Amsterdam University of Applied Sciences

Quantum Computing Introduction

Basic Quantum Algorithms (2)

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Deutsch–Jozsa's problem

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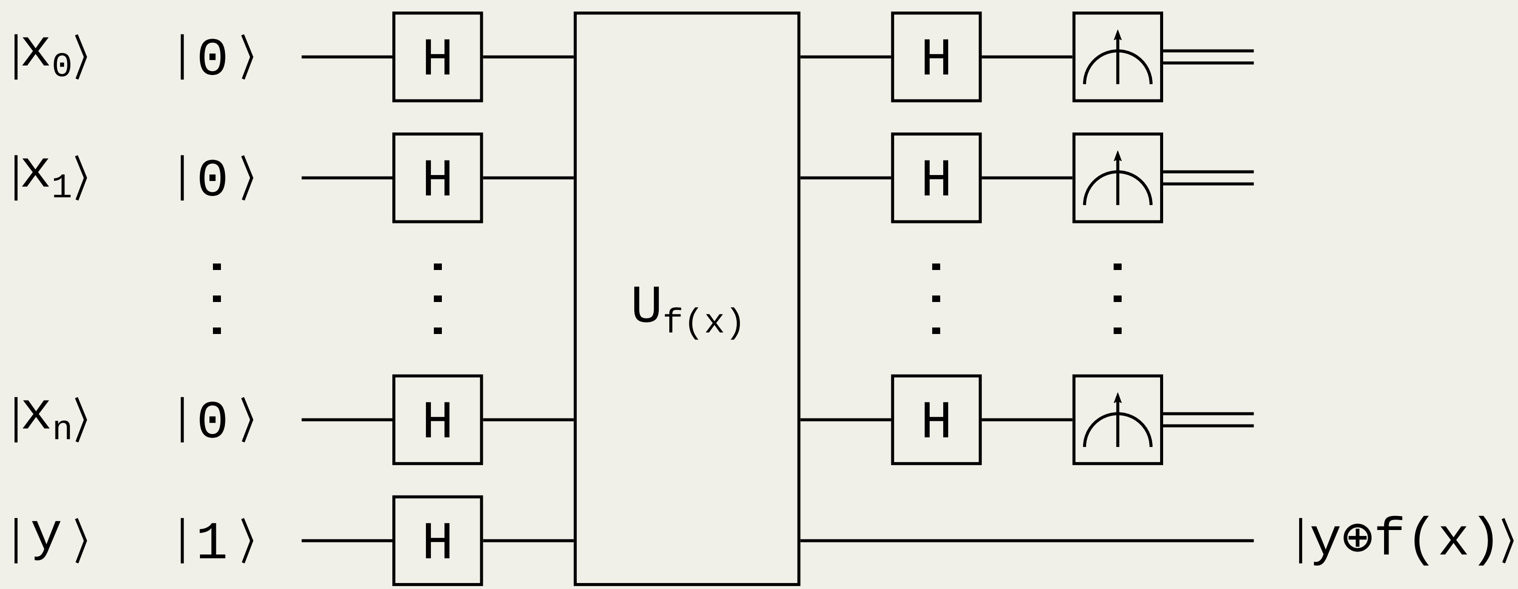
- Extension of Deutsch's problem to n–qubits input
- The function is either *constant* or *balanced*:
  - *Constant*: all outputs are 0s or all 1s
    - $f(x_1, x_2) = 0$ or $f(x_1, x_2) = 1$
  - *Balanced*: half of the outputs are 0s while the other half are 1s
    - $f(x_1, x_2) = x_1$, $f(x_1, x_2) = \widetilde{x_2}$ or $f(x_1, x_2) = x_1 \oplus x_2$

Deutsch–Jozsa's problem

- How many calls to the oracle are needed?
  - Classical: $2^{n-1} + 1$
  - Quantum: 1

$$
\begin{aligned}
f(000) &= 0\\
f(001) &= 0\\
f(010) &= 0\\
f(011) &= 0\\
f(100) &=\\
f(101) &=\\
f(110) &=\\
f(111) &=
\end{aligned}
$$

