1 Let A, B, and C be events. Use **Venn diagrams** to find expressions for the following:

(a) exactly one of the 3 events occurs

(b) exactly two of the events occur

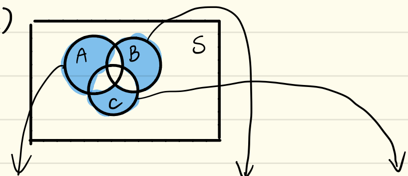
(c) one or more of the events occur

(d) two or more of the events occur

(e) all three events occur

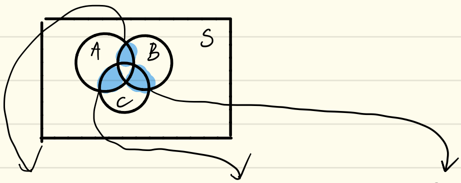
(f) none of the events occur

(a)



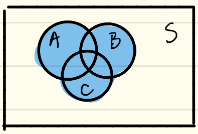
$(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)\cup(A^c\cap B^c\cap C)$

(b)



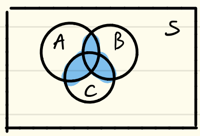
$(A\cap B\cap C^c)\cup(A\cap B^c\cap C)\cup(A^c\cap B\cap C)$

(c)



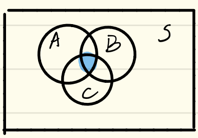
$A\cup B\cup C$

(d)



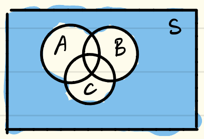
$(A\cap B)\cup(B\cap C)\cup(A\cap C)$

(e)



$A\cap B\cap C$

(f)



$(A\cup B\cup C)^c$

2 **(sequential experiments)** A coin is tossed until the first time that the same result appears twice in succession. To every outcome in which there are n tosses, assign the probability $\frac{1}{2^n}$.

(a) List elements of the sample space S.

(b) Check that indeed $P[S] = 1$.

(c) List the elements of the event $A =$ "the experiment ends before the fifth toss".

(d) What is the probability of the event A ?

(a) H: head T: tail

stop at 2 times : (HH), (TT)

stop at 3 times: (THH), (HTT)

stop at 4 times: (HTHH), (THTT)

⋮

$S = \{HH, TT, x_1x_2 \cdots x_{n-2}HH, y_1y_2 \cdots y_{n-2}TT\}$

where $x_1x_2 \cdots x_{n-2}$ is alternating ending with T

$y_1y_2 \cdots y_{n-2}$ is alternating ending with H

(b) For n tosses, the probability of each outcome is $\frac{1}{2^n}$

$$\Rightarrow P[S] = \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \cdots$$

$$= 2 \cdot \sum_{n=2}^{\infty} \frac{1}{2^n} = 2 \cdot \frac{\frac{1}{2^2} - 0}{1 - \frac{1}{2}} = 1$$

geometric sequence, with the constant ratio $\frac{1}{2}$

(c) $\{(HH), (TT), (THH), (HTT), (HTHH), (THTT)\}$

(d) $P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{7}{8}$

3 Consider two events E_1 and E_2.

(a) Suppose

$$P[\text{Event } E_1 \text{ doesn't occur}] = 0.9$$
$$P[\text{Event } E_2 \text{ doesn't occur}] = 0.8$$
$$P[\text{Event } E_1 \text{ or } E_2 \text{ occurs}] = 0.25$$

What is $P[E_1 \text{ and } E_2 \text{ both occur}]$?

(b) Suppose instead that

$$P[\text{Event } E_1 \text{ doesn't occur or event } E_2 \text{ doesn't occur}] = 0.7$$

Then what is $P[E_1 \text{ and } E_2 \text{ both occur}]$?

(a) $P(E_1^c) = 0.9$ $\quad P(E_2^c) = 0.8$ $\quad P(E_1 \cup E_2) = 0.25$ $\quad P(E_1 \cap E_2)$?

since $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$

$= (1 - P(E_1^c)) + (1 - P(E_2^c)) - P(E_1 \cup E_2)$

$= (1 - 0.9) + (1 - 0.8) - 0.25 = 0.05$

(b) $P(E_1^c \cup E_2^c) = 0.7$ $\quad P(E_1 \cap E_2)$?

