week 3

48. Compute the Fourier transforms of the following functions.

(a) $f_1(t) = \begin{cases} 0, & t < 0 \text{ and } t > 2 \\ -1, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \end{cases}$

(a) Solution1: $F_1(w) = \int_{-\infty}^{\infty} f_1(t) e^{-jwt}\, dt$

$$= \int_0^1 -e^{-jwt}\,dt + \int_1^2 e^{-jwt}\,dt$$

$$= \left[\frac{e^{-jwt}}{jw}\right]_0^1 + \left[\frac{e^{-jwt}}{-jw}\right]_1^2$$

$$= \frac{e^{-jw}-1}{jw} + \frac{e^{-jw2} - e^{-jw}}{-jw}$$

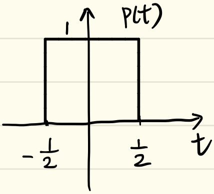
$$= \frac{e^{-j\frac{w}{2}}(e^{-j\frac{w}{2}} - e^{j\frac{w}{2}})}{jw} - \frac{e^{-j\frac{3w}{2}}(e^{-j\frac{w}{2}} - e^{j\frac{w}{2}})}{jw}$$

$$= \frac{e^{-j\frac{w}{2}}(-2j\sin\frac{w}{2})}{jw} - \frac{e^{-j\frac{3w}{2}}(-2j\sin\frac{w}{2})}{jw}$$

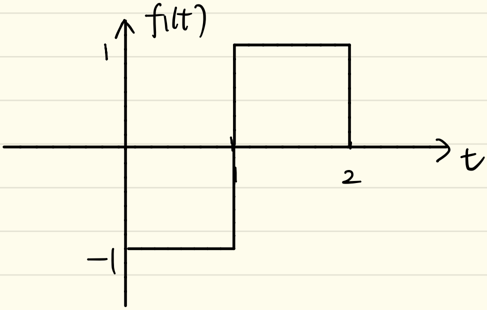
$$= \frac{2\sin\frac{w}{2}}{w} e^{-j\frac{3w}{2}} - \frac{2\sin\frac{w}{2}}{w} e^{-j\frac{w}{2}} \qquad \left(\operatorname{sinc}(aw) = \frac{\sin(a\pi w)}{a\pi w}\right)$$

$$= \operatorname{sinc}\left(\frac{w}{2\pi}\right) e^{-j\frac{3w}{2}} - \operatorname{sinc}\left(\frac{w}{2\pi}\right) e^{-j\frac{w}{2}}$$

Solution 2:



$$P(w) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-jwt}\,dt = \operatorname{sinc}\left(\frac{w}{2\pi}\right)$$



$$f_1(t) = -p\left(t-\tfrac{1}{2}\right) + p\left(t-\tfrac{3}{2}\right)$$

$$\Rightarrow F_1(w) = -P(w)e^{-j\frac{w}{2}} + P(w)\, e^{-j\frac{3}{2}w}$$

$$= -\operatorname{sinc}\left(\frac{w}{2\pi}\right)e^{-j\frac{w}{2}} + \operatorname{sinc}\left(\frac{w}{2\pi}\right)e^{-j\frac{3}{2}w}$$

(d) $f_4(t) = \begin{cases} 0, & |t| > 1 \\ e^{-|t|}, & |t| \leq 1 \end{cases}$

(d) $F_4(w) = \int_{-\infty}^{\infty} e^{-|t|} e^{-jwt} dt$

$= \int_{-1}^{1} e^{-|t|} e^{-jwt} dt$

$= \int_{-1}^{1} e^{-|t|} (\cos wt - j \sin wt) dt$

$= \int_{-1}^{1} e^{-|t|} \cos wt \, dt - j \int_{-1}^{1} e^{-|t|} \sin wt \, dt$

since $e^{-|t|}$ is an even function, $\cos wt$ is an even f, $\sin wt$ is an odd f.

