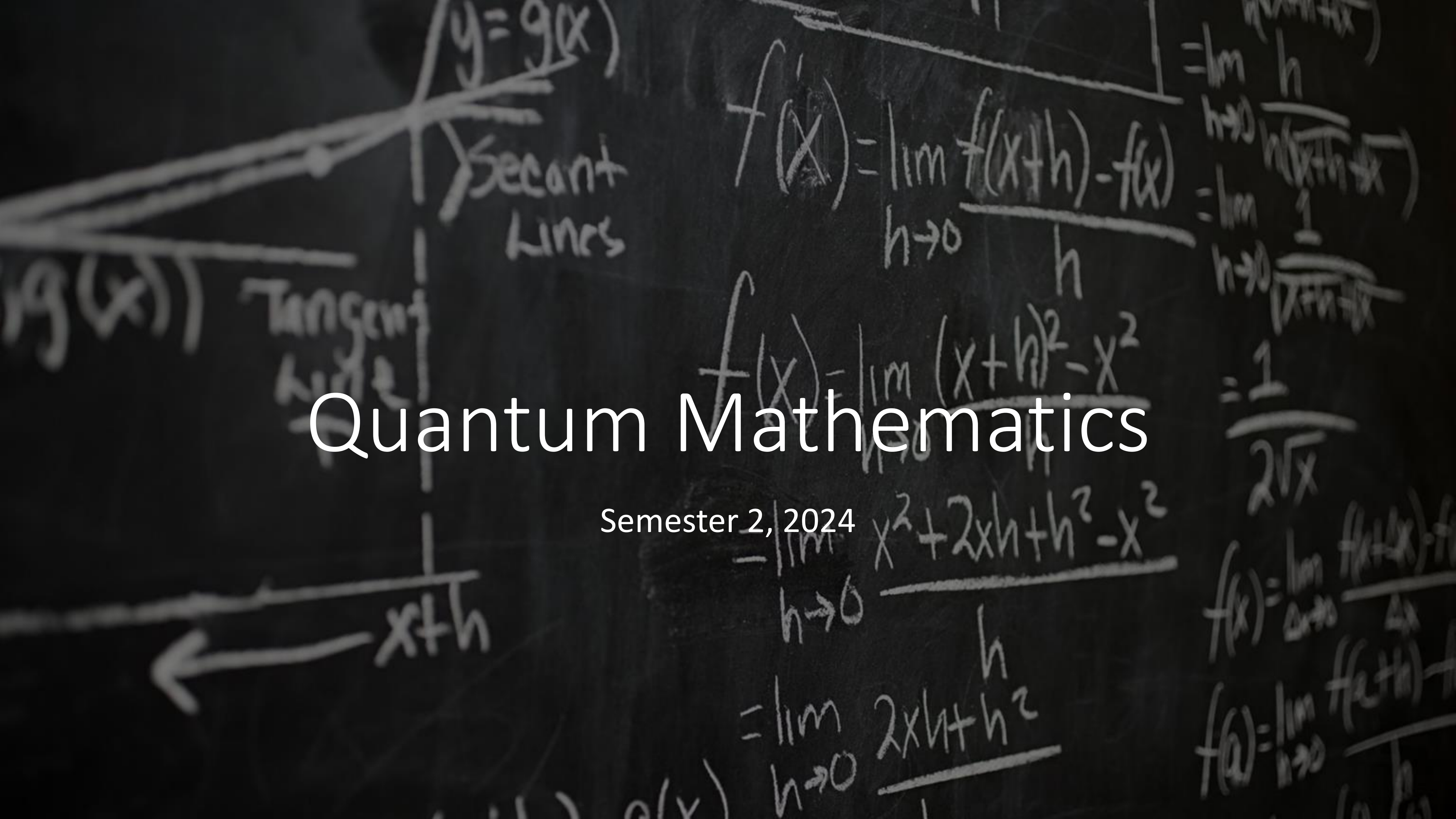
Quantum Mathematics

Semester 2, 2024



Mathematical tools

- Probability

- Complex vector space
  - Complex numbers

- All of quantum mechanics can be described by linear algebra
  - Vectors
  - Matrices
  - Lineair transformations

