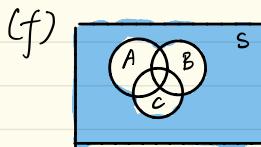
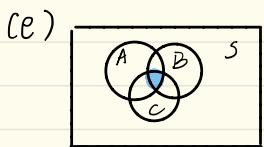
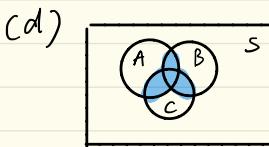
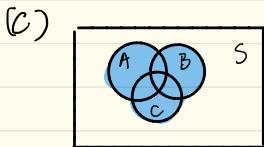
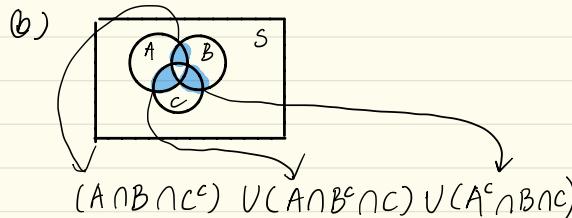
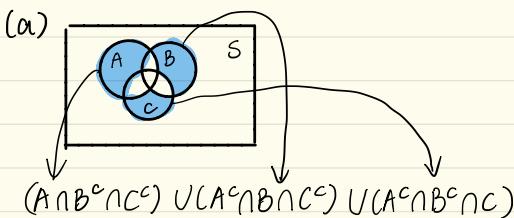


1 Let A , B , and C be events. Use **Venn diagrams** to find expressions for the following:

- (a) exactly one of the 3 events occurs
- (b) exactly two of the events occur
- (c) one or more of the events occur
- (d) two or more of the events occur
- (e) all three events occur
- (f) none of the events occur



2 (sequential experiments) A coin is tossed until the first time that the same result appears twice in succession. To every outcome in which there are n tosses, assign the probability $\frac{1}{2^n}$.

- (a) List elements of the sample space S .
- (b) Check that indeed $P[S] = 1$.
- (c) List the elements of the event $A = \text{"the experiment ends before the fifth toss"}$.
- (d) What is the probability of the event A ?

(a) H: head T: tail

stop at 2 times: (HH), (TT)

stop at 3 times: (THH), (HTT)

stop at 4 times: (HTHH), (THTT)

:

(b) For n tosses, the probability of each outcome is $\frac{1}{2^n}$

$$\Rightarrow P[S] = \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \dots \\ = 2 \cdot \sum_{n=2}^{\infty} \frac{1}{2^n} = 2 \cdot \frac{\frac{1}{2^2} - 0}{1 - \frac{1}{2}} = 1$$

geometric sequence, with the constant ratio $\frac{1}{2}$

(c) (HH), (TT), (THH), (HTT), (HTHH), (THTT)

$$(d) P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{7}{8}$$

3 Consider two events E_1 and E_2 .

(a) Suppose

$$P[\text{Event } E_1 \text{ doesn't occur}] = 0.9$$

$$P[\text{Event } E_2 \text{ doesn't occur}] = 0.8$$

$$P[\text{Event } E_1 \text{ or } E_2 \text{ occurs}] = 0.25$$

What is $P[E_1 \text{ and } E_2 \text{ both occur}]$?

(b) Suppose instead that

$$P[\text{Event } E_1 \text{ doesn't occur or event } E_2 \text{ doesn't occur}] = 0.7$$

Then what is $P[E_1 \text{ and } E_2 \text{ both occur}]$?

(a) $P(E_1^c) = 0.9 \quad P(E_2^c) = 0.8 \quad P(E_1 \cup E_2) = 0.25 \quad P(E_1 \cap E_2) ?$

$$\text{Since } P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$\begin{aligned} \Rightarrow P(E_1 \cap E_2) &= P(E_1) + P(E_2) - P(E_1 \cup E_2) \\ &= (1 - P(E_1^c)) + (1 - P(E_2^c)) - P(E_1 \cup E_2) \\ &= (1 - 0.9) + (1 - 0.8) - 0.25 = 0.05 \end{aligned}$$

