Quantum Computing Introduction

Fundamentals: A One–Qubit World

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Qubit

- **Qu**antum **Bit**
- System with 2 distinguishable states: $|0\rangle$ and $|1\rangle$
- Represented as a vector $|\psi\rangle$ in a 2–D space
  - $\rightarrow$ over the complex numbers
  - Basis: $|0\rangle$ and $|1\rangle$
- Indeterminate state (before measurement)
- A qubit can be in superposition:
  - 2 states at the same time
  - Linear combination of 2 states

Qubit

- Arbitrary quantum state $|\psi\rangle$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \rightarrow \text{Superposition}$$

  - α and β are complex numbers ($\alpha,\ \beta \in \mathbb{C}$)
  - α and β are called probability amplitudes

- A "ket" represents a column vector:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Quantum state now rewritten as a state vector:

$$\begin{aligned} |\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\ &= \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \end{aligned}$$

The Born rule

- The modulus squared of the amplitude of a state is the probability of that state resulting after measurement
  - $\mathbf{Pr}\left\{M(|\psi\rangle) = |0\rangle\right\} = \overline{\alpha}\alpha = |\alpha|^2$
  - $\mathbf{Pr}\left\{M(|\psi\rangle) = |1\rangle\right\} = \overline{\beta}\beta = |\beta|^2$
- Furthermore, $|\psi\rangle$ have to be normalized:

$$\begin{aligned} \langle\psi|\psi\rangle &= 1 \\ |\alpha|^2 + |\beta|^2 &= 1 \\ \overline{\alpha}\alpha + \overline{\beta}\beta &= 1 \end{aligned}$$

Question # 1

Question 1 *Given the quantum state* $|\psi\rangle = \sqrt{\frac{1}{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle$. *Is* $|\psi\rangle$ *a valid quantum state? Explain why.*

`Write down your solution here:`

Superposition

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- Linear combination of computational basis
- For instance:

$$|\psi_1\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$
$$|\psi_2\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle$$

  - Negative sign on $|1\rangle$ state $\rightarrow$ *relative phase*
- Relative phase:
  - Constructive and destructive interference

$$\begin{aligned}
|\psi\rangle &= |\psi_1\rangle + |\psi_2\rangle \\
&= \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle + \frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle \\
&= \frac{5}{6}\left(\frac{3}{5}|0\rangle + \frac{3}{5}|0\rangle\right) \\
&= \frac{5}{6}\left(\frac{6}{5}|0\rangle\right) \\
&= |0\rangle
\end{aligned}$$

Global phase

- Negative sign applied to the entire state
    - Not only to one term in the superposition
- Negative sign on $|\psi\rangle \rightarrow$ *global phase*
- Global phase:
    - No physical meaning
    - No impact on quantum measurement
    - Do not affect the amplitude of a complex number

Uniform superposition

- Special state
- Equal probability of collapsing to any basis vector
- All the basis vectors have the same probability amplitude

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

- 50% probability of collapsing to either state $|0\rangle$ or $|1\rangle$

Measurement

- Classical physics $\rightarrow$ No effect on the system
- Quantum mechanics $\rightarrow$ Significant effect
- Every measurable quantity is described by a corresponding Hermitian operator O (observable) acting on the state $|\psi\rangle$
  - Each possible outcome of the measurement is an eigenvalue λ_i of the observable
  - The state after measurement is the eigenvector associated with λ_i