- Dirac notation

Vectors again

- What about the dot product: $\vec{a}^T \cdot \vec{b}$?
- Assuming $\vec{a} = \begin{pmatrix} i \\ 0 \end{pmatrix}$

$$\|\vec{a}\| = \sqrt{\vec{a}^T \cdot \vec{a}}$$

$$= \sqrt{\begin{pmatrix} i & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix}}$$

$$= \sqrt{i^2}, \notin \mathbb{R}$$

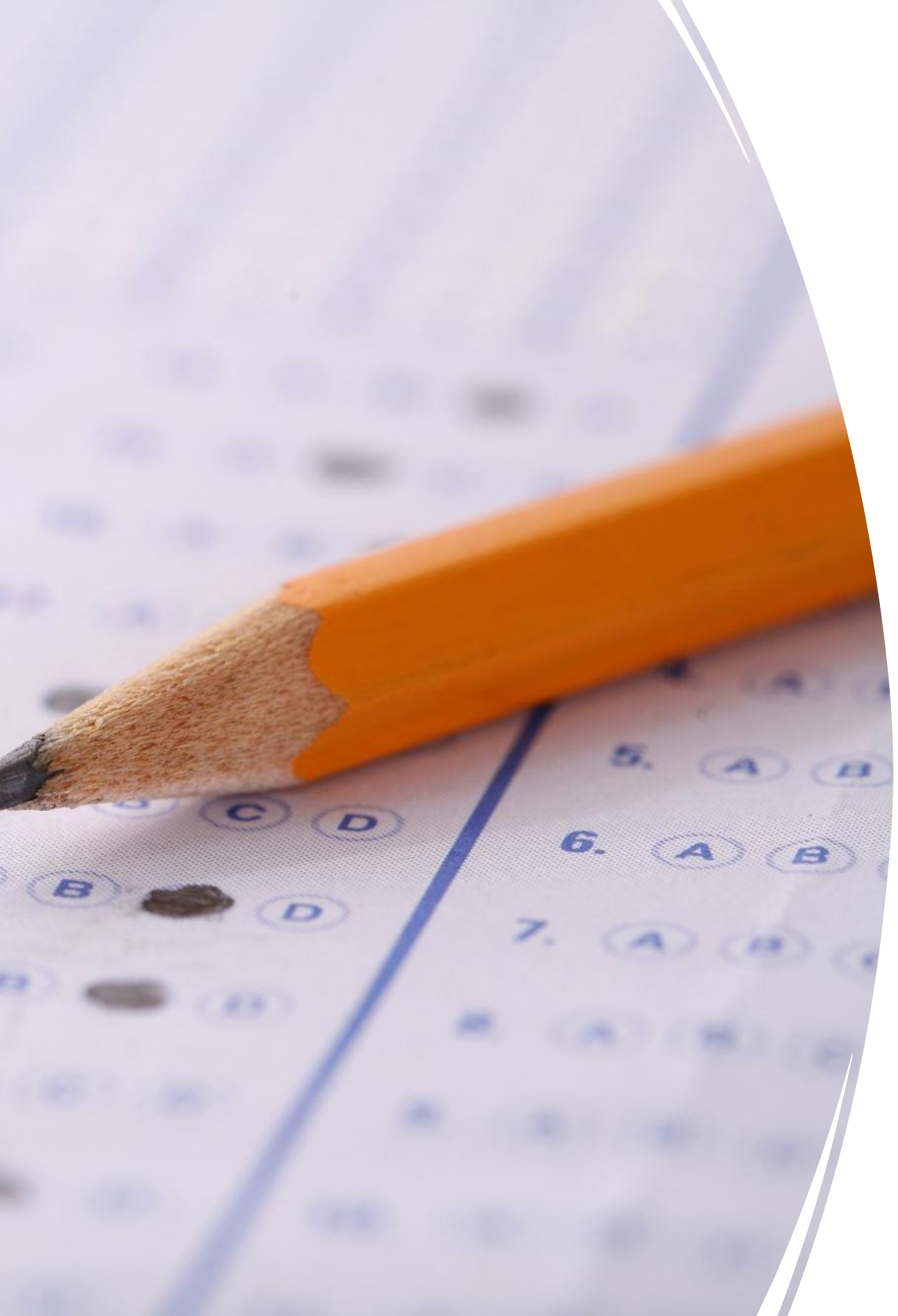
- Complex conjugate: $\bar{\vec{a}} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$

Vectors again

- Dot product: $\vec{a} \cdot \vec{b} = \overline{\vec{a}^T} \cdot \vec{b}$
- Assuming $\vec{a} = \begin{pmatrix} i \\ 0 \end{pmatrix}$

$$\begin{aligned} \|\vec{a}\| &= \sqrt{\overline{\vec{a}^T} \cdot \vec{a}} \\ &= \sqrt{\begin{pmatrix} -i & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix}} \\ &= \sqrt{-i^2} \\ &= 1 \end{aligned}$$

