Amsterdam University of Applied Sciences

Quantum Computing Introduction

Fundamentals: A Multi–Qubit World (2)

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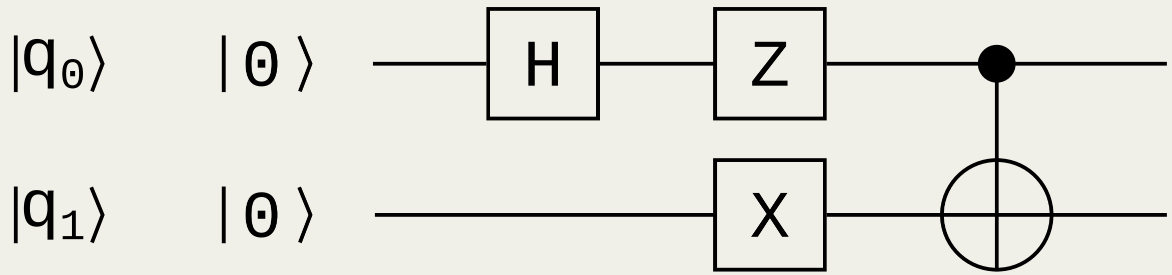
Useful transformations

- Cheat table with resulting transformations:

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$
$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$CNOT_{1,0}|00\rangle = |00\rangle$$
$$CNOT_{1,0}|01\rangle = |01\rangle$$
$$CNOT_{1,0}|10\rangle = |11\rangle$$
$$CNOT_{1,0}|11\rangle = |10\rangle$$
$$CZ_{0,1}|00\rangle = |00\rangle$$
$$CZ_{0,1}|01\rangle = |01\rangle$$
$$CZ_{0,1}|10\rangle = |10\rangle$$
$$CZ_{0,1}|11\rangle = -|11\rangle$$

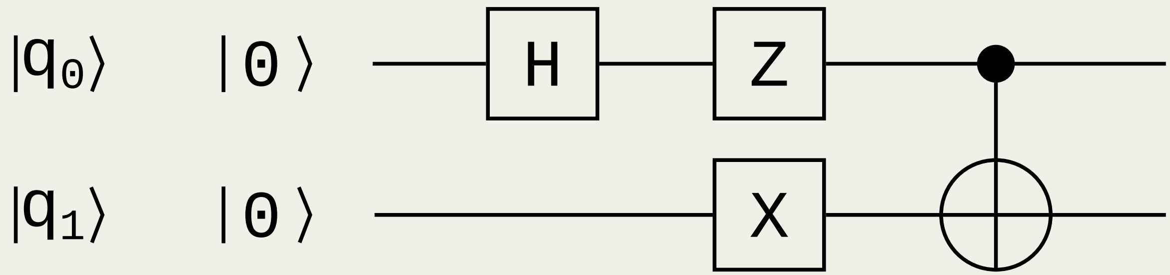
Example 1: Dirac



$$\mathit{CNOT}_{0,\,1} Z_0 X_1 H_0 \left|00\right\rangle$$

$$\begin{aligned}
\left|00\right\rangle &= \left|0\right\rangle \otimes \left|0\right\rangle \\
H_0 \rightarrow \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|01\right\rangle\right) &= \left|0\right\rangle \otimes \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) \\
&= \left|0\right\rangle \otimes \left|+\right\rangle \\
X_1 \rightarrow \frac{1}{\sqrt{2}}\left(\left|10\right\rangle + \left|11\right\rangle\right) &= \left|1\right\rangle \otimes \left|+\right\rangle \\
Z_0 \rightarrow \frac{1}{\sqrt{2}}\left(\left|10\right\rangle - \left|11\right\rangle\right) &= \left|1\right\rangle \otimes \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right) \\
&= \left|1\right\rangle \otimes \left|-\right\rangle \\
\mathit{CNOT}_{0,\,1} \rightarrow \frac{1}{\sqrt{2}}\left(\left|10\right\rangle - \left|01\right\rangle\right) &
\end{aligned}$$