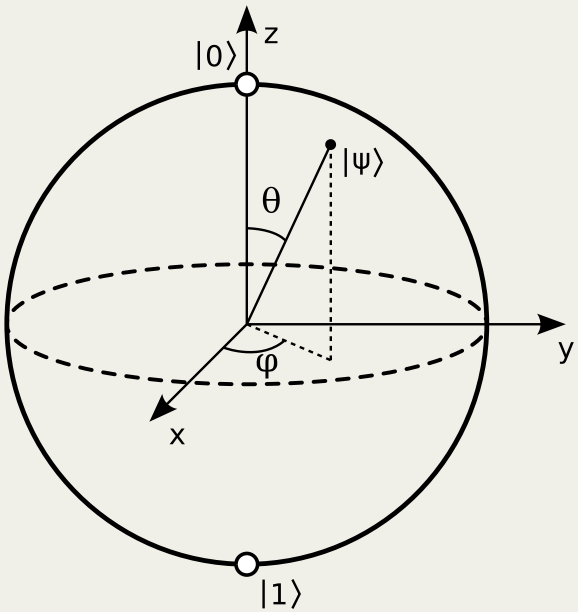
Measurement

- Possible to observe a qubit in superposition? NO!
- Upon measurement, it has to pick one state
- "Collapses", not arbitrarily, to $|0\rangle$ or $|1\rangle$
  - Probability amplitudes
- Reason for normalizing the state vector $|\psi\rangle$
  - Sum of probabilities must be 1
- Observation changes state!
- Measurement is irreversible

Bloch sphere

- Visual representation of a qubit:
- State vector $|\psi\rangle$ is represented by θ and ϕ:

$$|\psi\rangle = cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}sin\frac{\theta}{2}\,|1\rangle$$



  - Spherical coordinate system
  - θ and ϕ can describe every point on the sphere's surface