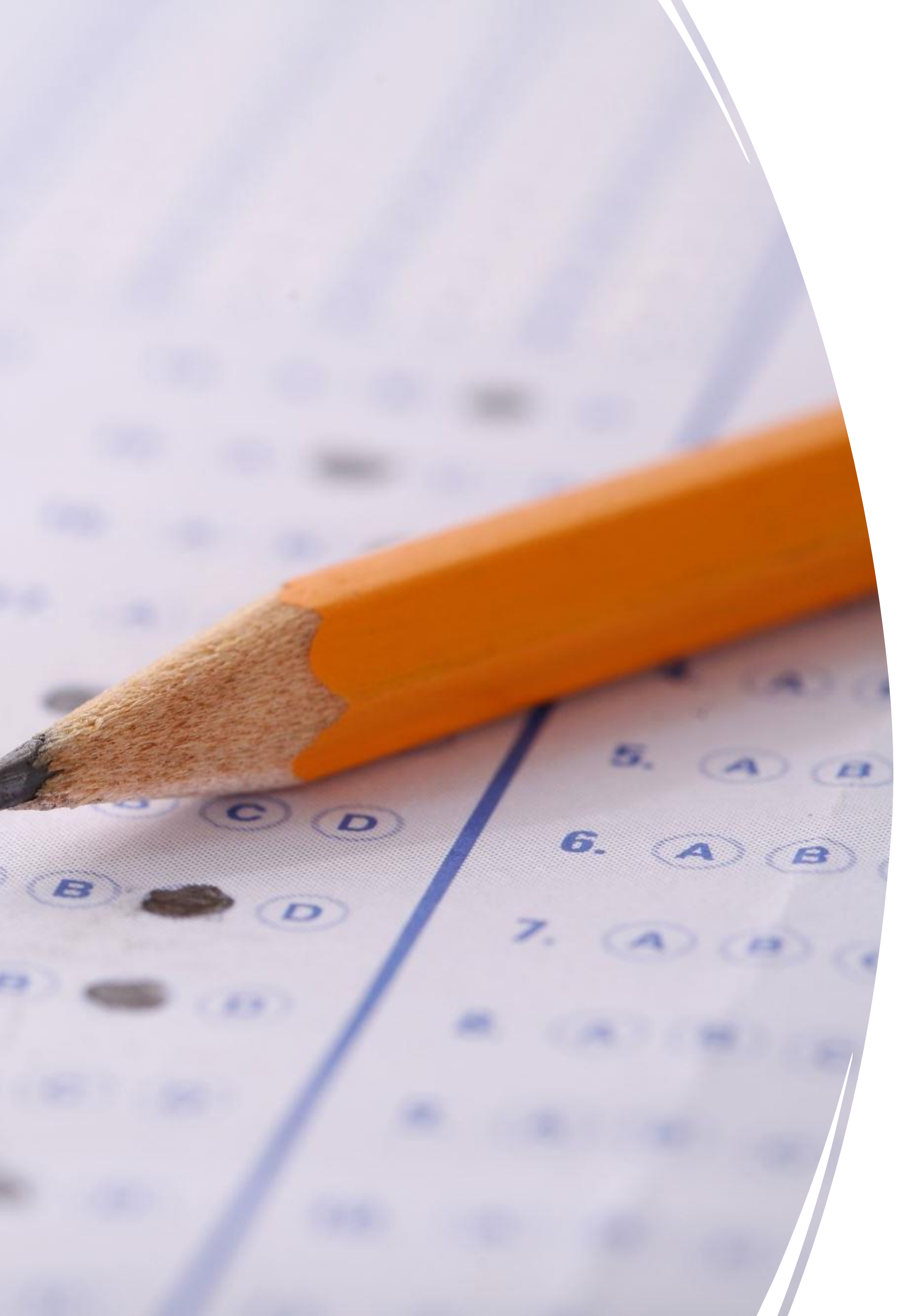
- Hermitian conjugate: $\vec{a}^\dagger = \overline{\vec{a}^T}$
- Inner product: $\vec{a}^\dagger \cdot \vec{b}$
- Outer product: $\vec{a} \cdot \vec{b}^\dagger$



Quiz

Given the vectors $\overrightarrow{v_1} = \begin{pmatrix} -i \\ 4+i \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 5 \\ 2-3i \end{pmatrix}$.

What is the result of $c = \overrightarrow{v_1} \cdot \overrightarrow{v_2}$?

