Question 9 3 pts

Consider the discrete time system given by the difference equation

$y[n] + \frac{1}{6}y[n-1] + \frac{1}{6}y[n-2] = 2v[n] + \frac{1}{6}v[n-1]$. Suppose that the output signal is given by $y[n] = \frac{11}{6}\cos(\pi n - \frac{\pi}{6})$. Then the input signal is

① Determine whether the system is stable

Taking the Z tranform on both side

$$Y(z) + \frac{1}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = 2V(z) + \frac{1}{6}z^{-1}V(z)$$

$$H(z) = \frac{Y(z)}{V(z)} = \frac{2+\frac{1}{6}z^{-1}}{1+\frac{1}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{2z^2+\frac{1}{6}z}{z^2+\frac{1}{6}z+\frac{1}{6}}$$

let $z^2+\frac{1}{6}z+\frac{1}{6}=0 \Rightarrow p_{1,2} = -\frac{1}{12} \pm j\frac{1}{12}\sqrt{23}$

since $|p_1|^2 = |p_2|^2 = \frac{24}{144} < 1 \Rightarrow$ the system is stable

② $$\cos(w_0 n+\phi) \longrightarrow \boxed{\begin{matrix}\text{Stable}\\ \text{LTI}\end{matrix}} \longrightarrow |H(e^{jw_0})|\cdot\cos(w_0 n+\phi+\angle H(e^{jw_0}))$$

For $w_0 = \pi$, we have $H(e^{j\pi}) = H(-1) = \dfrac{2-\frac{1}{6}}{1-\frac{1}{6}+\frac{1}{6}} = \frac{11}{6}$

$$y[n] = |H(e^{j\pi})|\cdot\cos(\pi n+\phi+\angle H(e^{j\pi}))$$

$$= \frac{11}{6}\cdot\cos(\pi n+\phi+0) \Rightarrow \phi = -\frac{\pi}{6}$$

$$\Rightarrow v[n] = \cos(w_0 n+\phi) = \cos(\pi n - \frac{\pi}{6})$$

Question 1: (26 points)
Consider the discrete-time system Σ_1 given by the following state representation:

$$x[n+1] = \overset{A}{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}} x[n] + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} v[n]$$
$$y[n] = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} x[n] \qquad D$$

(a) (3 points) Find the transfer function of the system Σ_1, showing all working.

(b) (2 points) Is the system Σ_1 stable? Explain your answer.

(c) (3 points) Describe the system Σ_1 by a difference equation of the form:

$$y[n+N] + \sum_{i=0}^{N-1} a_i y[n+i] = \sum_{i=0}^{M} b_i v[n+i],$$

assuming zero initial conditions. Show all working, making it clear what each of the coefficients a_i for all $i = 0, \ldots, N-1$ and b_j for all $j = 0, \ldots, M$ are.

(a) $H(s) = C(sI-A)^{-1}B + D \iff H(z) = C(zI-A)^{-1}B + D$

$$H(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z-2 & 1 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{(z-2)(z-1)} \begin{bmatrix} z-1 & -1 \\ 0 & z-2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{z-2}{(z-2)(z-1)}$$

$$= \frac{z-2}{z^2-3z+2}$$

(b) $P_1 = 2 \quad P_2 = 1$

$\Rightarrow P_1 = 2$ outside the unit circle $\rightarrow$ The system is not stable

(c) $H(z) = \dfrac{Y(z)}{V(z)} = \dfrac{z-2}{z^2-3z+2}$

$$Y(z)(z^2-3z+2) = V(z)(z-2)$$

Taking inverse z-tromstorm

$$y[n+2] - 3y[n+1] + 2y[n] = v[n+1] - 2v[n]$$

(d) (3 points) Now suppose that the input signal v is chosen to be a linear combination of the state signal components, as follows: for all $n \in \mathbf{Z}_+$ let

$$v[n] = \begin{bmatrix} K_1 & K_2 \end{bmatrix} x[n],$$

where K_1 and K_2 are constants. Let's now consider the system Σ_2 that results from inputting v to system Σ_1. Then Σ_2 can be described by a state representation of the following form:

$$x[n+1] = A_f x[n]$$
$$y[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} x[n]$$

Determine the matrix A_f, showing all workings.