- 2–D complex vector space $\rightarrow$ Surface of a unit sphere in a 3–D space

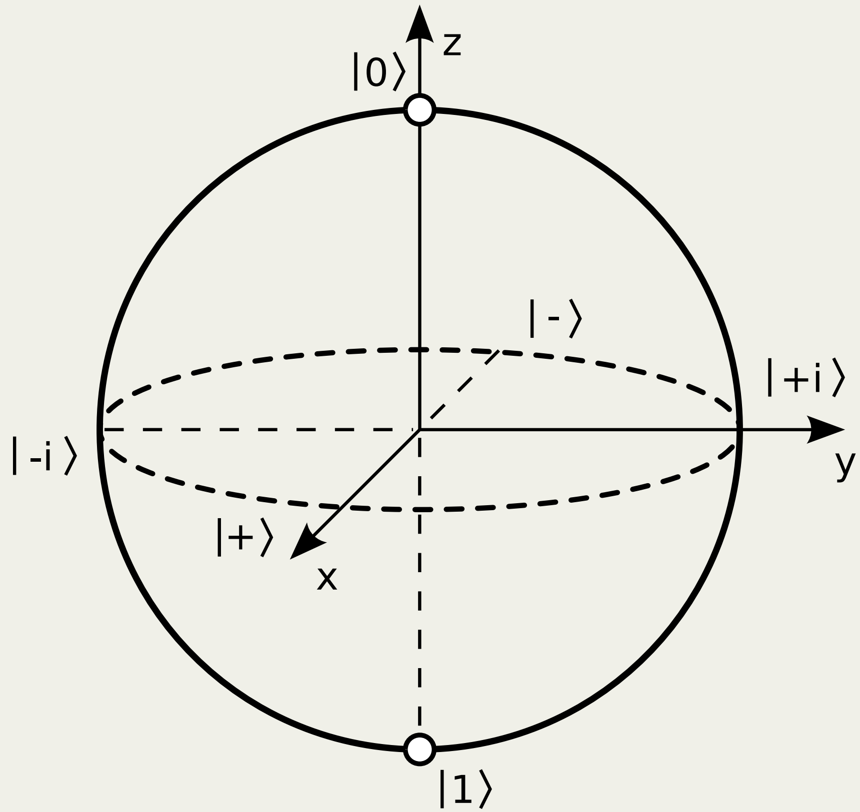
Bloch sphere

- Why $z = |0\rangle$?
  $\theta = 0$, $\phi = 0$

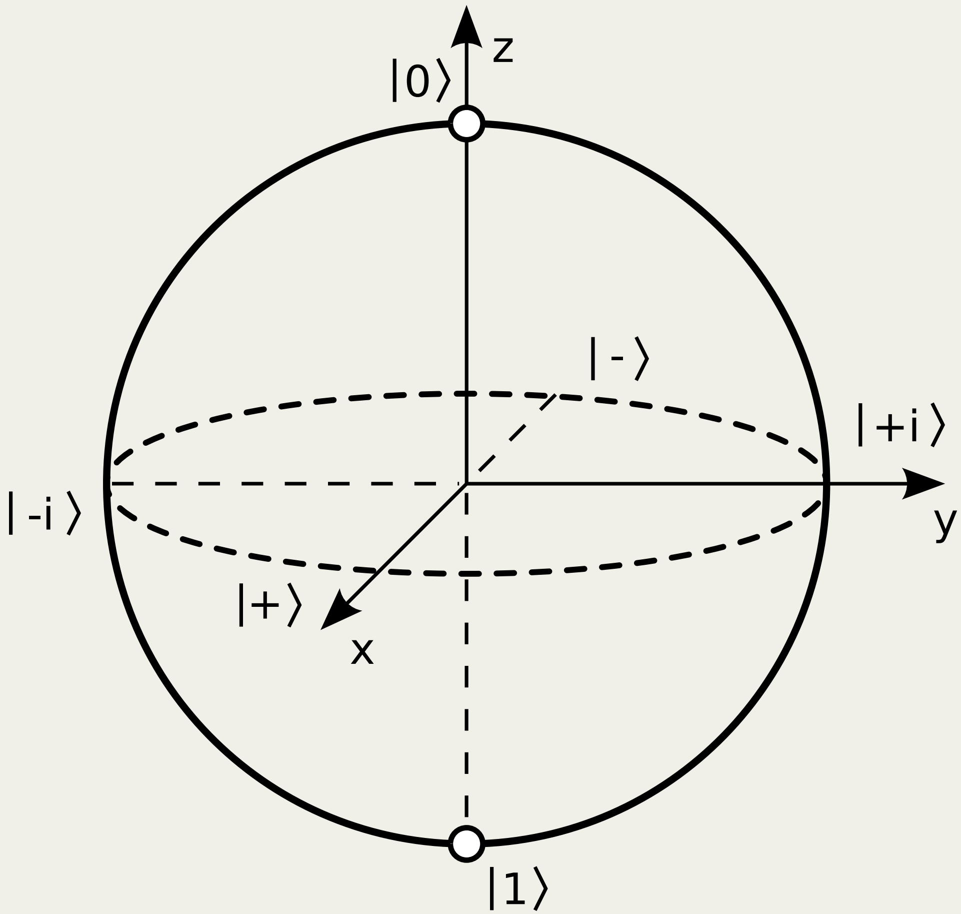
$$
\begin{aligned}
|\psi\rangle &= cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle \\
&= cos(0)|0\rangle + e^{0}sin(0)|1\rangle \\
&= |0\rangle
\end{aligned}
$$

- Why $-z = |1\rangle$?
  $\theta = \pi$, $\phi = 0$

$$
\begin{aligned}
|\psi\rangle &= cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle \\
&= cos\frac{\pi}{2}|0\rangle + e^{0}sin\frac{\pi}{2}|1\rangle \\
&= |1\rangle
\end{aligned}
$$



Bloch sphere



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

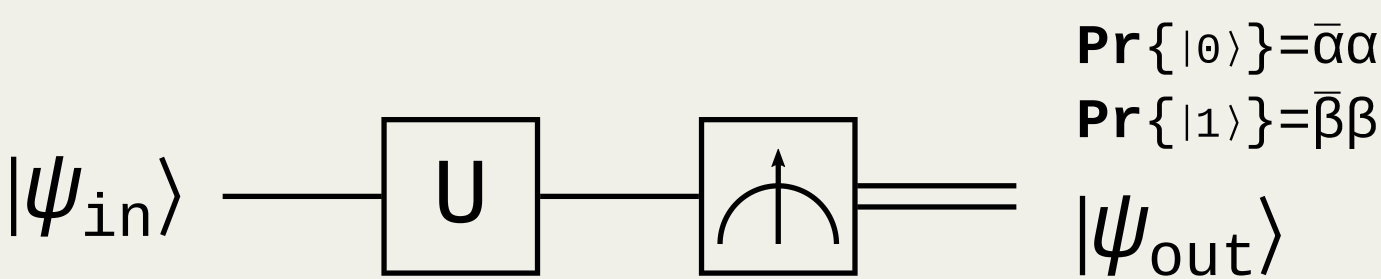
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|+i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Quantum Circuit Model