Example 1: Matrix–vector



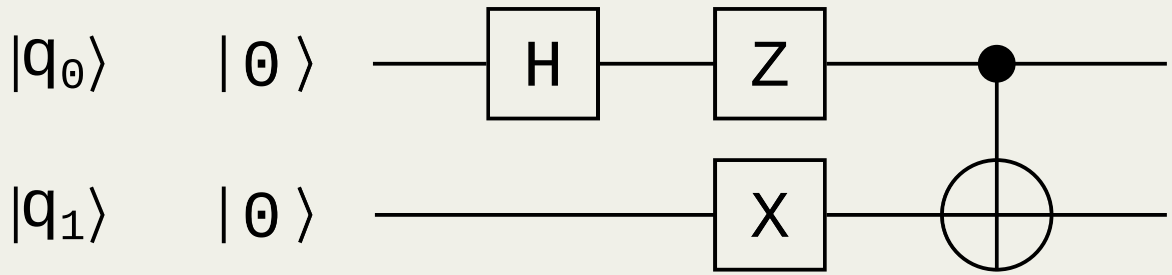
$$\mathit{CNOT}_{0,1} Z_0 X_1 H_0 \, |00\rangle$$

$$(I_1 \otimes H_0)\,|00\rangle \rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Example 1: Matrix–vector



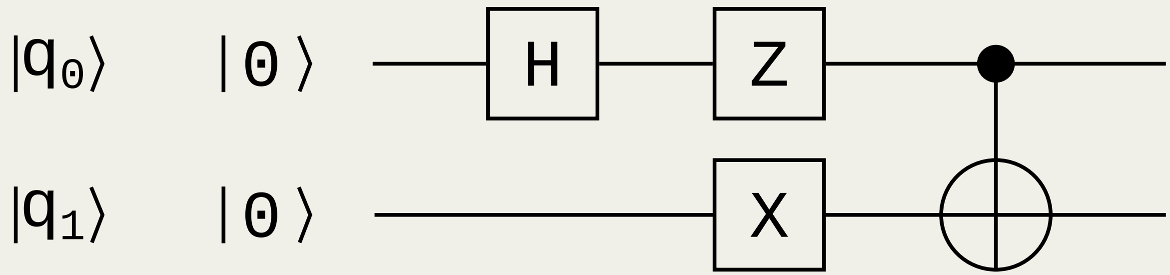
$$\mathit{CNOT}_{0,1} Z_0 X_1 H_0 \left|00\right\rangle$$

$$(X_1 \otimes I_0) H_0 \left|00\right\rangle \rightarrow \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Example 1: Matrix–vector



$$\mathit{CNOT}_{0,\,1} Z_0 X_1 H_0 \left|00\right\rangle$$

$$(I_1 \otimes Z_0) X_1 H_0 \left|00\right\rangle \rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Example 1: Matrix–vector

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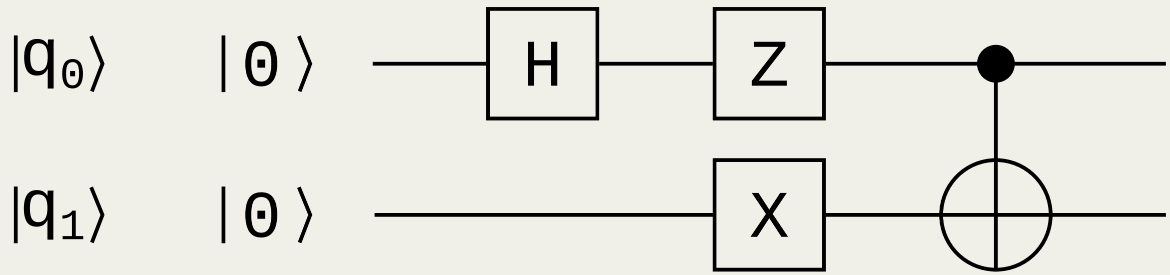


$$CNOT_{0,1} Z_0 X_1 H_0 \left|00\right\rangle$$

$$CNOT_{0,1} Z_0 X_1 H_0 \left|00\right\rangle \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

