93. Use Kirchoff's laws to find the state space model for the circuit below. Use state variables $x_1(t) = i_L(t)$, $x_2(t) = v_C(t)$ and with the output defined as $y(t) = i_L(t) + V_C(t)$.

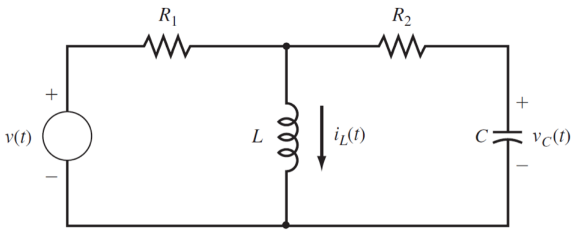


Figure 1: Diagram for Problem 93

$\dot{i}_L(t)$? $\dot{V}_c(t)$?

KVL: $V_L(t) = L\dot{i}_L(t) = V_c(t) + V_{R_2}(t)$

$= V_c(t) + C\dot{V}_c(t)\cdot R_2$ ①

KCL: $\frac{V(t) - L\dot{i}_L(t)}{R_1} = i_L(t) + C\dot{V}_c(t) \Rightarrow V(t) - L\dot{i}_L(t) = R_1 i_L(t) + CR_1\dot{V}_c(t)$ ②

From ②: $L\dot{i}_L(t) = V(t) - R_1 i_L(t) - CR_1\dot{V}_c(t)$ ③

take ③ into ①.

$V(t) - R_1 i_L(t) - CR_1\dot{V}_c(t) = V_c(t) + C\dot{V}_c(t)\cdot R_2$

$C(R_1+R_2)\dot{V}_c(t) = V(t) - R_1 i_L(t) - V_c(t)$

$\Rightarrow \dot{V}_c(t) = \frac{-R_1}{C(R_1+R_2)} i_L(t) + \frac{-1}{C(R_1+R_2)} V_c(t) + \frac{1}{C(R_1+R_2)} V(t)$

② × R_2: $CR_1R_2\dot{V}_c(t) = V(t)R_2 - LR_2\dot{i}_L(t) - R_1R_2 i_L(t)$ ④

① × R_1: $CR_1R_2\dot{V}_c(t) = LR_1\dot{i}_L(t) - R_1V_c(t)$ ⑤

combine ④ & ⑤

$L(R_1+R_2)\dot{i}_L(t) = V(t)R_2 - R_1R_2 i_L(t) + R_1V_c(t)$

$\Rightarrow \dot{i}_L(t) = \frac{-R_1R_2}{L(R_1+R_2)} i_L(t) + \frac{R_1}{L(R_1+R_2)} V_c(t) + \frac{R_2}{L(R_1+R_2)} V(t)$

$\Rightarrow \begin{bmatrix} \dot{i}_L(t) \\ \dot{V}_c(t) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} i_L(t) \\ V_c(t) \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix} V(t)$

95. (a) For the two-input two-output continuous-time system in Figure 2 shown below, find a state model that has the least number of state variables.

(b) Consider the corresponding discrete-time system, in which the integrator blocks are replaced by unit delays, and all the inputs and outputs are discrete-time signals. Find a state model for this discrete-time system.

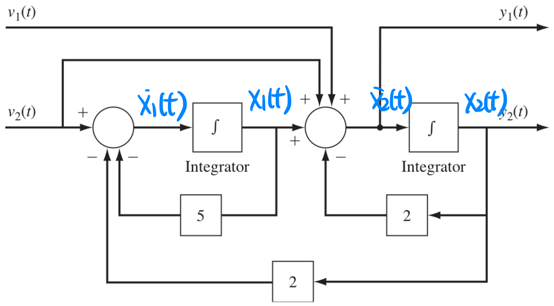


Figure 2: Diagram for Problem 95

$$\begin{cases} \dot{x}_1(t) = v_2(t) - 5x_1(t) - 2x_2(t) \\ \dot{x}_2(t) = x_1(t) + v_2(t) + v_1(t) - 2x_2(t) \end{cases} \qquad \begin{cases} y_1(t) = \dot{x}_2(t) \\ y_2(t) = x_2(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

96. A two-car system has state space representation

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}\frac{-k_f}{M} & 0 & 0\\ 0 & \frac{-k_f}{M} & 0\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{M} & 0\\ 0 & \frac{1}{M}\\ 0 & 0\end{bmatrix}\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}$$

Assume $k_f = 10$, $M = 1000$ and $x(0) = [60\ 60\ 100]^T$. Find the forces $f_1(t)$ and $f_2(t)$ that must be applied to the car to obtain $x(t) = [60\ 60\ 100]^T$ for all $t \geq 0$.

