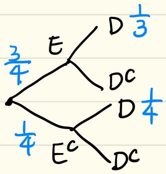
Solutions for Quiz on Week 4

Problem 1 *(Total Probability)* 2 points

Due to an Internet configuration error, packets sent from Sydney to Adelaide are routed via Melbourne with probability 3/4. If a packet is routed via Melbourne, it has conditional probability 1/3 of being dropped. If a packet is not routed via Melbourne, it has conditional probability 1/4 of being dropped. Find the probability that a packet is dropped.

E : routed via Melbourne D : packet is dropped

Given $P(E) = \frac{3}{4}$ $P(D|E) = \frac{1}{3}$, $P(D|E^c) = \frac{1}{4}$



$$P(D) = P(D|E)\cdot P(E) + P(D|E^c)\cdot P(E^c)$$

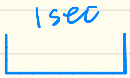
$$= \frac{1}{3}\cdot\frac{3}{4} + \frac{1}{4}\cdot\frac{1}{4} = \frac{5}{16}$$

Problem 2 *(Discrete RV)* 2+2=4 points

Let time be divided into slots of 1sec duration each. During every slot, the probability of "1 packet arriving" is 0.1, and the probability of "no packets arriving" is 0.9. Assume that packets arrive independently.

(a) In 10 seconds, what is the probability of 6 arrivals?

(b) In 1000 seconds, what is the probability of 100 arrivals?



Each second is a trial, a packet may arrive or it may not

$\Rightarrow \sim$ Bernoulli (0.1)

(a) 10 seconds $\rightarrow$ 10 trials $\rightarrow X \sim$ Binomial $(10, 0.1)$

$$P[X=6] = \binom{10}{6}(0.1)^6 \cdot (1-0.1)^4 \approx 1.38 \times 10^{-4}$$

(b) 1000 seconds $\rightarrow$ 1000 trials $\rightarrow X \sim$ Binomial $(1000, 0.1)$

$$P[X=100] = \binom{1000}{100}(0.1)^{100}(1-0.1)^{900} \approx 0.042$$

Problem 3 *(Continuous RV)* 2+2=4 points

Jimmy has decided to take a blood test. Assume that the waiting time for the result of the test is **exponentially distributed** with an average waiting time of 2 hours. Jimmy takes the blood test at 1pm.

(a) What is the probability that Jimmy waits less than 30 minutes?

(b) What is the probability that Jimmy waits between 1 and 3 hours?

let X: waiting time for the result $X \sim \exp(\lambda)$ $F_X(x) = 1 - e^{-\lambda x}$ $(x \geq 0)$

Given $E[X] = 2$ hours $\Rightarrow \lambda = \frac{1}{E[X]} = \frac{1}{2}$

(a) $P[X \leq 30 \text{ mins}] = P[X \leq \frac{1}{2} \text{ hours}] = F_X(0.5) = 1 - e^{-0.5\lambda}$

≈ 0.22

(b) $P[1 \leq X \leq 3] = P[X \leq 3] - P[X \leq 1]$

$= F_X(3) - F_X(1) = (1 - e^{-\frac{3}{2}}) - (1 - e^{-\frac{1}{2}})$

≈ 0.38

1 Consider a **geometric**$(\frac{1}{4})$ random variable (RV). Find an expression for its CDF.

$X \sim \text{Geometric}(\frac{1}{4})$, $P(\text{success}) = \frac{1}{4}$ $P(\text{failure}) = \frac{3}{4}$

count the number of times until first occurs

① $F_X(x) = P(X \le k) = 1 - P(X > k)$

$\downarrow$

first k trials are all failures

$= 1 - (\frac{3}{4})^k$

② $F_X(X) = P(X \le k) = P(X=1) + P(X=2) + \cdots + P(X=k)$

$= \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \cdots (\frac{3}{4})^{k-1} \cdot \frac{1}{4}$

$= \sum_{i=1}^{k} (\frac{3}{4})^{i-1} \cdot \frac{1}{4}$

$= \frac{1}{4} \sum_{i=1}^{k} (\frac{3}{4})^{i-1} = \frac{1}{4} \frac{1-(\frac{3}{4})^k}{1-\frac{3}{4}} = 1 - (\frac{3}{4})^k$

2 **(reading the question; translating the question into a more familiar question)** There are three servers, S_1, S_2, S_3, in a computer network. Each computer that wishes to use the service provided by these servers randomly and independently selects a server. The probability of choosing a particular server is the same for each computer, namely:

$$P(S_1) = 0.2$$
$$P(S_2) = 0.3$$
$$P(S_3) = 0.5$$

Suppose that the number of computers that log on is 5.