Deutsch–Jozsa's algorithm

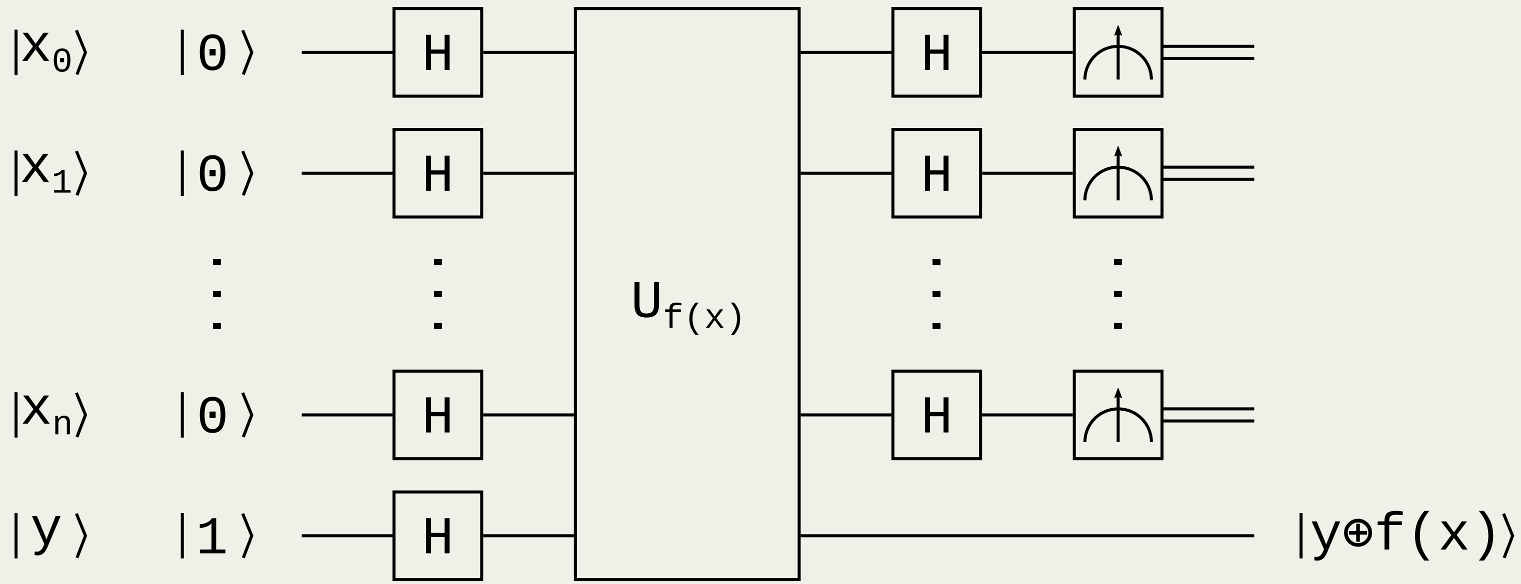


Input: $$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

Target: $$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Output: $$\frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{k=0}^{2^n-1} (-1)^{f(x)+k\cdot x} |k\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

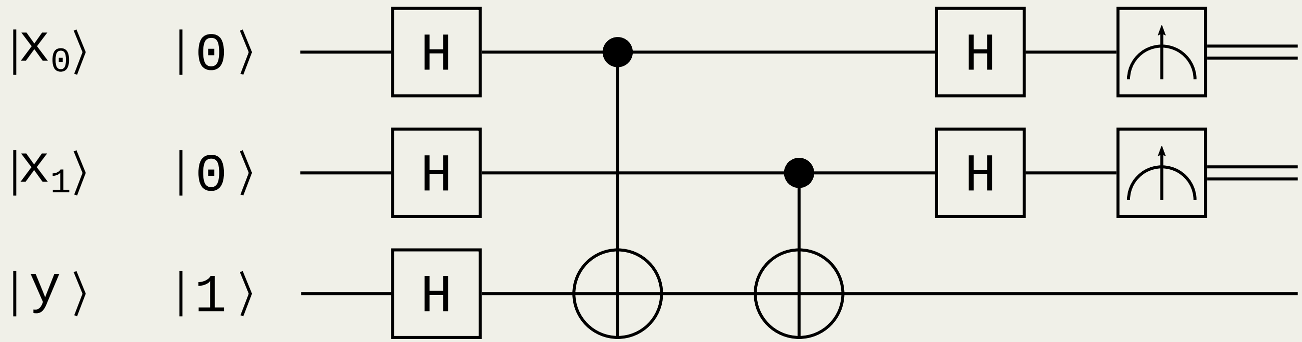
Deutsch–Jozsa's algorithm



Output: $\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1}(-1)^{f(x)+k\cdot x}\left|k\right\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

