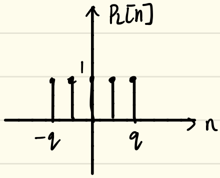
1. Let the discrete-time signal $x[n]$ be given by

$$x[n] = \begin{cases} 2, & \text{if } n = 0,1,2,3,4 \\ -2, & \text{if } n = 5,6,7 \\ 0, & \text{otherwise} \end{cases}$$

Find $X(e^{j\omega})$, the discrete-time Fourier transform (DTFT) of x.

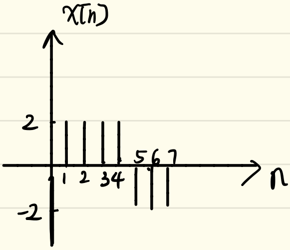


$$P_L[n] = \begin{cases} 1 & -q \le n \le q \\ 0 & \text{otherwise} \end{cases} \qquad L = 2q+1$$

$$P_L(e^{jw}) = \sum_{-q}^{q} 1 \cdot e^{-jwn} = e^{jwq} + e^{jw(q-1)} + \cdots + e^{-jw(q-1)} + e^{-jwq}$$

$$= e^{jwq}\left(\sum_{n=0}^{2q} e^{-jwn}\right)$$

$$= e^{jwq}\left(\frac{1-e^{-j(2q+1)w}}{1-e^{-jw}}\right) = \frac{\sin\left(\frac{2q+1}{2}w\right)}{\sin\left(\frac{w}{2}\right)}$$



Since $x[n] = 2 \cdot \underset{\uparrow \atop q=2}{P_L[n-2]} - 2 \cdot \underset{\uparrow \atop q=1}{P_L[n-6]}$

when $q=2$, $P_L(e^{jw}) = \dfrac{\sin\left(\frac{5}{2}w\right)}{\sin\left(\frac{w}{2}\right)}$

$q=1$, $P_L(e^{jw}) = \dfrac{\sin\left(\frac{3}{2}w\right)}{\sin\left(\frac{w}{2}\right)}$

$$\Rightarrow X(e^{jw}) = 2 \cdot \frac{\sin\left(\frac{5}{2}w\right)}{\sin\left(\frac{w}{2}\right)} \cdot e^{-j2w} - 2 \cdot \frac{\sin\left(\frac{3}{2}w\right)}{\sin\left(\frac{w}{2}\right)} \cdot e^{-j6w}$$

8. Compute the 4-point discrete Fourier transform (DFT) for each of the following signals, which are zero-valued except as specified:

(a) $x[0] = x[2] = 1$.

(b) $x[0] = x[1] = 1, x[2] = x[3] = -1$.

DFT: $a_n = \frac{1}{N} \sum_{k=-\infty}^{\infty} X[k] \cdot e^{-jn\left(\frac{2\pi}{N}\right) \cdot k}$ where $N = 4$

(a) $a_n = \frac{1}{4}\left[X[0] \cdot + X[2] \cdot e^{-jn \cdot \frac{2\pi}{4} \cdot 2}\right] = \left[X[0] + X[2] \cdot e^{-j\pi n}\right] \cdot \frac{1}{4}$

$a_0 = \left[X[0] + X[2] \cdot e^{0}\right] \cdot \frac{1}{4} = \frac{1}{2}$

$a_1 = \left[X[0] + X[2] \cdot e^{-j\pi}\right] \frac{1}{4} = \frac{1}{4}\left[1 + 1 \cdot (\cos\pi - j\sin\pi)\right] = 0$

$a_2 = \left[X[0] + X[2] \cdot e^{-j\pi \cdot 2}\right] \frac{1}{4} = \frac{1}{2}$

$a_3 = \left[X[0] + X[2] \cdot e^{-j\pi 3}\right] \frac{1}{4} = 0$

(b) $a_n = \left[X[0] + X[1] \cdot e^{-jn\frac{2\pi}{4} \cdot 1} + X[2] \cdot e^{-jn\frac{2\pi}{4} \cdot 2} + X[3] \cdot e^{-jn\frac{2\pi}{4} \cdot 3}\right] \frac{1}{4}$