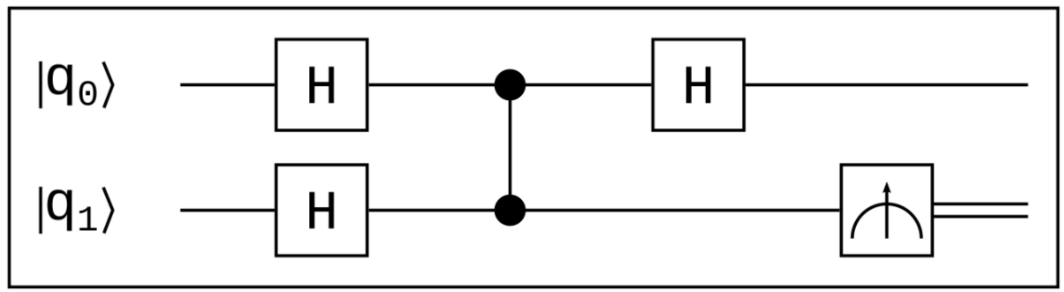
Question # 2

Question 2

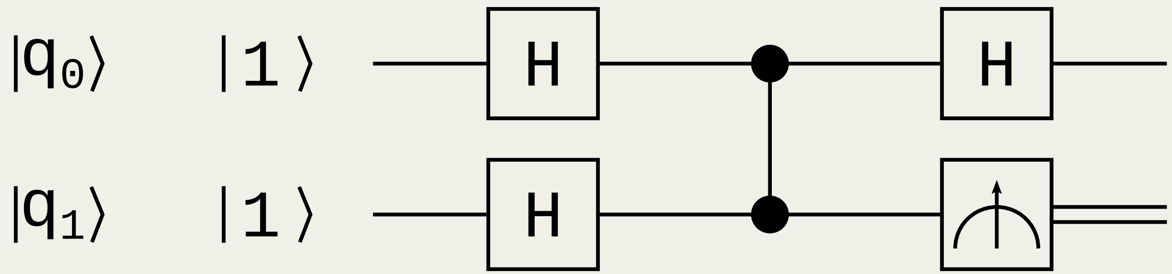
Preparing for the upcoming assessment, one of your classmates identified a quantum circuit supposed to prepare one of the Bell states (specifically $|\Phi^+\rangle$). As usual, you are not completely sure about its validity, hence your are going to implement such circuit in Qiskit to figure it out. Assume $|q_1 q_0\rangle = |00\rangle$.

Perform the following steps:

1. Implement the following circuit.
2. Execute the simulator.
3. Print the resulting vector state.
4. Does the proposed circuit prepare the corresponding Bell state?



Example 2: Dirac



$$H_0 \mathit{CZ}_{0,\,1} H_1 H_0 \left|11\right\rangle$$

$$\begin{aligned}
\left|11\right\rangle &= \left|1\right\rangle \otimes \left|1\right\rangle \\
H_0 \rightarrow \frac{1}{\sqrt{2}}\left(\left|10\right\rangle - \left|11\right\rangle\right) &= \left|1\right\rangle \otimes \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right) \\
&= \left|1\right\rangle \otimes \left|-\right\rangle \\
H_1 \rightarrow \frac{1}{2}\left(\left|00\right\rangle - \left|10\right\rangle - \left|01\right\rangle + \left|11\right\rangle\right) &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right) \otimes \left|-\right\rangle \\
&= \left|-\right\rangle \otimes \left|-\right\rangle \\
\mathit{CZ}_{0,\,1} \rightarrow \frac{1}{2}\left(\left|00\right\rangle - \left|10\right\rangle - \left|01\right\rangle - \left|11\right\rangle\right) &
\end{aligned}$$

Example 2: Dirac

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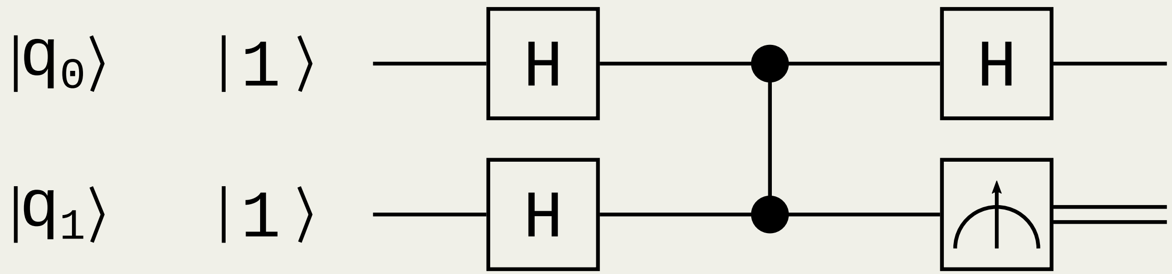
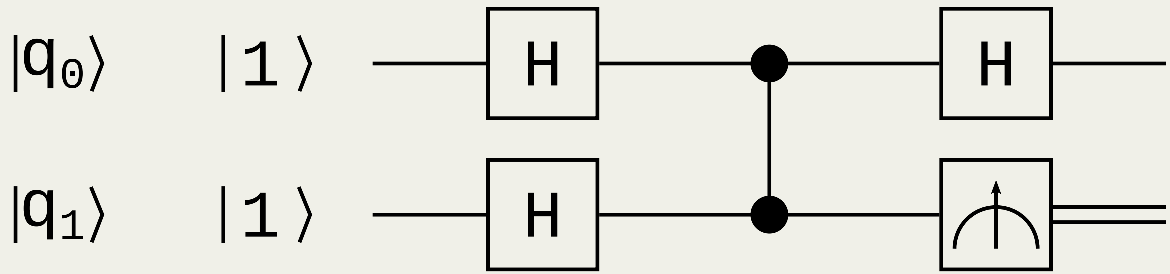