According to De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$

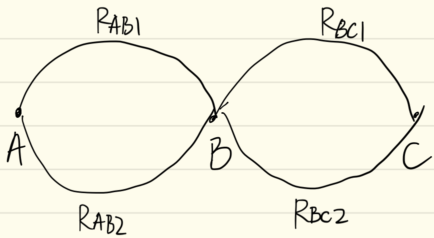
$\Rightarrow P(E_1^c \cup E_2^c) = P((E_1 \cap E_2)^c) = 0.7$

$\Rightarrow P(E_1 \cap E_2) = 1 - P((E_1 \cap E_2)^c) = 0.3$

4 **(reading the question; translating the question into a more familiar question)** Consider a communication network with three nodes A, B, and C, where the goal is to route a call from A to C. Note that:

1. Between A and B there are two routes that a call can use: R_{AB1} or R_{AB2}.
2. Between B and C there are two routes that a call can use: R_{BC1} or R_{BC2}.

Suppose that any of the routes $R_{AB1}, R_{AB2}, R_{BC1}, R_{BC2}$ can be blocked (due to congestion) and that the blocking occurs *independently* of each other. Suppose the blocking probability on each of the four routes is p. What is the probability that a call from A to C is blocked?



$P(A \to C \text{ blocked})$
$= 1 - P(A \to C \text{ unblocked})$

$A \to C$ unblocked $=$ $A \to B$ unblocked and $B \to C$ unblocked

$P(A \to B \text{ unblocked}) = 1 - P(A \to B \text{ blocked}) = 1 - P \cdot P$
$P(B \to C \text{ unblocked}) = 1 - P(B \to C \text{ blocked}) = 1 - P \cdot P$

$\Rightarrow P(A \to C \text{ blocked}) = 1 - P(A \to B \text{ unblocked}) \cdot P(B \to C \text{ unblocked})$

$= 1 - (1-P^2)(1-P^2) = 2P^2 - P^4$

5 **(sequential experiments)** A fair die is rolled and the number N_1 showing is observed. An integer N_2 is then selected at random from the range 1 to N_1 (each integer is equally likely with probability $1/N_1$).

(a) Find the sample space of outcomes.

(b) Find the set of outcomes corresponding to event "the die shows four"

(c) Find the set of outcomes corresponding to the event $\{N_2 = 3\}$

(d) Find the set of outcomes corresponding to the event $\{N_2 = 6\}$

(e) Compute the probability of the event $\{N_2 = 3\}$.

(f) Compute the probability of the event $\{N_1 = 4\}$ given $\{N_2 = 3\}$

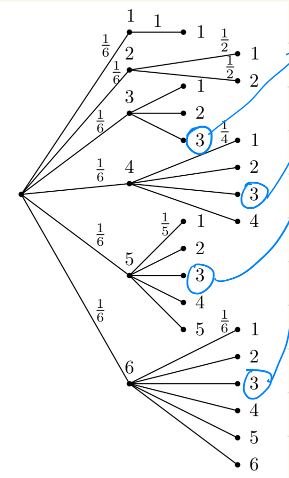
(g) Compute the probability of the event $\{N_1 = 4\}$ given $\{N_2 = 5\}$

(a) $S = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),$
$(5,3),(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

(b) Event (set of outcomes) that die shows 4 outcomes $= \{(4,1),(4,2),(4,3),(4,4)\}$.

(c) Event (set of outcomes) that $\{N_2 = 3\} = \{(3,3),(4,3),(5,3),(6,3)\}$.

(d) Event (set of outcomes) that $\{N_2 = 6\} = \{(6,6)\}$.



(e)

$$P(N_2=3) = P(N_2=3 \mid N_1=3)\cdot P(N_1=3) + P(N_2=3 \mid N_1=4)\cdot P(N_1=4) + P(N_2=3 \mid N_1=5)\cdot P(N_1=5) + P(N_2=3 \mid N_1=6)\cdot P(N_1=6)$$

$$= \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{6} + \frac{1}{5}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{19}{120}$$

(f) $$P(N_1=4 \mid N_2=3) = \frac{P(N_2=3 \mid N_1=4)\cdot P(N_1=4)}{P(N_2=3)} = \frac{\frac{1}{4}\cdot\frac{1}{6}}{\frac{19}{120}} = \frac{5}{19}$$

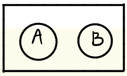
(g) $$P(N_1=4 \mid N_2=5) = \frac{\overset{=0}{P(N_2=5 \mid N_1=4)}\cdot P(N_1=4)}{P(N_2=5)} = 0$$

6 In an experiment, A, B, C, and D are events with probabilities:

$$P[A \cup B] = 3/4 \qquad P[A] = 3/8 \qquad P[C \cap D] = 1/4 \qquad P[C] = 1/2$$

A and B are disjoint, while events C and D are mutually independent.