- We have to check the first register (x)
  - $x = 0 \rightarrow$ *constant*
  - $x \neq 0 \rightarrow$ *balanced*

Deutsch–Jozsa's algorithm



$$\rightarrow |100\rangle$$

$$H_0 \rightarrow \frac{1}{\sqrt{2}}\left(|100\rangle + |101\rangle\right)$$

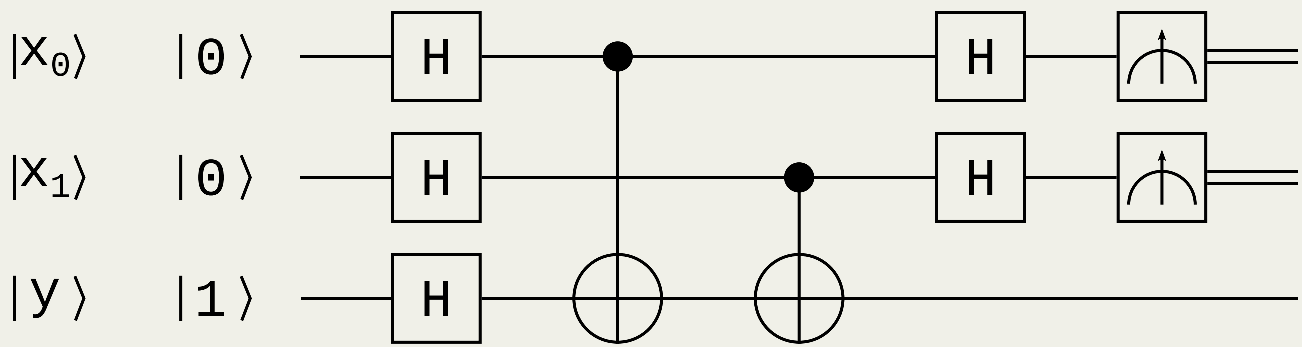
$$H_1 \rightarrow \frac{1}{2}\left(|100\rangle + |110\rangle + |101\rangle + |111\rangle\right)$$

$$H_2 \rightarrow \frac{1}{2\sqrt{2}}\left(|000\rangle - |100\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle + |011\rangle - |111\rangle\right)$$

$$\mathit{CNOT}_{0,\,2} \rightarrow \frac{1}{2\sqrt{2}}\left(|000\rangle - |100\rangle + |010\rangle - |110\rangle + |101\rangle - |001\rangle + |111\rangle - |011\rangle\right)$$

$$\mathit{CNOT}_{1,\,2} \rightarrow \frac{1}{2\sqrt{2}}\left(|000\rangle - |100\rangle + |110\rangle - |010\rangle + |101\rangle - |001\rangle + |011\rangle - |111\rangle\right)$$

Deutsch–Jozsa's algorithm



$$
\begin{aligned}
H_0 &\rightarrow \frac{1}{4}(|000\rangle + |001\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle - |010\rangle - |011\rangle + \\
&+ |100\rangle - |101\rangle - |000\rangle + |001\rangle + |010\rangle - |011\rangle - |110\rangle + |111\rangle) \\
&\rightarrow \frac{1}{4}(2\,|001\rangle - 2\,|101\rangle + 2\,|111\rangle - 2\,|011\rangle) \\
&\rightarrow \frac{1}{2}(|001\rangle - |101\rangle + |111\rangle - |011\rangle) \\
H_1 &\rightarrow \frac{1}{2\sqrt{2}}(|001\rangle + |011\rangle - |101\rangle - |111\rangle + |101\rangle - |111\rangle - |001\rangle + |011\rangle) \\
&\rightarrow \frac{1}{2\sqrt{2}}(2\,|011\rangle - 2\,|111\rangle) \\
&\rightarrow \frac{1}{\sqrt{2}}(|011\rangle - |111\rangle)
\end{aligned}
$$