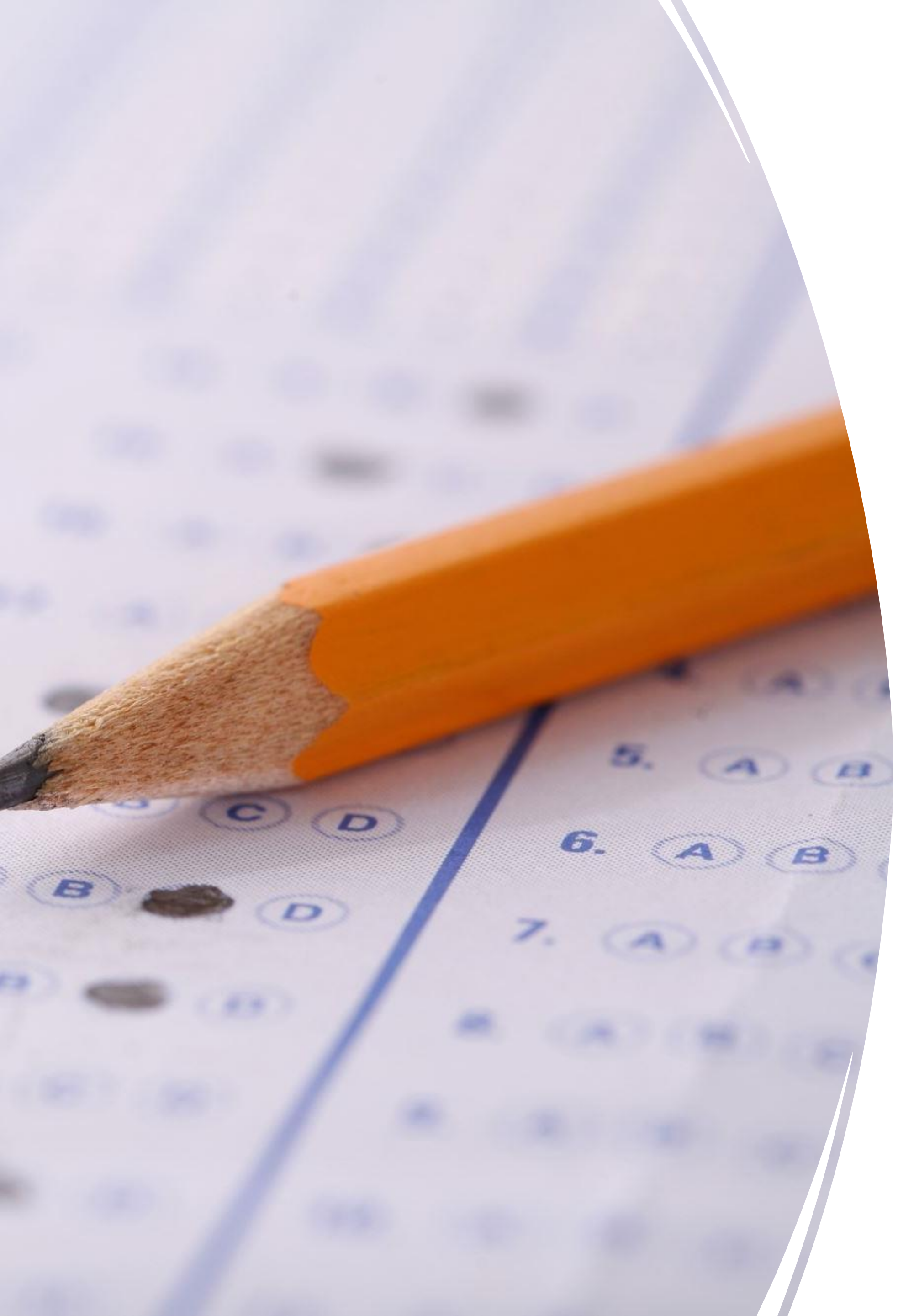


Quiz

Given the vectors $\overrightarrow{v_1} = \begin{pmatrix} -i \\ 4+i \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 5 \\ 2-3i \end{pmatrix}$.