(b) $P(E_1^c \cup E_2^c) = 0.7 \quad P(E_1 \cap E_2) ?$

According to De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$

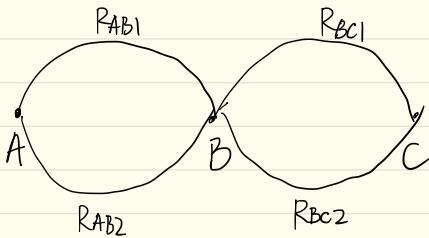
$$\Rightarrow P(E_1^c \cup E_2^c) = P((E_1 \cap E_2)^c) = 0.7$$

$$\Rightarrow P(E_1 \cap E_2) = 1 - P((E_1 \cap E_2)^c) = 0.3$$

- 4 (reading the question; translating the question into a more familiar question) Consider a communication network with three nodes A , B , and C , where the goal is to route a call from A to C . Note that:

- 1 Between A and B there are two routes that a call can use: R_{AB1} or R_{AB2} .
- 2 Between B and C there are two routes that a call can use: R_{BC1} or R_{BC2} .

Suppose that any of the routes $R_{AB1}, R_{AB2}, R_{BC1}, R_{BC2}$ can be blocked (due to congestion) and that the blocking occurs *independently* of each other. Suppose the blocking probability on each of the four routes is p . What is the probability that a call from A to C is blocked?



$$\begin{aligned} & P(A \rightarrow C \text{ blocked}) \\ &= 1 - P(A \rightarrow C \text{ unblocked}) \end{aligned}$$

$$\begin{aligned} A \rightarrow C \text{ unblocked} &= A \rightarrow B \text{ unblocked} \text{ and} \\ & B \rightarrow C \text{ unblocked} \end{aligned}$$

$$P(A \rightarrow B \text{ unblocked}) = 1 - P(A \rightarrow B \text{ blocked}) = 1 - P \cdot P$$

$$P(B \rightarrow C \text{ unblocked}) = 1 - P(B \rightarrow C \text{ blocked}) = 1 - P \cdot P$$

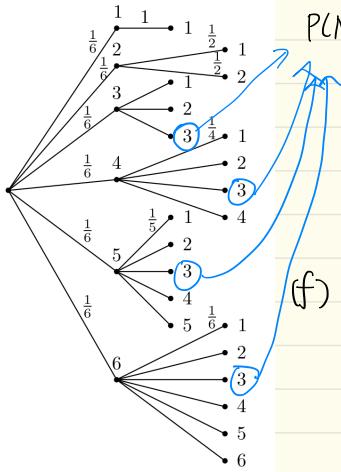
$$\begin{aligned} \Rightarrow P(A \rightarrow C \text{ blocked}) &= 1 - P(A \rightarrow B \text{ unblocked}) \cdot P(B \rightarrow C \text{ unblocked}) \\ &= 1 - (1 - P^2)(1 - P^2) = 2P^2 - P^4 \end{aligned}$$

- 5 (sequential experiments) A fair die is rolled and the number N_1 showing is observed. An integer N_2 is then selected at random from the range 1 to N_1 (each integer is equally likely with probability $1/N_1$).

- (a) Find the sample space of outcomes.
- (b) Find the set of outcomes corresponding to event "the die shows four"
- (c) Find the set of outcomes corresponding to the event $\{N_2 = 3\}$
- (d) Find the set of outcomes corresponding to the event $\{N_2 = 6\}$
- (e) Compute the probability of the event $\{N_2 = 3\}$.
- (f) Compute the probability of the event $\{N_1 = 4\}$ given $\{N_2 = 3\}$
- (g) Compute the probability of the event $\{N_1 = 4\}$ given $\{N_2 = 5\}$

- (a) $S = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$
- (b) Event (set of outcomes) that die shows 4 outcomes = $\{(4, 1), (4, 2), (4, 3), (4, 4)\}$.
- (c) Event (set of outcomes) that $\{N_2 = 3\} = \{(3, 3), (4, 3), (5, 3), (6, 3)\}$.
- (d) Event (set of outcomes) that $\{N_2 = 6\} = \{(6, 6)\}$.