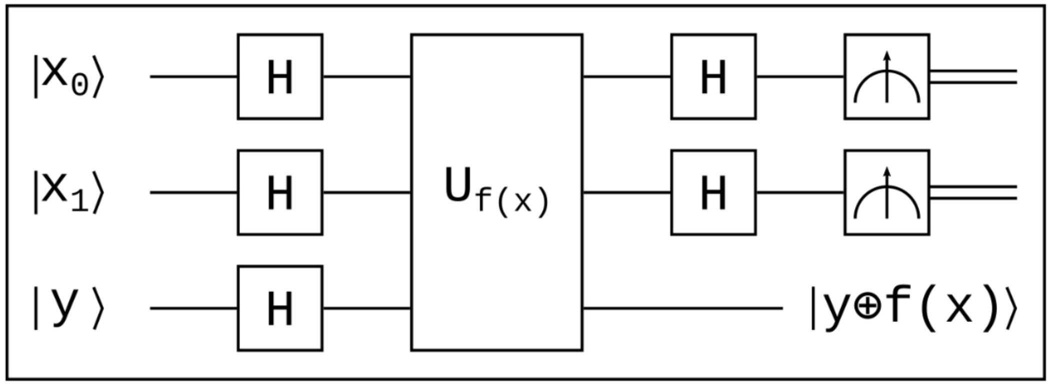
$$M(q_1 q_0) = 11 \Rightarrow Balanced$$

Question # 5

Question 5

During the lecture: "Basic Quantum Algorithms 2", I introduced the `Deutsch-Josza algorithm` (an extension to multiple input qubits of the `Deutsch algorithm`). I also identified all the possible constant and balanced functions for 2 qubits inputs. Moreover, I already solved an example for the `XOR` balanced function in the whiteboard.

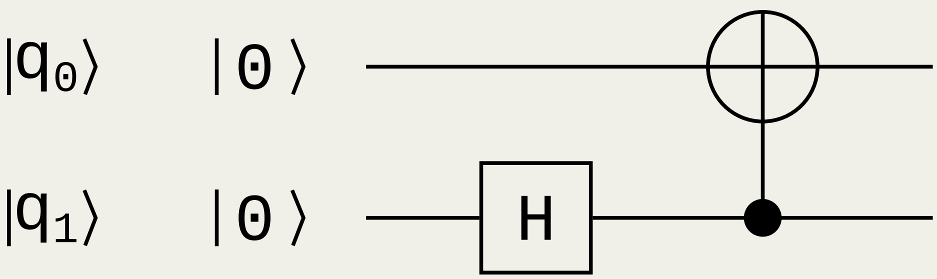
Now, you have to implement the Deutsch-Josza algorithm for the 2-qubits SET constant function and execute a single-shot simulation. Does your implementation correctly identify the function as constant?



NB: Remember that the algorithm states that the $|q_2\rangle$ register must be initialized to $|1\rangle$.

Bell states

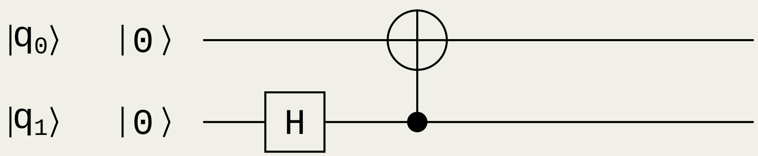
- Maximally entangled two–qubit states
- Variations in parity and phase
- How do we encode/prepare these states?



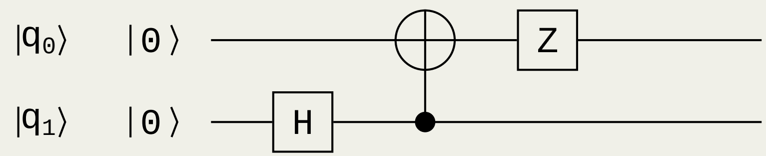
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell states

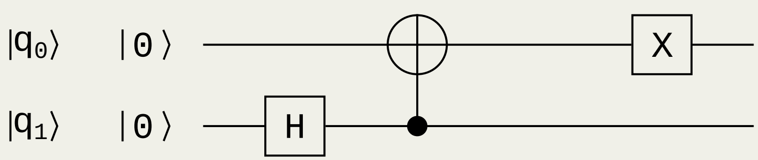
- Can be transformed by manipulating a single qubit



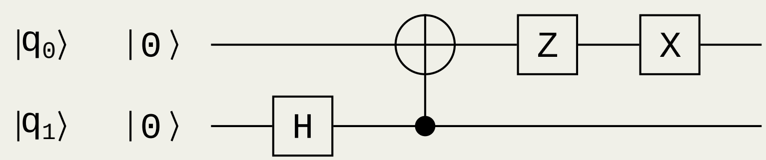
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$



