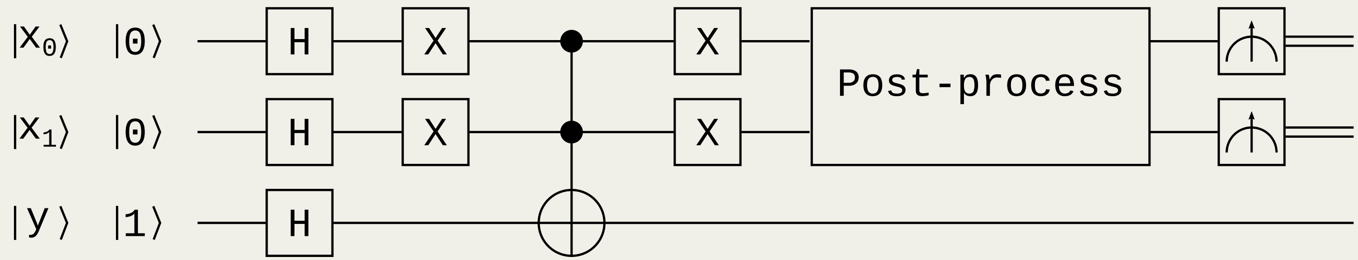
$$\rightarrow |100\rangle$$

$$H_{x_0} \rightarrow \frac{1}{\sqrt{2}} \left(|100\rangle + |101\rangle \right)$$

$$H_{x_1} \rightarrow \frac{1}{2} \left(|100\rangle + |110\rangle + |101\rangle + |111\rangle \right)$$

$$H_y \rightarrow \frac{1}{2\sqrt{2}} \left(|000\rangle - |100\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle + |011\rangle - |111\rangle \right)$$

Grover's algorithm



$$\rightarrow \frac{1}{2\sqrt{2}}\left(|000\rangle - |100\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle + |011\rangle - |111\rangle\right)$$

$$X_{x_0} \rightarrow \frac{1}{2\sqrt{2}}\left(|001\rangle - |101\rangle + |011\rangle - |111\rangle + |000\rangle - |100\rangle + |010\rangle - |110\rangle\right)$$

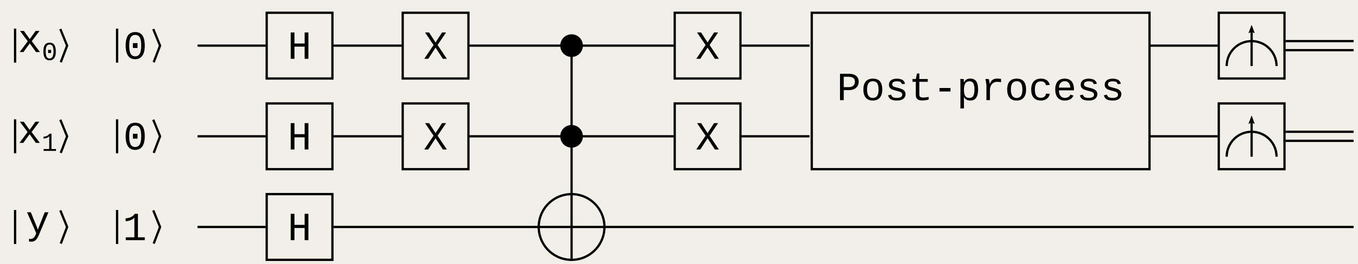
$$X_{x_1} \rightarrow \frac{1}{2\sqrt{2}}\left(|011\rangle - |111\rangle + |001\rangle - |101\rangle + |010\rangle - |110\rangle + |000\rangle - |100\rangle\right)$$

$$\mathit{CCNOT}_{x_0, x_1, y} \rightarrow \frac{1}{2\sqrt{2}}\left(|111\rangle - |011\rangle + |001\rangle - |101\rangle + |010\rangle - |110\rangle + |000\rangle - |100\rangle\right)$$

$$X_{x_0} \rightarrow \frac{1}{2\sqrt{2}}\left(|110\rangle - |010\rangle + |000\rangle - |100\rangle + |011\rangle - |111\rangle + |001\rangle - |101\rangle\right)$$

$$X_{x_1} \rightarrow \frac{1}{2\sqrt{2}}\left(|100\rangle - |000\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle + |011\rangle - |111\rangle\right)$$

Grover's algorithm



$$\rightarrow \frac{1}{2\sqrt{2}}\left(|100\rangle - |000\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle + |011\rangle - |111\rangle\right)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(- |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle - |111\rangle\right)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(\left(|0\rangle \otimes \left(- |00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right) + \left(|1\rangle \otimes \left(|00\rangle - |01\rangle - |10\rangle - |11\rangle\right)\right)\right)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(\left(|0\rangle \otimes \left(- |00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right) - \left(|1\rangle \otimes \left(- |00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right)\right)$$

$$\rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes \frac{1}{2}\left(- |00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