(e)



$$\begin{aligned}
 P(N_2=3) &= P(N_2=3 | N_1=3) \cdot P(N_1=3) + \\
 &\quad P(N_2=3 | N_1=4) \cdot P(N_1=4) + \\
 &\quad P(N_2=3 | N_1=5) \cdot P(N_1=5) + \\
 &\quad P(N_2=3 | N_1=6) \cdot P(N_1=6) \\
 &= \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{19}{120}
 \end{aligned}$$

$$\begin{aligned}
 (f) \quad P(N_1=4 | N_2=3) &= \frac{P(N_2=3 | N_1=4) \cdot P(N_1=4)}{P(N_2=3)} \\
 &= \frac{\frac{1}{4} \cdot \frac{1}{6}}{\frac{19}{120}} = \frac{5}{19}
 \end{aligned}$$

$$\begin{aligned}
 (g) \quad P(N_1=4 | N_2=5) &= \frac{P(N_2=5 | N_1=4) \cdot P(N_1=4)}{P(N_2=5)} \\
 &= 0
 \end{aligned}$$

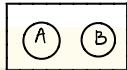
6 In an experiment, A, B, C , and D are events with probabilities:

$$P[A \cup B] = 3/4 \quad P[A] = 3/8 \quad P[C \cap D] = 1/4 \quad P[C] = 1/2$$

A and B are disjoint, while events C and D are mutually independent.

- (a) What is $P[A \cap B]$?
- (b) What is $P[B]$?
- (c) What is $P[A \cap B^C]$?
- (d) What is $P[A \cup B^C]$?
- (e) Is $P[A \cap B] = P[A]P[B]$ valid ?
- (f) What is $P[D]$?
- (g) What is $P[C \cup D]$?
- (h) What is $P[C \cap D^C]$?
- (i) What is $P[C \cup D^C]$?
- (j) What is $P[C^C \cap D^C]$?

(a) $P(A \cap B) = 0$



(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$

$$\Rightarrow P(B) = P(A \cup B) - P(A) = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

(c)
 $P(A \cap B^C) = P(A) = \frac{3}{8}$

(d)
 $P(A \cup B^C) = 1 - P(B) = \frac{5}{8}$

(e) NO $P(A \cap B) = 0 \neq P(A) \cdot P(B)$

(f) $P(C \cap D) = P(C) \cdot P(D) \Rightarrow P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1}{2}$

(g) $P(C \cup D) = P(C) + P(D) - P(C \cap D)$
 $= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(h)
 $P(C \cap D^C) = P(C) - P(C \cap D)$
 $= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(i) $P(C \cup D^C) = P(C) + P(D^C) - P(C \cap D^C)$
 $= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(j) De Morgan's Law

$$P(C^C \cap D^C) = P((C \cup D)^C) = 1 - P(C \cup D) = 1 - \frac{3}{4} = \frac{1}{4}$$

7 This problem has three **unrelated** parts.

(a) If $P[A|B] = 0.3$, $P[A^C|B^C] = 0.4$, and $P[B] = 0.7$,

- (i) What is $P[A|B^C]$?
- (ii) What is $P[A]$?
- (iii) What is $P[B|A]$?

(b) If $P[E] = 0.25$, $P[F|E] = 0.5$, $P[E|F] = 1/3$, what is $P[F]$?

(c) If $P[G] = P[H] = 2/3$, show that $P[G|H] \geq 1/2$.

$$(a) (i) P(A|B^C) = 1 - P(A^C|B^C) = 0.6$$

(ii) Total probability

$$P(A) = P(A|B^C) \cdot P(B^C) + P(A|B) \cdot P(B)$$

$$= 0.6 \cdot (1 - 0.7) + 0.3 \cdot 0.7 = 0.39$$

$$(iii) P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$= \frac{0.3 \cdot 0.7}{0.39} = \frac{7}{13}$$

$$(b) P(E|F) = \frac{P(F|E) \cdot P(E)}{P(F)} \Rightarrow P(F) = \frac{P(F|E) \cdot P(E)}{P(E|F)}$$
$$= \frac{0.5 \cdot 0.25}{\frac{1}{3}} = \frac{\frac{3}{8}}{\frac{1}{3}} = 0.375$$

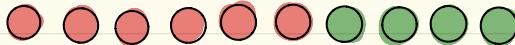
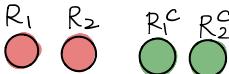
$$(c) P(G|H) = \frac{P(G \cap H)}{P(H)} = \frac{P(G) + P(H) - P(G \cup H)}{P(H)}$$

$$= \frac{\frac{4}{3} - P(G \cup H)}{\frac{2}{3}}$$

$$\text{Since } P(G \cup H) \leq 1 \Rightarrow P(G|H) \geq \frac{\frac{4}{3} - 1}{\frac{2}{3}} = \frac{1}{2}$$

- 8 (sequential experiments) An urn contains 6 red balls and 4 green balls. A ball is drawn at random from the urn, and then another ball is drawn at random **without replacing** the first ball. Let R_1 denote the event that the first ball is red, and let R_2 denote the event that the second ball is red.

