

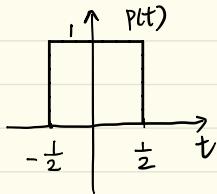
Week 3

48. Compute the Fourier transforms of the following functions.

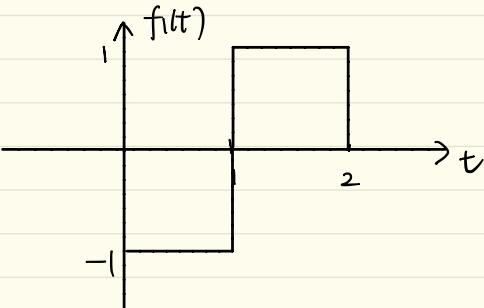
$$(a) f_1(t) = \begin{cases} 0, & t < 0 \text{ and } t > 2 \\ -1, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \end{cases}$$

$$\begin{aligned}
 (a) \text{ Solution 1: } F_1(W) &= \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt \\
 &= \int_0^1 -e^{-j\omega t} dt + \int_1^2 e^{-j\omega t} dt \\
 &= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^1 + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_1^2 \\
 &= \frac{e^{-j\omega} - 1}{-j\omega} + \frac{e^{-j\omega 2} - e^{-j\omega}}{-j\omega} \\
 &= \frac{e^{j\frac{W}{2}}(e^{-j\frac{W}{2}} - e^{j\frac{W}{2}})}{j\omega} - \frac{e^{-j\frac{3W}{2}}(e^{-j\frac{W}{2}} - e^{j\frac{W}{2}})}{j\omega} \\
 &= \frac{e^{j\frac{W}{2}}(-2j\sin\frac{W}{2})}{j\omega} - \frac{e^{-j\frac{3W}{2}}(-2j\sin\frac{W}{2})}{j\omega} \\
 &= \frac{2\sin\frac{W}{2}}{\omega} e^{-j\frac{3W}{2}} - \frac{2\sin\frac{W}{2}}{\omega} e^{-j\frac{W}{2}} \quad \left(\text{sinc}(aw) = \frac{\sin(a\pi w)}{a\pi w} \right) \\
 &= \text{sinc}\left(\frac{W}{2\pi}\right) e^{-j\frac{3W}{2}} - \text{sinc}\left(\frac{W}{2\pi}\right) e^{-j\frac{W}{2}}
 \end{aligned}$$

Solution 2 :



$$P(W) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j\omega t} dt = \text{sinc}\left(\frac{W}{2\pi}\right)$$



$$f_1(t) = -p(t - \frac{1}{2}) + p(t - \frac{3}{2})$$

$$\begin{aligned}
 \Rightarrow F_1(W) &= -p(W)e^{-j\frac{W}{2}} + p(W)e^{-j\frac{3W}{2}} \\
 &= -\text{sinc}\left(\frac{W}{2\pi}\right) e^{-j\frac{W}{2}} + \text{sinc}\left(\frac{W}{2\pi}\right) e^{-j\frac{3W}{2}}
 \end{aligned}$$

$$(d) f_4(t) = \begin{cases} 0, & |t| > 1 \\ e^{-|t|}, & |t| \leq 1 \end{cases}$$

$$\begin{aligned}(d) F_4(w) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt \\&= \int_{-1}^1 e^{-|t|} e^{-j\omega t} dt \\&= \int_{-1}^1 e^{-|t|} (\cos \omega t - j \sin \omega t) dt \\&= \int_{-1}^1 e^{-|t|} \cos \omega t dt - j \int_{-1}^1 e^{-|t|} \sin \omega t dt\end{aligned}$$

since $e^{-|t|}$ is an even function, $\cos \omega t$ is an even f, $\sin \omega t$ is an odd f.

$$= 2 \cdot \int_0^1 e^{-t} \cos \omega t dt$$

$$\text{Given } \int e^{at} \cos(bt) dt = \frac{e^{at}(a \cos bt + b \sin bt)}{a^2 + b^2}$$