$X(t) = [60\ 60\ 100]^T = X(0)$

$\Rightarrow$ The rate of the parameters are 0

$\Rightarrow$ $\dot{X}(t) = 0$ for $t \geq 0$

$$\begin{cases} \dot{X}_1(t) = -0.01\,X_1(t) + 0.001\,f_1(t) \\ \dot{X}_2(t) = -0.01\,X_2(t) + 0.001\,f_2(t) \\ \dot{X}_3(t) = -X_1(t) + X_2(t) \end{cases}$$

$$\begin{cases} \dot{X}_1(t) = -0.01 \times 60 + 0.001\,f_1(t) = 0 \\ \dot{X}_2(t) = -0.01 \times 60 + 0.001\,f_2(t) = 0 \\ \dot{X}_3(t) = -60 + 60 = 0 \end{cases} \Rightarrow \begin{cases} f_1(t) = 600 \\ f_2(t) = 600 \end{cases} \text{for all } t \geq 0$$

98. A linear time-invariant continuous-time system has a state space representation (A, B, C) where

$$A = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

(a) Compute the state transition matrix e^{At}.

(b) Compute the response y_{zi} arising from the zero input and initial state $x(0) = [1 \ -1]^T$.

(a) $\mathcal{L}[e^{At}] = \phi(S) = (SI - A)^{-1}$

$$(SI-A)^{-1} = \begin{bmatrix} S & 2 \\ 0 & S+1 \end{bmatrix}^{-1} = \frac{1}{S(S+1)} \begin{bmatrix} S+1 & 2 \\ 0 & S \end{bmatrix} = \begin{bmatrix} \frac{1}{S} & \frac{-2}{S(S+1)} \\ 0 & \frac{1}{S+1} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}[\phi(S)] = \begin{bmatrix} 1 & 2(1-e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$$

(b) zero input solution : $X(t) = e^{At} X(0)$

$$= \begin{bmatrix} 1 & 2(1-e^{-t}) \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2(1-e^{-t}) \\ -e^{-t} \end{bmatrix}$$

$$y_{zi}(t) = CX(t) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1-2(1-e^{-t}) \\ -e^{-t} \end{bmatrix} = 1-2(1-e^{-t}) - 3e^{-t}$$

$$= -1 - e^{-t} \quad \text{for } t \geq 0$$

100. The motion of a car moving on a level surface may be described by the differential equation

$$\frac{d^2y(t)}{dt} + \frac{k_f}{M}\frac{dy(t)}{dt} = \frac{1}{M}v(t)$$

where $y(t)$ is the position of the car at time t, $v(t)$ is the applied force and M and k_f are constants. Assume $M = 1$ and $k_f = 0.1$.

(a) Obtain a state representation (A, B, C) for the system in controller canonical form.

(b) Obtain the zero-input solution when the initial state is $x(0) = [y_0\ v_0]^T$, for some constants y_0 and v_0.

(c) The motion of the car is such that $y(10) = 0$ and $\dot{y}(10) = 55$. Find $y(0)$ and $\dot{y}(0)$.

(d) A force of $v(t) = 1$ is applied to the car for time $0 \le t \le 5$. The state $x(5) = [50\ 20]^T$. Find $x(0)$.

Note: parts (c) and (d) are unrelated questions.

$$\ddot{y} + 0.1\dot{y} = v$$

(a) let $x_1 = y$, $x_2 = \dot{y}$

$\dot{x}_1 = \dot{y} = x_2$, $\quad \dot{x}_2 = \ddot{y} = v - 0.1\dot{y} = v - 0.1x_2$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ 0 & -0.1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B} v$$

$$y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

(b) $X(t) = e^{At}X(0)$ $\qquad$ Note that: $\mathcal{L}[e^{At}] = (sI - A)^{-1}$

$$(sI-A)^{-1}X(0) = \begin{bmatrix}s-0 & -1\\ 0 & s+0.1\end{bmatrix}^{-1}\begin{bmatrix}y_0\\ v_0\end{bmatrix} = \frac{1}{s(s+0.1)}\begin{bmatrix}s+0.1 & 1\\ 0 & s\end{bmatrix}\begin{bmatrix}y_0\\ v_0\end{bmatrix}$$

$$= \frac{1}{s(s+0.1)}\begin{bmatrix}(s+0.1)y_0 + v_0\\ s v_0\end{bmatrix} = \begin{bmatrix}\frac{y_0}{s} + \frac{v_0}{s(s+0.1)}\\ \frac{v_0}{s+0.1}\end{bmatrix}$$