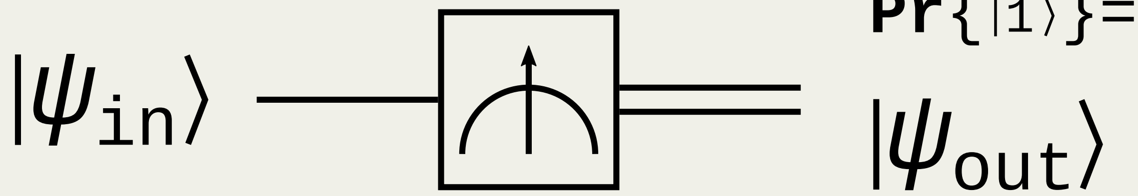
- Well–defined initial state (usually $|0\rangle$)
- State changes when passing through a “quantum gate”
  - Acts over one or two qubits
  - In a controlled manner
- Final state after the computation:
  - Trace what every quantum gate does to the qubit

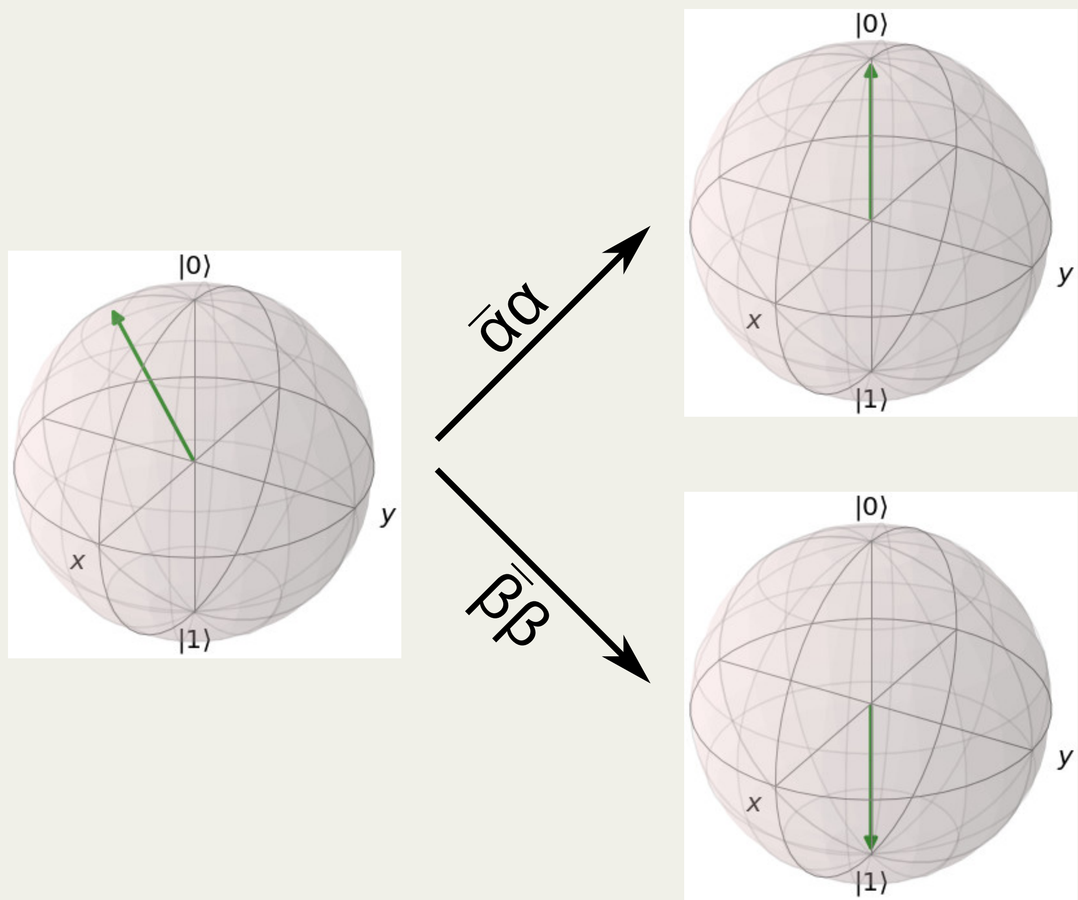


Measurement

$\mathbf{Pr}\{|0\rangle\} = \bar{\alpha}\alpha$

$\mathbf{Pr}\{|1\rangle\} = \bar{\beta}\beta$





Quantum gates

- Allow to perform operations on qubits
- Modeled by linear transformations:
  - Unitary transformations (reversible)
    - Its conjugate transpose ($U^\dagger$) is also its inverse: $UU^\dagger = U^\dagger U = I$
  - Preserve state normalization
- Counter–clockwise rotation around an axis of the Bloch sphere

Quantum gates

- Represented by circuit notation:

$$|\psi_{\mathtt{in}}\rangle \; \text{---}\boxed{\mathsf{U}}\text{---} \; |\psi_{\mathtt{out}}\rangle$$