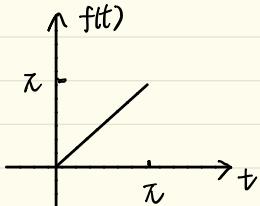
In this case, $a = -1$, $b = \omega$

$$\begin{aligned}&= 2 \cdot \left[\frac{e^{-t}(-\cos \omega t + \omega \sin \omega t)}{1 + \omega^2} \right]_0^1 \\&= 2 \cdot \left[\frac{e^{-1}(-\cos \omega + \omega \sin \omega) - e^0(-1 + 0)}{1 + \omega^2} \right] \\&= 2 \left[\frac{1 - e^{-1}(\cos \omega - \omega \sin \omega)}{1 + \omega^2} \right] \\&= \frac{2 - 2e^{-1}(\cos \omega - \omega \sin \omega)}{1 + \omega^2}\end{aligned}$$

49. Consider the signal $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$f(t) = t(u(t) - u(t - \pi))$$

- (a) Sketch the graph of $f(t)$ for $-2\pi \leq t \leq 2\pi$.
 (b) Use the definition of Fourier transform to find the Fourier transform of f .



$$\begin{aligned}
 & \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\
 &= \int_0^{\pi} t e^{-j\omega t} dt \\
 &= \int_0^{\pi} \frac{t}{-j\omega} d e^{-j\omega t} \\
 &= \left[\frac{t}{-j\omega} \cdot e^{-j\omega t} \right]_0^{\pi} - \int_0^{\pi} e^{-j\omega t} d \frac{t}{-j\omega} \\
 &= \left[\frac{t}{-j\omega} \cdot e^{-j\omega t} \right]_0^{\pi} - \left[\frac{e^{-j\omega t}}{(-j\omega)^2} \right]_0^{\pi} \\
 &= \frac{\pi e^{-j\omega \pi}}{-j\omega} - \frac{e^{-j\omega \pi}}{\omega^2} \Big|_0^{\pi} = \frac{1}{\omega^2} [(1 + j\pi\omega) e^{-j\omega \pi} - 1]
 \end{aligned}$$

50. Compute the inverse Fourier transforms of the following signals

(a) $X_1(\omega) = \cos(4\omega)$

$$X_1(\omega) = \frac{1}{2}[e^{-j4\omega} + e^{j4\omega}]$$

$$\begin{aligned}
 & \text{since } \mathcal{F}[\delta(t)] = 1, \quad \mathcal{F}[\delta(t - C)] = 1 \cdot e^{-j\omega C} \text{ (time shifting)} \\
 & \Rightarrow x_1(t) = \frac{1}{2} [\delta(t - 4) + \delta(t + 4)]
 \end{aligned}$$

52. Find the Fourier transform of the function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$g(t) = 2t(u(t) - u(t - \pi/2))$$

Hint: Firstly show that $u(at) = u(t)$ for any $a \geq 0$. Then show that $g(t) = f(2t)$, where f is given in Question 49.

① show that $u(at) = u(t)$

$$\mathcal{F}[u(at)] = \frac{1}{\alpha} \quad \mathcal{F}[u(t)] = \frac{1}{\alpha} U\left(\frac{\omega}{\alpha}\right) = \frac{1}{\alpha} \cdot \frac{1}{\frac{\omega}{\alpha}} = \frac{1}{\alpha}$$

$\Rightarrow u(at)$ and $u(t)$ are identical

② show that $g(t) = f(2t)$

$$\begin{aligned} f(2t) &= 2t(u(2t) - u(2t - \pi)) \\ &= 2t(u(t) - u(t - \frac{\pi}{2})) \quad (\text{since } u(at) = u(t)) \\ &= g(t) \end{aligned}$$

$$\begin{aligned} \text{Therefore, } G(w) &= \frac{1}{2} F\left(\frac{w}{2}\right) = \frac{1}{2} \cdot \frac{1}{\left(\frac{w}{2}\right)} [C + j\pi \frac{w}{2}] e^{-j\frac{\pi w}{2}} - 1 \\ &= \frac{2}{w^2} [C + j\frac{\pi w}{2}] e^{-j\frac{\pi w}{2}} - 1 \end{aligned}$$

54. Match the time-domain signals in Figure 1 to the appropriate amplitude spectra in Figure 2. In each case explain why the amplitude spectrum matches the time-domain signal.

