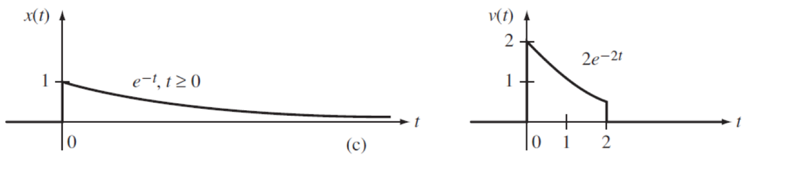
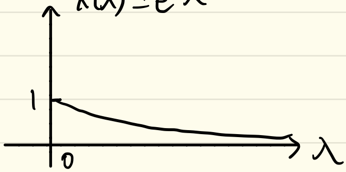
20. For the continuous-time signals shown in Figure 1 below, compute the convolution $(x \star v)(t)$ for all $t \geq 0$, and plot the resulting signal.



$$y(t) = x(t) * v(t) = \int_{-\infty}^{\infty} x(\lambda)\, v(t-\lambda)\, d\lambda$$

$x(\lambda) = e^{-\lambda}$

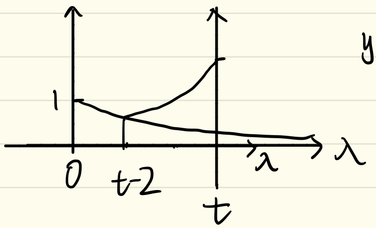


$0 \leq t \leq 2$:

$$y(t) = \int_0^t x(\lambda)\, v(t-\lambda)\, d\lambda$$
$$= \int_0^t e^{-\lambda}\, 2 \cdot e^{-2(t-\lambda)}\, d\lambda$$
$$= 2e^{-2t} \int_0^t e^{\lambda}\, d\lambda$$
$$= 2e^{-2t}(e^t - 1) = 2e^{-t}(1 - e^{-t})$$

$t \geq 2$

$$y(t) = \int_{t-2}^{t} e^{-\lambda} \cdot 2 \cdot e^{-2(t-\lambda)}\, d\lambda$$
$$= 2e^{-2t} \int_{t-2}^{t} e^{\lambda}\, d\lambda$$
$$= 2e^{-2t}(e^t - e^{t-2})$$
$$= 2e^{-t}(1 - e^{-2})$$



$$\Rightarrow (x * v)(t) = \begin{cases} 2e^{-t}(1 - e^{-t}), & 0 \leq t \leq 2 \\ 2e^{-t}(1 - e^{-2}), & t \geq 2 \end{cases}$$

22. A continuous-time linear time-invariant system has the input/output relationship

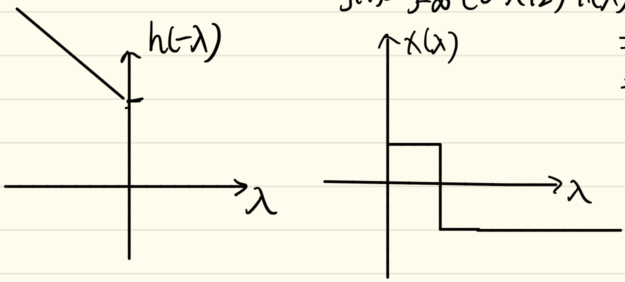
$$y(t) = \int_{-\infty}^{t} (t - \lambda + 2)x(\lambda)\, d\lambda$$

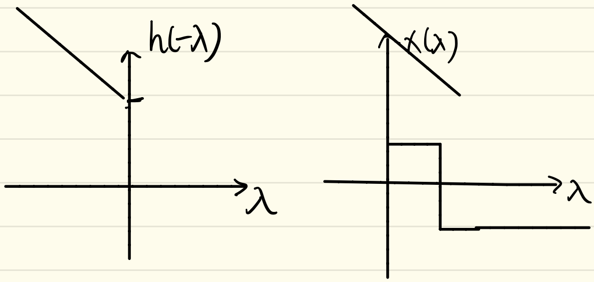
(b) For $1 \le t \le 2$, determine the output $y(t)$ arising from the input $x(t) = u(t) - 2u(t-1)$.