- Solution is marked with a phase of -1

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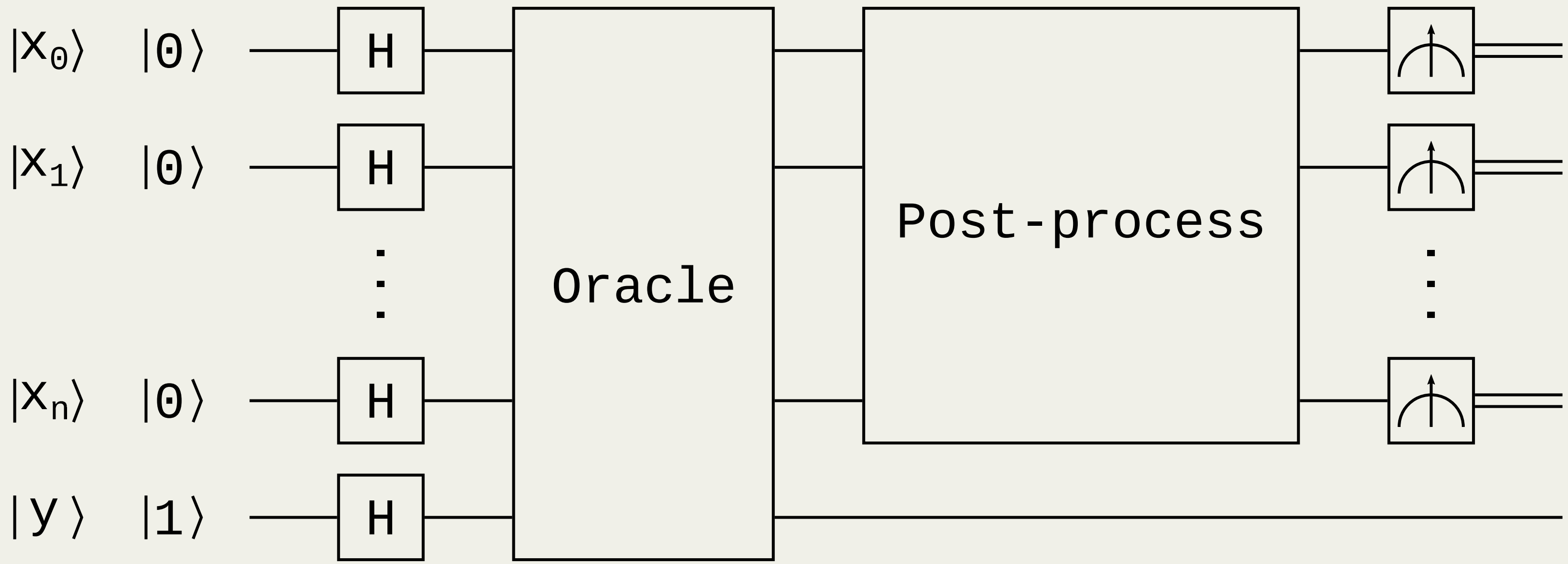
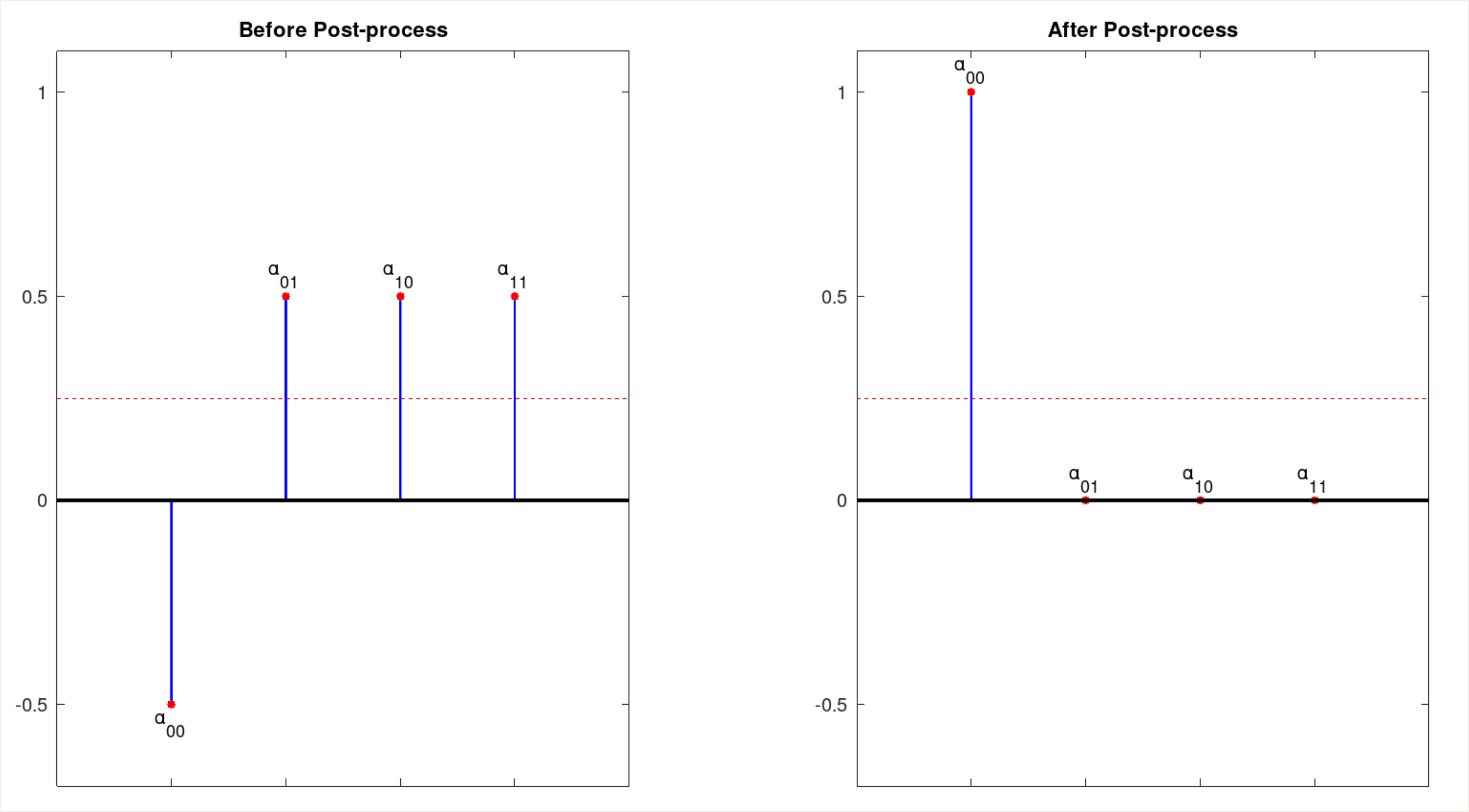
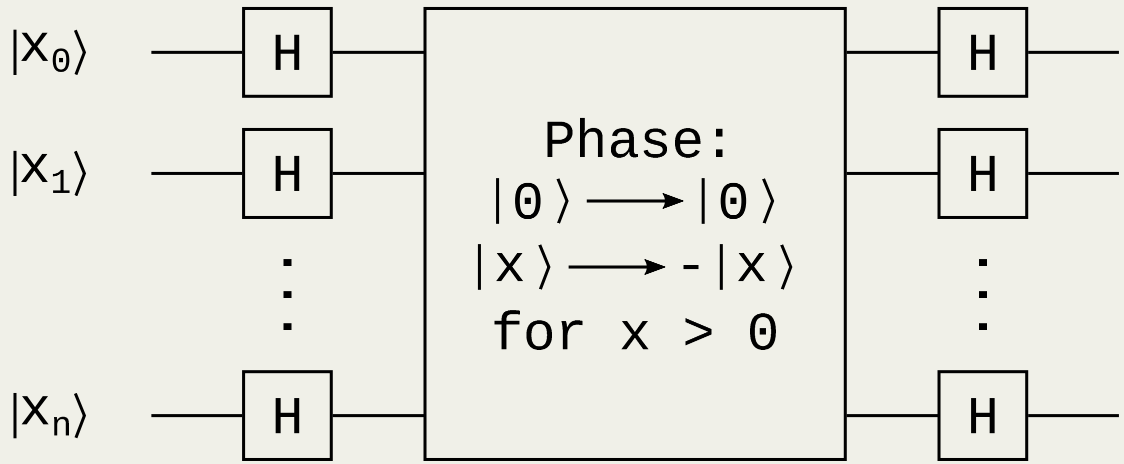