$$H_0 \, CZ_{0,1} \, H_1 H_0 \left|11\right\rangle$$

$$\begin{aligned}
H_0 \rightarrow \frac{1}{2}\left(\frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|01\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|10\right\rangle + \left|11\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|01\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|10\right\rangle - \left|11\right\rangle\right)\right) \\
\frac{1}{2\sqrt{2}}\left(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle - \left|11\right\rangle - \left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle + \left|11\right\rangle\right) \\
\frac{1}{2\sqrt{2}}\left(2\left|01\right\rangle - 2\left|10\right\rangle\right) \\
\frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right) \\
\left|\Psi^-\right\rangle
\end{aligned}$$

Example 2: Matrix–vector



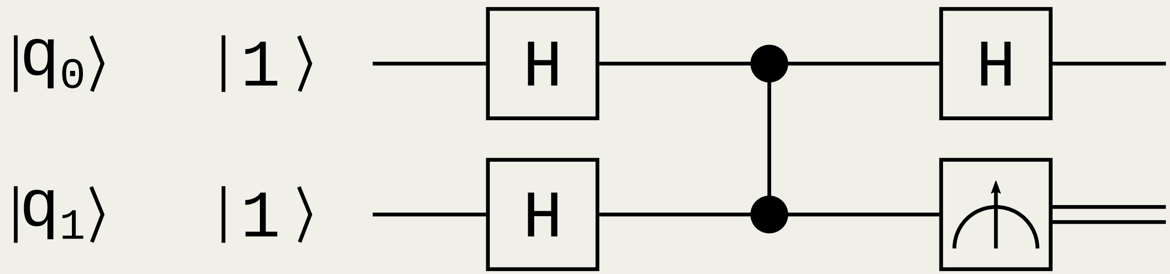
$$H_0 \mathit{CZ}_{0,1} H_1 H_0 \,|11\rangle$$

$$(I_1 \otimes H_0)\,|11\rangle \rightarrow \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\right)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\1 & -1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

Example 2: Matrix–vector



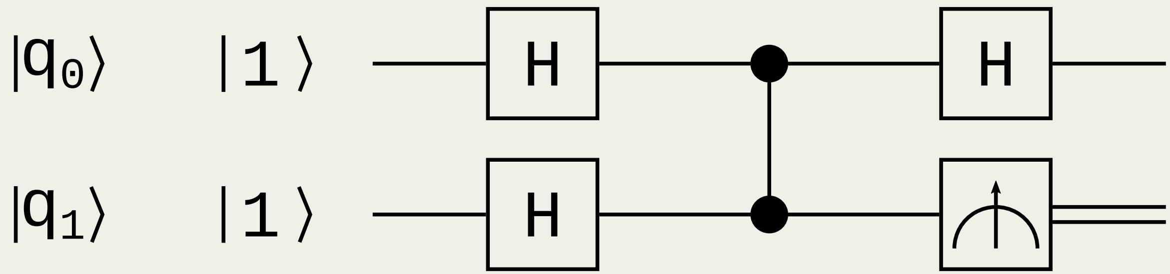
$$H_0 \mathit{CZ}_{0,1} H_1 H_0 \left|11\right\rangle$$

$$(H_1 \otimes I_0) H_0 \left|11\right\rangle \rightarrow \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Example 2: Matrix–vector

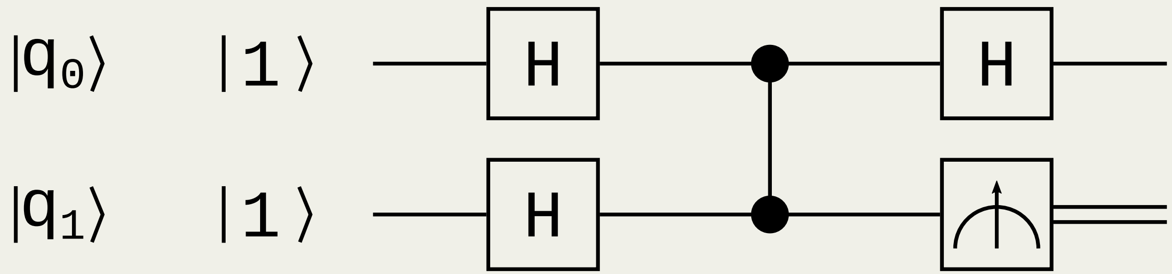


$$H_0 \mathit{CZ}_{0,1} H_1 H_0 |11\rangle$$

$$\mathit{CZ}_{0,1} H_1 H_0 |11\rangle \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\rightarrow \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Example 2: Matrix–vector



$$H_0 CZ_{0,1} H_1 H_0 \left|11\right\rangle$$

$$(I_1 \otimes H_0) CZ_{0,1} H_1 H_0 \left|11\right\rangle \rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Greenberger–Horne–Zeilinger (GHZ)