$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$



$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell states

- Can be transformed by manipulating a single qubit

$\|\psi_{in}\rangle$	Z_0	X_0	$Z_0 X_0$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)^\star$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)^\star$
$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)^\star$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)^\star$

* Global phase is not shown in the table since it can be ignored

Question # 9

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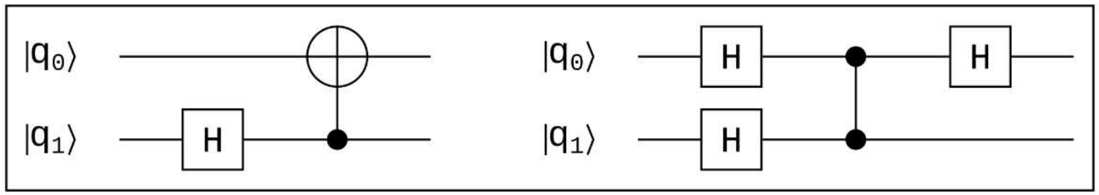
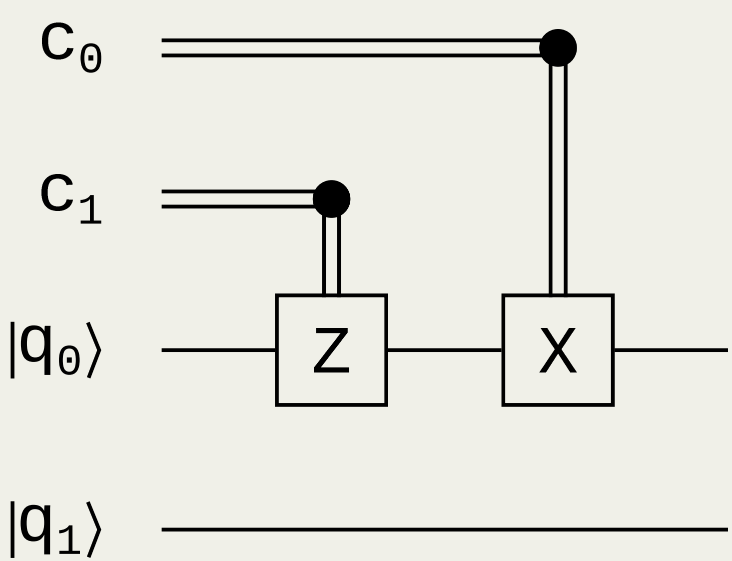
Question 9

During the lesson: "Basic Quantum Algorithms 2", I mentioned that the Bell states can be transformed by manipulating 1 single qubit. We even saw that process using the typical quantum circuit for preparing Bell states (left side of the image). Is it the same if we use a different quantum circuit? Test it by using the alternate circuit (show in the right side of the image). Implement the circuit to show the resulting Bell states.



Bell states

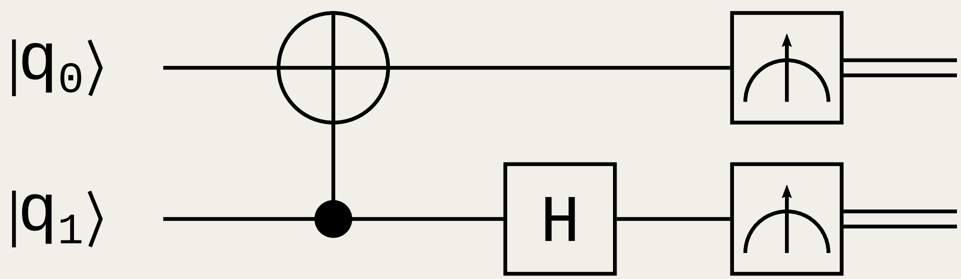
- Two bits can be encoded in a Bell state by manipulating one qubit



Input State: $\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	
I_0	00
X_0	01
Z_0	10
Z_0X_0	11

Bell states

- Can we "decode" the parity and phase?
  - Map parity and phase variations to separate computational basis states