- Post–process
  - Map solutions to computational basis states
  - Extract solutions marked with a phase of -1 by **inversion around mean**

Inversion around the mean



Inversion around the mean



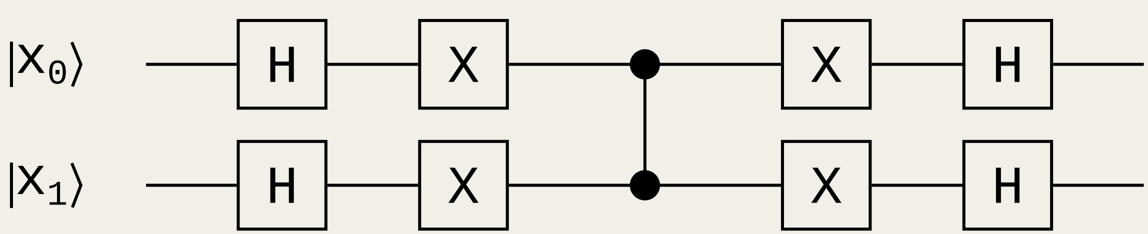
- Can be decomposed in:
    - H gates
    - A phase operation
    - H gates

Inversion around the mean

- For our two qubits example
  - The matrix is:

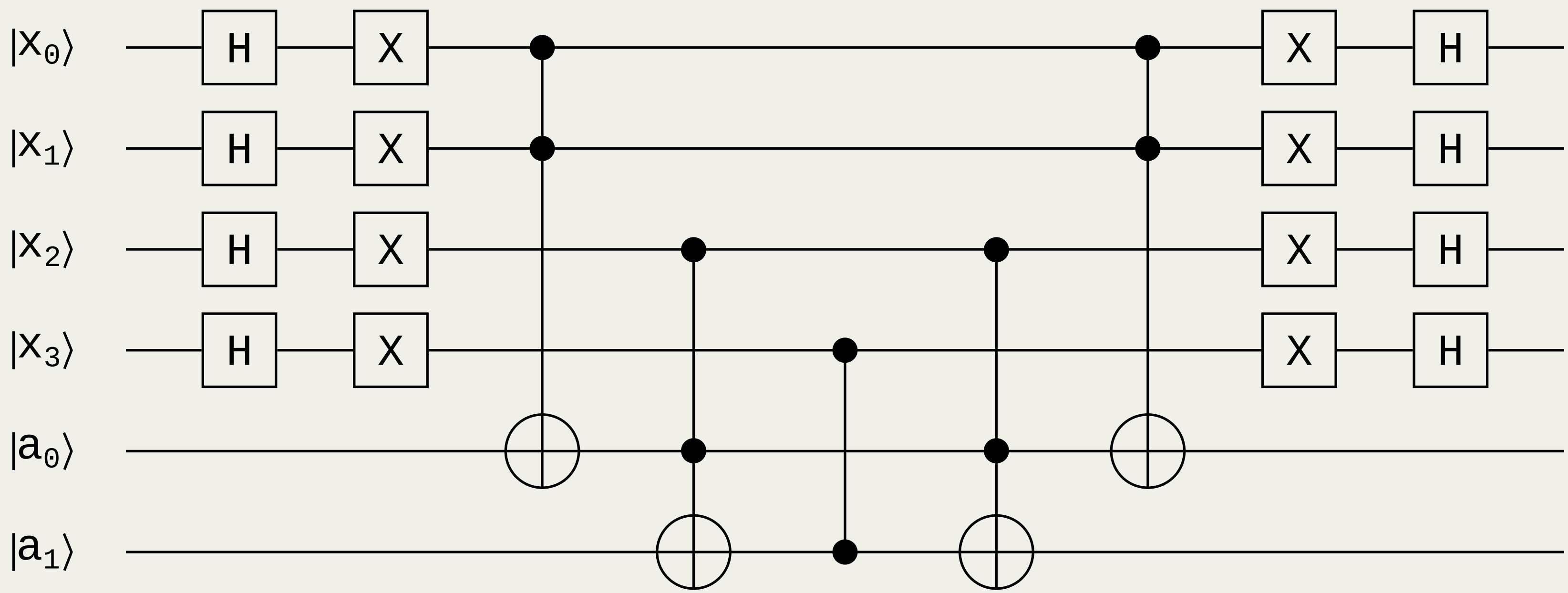
$$U_{IAM} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

  - The circuit is:



Inversion around the mean

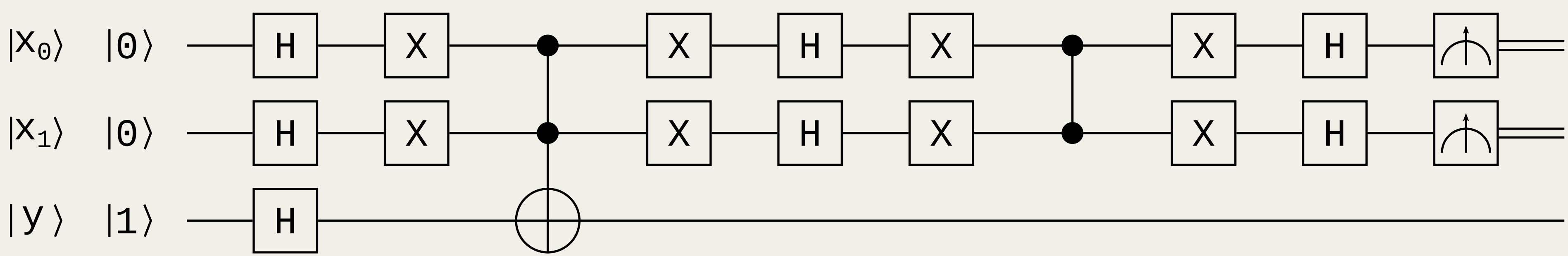
- What about a 4–qubits circuit?



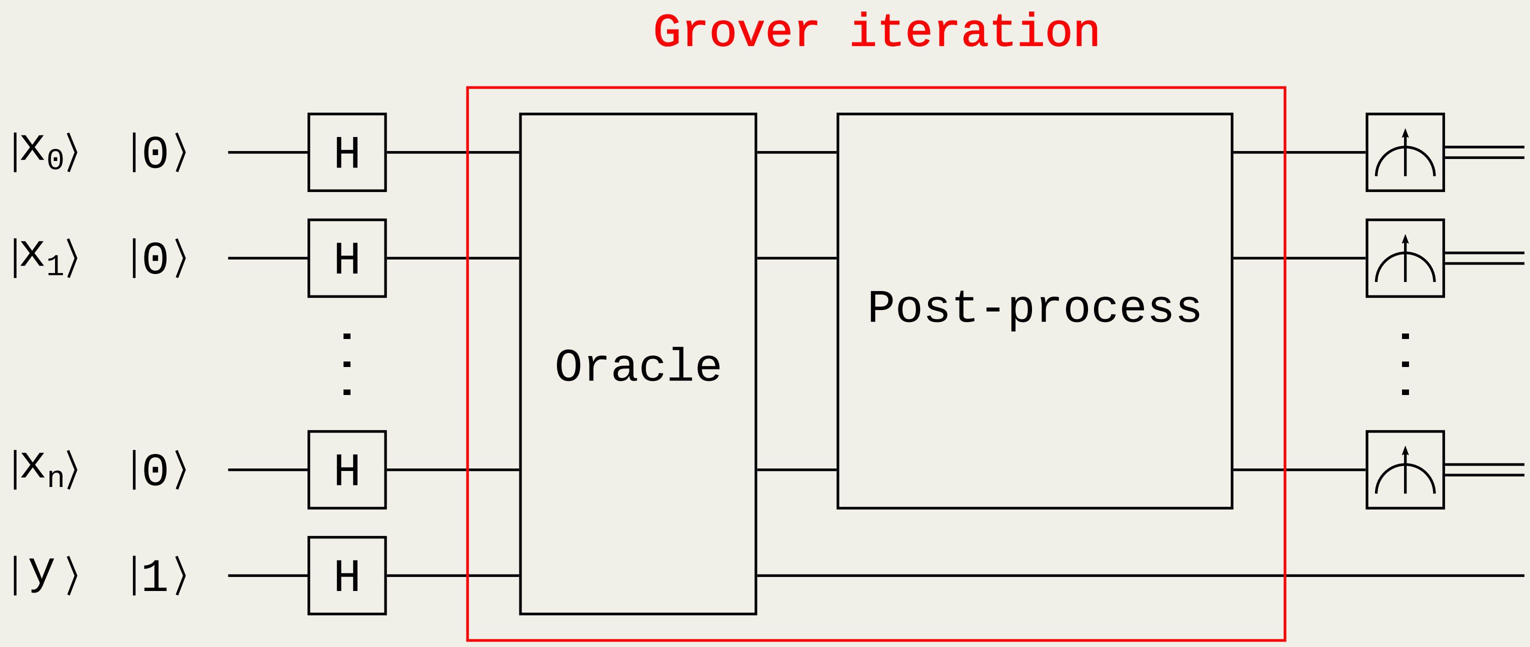
- For a generic n–qubits circuit:
    - Same approach: identify the pattern and continue it!

Complete circuit

- As usual, put the pieces together
  - Prepare input state
  - Apply oracle
  - Post–process
  - Measure
- Complete circuit for item $x^* = 00$:



Does Grover's algorithm scale?