- Or by unitary matrices:

$$U = \begin{bmatrix} u_{1,\,1} & u_{1,\,2} & \cdots & u_{1,\,n} \\ u_{2,\,1} & u_{2,\,2} & \cdots & u_{2,\,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n,\,1} & u_{n,\,2} & \cdots & u_{n,\,n} \end{bmatrix}$$

- The action of the gate on a quantum state is obtained by multiplying: $U\,|\psi\rangle$

NOT gate

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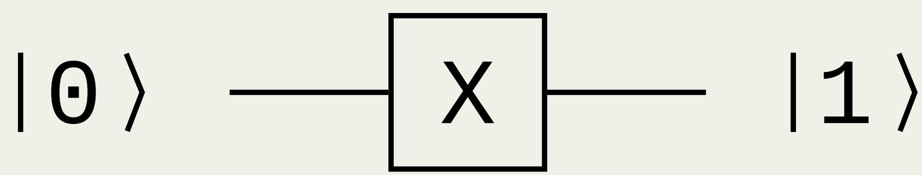
- Acts on a single qubit:
  - $|0\rangle \rightarrow |1\rangle$
  - $|1\rangle \rightarrow |0\rangle$
- Represented by the matrix $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Circuit notation:

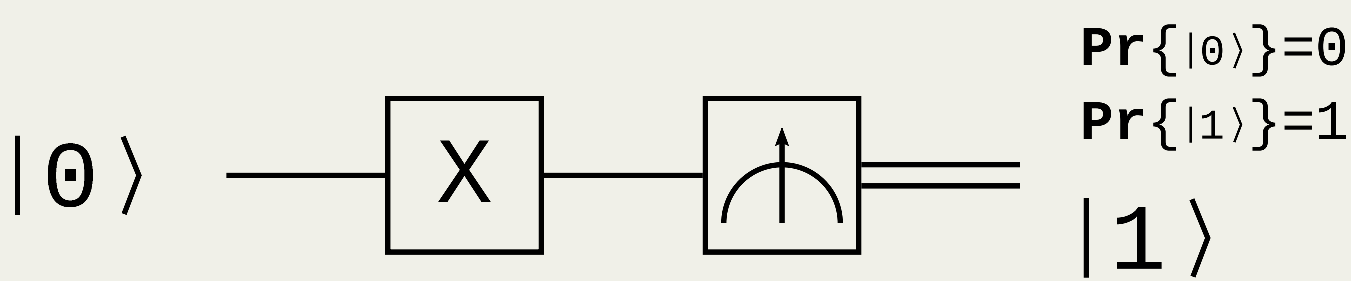


NOT gate

- Example:
  - Matrix notation:

$$
\begin{aligned}
|\psi_{out}\rangle &= X\,|0\rangle \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

