7.3. The frequency which, under the sampling theorem, must be exceeded by the sampling frequency is called the *Nyquist rate*. Determine the Nyquist rate corresponding to each of the following signals:

(a) $x(t) = 1 + \cos(2{,}000\pi t) + \sin(4{,}000\pi t)$

(b) $x(t) = \frac{\sin(4{,}000\pi t)}{\pi t}$

(c) $x(t) = \left(\frac{\sin(4{,}000\pi t)}{\pi t}\right)^2$

(a) $X(jw) = 2\pi\delta(w) + \pi[\delta(w+2000\pi) + \delta(w-2000\pi)]$
$+ j\pi[\delta(w+4000\pi) - \delta(w-4000\pi)]$

$\Rightarrow X(jw) = 0$ for $|w| > 4000\pi$

$\Rightarrow$ The Nyquist rate : $w_N = 2 \cdot 4000\pi = 8000\pi$

(b) From table 4.2,

$$\frac{\sin(w_1 t)}{\pi t} \xrightarrow{F} X(jw) = \begin{cases} 1 & |w| < w_1 \\ 0 & |w| > w_1 \end{cases}$$

since $w_1 = 4000\pi$, $w_N = 2 \cdot 4000\pi = 8000\pi$

(c) let $x_1(t) = \frac{\sin(4000\pi t)}{\pi t}$ $\quad x(t) = x_1(t) \cdot x_1(t)$

$\Rightarrow X(jw) = \frac{1}{2\pi} X_1(jw) * X_1(jw)$

$$= \begin{cases} \frac{1}{2\pi} & |w| < 2w_1 = 8000\pi \\ 0 & |w| > 8000\pi \end{cases}$$

$\Rightarrow w_N = 2 \cdot 8000\pi = 16000\pi$

7.5. Let $x(t)$ be a signal with Nyquist rate ω_0. Also, let

$$y(t) = x(t)p(t-1),$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ and } T < \frac{2\pi}{\omega_0}.$$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives $x(t)$ as its output when $y(t)$ is the input.

$$p(t) \xrightarrow{F} \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \delta\left(w - \frac{2k\pi}{T}\right)$$

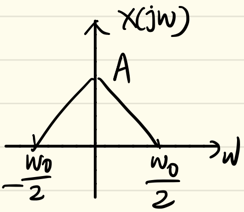
$$p(t-1) \xrightarrow{F} \frac{2\pi}{T} e^{-jw} \sum_{k\in\mathbb{Z}} \delta\left(w - \frac{2k\pi}{T}\right) = \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \delta\left(w - \frac{2k\pi}{T}\right)\cdot e^{-j\frac{2k\pi}{T}}$$

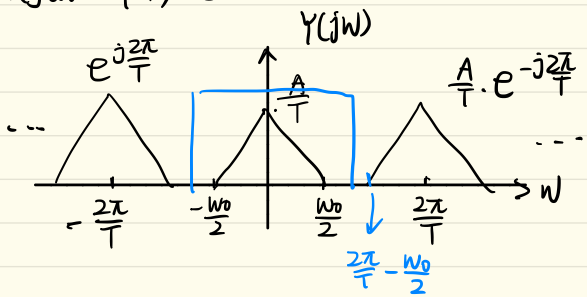
Since $y(t) = x(t)\cdot p(t-1)$, we have

$$Y(jw) = \frac{1}{2\pi}\left[X(jw) * \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \delta\left(w - \frac{2k\pi}{T}\right)\cdot e^{-j\frac{2k\pi}{T}}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{T}\cdot \sum_{k\in\mathbb{Z}} X\left(j\left(w - \frac{2k\pi}{T}\right)\right)\cdot e^{-j\frac{2k\pi}{T}}$$

$$= \frac{1}{T} \sum_{k\in\mathbb{Z}} X\left(j\left(w - \frac{2k\pi}{T}\right)\right)\cdot e^{-j\frac{2k\pi}{T}}$$





It is clear that if $H(jw) = \begin{cases} T & |w| < w_c \\ 0 & \text{otherwise} \end{cases}$

where $\frac{w_0}{2} < w_c < \frac{2\pi}{T} - \frac{w_0}{2}$

7.7. A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period T to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on the samples of $x(t)$; i.e.,

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)h_1(t - nT),$$

where $h_1(t)$ is the function shown in Figure P7.7. Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input.