Take the inverse Laplace transform

$$X(t) = \begin{bmatrix}y_0 + 10v_0(1 - e^{-0.1t})\\ v_0 e^{-0.1t}\end{bmatrix} \quad \text{for } t \ge 0$$

100. The motion of a car moving on a level surface may be described by the differential equation

$$\frac{d^2y(t)}{dt} + \frac{k_f}{M}\frac{dy(t)}{dt} = \frac{1}{M}v(t)$$

where $y(t)$ is the position of the car at time t, $v(t)$ is the applied force and M and k_f are constants. Assume $M = 1$ and $k_f = 0.1$.

(a) Obtain a state representation (A, B, C) for the system in controller canonical form.

(b) Obtain the zero-input solution when the initial state is $x(0) = [y_0\ v_0]^T$, for some constants y_0 and v_0.

(c) The motion of the car is such that $y(10) = 0$ and $\dot{y}(10) = 55$. Find $y(0)$ and $\dot{y}(0)$.

(d) A force of $v(t) = 1$ is applied to the car for time $0 \le t \le 5$. The state $x(5) = [50\ 20]^T$. Find $x(0)$.

Note: parts (c) and (d) are unrelated questions.

(c) $y(t) = CX(t) = [1 \quad 0]X(t) = y_0 + 10V_0(1 - e^{-0.1t}) \quad t \geq 0$

$\dot{y}(t) = V_0 e^{-0.1t}$

Given $y(10) = 0 \Rightarrow y_0 + 10V_0(1 - e^{-1}) = 0$

$\dot{y}(10) = 55 \Rightarrow V_0 e^{-1} = 55$

$\Rightarrow \begin{cases} V_0 = 55e \\ y_0 = 550(1-e) \end{cases}$

Therefore,

$y(0) = y_0 = 550(1-e) \qquad \dot{y}(0) = V_0 = 55e$

(d) $X(S) = (SI - A)^{-1}X(0) + (SI - A)^{-1}BU(S)$

$$(SI - A)^{-1}BU(S) = \frac{1}{S(S+0.1)}\begin{bmatrix} S+0.1 & 1 \\ 0 & S \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{S} = \begin{bmatrix} \frac{1}{S^2(S+0.1)} \\ \frac{1}{S(S+0.1)} \end{bmatrix}$$

Take the inverse laplace transform.

$$X(t) = \begin{bmatrix} y_0 + 10V_0(1 - e^{-0.1t}) + 10t - 100 + 100e^{-0.1t} \\ V_0 e^{-0.1t} + 10 - 10e^{-0.1t} \end{bmatrix}$$

$$X(5) = \begin{bmatrix} y_0 + 10V_0(1 - e^{-0.5}) + 50 - 100 + 100e^{-0.5} \\ V_0 e^{-0.5} + 10 - 10e^{-0.5} \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \end{bmatrix} \Rightarrow \begin{cases} V_0 = 26.49 \\ y_0 = -64.87 \end{cases}$$

$$\therefore X(0) = \begin{bmatrix} y_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} -64.87 \\ 26.49 \end{bmatrix}$$

101. (a) Let A and $\bar{A}$ and be square matrices such that

$$A = P\bar{A}P^{-1}$$

for some invertible matrix P. Show that A and $\bar{A}$ have the same eigenvalues.

(b) Let A and $\bar{A}$ be given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

Assume there exists a transformation P satisfying

$$A = P\bar{A}P^{-1}$$

Determine the values of a_0, a_1 and a_2.

(c) Find the matrix P.