(e) (6 points) Determine the values of K_1 and K_2 for which the system Σ_2 of part (d) has all its poles at the origin. Show all working.

(d) $x[n+1] = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} x[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[n]$

$= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} x[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \ K_2]\, x[n]$

$= \left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} \right) x[n] = \underbrace{\begin{bmatrix} 2 & -1 \\ K_1 & K_2+1 \end{bmatrix}}_{A_f} x[n]$

(e) The poles of a LTI discrete time system are those $z \in \mathbb{C}$ such that $\det(zI - A) = 0$

$\det(zI - A_f) = \det\left(\begin{bmatrix} z-2 & 1 \\ -K_1 & z-1-K_2 \end{bmatrix} \right)$

$= (z-2)(z-1-K_2) - (-K_1)$

$= z^2 - (K_2+3)z + (2+2K_2+K_1) = 0$

For all the poles $P_1, P_2 = 0 \iff \det(zI - A_f) = z^2$

$\Rightarrow \begin{cases} K_2 + 3 = 0 \\ 2 + 2K_2 + K_1 = 0 \end{cases} \Rightarrow \begin{cases} K_2 = -3 \\ K_1 = 4 \end{cases}$

poles at the origin. Show all working.

(f) (1 point) Assume that K_1 and K_2 have values as in part (e). Looking at your answer to part (b), what do you notice about the system Σ_2 of part (d), as compared to the system Σ_1?

(g) (3 points) Again assume that K_1 and K_2 have values as in part (e). Specify the output response $y[n]$ of the system Σ_2 of part (d) for all $n \geq 0$, using the initial condition

$$x[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Show all working.

(f) system Σ_2 is now stable, as all poles are inside the unit circle.

(g) since $K_1 = 4$, $K_2 = -3$, we have $A_f = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$

$$\begin{cases} x[1] = A_f\, x[0] = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ x[2] = A_f\, x[1] = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ x[3] = A_f\, x[2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \cdots \end{cases}$$

$$\begin{cases} y[0] = [0 \ 1]\, x[0] = 1 \\ y[1] = [0 \ 1]\, x[1] = 2 \\ y[2] = [0 \ 1]\, x[2] = 0 \end{cases} \qquad y[3] = 0 \ \cdots$$

Therefore, $y[n] = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 0, & n \geq 2 \end{cases}$

Show all working.

(h) (5 points) You have encountered an unknown discrete-time system, but then you run an experiment and observe its unit pulse response to be:

$$h[n] = \begin{cases} 0, & n \leq 1 \\ 3, & n = 2 \\ 0, & n \geq 3 \end{cases}$$

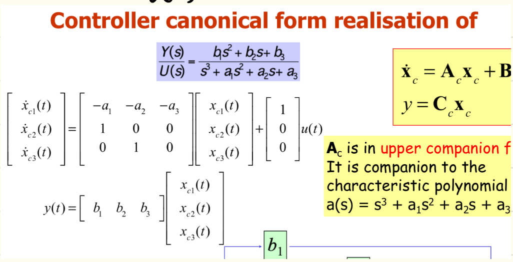
Obtain the state representation of this system in controller canonical form, showing all steps.

The unit pulse response: $h[n] = 3 \cdot \delta[n-2]$

① Taking the z-transform, $H(z) = 3 \cdot z^{-2}$

Since $H(z) = \frac{Y(z)}{U(z)}$, we have $\frac{Y(z)}{U(z)} = 3 \cdot z^{-2}$

② Since $\frac{Y(z)}{U(z)} = \frac{3}{z^2} = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$



$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

③ double check.

$\frac{Y(z)}{V(z)} = \frac{3}{z^2}$

$\Rightarrow y[n+2] = 3v[n]$

From the state representation

$y[n] = 3x_2[n]$

$\Rightarrow y[n+1] = 3x_2[n+1]$

$= 3x_1[n]$

$\Rightarrow y[n+2] = 3 \cdot x_1[n+1]$

$= 3 \cdot v[n]$ ✓