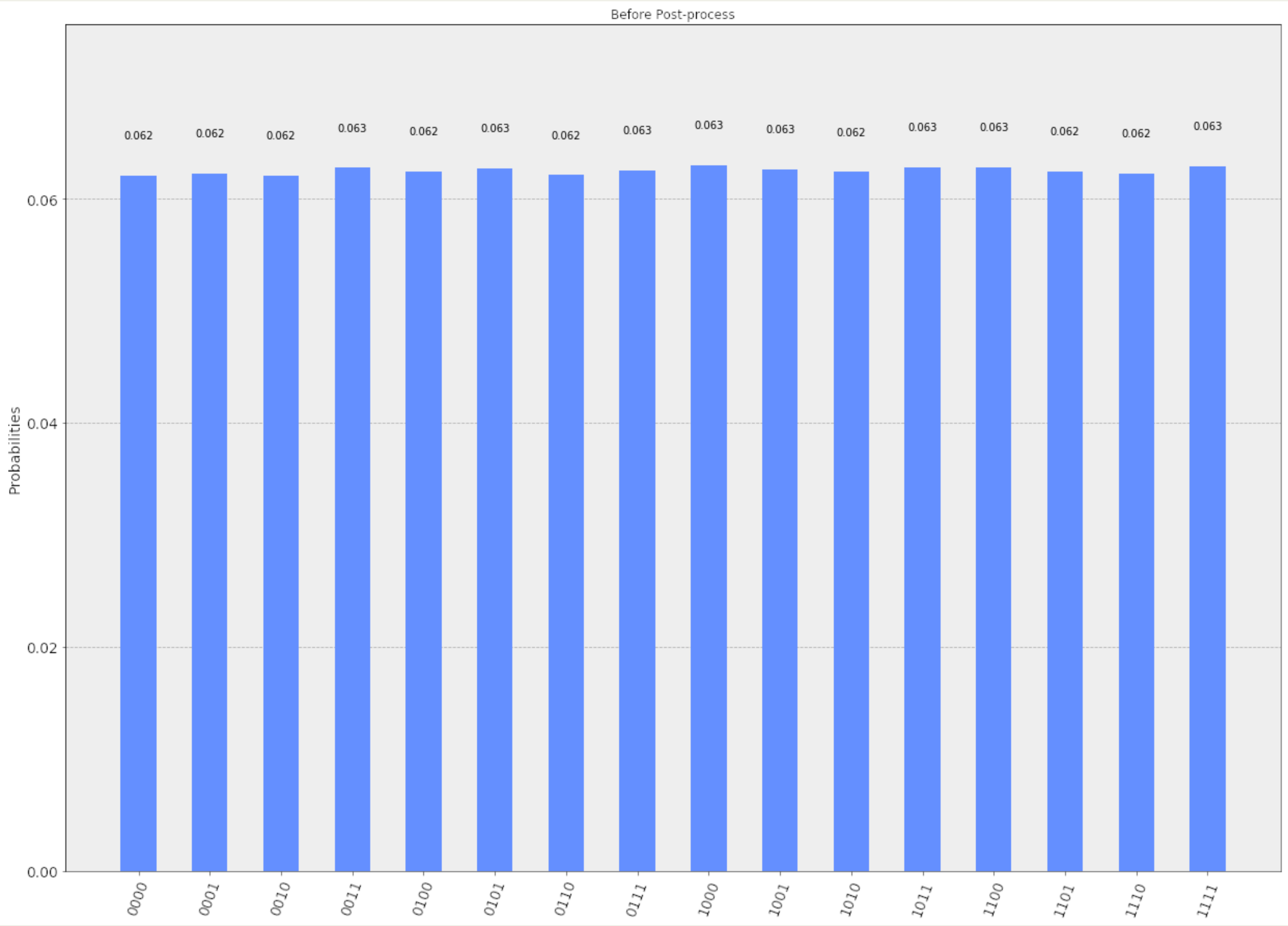


- Search space of N elements, exactly M solutions
- Repeat Grover iteration:

$$R \leq \left\lceil \frac{\pi}{4}\sqrt{\frac{N}{M}} \right\rceil$$

Example

- $N = 16$, $M = 1$, $x^* = |0100\rangle$ and $R \leq 4$



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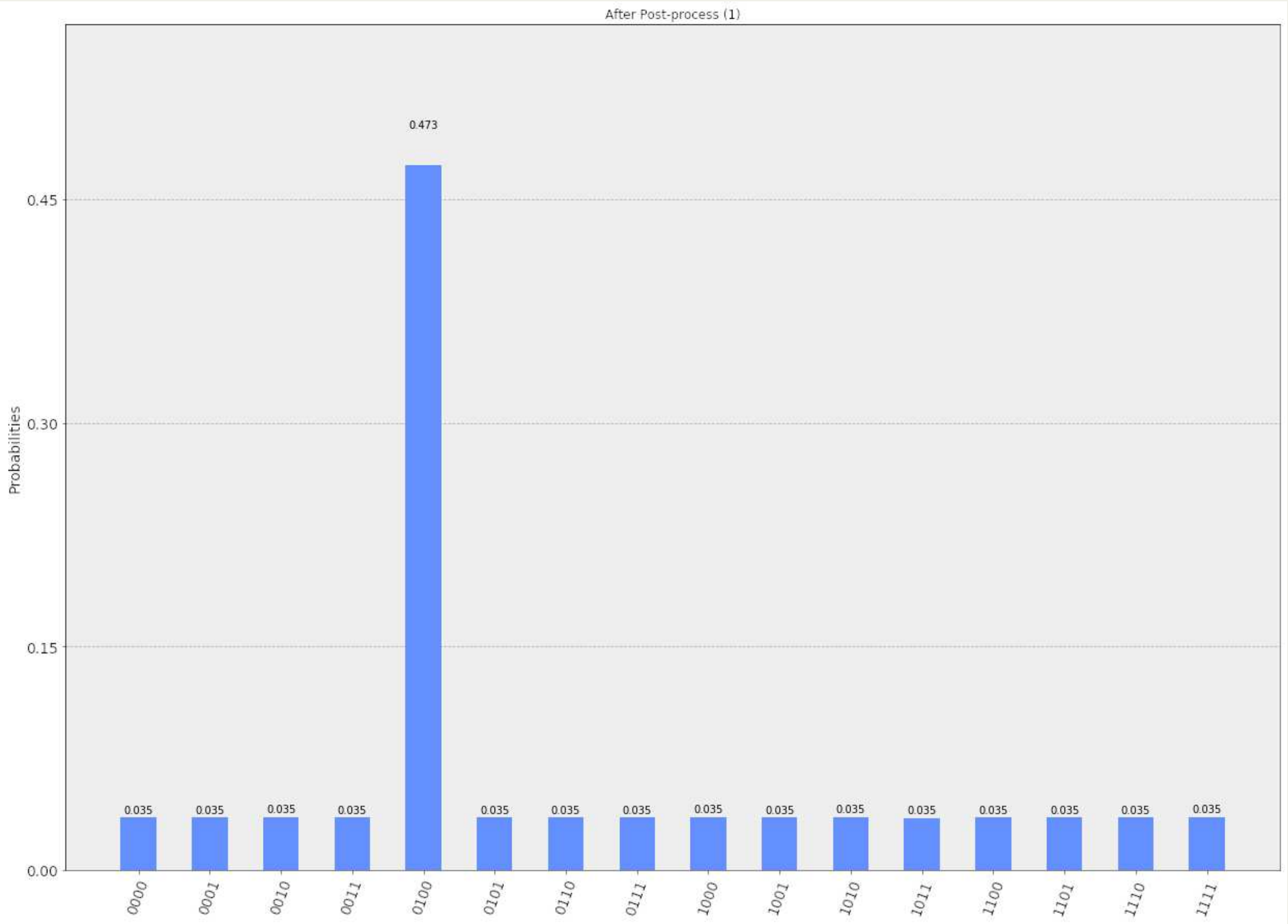

