week 1

34. Express the following signals in trigonometric form $a_k \cos(\omega t) + b_k \sin(\omega t)$:

(a) $2\cos(3t) - \cos(3t - \pi/4)$

(b) $\sin(2t - \pi/4) + 2\cos(2t - \pi/3)$

(c) $10\cos(\pi t + \pi/3) + 8\cos(\pi t - \pi/3)$

Hint: do NOT use Euler's formulae for the Fourier coefficients.

(a) $2\cos(3t) - \cos(3t - \frac{\pi}{4})$ $\qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= 2\cos(3t) - (\cos 3t \cos\frac{\pi}{4} + \sin 3t \sin\frac{\pi}{4})$

$= (2 - \frac{\sqrt{2}}{2})\cos 3t - \frac{\sqrt{2}}{2}\sin 3t$

$\approx 1.293\cos 3t - 0.707\sin 3t$

35. Express the signals in Question 34 in cosine-with-phase form $A_k \cos(\omega t + \theta_k)$.

$a_k = 1.293 \qquad b_k = -0.707$

$A_k = \sqrt{a_k^2 + b_k^2} \approx 1.473$

since $a_k > 0 \qquad \theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right) = \tan^{-1}\left(\frac{0.707}{1.293}\right) = 28.669°$

$28.669° \longrightarrow \frac{28.669°}{180°} \cdot \pi \approx 0.5$

40. Decide which of the following statements are true. For those that are true, give a proof. For those that are false, give an explicit counterexample.

(a) The product of two even functions is an even function.

(b) The product of two odd functions is an odd function.

(c) The product of an even function and an odd function is an even function.

(a) let $g(x)$, $f(x)$ are even function, let $h(x) = g(x)f(x)$

$h(-x) = g(-x)f(-x) = g(x)f(x) = h(x)$ → even function → True

(b) e.g. $g(x) = x$, $f(x) = x$ → False

(c) eg. $g(x) = x$, $f(x) = x^2$ → False

41. (a) Let $a > 0$ and let x and v both be even functions. Show that

$$\int_{-a}^{a} x(t)v(t)\, dt = 2\int_{0}^{a} x(t)v(t)\, dt$$

(b) Let $a > 0$, let x be an even function and let v an odd function. Show that

$$\int_{-a}^{a} x(t)v(t)\, dt = 0$$

(a) let $g(t) = x(t)v(t)$, $g(t)$ is also a even function

$\int_{-a}^{a} g(t)\,dt = \int_{-a}^{0} g(t)\,dt + \int_{0}^{a} g(t)\,dt$

let $t \to -m$,

$= \int_{a}^{0} g(-m)\,d(-m) + \int_{0}^{a} g(t)\,dt$

$= -\int_{a}^{0} g(m)\,dm + \int_{0}^{a} g(t)\,dt$

$= \int_{0}^{a} g(m)\,dm + \int_{0}^{a} g(t)\,dt$

let $m \to t$

$= \int_{0}^{a} g(t)\,dt + \int_{0}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt$

(a) Sketch the graph of the function showing at least three periods,

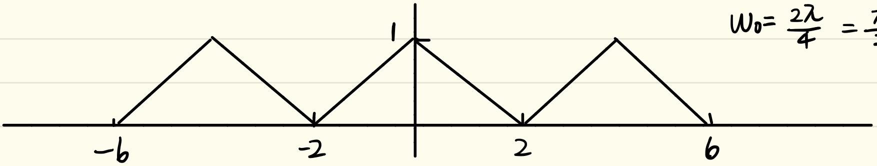
(b) Explain why the function satisfies the Dirichlet conditions.

(c) Use the definition of the trigonometric Fourier coefficients (a_k and b_k) to compute a general expression for these coefficients, and evaluate them for $k = 0, 1, 2$.

37. Repeat Question 36 for the function

$$f_2(t) = 1 - \left|\frac{t}{2}\right| \quad \text{for } -2 \le t < 2, \text{ and } f_2(t) = f_2(t+4).$$

(a)



$W_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

(c) $f_2(t)$ is a even function

$a_0 = \frac{1}{a}\int_{-a}^{a} f_2(t)\,dt = \frac{1}{4}\cdot 2\int_0^2 f_2(t)\,dt$

