Stability

31. Determine the stability of the following LTI systems.

(a) $H(s) = \dfrac{s+3}{s^2+3}$

(b) $H(s) = \dfrac{4s+8}{(s^2+4s+13)(s+4)}$

(c) $D\dfrac{d^3y}{dt^3} + E\dfrac{d^2y}{dt^2} + C\dfrac{dy}{dt} = kx(t)$, where the constants D, E, C and k are all positive.

32. Determine the stability of the LTI systems with the following impulse response functions.

(a) $h(t) = [2t^3 - 2t^2 + 3t - 2](u(t) - u(t-10))$

(b) $h(t) = \sin(2t)$

(c) $h(t) = e^{-t}\sin(2t)$

(d) $h(t) = e^{t}\sin(2t)$

Type 1: Given the transfer function $\Rightarrow$ check poles location

e.g. 31.(a)(b)

(a) $P_{1,2} = \pm\sqrt{3}j \Rightarrow$ Imaginary axis $\Rightarrow$ marginally stable

(b) $P_1 = -4 \quad P_{2,3} = \dfrac{-4 \pm \sqrt{16 - 4\times1\times13}}{2} = \dfrac{-4 \pm 6j}{2}$

All in left half plane $\Rightarrow$ stable

Type 2: Given the time domain function $\Rightarrow$ Apply laplace transform first

e.g. 31 (c)

$$DS^3Y(S) + ES^2Y(S) + CSY(S) = KX(S)$$

$$G(S) = \frac{Y(S)}{X(S)} = \frac{k}{DS^3 + ES^2 + CS}$$

$$DS^3 + ES^2 + CS = S(DS^2 + ES + C) = 0$$

$P_1 = 0 \quad P_{2,3} = \dfrac{-E \pm \sqrt{E^2 - 4DC}}{2D}$

$\downarrow$

Imaginary axis $\Rightarrow$ marginally stable

Type 3: Give the impulse response function $\Rightarrow$ check absolutely integrable

or Apply laplace transform

e.g. Q32

Q32.

(b) $H(s) = \frac{2}{s^2+4}$

$P_{1,2} = \pm 2j$ $\Rightarrow$ marginally stable

(c) $H(s) = \frac{2}{(s+1)^2+4}$

$(s+1)^2+4 = s^2+2s+1+4 = s^2+2s+5$

$P_{1,2} = \frac{-2 \pm \sqrt{4-4\times1\times5}}{2} = \frac{-2\pm4j}{2} = -1\pm2j$

All in left half plane $\Rightarrow$ stable

(d) $H(s) = \frac{2}{(s-1)^2+4}$

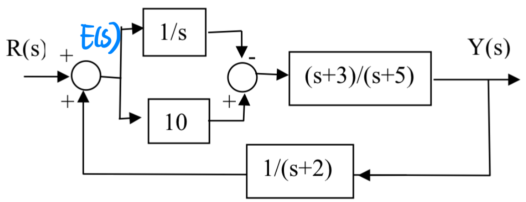
$(s-1)^2+4 = s^2-2s+1+4 = s^2-2s+5$

$P_{1,2} = \frac{2\pm\sqrt{4-4\times1\times5}}{2} = 1\pm2j$

All in right half plane $\Rightarrow$ unstable

Q1. Find the overall transfer functions of the following systems and discuss whether (a) and (b) are stable:

b)



open loop: (ignore feed back first)

$$E(S)\left(10-\frac{1}{S}\right)\cdot\left(\frac{S+3}{S+5}\right)=Y(S)$$

$$G_{ol}(S)=\frac{Y(S)}{E(S)}=\frac{10S-1}{S}\cdot\frac{S+3}{S+5}$$

feedback : $K(S)=\frac{1}{S+2}$

$\Rightarrow$ close loop:

$$\frac{Y(S)}{R(S)}=\frac{G_{ol}(S)}{1-K(S)G_{ol}(S)}=\frac{\frac{(10S-1)(S+3)}{S(S+5)}}{1-\frac{1}{S+2}\frac{(10S-1)(S+3)}{S(S+5)}}$$