$= 2 \cdot \int_{0}^{1} e^{-t} \cos wt \, dt$

Given $\int e^{at} \cos(bt) dt = \dfrac{e^{at}(a \cos bt + b \sin bt)}{a^2+b^2}$

In this case, $a = -1$, $b = w$

$$= 2 \cdot \left[\frac{e^{-t}(-\cos wt + w \sin wt)}{1+w^2} \right]_0^1$$

$$= 2 \cdot \left[\frac{e^{-1}(-\cos w + w \sin w) - e^0(-1+0)}{1+w^2} \right]$$

$$= 2 \left[\frac{1 - e^{-1}(\cos w - w \sin w)}{1+w^2} \right]$$

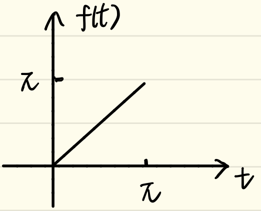
$$= \frac{2 - 2e^{-1}(\cos w - w \sin w)}{1+w^2}$$

49. Consider the signal $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(t) = t(u(t) - u(t-\pi))$$

(a) Sketch the graph of $f(t)$ for $-2\pi \leq t \leq 2\pi$.

(b) Use the definition of Fourier transform to find the Fourier transform of f.



$\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$= \int_0^{\pi} t e^{-j\omega t} dt$

$= \int_0^{\pi} \frac{t}{-j\omega} d e^{-j\omega t}$

$= \left[\frac{t}{-j\omega} \cdot e^{-j\omega t}\right]_0^{\pi} - \int_0^{\pi} e^{-j\omega t} d\frac{t}{-j\omega}$

$= \left[\frac{t}{-j\omega} \cdot e^{-j\omega t}\right]_0^{\pi} - \left[\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_0^{\pi}$

$= \frac{\pi e^{-j\omega\pi}}{-j\omega} - \frac{e^{-j\omega\pi}-1}{\omega^2} = \frac{1}{\omega^2}\left[(1+j\pi\omega)e^{-j\omega\pi}-1\right]$

50. Compute the inverse Fourier transforms of the following signals

(a) $X_1(\omega) = \cos(4\omega)$

$X_1(\omega) = \frac{1}{2}[e^{-j4\omega} + e^{j4\omega}]$

since $\mathcal{F}[\delta(t)] = 1$, $\mathcal{F}[\delta(t-c)] = 1 \cdot e^{-j\omega c}$ (time shifting)

$\Rightarrow x_1(t) = \frac{1}{2}[\delta(t-4) + \delta(t+4)]$

52. Find the Fourier transform of the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(t) = 2t(u(t) - u(t - \pi/2))$$

Hint: Firstly show that $u(at) = u(t)$ for any $a \geq 0$. Then show that $g(t) = f(2t)$, where is f is given in Question 49.

① show that $u(at) = u(t)$

$\mathcal{F}[u(t)] = \frac{1}{s}$ $\quad\quad$ $\mathcal{F}[u(at)] = \frac{1}{a} U(\frac{s}{a}) = \frac{1}{a} \cdot \frac{1}{\frac{s}{a}} = \frac{1}{s}$

$\Rightarrow$ $u(at)$ and $u(t)$ are identical

② show that $g(t) = f(2t)$

$f(2t) = 2t\,(u(2t) - u(2t - \pi))$

$= 2t\,(u(t) - u(t - \frac{\pi}{2}))$ $\quad\quad$ (since $u(at) = u(t)$)

$= g(t)$

Therefore, $G(w) = \frac{1}{2} F(\frac{w}{2}) = \frac{1}{2} \cdot \frac{1}{(\frac{w}{2})^2} [(1 + j\pi\frac{w}{2}) e^{-j\frac{\pi w}{2}} - 1]$

$= \frac{2}{w^2} [(1 + \frac{j\pi w}{2}) e^{-j\frac{\pi w}{2}} - 1]$

54. Match the time-domain signals in Figure 1 to the appropriate amplitude spectra in Figure 2. In each case explain why the amplitude spectrum matches the time-domain signal.