(a) What is $P[A \cap B]$?
(b) What is $P[B]$?
(c) What is $P[A \cap B^C]$?
(d) What is $P[A \cup B^C]$?
(e) Is $P[A \cap B] = P[A]P[B]$ valid ?
(f) What is $P[D]$?
(g) What is $P[C \cup D]$?
(h) What is $P[C \cap D^C]$?
(i) What is $P[C \cup D^C]$?
(j) What is $P[C^C \cap D^C]$?

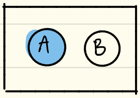


(a) $P(A \cap B) = 0$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$

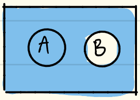
$\Rightarrow P(B) = P(A \cup B) - P(A) = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$

(c)



$P(A \cap B^c) = P(A) = \frac{3}{8}$

(d)



$P(A \cup B^c) = 1 - P(B) = \frac{5}{8}$

(e) NO $\quad P(A \cap B) = 0 \neq P(A) \cdot P(B)$

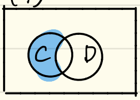
(f)

$P(C \cap D) = P(C) \cdot P(D) \Rightarrow P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1}{2}$

(g) $P(C \cup D) = P(C) + P(D) - P(C \cap D)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(h)



$P(C \cap D^c) = P(C) - P(C \cap D)$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(i)

$P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^c)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(j) De morgan's Law

$P(C^c \cap D^c) = P((C \cup D)^c) = 1 - P(C \cup D) = 1 - \frac{3}{4} = \frac{1}{4}$

7 This problem has three **unrelated** parts.

(a) If $P[A|B] = 0.3$, $P[A^C|B^C] = 0.4$, and $P[B] = 0.7$,

(i) What is $P[A|B^C]$?

(ii) What is $P[A]$?

(iii) What is $P[B|A]$?

(b) If $P[E] = 0.25$, $P[F|E] = 0.5$, $P[E|F] = 1/3$, what is $P[F]$?

(c) If $P[G] = P[H] = 2/3$, show that $P[G|H] \geq 1/2$.

(a)(i) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$

(ii) Total probability

$$P(A) = P(A|B^c)\cdot P(B^c) + P(A|B)\cdot P(B)$$
$$= 0.6\cdot(1-0.7) + 0.3\cdot 0.7 = 0.39$$

(iii) $$P[B|A] = \frac{P(A|B)\cdot P(B)}{P[A]}$$
$$= \frac{0.3\cdot 0.7}{0.39} = \frac{7}{13}$$

(b) $$P(E|F) = \frac{P(F|E)\cdot P(E)}{P(F)} \Rightarrow P(F) = \frac{P(F|E)\cdot P(E)}{P(E|F)}$$
$$= \frac{0.5\cdot 0.25}{\frac{1}{3}} = \frac{3}{8} = 0.375$$

(c) $$P[G|H] = \frac{P(G\cap H)}{P(H)} = \frac{P(G)+P(H)-P(G\cup H)}{P(H)}$$
$$= \frac{\frac{4}{3} - P(G\cup H)}{\frac{2}{3}}$$

since $P(G\cup H) \leq 1 \Rightarrow P(G|H) \geq \frac{\frac{4}{3}-1}{\frac{2}{3}} = \frac{1}{2}$

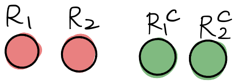
8 **(sequential experiments)** An urn contains 6 red balls and 4 green balls. A ball is drawn at random from the urn, and then another ball is drawn at random **without replacing** the first ball. Let R_1 denote the event that the first ball is red, and let R_2 denote the event that the second ball is red.