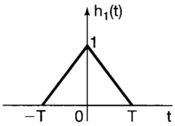


Figure P7.7

We note that:

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_1(t-nT) = h_1(t) * \underbrace{\left\{\sum_{n=-\infty}^{\infty} x(nT)\cdot \delta(t-nT)\right\}}_{x_p(t)}$$

Taking the Fourier Transform:

$$X_1(jw) = H_1(jw)\cdot X_p(jw)$$

The output of zero-order hold is:

$$x_0(t) = h_0(t) * \left\{\sum_{n=-\infty}^{\infty} x(nT)\cdot \delta(t-nT)\right\}$$

Taking the Fourier Transform:

$$X_0(jw) = H_0(jw)\cdot X_p(jw)$$

$$H(jw) = \frac{X_1(jw)}{X_0(jw)} = \frac{H_1(jw)}{H_0(jw)}$$

We know that: $H_1(jw) = \frac{1}{T}\left[\frac{\sin\left(\frac{wT}{2}\right)}{w/2}\right]^2$, $\frac{1}{H_0(jw)} = e^{j\frac{wT}{2}}\left[\frac{w}{2\sin\left(\frac{wT}{2}\right)}\right]$

$$H(jw) = \left(\frac{1}{T}\frac{\sin^2\left(\frac{wT}{2}\right)}{w^2/4}\right)\cdot e^{j\frac{wT}{2}}\left[\frac{w}{2\sin\left(\frac{wT}{2}\right)}\right]$$

$$= \frac{1}{T}e^{j\frac{wT}{2}}\frac{2\sin\left(\frac{wT}{2}\right)}{w}$$

7.8. Consider a real, odd, and periodic signal $x(t)$ whose Fourier series representation may be expressed as

$$x(t) = \sum_{k=0}^{5} \left(\frac{1}{2}\right)^k \sin(k\pi t).$$

Let $\hat{x}(t)$ represent the signal obtained by performing impulse-train sampling on $x(t)$ using a sampling period of $T = 0.2$.

(a) Does aliasing occur when this impulse-train sampling is performed on $x(t)$?

(b) If $\hat{x}(t)$ is passed through an ideal lowpass filter with cutoff frequency π/T and passband gain T, determine the Fourier series representation of the output signal $g(t)$.

(a) $x(t) = \sin(0) + (\frac{1}{2})\sin(\pi t) + \cdots + (\frac{1}{2})^5 \sin(5\pi t)$

$W_{max} = 5\pi \qquad W_s > 2W_{max} = 10\pi$

$\Rightarrow \frac{2\pi}{T} > 10\pi \qquad T < 0.25$

Yes, aliasing does occur in this case

(b) Since aliasing has already resulted in the loss of the $(\frac{1}{2})^5 \sin(5\pi t)$

the output will be:

$$y(t) = \sum_{k=0}^{4} (\tfrac{1}{2})^k \sin(k\pi t) = \sum_{k=-4}^{4} C_k e^{-j\frac{k\pi}{t}}$$

$$\begin{cases} C_k = \frac{1}{2}(a_k - jb_k) = -\frac{1}{2}j(\frac{1}{2})^k = -j(\frac{1}{2})^{k+1} & 1 \le k \le 4 \\ C_k = \frac{1}{2}(a_k + jb_k) = \frac{1}{2}j(\frac{1}{2})^{-k} = j(\frac{1}{2})^{-k+1} & -4 \le k \le -1 \\ C_0 = 0 \end{cases}$$

7.9. Consider the signal

$$x(t) = \left(\frac{\sin 50\pi t}{\pi t}\right)^2,$$

which we wish to sample with a sampling frequency of $\omega_s = 150\pi$ to obtain a signal $g(t)$ with Fourier transform $G(j\omega)$. Determine the maximum value of ω_0 for which it is guaranteed that

$$G(j\omega) = 75X(j\omega) \text{ for } |\omega| \le \omega_0,$$

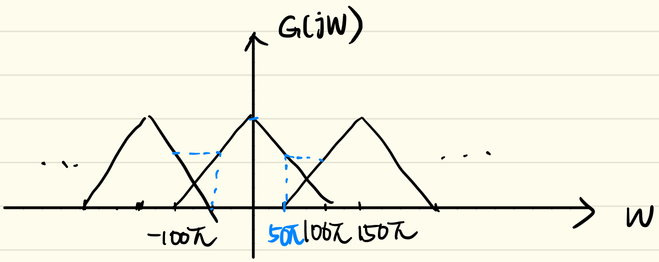
where $X(j\omega)$ is the Fourier transform of $x(t)$.

$X(t) = x_1(t)\cdot x_1(t)$ $\quad x_1(t) = \frac{\sin 50\pi t}{\pi t} \xrightarrow{F} X_1(jw) = \begin{cases} 1 & |w| < 50\pi \\ 0 & |w| > 50\pi \end{cases}$

$\Rightarrow X(jw) = \frac{1}{2\pi} X_1(jw) * X_1(jw) = \begin{cases} \frac{1}{2\pi} & |w| < 100\pi \\ 0 & |w| > 100\pi \end{cases}$

$G(jw) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \cdot X(j(w - \frac{2n\pi}{T}))$, where $\frac{2\pi}{T} = w_s = 150\pi \Rightarrow T = \frac{2\pi}{w_s} = \frac{1}{75}$

$\Rightarrow G(jw) = 75 \sum_{n\in\mathbb{Z}} X(j(w - n\cdot w_s))$



Clearly, $G(jw) = 75X(jw)$ for $|w| < 50\pi$