$= \left[1 + 1 \cdot e^{-j\frac{\pi}{2} \cdot n} - e^{-j\pi n} - e^{-j\frac{3\pi}{2} \cdot n}\right] \frac{1}{4}$

$a_0 = (1 + 1 \cdot e^{0} - e^{0} - e^{0}) \frac{1}{4} = 0$

$a_1 = (1 + e^{-j\frac{\pi}{2}} - e^{-j\pi} - e^{-j\frac{3\pi}{2}}) \frac{1}{4}$

$= [1 + (0 - j) - (-1) - (-(-j))] \frac{1}{4} = \frac{1}{2} - j\frac{1}{2}$

$a_2 = [1 + e^{-j\pi} - e^{-j2\pi} - e^{-j3\pi}] \frac{1}{4}$

$= 0$

$a_3 = [1 + e^{-j\frac{3}{2}\pi} - e^{-j3\pi} - e^{-j\frac{9\pi}{2}}] \frac{1}{4}$

$= \frac{1}{2} + j\frac{1}{2}$

12. An ideal lowpass digital filter has the frequency response function

$$H(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases} \quad \text{and} \quad H(e^{j\omega}) = H(e^{j(\omega+2\pi)}).$$

(a) Determine the unit-pulse response $h[n]$ of the filter.

(b) Compute the output response y when the input x is given by

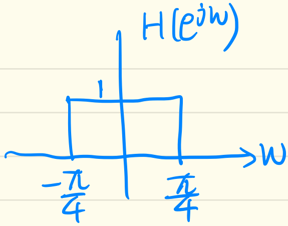
i. $x[n] = \cos(\pi n/8)$

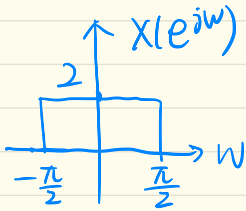
ii. $x[n] = \cos(3\pi n/4) + \cos(\pi n/16)$

iii. $x[n] = \mathrm{sinc}(n/2)$

You may use the following DTFT pair, for any $B > 0$:

$$\frac{B}{\pi}\,\mathrm{sinc}\left(\frac{Bn}{\pi}\right) \longleftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\omega + 2\pi k).$$





(a) inverse DTFT : $h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(e^{jw}) \cdot e^{jnw}\, dw$

$h[n] = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 \cdot e^{jnw}\, dw$

$= \frac{1}{2\pi}\left[\frac{e^{jnw}}{jn}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2\pi}\frac{1}{jn}\left[e^{j\frac{\pi}{4}n} - e^{j-\frac{\pi}{4}n}\right]$

$= \frac{1}{2\pi jn}\cdot\left[2j \cdot \sin\left(\frac{\pi}{4}n\right)\right) = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n} = \frac{1}{4}\,\mathrm{sinc}\left(\frac{n}{4}\right)$

(b) Lowpass filter only pass the freqnecy $w \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

i. $w = \frac{\pi}{8}$ $\rightarrow$ $y[n] = \cos\left(\frac{\pi n}{8}\right)$

ii. $w_1 = \frac{3\pi}{4}$, $w_2 = \frac{\pi}{16}$ $\rightarrow$ $y[n] = \cos\left(\frac{\pi n}{16}\right)$

iii. when $B = \frac{\pi}{2}$, $\frac{1}{2}\,\mathrm{sinc}\left(\frac{n}{2}\right) \Leftrightarrow \sum_{k=-\infty}^{\infty} \cdot P_{2B}^{\ \pi} \cdot (w + 2\pi k)$

$\Rightarrow X(e^{jw}) = 2 \cdot \sum_{k=-\infty}^{\infty} \cdot P_{\pi} \cdot (w + 2\pi k)$

$Y(e^{jw}) = X(e^{jw})\, H(e^{jw}) = \begin{cases} 2 & 0 \le |w| \le \frac{\pi}{4} \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow y[n] = 2 \cdot h[n] = \frac{1}{2}\,\mathrm{sinc}\left(\frac{n}{4}\right)$

13. Consider the discrete-time system given by the input-output difference equation

$$y[n+1]+0.9y[n]=1.9x[n+1].$$

(a) Use the input signal $x[n]=e^{j\omega n}$ to determine the system's frequency response $H(e^{j\omega})$.