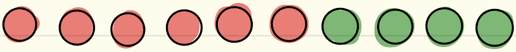
(a) What is $P[R_2|R_1]$?

(b) What is $P[R_2|R_1^C]$?

(c) What is $P[R_2]$?

(d) What is $P[R_1^C|R_2^C]$?





(a)

$$P(R_2|R_1) = \frac{5}{9}$$

(b) $P(R_2|R_1^c) = \frac{6}{9}$

(c) Theorem of total probability:

$$P(R_2) = P(R_2|R_1) \cdot P(R_1) + P(R_2|R_1^c) \cdot P(R_1^c)$$

$$= \frac{5}{9} \cdot \frac{6}{10} + \frac{6}{9} \cdot \frac{4}{10} = \frac{54}{90} = 0.6$$

(d) $P(R_1^c|R_2^c) = \dfrac{P(R_1^c) \cdot P(R_2^c|R_1^c)}{P(R_2^c)}$

$$= \frac{P(R_1^c)(1 - P(R_2|R_1^c))}{1 - P(R_2)} = \frac{\frac{4}{10}(1 - \frac{6}{9})}{1 - 0.6} = \frac{1}{3}$$

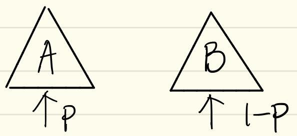
9 40 people are voting in an election where there are two possible candidates: Candidate A and Candidate B. Each voter must choose one of the two candidates, and decides to randomly select a candidate. Each voter chooses candidate A with probability p, and does so independently of the other voters. Express your answer for parts (a)-(d) in terms of mathematical expressions containing p.

(a) What is the probability that Candidate B obtains exactly 30 votes?

(b) What is the probability that Candidate B obtains 30 or more votes?

(c) Assume that there is only one voting machine, and the 40 voters are given a number from 1 to 40 which determines the order of voting. Let X = the number of the first person to vote for Candidate B (define $X = 0$ if no one votes for Candidate B). Find $P[X \leq 4]$.

(d) Let E be the event that Candidate A receives exactly 25 votes. Find $P[E|X \leq 4]$.



40 people

(a) $P(30 \text{ votes for } B) = \binom{40}{30}(1-p)^{30}\cdot p^{10} = \binom{40}{10}(1-p)^{30}\cdot p^{10}$

(b) $P(30 \text{ or more votes for } B) = P(30, 31, 32, \cdots 40 \text{ votes for } B)$

$= \binom{40}{30}(1-p)^{30}\cdot p^{10} + \binom{40}{31}(1-p)^{31}p^{9} + \cdots + \binom{40}{40}(1-p)^{40}$

$= \sum_{i=30}^{40}\binom{40}{i}(1-p)^{i}\cdot p^{40-i}$

(c) $P[X \leq 4] = P[X=0] + P[X=1] + P[X=2] + P[X=3] + P[X=4]$

$P[X=0] = P[\text{no one vote } B] = p^{40}$

$P[X=1] = P[\text{number 1 is the first person vote } B] = (1-p)$

$P[X=2] = P[\text{number 2 is the } \cdots] = p(1-p)$

$P[X=3] = P[\text{number 3 is the } \cdots] = p\cdot p(1-p)$

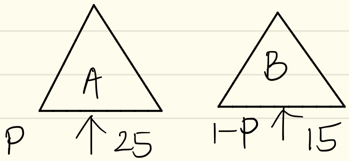
$P(X=4] = P[\text{number 4 is the } \cdots] = p\cdot p\cdot p(1-p)$

$= p^{40} + (1-p) + p(1-p) + p^2(1-p) + p^3(1-p)$

$= p^{40} + (1-p)(1+p+p^2+p^3)$

$= p^{40} + (1-p)\frac{1-p^3\cdot p}{1-p} = 1-p^4+p^{40}$

(d) Let E be the event that Candidate A receives exactly 25 votes. Find $P[E|X \leq 4]$.



$$P(E|X\leq 4) = \frac{P(E\cap(X\leq 4))}{P(X\leq 4)}$$ ← already know in (c)

$$= \frac{P(E\cap(X=0)) + P(E\cap(X=1)) + P(E\cap(X=2)) + P(E\cap(X=3)) + P(E\cap(X=4))}{P(X\leq 4)}$$