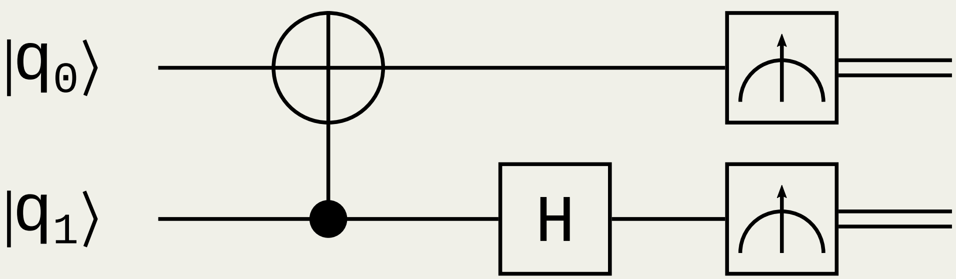


$$
\begin{aligned}
&\rightarrow |\Phi^+\rangle \\
&\rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
\textit{CNOT}_{1,0} &\rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
H_1 &\rightarrow \frac{1}{2}(|00\rangle + |10\rangle + |00\rangle - |10\rangle) \\
&\rightarrow \frac{1}{2}(2\,|00\rangle) \\
&\rightarrow |00\rangle
\end{aligned}
$$

- Decoding circuit is the reverse of the preparation circuit!

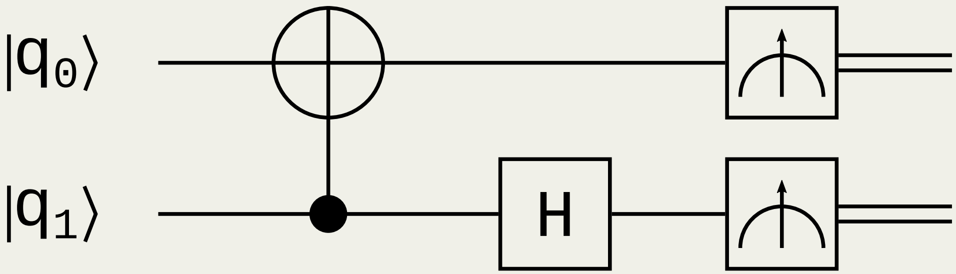
Bell states

- Can we “decode” the parity and phase?
  - Map parity and phase variations to separate computational basis states



$$
\begin{aligned}
& \rightarrow \left|\Phi^{-}\right\rangle \\
& \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
\mathit{CNOT}_{1,\,0} & \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\
H_1 & \rightarrow \frac{1}{2}(|00\rangle + |10\rangle - |00\rangle + |10\rangle) \\
& \rightarrow \frac{1}{2}(2\,|10\rangle) \\
& \rightarrow |10\rangle
\end{aligned}
$$

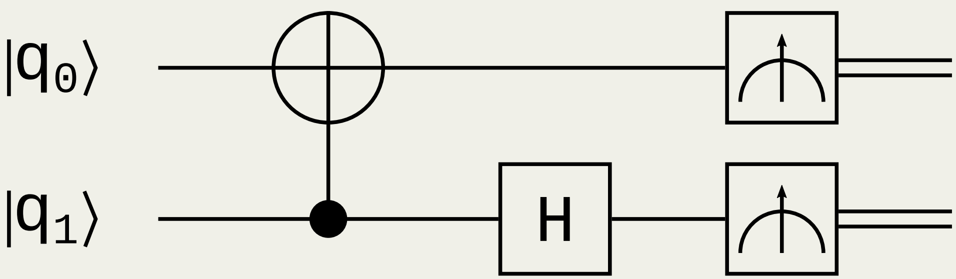
- Can we “decode” the parity and phase?
  - Map parity and phase variations to separate computational basis states



$$
\begin{aligned}
&\rightarrow |\Psi^{+}\rangle \\
&\rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
\mathit{CNOT}_{1,\,0} &\rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \\
H_1 &\rightarrow \frac{1}{2}(|01\rangle + |11\rangle + |01\rangle - |11\rangle) \\
&\rightarrow \frac{1}{2}(2\,|01\rangle) \\
&\rightarrow |01\rangle
\end{aligned}
$$

Bell states

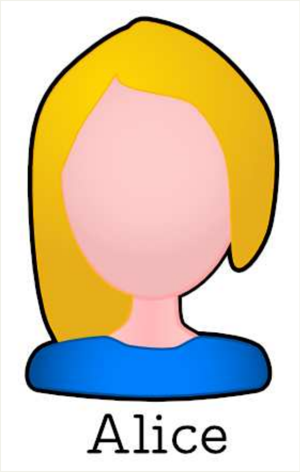
- Can we “decode” the parity and phase?
  - Map parity and phase variations to separate computational basis states



$$
\begin{aligned}
& \rightarrow |\Psi^-\rangle \\
& \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
\mathit{CNOT}_{1,0} & \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \\
H_1 & \rightarrow \frac{1}{2}(|01\rangle + |11\rangle - |01\rangle + |11\rangle) \\
& \rightarrow \frac{1}{2}(2\,|11\rangle) \\
& \rightarrow |11\rangle
\end{aligned}
$$