$\uparrow$ in L11

$$=\frac{(S+2)(10S-1)(S+3)}{(S+2)S(S+5)-(10S-1)(S+3)}$$

$$=\frac{10S^3+49S^2+55S-6}{S^3-3S^2-19S+3}$$

let $S^3-3S^2-19S+3=0$ $\Rightarrow$ use Matlab command roots([1 -3 -19 3])

$\Rightarrow$ Pole locations at $S=6.052,\ -3.21,\ 0.15$

At least one of the poles in RHP $\Rightarrow$ unstable

Q2. Find the time domain expressions of the following Laplace domain signals

a) U(s) = 10/s(s+5)

b) b) Y(s) = (4s+2)/(s² + 5s + 8)

(b) $Y(S) = \frac{4S+2}{S^2+5S+8}$ Try to complete the square term

$= \frac{4S+2}{(S+2.5)^2+\frac{7}{4}}$

According to table, $\frac{S+a}{(S+a)^2+b^2} \leftrightarrow e^{-at}\cos bt$

$\frac{b}{(S+a)^2+b^2} \leftrightarrow e^{-at}\sin bt$

Here $a = 2.5$ $b = \sqrt{\frac{7}{4}}$

$Y(S) = \frac{4(S+2.5)-8}{(S+2.5)^2+\frac{7}{4}} = 4\frac{(S+2.5)}{(S+2.5)^2+\frac{7}{4}} - \frac{\frac{8}{b}\cdot b}{(S+2.5)^2+\frac{7}{4}}$ $\frac{8}{b} = \frac{8}{\sqrt{\frac{7}{4}}} = \frac{16}{\sqrt{7}}$

$\Downarrow$

$y(t) = 4\cdot e^{-2.5t}\cos(\sqrt{\tfrac{7}{4}}t) - \frac{16}{\sqrt{7}}e^{-2.5t}\sin(\sqrt{\tfrac{7}{4}}t)$

Q3 a). If a signal u(t)=3 was input into a system with transfer function $G(s)=\dfrac{s+1}{s^2+3s+2}$ what would be the steady state output?

b) If a signal $u(t)=5-e^{-t}$ was input into a system with transfer function $G(s)=\dfrac{s+1}{s^2+6s+9}$ what would be the steady state output?

c) If u(t)=1 was input into a system with transfer function $G(s)=\dfrac{s+3}{s-2}$ what would be the steady state output?

Determine whether system is stable first

(b) $G(s)=\dfrac{s+1}{s^2+6s+9}$ let $s^2+6s+9=0$ $P_{1,2}=-3 \Rightarrow$ stable.

$$U(s)=\frac{5}{s}-\frac{1}{s+1}=\frac{4s+5}{s(s+1)}$$

$$Y(s)=U(s)G(s)=\frac{4s+5}{s(s+1)}\frac{s+1}{(s^2+6s+9)}$$

Final value Theorem: $y(\infty)=\lim_{s\to 0} s\cdot Y(s)=\dfrac{5\times 1}{1\times 9}=\dfrac{5}{9}$

(c) $G(s)=\dfrac{s+3}{s-2}$ $P_1=2 \Rightarrow$ unstable.

there will be no steady state output!

Q4: Without using inverse Laplace transforms, quantitatively describe (as appropriate) the response of the following systems to a unit step input:

d) $G(s) = \dfrac{2}{s^2 + 2s + 2}$

$$G(S) = \frac{2}{s^2+2s+2} = \frac{\omega_n^2}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$ (Second order system transfer function

$\Rightarrow$ natural frequency $\omega_n = \sqrt{2}$ rad/s, damping ratio $\varepsilon = \frac{1}{\sqrt{2}}$

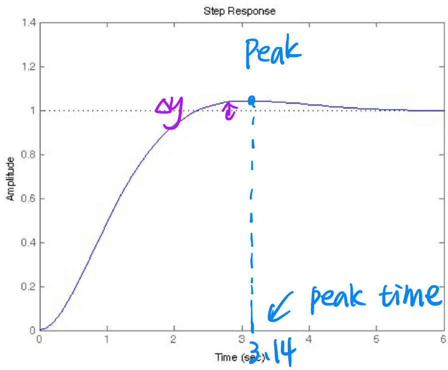
damped natural frequency: $\omega_d = \omega_n\sqrt{1-\varepsilon^2} = 1$ rad/s

peak time: $t_p = \frac{\pi}{\omega_d} = \pi \approx 3.14$ s

Maximum overshoot: $M_p = e^{\frac{-\varepsilon\pi}{\sqrt{1-\varepsilon^2}}} = e^{-\pi} \approx 4.0\%$