  - Circuit notation:



Pauli gates

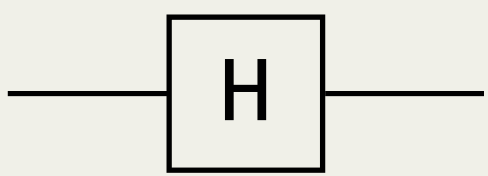
- The most simple quantum gates
- Rotation of π radians around the X, Y, or Z axis of the Bloch sphere
- Hence, they are called: Pauli–X, Pauli–Y and Pauli–Z
- The Pauli matrices are involutory:
  - Matrix that is its own inverse
  - $X^2 = Y^2 = Z^2 = I$

Pauli gates

Pauli–X	Pauli–Y	Pauli–Z
$\boxed{\mathrm{X}}$	$\boxed{\mathrm{Y}}$	$\boxed{\mathrm{Z}}$
$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Hadamard gate

- Creates superpositions from $|0\rangle$ and $|1\rangle$
- A measurement will have equal probabilities to return states $|1\rangle$ or $|0\rangle$
  - $|0\rangle \rightarrow |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
  - $|1\rangle \rightarrow |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- Represented by the matrix $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- H is also an unitary matrix
- Circuit notation:



Question # 3

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Question 3 *Assume that you can only use the quantum gates from the set* $\mathcal{Q} = \{I,\ X,\ Y,\ Z,\ H\}$. *Is it possible to create the quantum state* $|\psi_{out}\rangle = \frac{i}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$ *starting from the qubit* $|\psi_{in}\rangle = |0\rangle$. *Explain how?*

`Write down your solution here:`

Phase shift gate (R_ϕ)

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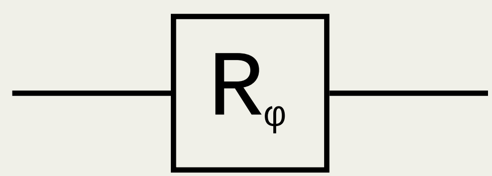
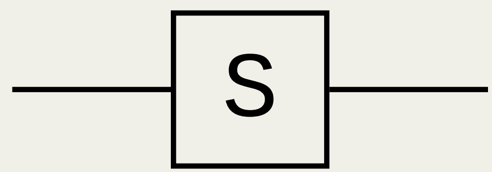
- Family of single–qubit gates
- Modifies the phase of the quantum state
  - $|0\rangle \rightarrow |0\rangle$
  - $|1\rangle \rightarrow e^{i\phi}|1\rangle$
- Represented by the matrix: $R_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$
- Circuit notation:



- Pauli–Z is a special case $\rightarrow \phi = \pi$

S gate

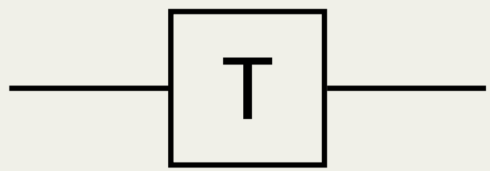
- Another special case of the R_ϕ gate:
  - $\phi = \frac{\pi}{2}$
- Represented by the matrix: $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- Circuit notation:



- The inverse of $S = S^\dagger$

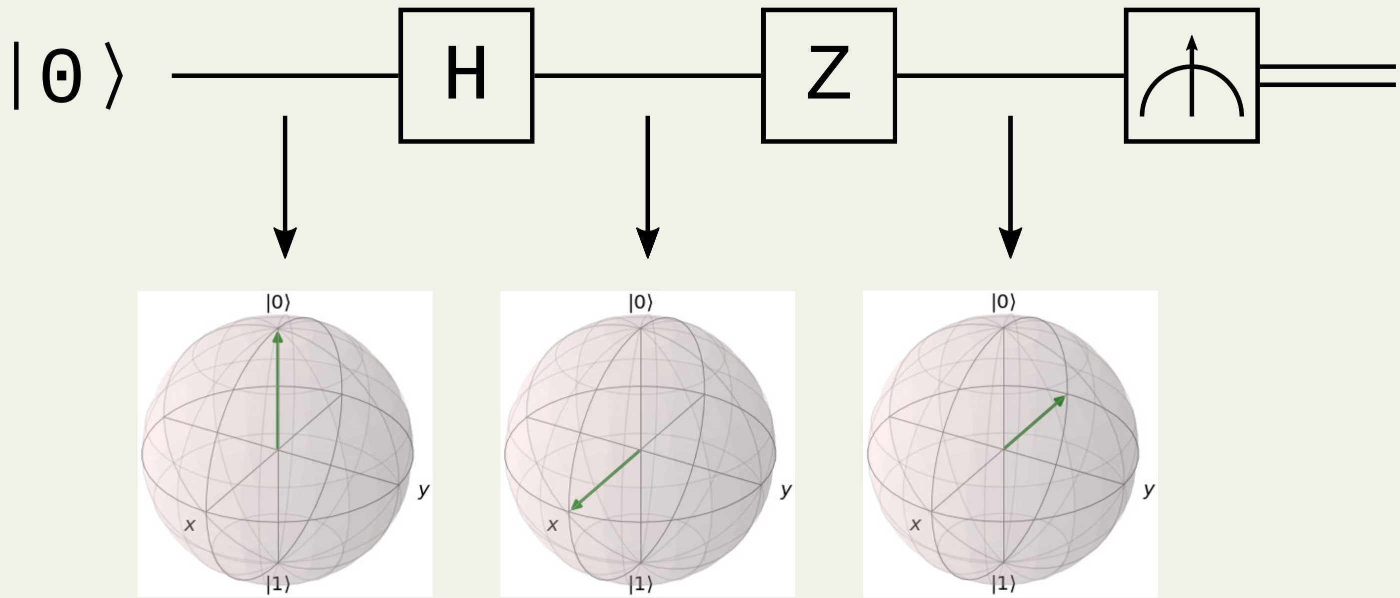
T gate

- Another special case of the R_ϕ gate:
  - $\phi = \frac{\pi}{4}$
- Represented by the matrix: $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$
- Circuit notation:



- Note: $S = T^2$

A simple circuit



Pr{|0⟩}=0.5

Pr{|1⟩}=0.5

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$
\begin{aligned}
&\rightarrow |0\rangle \\
H &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
Z &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}
$$

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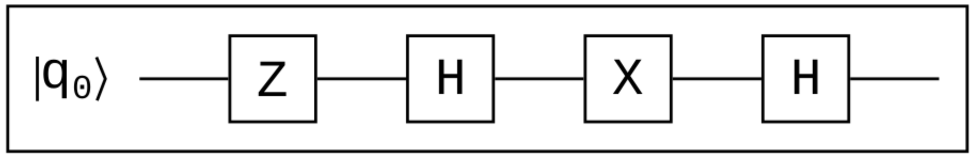

Figure 2: An arbitrary quantum circuit.

Question 9 *Consider the quantum circuit presented in Figure 2 and assume* $|q_0\rangle = |1\rangle$. *Determine, by using the Dirac notation, what is the state vector* $|\psi_{out}\rangle$*?*

`Write down your solution here:`

$$\rightarrow |1\rangle$$

What is next?

- Assignment is already available (check DLO)
- Bring your laptop for the next class
  - Please, with the software already installed
- ESK Wiskunde lectures and workshops
- Q & A, Discussion
  - Questions about the assignments or the material
  - Feel free to propose any discussion
  - You can always write me an email


Questions?