$P(E\cap(X=0)) = 0$ since $E\cap(X=0) = \emptyset$

$P(E\cap(X=1))$ = the first person vote B, rest of 39 people (14 vote B)
number 1 is

$$\Rightarrow P(E\cap(X=1)) = (1-p)\cdot\binom{39}{14}\cdot p^{25}\cdot(1-p)^{14} = \binom{39}{14}p^{25}(1-p)^{15}$$

$P(E\cap(X=2)$ = the first person vote B, rest of 38 people (14 vote B)
number 2 is

$$\Rightarrow P(E\cap(X=2)) = (1-p)\cdot\binom{38}{14}\cdot p^{25}\cdot(1-p)^{14} = \binom{38}{14}p^{25}(1-p)^{15}$$

$P(E\cap(X=))$ = the first person vote B, rest of 37 people (14 vote B)
number 3

$$\Rightarrow P(E\cap(X=3)) = (1-p)\cdot\binom{37}{14}\cdot p^{25}\cdot(1-p)^{14} = \binom{37}{14}p^{25}(1-p)^{15}$$

$P(E\cap(X=))$ = the first person vote B, rest of 36 people (14 vote B)
number 4

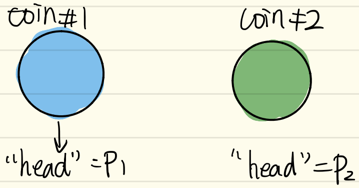
$$\Rightarrow P(E\cap(X=4)) = (1-p)\cdot\binom{36}{14}\cdot p^{25}\cdot(1-p)^{14} = \binom{36}{14}p^{25}(1-p)^{15}$$

$$\Rightarrow P(E|X\leq 4) = \frac{p^{25}(1-p)^{15}}{1-p^4+p^{40}}\cdot\left(\binom{36}{14}+\binom{37}{14}+\binom{38}{14}+\binom{39}{14}\right)$$

10 An experiment is defined as follows:
One of two coins is first selected at random (50% probability for choosing each) and then is tossed. Coin #1 comes up 'Heads' with probability p_1, while Coin #2 is 'Heads' with probability p_2. Express your answers below in terms of p_1 and p_2.

(a) **(prediction)** What is the probability that the outcome of the experiment is 'Heads'?

(b) **(inference)** What is the probability that Coin #2 was used given that a 'Heads' occurred?



$P(H | C_1) = P_1$

$P(H | C_2) = P_2$

(a) $P(H) = P(H|C_1)\cdot P(C_1) + P(H|C_2)\cdot P(C_2) = \frac{1}{2}P_1 + \frac{1}{2}P_2$

$= \frac{1}{2}(P_1 + P_2)$

(b) $P(C_2 | H) = \dfrac{P(H|C_2)\cdot P(C_2)}{P(H)}$

$= \dfrac{P_2 \cdot \frac{1}{2}}{\frac{1}{2}(P_1+P_2)} = \dfrac{P_2}{P_1+P_2}$

11 (***The Birthday Problem***) A room contains 30 people whose birthdays are not known. Each person's birthday has an equal probability of being an element from the set $\{1, 2, \ldots, 365\}$. These numbers represent the 365 days in a year (we will neglect 29th February). Consider the sample space S that contains all possible birthday outcomes for the 30 people (i.e., each outcome is a vector of 30 numbers).

(a) How many outcomes are in the sample space S? In other words, how many possible birthday combinations are there among 30 people?

(b) Find an expression for the probability that two or more people in the room have a birthday on the same day? Show your work!

(c) Now suppose that there are N (not necessarily 30) number of people in the room. What is the minimum value of N so that the probability that two or more people in the room have a birthday on the same day is higher than 0.5? Verify your answer via googling.....

(a) 30 people, each with 365 possible birthdays. $\Rightarrow (365)^{30}$ outcomes

(b) A = two or more people in the room have the same birthday (30 people)

B = No one have the same birthday in the room (30 people)

$\Rightarrow P(A) = 1 - P(B)$

$\rightarrow$ choose without replacement

$$P(B) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{336}{365}$$

Therefore $P(A) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{336}{365} \approx 0.706$

(c) Following similar procedure

$$1 - \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{(365-N+1)}{365} \geq 0.5$$

$\Rightarrow N_{min} = 23$