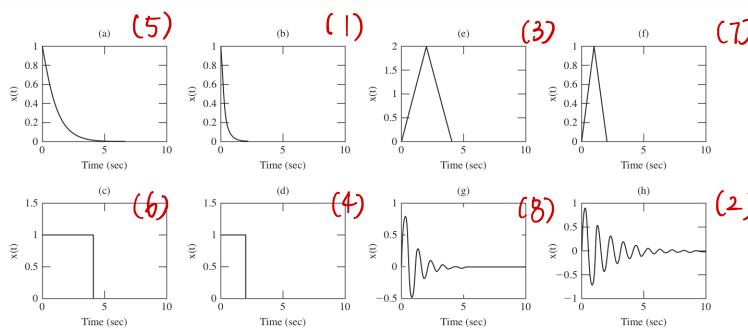


Figure 1: Time domain signals for Problem 54

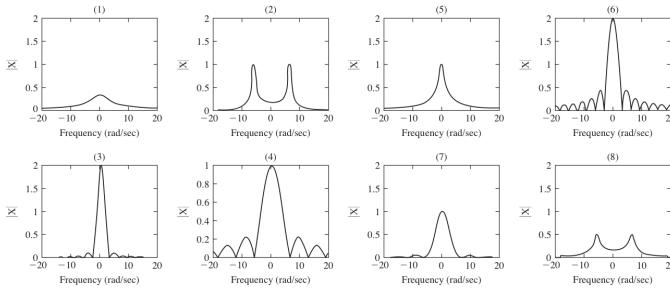


Figure 2: Fourier spectra for Problem 54

$$(a)(b): X(t) = e^{-bt} \sin(wt) \quad X(w) = \frac{1}{j\omega b} \quad |X| = \frac{1}{\sqrt{\omega^2 + b^2}}$$

$|X|$ has a peak in $w=0$, peak value depends on b

$$(c)(d): X(t) = \begin{cases} 1, & -\frac{\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases} \quad X(w) = \tau \operatorname{sinc}\left(\frac{\tau}{2\pi} w\right)$$

The amplitude of $|X|$ depends on τ

$$(e)(f): X(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & -\frac{\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases} \quad X(w) = \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau}{4\pi} w\right)$$

The amplitude of $|X|$ depends on τ

$$(g)(h)$$

$$X(t) = e^{-bt} \sin(wt) u(t)$$

$$X(w) \approx \frac{1}{j(w+b)} + \frac{1}{j(w-b)}$$

peak value depends on b .

As $b \uparrow$, $|X| \downarrow$

55. The continuous-time signal $x(t) = e^{-bt}u(t)$, with $b > 0$, has Fourier transform

$$X(\omega) = \frac{1}{j\omega + b}$$

where b is a constant. Use properties of the Fourier transform to obtain the transforms of following signals:

(a) $x_1(t) = x(5t - 4)$

(a) $X_1(t) = X(5t - \frac{4}{5})$

Given $x(t) \leftrightarrow X(\omega)$, $x(5t) \leftrightarrow \frac{1}{5}X(\frac{\omega}{5}) = \frac{1}{5} j\frac{1}{5}\omega + b = \frac{1}{5} j\omega + 5b$

let $x_2(t) = x(5t)$, then $X_1(t) = X_2(t - \frac{4}{5})$

According to time shifting, $X_1(\omega) = X_2(\omega) \cdot e^{-j\omega \frac{4}{5}}$

$$\Rightarrow X_1(\omega) = \frac{1}{j\omega + 5b} \cdot e^{-j\omega \frac{4}{5}}$$

(d) $x_4(t) = \frac{1}{jt - b}$

let $X(t) = \frac{1}{jt + b}$, $X_4(-t) = -X(t)$

Take Fourier Transform on both sides

$$X_4(-\omega) = -2\pi X(-\omega)$$

replace $-\omega$ by ω

$$X_4(\omega) = -2\pi X(\omega) = -2\pi \cdot e^{-b\omega} u(\omega)$$

56. Let $x(t)$ and $X(\omega)$ be a Fourier transform pair. Using the definition of Fourier transform, prove the following Fourier transform pairs:

$x(t - c)$	\longleftrightarrow	$X(\omega)e^{-j\omega c}$, $c \in \mathbf{R}$	(Time shift)
$x(-t)$	\longleftrightarrow	$X(-\omega)$	(Flipping Theorem)
$x(at)$	\longleftrightarrow	$\frac{1}{a}X\left(\frac{\omega}{a}\right)$, $a > 0$	(Time scaling)
$x(t)e^{j\omega_0 t}$	\longleftrightarrow	$X(\omega - \omega_0)$, $\omega_0 \in \mathbf{R}$	(Modulation)
$X(t)$	\longleftrightarrow	$2\pi x(-\omega)$	(Duality)

Prove = Duality

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

① let t replaced by $-t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

② let t replaced by w , w replaced by t

$$2\pi x(-w) = \int_{-\infty}^{\infty} X(t) e^{-jtw} dt$$

$$\downarrow \\ \mathcal{F}[X(t)]$$

$$\Rightarrow X(t) \Leftrightarrow 2\pi x(-w)$$

58. Consider the first order RC filter whose dynamics are described by the equations

$$RCV_c + V_c = V_{in}$$

$$V_{out} = V_c$$

(a) If $R = 100\text{Ohms}$ and $C = 0.2\text{mF}$ what is the time domain response if $V_{in} = 2\delta(t)$? What is the system time constant?

(b) If $V_{in}(t) = 2\sin(t)$ what is the time domain response?

(c) If the initial conditions of the system were nonzero so that $V_c(0) = 1$ and $V_{in}(t) = 1 \forall t \geq 0$ what would be the system response?

(d) How would your previous answer change if the input was delayed by 1 second, i.e. $V_{in}(t) = 1 \forall t \geq 1$?

$$(a) \quad RCV_{out} + V_{out} = V_{in} \quad \text{let } V_{out} = x, \quad V_{in} = u, \quad a = \frac{1}{RC}$$

$$\Rightarrow RC\dot{x} + x = u \Rightarrow \dot{x} + ax = au \quad (1)$$

- define a function $\beta(t)$, that satisfies $\dot{\beta} = ab \Rightarrow \beta = Ae^{at}$

- multiplying both sides of (1) by β

$$\beta\dot{x} + \dot{\beta}x = au\beta$$

$$\Rightarrow \beta x + C = \int au\beta dt \Rightarrow x(t) = \frac{\int au\beta dt - C}{\beta} = \frac{\int a e^{at} u dt + C_2}{e^{at}}$$

$$\text{if } u(t) = V_{in} = 2\delta(t)$$

$$x(t) = (a \int_0^t e^{at} \cdot 2\delta(t) dt + C_2) e^{-at} = (2a + C_2) e^{-at}$$

$$\text{Assuming } x(0) = 0, \Rightarrow C_2 = 0, \quad x(t) = 2a e^{-at} \quad (t > 0)$$

$$\text{time constant : } \frac{1}{a} = \frac{1}{RC} = 0.02$$

$$(b) \quad u(t) = 2\sin(t) = \frac{e^{jt} - e^{-jt}}{j}$$

$$x(t) = (a \int_0^t e^{at} \cdot \frac{e^{jt} - e^{-jt}}{j} dt + C_2) e^{-at}$$

$$= \left[\frac{a}{j} \left[\frac{e^{(a+j)t}}{a+j} \right]_0^t - \left[\frac{e^{(a-j)t}}{a-j} \right]_0^t + C_2 \right] e^{-at}$$

$$= \frac{2a^2 \sin t - 2a \cos t}{a^2 + 1} + C_3 e^{-at} \quad (\text{when } x(0) = 0, \quad C_3 = 0)$$

$$(c) \quad x(t) = (a \int_0^t e^{at} \cdot 1 dt + C_2) e^{-at} = (e^{at} - 1 + C_2) e^{-at}$$

$$= \underbrace{1 - e^{-at}}_{\text{particular}} + \underbrace{C_2 e^{-at}}_{\text{homogeneous solution}}$$

$$V_C(0) = x(0) = 1 - 1 + C_2 = 1 \Rightarrow C_2 = 1$$

$$(d) \quad x(t) = 1(t-1) - e^{-a(t-1)} + e^{-at} \quad (\text{input only impacts particular solution.})$$