Skolkovo Institute of Science and Technology

Exact Solutions of Some Nonconvex Quadratic Optimization Problems via SDP and SOCP Relaxations

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Team #2

Quadratic optimization problem (QOP)

$$
\left.\begin{array}{ll}
\text{minimize} & \boldsymbol{x}^T \boldsymbol{Q}_0 \boldsymbol{x} + 2\boldsymbol{q}_0^T \boldsymbol{x} \\
\text{subject to} & \boldsymbol{x}^T \boldsymbol{Q}_p \boldsymbol{x} + 2\boldsymbol{q}_p^T \boldsymbol{x} + \gamma_p \leq 0 \ (1 \leq p \leq m), \ \ \boldsymbol{x}^2 \in \mathcal{F}.
\end{array}\right\} \tag{1}
$$

Homogeneous quadratic optimization problem (HQOP):

$$
\boldsymbol{M}_p = \begin{pmatrix} \gamma_p, & \boldsymbol{q}_p^T \\ \boldsymbol{q}_p, & \boldsymbol{Q}_p \end{pmatrix} \ (0 \leq p \leq m) \ \ \text{and} \ \boldsymbol{M}_{m+1} = \begin{pmatrix} 1, & \boldsymbol{0}^T \\ \boldsymbol{0}, & \boldsymbol{O} \end{pmatrix}.
$$

$$
\left.\begin{array}{ll}
\text{minimize} & (x_0; \boldsymbol{x})^T \boldsymbol{M}_0 (x_0; \boldsymbol{x}) \\
\text{subject to} & (x_0; \boldsymbol{x})^T \boldsymbol{M}_p (x_0; \boldsymbol{x})^T \leq 0 \ (p = 1, \ldots, m), \\
& (x_0; \boldsymbol{x})^T \boldsymbol{M}_{m+1} (x_0; \boldsymbol{x})^T = 1, \ \ \boldsymbol{x}^2 \in \mathcal{F}.
\end{array}\right\} \tag{2}
$$

Notations: $Q_p \in S^n, q_p \epsilon \mathbb{R}^n, \gamma_p \in \mathbb{R} \, (0 \leq p \leq m)$

SDP (semidefinite programming) relaxation of QOP:

$$\left.\begin{array}{ll}\text{minimize} & \boldsymbol{x}^T \boldsymbol{Q}_0 \boldsymbol{x} + 2\boldsymbol{q}_0^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{Q}_p \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_p^T \boldsymbol{x} + \gamma_p \leq 0 \ (1 \leq p \leq m), \\ & \text{diag}\,(\bar{\boldsymbol{X}}) \in \mathcal{F}, \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathbb{S}_+^{1+n}. \end{array}\right\}$$

Using M_p, we rewrite SDP relaxation as

$$\left.\begin{array}{ll}\text{minimize} & \boldsymbol{M}_0 \bullet \boldsymbol{X} \\ \text{subject to} & \boldsymbol{M}_p \bullet \boldsymbol{X} \leq 0 \ (p = 1, \ldots, m), \ X_{00} = \boldsymbol{M}_{m+1} \bullet \boldsymbol{X} = 1, \\ & \text{diag}\,(\bar{\boldsymbol{X}}) \in \mathcal{F}, \ \boldsymbol{X} = \begin{pmatrix} X_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathbb{S}_+^{1+n}. \end{array}\right\} \qquad (3)$$

SOCP (second order cone programming)

$$\left.\begin{array}{ll}\text{minimize} & \boldsymbol{M}_0 \bullet \boldsymbol{X} \\ \text{subject to} & \boldsymbol{M}_p \bullet \boldsymbol{X} \leq 0 \ (1 \leq p \leq m), \ X_{00} = \boldsymbol{M}_{m+1} \bullet \boldsymbol{X} = 1, \\ & (X_{11}, X_{22}, \ldots, X_{nn})^T \in \mathcal{F}, \\ & \left\| \begin{pmatrix} X_{kk} - X_{jj} \\ 2X_{kj} \end{pmatrix} \right\| \leq X_{kk} + X_{jj} \ ((k, j) \in \Lambda). \end{array}\right\} \qquad (11)$$

where

$$\Lambda = \{(k, j) \ : \ 0 \leq k < j \leq n \ \text{ and } [\boldsymbol{M}_p]_{kj} \neq 0 \text{ for some } p\}.$$