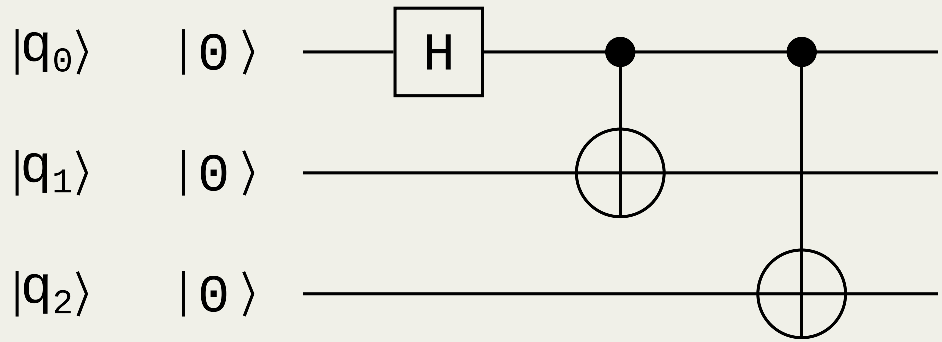
- A maximally entangled quantum state of $M > 2$ subsystems
- A quantum superposition of all subsystems being in state $|0\rangle$ with all of them being in state $|1\rangle$

$$|GHZ\rangle = \frac{|0\rangle^{\otimes M} + |1\rangle^{\otimes M}}{\sqrt{2}}$$

- For instance, if $M = 3$:

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

Greenberger–Horne–Zeilinger (GHZ)



$$
\begin{aligned}
&\rightarrow |000\rangle \\
H_0 &\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle + |001\rangle\right) \\
\mathit{CNOT}_{0,\,1} &\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) \\
\mathit{CNOT}_{0,\,2} &\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)
\end{aligned}
$$

Question # 7

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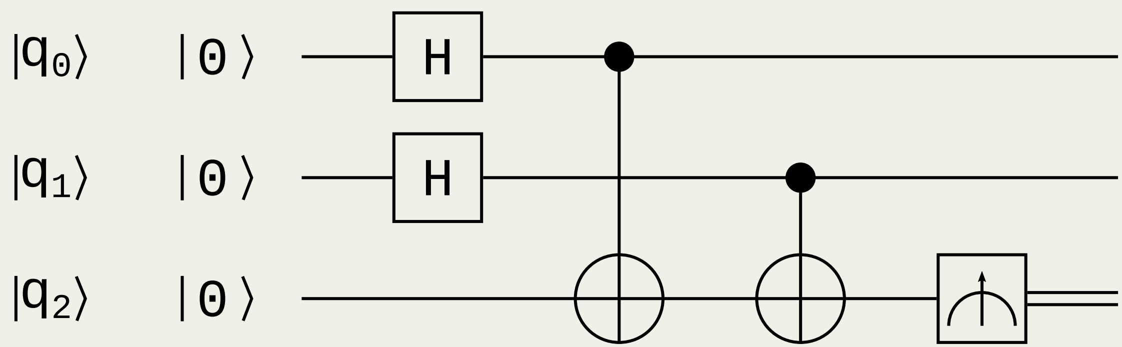
Question 7

During the lecture: "Fundamentals: A Multi-Qubit World (2)", I introduced the Greenberger-Horne-Zeilinger (GHZ) state for a 3-qubit quantum system. Perform the following steps:

1. Assemble a circuit that creates a 5-qubit GHZ state.
2. Draw the circuit.
3. Show the resulting state vector.

Parity: calculation



$$
\begin{aligned}
&\rightarrow |000\rangle \\
H_0 &\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle + |001\rangle\right) \\
H_1 &\rightarrow \frac{1}{2}\left(|000\rangle + |010\rangle + |001\rangle + |011\rangle\right) \\
\mathit{CNOT}_{0,\,2} &\rightarrow \frac{1}{2}\left(|000\rangle + |010\rangle + |101\rangle + |111\rangle\right) \\
\mathit{CNOT}_{1,\,2} &\rightarrow \frac{1}{2}\left(|000\rangle + |110\rangle + |101\rangle + |011\rangle\right)
\end{aligned}
$$