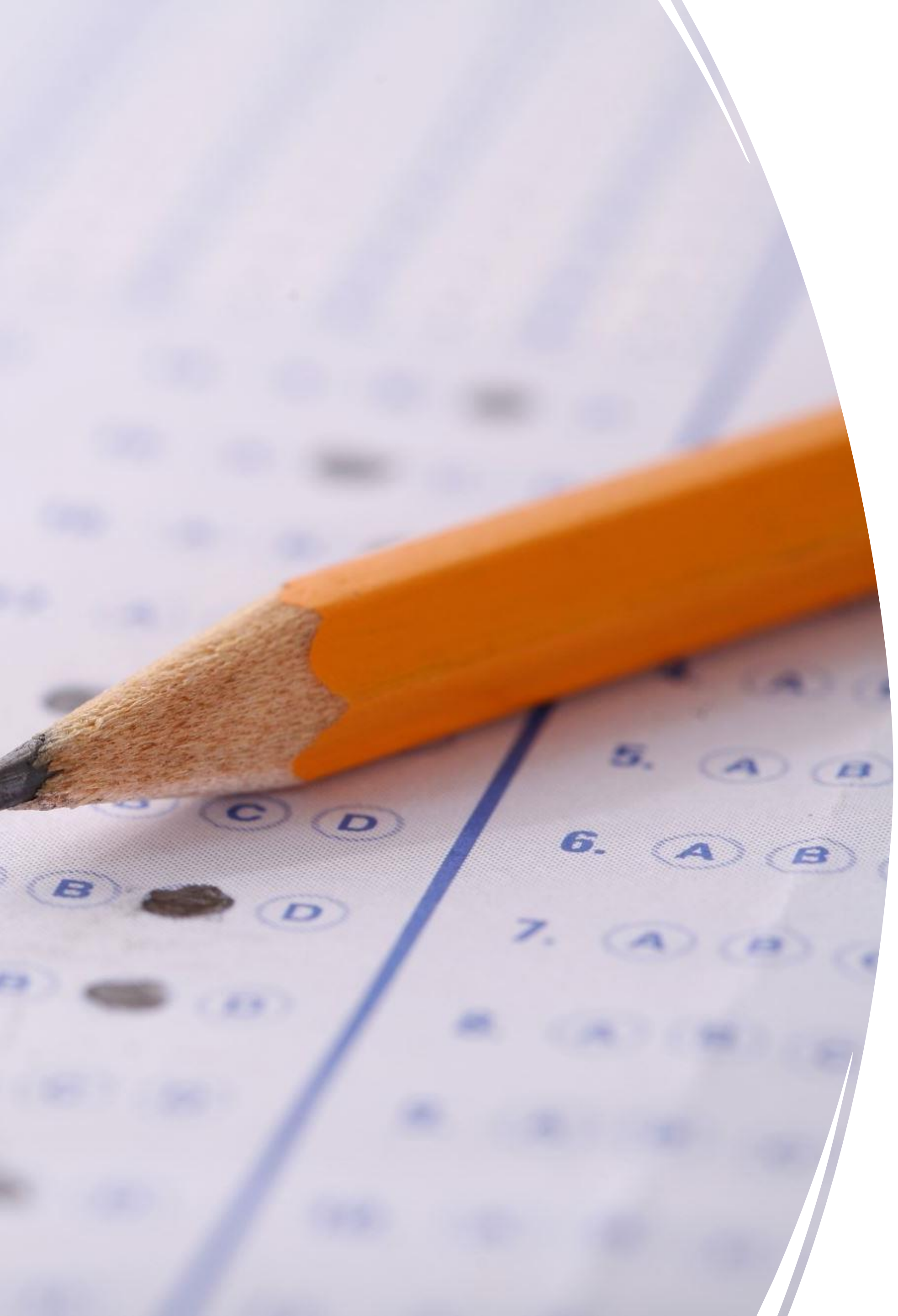
What is the result of $c = \overrightarrow{v_1} \cdot \overrightarrow{v_2}$?

$$c = 5 - 9i$$



Quiz

What is the length of the vector $\overrightarrow{v_1} = \begin{pmatrix} 3+2i \\ 2-i \end{pmatrix}$?



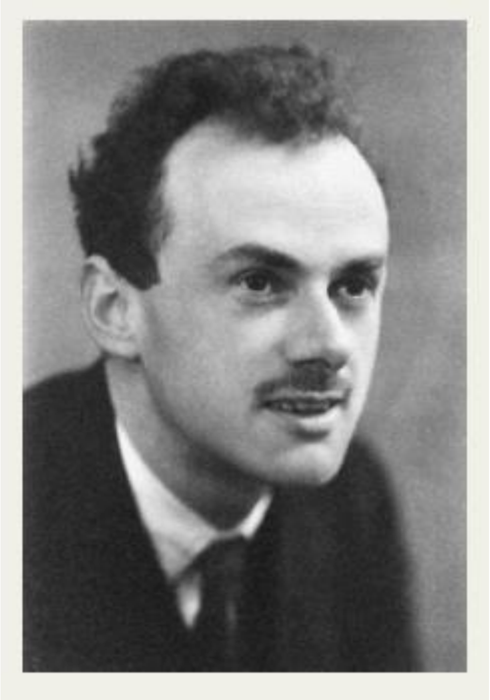
Quiz

What is the length of the vector $\overrightarrow{v_1} = \begin{pmatrix} 3+2i \\ 2-i \end{pmatrix}$?

$$\|\overrightarrow{v_1}\| = 3\sqrt{2}$$

More general approach to vectors

- Abstact mathematical object with some properties
- “Bra-ket” or “Dirac” notation:
  - $\vec{a} = |a\rangle$
  - $\vec{a}^{\dagger} = \langle a|$
  - $|a\rangle$, $|0\rangle$, $|\boxtimes\rangle$ are all vectors



More general approach to vectors

- Algebraic operations which obey the commutativity, associativity and distributivity laws:

$$|c\rangle = |a\rangle + |b\rangle$$

$$|a\rangle + |b\rangle = |b\rangle + |a\rangle$$

$$|c\rangle = a|d\rangle$$

$$a(|a\rangle + |b\rangle) = a|a\rangle + a|b\rangle$$

- Inner product: $\langle a|b\rangle$
- Outer product: $|a\rangle\langle b|$

Linear algebra

- Highly recommended video series
- Study of linear transformations and vectors
- **Linear combination**: sum of n vectors that have been scaled by numbers $\alpha_1|x_1\rangle + \alpha_2|x_2\rangle + \cdots + \alpha_n|x_n\rangle$
- **Linear independence**: A set of vectors is linear dependent if some vector in the set can be expressed as a linear combination of other vectors in the set

$$\mathcal{A} = \{|x_1\rangle, |x_2\rangle, |x_3\rangle\}$$

$$|x_3\rangle = \alpha_1|x_1\rangle + \alpha_2|x_2\rangle$$

- **Linear independence**: if a set of vectors is not linear dependent then it is linear independent