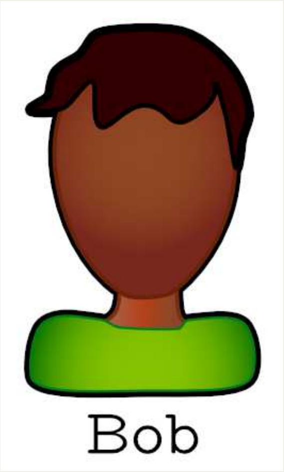
Superdense coding

- Communication protocol for classic bits:



1) Pre–share information

2) One bit communication



- Can we do better using qubits?
  - Communication protocol for classic bits using qubits
  - Pre–share quantum information
  - One qubit communication

Superdense coding

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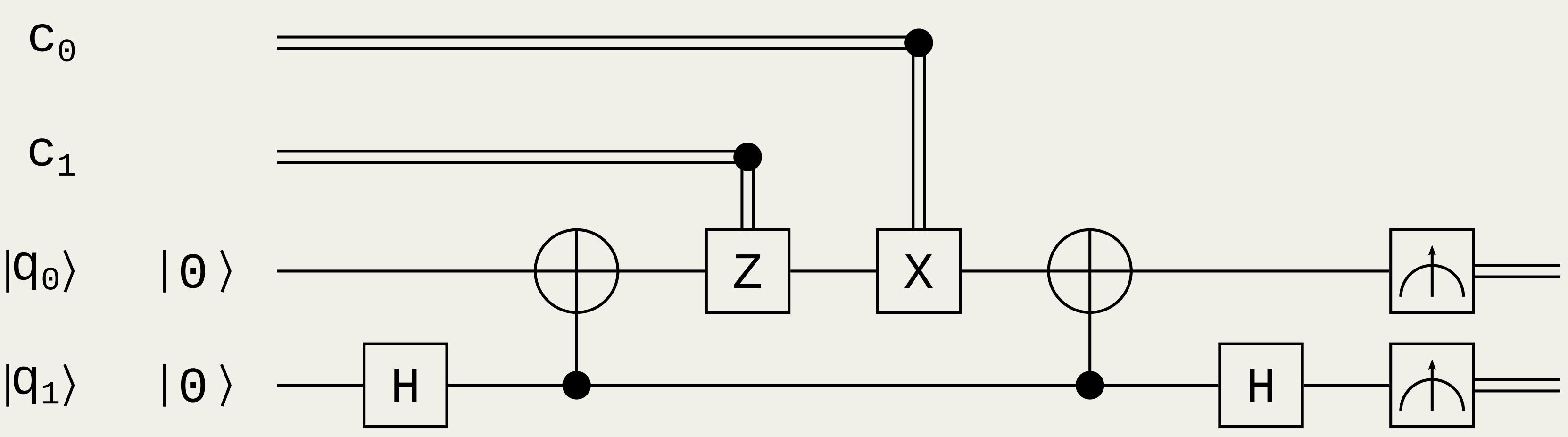
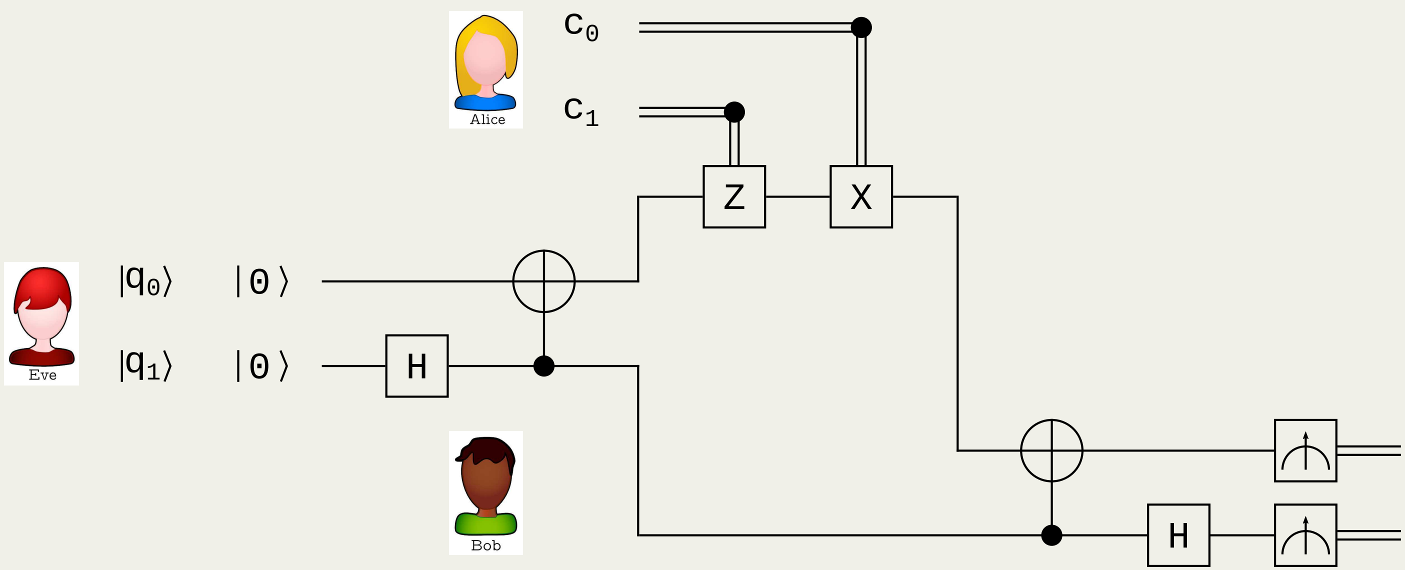
- Transmit two bits by sending one qubit