Parity: measurement

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- What is the state of q_0 and q_1 after measuring q_2?

$$\frac{1}{2}\left(|000\rangle + |110\rangle + |101\rangle + |011\rangle\right)$$

$$\frac{1}{2}\left(|000\rangle + |011\rangle + |110\rangle + |101\rangle\right)$$

$$\frac{1}{2}\left(\left(|0\rangle \otimes \left(|00\rangle + |11\rangle\right)\right) + \left(|1\rangle \otimes \left(|10\rangle + |01\rangle\right)\right)\right)$$

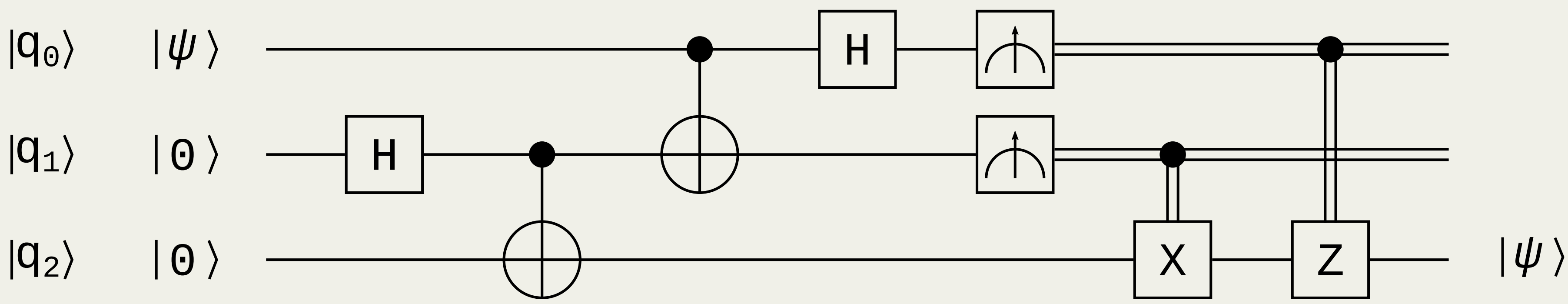
$$\mathbf{Pr}\left\{|0\rangle\right\} = \mathbf{Pr}\left\{|1\rangle\right\} = 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

$M(q_2) = 0 \rightarrow |0\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ Zero parity states are left

$M(q_2) = 1 \rightarrow |1\rangle \otimes \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ One parity states are left

State collapsed to the given parity measurement!

Teleportation



- Transmit the state of a qubit over any given distance:
  - In a completely secure way
- Enables secure communication
- Treats entangled states as a *resource*:
  - Use them to perform tasks that cannot be accomplished by classical means

Teleportation

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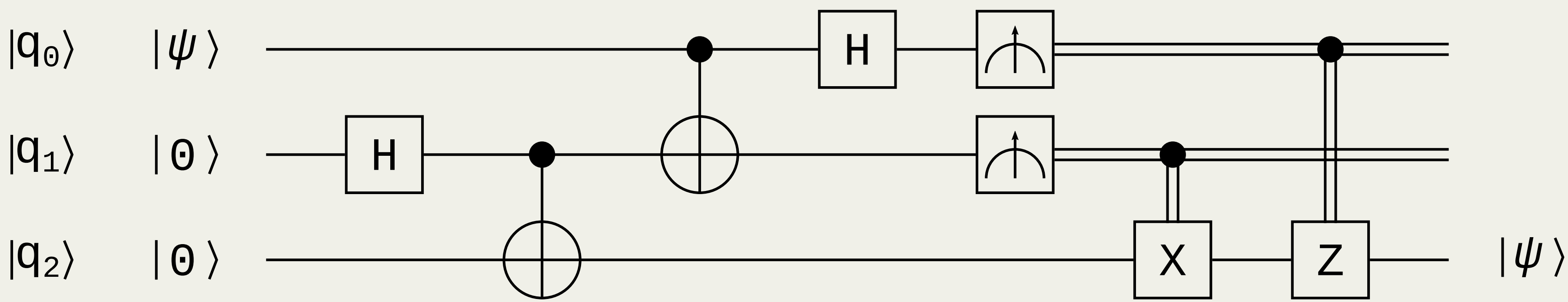
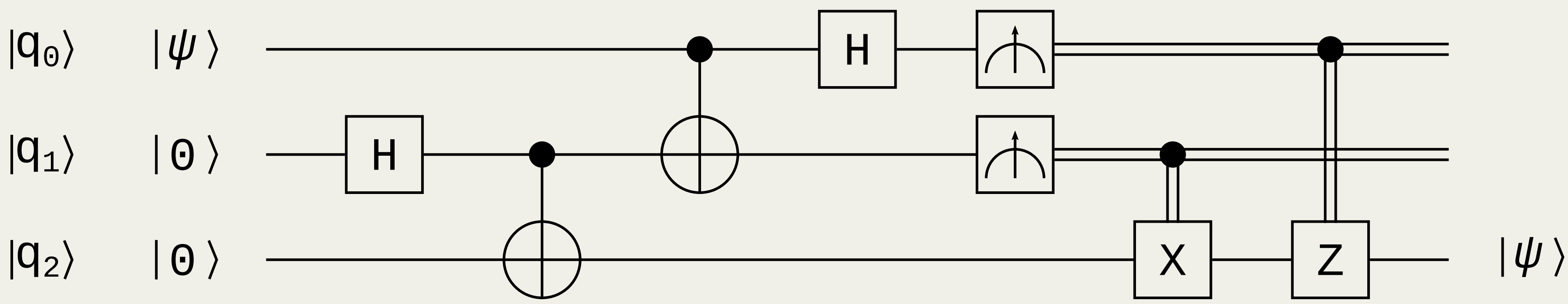