- What is $P[R_2|R_1]$?
- What is $P[R_2|R_1^C]$?
- What is $P[R_2]$?
- What is $P[R_1^C|R_2^C]$?



(a)

$$P(R_2|R_1) = \frac{5}{9}$$



$$(b) P(R_2|R_1^C) = \frac{6}{9}$$



(c) Theorem of total probability:

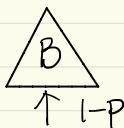
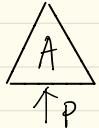
$$\begin{aligned} P(R_2) &= P(R_2|R_1) \cdot P(R_1) + P(R_2|R_1^C) \cdot P(R_1^C) \\ &= \frac{5}{9} \cdot \frac{6}{10} + \frac{6}{9} \cdot \frac{4}{10} = \frac{54}{90} = 0.6 \end{aligned}$$

$$(d) P(R_1^C|R_2^C) = \frac{P(R_1^C) \cdot P(R_2^C|R_1^C)}{P(R_2^C)}$$

$$= \frac{\frac{4}{10} (1 - P(R_2|R_1^C))}{1 - P(R_2)} = \frac{\frac{4}{10} (1 - \frac{6}{9})}{1 - 0.6} = \frac{1}{3}$$

- 9 40 people are voting in an election where there are two possible candidates: Candidate A and Candidate B. Each voter must choose one of the two candidates, and decides to randomly select a candidate. Each voter chooses candidate A with probability p , and does so independently of the other voters. Express your answer for parts (a)-(d) in terms of mathematical expressions containing p .

- What is the probability that Candidate B obtains exactly 30 votes?
- What is the probability that Candidate B obtains 30 or more votes?
- Assume that there is only one voting machine, and the 40 voters are given a number from 1 to 40 which determines the order of voting. Let X = the number of the first person to vote for Candidate B (define $X = 0$ if no one votes for Candidate B). Find $P[X \leq 4]$.
- Let E be the event that Candidate A receives exactly 25 votes. Find $P[E|X \leq 4]$.



40 people

$$(a) P(30 \text{ votes for } B) = \binom{40}{30} (1-p)^{30} \cdot p^{10} = \binom{40}{10} (1-p)^{30} \cdot p^{10}$$

$$(b) P(30 \text{ or more votes for } B) = P(30, 31, 32, \dots, 40 \text{ Votes for } B)$$

$$\begin{aligned} &= \binom{40}{30} (1-p)^{30} \cdot p^{10} + \binom{40}{31} (1-p)^{31} \cdot p^9 + \dots + \binom{40}{40} (1-p)^{40} \\ &= \sum_{i=30}^{40} \binom{40}{i} (1-p)^i \cdot p^{40-i} \end{aligned}$$

$$(c) P[X \leq 4] = P[X=0] + P[X=1] + P[X=2] + P[X=3] + P[X=4]$$

$$P[X=0] = P[\text{no one votes for } B] = p^{40}$$

$$P[X=1] = P[\text{number 1 is the first person to vote for } B] = (1-p)$$

$$P[X=2] = P[\text{number 2 is the } \dots] = P(1-p)$$

$$P[X=3] = P[\text{number 3 is the } \dots] = P \cdot P(1-p)$$

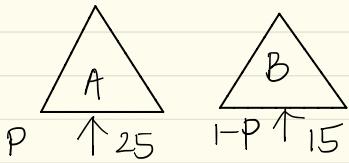
$$P[X=4] = P[\text{number 4 is the } \dots] = P \cdot P \cdot P(1-p)$$

$$= p^{40} + (1-p) + P(1-p) + P^2(1-p) + P^3(1-p)$$

$$= p^{40} + (1-p)(1+p + p^2 + p^3)$$

$$= p^{40} + (1-p) \frac{1-p^3 \cdot p}{1-p} = 1-p^4 + P^{40}$$

- (d) Let E be the event that Candidate A receives exactly 25 votes. Find $P[E|X \leq 4]$.