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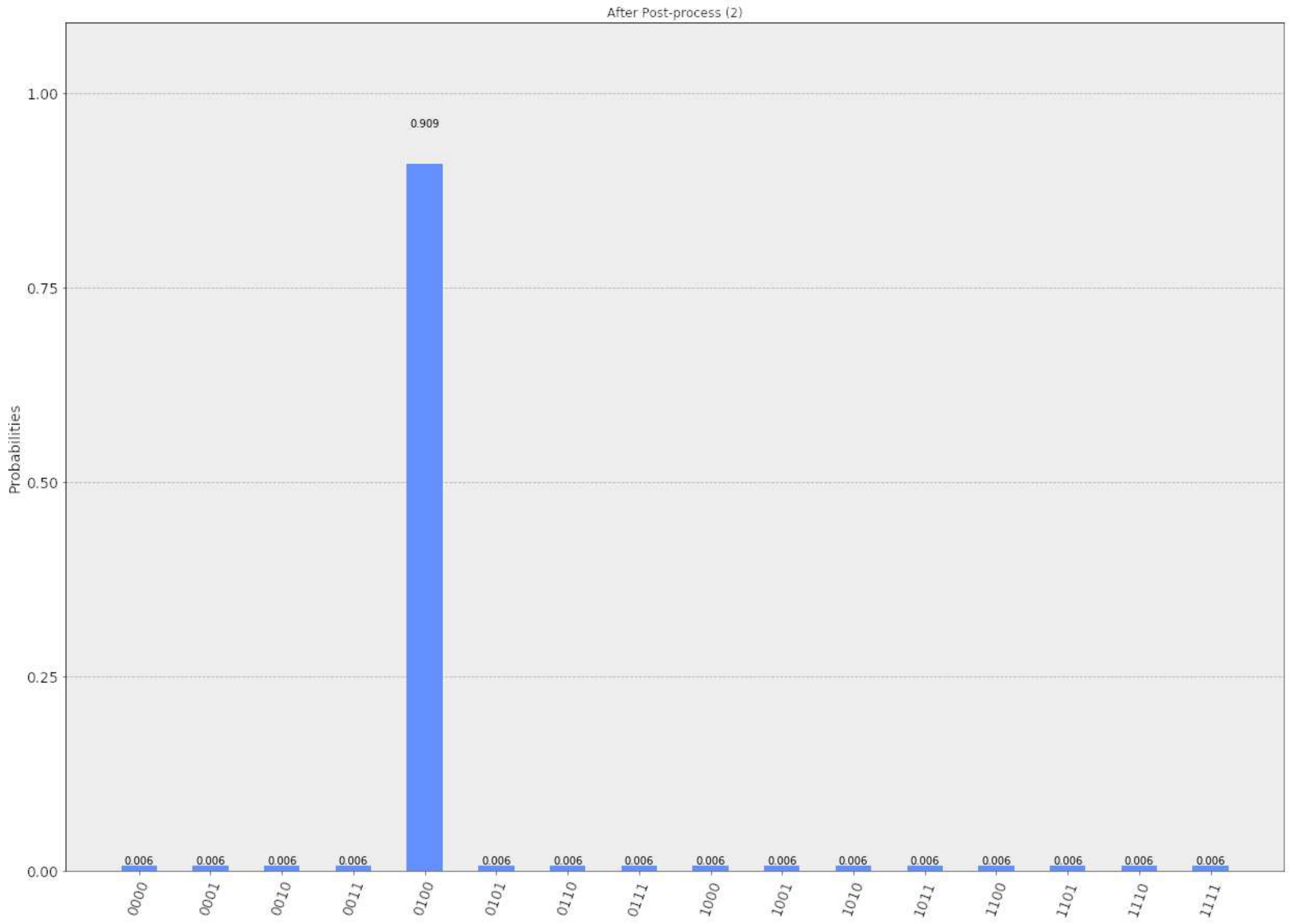