(a) $AV = \underset{\uparrow\text{ eigenvalue}}{\lambda} V \leftarrow$ eigenvector

$\Rightarrow P\bar{A}P^{-1}V = \lambda V$

Multiply P^{-1} in the left: $P^{-1}P\bar{A}P^{-1}V = \lambda P^{-1}V$

$\Rightarrow \bar{A}P^{-1}V = \lambda P^{-1}V$

Let $\tilde{V} = P^{-1}V \Rightarrow \bar{A}\tilde{V} = \lambda\tilde{V}$

$\Rightarrow \lambda$ also are eigenvalues of $\bar{A}$

(b) $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ since A is diagonal matrix $\Rightarrow$ eigenvalues $\lambda = -2, -1, 1$

The determinant of $(\bar{A} - \lambda I)$ is

$$\det(\bar{A} - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2-\lambda \end{bmatrix}$$

$$= \lambda^2(-a_2-\lambda) + (-a_0) + 0 - 0 - 0 - a_1\lambda$$

$\lambda = -2$: $4(-a_2+2) - a_0 + 2a_1 = 0$

$\lambda = -1$: $(-a_2+1) - a_0 + a_1 = 0$

$\lambda = 1$: $(-a_2-1) - a_0 - a_1 = 0$

$\Rightarrow \begin{cases} a_0 = -2 \\ a_1 = -1 \\ a_2 = 2 \end{cases}$

101. (a) Let A and $\bar{A}$ and be square matrices such that

$$A = P\bar{A}P^{-1}$$

for some invertible matrix P. Show that A and $\bar{A}$ have the same eigenvalues.

(b) Let A and $\bar{A}$ be given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

Assume there exists a transformation P satisfying

$$A = P\bar{A}P^{-1}$$

Determine the values of a_0, a_1 and a_2.

(c) Find the matrix P.

(c)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \Rightarrow \text{eigenvalues } \lambda = -2, -1, 1$$

$A V_1 = \lambda_1 V_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \begin{cases} y_1 = -2x_1 \\ z_1 = -2y_1 \\ 2x_1 + y_1 - 2z_1 = -2z_1 \end{cases}$$

Assume $x_1 = -1$, $\Rightarrow y_1 = 2 \Rightarrow z_1 = -4$

$A V_2 = \lambda_2 V_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = -1\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \qquad \begin{cases} y_2 = -x_2 \\ z_2 = -y_2 \\ 2x_2 + y_2 - 2z_2 = -z_2 \end{cases}$$

Assume $x_2 = -1$, $\Rightarrow y_2 = 1$, $z_3 = -1$

$A V_3 = \lambda_3 V_3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_4 \\ z_3 \end{bmatrix} = 1\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \qquad \begin{cases} y_3 = x_3 \\ z_3 = y_3 \\ 2x_3 + y_3 - 2z_3 = z_3 \end{cases}$$

Assume $x_3 = -1 \Rightarrow y_3 = z_3 = -1$

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & -1 \\ -4 & -1 & -1 \end{bmatrix}^{-1}$$

105. Suppose a system has input-output transfer function

$$H(s) = \frac{32}{s^2 + 8s + 16}$$

(a) Give an input-output differential equation that describes the system. Use v and y as the input and output variables, respectively.

(b) Determine whether the system is critically damped, overdamped or underdamped.

(c) Find the steady-state value of the step response. Hence plot the graph of the step response of the system assuming zero initial conditions. (You do not need to solve for $y(t)$).

(d) Find the steady-state sinusoidal response when the input is $v(t) = \cos(4t + \pi)$.

(a) $H(S) = \frac{Y(S)}{V(S)} = \frac{32}{S^2+8S+16}$

$\Rightarrow S^2Y(S) + 8SY(S) + 16Y(S) = 32V(S)$ Take the inverse Laplace Transform

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y(t) = 32v(t)$$

(b) Damping ratio ε

$H(S) = \frac{32}{S^2+8S+16} = \frac{\omega_n^2}{S^2+2\varepsilon\omega_n S+\omega_n^2}$ $\Rightarrow \omega_n^2 = 16 \Rightarrow \omega_n = 4$

$\varepsilon = \frac{8}{2\omega_n} = 1$ $\Rightarrow$ critically damped

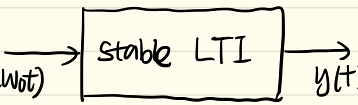
(c) Determine whether the system is stable

$P_1, P_2 = -4 \Rightarrow$ stable

Final value theorem $y(\infty) = \lim_{S\to 0} SY(S) = \lim_{S\to 0} H(S) = \frac{32}{16} = 2$

(d)

$v(t) = \cos(\omega_0 t)$



$y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$

$\omega_0 = 4$ $\quad H(j4) = \frac{32}{-16+8\times 4j+16} = \frac{1}{j} = 1\angle -\frac{\pi}{2} = |H(j\omega_0)| \angle H(j\omega_0)$

$\Rightarrow y(t) = 1\times\cos(4t+\pi-\frac{\pi}{2}) = \cos(4t+\frac{\pi}{2})$