Linear algebra

- Given the set $\mathcal{A} = \left\{ |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$
- $\mathcal{A}$ is linear independent
  - Any other vector can be expressed as a linear combination of $|x\rangle$ and $|y\rangle$

$$|a\rangle = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$|a\rangle = 3|x\rangle + 2|y\rangle$$
$$|a\rangle = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Linear algebra

- What if $\mathcal{B} = \left\{ |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |z\rangle = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$
  - $\mathcal{B}$ is NOT linear independent
    - $|x\rangle = \frac{1}{3}|y\rangle + \frac{1}{3}|z\rangle$
    - $|y\rangle = 3|x\rangle - |z\rangle$
    - $|z\rangle = 3|x\rangle - |y\rangle$

- $3+$ vectors in the plane are linear dependent
- At most 2 vectors in the plane can be linear independent

Linear algebra

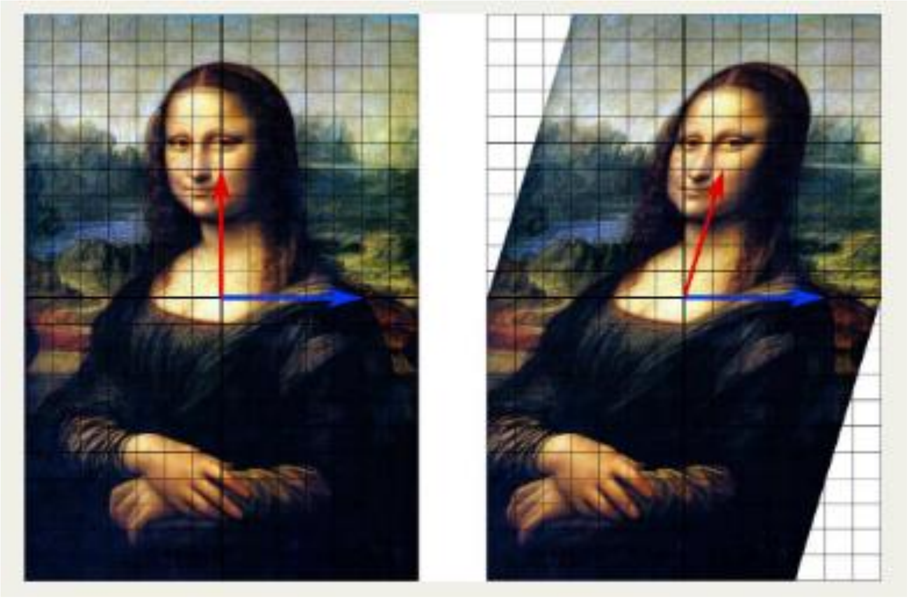
- Given $\mathcal{B} = \left\{ |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |z\rangle = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$, the span of $\mathcal{B}$ is the set of all vectors that can be expressed as a linear combination of the vectors in $\mathcal{B}$
- The span of $\mathcal{B} = \mathbb{R}^2$
- Redundant → $\{\mathcal{B} \setminus |z\rangle\}$ also spans $\mathbb{R}^2$

Linear algebra

- A **basis** is a set of linearly independent vectors that span a vector space
- Minimal set of vectors
- $\mathcal{A} = \left\{ |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$
  - $\mathcal{A}$ is linear independent
  - $\mathcal{A}$ spans $\mathbb{R}^2$
  - $\mathcal{A}$ is a basis for $\mathbb{R}^2$
- The cardinality of the basis represents the **dimension** of the vector space

Eigenvectors and eigenvalues

- A nonzero vector that changes only by a scalar factor when the linear transformation is applied.
- The eigenvalue is the scaling factor
- Geometrically:
  - Vector pointing in a direction
  - Stretched by the transformation in that direction
  - The stretching factor is the eigenvalue



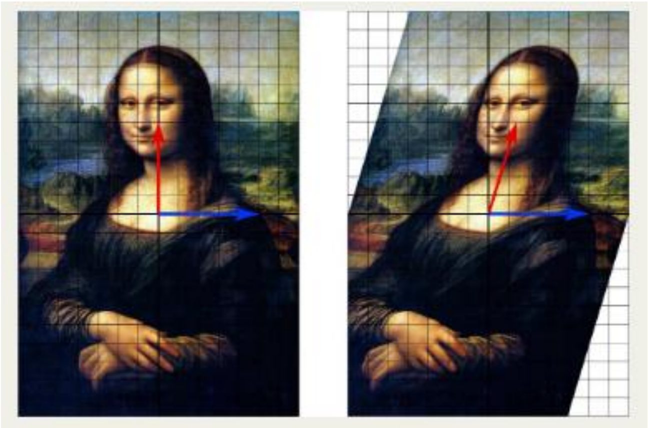
Eigenvectors and eigenvalues

- Can be used to represent a large dimensional matrix
- Mathematically expressed as:

$$A|v\rangle = \lambda|v\rangle$$

- Well-defined methods to compute both eigenvectors and eigenvalues:

$$det(A - \lambda I) = 0$$



Eigenvectors and eigenvalues

- Example: given the matrix $A = \begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix}$. Then:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\left(\begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\left(\begin{pmatrix} 3 - \lambda & -1 \\ 0 & -1 - \lambda \end{pmatrix}\right) = 0$$