$$y(t) = \int_{-\infty}^{t} (t-\lambda+2)x(\lambda)d\lambda = \int_{-\infty}^{t} h(t-\lambda)x(\lambda)\, d\lambda$$

$$\Rightarrow h(t-\lambda) = t-\lambda+2$$

$$\Rightarrow h(t) = t+2$$





for $1 \le t \le 2$, $y(t) = \int_0^1 (t-\lambda+2)\cdot 1\, d\lambda + \int_1^t (t-\lambda+2)\cdot(-1)d\lambda$

$$= \left[(t+2)\lambda - \frac{\lambda^2}{2}\right]_0^1 - \left[(t+2)\lambda - \frac{\lambda^2}{2}\right]_1^t$$

$$= -\frac{t^2}{2} + 3$$

24. Assume x and h are continuous-time signals such that

$$x(t) = 0, \text{for all } |t| \geq T_1, \quad h(t) = 0, \text{for all } |t| \geq T_2$$

for some positive real numbers T_1 and T_2. Show that

$$(x \star h)(t) = 0, \text{for all } |t| \geq T_3$$

where $T_3 = T_1 + T_2$.

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$x(\lambda) \neq 0$ in $-T_1 \leq \lambda \leq T_1$ $\qquad$ $h(t-\lambda) \neq 0$ in $-T_2 \leq t-\lambda \leq T_2$

$\Rightarrow \lambda - T_2 \leq t \leq \lambda + T_2$

Need to maximize the range

$\Rightarrow -T_1 - T_2 \leq t \leq T_1 + T_2$

$\Rightarrow -T_3 \leq t \leq T_3$ $\qquad$ $(x*h)(t) \neq 0$

26. (a) Use the definition of Laplace transform to show that

$$te^{-bt}u(t) \longleftrightarrow \frac{1}{(s+b)^2}$$

is a Laplace transform pair, for any $b \in \mathbf{R}$.

(b) Hence use the definition of the Laplace transform and Mathematical Induction to show that, for any integer $N \geq 1$,

$$t^N e^{-bt}u(t) \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

is a Laplace transform pair.

(a) $\int_0^\infty t e^{-bt} u(t) e^{-st} dt$

$= \int_0^\infty \frac{t}{-(b+s)} d e^{-(b+s)t} = \left[\frac{t}{-(b+s)} e^{-(b+s)t}\right]_0^\infty - \int_0^\infty e^{-(b+s)t} d \frac{t}{-(b+s)}$

$= 0 - \left[\frac{e^{-(b+s)t}}{(-(b+s))^2}\right]_0^\infty = \frac{1}{(s+b)^2}$

(b) when $N=1$, satisfied

Assume $N=n-1$, satisfied, we have $t^{n-1}e^{-bt}u(t) \Longleftrightarrow \frac{(n-1)!}{(s+b)^n}$

Now, we need to prove that $N=n$ also satisfied

Time Multiplication: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$

let $x(t) = t^{n-1}e^{-bt}u(t)$ $\quad y(t) = t \cdot x(t) = t^n e^{-bt} u(t)$

$\mathcal{L}[y(t)] = \mathcal{L}[t\,x(t)] = -\frac{d}{ds}X(s)$

$= -\frac{d}{ds}\left(\frac{(n-1)!}{(s+b)^n}\right)$

$= -1 \cdot (n-1)! \cdot (-n) \cdot \frac{1}{(s+b)^{n+1}}$

$= \frac{n!}{(s+b)^{n+1}}$

27. A continuous-time signal x has Laplace transform

$$X(s) = \frac{s+1}{s^2+5s+7}$$

In each case, find the LT of v:

(a) $v(t) = x(3t-4)u(3t-4)$

(b) $v(t) = tx(t)$

(c) $v(t) = \frac{d^2x}{dt^2}$, assuming $x(0^-) = 1$, $\dot{x}(0^-) = -4$.