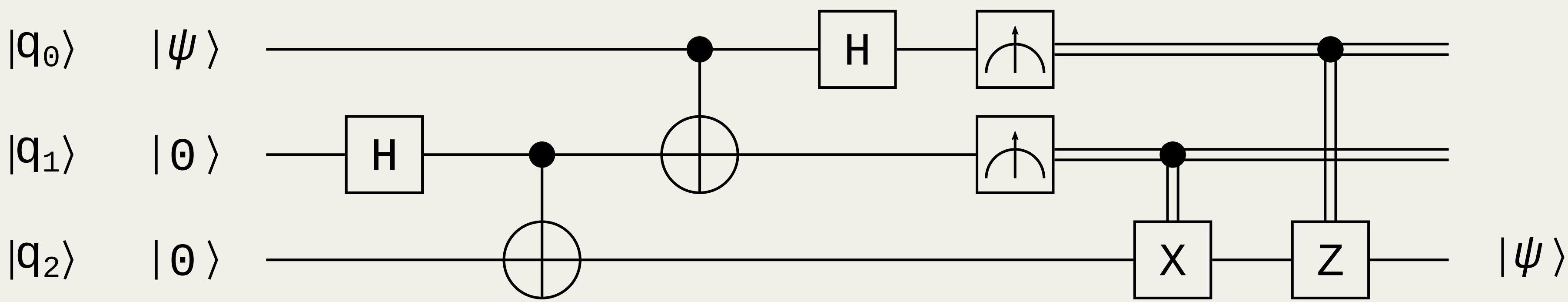
1. Prepare state to be teleported
2. Add two qubits
3. Prepare Bell state
4. Entangle state to Bell pair
5. Measure
6. Correct collapsing “errors”

Teleportation



$$
\begin{aligned}
& \rightarrow |00\psi\rangle \\
& \rightarrow |00\rangle \otimes |\psi\rangle \\
& \rightarrow |00\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle) \\
& \rightarrow \alpha\,|000\rangle + \beta\,|001\rangle \\
H_1 & \rightarrow \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|010\rangle + \beta\,|001\rangle + \beta\,|011\rangle) \\
\mathit{CNOT}_{1,\,2} & \rightarrow \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|110\rangle + \beta\,|001\rangle + \beta\,|111\rangle) \\
\mathit{CNOT}_{0,\,1} & \rightarrow \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|110\rangle + \beta\,|011\rangle + \beta\,|101\rangle)
\end{aligned}
$$