(a) What is the probability that no computer connects to server 1?

(b) What is the probability that one computer connects to S_1, one computer to S_2, and three connect to S_3?

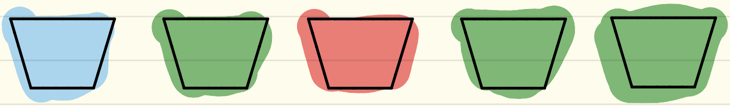
(a) either "S1" or "not S1" (reformulate to the simple situation)

Define X : the number of computers that connect to S1

$X \sim \text{Binomial}(5, 0.2)$

$$P[X=0] = (1-0.2)^5 = 0.8^5 \approx 0.328$$

(b) multinomial experiment (more than two outcomes)



Define X_j: the number of computers that connect to S_j

$$P[X_1=1, X_2=1, X_3=3] = \binom{5}{1,1,3} 0.2^1 0.3^1 0.5^3$$

$$= \frac{5!}{1!1!3!} 0.2^1 0.3^1 0.5^3 = 0.15$$

3 **(Single Server; reading the question; translating the question into a more familiar question)** Voice calls arrive at a private branch exchange (PBX) in such a way that the number of arrivals in 10 seconds is **Poisson**(5). Suppose the exchange starts with zero calls, and has 15 available circuits (one circuit per voice call). If 15 circuits are in use, then the exchange must block new arrivals.

(a) What is the average rate of calls arriving to the PBX in calls per second? What is the average rate in calls per minute?

(b) Compute the probability that the exchange has to block some voice calls during the first 10 second period. Assume that connected calls last much longer than 10 seconds.

(a) N = the number of calls arrival in 10 seconds

$N \sim \text{poisson}(5)$

$\Rightarrow$ the average number is 5 calls in 10 seconds

$\Rightarrow$ 0.5 calls per second

$\Rightarrow$ 30 calls per minute

(b) If $N \geq 16$, the voice call will be blocked (since we only have 15 available circuits)

$P[N \geq 16] = 1 - P[N \leq 15]$

$= 1 - [P[N=0] + P[N=1] + \cdots + P[N=15]]$

Since $P[N=k] = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{5^k}{k!} e^{-5}$

$= 1 - \sum_{k=0}^{15} e^{-5} \frac{5^k}{k!}$

4 Jimmy has decided to take a blood test. Assume that the waiting time for the result of the test is **exponentially distributed** with an average waiting time of 2 hours. Jimmy takes the blood test at 1pm.

(a) What is the probability that Jimmy waits less than 30 minutes?

(b) What is the probability that Jimmy waits between 1 and 3 hours?

(c) It is 5pm and Jimmy still hasn't gotten his result yet. Find the probability that Jimmy waits less than 30 more minutes. Compare your result to part (a).

(c) Given that Jimmy has already waited 4 hour,

$$P[X \le 4.5 \mid X \geq 4] = \frac{P((X \geq 4) \cap (X \le 4.5))}{P(X \geq 4)}$$

$$= \frac{P(4 \le X \le 4.5)}{P(X \geq 4)}$$

$$= \frac{F_X(4.5) - F_X(4)}{1 - F_X(4)} = \frac{(1-e^{-\frac{4.5}{2}}) - (1-e^{-\frac{4}{2}})}{1-(1-e^{-\frac{4}{2}})}$$

$$\approx 0.22$$

$\Rightarrow$ exactly same as the result in (a), since exponential rv are memoryless

5 Given a **Gaussian** random variable $X \sim \mathcal{N}(0, \sigma^2)$, evaluate the following probabilities in terms of the $Q(\cdot)$ function and σ. Please simplify all expressions so that there are **no negative-valued arguments** within the $Q(\cdot)$ function.