Superdense coding



$$\rightarrow |00\rangle$$

$$H_1 \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\mathit{CNOT}_{1,0} \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Assume Alice chooses 10

$$Z_1 \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Bob receives the message

$$\mathit{CNOT}_{1,0} \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

$$H_1 \rightarrow \frac{1}{2}(|00\rangle + |10\rangle - |00\rangle + |10\rangle)$$

$$\rightarrow \frac{1}{2}(2\,|10\rangle)$$

$$\rightarrow |10\rangle$$

Superdense coding

- Allows to transmit two classic bits of information by sending a single qubit
  - With the assumption that a Bell state was shared beforehand
- Often seen as the opposite of teleportation
  - Which requires sending two classic bits to teleport one quantum state
- Basis for secure quantum secret coding
  - Impossible to eavesdrop when the Bell state was shared in a secure way

Question # 10

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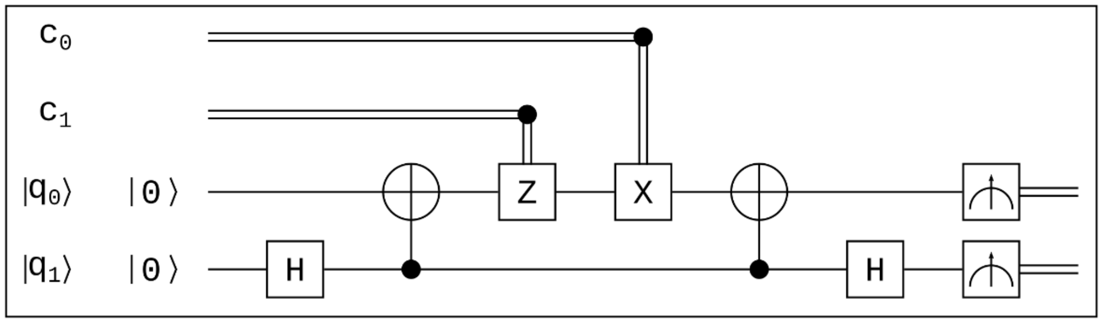
Question 10

During the lesson: "Basic Quantum Algorithms 2" I introduced the Superdense Coding algorithm. It allows you to transmit two bits by sending one qubit. Write a program that allows the user to enter the encoded message (0, 1, 2 or 3), implement the quantum circuit for the superdense coding (encoding the message provided by the user) and transmit it. Perform the measurement and show the message encoded by the user.



What is next?

- Assignment is already available (check DLO)
- ESK Wiskunde lectures and workshops
- Q & A, Discussion
  - Questions about the assignment or the material
  - Feel free to propose any discussion
  - You can always write me an email


Questions?