(d) $v(t) = x(t)\sin(2t)$

(a) $V(t) = x(3(t-\frac{4}{3}))u(3(t-\frac{4}{3}))$

$$V(s) = \frac{1}{3}X(\frac{s}{3})e^{-\frac{4}{3}s} = \frac{1}{3}\cdot\frac{\frac{s}{3}+1}{\frac{s^2}{9}+\frac{5}{3}s+63}\cdot e^{-\frac{4}{3}s}$$

$$= \frac{1}{3}\cdot\frac{3s+9}{s^2+15s+63}\cdot e^{-\frac{4}{3}s}$$

$$= \frac{s+3}{s^2+15s+63}\cdot e^{-\frac{4}{3}s}$$

(b) $V(s) = -\frac{d}{ds}X(s)$

(c) $V(s) = s^2X(s) - sx(0^-) - \dot{x}(0^-)$

$$= s^2X(s) - s + 4$$

(d) $V(t) = x(t)\cdot\frac{1}{2j}(e^{j2t} - e^{-j2t})$

$$V(s) = \frac{1}{2j}[X(s-2j) - X(s+2j)]$$

28. Let $p = \sigma + j\omega \in \mathbf{C}$ for some real numbers σ and ω. Let $c \in \mathbf{C}$ and $t \in \mathbf{R}$. Show that

$$ce^{pt} + \bar{c}e^{\bar{p}t} = 2|c|e^{\sigma t}\cos(\omega t + \angle c)$$

Assume $c = a + jb = |c|e^{j\angle c}$, a, b are real numbers

$\Rightarrow a = |c|\cos\angle c \qquad b = |c|\sin\angle c$

$$ce^{pt} + \bar{c}e^{\bar{p}t} = (a+jb)e^{(\sigma+j\omega)t} + (a-jb)e^{(\sigma-j\omega)t}$$

$$= ae^{\sigma t}(e^{j\omega t} + e^{-j\omega t}) + jbe^{\sigma t}(e^{j\omega t} - e^{-j\omega t})$$

$$= ae^{\sigma t} \cdot 2\cos\omega t + jbe^{\sigma t}\, 2j\sin\omega t$$

$$= 2 \cdot e^{\sigma t}(a\cos\omega t - b\sin\omega t)$$

$$= 2 \cdot e^{\sigma t}(|c|\cos\angle c\cos\omega t - |c|\sin\angle c\sin\omega t)$$

$$= 2|c|e^{\sigma t}\cos(\omega t + \angle c)$$

30. Use partial fractions to obtain inverse Laplace transforms for the following functions.

(c) $X(s) = \dfrac{3s^2+2s+1}{(s+1)(s+2)^2}$

(c) $X(s) = \dfrac{C_1}{(s+1)} + \dfrac{C_2}{(s+2)^2} + \dfrac{C_3}{(s+2)}$

$= \dfrac{C_1(s+2)^2 + C_2(s+1) + C_3(s+1)(s+2)}{(s+1)(s+2)^2}$

Numerator: $C_1(s^2+4s+4) + C_2(s+1) + C_3(s^2+3s+2)$

$= (C_1+C_3)s^2 + (4C_1+C_2+3C_3)s + 4C_1+C_2+2C_3$

$= 3s^2+2s+1$

$\Rightarrow C_1 = 2 \quad C_2 = -9 \quad C_3 = 1$

$X(s) = \dfrac{2}{s+1} + \dfrac{-9}{(s+2)^2} + \dfrac{1}{(s+2)}$

$\Rightarrow x(t) = [2e^{-t} - 9te^{-2t} + e^{-2t}]\,u(t)$

32. Determine the stability of the LTI systems with the following impulse response functions.

(a) $h(t) = [2t^3 - 2t^2 + 3t - 2](u(t) - u(t-10))$

(b) $h(t) = \sin(2t)$

(c) $h(t) = e^{-t}\sin(2t)$

(d) $h(t) = e^{t}\sin(2t)$

$\int_0^\infty |h(t)|\,dt < \infty \Rightarrow$ h is absolutely integrable $\Rightarrow$ stable

(a) $\int_0^\infty |h(t)|\,dt = \int_0^{10} 2t^3-2t^2+3t-2\,dt$ = constant

$\Rightarrow$ stable

(b) $\int_0^\infty |h(t)|\,dt = \int_0^\infty |\sin(2t)|\,dt$ is bounded but not absolutely integrable

$\Rightarrow$ marginally stable

(c) $\int_0^\infty |h(t)|\,dt = \int_0^\infty e^{-t}|\sin(2t)|\,dt$ = constant

$\Rightarrow$ stable

(d) $\int_0^\infty e^{t}|\sin(2t)|\,dt \to \infty$

$\Rightarrow$ unstable