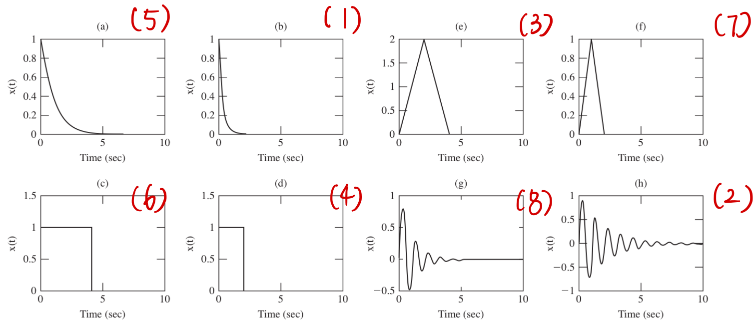


Figure 1: Time domain signals for Problem 54

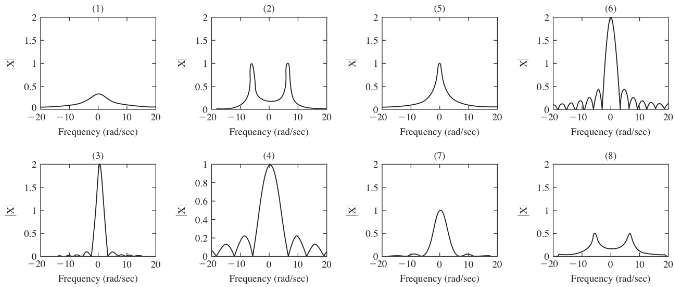


Figure 2: Fourier spectra for Problem 54

(a)(b): $x(t) = e^{-bt}u(t)$ $\quad X(w) = \frac{1}{jw+b}$ $\quad |X| = \frac{1}{\sqrt{w^2+b^2}}$

$|X|$ has a peak in $w=0$, peak value depends on b

(c)(d): $x(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$ $\quad X(w) = \tau \,\text{sinc}\left(\frac{\tau}{2\pi} w\right)$

The amplitude of $|X|$ depends on τ

(e)(f): $x(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$ $\quad X(w) = \frac{\tau}{2} \,\text{sinc}^2\left(\frac{\tau}{4\pi} w\right)$

The amplitude of $|X|$ depends on τ

(g)(h)

$x(t) = e^{-bt}\sin(w_0 t)\, u(t)$

$X(w) \approx \frac{1}{j(w+w_0)+b} + \frac{1}{j(w-w_0)+b}$

Peak value depends on b.

As $b \uparrow$, $|X| \downarrow$

55. The continuous-time signal $x(t) = e^{-bt}u(t)$, with $b > 0$, has Fourier transform

$$X(\omega) = \frac{1}{j\omega + b}$$

where b is a constant. Use properties of the Fourier transform to obtain the transforms of following signals:

(a) $x_1(t) = x(5t-4)$

(a) $x_1(t) = x(5(t-\frac{4}{5}))$

Given $x(t) \leftrightarrow X(w)$, $x(5t) \Leftrightarrow \frac{1}{5}X(\frac{w}{5}) = \frac{1}{5}\frac{1}{j\frac{w}{5}+b} = \frac{1}{jw+5b}$

let $x_2(t) = x(5t)$, then $x_1(t) = x_2(t-\frac{4}{5})$

Auording to time shifting. $X_1(w) = X_2(w)\cdot e^{-jw\frac{4}{5}}$

$\Rightarrow X_1(w) = \frac{1}{jw+5b}\cdot e^{-jw\frac{4}{5}}$

(d) $x_4(t) = \frac{1}{jt-b}$

let $X(t) = \frac{1}{jt+b}$, $x_4(-t) = -X(t)$

Take Fourier Transform on both sides

$X_4(-w) = -2\pi x(-w)$

replace $-w$ by w

$X_4(w) = -2\pi x(w) = -2\pi\cdot e^{-bw}u(w)$

56. Let $x(t)$ and $X(\omega)$ be a Fourier transform pair. Using the definition of Fourier transform, prove the following Fourier transform pairs:

$$
\begin{array}{llll}
x(t-c) & \longleftrightarrow & X(\omega)e^{-j\omega c},\ c \in \mathbf{R} & \text{(Time shift)} \\
x(-t) & \longleftrightarrow & X(-\omega) & \text{(Flipping Theorem)} \\
x(at) & \longleftrightarrow & \frac{1}{a}X\left(\frac{\omega}{a}\right),\ a > 0 & \text{(Time scaling)} \\
x(t)e^{j\omega_0 t} & \longleftrightarrow & X(\omega-\omega_0),\ \omega_0 \in \mathbf{R} & \text{(Modulation)} \\
X(t) & \longleftrightarrow & 2\pi x(-\omega) & \text{(Duality)}
\end{array}
$$

Prove : Duality

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}\,dw$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(w)e^{jwt}\,dw$$

① let t replaced by $-t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(w)e^{-jwt}\,dw$$

② let t replaced by w, w replaced by t

$$2\pi x(-w) = \underbrace{\int_{-\infty}^{\infty} X(t)e^{-jtw}\,dt}_{\mathcal{F}[X(t)]}$$

$\Rightarrow X(t) \Leftrightarrow 2\pi x(-w)$

58. Consider the first order RC filter whose dynamics are described by the equations

$$RC\dot{V}_c + V_c = V_{in}$$
$$V_{out} = V_c$$

(a) If $R = 100 Ohms$ and $C = 0.2mF$ what is the time domain response if $V_{in} = 2\delta(t)$? What is the system time constant?

(b) If $V_{in}(t) = 2\sin(t)$ what is the time domain response?

(c) If the initial conditions of the system were nonzero so that $V_c(0) = 1$ and $V_{in}(t) = 1 \forall t \geq 0$ what would be the system response?

(d) How would your previous answer change if the input was delayed by 1 second, i.e. $V_{in}(t) = 1 \forall t \geq 1$?

(a) $RC\,\dot{V}_{out} + V_{out} = V_{in}$ let $V_{out} = x$, $V_{in} = u$, $a = \frac{1}{RC}$

$\Rightarrow RC\dot{x} + x = u \Rightarrow \dot{x} + ax = au$ ①

- define a function $\beta(t)$, that satisfies $\dot{\beta} = a\beta \Rightarrow \beta = Ae^{at}$
- multiplying both sides of ① by β

$$\beta\dot{x} + \dot{\beta}x = au\beta$$

$$\Rightarrow \beta x + c = \int au\beta\, dt \Rightarrow x(t) = \frac{\int au\beta dt - c}{\beta} = \frac{\int a e^{at} u dt + C_2}{e^{at}}$$

if $u(t) = V_{in} = 2\delta(t)$

$$x(t) = (a\int_0^t e^{at}\cdot 2\delta(t)\,dt + C_2)e^{-at} = (2a + C_2)e^{-at}$$

time constant : $\frac{1}{a} = RC = 0.02$

(b) $u(t) = 2\sin(t) = \frac{e^{jt} - e^{-jt}}{j}$

$$x(t) = (a\int_0^t e^{at}\cdot\frac{e^{jt}-e^{-jt}}{j}dt + C_2)e^{-at}$$

$$= \left[\frac{a}{j}\left[\frac{e^{(a+j)t}}{a+j}\right]_0^t - \left[\frac{e^{(a-j)t}}{a-j}\right]_0^t + C_2\right]e^{-at}$$

$$= \frac{2a^2\sin t - 2a\cos t}{a^2+1} + C_3 e^{-at}$$

(c) $x(t) = (a\int_0^t e^{at}\cdot 1\,dt + C_2)e^{-at} = (e^{at} - 1 + C_2)e^{-at}$

$= 1 - e^{-at} + C_2e^{-at}$

$V_c(0) = x(0) = 1 - 1 + C_2 = 1 \Rightarrow C_2 = 1$

(d) $x(t) = (b\int_1^t e^{at}\cdot 1\,dt + C_2)e^{-at}$