(b) Obtain the unit pulse response for the system and use it to confirm your answer for part (a).

(a) $x[n]=e^{jwn}$ $\quad x[n+1]=e^{jw(n+1)}$

Assume the system is stable LTI.

$y[n]=H(e^{jw})\cdot e^{jwn}$ $\quad y[n+1]=H(e^{jw})\cdot e^{jw(n+1)}$

$$\Rightarrow H(e^{jw})\cdot e^{jw(n+1)}+0.9\,H(e^{jw})\cdot e^{jwn}=1.9\cdot e^{jw(n+1)}$$

$$H(e^{jw})=\frac{1.9e^{jw(n+1)}}{e^{jw(n+1)}+0.9e^{jwn}}=\frac{1.9e^{jw}}{e^{jw}+0.9}$$

(b) Telescope:

$$y[n+1]=1.9x[n+1]-0.9y[n]$$
$$=1.9\delta[n+1]-0.9y[n]$$

since the system is causal system, $y[n]=0$ when $n<0$

when $n=-1$, $y[0]=1.9\delta[0]-0.9y[-1]=1.9$

$n=0$, $y[1]=1.9\delta[1]-0.9y[0]=(-0.9)\cdot 1.9$

$n=1$, $y[2]=1.9\delta[2]-0.9y[1]=(-0.9)^2\cdot 1.9$

$\vdots$

$\Rightarrow y[n]=(-0.9)^n\cdot 1.9$ when $n\geq 0$

$\Rightarrow h[n]=1.9\cdot(-0.9)^n\cdot u[n]$

$$\text{DTFT}:\ H(e^{jw})=\sum_{n=0}^{\infty}1.9(-0.9)^n e^{-jwn}$$

$$=1.9\cdot\sum_{n=0}^{\infty}(-0.9e^{-jw})^n=1.9\cdot\frac{1}{1-(-0.9e^{-jw})}$$

$$=1.9\,\frac{1}{1+0.9e^{-jw}}=\text{answer in (a)}$$

14. An LTI discrete-time system with unit pulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in cascade with another causal LTI discrete-time system with unit pulse response $h_2[n]$. The resulting cascaded connection has frequency response

$$H(e^{j\omega}) = \frac{3\sin(1.5\omega)}{(3-e^{-j\omega})\sin(\omega/2)}.$$

Find $h_2[n]$.

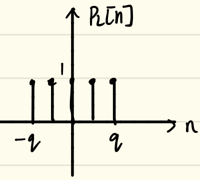
$$H_1(e^{jw}) = \sum_{n=-\infty}^{\infty} h_1[n]e^{-jwn} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\cdot e^{-jw}\right)^n$$

$$= \frac{1}{1-\frac{1}{3}e^{-jw}} = \frac{3}{3-e^{-jw}}$$

$$H(e^{jw}) = H_1(e^{jw})\,H_2(e^{jw}) \qquad \text{Cascade}$$

$$H_2(e^{jw}) = \frac{H(e^{jw})}{H_1(e^{jw})} = \frac{\sin(1.5w)}{\sin(\frac{w}{2})}$$

Recall:



$$P_L[n] = \begin{cases} 1 & -q \le n \le q \\ 0 & \text{otherwise} \end{cases}$$

$$P_L(e^{jw}) = \sum_{-q}^{q} 1\cdot e^{-jwn} = e^{jwq} + e^{jw(q-1)} + \dots + e^{-jw(q-1)} + e^{-jwq}$$

$$= e^{jwq}\left(\sum_{n=0}^{2q} e^{-jwn}\right)$$

$$= e^{jwq}\left(\frac{1-e^{-j(2q+1)w}}{1-e^{-jw}}\right) = \frac{\sin(\frac{2q+1}{2}w)}{\sin(\frac{w}{2})}$$

$$H_2(e^{jw}) = \frac{\sin(\frac{3}{2}w)}{\sin(\frac{w}{2})} \rightarrow q = 1$$

$$h_2[n] = P_L[n] = \begin{cases} 1 & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta[n+1] + \delta[n] + \delta[n-1]$$