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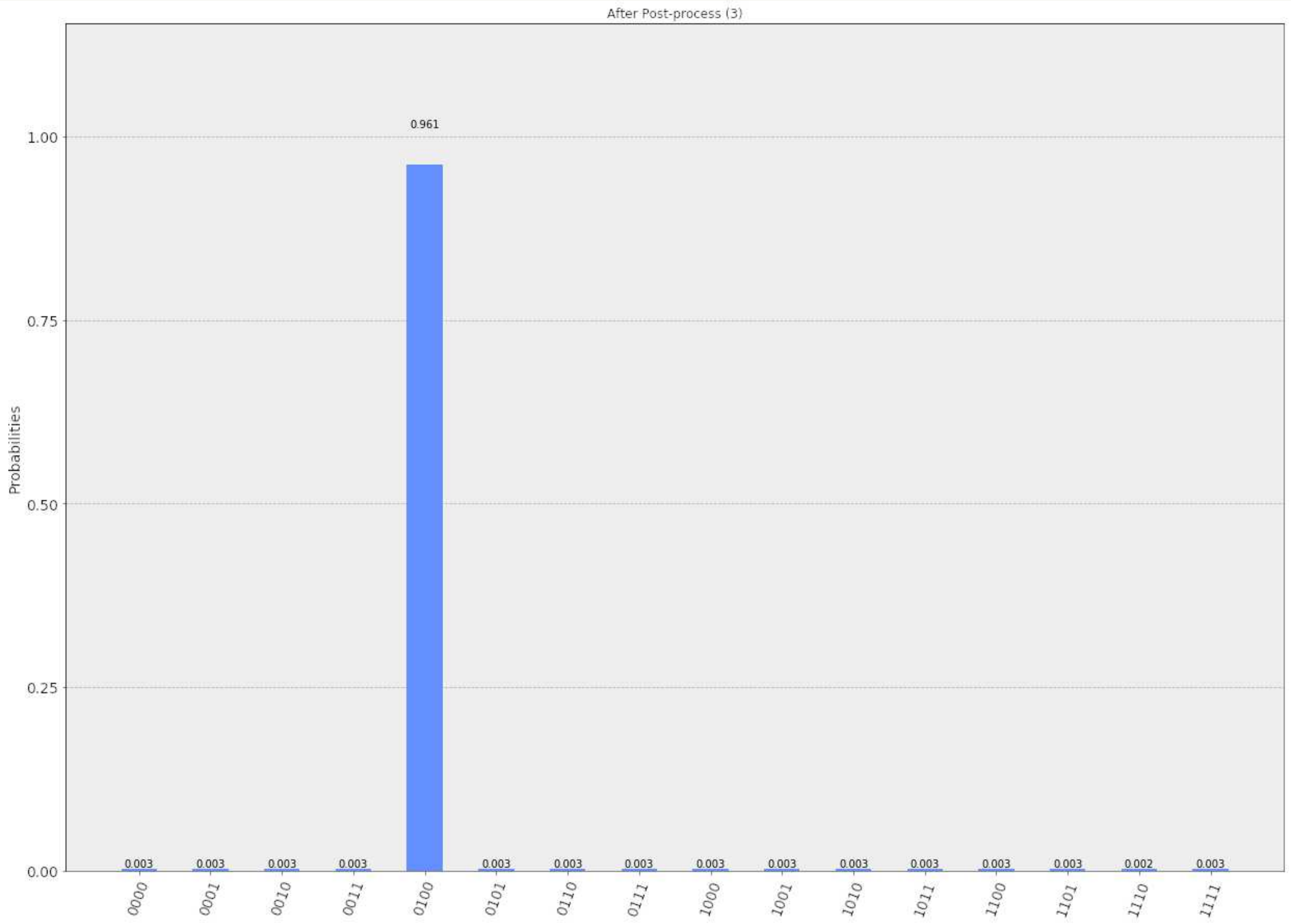

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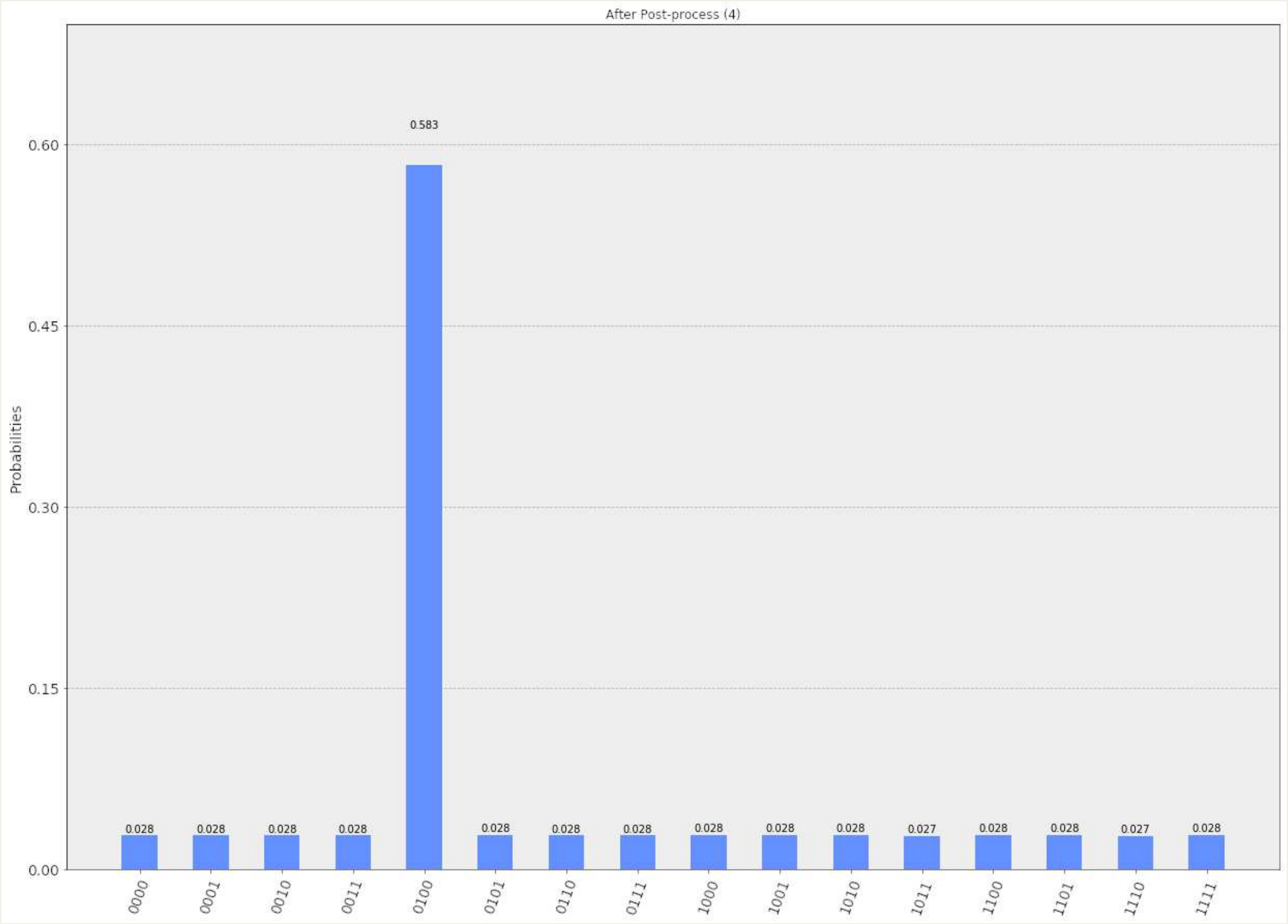