$$\begin{aligned} P(E|X \leq 4) &= \frac{P(E \cap (X \leq 4))}{P(X \leq 4)} \leftarrow \text{already know in (c)} \\ &= \frac{P(E \cap (X=0)) + P(E \cap (X=1)) + P(E \cap (X=2)) + P(E \cap (X=3)) + P(E \cap (X=4))}{P(X \leq 4)} \end{aligned}$$

$$P(E \cap (X=0)) = 0 \quad \text{since } E \cap (X=0) = \emptyset$$

$P(E \cap (X=1)) =$ the first person vote B, rest of 39 people (14 vote B)
number 1 is

$$\Rightarrow P(E \cap (X=1)) = (1-P) \cdot \binom{39}{14} \cdot P^{25} \cdot (1-P)^{14} = \binom{39}{14} P^{25} (1-P)^{15}$$

$P(E \cap (X=2)) =$ the first person vote B, rest of 38 people (14 vote B)
number 2 is

$$\Rightarrow P(E \cap (X=2)) = (1-P) \cdot \binom{38}{14} \cdot P^{25} \cdot (1-P)^{14} = \binom{38}{14} P^{25} (1-P)^{15}$$

$P(E \cap (X=3)) =$ the first person vote B, rest of 37 people (14 vote B)
number 3

$$\Rightarrow P(E \cap (X=3)) = (1-P) \cdot \binom{37}{14} \cdot P^{25} \cdot (1-P)^{14} = \binom{37}{14} P^{25} (1-P)^{15}$$

$P(E \cap (X=4)) =$ the first person vote B, rest of 36 people (14 vote B)
number 4

$$\Rightarrow P(E \cap (X=4)) = (1-P) \cdot \binom{36}{14} \cdot P^{25} \cdot (1-P)^{14} = \binom{36}{14} P^{25} (1-P)^{15}$$

$$\Rightarrow P(E|X \leq 4) = \frac{P^{25} (1-P)^{15}}{1-P^4 + P^{40}} \cdot \left(\binom{36}{14} + \binom{37}{14} + \binom{38}{14} + \binom{39}{14} \right)$$

- 10 An experiment is defined as follows:

One of two coins is first selected at random (50% probability for choosing each) and then is tossed. Coin #1 comes up 'Heads' with probability p_1 , while Coin #2 is 'Heads' with probability p_2 . Express your answers below in terms of p_1 and p_2 .

(a) (prediction) What is the probability that the outcome of the experiment is 'Heads'?

(b) (inference) What is the probability that Coin #2 was used given that a 'Heads' occurred?

coin #1



$$\text{"head"} = p_1$$

coin #2



$$\text{"head"} = p_2$$

$$P(H|C_1) = p_1$$

$$P(H|C_2) = p_2$$

$$(a) P(H) = P(H|C_1) \cdot P(C_1) + P(H|C_2) \cdot P(C_2) = \frac{1}{2}p_1 + \frac{1}{2}p_2 \\ = \frac{1}{2}(p_1 + p_2)$$

$$(b) P(C_2|H) = \frac{P(H|C_2) \cdot P(C_2)}{P(H)}$$

$$= \frac{p_2 \cdot \frac{1}{2}}{\frac{1}{2}(p_1 + p_2)} = \frac{p_2}{p_1 + p_2}$$

11 (*The Birthday Problem*) A room contains 30 people whose birthdays are not known. Each person's birthday has an equal probability of being an element from the set $\{1, 2, \dots, 365\}$. These numbers represent the 365 days in a year (we will neglect 29th February). Consider the sample space S that contains all possible birthday outcomes for the 30 people (i.e., each outcome is a vector of 30 numbers).

- (a) How many outcomes are in the sample space S ? In other words, how many possible birthday combinations are there among 30 people?
- (b) Find an expression for the probability that two or more people in the room have a birthday on the same day? Show your work!
- (c) Now suppose that there are N (not necessarily 30) number of people in the room. What is the minimum value of N so that the probability that two or more people in the room have a birthday on the same day is higher than 0.5? Verify your answer via googling.....

(a) 30 people, each with 365 possible birthdays. $\Rightarrow (365)^{30}$ outcomes

(b) $A = \text{two or more people in the room have the same birthday}$ (30 people)
 $B = \text{No one have the same birthday in the room}$ (30 people)

$$\Rightarrow P(A) = 1 - P(B)$$

\rightarrow choose without replacement

$$P(B) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{336}{365}$$

$$\text{Therefore } P(A) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{336}{365} \approx 0.706$$

(c) Following similar procedure

$$1 - \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{(365-N+1)}{365} \geq 0.5$$

$$\Rightarrow N_{\min} = 23$$