Checking using Matlab (>>step([2], [1 2 2])):

$\Delta y = y_{final} \cdot M_p = 4.0\%$
$= 0.04$



Q6. The input torque (u) to output angle (θ) relationship of a simple rigid damped pendulum undergoing friction about small angles can be approximated by the equation:

$$I\ddot{\theta} = -b\dot{\theta} - \frac{g}{l}\theta + u$$

Here I is the inertia, b is a friction coefficient, g is acceleration due to gravity and l is the length of the pendulum.

a) Find the transfer function of the system
b) If I = 1, b = 3 and g = 9.8, what length pendulum should be used to give the impulse response with a frequency of 1 rad/s?
c) What range of lengths would result in the impulse response having a frequency < 2 rad/s?
d) Identify the settling time for your answer to b
e) What would be the peak overshoot if a step input of size 2 was input to the system, ie u(t) = 2.
f) At what time would the maximum overshoot occur?
g) Roughly sketch the response.

(a) $IS^2\theta(S) = -bs\,\theta(s) - \frac{g}{l}\theta(s) + U(S)$

$$\Rightarrow \frac{\theta(S)}{U(S)} = \frac{1}{IS^2+bS+\frac{g}{l}}$$

(b) $G(S) = \frac{1}{S^2+3S+\frac{9.8}{l}}$ $\quad W_d = 1$ rad/s

$$= \frac{W_n^2}{S^2+2\xi W_n S+W_n^2} \Rightarrow W_n = \sqrt{\frac{9.8}{l}} \qquad \xi = \frac{3}{2W_n} = \frac{3}{2\sqrt{\frac{9.8}{l}}}$$

$$W_d = W_n\sqrt{1-\xi^2} = \sqrt{\frac{9.8}{l}}\sqrt{1-\frac{9l}{4\times 9.8}} = \sqrt{\frac{9.8}{l}-\frac{9}{4}} = 1 \quad \Rightarrow \frac{9.8}{l} = \frac{9}{4} \Rightarrow l = 3.0154\text{m}$$

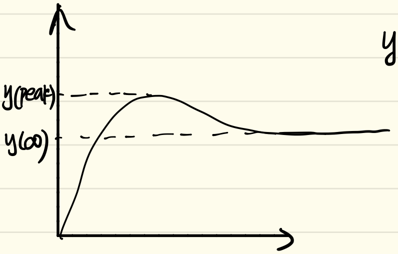
(c) $W_d = \sqrt{\frac{9.8}{l}-\frac{9}{4}} < 2$

$$\Rightarrow \frac{9.8}{l}-\frac{9}{4} < 4 \Rightarrow l > 1.56\text{ m}$$

(d) $t_s = \frac{-\ln(\%\text{tatol})}{\sigma} = \frac{-\ln(0.01)}{\xi W_n} = \frac{4.6}{1.5} = 3.1S$

e) What would be the peak overshoot if a step input of size 2 was input to the system, ie u(t) = 2. (use $\ell = 3.0154m$)
f) At what time would the maximum overshoot occur?
g) Roughly sketch the response.

(e) $$M_p = e^{\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)} = e^{-\frac{\pi\times 0.83}{\sqrt{1-0.83^2}}} = 0.93\%$$



$$y(peak) = y(\infty)\cdot M_p$$

$$Y(s) = G(s)\cdot U(s) = \frac{1}{s^2+3s+3.25}\cdot\frac{2}{s}$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \frac{2}{3.25} = 0.6154$$

$$y(peak) = 0.6154 \times 0.93\% = 0.0057$$

(f) $$t_p = \frac{\pi}{\omega_d} = \pi \text{ sec}$$

d from the previous sections should end up with something that looks l