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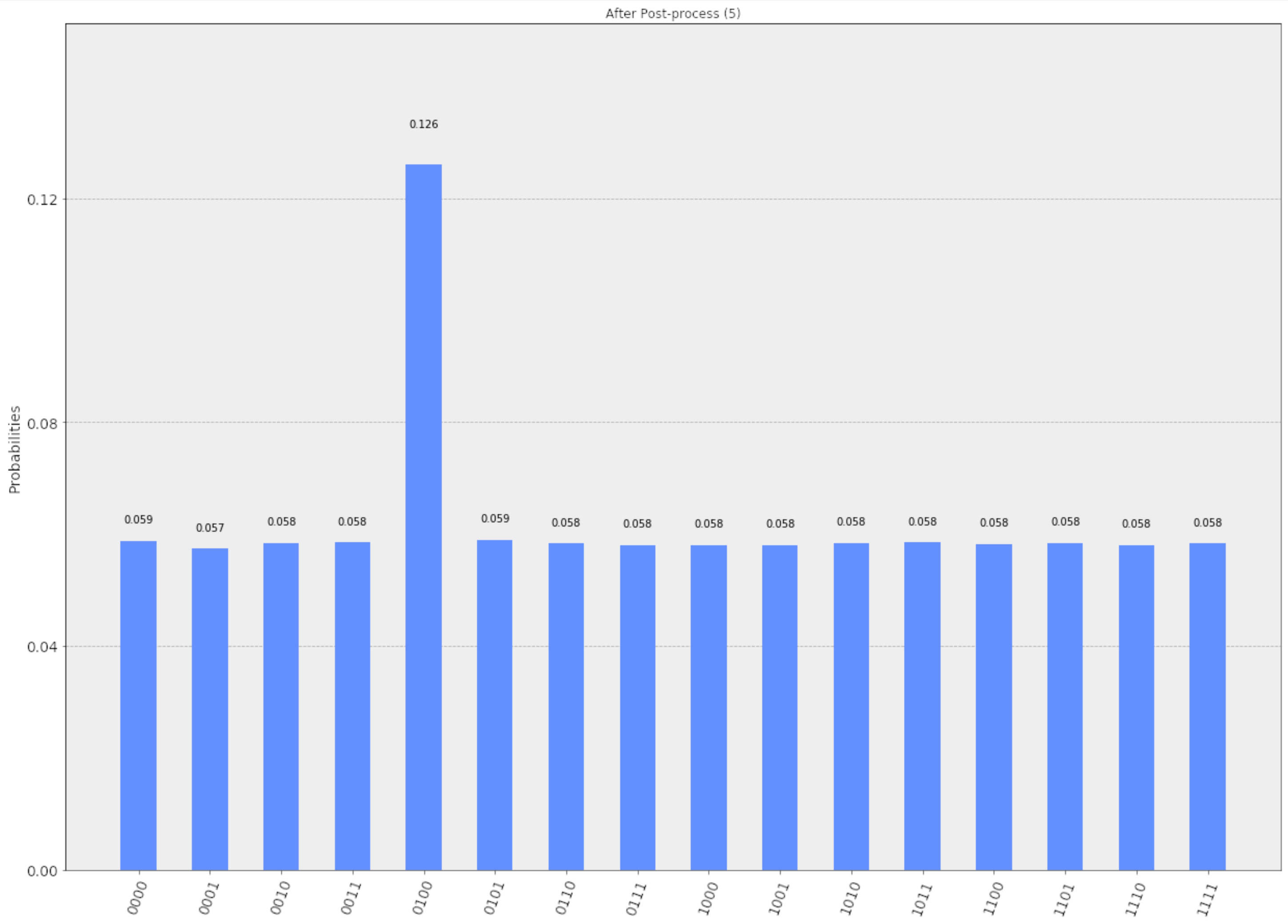


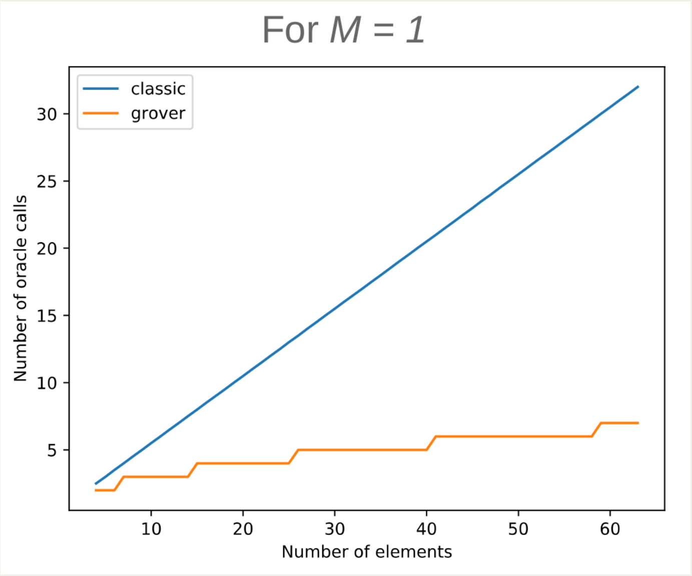
Complexity of Grover's algorithm

- Number of oracle calls required:

$$\mathcal{O}_{\text{c}}\left(\frac{N}{M}\right)$$

$$\mathcal{O}_{\text{g}}\left(\sqrt{\frac{N}{M}}\right)$$

- "Only" quadratic speedup
  - Still a huge improvement!




Questions?