Eigenvectors and eigenvalues

- Example: given the matrix $A = \begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix}$. Then:

$$\det\left(\begin{pmatrix} 3-\lambda & -1 \\ 0 & -1-\lambda \end{pmatrix}\right) = 0$$

$$(3-\lambda)(-1-\lambda) - (-1 \cdot 0) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

- Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$

Eigenvectors and eigenvalues

- Now, calculate the corresponding eigenvectors. For $\lambda_1 = -1$:

$$A|v\rangle = \lambda|v\rangle$$

$$\begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x - y = -x \\ 0 \cdot x - y = -y \end{cases}$$

$$\begin{cases} 3x - y = -x \\ y = y \end{cases}$$

Eigenvectors and eigenvalues

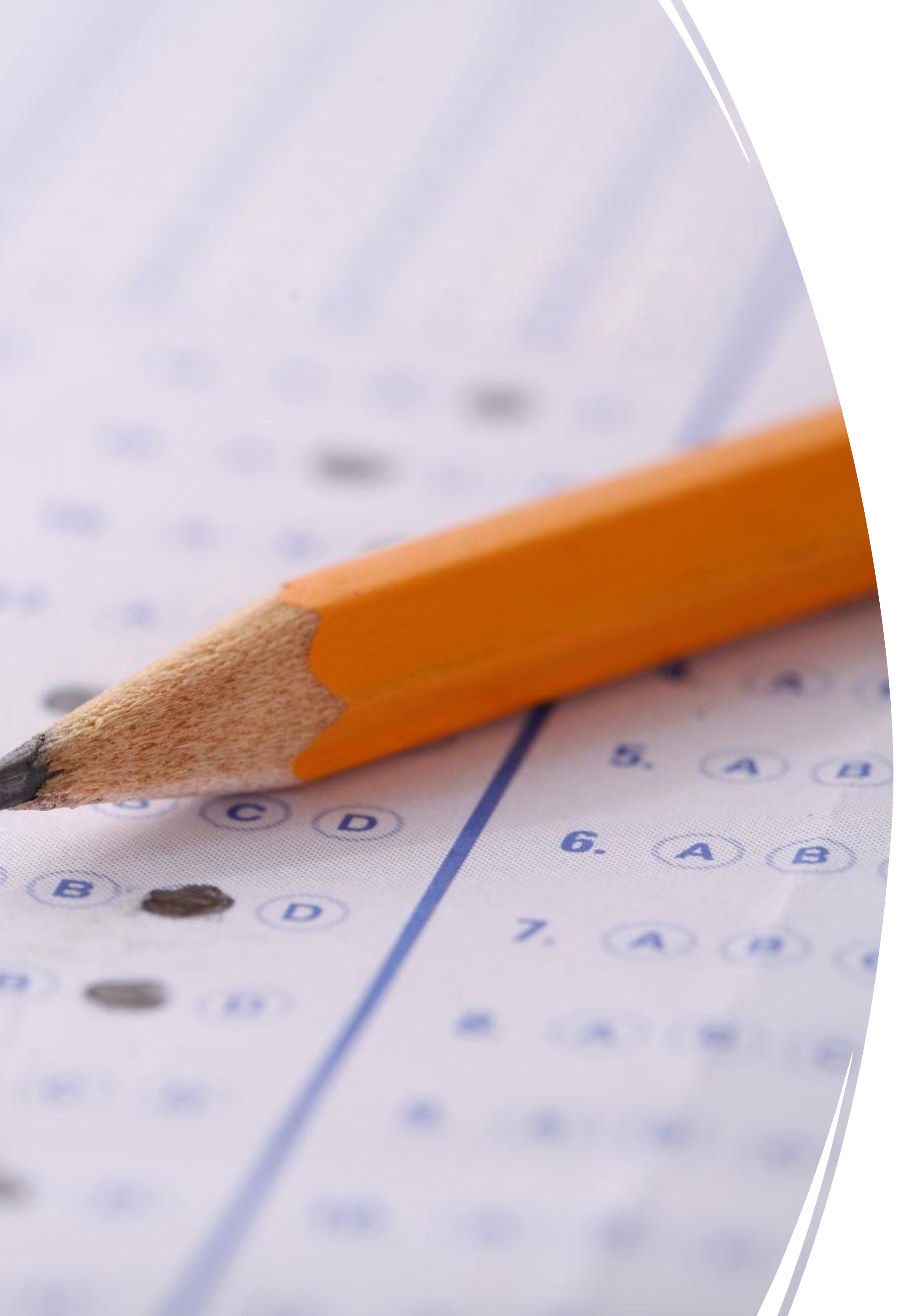
- Calculate the corresponding eigenvectors. For $\lambda_1 = -1$:

$$\begin{cases} 3x - y = -x \\ y = y \end{cases}$$

- Now, choose wisely! Lets say $y = 4$, then $x = 1$. So the corresponding eigenvector:

$$|v_1\rangle = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

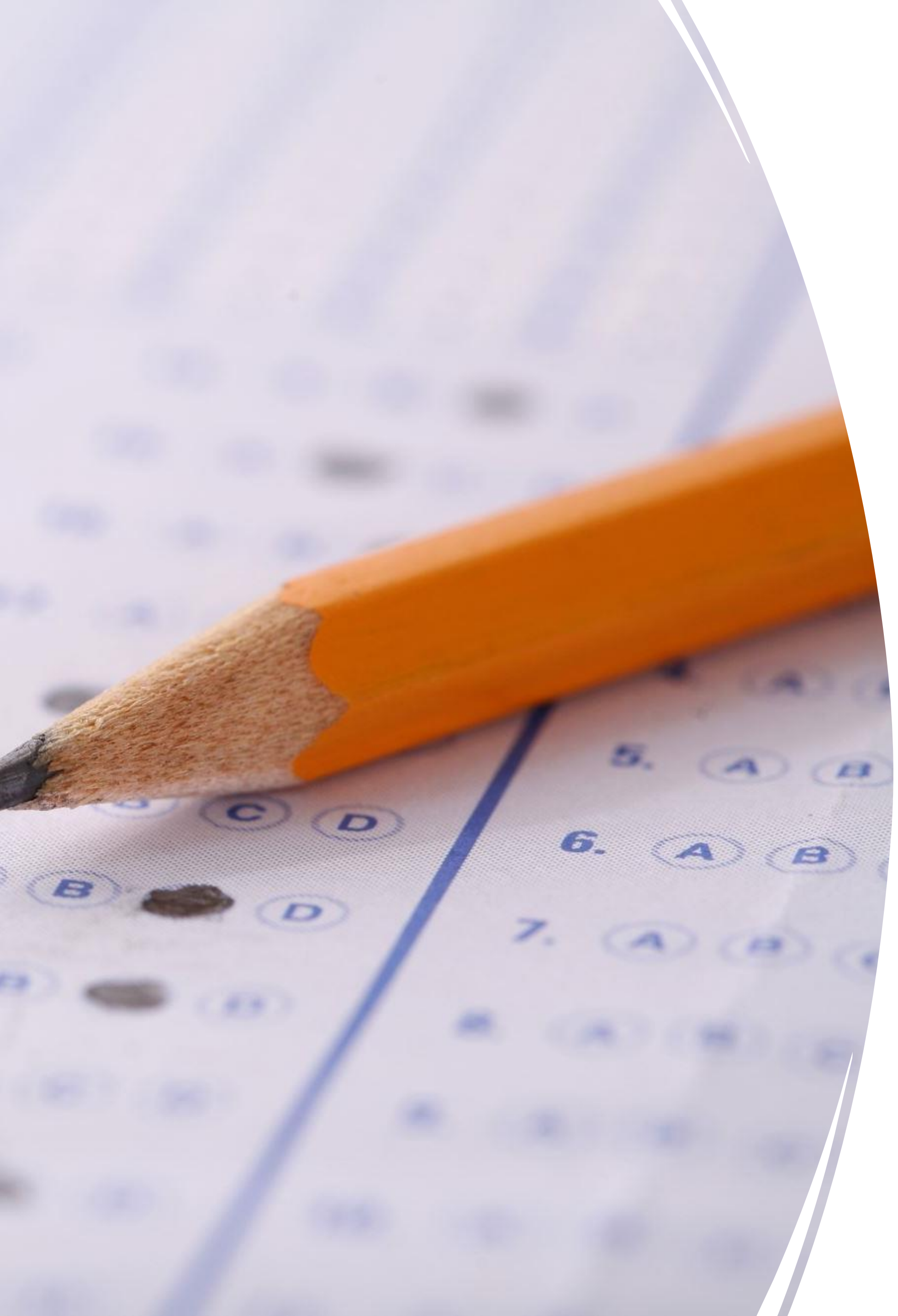
- Make the same calculations for $\lambda_2 = 3$



Quiz

Given the linear transformation $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Calculate the eigenvalues of Z.



Final Quiz

Given the linear transformation $\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Calculate the eigenvectors of X.