Teleportation



$$
\begin{aligned}
&\rightarrow \frac{1}{\sqrt{2}}\left(\alpha\left|000\right\rangle + \alpha\left|110\right\rangle + \beta\left|011\right\rangle + \beta\left|101\right\rangle\right) \\
H_0 &\rightarrow \frac{1}{2}\left(\alpha\left|000\right\rangle + \alpha\left|001\right\rangle + \alpha\left|110\right\rangle + \alpha\left|111\right\rangle + \beta\left|010\right\rangle - \beta\left|011\right\rangle + \beta\left|100\right\rangle - \beta\left|101\right\rangle\right) \\
&\rightarrow \frac{1}{2}\left(\alpha\left|000\right\rangle + \beta\left|100\right\rangle + \alpha\left|001\right\rangle - \beta\left|101\right\rangle + \alpha\left|110\right\rangle + \beta\left|010\right\rangle + \alpha\left|111\right\rangle - \beta\left|011\right\rangle\right) \\
&\rightarrow \frac{1}{2}\left(\left(\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right)\otimes\left|00\right\rangle\right) + \left(\left(\alpha\left|0\right\rangle - \beta\left|1\right\rangle\right)\otimes\left|01\right\rangle\right) + \left(\left(\alpha\left|1\right\rangle + \beta\left|0\right\rangle\right)\otimes\left|10\right\rangle\right)\right. \\
&+ \left.\left(\left(\alpha\left|1\right\rangle - \beta\left|0\right\rangle\right)\otimes\left|11\right\rangle\right)\right)
\end{aligned}
$$

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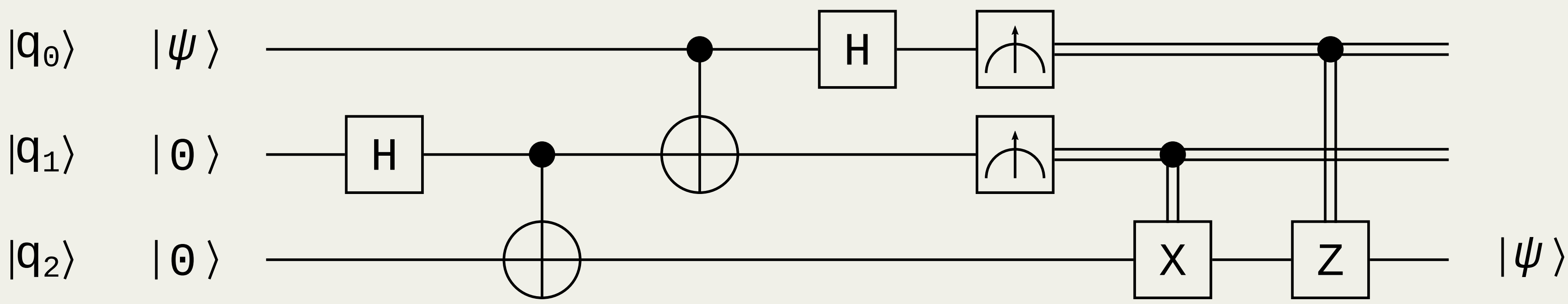

- What about the outcomes when we measure?

$$\rightarrow \frac{1}{2}\left(\left((\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle\right) + \left((\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle\right) + \left((\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle\right) + \left((\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\right)\right)$$

If $M(q_1) = 0 \wedge M(q_0) = 0 \rightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle$ No corrections required

If $M(q_1) = 0 \wedge M(q_0) = 1 \rightarrow (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle$ Z_2 correction

If $M(q_1) = 1 \wedge M(q_0) = 0 \rightarrow (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle$ X_2 correction

If $M(q_1) = 1 \wedge M(q_0) = 1 \rightarrow (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle$ Z_2X_2 correction

Question # 10

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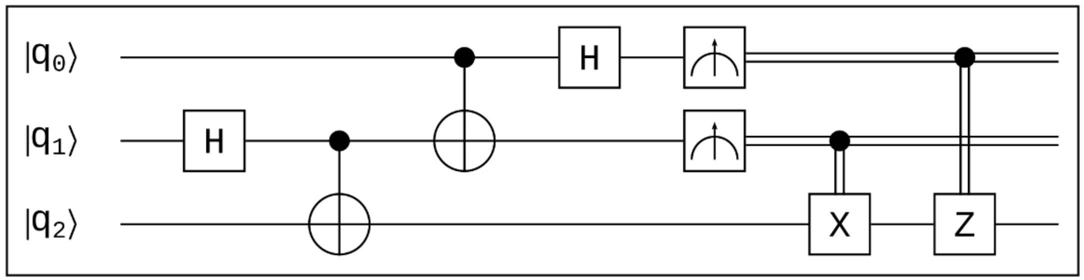
Question 10

During the lecture: "Fundamentals: A Multi-Qubit World (2)", I introduced the Quantum Teleportation algorithm.

1. Implement, simulate and draw the teleportation quantum circuit.
2. Show the initial and the resulting vector state (you can use the Bloch sphere).
3. Explain how do you recognize that the algorithm works.



What is next?

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- Assignment is already available (check DLO)
- ESK Wiskunde lectures and workshops
- Q & A, Discussion
  - Questions about the assignment or the material
  - Feel free to propose any discussion
  - You can always write me an email


Questions?