$= \frac{1}{2}\int_0^2 1 - \frac{t}{2}\,dt$

$= \frac{1}{2}\left[t - \frac{t^2}{4}\right]_0^2 = \frac{1}{2}(2-1) = \frac{1}{2}$

even even

$a_k = \frac{2}{a}\int_{-a}^{a} f_2(t)\cos(kW_0t)\,dt$

$= \frac{2}{4}\cdot 2\int_0^a f_2(t)\cos(kW_0t)\,dt$

$= \int_0^2 \left(1 - \frac{t}{2}\right)\cos(kW_0t)\,dt$

$= \int_0^2 \cos(kW_0t)\,dt - \underline{\int_0^2 \frac{t}{2}\cos(kW_0t)\,dt}$

$= \left[\frac{\sin kW_0t}{kW_0}\right]_0^2 - \int_0^2 \frac{t}{2kW_0}\,d\sin(kW_0t)$

$= \frac{\sin 2kW_0}{kW_0} - \frac{1}{kW_0}\sin(2kW_0) + \frac{1-\cos(2kW_0)}{2(kW_0)^2}$

$= \frac{1-\cos(2kW_0)}{2(kW_0)^2}$

$\int_0^2 \frac{t}{2kW_0}\,d\sin(kW_0t)$

$= \left[\frac{t}{2kW_0}\sin(kW_0t)\right]_0^2 - \int_0^2 \sin(kW_0t)\,d\frac{t}{2kW_0}$

$= \left[\frac{t}{2kW_0}\sin(kW_0t)\right]_0^2 - \left[\frac{-\cos(kW_0t)}{2(kW_0)^2}\right]_0^2$

$= \frac{1}{kW_0}\sin(2kW_0) - \left(\frac{-\cos(2kW_0)+1}{2(kW_0)^2}\right)$

since $W_0 = \frac{\pi}{2}$

$a_k = \frac{1-\cos(k\pi)}{2(k\cdot\frac{\pi}{2})^2} = 2\,\frac{1-\cos(k\pi)}{k^2\pi^2}$

If k is even, $\cos(k\pi) = 1$, $a_k = 0$

k is odd, $\cos(k\pi) = -1$, $a_k = \frac{4}{k^2\pi^2}$

even odd

$b_k = \frac{2}{a}\int_{-a}^{a} f_2(t)\sin(kW_0t)\,dt = 0$

47. Suppose the periodic function f has fundamental frequency ω_0 and complex Fourier coefficients $\{c_k : k \in \mathbf{Z}\}$. Let $t_0 \in \mathbf{R}$ and define

$$f_1(t) = f(t - t_0), \quad f_2(t) = f(-t),$$

Assume f : period T_0, $W_0 = \frac{2\pi}{T_0}$

$f_1(t+T_0) = f(t+T_0-t_0) = f(t-t_0) = f_1(t)$

$\Rightarrow f_1$: period $T_0 \Rightarrow w_0 = \frac{2\pi}{T_0}$

similarly: $f_2(t+T_0) = f(-(t+T_0)) = f(-t) = f_2(t)$

$\Rightarrow f_2$: period $T_0 \Rightarrow w_0 = \frac{2\pi}{T_0}$

For $f_1(t)$: $\tilde{C}_k = \frac{1}{T}\int_0^T f(t-t_0)\, e^{-jkW_0t}\, dt$

let $t - t_0 = t'$ $\Rightarrow t = t' + t_0$

$= \frac{1}{T}\int_{-t_0}^{T-t_0} f(t')\, e^{-jkW_0(t'+t_0)}\, d(t'+t_0)$

$= e^{-jkW_0t_0} \cdot \frac{1}{T}\int_0^T f(t')\, e^{-jkW_0t'}\, dt'$

$= e^{-jkW_0t_0} \cdot C_k$

For $f_2(t)$: $\hat{C}_k = \frac{1}{T}\int_0^T f(-t)\, e^{-jkW_0t}\, dt$

let $t' = -t$ $\Rightarrow t = -t'$

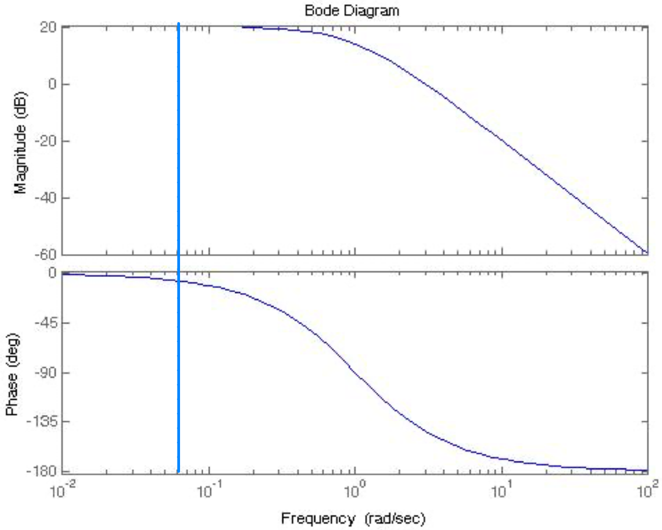
$= \frac{1}{T}\int_0^{-T} f(t')\, e^{jkW_0t'}\, d(-t')$

$= \frac{1}{T}\int_T^0 f(t')\, e^{jkW_0t'}\, dt'$

$= C_{-k}$

week 2

Q1. The Bode plots for a plant are shown below.