(a) $P(X \leq 7)$

(b) $P(X > 5)$

(c) $P(-7 < X \leq 5)$

(d) $P(|X-2| \leq 6)$

(e) $P(1 \leq |X-2| \leq 6)$

$$\frac{X-\mu}{\sigma} = \frac{X}{\sigma} \sim N(0,1)$$

$$Q(x) = P[X \geq x]$$

$$Q(-x) = 1 - Q(x)$$

(a) $P(X \leq 7) = 1 - P(X > 7) = 1 - Q\left(\frac{7-0}{\sigma}\right)$

(b) $P(X > 5) = Q\left(\frac{5}{\sigma}\right)$

(c) $P(-7 < X \leq 5) = P[X > -7] - P[X > 5] = Q\left(\frac{-7}{\sigma}\right) - Q\left(\frac{5}{\sigma}\right)$

$= 1 - Q\left(\frac{7}{\sigma}\right) - Q\left(\frac{5}{\sigma}\right)$

(d) $P(|X-2| \leq 6) = P[-6 \leq X-2 \leq 6]$

$= P(-4 \leq X \leq 8) = 1 - Q\left(\frac{4}{\sigma}\right) - Q\left(\frac{8}{\sigma}\right)$

(e) $P(1 \leq |X-2| \leq 6) = P[1 \leq X-2 \leq 6] + P[-1 \geq X-2 \geq -6]$

$= P[(-4 \leq X \leq 1) \cup (3 \leq X \leq 8)]$

$= 1 - Q\left(\frac{4}{\sigma}\right) - Q\left(\frac{1}{\sigma}\right) + Q\left(\frac{3}{\sigma}\right) - Q\left(\frac{8}{\sigma}\right)$

6 Repeat part (a) of the previous question for $X \sim \mathcal{N}(3, \sigma^2)$.

(a) $P(X \leq 7) = 1 - P(X > 7) = 1 - Q\left(\frac{7-3}{\sigma}\right)$

$= 1 - Q\left(\frac{4}{\sigma}\right)$

7 Let X be a **Poisson** random variable (rv) with an unknown parameter. You are then told that $P(X = 0) = \frac{1}{4}$. What is $P(X = 3)$?

Assume $X \sim \text{Poisson}(\lambda)$, $P[X=k] = \frac{\lambda^k}{k!} e^{-\lambda}$

$$P[X=0] = \frac{\lambda^0}{0!} e^{-\lambda} = \frac{1}{4} \Rightarrow \lambda = -\ln\frac{1}{4} = \ln 4$$

$$\Rightarrow P[X=3] = \frac{\lambda^3}{3!} e^{-\lambda}$$

$$= \frac{(\ln 4)^3}{3!} \cdot \frac{1}{4} = \frac{(\ln 4)^3}{24} \approx 0.111$$

8 Let X be a **Binomial**(7000,0.002) rv. Estimate $P(20 \le X \le 23)$.

since n is quite large and p is quite small, we can use poisson approximation to calculate the probability

$\lambda = np = 7000 \times 0.002 = 14$ $\quad Y \sim \text{poisson}(14)$

$$P(20 \le X \le 23) \approx P(20 \le Y \le 23)$$

$$= P(Y=20) + P(Y=21) + P(Y=22) + P(Y=23)$$

$$= e^{-14}\left(\frac{14^{20}}{20!} + \frac{14^{21}}{21!} + \frac{14^{22}}{22!} + \frac{14^{23}}{23!}\right) = 0.067$$

9 Consider the function

$$g(x) = \begin{cases} \frac{2}{3} - \frac{2x}{9}, & 0 \leq x \leq 3 \\ 0, & \text{else} \end{cases}$$

Check whether or not $g(\cdot)$ is a valid PDF. If it is, find the cumulative distribution function (CDF) and plot it.

① check $g(x)$ => check whether its integral equals to 1

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_0^3 \left(\frac{2}{3} - \frac{2x}{9}\right) dx$$

$$= \left[\frac{2}{3}x - \frac{x^2}{9}\right]_0^3 = 1$$

② For $x<0$, $g(x)=0$, $\Rightarrow F_X(x) = 0$

For $0 \leq x \leq 3$, $F_X(x) = \int_{-\infty}^{x} g(v)\,dv$

$$= \int_0^x \left(\frac{2}{3} - \frac{2v}{9}\right) dv$$

$$= \left[\frac{2}{3}v - \frac{v^2}{9}\right]_0^x = \frac{2}{3}x - \frac{x^2}{9}$$

For $x>3$, $F_X(x) = 1$

In summary,

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2x}{3} - \frac{x^2}{9} & 0 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$