Toolbox

	Article	Our project
Software	MATLAB	Python3
Library	SDPT3	CVXPY
Description	SDPT3 is a Matlab package for solving convex optimization problems involving linear equations and inequalities, second-order cone constraints, and semidefinite constraints (linear matrix inequalities).	CVXPY is a Python-embedded modeling language for convex optimization problems. It allows to express your problem in a natural way that follows the math, rather than in the restrictive standard form required by solvers.
Solver	Sparse Cholesky solver	MOSEK

Choosing a solver: CVXPY

	LP	QP	SOCP	SDP	EXP	MIP
CBC	X					X
GLPK	X					
GLPK_MI	X					X
OSQP	X	X				
CPLEX	X	X	X			X
ECOS	X	X	X		X	
ECOS_BB	X	X	X		X	X
GUROBI	X	X	X			X
MOSEK	X	X	X	X	X	X*
CVXOPT	X	X	X	X		
SCS	X	X	X	X	X	

Comparison of SDP and SOCP solutions

	n	m	density	Time_SDP	value_SDP	Time_SOCP	value_SOCP
0	50	50	0.1	2.621127	-31769.431842	0.410862	-31769.431181
1	70	50	0.1	14.175238	-138963.764597	1.712452	-138963.763981
2	100	50	0.1	98.355543	-211397.366271	3.804123	-211397.365277

Reproducibility of the results

5 tries with n=100 and m=50:

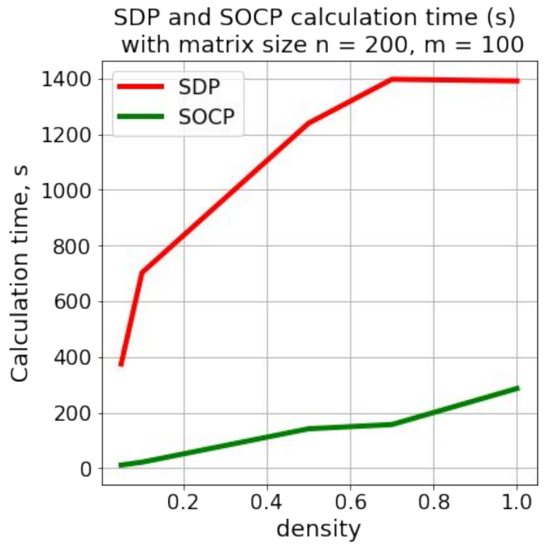
	n	m	density	Time_SDP	Time_SOCP
0	100	50	0.1	93.350900	3.857503
1	100	50	0.1	96.636242	3.724648
2	100	50	0.1	85.957319	3.734380
3	100	50	0.1	88.481009	3.749798
4	100	50	0.1	91.458841	3.691620

Results

Comparing performance of SDP and SOCP:

1. QOPs with <u>n=200</u>, <u>m=100</u> and varying *sparsity*

	Our results			Paper (table 3)		
density	Time SDP	Time SOCP	Ratio	Time SDP	Time SOCP	Ratio
0,05	374,76	11,69	32,05	198,80	15,10	13,17
0,10	702,42	22,12	31,76	290,40	28,80	10,08
0,50	1239,14	142,04	8,72	1430,70	173,10	8,27
0,70	1396,74	157,27	8,88	1858,30	212,40	8,75
1,00	1390,85	286,56	4,85	2282,80	342,50	6,67

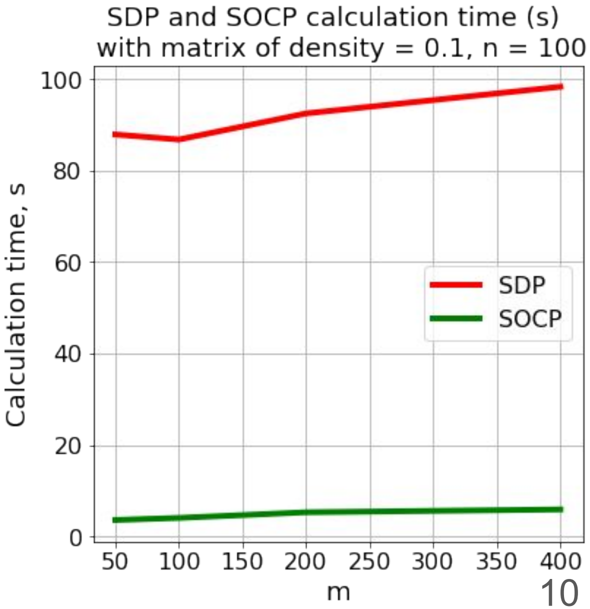


Results

Comparing performance of SDP and SOCP:

2. QOPs with n=100, density=0,1 and varying *m*

	Our results			Paper (table 4)		
m	Time SDP	Time SOCP	Ratio	Time SDP	Time SOCP	Ratio
50	87,91	3,62	24,26	18,30	2,30	7,96
100	86,80	4,08	21,28	42,10	6,50	6,48
200	92,51	5,31	17,43	125,40	17,70	7,08
400	98,34	5,93	16,58	733,10	95,60	7,67

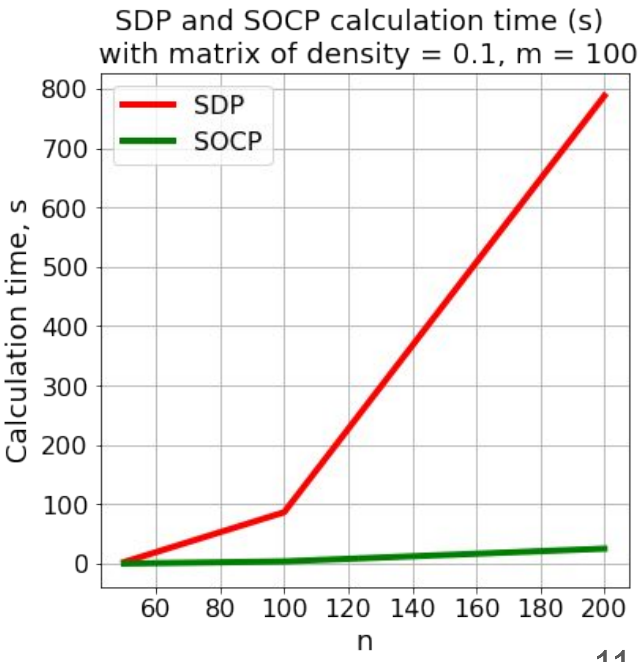


Results

Comparing performance of SDP and SOCP:

3. QOPs with m=100, density=0,1 and varying *n*

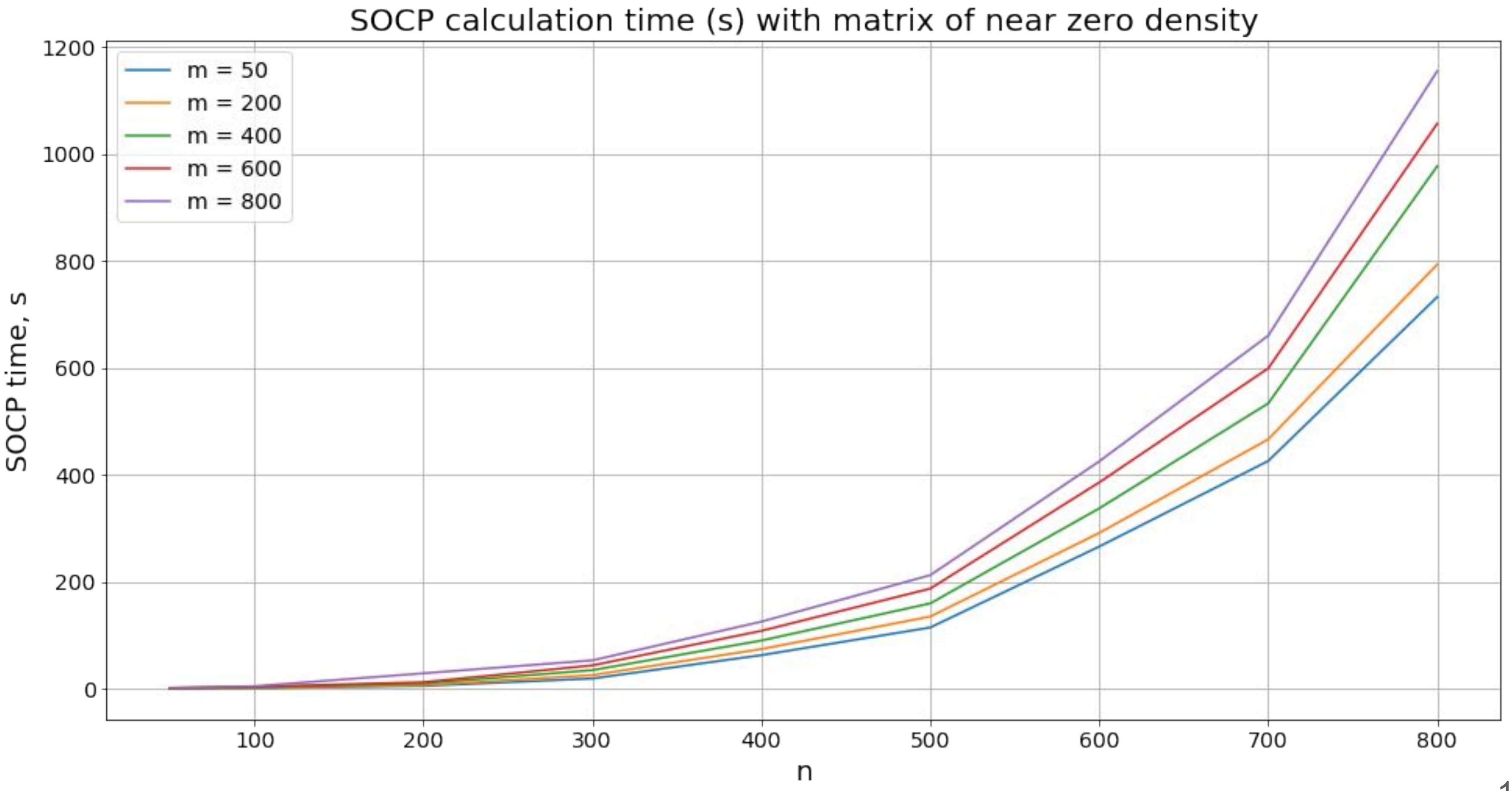
	Our results			Paper (table 5)		
n	Time SDP	Time SOCP	Ratio	Time SDP	Time SOCP	Ratio
50	2,85	0,40	7,17	12,60	1,30	9,69
100	86,80	4,08	21,28	42,10	6,50	6,48
200	787,80	25,57	30,82	290,40	28,80	10,08
400				3910,40	236,70	16,52



Results

4. Diagonal QOPs with varying *n* and *m*

		Our results			Paper (table 6)		
n	m	Time SDP	Time, SOCP	Ratio	Time SDP	Time SOCP	Ratio
100	50	48,91	1,09	44,87	25,70	3,80	6,76
100	100	47,10	1,27	37,23	42,10	10,40	4,05
200	100	2100,38	5,38	390,65	247,90	17,50	14,17
200	200	2130,48	6,60	322,88	376,70	43,40	8,68



SOCP calculation time (s) with matrix of near zero density
m = 50
m = 200
m = 400
m = 600
m = 800
SOCP time, s
n
0
200
400
600
800
1000
1200
100
200
300
400
500
600
700
800

Conclusions

1. Numerically, the proposed SOCP relaxation has proven to be more efficient than the SDP relaxation in all of the test problems in the previous section, and we have concluded that the proposed SOCP relaxation is a much better approach to the class of QOPs with almost OD-nonpositive coefficient matrices than the SDP relaxation.
2. Despite the fact that trend of time ratio between SDP and SOCP approximately the same as in the original paper, the absolute values of calculation time are sufficiently lower for SOCP, whereas for SDP it is varying.
3. For high dimensionality problems SDP relaxation is much more time consuming comparing to the article.
4. In case of diagonal QOPs we obtained much better results for SOCP than in the paper.

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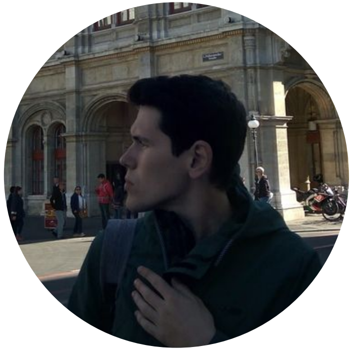

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