Use the Bode plot to draw the input and output of the plant if the input is equal to

a) $u(t) = \sin(0.02\pi t)$

$0.02\pi \approx 0.06$ rad/s

from bode diagram, we have gain = 20dB, phase $\approx -10^\circ$

dB gain → linear gain

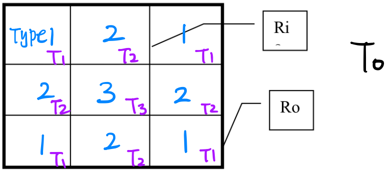
20dB → 10

The output will be $y(t) = 10 \cdot \sin(0.02\pi t - 10^\circ)$

Q5:

An air conditioner supplies cold air at the same temperature to each room on a floor of a building. The floor plan is shown in the figure below. The cold air flow produces an equal amount of heat flow *q* out of each room. Write a set of differential equations governing the temperature of each room where:

To = Temperature outside the building
Ro = Thermal resistance of outside walls
Ri = Thermal resistance of inside walls



Assume that:

1. All the rooms are perfect squares
2. There is no heat flow through the floors or ceilings
3. The temperature is uniform throughout each room

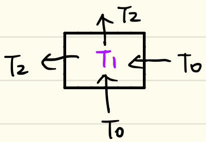
Take advantage of symmetry to reduce the number of differential equations to three.

Intuitive: $T_o > T_1 > T_2 > T_3$

heat transfer through wall: $q = \dfrac{\Delta T_{wall}}{R_{wall}}$

heat capacity: $C = \dfrac{Q}{\Delta T}$

For type 1:

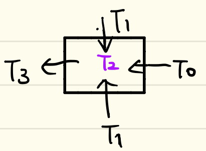


$$\left.\begin{aligned} q_{out} &= q + \frac{T_1 - T_2}{R_i} \times 2 \\ q_{in} &= \frac{T_o - T_1}{R_o} \times 2 \end{aligned}\right\} \quad Q = q_{in} - q_{out}$$

Assume $\Delta t = 1$ unit $\qquad \dot{T}_1 = \dfrac{\Delta T_1}{\Delta t} = \Delta T_1$

$$\Rightarrow \Delta T_1 = \dot{T}_1 = \frac{1}{C} Q = \frac{1}{C}\left(\frac{2}{R_o}(T_o - T_1) - \frac{2}{R_i}(T_1 - T_2) - q\right)$$

For type 2:



$$\left.\begin{aligned} q_{out} &= q + \frac{T_2 - T_3}{R_i} \\ q_{in} &= \frac{T_o - T_2}{R_o} + \frac{T_1 - T_2}{R_i} \times 2 \end{aligned}\right\} \quad Q = q_{in} - q_{out}$$

Assume $\Delta t = 1$ unit $\qquad \dot{T}_2 = \dfrac{\Delta T_2}{\Delta t} = \Delta T_2$

$$\Rightarrow \Delta T_2 = \dot{T}_2 = \frac{1}{C} Q = \frac{1}{C}\left(\frac{1}{R_o}(T_o - T_2) + \frac{2}{R_i}(T_1 - T_2) - \frac{1}{R_i}(T_2 - T_3) - q\right)$$

Q5:
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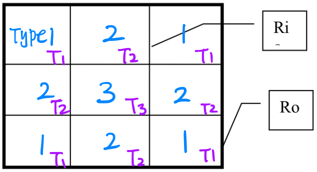

To

Assume that:
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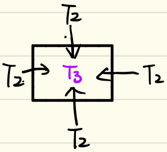
2. There is no heat flow through the floors or ceilings
3. The temperature is uniform throughout each room

Take advantage of symmetry to reduce the number of differential equations to three.

For type 3:



$$\left.\begin{aligned} q_{out} &= q \\ q_{in} &= \frac{T_2 - T_3}{R_i} \times 4 \end{aligned}\right\} \quad Q = q_{in} - q_{out}$$

Assume $\Delta t = 1$ unit $\qquad \dot{T}_3 = \frac{\Delta T_3}{\Delta t} = \Delta T_3$

$$\Rightarrow \Delta T_3 = \dot{T}_3 = \frac{1}{c} Q = \frac{1}{c}\left(\frac{4}{R_i}(T_2 - T_3) - q\right)$$

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} q \\ T_o \end{bmatrix}$$